Section 4.4

Exercise 1

Use the method of Example 3 to show that the following set of veectors forms a basis for $\mathbb{R}^2$.

$$\{(2,1),(3,0)\}$$

Answer

The given set is basis for $\mathbb{R}^2$ if and only if the vectors are linearly independent and they span the rest of $\mathbb{R}^2$.

Linear independence (without loss of generality) is satisfied if for the next equation

$$k_1(2,1)+k_2(3,0)=(0,0)$$

which can be rewritten as the following system of equations:

$$\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the answers are only $k_1=0$ and $k_2=0$. An equivalent statement is that the determinant of the coefficient matrix of the given system is nonzero. Therefore, if the determinant of the coefficient matrix is 0, then the column vectors are linearly dependent. Since $2(0)-3=-1\neq 0$, the the column vectors span $\mathbb{R}^2$.

The given basis spans the given vector space if $\exists k_1, k_2 \in \mathbb{R}$ such that, $\forall (b_1, b_2) \in \mathbb{R}^2$.

$$k_1(2,1)+k_2(3,0)=(b_1,b_2)$$

Similar to before, an equivalent statement is that the coefficient matrix of the given system is nonzero, which we already proved. ■

Exercise 3

Show that the following polynomials form a basis for $P-2$.

$$1+x,\ x^2-1,\ 2x-1$$

Answer

We can test linear independence and span by evaluating the determinant of the corresponding Wronskian to this basis.

$$\begin{aligned}\det(W)&=\begin{vmatrix} 1+x & x^2-1 & 2x-1 \\ 1 & 2x & 2 \\ 0 & 2 & 0 \end{vmatrix}\\ &=-2(2+2x-2x+1)\\ &=-2(3)\\ &=-6\end{aligned}$$

Since the determinant is nonzero, this conforms a basis.

Exercise 7

In each part, show that the set of vectors is not a basis for $\mathbb{R}^3$.

(a) $\{(2,-3,1),(4,1,1),(0,-7,1)\}$

(b) $\{(1,6,4),(2,4,-1),(-1,2,5)\}$

Answer

(a) $\begin{vmatrix} 2 & 4 & 0 \\ -3 & 1 & -7 \\ 1 & 1 & 1 \end{vmatrix} = 7(2-4) + (2+12) = 0.$

(b) $\begin{vmatrix} 1 & 2 & -1 \\ 6 & 4 & 2 \\ 4 & -1 & 5 \end{vmatrix} = (20+2) - 2(30-8) - 1(-6-16) = 0.$

Exercise 9

Show that the following matrices do not form a basis for M_{22}:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

Answer

These matrices are linarly independent if $\nexists (k_1, k_2, k_3, k_4) \in \mathbb{R}^4$ such that

$$k_1 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + k_2 \begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix} + k_3 \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} + k_4 \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and $(k_1, k_2, k_3, k_4) \neq \mathbf{0}$. We can rearrange this system as follows:

$$\begin{bmatrix} k_1 + 2k_2 + k_3 \\ -k_2 - k_3 - k_4 \\ k_1 + 3k_2 + k_3 + k_4 \\ k_1 + 2k_2 + k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which, can be expressed as the following linear transformation:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & -1 \\ 1 & 3 & 1 & 1 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the determinant of the cofficient matrix is 0, the given set is not a basis for $\mathbb{R}^4$.

Exercise 11

Exercise 13

Exercise 15

Exercise 17

Exercise 21