Section 4.4

Exercise 1

Use the method of Example 3 to show that the following set of veectors forms a basis for $\mathbb{R}^2$.

$$\{(2,1),(3,0)\}$$

Answer

The given set is basis for $\mathbb{R}^2$ if and only if the vectors are linearly independent and they span the rest of $\mathbb{R}^2$.

Linear independence (without loss of generality) is satisfied if for the next equation

$$k_1(2,1)+k_2(3,0)=(0,0)$$

which can be rewritten as the following system of equations:

$$\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} & k_1 \\ k_2 & \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the answers are only $k_1=0$ and $k_2=0$. An equivalent statement is that the determinant of the coefficient matrix of the given system is nonzero. Therefore, if the determinant of the coefficient matrix is 0, then the column vectors are linearly dependent. Since $2(0)-3=-1\neq 0$, the the column vectors span $\mathbb{R}^2$.

The given basis spans the given vector space if $\exists k_1, k_2 \in \mathbb{R}$ such that, $\forall (b_1,b_2)\in\mathbb{R}^2$.

$$k_1(2,1)+k_2(3,0)=(b_1,b_2)$$

Similar to before, an equivalent statement is that the coefficient matrix of the given system is nonzero, which we already proved. ∎

Exercise 3

Show that the following polynomials form a basis for $P-2$.

$$1+x,\ x^2-1,\ 2x-1$$

Answer

We can test linear independence and span by evaluating the determinant of the corresponding Wronskian to this basis.

$$\begin{aligned}\det(W)&=\begin{vmatrix} 1+x & x^2-1 & 2x-1 \\ 1 & 2x & 2 \\ 0 & 2 & 0 \end{vmatrix}\\ &=-2(2+2x-2x+1)\\ &=-2(3)\\ &=-6\end{aligned}$$

Since the determinant is nonzero, this conforms a basis.

Exercise 7

In each part, show that the set of vectors is not a basis for $\mathbb{R}^3$.

(a) $\{(2,-3,1),(4,1,1),(0,-7,1)\}$

(b) $\{(1,6,4),(2,4,-1),(-1,2,5)\}$

Answer

(a) $\begin{vmatrix} 2 & 4 & 0 \\ -3 & 1 & -7 \\ 1 & 1 & 1 \end{vmatrix} = 7(2-4)+(2+12)=0.$

(b) $\begin{vmatrix} 1 & 2 & -1 \\ 6 & 4 & 2 \\ 4 & -1 & 5 \end{vmatrix} = (20+2)-2(30-8)-1(-6-16)=0.$

Exercise 9

Show that the following matrices do not form a basis for M_{22}:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} & 2 & -2 \\ 3 & 2 & \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

Answer

These matrices are linarly independent if $\nexists (k_1,k_2,k_3,k_4) \in \mathbb{R}^4$ such that

$$k_1 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + k_2 \begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix} + k_3 \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} + k_4 \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and $(k_1,k_2,k_3,k_4) \neq \mathbf{0}$. We can rearrange this system as follows:

$$\begin{bmatrix} k_1+2k_2+k_3 \\ -k_2-k_3-k_4 \\ k_1+3k_2+k_3+k_4 \\ k_1+2k_2+k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which, can be expressed as the following linear transformation:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & -1 \\ 1 & 3 & 1 & 1 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} & 0 \\ 0 & \\ 0 & \\ 0 & \end{bmatrix}$$

Since the determinant of the cofficient matrix is 0, the given set is not a basis for $\mathbb{R}^4$.

Exercise 11

Find the coordinate vector of $\boldsymbol{w}$ relative to the basis $S=\{\boldsymbol{u}_1,\boldsymbol{u}_2\}$ for $\mathbb{R}^2$.

(a) $\boldsymbol{u}_1=(2,-4), \boldsymbol{u}_2=(3,8), \boldsymbol{w}=(1,1)$

(b) $\boldsymbol{u}_1=(1,1), \boldsymbol{u}_2=(0,2), \boldsymbol{w}=(a,b)$

Answer

(a) $\begin{bmatrix} 2 & 3 \\ -4 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} \frac{2}{7} & -\frac{3}{28} \\ \frac{1}{7} & \frac{1}{14} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{5}{28} \\ \frac{3}{14} \end{bmatrix} \end{aligned}$$

(b) $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} a \\ \frac{a-b}{2} \end{bmatrix} \end{aligned}$$

Exercise 13

Find the coordinate vector of $\boldsymbol{v}$ relative to the basis $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ for $\mathbb{R}^3$.

