MATH 2210 HOMEWORK WORKSHEET 8

Name: KEY

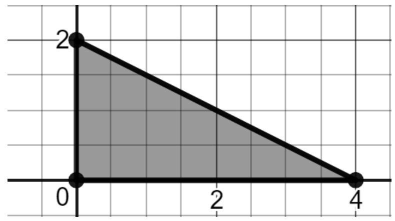
The Extreme Value Theorem

1. Find the absolute maximum and minimum values of

$$f(x, y) = x + y - xy$$

on the closed triangular region with vertices $(0, 0)$**,** $(0, 2)$**, and** $(4, 0)$**.**

The triangular region is the following



The lines that make up the edges are $x = 0$ for $0 \leq y \leq 2$, $y = 0$ for $0 \leq x \leq 4$ and $y = -\frac{1}{2}x + 2$ for $0 \leq x \leq 4$.
First, we need to find the critical points of $f(x, y)$. That is, where $f_x = f_y = 0$.

$$f_x = 1 - y = 0, \qquad f_y = 1 - x = 0$$

and so the only point that makes both equations true is $(1, 1)$, which is in the interior of the region.
Next we need to restrict $f(x, y)$ to each of the edges and determine if there are any critical points of these restricted functions between the corners. That is, we need to look for critical points of the functions $f(0, y)$, $f(x, 0)$, and $f\left(x, -\frac{1}{2}x + 2\right)$.

$$f(0, y) = y \qquad\qquad \frac{d}{dy}f(0, y) = 1 \neq 0$$

$$f(x, 0) = x \qquad\qquad \frac{d}{dx}f(x, 0) = 1 \neq 0$$

$$\begin{aligned} f\left(x, -\frac{1}{2}x + 2\right) &= x + -\frac{1}{2}x + 2 - x\left(-\frac{1}{2}x + 2\right) \\ &= \frac{1}{2}x^2 - \frac{3}{2}x + 2 \end{aligned} \qquad\qquad \frac{d}{dx}f\left(x, -\frac{1}{2}x + 2\right) = x - \frac{3}{2}$$

Thus the only additional critical point we gain is $x = \frac{3}{2}$ on the line $y = -\frac{1}{2}x + 2$, that is, $\left(\frac{3}{2}, \frac{5}{4}\right)$. We also must consider the corners. Thus we compute f at each of the critical points and corners to find

$$f(0, 0) = 0 \qquad f(0, 2) = 2 \qquad f(4, 0) = 4 \qquad f(1, 1) = 1 \qquad f\left(\frac{3}{2}, \frac{5}{4}\right) = \frac{7}{8}$$

Thus the absolute minimum value 0 occurs at $(0, 0)$ and the absolute maximum value 4 occurs at $(4, 0)$.

Lagrange Multipliers

2. Each of these extreme value problems has a solution with both a maximum value and a minimum value. Use Lagrange multipliers to find the extreme values of the function subject to the given constraint.

(a) $f(x,y,z) = \mathrm{e}^{xyz}; \quad 2x^2 + y^2 + z^2 = 24$

So that the constraint has the form $g(x,y,z) = k$, we have that

$$g(x,y,z) = 2x^2 + y^2 + z^2, \qquad \nabla f = \langle yz\mathrm{e}^{xyz},\ xz\mathrm{e}^{xyz},\ xy\mathrm{e}^{xyz}\rangle, \qquad \nabla g = \langle 4x,\ 2y,\ 2z\rangle$$

So then $\nabla f = \lambda \nabla g$ and $g(x,y,z) = 24$ yields the system of equations

$$yz\mathrm{e}^{xyz} = 4\lambda x, \qquad xz\mathrm{e}^{xyz} = 2\lambda y, \qquad xy\mathrm{e}^{xyz} = 2\lambda z, \qquad 2x^2 + y^2 + z^2 = 24.$$

Note that if any of x, y, z, λ are zero, then at least two of the coordinates is zero by the first three equations (just plug in the 0 and see). The fourth equation will not let all 3 coordinates be 0 and so can be used to obtain the value of the third. This yields the 6 points $(0, 0, \pm 2\sqrt{6})$, $(0, \pm 2\sqrt{6}, 0)$, $(\pm 2\sqrt{3}, 0, 0)$. The value of f for any of these is $\mathrm{e}^0 = 1$.
If none of x, y, z, λ are 0, then we solve for e^{xyz} and set each equal to each other to obtain

$$\frac{4\lambda x}{yz} = \frac{2\lambda y}{xz} = \frac{2\lambda z}{xy} \quad \Rightarrow \quad \frac{2x}{yz} = \frac{y}{xz} = \frac{z}{xy} \quad \Rightarrow \quad 2x^2 = y^2 = z^2$$

Plugging this into $2x^2 + y^2 + z^2 = 24$ implies that $y^2 = 8$ or that $y = \pm 2\sqrt{2}$. This yields the 8 points $(\pm 2, \pm 2\sqrt{2}, \pm 2\sqrt{2})$ (8 because each sign combination is possible). The value of f at these points is $\mathrm{e}^{\pm 16}$ where the sign depends on how many negatives are in the product xyz. Thus the maximum value subject to the constraint is e^{16} and the minimum is e^{-16} since 1 is between them.

