Homework 7

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Directional Derivatives and the Gradient Vector

Problem 1

Find the directional derivative of the function at the given point P in the direction of the vector $\boldsymbol{v}$.

1. $f(x, y) = \frac{x}{x^2+y^2}, P = (1, 2), \boldsymbol{v} = \langle 3, 5 \rangle$
2. $f(x, y, z) = xy^2 \tan^{-1} z, P = (2, 1, 1), \boldsymbol{v} = \langle 1, 1, 1 \rangle$

Answer

Let $\boldsymbol{u}$ be the unit vector in $\mathbb{R}^n$ with the same direction as $\boldsymbol{v}$. Let u_i be the ith component of $\boldsymbol{u}$. Then

$$D_{\boldsymbol{u}} f(\boldsymbol{x}) = \sum_{i=1}^{n} u_i f_{x_i}(\boldsymbol{x})$$

1.

$$\begin{aligned}
\boldsymbol{u} &= \frac{\boldsymbol{v}}{|\boldsymbol{v}|} \\
&= \frac{1}{\sqrt{3^2 + 5^2}} \langle 3, 5 \rangle \\
&= \frac{1}{\sqrt{34}} \langle 3, 5 \rangle \\
&= \langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \rangle
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial x}{\partial x} \frac{1}{x^2 + y^2} + x \frac{\partial}{\partial x} \left(x^2 + y^2\right)^{-1} \\
&= \frac{1}{x^2 + y^2} - \frac{2x^2}{\left(x^2 + y^2\right)^2} \\
&= \frac{x^2 + y^2 - 2x^2}{\left(x^2 + y^2\right)^2} \\
&= \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial x}{\partial y} \frac{1}{x^2 + y^2} + x \frac{\partial}{\partial y} \left(x^2 + y^2\right)^{-1} \\
&= -\frac{2xy}{\left(x^2 + y^2\right)^2}
\end{aligned}$$

$$\begin{aligned}
D_{\boldsymbol{u}} f(\boldsymbol{x}) &= \frac{3}{\sqrt{34}} \left(\frac{y^2 - x^2}{\left(x^2 + y^2\right)^2} \right) - \frac{5}{\sqrt{34}} \left(\frac{2xy}{\left(x^2 + y^2\right)^2} \right) \\
&= \frac{1}{\sqrt{34} \left(x^2 + y^2\right)^2} \left(3\left(y^2 - x^2\right) - 5(2xy)\right) \\
&= \frac{3y^2 - 10xy - 3x^2}{\sqrt{34} \left(x^2 + y^2\right)^2}
\end{aligned}$$

$$
\begin{aligned}
D_{\boldsymbol{u}}f(P) &= \frac{3(2^2) - 10(1)(2) - 3(1^2)}{\sqrt{34}(1^2 + 2^2)^2} \\
&= \frac{3(4) - 10(2) - 3}{\sqrt{34}(1+4)^2} \\
&= \frac{12 - 20 - 3}{25\sqrt{34}} \\
&= -\frac{11}{25\sqrt{34}}
\end{aligned}
$$

2.

$$
\begin{aligned}
\boldsymbol{u} &= \frac{1}{\sqrt{1^2 + 1^2 + 1^2}}\langle 1, 1, 1\rangle \\
&= \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\rangle
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= y^2 \tan^{-1}(z) \\
\frac{\partial f}{\partial y} &= 2xy \tan^{-1}(z) \\
\frac{\partial f}{\partial z} &= \frac{xy^2}{1+z^2}
\end{aligned}
$$

$$
D_{\boldsymbol{u}}f(\boldsymbol{x}) = \frac{y}{\sqrt{3}}\left(y \tan^{-1} z + 2x \tan^{-1} z + \frac{xy^2}{1+z^2}\right)
$$

$$
\begin{aligned}
D_{\boldsymbol{u}}f(2,1,1) &= \frac{1}{\sqrt{3}}\left(1 \tan^{-1}(1) + 2(2) \tan^{-1}(1) + \frac{2(1^2)}{1+1^2}\right) \\
&= \frac{1}{\sqrt{3}}\left(\frac{\pi}{4} + (4)\frac{\pi}{4} + \frac{2}{2}\right) \\
&= \frac{1}{\sqrt{3}}\left(\frac{\pi}{4} + \pi + 1\right)
\end{aligned}
$$

Problem 2

Find the maximum rate of change of

$$f(x, y, z) = x \ln(yz)$$

at the point $P = \left(1, 2, \frac{1}{2}\right)$ at the direction in which it occcurs.