(a) $\boldsymbol{v} = (2, -1, 3), \boldsymbol{v}_1 = (1, 0, 0), \boldsymbol{v}_2 = (2, 2, 0), \boldsymbol{v}_3 = (3, 3, 3)$

(b) $\boldsymbol{v} = (5, -12, 3), \boldsymbol{v}_1 = (1, 2, 3), \boldsymbol{v}_2 = (-4, 5, 6), \boldsymbol{v}_3 = (7, -8, 9)$

Answer

(a)
$$\begin{aligned} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \\ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ 1 & -1 & -1 \\ 0 & \frac{1}{2} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \end{aligned}$$

(b)
$$\begin{aligned} \begin{bmatrix} 1 & -4 & 7 \\ 2 & 5 & -8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 5 \\ -12 \\ 3 \end{bmatrix} \\ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} \frac{31}{80} & \frac{13}{40} & -\frac{1}{80} \\ -\frac{7}{40} & -\frac{1}{20} & \frac{11}{120} \\ -\frac{1}{80} & -\frac{3}{40} & \frac{13}{240} \end{bmatrix} \begin{bmatrix} 5 \\ -12 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Exercise 15

First show that the set $S = \{A_1, A_2, A_3, A_4\}$ is a basis for M_{22}, then express A as a linear combination of the vectors in S, and then find the coordinate vector of A relative to S.

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Answer

Solved in MATLAB using the following script:

```
A = [1,0;1,0];
A_1 = [1,1;1,1]; A_2 = [0,1;1,1]; A_3 = [0,0;1,1]; A_4 = [0,0;0,1];

function vectorized = vectorize(A)
    vectorized = reshape(A, [4,1])
end

function coordinates = verify_and_solve(A, A_1, A_2, A_3, A_4)
    basis = horzcat(
        vectorize(A_1),
        vectorize(A_2),
        vectorize(A_3),
        vectorize(A_4)
    )

    if det(basis) != 0
        coordinates = reshape(basis \ vectorize(A), [2,2])
      end
end

answers = verify_and_solve(A, A_1, A_2, A_3, A_4)
```

Answer is $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$.

Exercise 17

First show that $S = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3\}$ is a basis for $\mathbb{P}_2$, then express $\boldsymbol{p}$ as a linear combination of the vectors in S, and then find the coordinate vector of $\boldsymbol{p}$ relative to S.

$$\boldsymbol{p}_1 = 1 + x + x^2, \quad \boldsymbol{p}_2 = x + x^2, \quad \boldsymbol{p}_3 = x^2, \quad \boldsymbol{p} = 7 - x + 2x^2$$

Answer

We can express this system as follows:

$$k_1\boldsymbol{p}_1 + k_2\boldsymbol{p}_2 + k_3\boldsymbol{p}_3 = \boldsymbol{p}$$

Let $f : \mathbb{R}^3 \to \mathbb{R}$ such that $f(\boldsymbol{a}) = \boldsymbol{a} \cdot (1, x, x^2)$. Then

$$
\begin{aligned}
k_1\boldsymbol{p}_1 + k_2\boldsymbol{p}_2 + k_3\boldsymbol{p}_3 &= k_1 f(1,1,1) + k_2 f(0,1,1) + k_3 f(0,0,1) \\
&= [f(1,1,1) \;\; f(0,1,1) \;\; f(0,0,1)] \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ x & x & 0 \\ x^2 & x^2 & x^2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}
\end{aligned}
$$

and,

$$
\begin{aligned}
7 - x + 2x^2 &= f(7,-1,2) \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x^2 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \\ 2 \end{bmatrix}
\end{aligned}
$$

We can simplify the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x^2 \end{bmatrix}$ from both sides,

$$
\begin{aligned}
\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} &= \begin{bmatrix} 7 \\ -1 \\ 2 \end{bmatrix} \\
\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \\ 2 \end{bmatrix} \\
\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} &= \begin{bmatrix} 8 \\ -3 \\ 2 \end{bmatrix}
\end{aligned}
$$

Therefore, $k_1 = 8, k_2 = -3, k_3 = 2$.

Exercise 21

In eatch part, let $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ be a multiplication by A, and let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be the standard basis for $\mathbb{R}^3$. Determine whether the set $\{T_A(\boldsymbol{e}_1), T_A(\boldsymbol{e}_2), T_A(\boldsymbol{e}_3)\}$ is linearly independent in $\mathbb{R}^3$.

(a)
$$
A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ -1 & 2 & 0 \end{bmatrix}
$$

(b)
$$
A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}
$$

Answer

Note that $\det(I_3) = 1$. Therefore, the set resulting from transforming such basis is linearly independent if and only if the basis of the linear transformation is linearly independent.

(a) $\det(A) = 10$. Therefore, it is linearly independent.

