Homework 2

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Section 1.4

9 Exercises attempted

9

Find the inverse of

$$\begin{bmatrix} \frac{1}{2}(e^x + e^{-x}) & \frac{1}{2}(e^x - e^{-x}) \\ \frac{1}{2}(e^x - e^{-x}) & \frac{1}{2}(e^x + e^{-x}) \end{bmatrix}$$

Answer

Let A be the given matrix. Then

$$\begin{aligned} det(A) &= \left[\frac{1}{2}(e^x + e^{-x})\right]^2 - \left[\frac{1}{2}(e^x - e^{-x})\right]^2 \\ &= \frac{1}{4}\left[(e^x + e^{-x})^2 - (e^x - e^{-x})^2\right] \\ &= \frac{1}{4}\left[(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})\right] \\ &= \frac{1}{4}\left(e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}\right) \\ &= \frac{1}{4}(4) \\ &= 1 \end{aligned}$$

And given $A^{-1} = \frac{1}{det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, then

$$A^{-1} = (1) \begin{bmatrix} \frac{1}{2}(e^x + e^{-x}) & -\frac{1}{2}(e^x - e^{-x}) \\ -\frac{1}{2}(e^x - e^{-x}) & \frac{1}{2}(e^x + e^{-x}) \end{bmatrix}$$

11

Verify that $(A^T)^{-1} = (A^{-1})^T$ for $A = \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$

Answer

$$A^T = \begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix}$$

and

$$A^{-1} = \frac{1}{20} \begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix}$$

Then

$$\begin{aligned} (A^T)^{-1} &= \begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix}^{-1} \\ &= \frac{1}{20} \begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix} \end{aligned}$$

And

$$\begin{aligned} (A^{-1})^T &= \frac{1}{20} \begin{bmatrix} 4 & -3 \\ 4 & 2 \end{bmatrix}^T \\ &= \frac{1}{20} \begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix} \end{aligned}$$

13

Verify that the equation $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ for $A = \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Answer

$$
\begin{aligned}
(ABC)^{-1} &= \left(\begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right)^{-1} \\
&= \left(\begin{bmatrix} 2(3)+(-3)5 & 2(1)+(-3)2 \\ 4(3)+4(5) & 4(1)+4(2) \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right)^{-1} \\
&= \left(\begin{bmatrix} 6-15 & 2-6 \\ 12+20 & 4+8 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right)^{-1} \\
&= \left(\begin{bmatrix} -9 & -4 \\ 32 & 12 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right)^{-1} \\
&= \left(\begin{bmatrix} -18 & -8 \\ 96 & 36 \end{bmatrix}\right)^{-1} \\
&= \frac{1}{120} \begin{bmatrix} 36 & 8 \\ -96 & -18 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
C^{-1}B^{-1}A^{-1} &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}^{-1} \\
&= \frac{1}{6}\left(\frac{1}{20}\right) \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix} \\
&= \frac{1}{120} \begin{bmatrix} 6 & -3 \\ -10 & 6 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix} \\
&= \frac{1}{120} \begin{bmatrix} 36 & 8 \\ -96 & -18 \end{bmatrix}
\end{aligned}
$$

19

Given $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$, compute the following

a) A^3

b) A^{-3}

c) $A^2 - 2A - I$

Answer

a)

$$
\begin{aligned}
A^3 &= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 11 & 4 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 41 & 15 \\ 30 & 11 \end{bmatrix}
\end{aligned}
$$

b)

$$\begin{aligned} A^{-3} &= (A^3)^{-1} \\ &= \begin{bmatrix} 41 & 15 \\ 30 & 11 \end{bmatrix}^{-1} \\ &= \frac{1}{(41)(11)-(15)(30)} \begin{bmatrix} 11 & -15 \\ -30 & 41 \end{bmatrix} \\ &= \begin{bmatrix} 11 & -15 \\ -30 & 41 \end{bmatrix} \end{aligned}$$

c)

$$\begin{aligned} A^2 - 2A - I &= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} - 2 \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 4 \\ 8 & 3 \end{bmatrix} - \begin{bmatrix} 6 & 2 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 2 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 2 \\ 4 & 0 \end{bmatrix} \end{aligned}$$

23

Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, find all the values of A for which A and B commute.

Answer

$$AB = BA$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$$

Which implies that $a = d$, $b \in R$ and $c = 0$.

