Project 1 TMA4265

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Exercise 1

a)
It does not matter how many days a susceptible individual has lasted without getting infected, the probability of getting infected the next day is still going to be β. The same can be said for an infected individual, the probability of recovering and gaining an immunity is always going to be γ. Finally, a recovered and immune individual is always going to stay in that state, no matter what happened previously. This means that X_n satisfies the Markov property for all time points n. That is,

$$Pr\{X_n = j | X_o = i_0, ..., X_{n-1} = i_{n-1}, X_n = i\} = Pr\{X_n = j | X_n = i\}$$

For all states $i_0, ..., i_n, i, j$. Since X_n is a stochastic process with a finite state space (0, 1 and 2) that satisfies the Markov property, X_n is per definition a discrete-time Markov chain.
A discrete-time Markov chain can uniquely be modeled by a transition probability matrix. This matrix, lets denote it P, contains all the different transition probabilities for the Markov chain. The probability of transitioning from state i to state j is the element in the ith row and the jth column of P. P must also have the property that all rows sum to one. For X_n, it is clear that

$$P = \begin{pmatrix} 1-\beta & \beta & 0 \\ 0 & 1-\gamma & \gamma \\ 0 & 0 & 1 \end{pmatrix}$$

is the transition probability matrix.

b)
From state S there are two possible ways to transition. Either you stay in S, or you transition to I. Then, form state I, you either stay in I or transition to state R. However, once you reach R, there is no escape, you will stay there forever. We say that R is an absorbing state. A Markov chain is irreducible if from whatever state, there is a way to transition to any state in the chain. This is not the case here, because there is no way to get from state I to S, nor from state R to S or I. Therefore, X_n is reducible, and can be reduced to the following equivalence classes: $\{S\}, \{I\}$ and $\{R\}$. Both state S and I are transient, because there is a positive probability that you end up in state R, thus never returning to the original state (for state S its sufficient to reach state I). State R is recurrent, you are even guaranteed to return in one step. For each of the states there is always a chance you stay in that state. The period of each of the states is therefore one.

c)
We want to model the first instance where a susceptible individual gets infected. Each day is independent of the next and there are only two outcomes: either you get infected or you do not. We recognise this as the geometric distribution with parameter β. The expected value of a geometric distribution is $\frac{1}{\beta}$. The expected time until a susceptible individual becomes infected is therefore equal to $\frac{1}{\beta} = \frac{1}{0.05} = 20$. The time until an infected individual becomes recovered, can also be modeled by a geometric distribution, with γ as the parameter. The expected time is $\frac{1}{\gamma} = \frac{1}{0.20} = 5$.

d)
We simulated the Markov process X_n in Python, and the values are close enough to their expected values.

The average time in state 0 is: 20.015
The average time in state 1 is: 5.033

e)

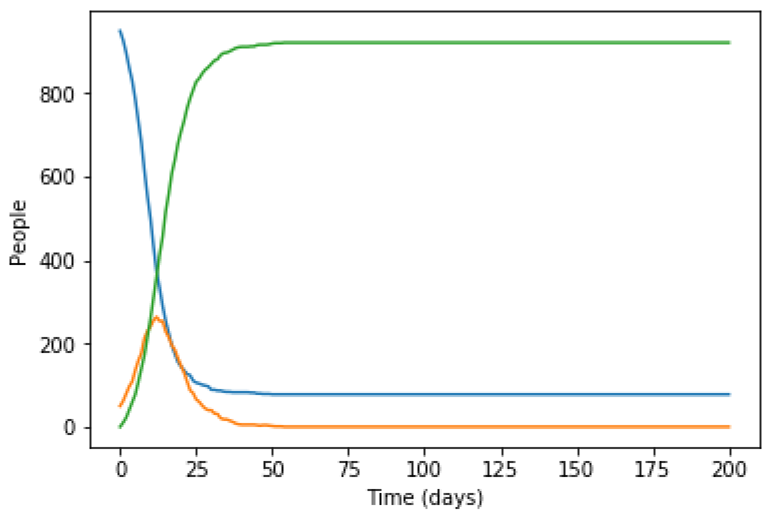

Figure 1: Here we see one realization of Y_n. The blue graph is the number of susceptible, the orange graph is the number of infected and the green graph is the number of immune. Note that once there are no infected people remaining, no more transitions will occur.

