# Section 2.3

## Problem 3

Verify that $\det(kA) = k^n \det(A)$.

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{bmatrix}; k = -4$$

### Answer

$$kA = \begin{bmatrix} -8 & 4 & -12 \\ -12 & -8 & -4 \\ -4 & -16 & -20 \end{bmatrix}$$

$$\begin{aligned} \det(kA) &= \begin{vmatrix} -8 & 4 & -12 \\ -12 & -8 & -4 \\ -4 & -16 & -20 \end{vmatrix} \\ &= -8(160 - 64) - 4(240 - 16) - 12(10)(16) \\ &= -8(96) - 4(224) - 12(10)(16) \\ &= -2^7(6 + 7 + 15) \\ &= -2^9(7) \end{aligned}$$

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$$\begin{aligned} \det(A) &= \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & 4 & 5 \end{vmatrix} \\ &= 56 \end{aligned}$$

$$\begin{aligned} 4^3 \det(A) &= 56(-2^6) \\ &= 7(-2^9) \end{aligned}$$

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## Problem 5

Verify that $\det(AB) = \det(BA)$ and determine whether the equality $\det(A + B) = \det(A) + \det(B)$ holds.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix}$$

### Answer

$$\begin{aligned} \det(A) &= 16 - 6 \\ &= 10 \end{aligned}$$

$$\begin{aligned} \det(B) &= 1 + (7 - 10) + 3(-5) \\ &= 1 - 3 - 15 \\ &= -17 \end{aligned}$$

$$\det(A)\det(B) = -170$$

$$
\begin{aligned}
AB &= \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 7 & 1 & 2 \\ 5 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2+7 & -2+1 & 6+2 \\ 3+28 & -3+4 & 9+8 \\ 10 & 0 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\det(AB) &= 18 + (62 - 170) + 8(-10) \\
&= 18 - 108 - 80 \\
&= -170
\end{aligned}
$$

---

Use determinants to decide whether the given matrix is invertible.

## Problem 7

$$
A = \begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}
$$

**Answer**

$$
\begin{aligned}
\det(A) &= -6 - 5(-3) + 5(-4+2) \\
&= -6 + 15 - 10 \\
&= -1
\end{aligned}
$$

Therefore, $A$ is invertible.

## Problem 13

$$
A = \begin{bmatrix} 2 & 0 & 0 \\ 8 & 1 & 0 \\ -5 & 3 & 6 \end{bmatrix}
$$

**Answer**

$$
\det(A) = 0
$$

Therefore, $A$ is singular.

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## Problem 17

Find the values for $k$ for which the matrix $A$ is invertible.

$$
A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \\ k & 3 & 2 \end{bmatrix}
$$

**Answer**

We know that

$$
\begin{aligned}
\det(A) &= (2-6) - 3(4-12) + k(12-4) \\
&= -4 + 24 + 8k \\
&= 20 + 8k
\end{aligned}
$$

Let $\det(A) \neq 0$. Then:

$$
\begin{aligned}
0 &\neq 8k + 20 \\
-\frac{20}{8} &\neq k \\
-\frac{5}{2} &\neq k
\end{aligned}
$$

Therefore, $\forall k \neq -\frac{5}{2}$, $A$ is invertible

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## Problem 19

Decide whether the matrix is invertible, and if so, use the adjoint method to find its inverse.

$$
A = \begin{bmatrix} 2 & 5 & 5 \\ -1 & -1 & 0 \\ 2 & 4 & 3 \end{bmatrix}
$$

### Answer

From *Problem 2.3.7*, we know that $\det(A) = -1$. Therefore, the matrix is invertible.

From computation, we get that the adjoint of $A$ is:

$$
adj(A) = \begin{bmatrix} -3 & 3 & -2 \\ 5 & -4 & 2 \\ 5 & -5 & 3 \end{bmatrix}
$$

Therefore,

$$
\begin{aligned}
A^{-1} &= -\begin{bmatrix} -3 & 3 & -2 \\ 5 & -4 & 2 \\ 5 & -5 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 3 & -3 & 2 \\ -5 & 4 & 2 \\ -5 & 5 & -3 \end{bmatrix}
\end{aligned}
$$

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## Problem 25

Solve by Cramer's rule, if it applies.

$$\begin{aligned} 4x + 5y &= 2 \\ 11x + y + 2z &= 3 \\ x + 5y + 2z &= 1 \end{aligned}$$

**Answer**

Let

$$A = \begin{bmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix}$$

and

$$\boldsymbol{b} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

Then

$$\begin{aligned} \det(A) &= -2(20-5) + 2(11-1) \\ &= -30 + 20 \\ &= -10 \end{aligned}$$

