Homework 3 ,

Carlos Rubio

January 21th, 2025

Vector Functions and Space Curves

Problem 1

Let $\hat{r}(t) = \langle \sqrt{2-t}, \frac{e^t-1}{t}, \ln(1+t) \rangle$

1. Find the domain of $\hat{r}(t)$
2. Find $\lim_{t\to 0} \hat{r}(t)$

Answer:

1. Originally, I thought that the z-component ($\ln(1+t)$) would influence the

domain such that we could discard all the numbers in $\mathbb{R}^-$ (therefore, $t \in (0, 2]$). This is mistaken, since $\ln(1+t) \Longrightarrow 1+t \geq 0 \Longrightarrow t \geq -1$. Therefore, the correct domain is

$$t \in (-1, 0) \cup (0, 2)$$

2. I made a mistake while computating the limit and applying L'hopital's rule

mentally, and concluded that $\lim_{t\to 0} \frac{e^t-1}{t} = \lim_{t\to 0}(e^t) = 0$ instead of 1. Of course, this is a very dumb mistake. Here is the actual limit:

$$\lim_{t\to 0} \hat{r}(t) = \langle \sqrt{2}, 1, 0 \rangle$$

Problem 2

Sketch the curve with the equation

$$\hat{r}(t) = \langle \sin(\pi t), t, \cos(\pi t) \rangle$$

and indicate with an arrow the direction in which t increases.

Answer:

This is a helix with radius 1. When looked from the y-axis, the helix casts a circle shadow over the xz-plane, that is maped in a counter-clockwise motion with respect to t. The helix progreses in the positive direction along the y-axis.

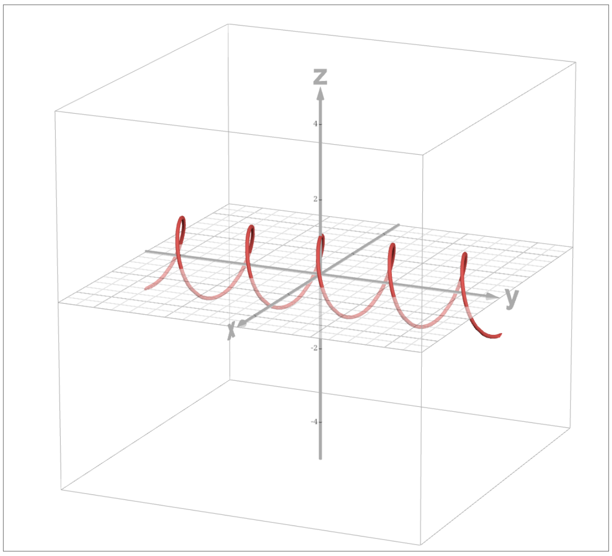

Figure 1: Nice image of a the given function, powered by Desmos 3D

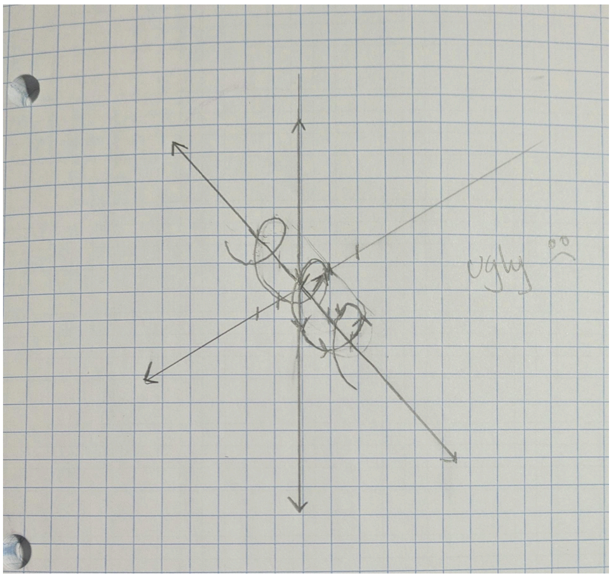

Figure 2: My attempt to make the helix look like the one in the solution sheet

Problem 3

Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 16$ and the plane $x + z = 5$

Answer:

$x^2 + y^2 = 16 \implies x(t) = 4\cos(t)$ and $y(t) = 4\sin(t), 0 \leq t \leq 4\pi$.

Then, $x + z = 5 \implies z = 5 - 4\cos(t)$.

The vector function for the curve of the intersection is $\hat{r}(t) = \langle 4\cos(t), 4\sin(t), 5 - 4\cos(t)\rangle$.

Derivatives and Integral of Vector Fuctions

Problem 4

Let $\hat{r}(t) = \langle\sqrt{2-t}, \frac{e^t-1}{t}, \ln(1+t)$. Find $\hat{r}'(t)\rangle$.

Answer:

$$\begin{aligned}
\frac{d\hat{r}(t)}{dt} &= \langle\frac{d\sqrt{2-t}}{dt}, \frac{d\frac{e^t-1}{t}}{dt}, \frac{d\ln(1+t)}{dt}\rangle \\
&= \langle\frac{d(2-t)^{\frac{1}{2}}}{dt}, \frac{d\frac{e^t-1}{t}}{dt}, \frac{d\ln(1+t)}{dt}\rangle \\
&= \langle-\frac{1}{2\sqrt{2-t}}, \frac{e^t}{t} + \frac{e^t+1}{t^2}, \frac{1}{1+t}\rangle
\end{aligned}$$

Problem 5

Consider the curve given by $\hat{r}'(t) = \langle\sin^3(t), \cos^3(t), \sin^2(t)\rangle, 0 \leq t \leq \frac{\pi}{2}$. Find the unit tangent vector.