f)
We estimated the values of $E[max\{I_0, I_1, ..., I_{200}\}]$ and $E[min\{\text{arg max}_{\{n \leq 200\}}\{I_n\}\}]$.,This is the output we get from running our code in Python:

Max number of infected: 277.367
Time until max: 10.412

Exercise 2

a)
We know from the lectures that

$$X(t) - X(0) \sim Poisson(\lambda * t)$$

So

$$P(X(59) - X(0) \geq 100) = P\left(Z \geq \frac{100 - 1.5 * 59}{\sqrt{1.5 * 59}}\right) = P(Z \geq 1.222) = 0.1109$$

Because of the relatively high number of samples (days), we felt that our transformation to a standard normal variable was justified. We verified this by simulating $X(t)$ 1000 times in python. The following output agrees with our calculation:

The percentage of iterations larger than 100 is: 0.112

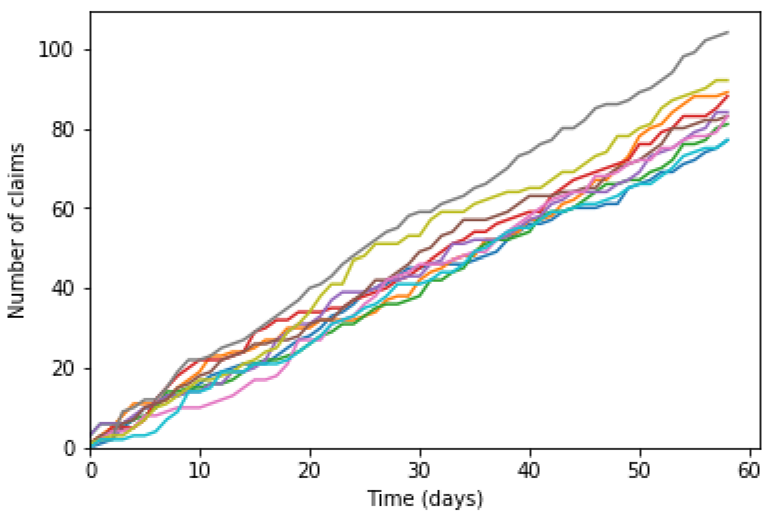

Figure 2: Ten realizations of $X(t)$

b)
We want to calculate E[Z(t)]. We start by using the law of total expectation.

$$E[Z(t)] = E\left[\sum_{i=1}^{X(t)} C_i\right] = E\left[E\left[\sum_{i=1}^{X(t)} C_i \middle| X(t)\right]\right] \tag{1}$$

C_i and X(t) are independent, so

$$E\left[\sum_{i=1}^{X(t)} C_i \middle| X(t)\right] = E\left[\sum_{i=1}^{X(t)} C_i\right] = \sum_{i=1}^{X(t)} E[C_i] = \sum_{i=1}^{X(t)} \frac{1}{\beta} = \frac{1}{\beta} X(t) \tag{2}$$

If we insert this into (1), we get

$$E[Z(t)] = E\left[\frac{1}{\beta} X(t)\right] = \frac{1}{\beta} \lambda * t = \frac{1}{10} 1.5 * 59 = 8.85$$

To calculate Var(Z(t)), we use the law of total variance.

$$Var(Z(t)) = E\left[Var\left(\sum_{i=1}^{X(t)} C_i \middle| X(t)\right)\right] + Var\left(E\left[\sum_{i=1}^{X(t)} C_i \middle| X(t)\right]\right) \tag{3}$$

We then calculate

$$Var\left(\sum_{i=1}^{X(t)} C_i \middle| X(t)\right) = Var\left(\sum_{i=1}^{X(t)} C_i\right) = \sum_{i=1}^{X(t)} Var(C_i) = \sum_{i=1}^{X(t)} \frac{1}{\beta^2} = \frac{1}{\beta^2}X(t)$$

If we insert this and (2) into (3), we get the following

$$Var(Z(t)) = E\left[\frac{1}{\beta^2}X(t)\right] + Var\left(\frac{1}{\beta}X(t)\right) = \frac{1}{\beta^2}\lambda * t + \frac{1}{\beta^2}\lambda * t = \frac{2}{\beta^2}\lambda * t = 1.77$$

So Var(Z(t)) = 1.77. Once again we verified our results through simulation in Python, and observe that they are pretty close to the true values.

The expected value of the claims is: 8.853344620545165
The estimated variance of the claims is: 1.8349499199890555