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$$\begin{aligned} \det(A_1) &= \begin{vmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} \\ &= -2(10-5) + 2(15-1) \\ &= -2(5) + 2(14) \\ &= -10 + 28 \\ &= 18 \end{aligned}$$

$$\boxed{\frac{\det(A_1)}{\det(A)} = -\frac{18}{10} = -\frac{9}{5}}$$

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$$\begin{aligned} \det(A_2) &= \begin{vmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix} \\ &= -2(4-2) + 2(12-12) \\ &= -4 \end{aligned}$$

$$\boxed{\frac{\det(A_2)}{\det(A)} = -\frac{4}{10} = -\frac{2}{5}}$$

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$$\begin{aligned}
\det(A_3) &= \begin{vmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{vmatrix} \\
&= 2(11-5) - 3(20-5) + (4-55) \\
&= 2(6) - 3(15) - 51 \\
&= 12 - 45 - 51 \\
&= -82
\end{aligned}$$

$$\boxed{\frac{\det(A_3)}{\det(A)} = -\frac{41}{5}}$$

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## Problem 33

Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Assuming that $\det(A) = -7$, find

a)

$$\det(3A)$$

b)

$$\det(A^{-1})$$

c)

$$\det(2A^{-1})$$

d)

$$\det((2A)^{-1})$$

e)

$$\det\left(\begin{bmatrix} a & g & d \\ b & h & e \\ c & i & f \end{bmatrix}\right)$$

### Answer

a)

$$\det(3A) = 27(-7) = -189$$

b)

$$\det(A^{-1}) = \frac{1}{\det(A)} = -\frac{1}{7}$$

c)

$$\det(2A^{-1}) = 8\det(A^{-1}) = \left(-\frac{8}{7}\right)$$

d)

$$\det((2A)^{-1}) = \frac{1}{\det(2A)} = \frac{1}{8\det(A)} = -\frac{1}{56}$$

d)

$$\det\left(\begin{bmatrix} a & g & d \\ b & h & e \\ c & i & f \end{bmatrix}\right) = \det(A^T) = \det(A) = -7$$

## Problem 35

Find the determinant $\det(-A)$, given that $A$ is a $4 \times 4$ matrix for which $\det(A) = -2$.

### Answer

$$\det(-A) = (-1)^4 \det(A) = \det(A) = -2$$

# Section 3.1

## Exercise 3

Find the components of the vector $\overline{P_1 P_2}$.

1. $P_1(3,5), P_2(2,8)$
2. $P_1(5,-2,1), P_2(2,4,2)$

### Answer

1. $\overline{P_1 P_2} = (-1,3)$
2. $\overline{P_1 P_2} = (-3,6,1)$

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## Exercise 5

1. Find the terminal point of the vector that is equivalent to $\boldsymbol{u} = (1,2)$

and whose and whose initial point is $A(1,1)$.

1. Find the initial point of the vector that is equivalent to $\boldsymbol{u} = (1,1,3)$

and whose and whose initial point is $B(-1,-1,2)$.

### Answer

Let $\boldsymbol{v}$ be the vector of interest.

1. $\boldsymbol{v} = \boldsymbol{u} + A = (1,2) + (1,1) = (2,3)$.
2. $\boldsymbol{v} = B - \boldsymbol{u} = (-1,-1,2) - (1,1,3)$

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## Exercise 11

Attempted!!!

Let $\boldsymbol{u} = (-3,2,1,0), \boldsymbol{v} = (4,7,-3,2)$ and $\boldsymbol{w} = (5,-2,8,1)$. Find the components of

1. $\boldsymbol{v} - \boldsymbol{w}$
2. $-\boldsymbol{v} + (\boldsymbol{v} - 4\boldsymbol{w})$
3. $6(\boldsymbol{u} - 3\boldsymbol{v})$
4. $6(\boldsymbol{v} - \boldsymbol{w}) - (4\boldsymbol{u} + \boldsymbol{v}$

### Answer

1. $$\begin{aligned} \boldsymbol{v} - \boldsymbol{w} &= (4,7,-3,2) - (5,-2,8,1) \\ &= (-1,9,-11,-1) \end{aligned}$$
2. $$\begin{aligned} -\boldsymbol{u} + (\boldsymbol{v} - 4\boldsymbol{w}) &= -(-3,2,1,0) + ((4,7,-3,2) - 4(5,-2,8,1)) \\ &= (3,-2,-1,0) + ((4,7,-3,2) + (-20,8,-32,-4)) \\ &= (3,-2,-1,0) + (-16,15,-35,-2) \\ &= (-13,13,-36,-2) \end{aligned}$$