Answer

The maximum rate of change of f at P is given by $|\nabla f(P)|$, and it's direction is $\frac{\nabla f}{|\nabla f|}$.

$$\begin{aligned} f_x &= \ln(yz) \\ f_y &= \frac{x}{y} \\ f_z &= \frac{x}{z} \end{aligned}$$

$$\begin{aligned} \nabla f(P) &= \langle \ln\left(\frac{2}{2}\right), \frac{1}{2}, 2\rangle \\ &= \langle 0, \frac{1}{2}, 2\rangle \end{aligned}$$

$$\begin{aligned} |\nabla f(P)| &= |\langle 0, \frac{1}{2}, 2\rangle| \\ &= |\sqrt{\frac{1}{4} + 4} \\ &= |\sqrt{\frac{17}{4}} \end{aligned}$$

$$\begin{aligned} \frac{\nabla f(P)}{|\nabla f(P)|} &= \frac{2}{\sqrt{17}} \langle 0, \frac{1}{2}, 2\rangle \\ &= \langle 0, \frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}\rangle \end{aligned}$$

Problem 3

Find equations of

1. the tangent plane and
2. the normal line to the surface

$$x = y^2 + z^2 + 1$$

at the point $P = (3, 1, -1)$. *Hint: Recall that the normal line is the line through the point that is orthogonal to the surface.*

Answer

1. The equation for the tangent planes to the equation is given by

$$\nabla f(x, y, z) \cdot \boldsymbol{r}'(t)$$

Then

$$\nabla f(x, y, z) = \langle 1, -2y, -2z \rangle$$
$$\nabla f(P) = \langle 1, -2, 2 \rangle$$

Which is the normal vector to the tangent plane of f at P. The equation of the tangent plane is:

$$x - 3 - 2(y - 1) + 2(z + 1) = 0$$

Maximum and Minimum values

Problem 4

Suppose $(0, 2)$ is a critical point of a function g with continuous second derivative. In each case below, what can you say about g? Explain your reasoning.

1. $g_{xx}(0, 2) = -1$, $g_{xy}(0, 2) = 6$, $g_{xy}(0, 2) = 1$
2. $g_{xx}(0, 2) = -1$, $g_{xy}(0, 2) = 2$, $g_{xy}(0, 2) = -8$
3. $g_{xx}(0, 2) = 4$, $g_{xy}(0, 2) = 6$, $g_{xy}(0, 2) = 9$

Answer

We can use the determinant of Hessian matrix $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$ to reach conclusions about the point and the function.

1.

$$\begin{aligned} f_{xx}(0, 2) f_{yy}(0, 2) - \left[f_{xy}(0, 2)\right]^2 &= -1(1) - 6^2 \\ &= -37 < 0 \end{aligned}$$

Therefore, $(0, 2)$ is a saddle point.

2.

$$\begin{aligned} f_{xx}(0, 2) f_{yy}(0, 2) - \left[f_{xy}(0, 2)\right]^2 &= -1(-8) - 2^2 \\ &= 8 - 4 \\ &= 4 > 0 \end{aligned}$$

Also, $f_{xx} > 0$. Therefore, $(0, 2)$ is a local maximum at f.

3.

$$\begin{aligned} f_{xx}(0, 2) f_{yy}(0, 2) - \left[f_{xy}(0, 2)\right]^2 &= 4(9) - 6^2 \\ &= 36 - 36 \\ &= 0 \end{aligned}$$

Therefore, the test is inconclusive about the relation between $(0, 2)$ and f.

Problem 5

Find the local extrema and the saddle point(s) of the given function.

1. $f(x, y) = y(e^x - 1)$
2. $f(x, y) = xye^{-\frac{x^2+y^2}{2}}$

Answer

Given $f : \mathbb{R}^n \to \mathbb{R}, \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{v} \in \mathbb{R}^n$, the critical points of $f(\boldsymbol{x})$ are the points for which $\forall \boldsymbol{u} = \frac{1}{|\boldsymbol{v}|}\boldsymbol{v}, D_{\boldsymbol{u}} f(\boldsymbol{x}) = 0$. Then, we can use the second derivative test to infer about the given critical points.

