MATH 2210 HOMEWORK WORKSHEET 2 SOLUTIONS

Name: KEY

The Cross Product

1. Find two unit vectors that are orthogonal to both $\mathbf{j} + 2\mathbf{k}$ and $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

To do this we find the cross product, scale it to a unit vector and then take its negative too. Let $\mathbf{u} = \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 1 & -2 & 3 \end{vmatrix} = (1(3) - 2(-2))\mathbf{i} - (0 - 2(1))\mathbf{j} + (0 - 1(1))\mathbf{k} = 7\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

The length of this vector is

$$\sqrt{7^2 + 2^2 + (-1)^2} = \sqrt{49 + 4 + 1} = \sqrt{54} = 3\sqrt{6}$$

Then the two unit vectors are

$$\frac{7}{3\sqrt{6}}\mathbf{i} + \frac{2}{3\sqrt{6}}\mathbf{j} - \frac{1}{3\sqrt{6}}\mathbf{k}, \qquad -\frac{7}{3\sqrt{6}}\mathbf{i} - \frac{2}{3\sqrt{6}}\mathbf{j} + \frac{1}{3\sqrt{6}}\mathbf{k}$$

2. Suppose that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 2$. Find $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w})$.

By the properties of the cross product and dot product

$$\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w}) = -\mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = -(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = -\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = -2$$

3. Let $\mathbf{u}$ and $\mathbf{v}$ be any nonzero, non-parallel vectors in $\mathbb{R}^3$. Compute $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v}$ and explain why your answer is right.

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$ because $\mathbf{u} \times \mathbf{v}$ is always orthogonal to $\mathbf{v}$ (and $\mathbf{u}$).

Equations of Lines and Planes

4. Find the vector equation, parametric equations, and symmetric equations for the line in $\mathbb{R}^3$ that passes through the points $(4, -1, 2)$ and $(1, 1, 5)$.

We need a vector in the direction of the line to find a vector equation of the line. To find that, we take the difference of the two points and find

$$\mathbf{v} = (1, 1, 5) - (4, -1, 2) = \langle -3, 2, 3 \rangle$$

We use one of these points as r_0, namely, $r_0 = (4, -1, 2)$, thus a vector equation for this line is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = \langle 4 - 3t, -1 + 2t, 2 + 3t \rangle$$

The parametric equations are just each component so that

$$x(t) = 4 - 3t, \qquad y(t) = -1 + 2t, \qquad z(t) = 2 + 3t$$

and the symmetric equations are found by solve each parametric equation for t and setting them equal to each other to obtain

$$\frac{x - 4}{-3} = \frac{y + 1}{2} = \frac{z - 2}{3}$$

5. Find a vector parallel to the line whose symmetric equations are

$$\frac{x - 4}{3} = \frac{y}{2} = z + 2.$$

A vector in the direction of the line is found by the denominators of the symmetric equations. Thus a vector parallel is

$$\langle 3, 2, 1 \rangle.$$

6. Find an equation for the plane through $(3, -1, 1)$, $(4, 0, 2)$, and $(6, 3, 1)$.

To compute the equation of a plane we need a point and an orthogonal vector. To find an orthogonal vector, we take the cross product of two vectors in the plane. The two vectors come from differences of points in the plane, thus let

$$\mathbf{u} = (4, 0, 2) - (3, -1, 1) = \langle 1, 1, 1 \rangle$$
$$\mathbf{v} = (6 - 3, 3 - (-1), 1 - 2) = \langle 3, 4, 0 \rangle$$

The orthogonal vector $\mathbf{n}$ is $\mathbf{u} \times \mathbf{v}$,

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 3 & 4 & 0 \end{vmatrix} = (0 - 1(4))\mathbf{i} - (0 - 1(3))\mathbf{j} + (1(4) - 1(3))\mathbf{k} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k} = \langle -4, 3, 1 \rangle.$$

Then the equation of the plane is

$$-4(x - 3) + 3(y + 1) + (z - 1) = 0.$$

7. Find the distance from the point $(-6, 3, 5)$ to the plane $x - 2y - 4z = 8$.

Recall tha tthe distance between (x_0, y_0, z_0) and the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Then in this case, the distance between$(-6, 3, 5)$ and $x - 2y - 4z - 8 = 0$ is

$$D = \frac{|1(-6) - 2(3) - 4(5) - 8|}{\sqrt{1^2 + (-2)^2 + (-4)^2}} = \frac{|-6 - 6 - 20 - 8|}{\sqrt{21}} = \frac{40}{\sqrt{21}}$$

Cylinders and Quadric Surfaces

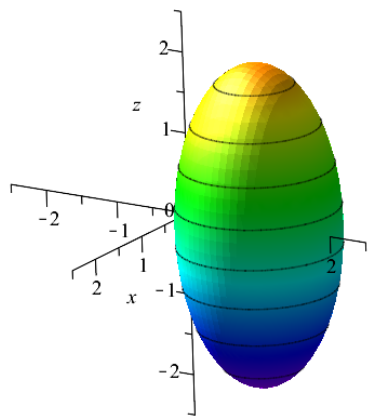
8. Identify and sketch the graph of the surface defined by

$$4x^2 + 4y^2 - 8y + z^2 = 0.$$

We complete the square on y by adding 4 to both sides to obtain

$$\begin{aligned} 4x^2 + 4y^2 - 8y + 4 + z^2 &= 4 \\ 4x^2 + 4(y^2 - 2y + 1) + z^2 &= 4 \\ 4x^2 + 4(y-1)^2 + z^2 &= 4 \\ x^2 + (y-1)^2 + \frac{z^2}{4} &= 1 \end{aligned}$$

which is an ellipsoid centered at $(0,1,0)$ with x-radius 1, y-radius 1, and z-radius 2. The graph of this is below



9. Find an equation for the surface consisting of all points that are equidistant from the point $(-1,0,0)$ and the plane $x = 1$. Identify the surface.

Lett (x,y,z) be any point on the surface. Note that the plane $x = 1$ also has equation $x - 1 = 0$. Then the distance from (x,y,z) to the point $(-1,0,0)$ is

$$d = \sqrt{(x+1)^2 + y^2 + z^2}$$

and the distance from the point to the surface is

$$D = \frac{|ax + by + cz + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|x-1|}{\sqrt{1^2 + 0^2 + 0^2}} = |x - 1|$$

Then setting $d = D$ or equivalently $d^2 = D^2$ yields

$$\begin{aligned} (x-1)^2 &= (x+1)^2 + y^2 + z^2 \\ x^2 - 2x + 1 &= x^2 + 2x + 1 + y^2 + z^2 \\ -4x &= y^2 + z^2 \end{aligned}$$

which is a paraboloid.