Homework 7

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Extreme Value Theorem

Problem 1

Find the absolute maximum and minimum values of

$$f(x, y) = x + y - xy$$

on the closed triangular region with vertices $(0, 0), (0, 2), (4, 0)$.

Answer

To find the absolute extrema in the given area, we need to compute all critical poits of f in the given area, incluiding the bounderies. Note that the bounderies are paths in $\mathbb{R}^2$, which we can compute as follows:

$$\boldsymbol{r}_1 = (0, 0) + t(4, 0) \Longrightarrow \forall x \in [4, 0], y = 0$$
$$\boldsymbol{r}_2 = (0, 0) + t(0, 2) \Longrightarrow \forall y \in [0, 2], x = 0$$
$$\boldsymbol{r}_3 = (0, 2) + t(4, -2) \Longrightarrow y = 2 - \frac{x}{2}$$

Thus, as we search for the critical points of f, we will consider the following subdomains of f:

1. $f(x, y) \forall (x, y) \in \mathbb{R}^2$
2. $f(x, 0) \forall x \in \mathbb{R}$
3. $f(0, y) \forall y \in \mathbb{R}$
4. $f\left(x, 2 - \frac{x}{2}\right) \forall x \in \mathbb{R}$

From computation we get:

1. $f_x(x, y) = 1 - y \wedge f_y(x, y) = 1 - x \Longrightarrow (1, 1)$ is a critical point.
2. $f_x(x, 0) = 1 \Longrightarrow$ no critical points in this path.
3. Similarly, $f_y(0, y) = 1 \Longrightarrow$ no critical points in this path.
4. Similarly, $f_x\left(x, 2 - \frac{x}{2}\right) = x - \frac{3}{2} \Longrightarrow \left(\frac{3}{4}, \frac{5}{4}\right)$ is a critical point.

If we evaluate the function at the given vertices and the critical points and compare them, we will find the absolute maxmimum and minimum in the area, which in this case corresponds to $f(4, 0) = 4$ and $f(0, 0) = 0$ respectively.

Lagrange Multipliers

Problem 2

Each of these extreme value problems has a solution with both a maximum value and a minimum value. Use Lagrange multipliers to find the extreme values of the function subject to the given constraint.

(a) $f(x,y,z) = e^{xyz}, 2x^2 + y^2 + z^2 = 24$

(b) $f(x,y,z) = x^4 + y^4 + z^4, x^2 + y^2 + z^2 = 1$

Answer

(a)

$$\begin{aligned} \nabla f &= \langle yze^{xyz}, xze^{xyz}, xye^{xyz} \rangle \\ \nabla g &= \langle 4x, 2y, 2z \rangle \end{aligned}$$

Solving for the given system:

(i) $yze^{xyz} = \lambda 4x,$

(ii) $xze^{xyz} = \lambda 2y$

(iii) $xye^{xyz} = \lambda 2z$

If we multiply (i) with x and (ii) with y, we get equivalent expressions we can simplify:

$$\begin{aligned} \lambda 4x^2 &= \lambda 2y^2 \\ 2x^2 &= y^2 \\ \sqrt{2}x &= y \end{aligned}$$

We can substitute y in the (ii) and (iii) in the following manner:

(ii)

$$\begin{aligned} xze^{xyz} &= \lambda 2y \\ &= \lambda 2\sqrt{2}x \\ ze^{xyz} &= \lambda 2\sqrt{2} \\ e^{xyz} &= \frac{\lambda 2\sqrt{2}}{z} \end{aligned}$$

(iii)

$$\begin{aligned} xye^{xyz} &= \lambda 2z \\ \sqrt{2}x^2 e^{\sqrt{2}x^2 z} &= \lambda 2z \\ \left(\sqrt{2}x^2\right)\left(\frac{\lambda 2\sqrt{2}}{z}\right) &= \lambda 2z \\ \left(\sqrt{2}x^2\right)\sqrt{2} &= z^2 \\ 2x^2 &= z^2 \\ \sqrt{2}x &= z \end{aligned}$$

We can solve for $x = t, y = \sqrt{2}t, z = \sqrt{2}t$ in g:

$$
\begin{aligned}
2t^2 + 2t^2 + 2t^2 &= 24 \\
6t^2 &= \\
t^2 &= 4 \\
t &= \pm 2
\end{aligned}
$$

By pluggin $t = \pm 2$ into $f\left(t, \sqrt{2}t, \sqrt{2}t\right)$:

$$e^{\pm\left(2 \cdot 2\sqrt{2} \cdot 2\sqrt{2}\right)} = e^{\pm 16}$$

Therefore, $f\left(2, 2\sqrt{2}, 2\sqrt{2}\right) = e^{16}$ and $f\left(-2, -2\sqrt{2}, -2\sqrt{2}\right) = \frac{1}{e^{16}}$ are the absolute maximum and minimum at the level curve, respectively.

