MATH 2210 HOMEWORK WORKSHEET 4 SOLUTIONS

Name: KEY

Arc Length and Curvature

1. Find the length of the curve $\mathbf{r}(t) = \langle 2t^{2/3},\ \cos(2t),\ \sin(2t)\rangle$, $0 \le t \le 1$.

First note that

$$\mathbf{r}'(t) = \left\langle \frac{4}{3}t^{-1/3},\ -2\sin(2t),\ 2\cos(2t) \right\rangle.$$

Then

$$\begin{aligned} |\mathbf{r}'(t)| &= \sqrt{\left(\frac{4}{3}t^{-1/3}\right)^2,\ (-2\sin(2t))^2 + (2\cos(2t))^2} \\ &= \sqrt{\frac{16}{9}t^{-2/3} + 4\sin^2(2t) + 4\cos^2(2t)} \\ &= \sqrt{\frac{16}{9}t^{-2/3} + 4} \\ &= \sqrt{\frac{16 + 36t^{2/3}}{9t^{2/3}}} \\ &= \frac{2}{3t^{1/3}}\sqrt{4 + 9t^{2/3}} \end{aligned}$$

Then the arc length is

$$\begin{aligned} L &= \int_0^1 |\mathbf{r}'(t)|\ dt \\ &= \int_0^1 \frac{2}{3t^{1/3}}\sqrt{4 + 9t^{2/3}}\, dt && u = 4 + 9t^{2/3}, \quad du = 6t^{-1/3} = 9\left(\frac{2}{3t^{1/3}}\right) \\ &= \frac{1}{9}\int_4^{13} \sqrt{u}\, du && u(0) = 4, \quad u(1) = 13 \\ &= \frac{1}{9}\left(\frac{2}{3}u^{3/2}\right)\Big|_4^{13} \\ &= \frac{2}{27}(13^{3/2} - 8) \\ &= \frac{2(13^{3/2} - 8)}{27} \end{aligned}$$

2. Reparameterize the curve

$$\mathbf{r}(t) = \mathrm{e}^t\,\mathbf{i} + \mathrm{e}^t \sin t\,\mathbf{j} + \mathrm{e}^t \cos t\,\mathbf{k}$$

with respect to arc length measured from the point $(1, 0, 1)$ in the direction of increasing t.

Note that $(1, 0, 1) = \mathbf{r}(0)$. Then

$$\mathbf{r}'(t) = \langle \mathrm{e}^t,\ \mathrm{e}^t \sin t + \mathrm{e}^t \cos t,\ \mathrm{e}^t \cos t - \mathrm{e}^t \sin t \rangle$$

$$\begin{aligned} |\mathbf{r}'(t)| &= \sqrt{\mathrm{e}^{2t} + \mathrm{e}^{2t}(\sin^2 t + 2\sin t\cos t + \cos^2 t) + \mathrm{e}^{2t}(\cos^2 t - 2\sin t\cos t + \sin^2 t)} \\ &= \sqrt{\mathrm{e}^{2t} + 2\mathrm{e}^{2t}} \\ &= \mathrm{e}^t\sqrt{3} \end{aligned}$$

$$\begin{aligned} s(t) &= \int_0^t \mathrm{e}^\beta \sqrt{3}\, d\beta \\ &= \mathrm{e}^\beta \sqrt{3}\Big|_0^t \\ s(t) &= \mathrm{e}^t\sqrt{3} - \sqrt{3} \end{aligned}$$

Solving for t yields

$$t = \ln\left(\frac{s + \sqrt{3}}{\sqrt{3}}\right)$$

and hence

$$\mathbf{r}(s) = \left(\frac{s+\sqrt{3}}{\sqrt{3}}\right)\mathbf{i} + \left(\frac{s+\sqrt{3}}{\sqrt{3}}\right)\sin\left(\ln\left(\frac{s+\sqrt{3}}{\sqrt{3}}\right)\right)\mathbf{j} + \left(\frac{s+\sqrt{3}}{\sqrt{3}}\right)\cos\left(\ln\left(\frac{s+\sqrt{3}}{\sqrt{3}}\right)\right)\mathbf{k}$$

3. Consider the curve given by $\mathbf{r}(t) = \langle \sin^3 t,\ \cos^3 t,\ \sin^2 t \rangle$, $0 \le t \le \pi/2$.

(a) Find the unit tangent vector. *Note: This question was asked on the previous homework as well.*

From Homework 13,

$$\mathbf{T}(t) = \left\langle \frac{3}{\sqrt{13}} \sin t,\ -\frac{3}{\sqrt{13}} \cos t,\ \frac{2}{\sqrt{13}} \right\rangle$$

(b) Find the unit normal vector.

$$\begin{aligned} \mathbf{T}'(t) &= \left\langle \frac{3}{\sqrt{13}} \cos t,\ \frac{3}{\sqrt{13}} \sin t,\ 0 \right\rangle \\ |\mathbf{T}'(t)| &= \sqrt{\frac{9}{13}\cos^2 t + \frac{9}{13}\sin^2 t} = \sqrt{\frac{9}{13}} = \frac{3}{\sqrt{13}} \\ \mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \langle \cos t,\ \sin t,\ 0 \rangle \end{aligned}$$

(c) Find the unit binormal vector.