3. $$\begin{aligned} 6(\boldsymbol{u} - 3\boldsymbol{v}) &= 6((-3,2,1,0) - 3(4,7,-3,2)) \\ &= 6((-3,2,1,0) + (-12,-21,9,-6)) \\ &= 6(-15,-19,19,-6) \\ &= (-90,-114,60,-36) \end{aligned}$$
4. $$\begin{aligned} 6(\boldsymbol{v} - \boldsymbol{w}) - (4\boldsymbol{u} + \boldsymbol{v}) &= 6((4,7,-3,2) + (-5,2,-8,-1)) - (4(-3,2,1,0) - (4,7,-3,2)) \\ &= 6(-1,9,-11,1) - ((-12,8,4,0) + (-4,-7,3,-2)) \\ &= (-6,54,-66,6) - (-16,1,7,-2) \\ &= (-22,53,-59,8) \end{aligned}$$

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## Exercise 15

Which of the following vectors in $\mathbb{R}^6$, if any, are parallel to $\boldsymbol{u} = (-2,1,0,3,5,1)$?

1. $(4,2,0,6,10,2)$
2. $(4,-2,0,-6,-10,-2)$
3. $(0,0,0,0,0,0)$

**Answer**

$(4,-2,0,-6,-10,-2) = -1\boldsymbol{u}$. Therefore, it is parallel to $\boldsymbol{u}$.

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## Exercise 17

Let $\boldsymbol{u} = (1,-1,3,5)$ and $\boldsymbol{v} = (2,1,0,-3)$. Find scalars $a$ and $b$ so that $a\boldsymbol{u} + b\boldsymbol{v} = (1,-4,9,18)$.

**Answer**

From the given assumptions, it follows that

$$\begin{aligned} a\boldsymbol{u} + b\boldsymbol{v} &= (1,-4,9,18) \\ (a,-a,3a,5a) + (2b,b,0,-3b) &= (1,-4,9,18) \\ (a,-a,3a,5a) &= (1,-4,9,18) - (2b,b,0,-3b) \\ (a,-a,3a,5a) &= (1-2b,-4-b,9,18+3b) \\ (a,-a,3a,5a) &= (1-2b,-4-b,9,18+3b) \end{aligned}$$

It follows that $a = 3$, and $3 = 1 - 2b \iff b = -1$.

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## Exercise 19

Find scalars $c_1, c_2$ and $c_3$ for which

$$c_1(1,-1,0) + c_2(3,2,1) + c_3(0,1,4) = (-1,1,19)$$

is satisfied.

**Answers**

From the given assumptions, it follows that

$$\begin{aligned} (-1,1,19) &= c_1(1,-1,0) + c_2(3,2,1) + c_3(0,1,4) \\ &= (c_1,-c_1,0) + (3c_2,2c_2,c_2) + (0,c_3,4c_3) \end{aligned}$$

from which we get the following system of equations:

$$
\begin{aligned}
c_1 + 3c_2 &= -1 \\
-c_1 + 2c_2 + c_3 &= 1 \\
c_2 + 4c_3 &= 19
\end{aligned}
$$

For which the solutions is at $c_1 = 2, c_2 = -1$ and $c_3 = 5$.

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### Exercise 21

Show that there do not exist scalars $c_1, c_2$ and $c_3$ such that

$$c_1(-2, 9, 6) + c_2(-3, 2, 1) + c_3(1, 7, 5) = (0, 5, 4)$$

#### Answers

We can prove this by contradiction: Let's assume that the given equation can be express as a consistent system of linear equations. Then, from the given assumptions, it follows that

$$
\begin{aligned}
(0, 5, 4) &= c_1(-2, 9, 6) + c_2(-3, 2, 1) + c_3(1, 7, 5) \\
&= (-2c_1, 9c_1, 6c_1) + (-3c_2, 2c_2, c_2) + (c_3, 7c_3, 5c_3)
\end{aligned}
$$

from which we get the following system of equations:

$$
\begin{aligned}
-2c_1 - 3c_2 + c_3 &= 0 \\
9c_1 + 2c_2 + 7c_3 &= 5 \\
6c_1 + c_2 + 5c_3 &= 4
\end{aligned}
$$

But this system implies $c_3 = 2c_1 + 3c_2 \Longrightarrow c_1 = \frac{1}{4} - c_2 \Longrightarrow 0c_2 \frac{11}{2} = 5$, which is a contradiction. ∎

## Section 3.2

## Section 3.3