Section 1.6

Solve the following systems by inverting the coefficient matrix and using *Theorem 1.6.2*.

Problem 1

$$\begin{aligned} x_1 + x_2 &= 2 \\ 5x_1 + 6x_2 &= 9 \end{aligned}$$

Answer

$$A = \begin{bmatrix} 1 & 1 \\ 5 & 6 \end{bmatrix}$$

$$\hat{b} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$$

$$\det(A) = 1$$

$$\begin{aligned} A^{-1}\hat{b} &= \begin{bmatrix} 6 & -1 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 9 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ -1 \end{bmatrix} \end{aligned}$$

Problem 3

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 3 \end{aligned}$$

Answer

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

$$\hat{b} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= (2-3) - 3(0) + (6-4) \\ &= 3 \end{aligned}$$

$$\begin{aligned} A^{-1} &= \frac{1}{\det(A)} \operatorname{adj}(A) \\ &= \begin{bmatrix} -\frac{1}{3} & 0 & \frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{4}{3} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A^{-1}\hat{b} &= \begin{bmatrix} -\frac{1}{3} & 0 & \frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{4}{3} \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{3} \\ \frac{4}{3} \\ \frac{5}{3} \end{bmatrix} \end{aligned}$$

For the following exercises, solve the linear system together by reducing the appropriate augmented matrix.

Problem 9

$$x_1 - 5x_2 = b_1$$
$$3x_1 + 2x_2 = b_2$$

1. $b_1 = 1, b_2 = 4$
2. $b_1 = -2, b_2 = 5$

Answer

1. $\left[\begin{array}{cc|c} 1 & -5 & 1 \\ 3 & 2 & 4 \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & -5 & 1 \\ 0 & 17 & 1 \end{array}\right] \implies$

$\left[\begin{array}{cc|c} 1 & -5 & 1 \\ 0 & 1 & \frac{1}{17} \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & 0 & \frac{23}{17} \\ 0 & 1 & \frac{1}{17} \end{array}\right]$

2. $\left[\begin{array}{cc|c} 1 & -5 & -2 \\ 3 & 2 & 5 \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & -5 & -2 \\ 0 & 17 & 20 \end{array}\right] \implies$

$\left[\begin{array}{cc|c} 1 & -5 & 1 \\ 0 & 1 & \frac{20}{17} \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & 0 & \frac{117}{17} \\ 0 & 1 & \frac{1}{17} \end{array}\right]$

Problem 11

$$4x_1 - 7x_2 = b_1$$
$$x_1 + 2x_2 = b_2$$

1. $b_1 = 0, b_2 = 1$
2. $b_1 = -4, b_2 = 6$
3. $b_1 = -1, b_2 = 3$
4. $b_1 = -5, b_2 = 1$

Answer

1. $\left[\begin{array}{cc|c} 4 & -7 & 1 \\ 3 & 2 & 4 \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & -\frac{7}{4} & \frac{1}{4} \\ 3 & 2 & 4 \end{array}\right] \implies$

$\left[\begin{array}{cc|c} 1 & -\frac{7}{4} & \frac{1}{4} \\ 0 & \frac{29}{4} & \frac{13}{4} \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & -\frac{7}{4} & \frac{1}{4} \\ 0 & 1 & \frac{13}{29} \end{array}\right] \implies$

$\left[\begin{array}{cc|c} 1 & 0 & \frac{81}{29(4)} \\ 0 & 1 & \frac{13}{29} \end{array}\right]$

2. $\left[\begin{array}{cc|c} 4 & -7 & -4 \\ 3 & 2 & 6 \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & -\frac{7}{4} & -1 \\ 3 & 2 & 6 \end{array}\right] \implies$

$\left[\begin{array}{cc|c} 1 & -\frac{7}{4} & -1 \\ 0 & \frac{29}{4} & 9 \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & -\frac{7}{4} & -1 \\ 0 & 1 & \frac{36}{29} \end{array}\right] \implies$

$\left[\begin{array}{cc|c} 1 & 0 & \frac{55}{29} \\ 0 & 1 & \frac{36}{29} \end{array}\right]$

3. $\left[\begin{array}{cc|c} 4 & -7 & -1 \\ 3 & 2 & 3 \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & -\frac{7}{4} & -\frac{1}{4} \\ 3 & 2 & 3 \end{array}\right] \implies$

$\left[\begin{array}{cc|c} 1 & -\frac{7}{4} & -\frac{1}{4} \\ 0 & \frac{29}{4} & 6 \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & -\frac{7}{4} & -\frac{1}{4} \\ 0 & 1 & \frac{24}{29} \end{array}\right] \implies$

$\left[\begin{array}{cc|c} 1 & 0 & \frac{47}{29} \\ 0 & 1 & \frac{24}{29} \end{array}\right]$

4. $\left[\begin{array}{cc|c} 4 & -7 & 0 \\ 3 & 2 & 1 \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & -\frac{7}{4} & 0 \\ 3 & 2 & 1 \end{array}\right] \implies$

$\left[\begin{array}{cc|c} 1 & -\frac{7}{4} & 0 \\ 0 & \frac{29}{4} & 1 \end{array}\right] \implies \left[\begin{array}{cc|c} 1 & -\frac{7}{4} & 0 \\ 0 & 1 & \frac{4}{29} \end{array}\right] \implies$

$\left[\begin{array}{cc|c} 1 & 0 & \frac{7}{29} \\ 0 & 1 & \frac{24}{29} \end{array}\right]$

For the following exercises, determine the conditions on the b_i's, if any, in order to guarantee that the linear system is consistent.