29

If a polynomial $p(x)$ can be factord as a product of lower degree polynomials, say

$$p(x) = p_1(x)p_2(x)$$

and if A is a square matrix, then it can be proved that

$$p(A) = p_1(A)p_2(A)$$

Verify the next statements for $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$

$$p(x) = x^2 - 9 \quad p_1(x) = x + 3 \quad p_2(x) = x - 3$$

Answer

$$\begin{aligned} p_1(A) &= A + 3I \\ &= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 1 \\ 2 & 4 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} p_2(A) &= A - 3I \\ &= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} p_1(A)p_2(A) &= (A + 3I)(A - 3I) \\ &= \begin{bmatrix} 6 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 \\ 8 & -6 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} p(A) &= A^2 - 9I \\ &= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 4 \\ 8 & 3 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 \\ 8 & -6 \end{bmatrix} \end{aligned}$$

Thus, $p(A) = p_1(A)p_2(A)$. ■

35

Can a matrix with a row of zeroes or a column of zeros have an inverse? Explain.

Answer

No, as the determinant would be 0. For example, let $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$. Then $det(A) = 1(0) - 0(2) = 0$, then $A^{-1} = \frac{1}{0}\begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$, but $\frac{1}{0} =$ undefined. Therefore, the inverse does not exist.

43

(a) Show that if A is invertible and $AB = AC$, then $B = C$.

(b) Explain why part (a) and Example 3 do not contradict each other.

Answer (Use row reduction instead)

(a) We can multiply both sides with A^{-1}. Then:

$$\begin{aligned} A^{-1}AB &= A^{-1}AC \\ B &= C \end{aligned}$$

(b) The difference is that the matrix A is nonsingular, meaning that the product of itself and its inverse is the identity matrix. This implies that the product of a matrix A and the product of its inverse $A^{-}-1$ with any other matrix B is the product of B and the identity matrix I, which is B. Whereas, the Example 3 is a singular matrix.

45

(a) Show that if A, B and $A + B$ are invertible matrice swith the same size, then

$$A(A^{-1} + B^{-1})B(A + B)^{-1} = I$$

(b) What does the result in the part (a) tell you about the matrix $A^{-1} + B^{-1}$?

Answer

(a) This can be shown using basic theorems of matrices

$$\begin{aligned} I &= A(A^{-1} + B^{-1})B(A + B)^{-1} \\ &= (AA^{-1} + AB^{-1})B(A + B)^{-1} \\ &= (I + AB^{-1})B(A + B)^{-1} \\ &= (B + A)(A + B)^{-1} \\ &= I \end{aligned}$$

(b) $A^{-1} + B^{-1} = (A + B)^{-1}$

Section 1.5

3

Find a row operation and the corresponding elementary matrix that will restore the given elementary matrix to the identity matrix.

a) $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

b) $\begin{bmatrix} -7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$

d) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Answer

a) Adding $3R_2$ to R_1. $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

b) Multiplying $-\frac{1}{7}R_1$. $\begin{bmatrix} -\frac{1}{7} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c) Adding $-5R_1$ to R_3. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$

d) Exchanging R_1 and R_3. Multiplying it by itself suffices. $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

5

Identify the row operation corresponding to E and verify the product EA results from applying the row operation to A

a) $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A = \begin{bmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{bmatrix}$

b) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix}$

c) $E = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Answer

a)

$$\begin{aligned} EA &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -2 & 5 & -1 \\ 3 & -6 & -6 & -6 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -6 & -6 & -6 \\ -1 & -2 & 5 & -1 \end{bmatrix} \end{aligned}$$

The corresponding row operation is swapping R_1 and R_2. As it is trivial, we won't write it down.

b)

$$\begin{aligned} EA &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 & 0 & -4 & -4 \\ 1 & -3 & -1 & 5 & 3 \\ -1 & 9 & 4 & -12 & -10 \end{bmatrix} \end{aligned}$$

The corresponding row operation is adding $-3R_2$ to R_3.

c) $E = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

d)

$$EA = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 13 & 28 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

The corresponding row operation is adding $4R_3$ to R_1.