1. $f(x, y) = y(e^x - 1)$

$$\begin{aligned} f_x &= \frac{\partial}{\partial x} ye^x - y \\ &= y\frac{\partial}{\partial x}e^x - \frac{\partial y}{\partial x} \\ &= ye^x \end{aligned}$$

If $y = 0$, then any $f_x(x, y) = 0$. If $y \neq 0$, then

$$\begin{aligned} ye^x &= 0 \\ e^x &= -y \\ x &= \ln(-y) \end{aligned}$$

Therefore, $\left(f_{x(x,y)} = 0\right) \iff (y < 0 \wedge x = \ln(-y)) \vee (y = 0)$.

$$\begin{aligned} f_y &= \frac{\partial}{\partial y} ye^x - y \\ &= e^x\frac{\partial}{\partial y}y - \frac{\partial y}{\partial y} \\ &= e^x - 1 \end{aligned}$$

$$\begin{aligned} e^x - 1 &= 0 \\ e^x &= 1 \\ x &= 0 \end{aligned}$$

Therefore, $\forall y, x = 0 \Longrightarrow f_{y(x,y)} = 0$.

Since f is a continuous function, and the conditions for critical point require $x = 0$, and $\nexists y \neq 0 \ (\ln(-y) = 0)$, then $(0, 0)$ is only critical point at f.

If we use the determinant of the Hessian matrix in the second derivative test:

$$\begin{aligned} f_{xx}(x, y) &= ye^x \\ f_{yx}(x, y) &= e^x \\ f_{yx}(x, y) &= 0 \end{aligned}$$

$$
\begin{aligned}
\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} &= \begin{vmatrix} ye^x & e^x \\ e^x & 0 \end{vmatrix} \\
&= (0)ye^x - (e^x)^2 \\
&= -e^{2x}
\end{aligned}
$$

Since $-e^{2x}|_{x=0,y=0} = -1$, $(0,0)$ is a saddle point in f.

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2. $f(x,y) = xye^{-\frac{x^2+y^2}{2}}$

$$
\begin{aligned}
f_x &= \frac{\partial}{\partial x}\left(xye^{-\frac{x^2+y^2}{2}}\right) \\
&= y\left(\frac{\partial x}{\partial x}e^{-\frac{x^2+y^2}{2}} + x\frac{\partial}{\partial x}\left(e^{-\frac{x^2+y^2}{2}}\right)\right) \\
&= y\left(e^{-\frac{x^2+y^2}{2}} - x^2e^{-\frac{x^2+y^2}{2}}\right) \\
&= (1-x^2)ye^{-\frac{x^2+y^2}{2}}
\end{aligned}
$$

Here, I made the mistake of trying to solve the derivative at 0. When it got too complicated, I checked the solution sheet to see what method was employed, when I saw that the e factor could never be equal to 0. This means that $\forall x \in \mathbb{R}, y = 0 \iff f_x = 0$ and $\forall y \in \mathbb{R}, x = \pm 1 \iff f_x = 0$.

$$
\begin{aligned}
f_y &= \frac{\partial}{\partial y}\left(xye^{-\frac{x^2+y^2}{2}}\right) \\
&= x\left(\frac{\partial y}{\partial y}e^{-\frac{x^2+y^2}{2}} + y\frac{\partial}{\partial y}\left(e^{-\frac{x^2+y^2}{2}}\right)\right) \\
&= y\left(e^{-\frac{x^2+y^2}{2}} - y^2e^{-\frac{x^2+y^2}{2}}\right) \\
&= (1-y^2)xe^{-\frac{x^2+y^2}{2}}
\end{aligned}
$$

In this case, we have a similar argument to the solutions of f_x, except that now $x = 0$ and $y = \pm 1$. Or, more formally, $\forall y \in \mathbb{R}, x = 0 \iff f_y = 0$ and $\forall x \in \mathbb{R}, y = \pm 1 \iff f_y = 0$

Let's check a couple of cases where a point (x, y) could be a critical point.

- Let $x = 0$, implying $f_y = 0$. Then, for (x, y) to be a critical point, $f_x = 0 \implies x = \pm 1$ or $y = 0$. Since we established $x = 0$, then $y = 0$. The same happens if we let $y = 0$ and then try to find a suitable value for x.
- Let $x = \pm 1$, implying $f_x = 0$. Then, for (x, y) to be a critical point, $f_y = 0 \implies y = \pm 1$ or $x = 0$. Since we established $x = \pm 1$, then $y = \pm 1$.