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{3}{\sqrt{13}} \sin t & -\frac{3}{\sqrt{13}} \cos t & \frac{2}{\sqrt{13}} \\ \cos t & \sin t & 0 \end{vmatrix}$$

$$= \left\langle -\frac{2}{\sqrt{13}} \sin t,\ \frac{2}{\sqrt{13}} \cos t,\ \frac{3}{\sqrt{13}} (\sin^2 t + \cos^2 t) \right\rangle = \left\langle -\frac{2}{\sqrt{13}} \sin t,\ \frac{2}{\sqrt{13}} \cos t,\ \frac{3}{\sqrt{13}} \right\rangle$$

(d) Find the curvature.

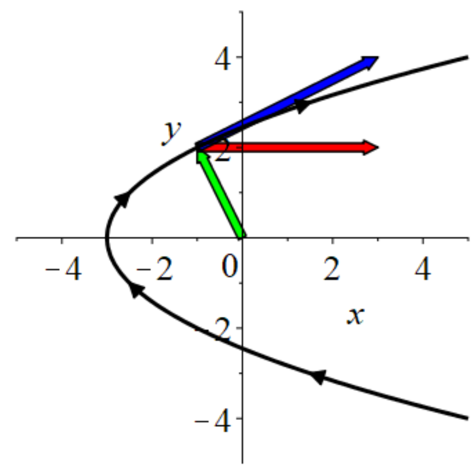
$$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{\frac{3}{\sqrt{13}}}{\sqrt{13} \sin t \cos t} = \frac{3}{13 \sin t \cos t}$$

Motion in Space: Velocity and Acceleration

4. Find the velocity, speed, and acceleration of a particle moving with position function

$$\mathbf{r}(t) = (2t^2 - 3)\mathbf{i} + 2t\mathbf{j}.$$

Sketch the path of the particle on the axes below and draw the position, velocity, and acceleration vectors for $t = 1$.



The velocity, speed, and acceleration are the following functions.

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 4t, 2\rangle$$

$$s(t) = |\mathbf{v}| = \sqrt{(4t)^2 + 2^2} = 2\sqrt{4t^2 + 1}$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 4, 0\rangle$$

At $t = 1$, we have that

position: $\mathbf{r}(1) = \langle -1, 2\rangle$, velocity: $\mathbf{v}(1) = \langle 4, 2\rangle$, acceleration: $\mathbf{a}(1) = \langle 4, 0\rangle$.

These are plotted above in green (position), blue (velocity), and red (acceleration).

5. Find the tangential and normal components of the acceleration vector of the curve

$$\mathbf{r}(t) = t\,\mathbf{i} + 2\mathrm{e}^t\,\mathbf{j} + \mathrm{e}^{2t}\,\mathbf{k}.$$

$$\mathbf{v} = \mathbf{r}'(t) = \langle 1,\ 2\mathrm{e}^t,\ 2\mathrm{e}^{2t}\rangle$$

$$\mathbf{a} = \mathbf{r}''(t) = \langle 0,\ 2\mathrm{e}^t,\ 4\mathrm{e}^{2t}\rangle$$

$$\begin{aligned} |\mathbf{r}'(t)| &= \langle\sqrt{1 + (2\mathrm{e}^t)^2 + (2\mathrm{e}^{2t})^2} = \sqrt{1 + 4\mathrm{e}^{2t} + 4\mathrm{e}^{4t}} \\ &= \sqrt{(1 + 2\mathrm{e}^{2t})^2} = 1 + 2\mathrm{e}^{2t} \end{aligned}$$

$$\begin{aligned} \mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2\mathrm{e}^t & 2\mathrm{e}^{2t} \\ 0 & 2\mathrm{e}^t & 4\mathrm{e}^{2t} \end{vmatrix} \\ &= (8\mathrm{e}^{3t} - 4\mathrm{e}^{3t})\,\mathbf{i} - (4\mathrm{e}^{2t} - 0)\,\mathbf{j} + (2\mathrm{e}^t - 0)\,\mathbf{k} \\ &= 4\mathrm{e}^{3t}\,\mathbf{i} - 4\mathrm{e}^{2t}\,\mathbf{j} + 2\mathrm{e}^t\,\mathbf{k} \end{aligned}$$

$$\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$$

$$\begin{aligned} a_{\mathbf{T}} &= \frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}|} = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} \\ &= \frac{4\mathrm{e}^{2t} + 8\mathrm{e}^{4t}}{1 + 2\mathrm{e}^{2t}} \\ &= \frac{4\mathrm{e}^{2t}(1 + 2\mathrm{e}^{2t})}{1 + 2\mathrm{e}^{2t}} \\ a_{\mathbf{T}} &= 4\mathrm{e}^{2t} \end{aligned}$$

$$\begin{aligned} a_{\mathbf{N}} &= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} \\ &= \frac{\sqrt{(4\mathrm{e}^{3t})^2 + (-4\mathrm{e}^{2t})^2 + (2\mathrm{e}^2)^2}}{\sqrt{1^2 + (2\mathrm{e}^t)^2 + (2\mathrm{e}^{2t})^2}} \\ &= \frac{2\mathrm{e}^t\sqrt{4\mathrm{e}^{4t} + 4\mathrm{e}^{2t} + 1}}{\sqrt{1 + 4\mathrm{e}^{2t} + 4\mathrm{e}^{4t}}} \\ a_{\mathbf{N}} &= 2\mathrm{e}^t \end{aligned}$$