9

Use Theorem 1.4.5 and then use the inversion algorithm to find A^{-1}, if it exists.

a) $A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$

b) $A = \begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix}$

Answer

(a) Using **Theorem 1.4.5**:

$$det(A) = 1(7) - 4(2) = -1$$

$$\begin{aligned} \frac{1}{det(A)} A^T &= -1 \begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -7 & 4 \\ 2 & -1 \end{bmatrix} \end{aligned}$$

Using the inversion algorithm:

0. $\begin{bmatrix} 1 & 4 & | & 1 & 0 \\ 2 & 7 & | & 0 & 1 \end{bmatrix}$

1. Add $-2R_1$ to R_2: $\begin{bmatrix} 1 & 4 & | & 1 & 0 \\ 0 & -1 & | & -2 & 1 \end{bmatrix}$

2. Multiply $-R_2$. $\begin{bmatrix} 1 & 4 & | & 1 & 0 \\ 0 & 1 & | & 2 & -1 \end{bmatrix}$

3. Add $-4R_2$ to R_1. $\begin{bmatrix} 1 & 0 & | & -7 & 4 \\ 0 & 1 & | & 2 & -1 \end{bmatrix}$

b) A is not invertible because $det(A) = 0$.

13

Use the inversion algorithm to find the inverse of the matrix (if the inverse exists).

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Answer

0. $\begin{bmatrix} 1 & 0 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 0 & 1 & 0 \\ 1 & 1 & 0 & | & 0 & 0 & 1 \end{bmatrix}$

1. Add $-R_1$ to R_2. $\begin{bmatrix} 1 & 0 & 1 & | & 1 & 0 & 0 \\ -1 & 1 & 0 & | & -1 & 1 & 0 \\ 1 & 1 & 0 & | & 0 & 0 & 1 \end{bmatrix}$

2. Add $-R_1$ to R_3. $\begin{bmatrix} 1 & 0 & 1 & | & 1 & 0 & 0 \\ -1 & 1 & 0 & | & -1 & 1 & 0 \\ 0 & 1 & -1 & | & -1 & 0 & 1 \end{bmatrix}$

3. Add R_3 to R_1. $\begin{bmatrix} 1 & 1 & 0 & | & 0 & 0 & 1 \\ -1 & 1 & 0 & | & -1 & 1 & 0 \\ 0 & 1 & -1 & | & -1 & 0 & 1 \end{bmatrix}$

4. Add R_2 to R_1. $\begin{bmatrix} 0 & 2 & 0 & | & -1 & 1 & 1 \\ -1 & 1 & 0 & | & -1 & 1 & 0 \\ 0 & 1 & -1 & | & -1 & 0 & 1 \end{bmatrix}$

5. Multiply $\frac{1}{2}R_1$. $\begin{bmatrix} 0 & 1 & 0 & | & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -1 & 1 & 0 & | & -1 & 1 & 0 \\ 0 & 1 & -1 & | & -1 & 0 & 1 \end{bmatrix}$

6. Add $-R_1$ to R_2. $\begin{bmatrix} 0 & 1 & 0 & | & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -1 & 0 & 0 & | & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -1 & | & -1 & 0 & 1 \end{bmatrix}$

7. Multiply $-R_2$. $\begin{bmatrix} 0 & 1 & 0 & | & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & | & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & | & -1 & 0 & 1 \end{bmatrix}$

8. Add $-R_1$ to R_3. $\begin{bmatrix} 0 & 1 & 0 & | & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & | & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -1 & | & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

9. Multiply $-R_3$. $\begin{bmatrix} 0 & 1 & 0 & | & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & | & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & | & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

10. Exchage R_1 and R_2. $\begin{bmatrix} 1 & 0 & 0 & | & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & | & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & | & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

15

Use the inversion algorithm to find the inverse of the matrix $\begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$, if it exists.

Answer

0) $\begin{bmatrix} 2 & 6 & 6 & | & 1 & 0 & 0 \\ 2 & 7 & 6 & | & 0 & 1 & 0 \\ 2 & 7 & 7 & | & 0 & 0 & 1 \end{bmatrix}$

1) Add $-1R_1$ to R_2: $\begin{bmatrix} 2 & 6 & 6 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -1 & 1 & 0 \\ 2 & 7 & 7 & | & 0 & 0 & 1 \end{bmatrix}$

2) Add $-1R_1$ to R_3: $\begin{bmatrix} 2 & 6 & 6 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -1 & 1 & 0 \\ 0 & 1 & 1 & | & -1 & 0 & 1 \end{bmatrix}$

