Homework 5

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Functions of Several Variables

Problem 1

A company makes three sizes of cardboard boxes: small, medium, and large. It costs $2.50 to make a small box, $4.00 to make a medium box, and $4.50 for a large box. Fixed costs are $8000.

1. Express the cost of making x small boxes, y medium boxes, and z large

boxes as a function of three variables $C = f(x, y, z)$.

1. Find $f(3000, 5000, 4000)$ and interpret it.

2. What is the domain of f?

Answer

1. $f(x, y, z) = 2.5x + 4y + 4.5z + 8000$
2. $$\begin{aligned} f(3000, 5000, 4000) &= 2.5(3000) + 4(5000) + 4.5(4000) + 8000 \\ &= 7500 + 20000 + 18000 + 8000 \\ &= (7.5 + 20 + 18 + 8)10^3 \\ &= 53500 \end{aligned}$$

It costs $53,500 to make 3000 small boxes, 5000 medium boxes and 4000 large boxes.

3. The domain of f is $\{(x, y, z) \in \mathbb{R}^3 |\ (x \geq 0, y \geq 0, z \geq 0)\}$. From numeric perspective, f can map all vectors or points in $\mathbb{R}^3$. However, if we consider the application context, it's obvious that the number of boxes of a any type requested must be (at least, as noted later) a positive real number.

After more consideration, there are two more assumptions we could make: x, y, z must be all positive integers (only whole boxes can be requested, and their number is discrete), and this function does not apply for $x = y = z = 0$, which would be equivalent to not making a petition. If this assumptions are true, then we must restrict the domain even further into $\{(x, y, z) \in \mathbb{R}^3 |\ (x \geq 0, y \geq 0, z \geq 0) \text{ and } (x, y, z) \neq (0, 0, 0) \text{ and } x, y, z \in \mathbb{Z}\}$.

Problem 2

Find and sketch the domain of the function

$$f(x, y) = \sqrt{x^2 + y^2 - 4}$$

Answer

Domain of f is all $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 4\}$. Note that $\sqrt{x^2 + y^2} \geq 2$. We know that $x^2 + y^2 < 4$ corresponds to the set of vectors $\boldsymbol{u}$ such that their norm $|\boldsymbol{u}| < 2$, or all the dots within a circle of radius $r = 2$, excluding the circumference. Therefore, the domain is $\mathbb{R}^2 - \{\boldsymbol{u} \in \mathbb{R}^2 : |\boldsymbol{u}| < 2\}$. This can be graphed as the complete xy-plane with a circular hole of radius $r < 2$.

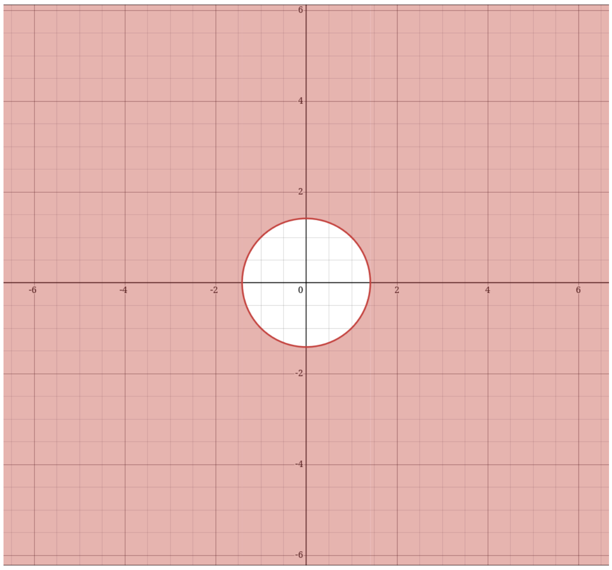

Figure 1: Note that all the vectors that fall in the circumference of the circle is in the domain of f.

Problem 3

Match the function with its graph (labeled I-IV and listed on the following page). Give reasons for your choices.

1. $f(x, y) = \frac{1}{1+x^2+y^2}$
2. $f(x, y) = \frac{1}{1+x^2y^2}$
3. $f(x, y) = \ln(x^2 + y^2)$
4. $f(x, y) = \cos\left(\sqrt{x^2 + y^2}\right)$

[Graphs given in the worksheet]

Answer

1. II. Note that $\forall x, y \in \mathbb{R}^2, (x^2 + y^2 \geq 0) \implies (1 + x^2 + y^2 \geq 1)$. Therefore, the domain of f is $\{(x, y) |\ (x, y) \in \mathbb{R}^2\}$, which means that the function doesn't have any holes through the xy-plane. Note now that $\lim_{(x,y)\to\infty} f(x, y) = 0$, and $f(0, 0) = 1$. Therefore, the range of this function is $(0, 1]$: never crossing above the $z = 1$ plane, nor below the $z = 0$ plane.

