Homework 1

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January 7th, 2025

Three dimensional Coordinate Systems

Problem 1

Plot points $(2, -2, -3)$ and $(3, 4, 2)$.

Answer

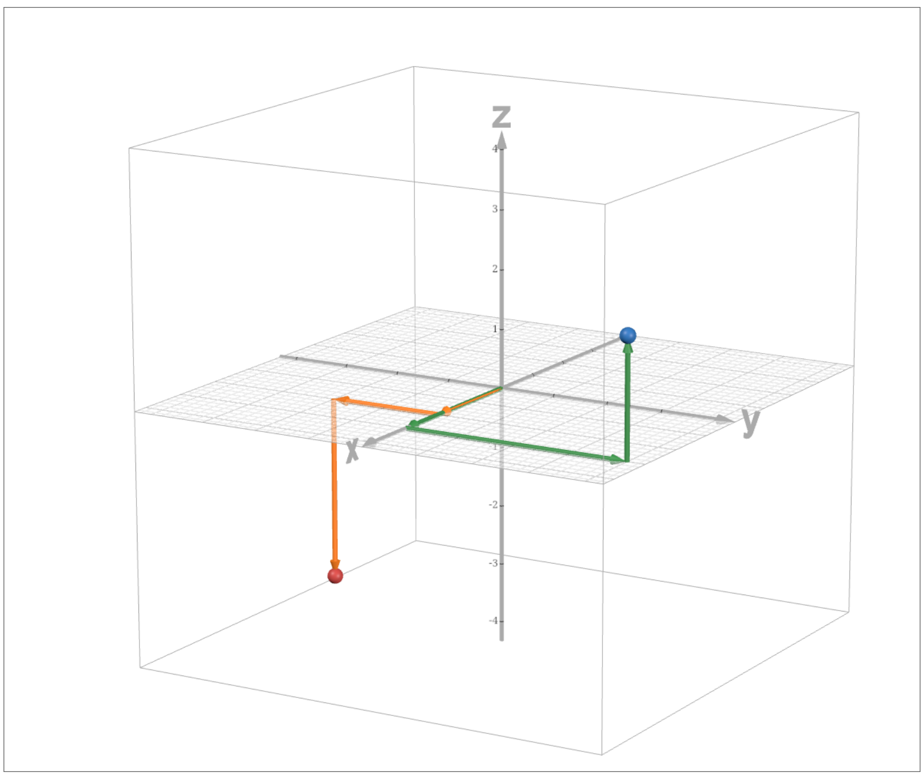

Figure 1: Points in 3D

Problem 2

Describe the surface defined by the equation $x^2 + y^2 + z^2 = 9$ and then graph it on the axes below.

Answer

The surface defined by the given equation corresponds to the surface of a sphere with radius 3.