Answer:

$$\begin{aligned}
\hat{r}'(t) &= \langle\frac{d\sin^3(t)}{dt}, \frac{d\cos^3(t)}{dt}, \frac{d\sin^2(t)}{dt}\rangle \\
&= \langle 3\sin^2 t\cos t, -3\cos^2 t\sin t, 2\sin t\cos t\rangle
\end{aligned}$$

By Pythagoras' Theorem, the norm of $\hat{r}'(t)$ at $t \in \mathbb{R}$:

$$\begin{aligned}
|\hat{r}(t)| &= \sqrt{\left(3\sin^2(t)\cos(t)\right)^2 + \left(-3\cos^2 t\sin t\right)^2 + (2\sin t\cos t)^2} \\
&= \sqrt{9\sin^4 t\cos^2 t + 9\cos^4 t\sin^2 t + 4\sin^2 t\cos^2 t} \\
&= \sqrt{(\sin^2 t\cos^2 t)(9\sin^2 t + 9\cos^2 t + 4)} \\
&= (\sin t\cos t)\sqrt{9(\sin^2 t + \cos^2 t) + 4)} \\
&= (\sin t\cos t)\sqrt{9+4} \\
&= (\sin t\cos t)\sqrt{13}
\end{aligned}$$

Then, the unit tangent vector is

$$\begin{aligned}
\frac{1}{|\hat{r}'(t)|}\hat{r}'(t) &= \frac{1}{\sqrt{13}(\sin t\cos t)}\langle 3\sin^2 t\cos t, -3\cos^2 t\sin t, 2\sin t\cos t\rangle \\
&= \langle\frac{3}{\sqrt{13}}\sin t, -\frac{3}{\sqrt{13}}\cos t, \frac{2}{\sqrt{13}}\rangle
\end{aligned}$$

Problem 6

Find the parametric equations for the tangent line to the curve

$$x = t^2 + 1, y = 4\sqrt{t}, z = e^{t^2-t}$$

at the point $(2, 4, 1)$.

Answer:

We can get the derivative of every component. For x-component:

$$\begin{aligned}\frac{dx}{dt} &= \frac{d(t^2+1)}{dt} \\ &= 2t\end{aligned}$$

For the y-component:

$$\begin{aligned}\frac{dy}{dt} &= \frac{d\left(4\sqrt{t}\right)}{dt} \\ &= \frac{4}{2}\sqrt{t} \\ &= \frac{2}{\sqrt{t}}\end{aligned}$$

For the z-component:

$$\begin{aligned}\frac{dz}{dt} &= \frac{d\left(e^{t^2-t}\right)}{dt} \\ &= (2t-1)e^{t^2-t} \\ &= 2te^{t^2-t} - e^{t^2-t}\end{aligned}$$

Here, I made the mistake of plugging the values from the given point as follows:

$$\langle 2(2), \frac{2}{\sqrt{4}}, (2-1)e^{1^2-1}\rangle = \langle 4, 1, 1\rangle$$

I assumed this vector to be the value of the slope of the tangent at the given point. After checking the answer key, I realized that now we want to find a colinear vector to our tangent so we can construct a vector equation for the line. First, $\hat{r}(t) = \langle 2, 4, 1\rangle \implies t = 1$. Therefore, the slope of the tangent at that point is

$$\begin{aligned}\hat{r}(1) &= \langle 2(1), \frac{2}{\sqrt{1}}, (2-1)e^{1^2-1}\rangle \\ &= \langle 2, \frac{2}{1}, e^0(1)\rangle \\ &= \langle 2, 2, 1\rangle\end{aligned}$$

Then, we can solve for the equation of the tangent as follows:

$$x = 2t + 2$$

$$y = 2t + 4$$

$$z = t + 1$$

Problem 7

Evaluate the integral

$$\int_0^{\frac{\pi}{4}} \left(\sec(t)\tan(t)\hat{\imath} + t\cos(2t)\hat{\jmath} + \sin^2(2t)\cos(2t)\hat{k}\right)dt$$

Answer:

$$\int_0^{\frac{\pi}{4}} \left(\sec(t)\tan(t)\hat{\imath} + t\cos(2t)\hat{\jmath} + \sin^2(2t)\cos(2t)\hat{k}\right)dt$$

$$= \int_0^{\frac{\pi}{4}} \left(\sec(t)\tan(t)\hat{\imath}\right)dt + \int_0^{\frac{\pi}{4}} \left(t\cos(2t)\hat{\jmath}\right)dt + \int_0^{\frac{\pi}{4}} \left(\sin^2(2t)\cos(2t)\hat{k}\right)dt$$

$$= \sec(t)\hat{\imath} \mid_0^{\frac{\pi}{4}} + \left(\frac{1}{2}\right)\left(t\sin 2t + \frac{1}{2}\cos 2t\right)\hat{\jmath} \mid_0^{\frac{\pi}{4}} + \left(\frac{1}{6}\right)(\sin^3 2t)\hat{k} \mid_0^{\frac{\pi}{4}}\ dt$$

$$= \left(\sqrt{2} - 1\right)\hat{\imath} + \left(\frac{\pi}{8} - \frac{1}{4}\right)\hat{\jmath} + \left(\frac{1}{6}\right)\hat{k}$$