Problem 13

$$\begin{aligned} x_1 + 3x_2 &= b_1 \\ -2x_1 + x_2 &= b_2 \end{aligned}$$

Answer

This system should be consistent for all values of (b_1, b_2) as its determinant is non-zero.

Problem 15

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= b_1 \\ 4x_1 - 5x_2 + 8x_3 &= b_2 \\ -3x_1 + 3x_2 - 3x_3 &= b_3 \end{aligned}$$

Answer

$$\begin{bmatrix} 1 & -2 & 5 & | & b_1 \\ 4 & -5 & 8 & | & b_2 \\ -3 & 3 & -3 & | & b_3 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -2 & 5 & | & b_1 \\ 4 & -5 & 8 & | & b_2 \\ 1 & -2 & 5 & | & b_3 + b_2 \end{bmatrix} \Longrightarrow$$

$$\begin{bmatrix} 1 & -2 & 5 & | & b_1 \\ 4 & -5 & 8 & | & b_2 \\ 0 & 0 & 0 & | & b_3 + b_2 - 1b_1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -2 & 5 & | & b_1 \\ 0 & 3 & -12 & | & b_2 - 4b_1 \\ 0 & 0 & 0 & | & b_3 + b_2 - 1b_1 \end{bmatrix} \Longrightarrow$$

$$\begin{bmatrix} 1 & -2 & 5 & | & b_1 \\ 0 & 1 & -4 & | & \frac{1}{3}(b_2 - 4b_1) \\ 0 & 0 & 0 & | & b_3 + b_2 - 1b_1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & -3 & | & \frac{2}{3}b_2 - \frac{5}{3}b_1 \\ 0 & 1 & -4 & | & \frac{1}{3}(b_2 - 4b_1) \\ 0 & 0 & 0 & | & b_3 + b_2 - b_1 \end{bmatrix}$$

System is consistent for all $b_1 = b_2 + b_3$.

Problem 19

Solve the matrix equation for X.

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix} X = \begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & 1 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 & 3 \\ -2 & 1 & -2 \\ -4 & 2 & -5 \end{bmatrix}$$

$$\begin{aligned} X &= \begin{bmatrix} 3 & -1 & 3 \\ -2 & 1 & -2 \\ -4 & 2 & -5 \end{bmatrix} \begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 8 & 12 & -3 & 27 & 26 \\ -6 & -8 & 1 & -18 & -17 \\ -15 & -21 & 9 & -38 & 35 \end{bmatrix} \end{aligned}$$

Section 1.7

Problem 3

Find the product by inspection of

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -4 & 1 \\ 2 & 5 \end{bmatrix}$$

Answer

$$\begin{bmatrix} 6 & 3 \\ 4 & -1 \\ 4 & 10 \end{bmatrix}$$

Problem 7

Given

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

Find A^2, A^{-2} and A^{-k} (where k is any integer) by inspection.

Answer

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$A^{-2} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

$$A^{-k} = \begin{bmatrix} 1 & 0 \\ 0 & \left(-\frac{1}{2}\right)^k \end{bmatrix}$$

Problem 11

Given

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Comput the product by inspection

Answer

The product is the zero matrix.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 13

Compute the indicated quantity

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{39}$$

Answer

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Problem 17

Create a symmetric matrix by ubstituting approprite numbers for the $\times$'s.

1. $\begin{bmatrix} 2 & -1 \\ \times & 3 \end{bmatrix}$
2. $\begin{bmatrix} 1 & \times & \times & \times \\ 3 & 1 & \times & \times \\ 7 & -8 & 0 & \times \\ 2 & -3 & 9 & 0 \end{bmatrix}$

Answer

1. $\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 3 & 7 & 2 \\ 3 & 1 & -8 & -3 \\ 7 & -8 & 0 & 9 \\ 2 & -3 & 9 & 0 \end{bmatrix}$

For the following exercises, determine by inspection whether the matrix is invertible.

Problem 19

$$\begin{bmatrix} 0 & 6 & -1 \\ 0 & 7 & -4 \\ 0 & 0 & -2 \end{bmatrix}$$

Answer

Matrix is singular.

Problem 21

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -5 & 0 & 0 \\ 4 & -3 & 4 & 0 \\ 1 & -2 & 1 & 3 \end{bmatrix}$$

Answer

Invertible.

Problem 23

Find the diagonal entries of AB by inspection.

$$A = \begin{bmatrix} 3 & 2 & 6 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 2 & 7 \\ 0 & 5 & 3 \\ 0 & 0 & 6 \end{bmatrix}$$

Answer

$$AB = \begin{bmatrix} -3 & 9 & 26 \\ 0 & 5 & -13 \\ 0 & 0 & -6 \end{bmatrix}$$

Problem 25

Find all the values of the unknown constant(s) for which A is symmetric.

$$A = \begin{bmatrix} 4 & -3 \\ a+5 & -1 \end{bmatrix}$$

Answer

$$a = -8$$

Problem 27

Find all the values of x for which A is invertible.

$$A = \begin{bmatrix} x-1 & x^2 & x^4 \\ 0 & x+2 & x^3 \\ 0 & 0 & x-4 \end{bmatrix}$$

Answer

Invertible for $x \in (\infty, -2) \cup (-2, 1) \cup (1, 4) \cup (4, \infty)$.

Problem 31

Find a diagonal matrix A that satisfies

$$A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Answer

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$