Now, note that

$$\frac{\partial(f)}{\partial(x)} = -\frac{2x}{(1 + x^2 + y^2)^2}$$

and

$$\frac{\partial(f)}{\partial(x)} = -\frac{2x}{\left(1 + x^2 + y^2\right)^2}$$

This implies that, when projected to the xz plane such that $f(x, 0) = -\frac{2x}{\left(1+x^2\right)^2}$, the function is increasing from $(-\infty, 0)$ and decreasing over $(0, \infty)$. Similar argument for y. The next function, which presents similar behaviours in domain and range, differs in this way.

2. I. There are similar arguments to the last function for this function: $\forall x, y \in \mathbb{R}^2, \left(x^2 y^2 \geq 0\right) \Longrightarrow \left(1 + x^2 y^2 \geq 1\right)$ implies that the graph of f doesn't have holes through the xy-plane. And $\lim_{(x,y) \to \infty} f(x, y) = 0$, and $f(0, 0) = 1$.

Now,

$$\frac{\partial(f)}{\partial(x)} = -\frac{2x}{\left(1 + x^2 + y^2\right)^2}$$

and

$$\frac{\partial(f)}{\partial(x)} = -\frac{2x}{\left(1 + x^2 + y^2\right)^2}$$

This means that, when projected to the xz-plane, $f(x, 0) = 0$ which implies that the function is constant. Similar argument for y.

3. III. Note that $\lim_{(x,y) \to (0,0)} \ln\left(x^2 + y^2\right) = -\infty$ and, as mentioned before $\forall x, y \in \mathbb{R}^2, \left(x^2 + y^2 \geq 0\right)$. Therefore, the function presents an asymptote at $(x, y) = (0, 0)$. For all other directions, the function grows without bound.

4. IV. I'm not going to use calculus to justify this one: simply state that this function does look like a cosine function when projected over either one of xz and yz planes.

Limits and Continuity

Problem 4

Find the limit if it exists, or show that the limit does not exists.

1. $\lim_{(x,y)\to(3,2)}(x^2y^2 - 4y^2)$
2. $\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^4+y^4}$
3. $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$

Answer

1. Since this a polynomial function, we can extend the limit rules from $\mathbb{R}$ to

$\mathbb{R}^2$:

$$\begin{aligned}\lim_{(x,y)\to(3,2)} x^2y^2 - 4y^2 &= 9(4) - 4(4) \\ &= -7\end{aligned}$$

.

2. Note that $\forall x, y \in \mathbb{R}^2$

$$(x > 0) \Longrightarrow \left(0 \leq \frac{xy^4}{x^4+y^4} \leq xy^4\right),$$

$$(x < 0) \Longrightarrow \left(0 \geq \frac{xy^4}{x^4+y^4} \geq xy^4\right) \text{ and}$$

$$\lim_{(x,y)\to(0,0)} xy^4 = 0$$

Without loss of generality, by *Squeeze Theorem*, $\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^4+y^4} = 0$.

3. Honestly, I was about to just apply the *Squeeze Theorem* again. But after reading the solution sheet, I've added the "mismatch of exponents" clue to my arsenal.

Note that, examined accross the line $x = y$,

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} &= \lim_{y\to 0} \frac{y^4}{y^2+y^6} \\ &= 0\end{aligned}$$

But from the curve $x = y^3$,

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} &= \lim_{y\to 0} \frac{y^6}{2y^6} \\ &= \frac{1}{2}\end{aligned}$$

Therefore, the limit does not exists.

Problem 5

Determine the set of points at which the function is continous.

$$F(x, y) = \frac{1 + x^2 + y^2}{1 - x^2 - y^2}$$

Answer

The function is continuous $\forall x, y \in \mathbb{R}^2 | \ (x^2 + y^2 \neq 1)$.

Problem 6

Use polar coordinates to find the limit. [If (r, θ) are polar coordinates of the point (x, y) with $r \geq 0$, note that $r \to 0^+$ as $(x, y) = (0, 0)$.]

$$\lim_{(x,y)\to(0,0)} (x^2 + y^2) \ln(x^2 + y^2)$$

Answer

Let $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Then

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} (x^2 + y^2) \ln(x^2 + y^2) &= \lim_{(r,\theta)\to(0,\theta_0)} (r^2\cos^2(\theta) + r^2\sin^2(\theta)) \ln(r^2\cos^2(\theta) + r^2\sin^2(\theta)) \\
&= \lim_{(r,\theta)\to(0,\theta_0)} r^2 \ln(r^2) \\
&= \lim_{r\to 0} r^2 \ln(r^2) \\
&= \lim_{r\to 0} 2\frac{\ln(r)}{r^{-2}} \\
&= \lim_{r\to 0} \frac{2}{r}\left(-\frac{r^3}{2}\right) \\
&= \lim_{r\to 0} -r^2 \\
&= 0
\end{aligned}$$