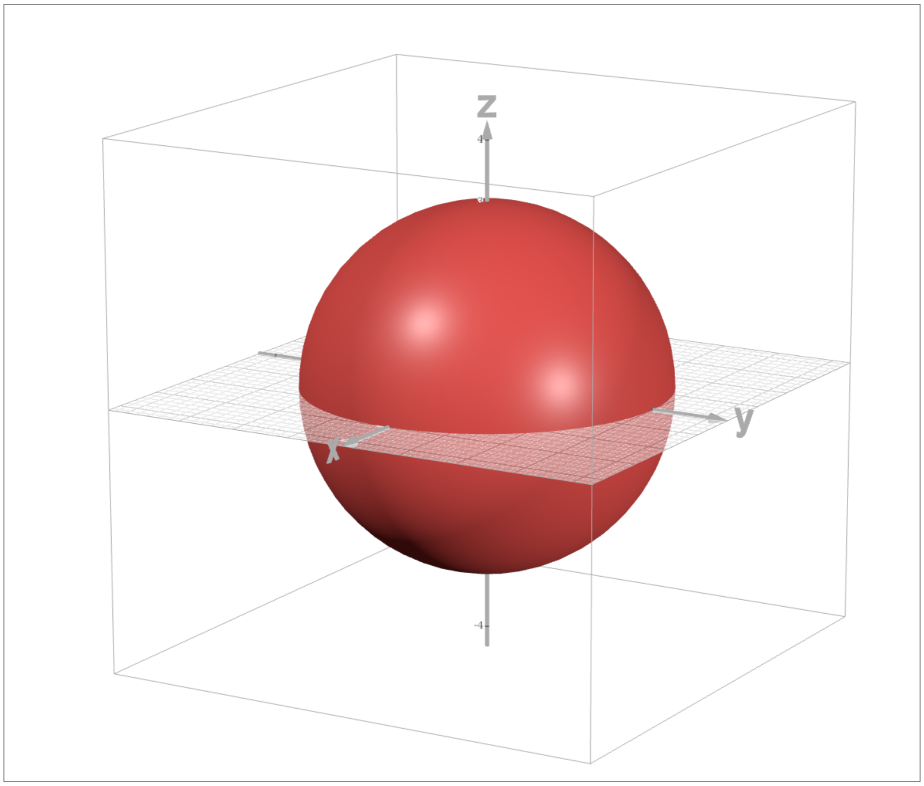

Figure 2: Surface of a Sphere

Vectors

Problem 3

Consider the vector $(-2, 4, \sqrt{5})$. Find a unit vector in the same direction as this vector; then find a vector of length 10 in the same direction of this vector.

Answer

The unit vector in the same direction of $\hat{x}$ is equal to the scalar product of the reciprocal of the norm of the given vector and the vector. Given $\hat{x} = (-2, 4, \sqrt{5})$, the norm $|\hat{x}|$ is

$$\begin{aligned}\sqrt{(-2)^2 + 4^2 + (\sqrt{5})^2} &= \sqrt{4 + 16 + 5} \\ &= \sqrt{25} \\ &= 5\end{aligned}$$

Thus, the unit vector is $\left(\frac{-2}{5}, \frac{4}{5}, \frac{\sqrt{5}}{5}\right)$.

A vector of length 10 in the same direction is the scalar product $10\hat{x} = (-4, 8, 2\sqrt{5})$.

Problem 4

Let $a = 8i + j - 4k$ and $b = 5i - 2j + k$. Find

1) $a + b$
2) $4a - 2b$
3) $|a|$
4) $|a - b|$

Answer

As a remainder, $i = \langle 1, 0, 0 \rangle, j = \langle 0, 1, 0 \rangle, k = \langle 0, 0, 1 \rangle$, or the basis of R^3.

1)

$$\begin{aligned}a + b &= (8 + 5)i + (1 - 2)j + (1 - 4)k \\ &= 13i - j - 3k \\ &= \langle 13, -1, -3 \rangle\end{aligned}$$

2)

$$\begin{aligned}
4a - 2b &= 4(8i + j - 4k) - 2(5i - 2j + k) \\
&= (32i + 4j - 16k) - (10i - 4j + 2k) \\
&= (32 - 10)i + (4 - 4)j - (16 + 2)k \\
&= 22i + 0j - 18k \\
&= \langle 22, 0, -18 \rangle
\end{aligned}$$

As it turns out, I miscalulated the product of the $\hat{j}$ component of $\hat{b}$ in the third line and the calculation should look like this

$$\begin{aligned}
4a - 2b &= 4(8i + j - 4k) - 2(5i - 2j + k) \\
&= (32i + 4j - 16k) - (10i - 4j + 2k) \\
&= (32 - 10)i + (4 + 4)j - (16 + 2)k \\
&= 22i + 8j - 18k \\
&= \langle 22, 8, -18 \rangle
\end{aligned}$$

3)

$$\begin{aligned}
|a| &= |\langle 8, 1, -4 \rangle| \\
&= \sqrt{8^2 + 1^2 + (-4)^2} \\
&= \sqrt{64 + 1 + 16} \\
&= \sqrt{81} \\
&= 9
\end{aligned}$$