(b) $f(x,y,z) = x^4 + y^4 + z^4, \quad x^2 + y^2 + z^2 = 1$

So that the constraint has the form $g(x,y,z) = k$, we have that

$$g(x,y,z) = x^2 + y^2 + z^2, \qquad \nabla f = \left\langle 4x^3, 4y^3, 4z^3\right\rangle, \qquad \nabla g = \langle 2x,\ 2y,\ 2z\rangle$$

So then $\nabla f = \lambda \nabla g$ and $g(x,y,z) = 1$ yields the system of equations

$$4x^3 = 2\lambda x, \qquad 4y^3 = 2\lambda y, \qquad 4z^3 = 2\lambda z, \qquad x^2 + y^2 + z^2 = 1.$$

If all x, y, z are nonzero, then $\lambda = 2x^2 = 2y^2 = 2z^2$ or that $x^2 = y^2 = z^2$ and since they add to 1, each is $\frac{1}{3}$. And hence two possibilities $\pm\frac{1}{\sqrt{3}}$ for each variable. That is 8 points but they all produce the same f value of $\frac{1}{3}$.
If exactly one of x, y, z is zero, then by the same reasoning above, the other two would have equal squared values of $\frac{1}{2}$ (plug in a 0 for one of the variables, solve for λ and plug into the last equation). That is, the values of the other two variables would be $\pm\frac{1}{\sqrt{2}}$. This produces 4 points for each of the 3 variables being 0, that is, a total of 12 points, but they all produce the same f value of $\frac{1}{2}$.
If exactly two of x, y, z are zero, then the other coordinate would square to 1. That is, its value would be ± 1. This produces 2 points for each of the 3 variables being nonzero, that is, a total of 6 points. Each of these produce the same f value of 1.
Thus subject to the constraint the maximum value of the function is 1 and the minimum value is $\frac{1}{3}$.

3. Find the extreme values of the function

$$f(x,y) = 2x^2 + 3y^2 - 4x - 5$$

on the region described by $x^2 + y^2 \leq 16$**.**

First we compute the critical points of f. Since it is a polynomial, it is differentiable everywhere and so the only critical points will be of the type $f_x = f_y = 0$.

$$f_x = 4x - 4, \qquad f_y = 6y$$

Thus the only point that makes both 0 is the point $(1,0)$ which is in the region in question.
The only other place where a maximum or minimum can occur is on the boundary. So we use Lagrange multipliers to maximize $f(x,y)$ subject to the constraint $x^2 + y^2 = 16$. This means that $g(x,y) = x^2 + y^2$ and hence

$$\nabla f = \langle 4x - 4,\ 6y \rangle, \qquad \nabla g = \langle 2x,\ 2y \rangle.$$

Then we solve the system $\nabla f = \lambda \nabla g$ and the constraint, together which are

$$4x - 4 = 2\lambda x, \qquad 6y = 2\lambda y, \qquad x^2 + y^2 = 16.$$

The second equation requires either $y = 0$ or $\lambda = 3$.
If $y = 0$, then $x = \pm 4$ by the last equation and by the first equation $\lambda = \frac{3}{2}$ or $\lambda = -\frac{5}{2}$, which is only important to make sure that a solution exists. This produces the two points $(\pm 4, 0)$.
If $y \neq 0$, then $\lambda = 3$, which by the first equation makes $x = -2$. Then by the third equation $y^2 = 12$ or $y = \pm 2\sqrt{3}$. This yields two points $(-2, \pm 2\sqrt{3})$.
We now only need to compute $f(x,y)$ at each of these points, the ones on the boundary and the critical point in the middle. The largest is the maximum and the smallest is the minimum.

$$\begin{array}{ccc} f(4,0) = 11 & f(-4,0) = 43 & f(1,0) = -7 \\ f(-2,2\sqrt{3}) = 47 & f(-2,2\sqrt{3}) = 47 & \end{array}$$

Thus the maximum value of the function on this region is 47 and the minimum value is -7.