3) Add $-1R_2$ to R_3: $\begin{bmatrix} 2 & 6 & 6 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -1 & 1 & 0 \\ 0 & 0 & 1 & | & -1 & -1 & 1 \end{bmatrix}$

4) Add $-6R_3$ to R_1: $\begin{bmatrix} 2 & 6 & 0 & | & 7 & 6 & -6 \\ 0 & 1 & 0 & | & -1 & 1 & 0 \\ 0 & 0 & 1 & | & -1 & -1 & 1 \end{bmatrix}$

5) Add $-6R_2$ to R_1: $\begin{bmatrix} 2 & 0 & 0 & | & 13 & 0 & -6 \\ 0 & 1 & 0 & | & -1 & 1 & 0 \\ 0 & 0 & 1 & | & -1 & -1 & 1 \end{bmatrix}$

6) Multiply $\frac{1}{2}R_1$: $\begin{bmatrix} 1 & 0 & 0 & | & \frac{13}{2} & 0 & -3 \\ 0 & 1 & 0 & | & -1 & 1 & 0 \\ 0 & 0 & 1 & | & -1 & -1 & 1 \end{bmatrix}$

23

Express the matrix and its inverse as a product of elementary matrices.

$$\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix}$$

Answer

Let A be the given matrix.

$$det(A) = -3(2) - (2) = -8$$

$$\begin{aligned} A^{-1} &= -\frac{1}{8}\begin{bmatrix} 2 & -1 \\ -2 & -3 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{4} & \frac{1}{8} \\ \frac{1}{4} & \frac{3}{8} \end{bmatrix} \end{aligned}$$

If we apply the inversion algorithm to A:

0) $\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix}$

1) Multiply $-\frac{1}{3}R_1$: $\begin{bmatrix} 1 & -\frac{1}{3} \\ 2 & 2 \end{bmatrix}$

2) Add $-2R_1$ to R_2: $\begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & \frac{8}{3} \end{bmatrix}$

3) Multiply $\frac{3}{8}R_2$: $\begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 1 \end{bmatrix}$

4) Add $\frac{1}{3}R_2$ to R_1: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Therefore,

$$A = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{8}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 1 \end{bmatrix}$$

If we apply the inversion algorithm to A^{-1}:

0) $\begin{bmatrix} -\frac{1}{4} & \frac{1}{8} \\ \frac{1}{4} & \frac{3}{8} \end{bmatrix}$

1) Multiply $-4R_1$: $\begin{bmatrix} 1 & -2 \\ \frac{1}{4} & \frac{3}{8} \end{bmatrix}$

2) Add $-\frac{1}{4}R_1$ to R_2: $\begin{bmatrix} 1 & -2 \\ 0 & \frac{7}{8} \end{bmatrix}$

3) Multiply $\frac{8}{7}R_2$: $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

4) Add $-2R_2$ to R_1: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Therefore,

$$A = \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{7}{8} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

25

Express the matrix and its inverse as a product of elementary matrices.

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer

Let A be the given matrix.

$$det(A) = 4$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix}4 & 0 & 0\\ 0 & 1 & 0\\ 8 & -3 & 4\end{bmatrix}$$
$$= \begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{4} & 0\\ 2 & -\frac{3}{4} & 1\end{bmatrix}$$

If we apply the inversion algorithm to A:

0) $\begin{bmatrix}1 & 0 & -2\\ 0 & 4 & 3\\ 0 & 0 & 1\end{bmatrix}$

1) Add $-3R_3$ to R_2: $\begin{bmatrix}1 & 0 & -2\\ 0 & 4 & 0\\ 0 & 0 & 1\end{bmatrix}$

2) Add $2R_3$ to R_1: $\begin{bmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 1\end{bmatrix}$

3) Multiply $\frac{1}{4}R_2$: $\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{4} & 0\\ 0 & 0 & 1\end{bmatrix}$

Therefore,

$$A = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 3\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & -2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 1\end{bmatrix}$$

If we apply the inversion algorithm to A^{-1}:

Therefore,

$$A = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 2 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -3 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{1}{4} & 0\\ 0 & 0 & 1\end{bmatrix}$$

31

Prove that if A and B are $m \times n$ matrices, then A and B are row equivalent if and only if A and B have the same reduced row echelon form.

Answer

$$XA = B \iff XCrref(A) = B \iff rref(A) = C^{-1}X^{-1}B$$