4)

$$\begin{aligned}
|a - b| &= |(8 - 5)i + (1 + 2)j - (4 + 1)k \\
&= |3i + 3j - 5k| \\
&= |\langle 3, 3, -5 \rangle| \\
&= \sqrt{3^2 + 3^2 + (-5)^2} \\
&= \sqrt{9 + 9 + 25} \\
&= \sqrt{43}
\end{aligned}$$

Problem 5

A crane suspends a 500 lbs steel beam horizontally by support cables (with negligible weight) attached from a hook to each end of the beam. The support cables each make an angle of 60° with the beam. Find the tension in each support cable and the magnitude of each tension.

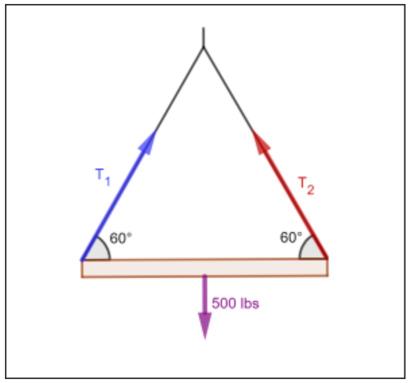

Figure 3: Vector representation

Answer

Student note: At the end, I ended asking help to a math tutor. But this was such a fun problem that reminded me of the importance of stepping back from the context of the class.

As a remainder, given the norm of a vector $\hat{a} \in R^2$ and the angle θ it makes with the x-axis, the vector can be expressed as a linear combination $(|\hat{a}| \cos\theta)\hat{i} + (|\hat{a}| \sin\theta)\hat{j}$.

We know from classic mechanics, that given a force in a system, there must be a reaction of the same norm in the opposite direction. In the present system, the crane is an object being pulled down by a gravitational force that corresponds to the vector $-500\hat{j}$ from our referential point. Therefore, the system also features an opposite vector of $500\hat{j}$. We also know that this force is *equally* distributed between the cables, each forming angles of $\theta = 60° = \frac{\pi}{3}$ with respect to the crane. This means that there are forces (represented by vectors T_1 and T_2) acting along the cables. Mathematically, we can express this as following:

$$T_1 + T_2 = 500\hat{j}$$
$$T_1 = \langle |T_1| \cos\frac{\pi}{3}, |T_1| \sin\frac{\pi}{3} \rangle$$
$$T_2 = \langle -|T_2| \cos\frac{\pi}{3}, |T_2| \sin\frac{\pi}{3} \rangle$$

In order to find the components of the vectors, we need to find their norm, which is equal for both of them ($|T_1| = |T_2| = |T|$). And we know that the sums of of their y-components is equal to 500 ($|T_1 + T_2| \sin\theta$). Therefore:

$$
\begin{aligned}
2|T|\sin\theta &= 500 \\
|T| &= \frac{500}{2\sin\theta} \\
&= \frac{(500)(2)}{2\sqrt{3}} \\
&= \frac{500}{\sqrt{3}}
\end{aligned}
$$

Now, we can figure out the components of both vectors. Note that $|T_1|\cos\theta = -|T_2|\cos\theta$, as the x-component of both vectors go in opposite directions.

$$
\begin{aligned}
T_1 &= \langle |T_1|\cos\theta, |T_1|\sin\theta\rangle \\
&= \langle \frac{500}{\sqrt{3}} \cdot \frac{1}{2}, \frac{500}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2}\rangle \\
&= \langle \frac{250}{\sqrt{3}}, 250\rangle
\end{aligned}
$$

and

$$T_2 = \langle -\frac{250}{\sqrt{3}}, 250\rangle$$

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The Dot Product

Problem 6

Let $\hat{u} = \langle 0, 1, -1 \rangle$ and $\hat{v} = \langle -1, a, 1 \rangle$ for some Real number a. What value of a will make $\hat{u}$ and $\hat{v}$ orthogonal? What value of a will produce an angle of $\frac{\pi}{3}$ between them?