(b) $\det(A) = 0$. Therefore, it is linearly *dependent*.

Section 4.5 (8 Exercises)

In Exercises 3 and 5, find a basis for the solution space of the homogenous linear system, and find the dimension of that space.

Exercise 4.5.3

$$
\begin{aligned}
2x_1 + x_2 + 3x_3 &= 0 \\
x_1 + 5x_3 &= 0 \\
x_2 + x_3 &= 0
\end{aligned}
$$

Answer

Let A be the coefficient matrix corresponding the given system. We can find the solution space by row reducing the matrix.

$$\operatorname{rref}\left(\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 5 \\ 0 & 1 & 1 \end{bmatrix}\right) = I$$

Given by theorem, the basis of the solution space of is $\{\mathbf{0}\}$.

Exercise 4.5.5

$$
\begin{aligned}
x_1 - 3x_2 + x_3 &= 0 \\
2x_1 - 6x_2 + 2x_3 &= 0 \\
3x_1 - 9x_1 + 3x_3 &= 0
\end{aligned}
$$

Answer

Similar to *4.3.3*.

$$\operatorname{rref}\left(\begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ 3 & -9 & 3 \end{bmatrix}\right) = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This means that any solution to the homogenous system is a scalar multiple of $(3t + s, t, s)$. We can express this vector as the linear combination $t(3, 1, 0) + s(1, 0, 1)$.

Therefore, the basis of the solution space is $\{(3, 1, 0), (1, 0, 1)\}$.

Exercise 4.5.7

In each part, find a basis for the given subspace of $\mathbb{R}^3$, and state its dimensions.

(a) The plane $3x - 2y + 5z = 0$.

(b) The plane $x - y = 0$.

(c) The plane $x = 2t, y = -t, z = 4t$.

(d) All vectors of the form (a, b, c), where $b = a + c$.

Answer

(a) $\left\{\left(\frac{2}{3}, 1, 0\right), \left(-\frac{5}{3}, 0, 1\right)\right\}$. 2 dimensions.

(b) $\{(1, 1, 0), (0, 0, 1)\}$. 2 dimensions.

(c) $\{(2, -1, 4)\}$. 1 dimension.

(d) $\{(1, 1, 0), (0, 1, 1)\}$. 2 dimensions.

Exercise 4.5.9

Find the dimension of each of the following vector spaces:

(a) The vector space of all diagonal $n \times n$ matrices.

(b) The vector space of all symmetric $n \times n$ matrices.

(c) The vector space of all upper triangular $n \times n$ matrices.

Answer

(a) n dimensions.

(b) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ dimensions.

(c) Identical to above.

Exercise 4.5.13

Find a standard basis vector for $\mathbb{R}^4$ that can be add to the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2)$ to produce a basis for $\mathbb{R}^4$.

(a) $\boldsymbol{v}_1 = (1, -4, 2, -3), \boldsymbol{v}_2 = (-3, 8, -4, 6)$

Answer

(a) The solution space of span of the given vectors is $\left\{\left(0, \frac{3}{4}, 1, 0\right), \left(0, -\frac{1}{2}, 0, 1\right)\right\}$. Adding such vectors to the basis will make a basis to $\mathbb{R}^4$.

Exercise 4.5.15

The vectors $\boldsymbol{v}_1 = (1, -2, 3)$ and $\boldsymbol{v}_2 = (0, 5, -3)$ are linearly independent. Enlarge $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

Answer

We can find a third vector to complete the basis by using the cross product.

$$\boldsymbol{v}_1 \times \boldsymbol{v}_2 = (-2, 8, -5)$$

Exercise 4.5.17

Find a basis for the subspace of $\mathbb{R}^3$ that is spanned by the vectors

$$\boldsymbol{v}_1 = (1, 0, 0), \boldsymbol{v}_2 = (1, 0, 1), \boldsymbol{v}_3 = (2, 0, 1), \boldsymbol{v}_4 = (0, 0, -1)$$

Answer

Since $\boldsymbol{v}_3 = \boldsymbol{v}_1 + \boldsymbol{v}_2$ and $\boldsymbol{v}_4 = \boldsymbol{v}_1 - \boldsymbol{v}_2$, $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is the basis for the spanned subspace.

Exercise 4.5.19

In each part, let $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ be a multiplication by A and find the dimension of the subspace of $\mathbb{R}^3$ consisting of all vectors $\boldsymbol{x}$ for which $T_A(\boldsymbol{x}) = \boldsymbol{0}$.

(a)
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

(b)
$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

(c)
$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Answer

(a) 1 dimension.

(b) 2 dimensions.

(c) 0 dimensions.