(b)

$$
\begin{aligned}
\nabla f &= \langle 4x^3, 4y^3, 4z^3 \rangle \\
\nabla g &= \langle 2x, 2y, 2z \rangle
\end{aligned}
$$

Note that $\forall x_i \neq 0 \in \boldsymbol{x}, \left(f_{x_i} - \lambda g_{x_i} = 0\right) \iff \left(\lambda = 2x_i^2\right)$. In other words, there are three cases we must consider: (i) $\exists x_i = 0$, (ii) $\exists x_i \exists x_j (x_i = 0), \left(x_j = 0\right)$ and (iii) $\nexists x_i = 0$.

(i) Suppose $\forall x_i \neq 0$. Then $\left(\lambda = 2x_i^2\right) \iff (x = y = z = t)$. If we plug (t, t, t) to $g(x, y, z)$, then

$$
\begin{aligned}
g(t, t, t) &= 1 \\
3t^2 &= 1 \\
t^2 &= \frac{1}{3} \\
t &= \pm\sqrt{\frac{1}{3}}
\end{aligned}
$$

Which, if plugged into f:

$$
\begin{aligned}
f\left(\boldsymbol{r}\left(\pm\sqrt{\frac{1}{3}}\right)\right) &= 3\left(\pm\sqrt{\frac{1}{3}}\right)^4 \\
&= 3\left(\frac{1}{9}\right) \\
&= \frac{1}{3}
\end{aligned}
$$

(ii) I owe the next two parts of the work to Rhett, who pointed out that these cases exists. Suppose $\exists x_i = 0$. Without loss of generality, let $x_i = x = 0$. Then

$$
\begin{aligned}
g(0, t, t) &= 1 \\
2t^2 &= 1 \\
t &= \pm\sqrt{\frac{1}{2}}
\end{aligned}
$$

Which, if plugged into f:

$$
\begin{aligned}
f\left(\boldsymbol{r}\left(\pm\sqrt{\frac{1}{3}}\right)\right) &= 3\left(\pm\sqrt{\frac{1}{2}}\right)^4 \\
&= \frac{3}{8}
\end{aligned}
$$

(iii) Suppose $\exists(x_i = 0), \exists\left(x_j = 0\right)$. Let $x_i = x = 0$ and $x_j = y = 0$. Then

$$
\begin{aligned}
g(0, 0, t) &= 1 \\
t^2 &= 1 \\
t &= \pm 1
\end{aligned}
$$

Which, if plugged into f:

$$
\begin{aligned}
f(\boldsymbol{r}(\pm 1)) &= 3(\pm 1)^4 \\
&= 1
\end{aligned}
$$

Therefore, $f\left(\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right) = \frac{1}{3}$ and $f(1, 0, 0) = 1$ are the minimum and maximum at the given surface level, respectively.

Problem 3

Find the extreme values of the function

$$f(x, y) = 2x^2 + 3y^2 - 4x - 5$$

on the region described by $x^2 + y^2 \leq 16$.

Answer

$$\nabla f = \langle 4(x-1), 6y \rangle$$
$$\nabla g = \langle 2x, 2y \rangle$$

Note the solving for the critical points in f without bounds to the domain returns $(1, 0)$ as a critical point. Also, $g(1, 0) = 1 < 16$. Therefore, $f(1, 0) = -7$ is the only extrema inside the bounded region of the function.

To find the critical points at the bounderies, we can use Lagrange's method, were our constraint is redifined as a strict equality $g(x, y) = x^2 + y^2 = 16$. Note that:

$$\begin{aligned} f_y &= \lambda g_y \\ 6y &= 2\lambda y \\ 3y &= \lambda y \end{aligned}$$

implies that there are two cases: (i) $y \neq 0$ and (ii) $y = 0$.

(i) Suppose $y \neq 0$. Then

$$f_y = \lambda g_y \Longleftrightarrow 3 = \lambda$$

Then, if we plug $\lambda = 3$ into $f_x = \lambda g_x$:

$$\begin{aligned} f_x &= \lambda g_x \\ 4(x-1) &= 6x \\ 2x - 2 &= 3x \\ 2x - 3x &= 2 \\ -x &= 2 \\ x &= -2 \end{aligned}$$

Then, we can plug $x = -2$ to our constraint to find y:

$$\begin{aligned} g(-2, y) &= 16 \\ 4 + y^2 &= 16 \\ y^2 &= 12 \\ y &= \pm 2\sqrt{3} \end{aligned}$$

which implies that $f\left(-2, \pm 2\sqrt{3}\right) = 47$ is another potential extrema in the boundaries.

Suppose $y = 0$. We can solve for x using just the constrain:

$$\begin{aligned} g(x, 0) &= 16 \\ x^2 &= 16 \\ x &= \pm 4 \end{aligned}$$

which implies that $f(-4, 0) = 43$ and $f(4, 0) = 11$ are potential extremas in the boundaries.

By inspection, $-7 < 19 < 43 < 47$, then $f(1, 0) = -7$ and $f\left(-2, \pm 2\sqrt{3}\right) = 47$ are the absolute minimum and maximum within the bounded region of the function.