Homework 2 ,

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The Cross Product

Problem 1

Find two unit vectors that are orthogonal to both $\hat{\jmath} + 2\hat{k}$ and $\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$.

Answer

Let $\hat{a} = \hat{\jmath} + 2\hat{k}$ and $\hat{b} = \hat{\imath} - \hat{\jmath} + \hat{k}$. Then $\hat{a} = \langle 0, 1, 2 \rangle$ and $\hat{b} = \langle 1, -2, 3 \rangle$ Then

$$\begin{aligned} \hat{a} \times \hat{b} &= \langle 1(3) - 2(-2), 2(1) - 0(3), 0(-2) - 1(1) \rangle \\ &= \langle 3 + 3, 2 - 0, 0 - 1 \rangle \\ &= \langle 7, 2, -1 \rangle \end{aligned}$$

Since $\hat{b} \times \hat{a} = -\hat{a} \times \hat{b}$ is also an orthogonal to $\hat{a}$ and $\hat{b}$, we can change the sign of the components to find such vector. The length of both vectors is:

$$\begin{aligned} \parallel \hat{a} \times \hat{b} \parallel &= \sqrt{7^2 + 2^2 + (-1)^2} = \sqrt{49 + 4 + 1} \\ &= \sqrt{54} \\ &= \sqrt{6(9)} \\ &= 3\sqrt{6} \end{aligned}$$

Therefore, the unit vectors orthogonal to $\hat{u}$ and $\hat{v}$ are

$$\langle \frac{7}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}, -\frac{1}{3\sqrt{3}} \rangle$$

and

$$\langle -\frac{7}{3\sqrt{3}}, -\frac{2}{3\sqrt{3}}, \frac{1}{3\sqrt{3}} \rangle$$

Problem 2

Suppose that $\hat{u} \cdot (\hat{v} \times \hat{w}) = 2$. Find $\hat{v} \cdot (\hat{u} \times \hat{w})$.

Answer

$$\begin{aligned} \hat{v} \cdot (\hat{u} \times \hat{w}) &= \hat{v} \cdot -(\hat{w} \times \hat{u}) \\ &= -(\hat{v} \times \hat{w}) \cdot \hat{u} \\ &= -(\hat{v} \times \hat{w}) \cdot \hat{u} \\ &= -\hat{u} \cdot (\hat{v} \times \hat{w}) \\ &= -2 \end{aligned}$$

Problem 3

Let $\hat{u}$ and $\hat{v}$ be any nonzero, non-parallel vectors in $\mathbb{R}^3$. Compute $(\hat{u} \times \hat{v}) \cdot \hat{v}$ and explain why your answer is right.

Answer

This is problem is a basic corollary from the definition of the cross product. Let $\hat{u}, \hat{v}, \hat{a} \in \mathbb{R}^3$. Since $\hat{u} \times \hat{v}$ is a vector orthogonal to both $\hat{u}$ and $\hat{v}$, and since $\hat{a} \cdot \hat{v} = 0 \iff \hat{a}$ is colinear (therefore, non-orthogonal) to $\hat{v}$, then $\hat{v} \times \hat{v} \cdot \hat{v} = 0$.

Equations of Lines and Planes

Problem 4

Find the vector equation, parametric equations, an symmetric equations for the line in $\mathbb{R}^3$ that passes through the points $(4, -1, 2)$ and $(1, 1, 5)$.

Answer

Given two vectors $\hat{r}_0$ and $\hat{r} \in \mathbb{R}^n$, the **vector equation** of a line is $r_0 + t\hat{r}$, where $t \in \mathbb{R}^1$. Let $\hat{r}_0 = \langle 1, 1, 5 \rangle$ and $\hat{r} = (4, -1, 2) - (1, 1, 5) = \langle 3, -2, -3 \rangle$, the equation of the line that passes through both points could be expressed as

$$\langle 1, 1, 5 \rangle + t\langle 3, -2, -3 \rangle$$

or

$$\langle 1, 1, 5 \rangle + t\langle \frac{3}{\sqrt{21}}, -\frac{2}{\sqrt{21}}, -\frac{3}{\sqrt{21}} \rangle$$

as well as other combinations.

The **parametric equations**, which are just the components of the vector resulting of the vector equation:

$$x = 3t + 1$$

$$y = 1 - 2t$$

$$z = 5 - 3t$$

The **symmetric equations** are found by solving all equations for t.

$$x = 3t + 1 \Longrightarrow x - 1 = 3t \Longrightarrow \frac{x - 1}{3} = t$$

$$y = 1 - 2t \Longrightarrow y - 1 = -2t \Longrightarrow \frac{1 - y}{2} = t$$

$$z = 5 - 3t \Longrightarrow z - 5 = -3t \Longrightarrow \frac{5 - z}{3} = t$$

Therefore,

$$\frac{x - 1}{3} = \frac{1 - y}{2} = \frac{5 - z}{3}$$

Problem 5

Find a vector parallel to the line whose symmetric equations are

$$\frac{x - 4}{3} = \frac{y}{2} = z + 2$$

Answer

$$\frac{x-4}{3} = t \Longrightarrow x - 4 = 3t \Longrightarrow x = 3t + 4$$

$$\frac{y}{2} = t \Longrightarrow y = 2t$$

$$z + 2 = t \Longrightarrow z = t - 2$$

The parametric equations to the corresponding line are $\frac{x-4}{3} = \frac{y}{2} = z + 2 \Longrightarrow x = 3t + 4, y = 2t, z = t - 2$. Written as a vector equation,

$$\langle 4, 0, -2\rangle + t\langle 3, 2, 1\rangle$$

So any vector of the form $t\langle 3, 2, 1\rangle$ is parallel to the line. In fact, this are the only vectors we can define, if all the vectors start at the origin.