Answer

Note that in general

$$\begin{aligned} \hat{u} \cdot \hat{v} &= 0(-1) + 1a + (1)(-1) \\ &= a - 1 \end{aligned}$$

From this and $\forall n \in N \geq 2$, $\forall \hat{a}, \hat{b} \in R^n$, $\hat{a} \cdot \hat{b} = 0 \iff \hat{a} \perp \hat{b}$, it follows that $a = 1$ will satisfy orthogonality between $\hat{u}$ and $\hat{v}$.

$$\begin{aligned} \hat{u} \cdot \hat{v} = a - 1 &= 0 \\ a &= 1 \end{aligned}$$

For the second question, I misunderstood the question and the relation between the dot product and the angle between vectors. In an embarrassing episode, I assumed that $\hat{a} \cdot \hat{b} = \theta$ and $\hat{u} \cdot \hat{v} = \frac{\pi}{3}$ implying that $a = \frac{\pi}{3} + 1$ (faulty computation as follows):

$$\begin{aligned} \frac{\pi}{3} &= 0(-1) + 1a + 1(-1) \\ &= a - 1 \\ \implies \frac{\pi}{3} + 1 &= a \end{aligned}$$

As it turns out, and what I figured out after peeking quickly at the solution sheet, $\hat{u} \cdot \hat{v} = |\hat{u}||\hat{v}| \cos \theta$. Let's work with that:

Let $\theta = \frac{\pi}{3}$. Therefore, $\cos \theta = \frac{1}{2}$, and

$$
\begin{aligned}
\frac{1}{2} &= \frac{\hat{u} \cdot \hat{v}}{|\hat{u}||\hat{v}|} \\
&= \frac{a-1}{\sqrt{2}\sqrt{2+a^2}} \\
&= \frac{a-1}{\sqrt{4+2a^2}} \\
\frac{\sqrt{4+2a^2}}{2} &= a-1 \\
\sqrt{4+2a^2} &= 2a-2 \\
4+2a^2 &= (2a-2)^2 \\
4+2a^2 &= 4a^2-8a+4 \\
2a^2 &= 4a^2-8a \\
a^2 &= 2a^2-4a \\
0 &= a^2-4a \\
&= a(a-4)
\end{aligned}
$$

Which implies $a \in \{0, 4\}$. ■

Problem 7

Show that the vector $\text{orth}_{\hat{a}}\hat{b} = \hat{b} - \text{proj}_{\hat{a}}\hat{b}$ is orthogonal to $\hat{a}$. (It is called the **orthogonal projection** of $\hat{b}$.)

Answer

We can prove the orthogonality if the dot product of the orthogonal projection and $\hat{a}$ is 0.

$$
\begin{aligned}
\hat{a} \cdot \mathrm{orth}_{\hat{a}} \hat{b} &= \hat{a} \cdot (\hat{b} - \mathrm{proj}_{\hat{a}} \hat{b}) \\
&= \hat{a} \cdot (\hat{b} - \frac{\hat{a} \cdot \hat{b}}{|\hat{a}|^2} \hat{a}) \\
&= \hat{a} \cdot \hat{b} - \frac{\hat{a} \cdot \hat{b}}{|\hat{a}|^2} \hat{a} \cdot \hat{a} \\
&= (\hat{a} \cdot \hat{b})(1 - \frac{1}{|\hat{a}|^2} (\hat{a} \cdot \hat{a})) \\
&= (\hat{a} \cdot \hat{b})(1 - \frac{|\hat{a}|^2}{|\hat{a}|^2}) \\
&= (\hat{a} \cdot \hat{b})(1 - 1) \\
&= (\hat{a} \cdot \hat{b})(0) \\
&= 0
\end{aligned}
$$

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