Therefore, $\{(-1,-1), (-1,1), (0,0), (1,-1), (1,1)\}$ is the set of critical points of f.

$$
\begin{aligned}
f_{xx}(x,y) &= \frac{\partial}{\partial x}f_x \\
f_{xx}(x,y) &= ye^{-\frac{x^2+y^2}{2}}\frac{\partial}{\partial x}(1-x^2) + (1-x^2)\frac{\partial}{\partial x}\left(ye^{-\frac{x^2+y^2}{2}}\right) \\
f_{xx}(x,y) &= -2xe^{-\frac{x^2+y^2}{2}} + (1-x^2)\left(ye^{-\frac{x^2+y^2}{2}}\right)\frac{\partial}{\partial x}\left(-\frac{x^2+y^2}{2}\right) \\
f_{xx}(x,y) &= -2xe^{-\frac{x^2+y^2}{2}} + (1-x^2)\left(ye^{-\frac{x^2+y^2}{2}}\right)(-x) \\
f_{xx}(x,y) &= xy(x^2-3)e^{\frac{-x^2+y^2}{2}}
\end{aligned}
$$

(Note that computations for f_{xx} and f_{yy} are similar.)

$$\begin{aligned} f_{yy}(x,y) &= \frac{\partial}{\partial y} f_y \\ &= xy(y^2-3)e^{\frac{-x^2+y^2}{2}} \end{aligned}$$

$$\begin{aligned} f_{xy}(x,y) &= \frac{\partial}{\partial y} f_x \\ &= (1-x^2)\frac{\partial}{\partial y}\left(ye^{-\frac{x^2+y^2}{2}}\right) \\ &= (1-x^2)\frac{\partial}{\partial y}\left(ye^{-\frac{x^2+y^2}{2}}\right) \\ &= (1-x^2)\left(\frac{\partial y}{\partial y}e^{-\frac{x^2+y^2}{2}} + y\frac{\partial}{\partial y}e^{-\frac{x^2+y^2}{2}}\right) \\ &= (1-x^2)\left(e^{-\frac{x^2+y^2}{2}} - y^2e^{-\frac{x^2+y^2}{2}}\right) \\ &= (1-x^2)\left(e^{-\frac{x^2+y^2}{2}}\right)(1-y^2) \end{aligned}$$

The determinant of the Hessian Matrix is:

$$\begin{aligned} \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} &= \left(xye^{-\frac{x^2+y^2}{2}}\right)^2(x^2-3)(y^2-3) - \left((1-x^2)(1-y^2)\left(e^{-\frac{x^2+y^2}{2}}\right)\right)^2 \\ &= (x^2-3)(y^2-3)x^2y^2\left(e^{-\frac{x^2+y^2}{2}}\right)^2 - (1-x^2)^2(1-y^2)^2\left(e^{-\frac{x^2+y^2}{2}}\right)^2 \\ &= e^{-(x^2+y^2)}\left((x^2-3)(y^2-3)x^2y^2 - (1-x^2)^2(1-y^2)^2\right) \end{aligned}$$

and if we evaluated the Hessian determinant at the critical points:

1. At $(0,0)$:

$$\begin{aligned} e^{-x^2-y^2}\left((x^2-3)(y^2-3)x^2y^2 - (1-x^2)^2(1-y^2)^2\right)|_{0,0} \\ = e^0((-3)(-3)(0^2)(0^2) - (1)^2(1)^2) \\ = -1 < 0 \end{aligned}$$

Therefore, $(0,0)$ is a saddle point.

2. At $(-1,-1)$:

$$\begin{aligned} e^{-x^2-y^2}\left((x^2-3)(y^2-3)x^2y^2 - (1-x^2)^2(1-y^2)^2\right)|_{-1,-1} \\ = e^{-2}((-2)(-2) - (1-1)(1-1)) \\ = \frac{4-0}{e^2} = \frac{4}{e^2} > 0 \end{aligned}$$

and

$$f_{xx}(0,0) = (-1)(-1)((-1)^2 - 3)e^{-\frac{(-1)^2+(-1)^2}{2}} = (-2)e^{-1} = -\frac{2}{e} < 1$$

Therefore, $(-1,-1)$ is a local minimum.

Note that, for $(|x|,|y|) = (1,1)$, the determinant of the Hessian is always $\frac{4}{e^2}$. I'll skip computation for brevity.

3. At $(-1,1)$:

$$f_{xx}(-1,1) = (-1)(1)((-1)^2 - 3)e^{-\frac{(-1)^2+(1)^2}{2}} = (2)e^{-1} = \frac{2}{e} > 1$$

Therefore, $(-1,1)$ is a local maximum.

4. At $(1,-1)$:

$$f_{xx}(1,-1) = (1)(-1)((1)^2 - 3)e^{-\frac{(1)^2+(-1)^2}{2}} = (2)e^{-1} = \frac{2}{e} > 1$$

Therefore, $(1,-1)$ is a local maximum.

5. At $(1,1)$:

$$f_{xx}(1,1) = (1)(1)((1)^2 - 3)e^{-\frac{(1)^2+(1)^2}{2}} = (-2)e^{-1} = -\frac{2}{e} < 1$$

Therefore, $(1,1)$ is a local minimum.