Problem 6

Find an equation for the plain through $(3, -1, 1)$, $(4, 0, 2)$, and $(6, 3, 1)$.

Answer

Let's define two vectors in $\mathbb{R}^3$ from the given points, namely $\hat{u}$ and $\hat{v}$:

$$\begin{aligned}\hat{u} &= \hat{0} + (3, -1, 1)\\ &= \langle 3, -1, 1\rangle\end{aligned}$$

$$\begin{aligned}\hat{v} &= (3, -1, 1) - (4, 0, 2)\\ &= \langle -1, -1, -1\rangle\end{aligned}$$

$$\begin{aligned}\hat{w} &= (3, -1, 1) - (6, 3, 1)\\ &= \langle -3, -4, 0\rangle\end{aligned}$$

The equation of a plane is given as all the points in in the vector space that are orthogonal to a vector $\hat{n}$. Since we want to find a plane containing vectors $\hat{u}$, $\hat{v} + \hat{u}$ and $\hat{w} + \hat{u}$, we can find the cross product $\hat{v} \times \hat{w}$ to find such normal.

$$\begin{aligned}\hat{v} \times \hat{w} &= \langle -1(-4), 1(-3), -1(-4) - (-1)(-3)\rangle\\ \hat{v} \times \hat{w} &= \langle 4, -3, 1\rangle\end{aligned}$$

Therefore, the equation for the plane is $4(x + 3) - 3(y - 1) + z + 1 = 0$.

Problem 7

Find the distance from the point $(-6, 3, 5)$ to the plane $x - 2y - 4z = 8$.

Answer

$$\begin{aligned}\frac{|-6(1) + 3(-2) + 5(-4) - 8|}{\sqrt{1^2 + (-2)^2 + (-4)^2}} &= \frac{|-6 - 6 - 20 - 8|}{\sqrt{1 + 4 + 16}}\\ &= \frac{40}{\sqrt{21}}\end{aligned}$$

Cylinders and Quadric Surfaces

Problem 8

Identify and sketch the graph of the surface defined by $4x^2 + 4y^2 - 8y + z^2 = 0$

Answer

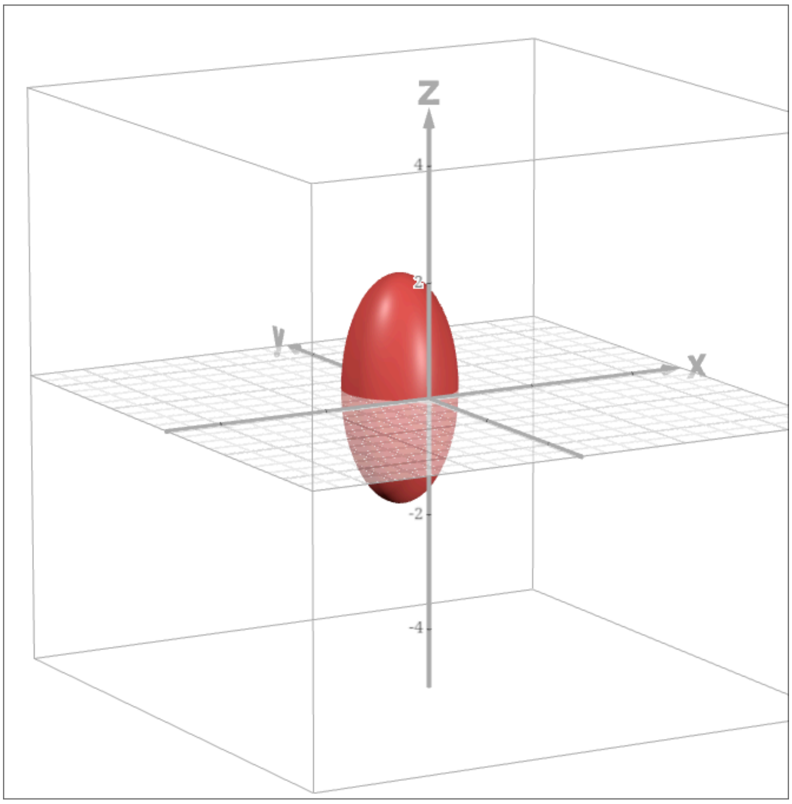

Figure 1: Vertical elipsoid

This is an elipsoid. Honestly, I couldn't figure out how to find the equation to identify it, as I didn't think of completing the square $y^2 - 2y$. With that, it's pretty intuitive;

$$\begin{aligned} 0 &= 4x^2 + 4y^2 - 8y + z^2 \\ &= x^2 + y^2 - 2y + \frac{z^2}{4} \\ 1 &= x^2 + y^2 - 2y + 1\frac{z^2}{4} \\ &= x^2 + (y-1)^2 + \frac{z^2}{4} \end{aligned}$$

Problem 9

Find an equation for the surface consisting of all points that are equidistant from the point $(-1, 0, 0)$ and the plane $x = 1$. Identify the surface.

Answer

I couldn't figure out this one either, and it became to late. I guess this i where I bend. D: