MATH 2210 HOMEWORK WORKSHEET 7

Name: KEY

Directional Derivatives and the Gradient Vector

1. Find the directional derivative of the function at the given point in the direction of the vector v.

(a) $f(x,y) = \dfrac{x}{x^2+y^2}$, $(1,2)$, $\mathbf{v} = \langle 3,5\rangle$.

We first need the unit vector in the direction of $\mathbf{v}$. Now $|\mathbf{v}| = \sqrt{3^2+5^2} = \sqrt{34}$. Thus the unit vector is $\hat{\mathbf{v}} = \left\langle \frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \right\rangle$. Next we need the gradient vector $\nabla f = \langle f_x, f_y\rangle$. The first partials, gradient, and directional derivative are then

$$f_x = \frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}, \qquad f_y = \frac{(x^2+y^2)(0)-x(2y)}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2}$$

$$\nabla f = \left\langle \frac{y^2-x^2}{(x^2+y^2)^2}, -\frac{2xy}{(x^2+y^2)^2} \right\rangle$$

$$D_{\mathbf{v}}f(1,2) = \nabla f \cdot \hat{\mathbf{v}} = \frac{3}{\sqrt{34}}\left(\frac{y^2-x^2}{(x^2+y^2)^2}\right) + \frac{5}{\sqrt{34}}\left(-\frac{2xy}{(x^2+y^2)^2}\right)\Bigg|_{(x,y)=(1,2)} = -\frac{11}{25\sqrt{34}}$$

(b) $f(x,y,z) = xy^2\tan^{-1} z$, $(2,1,1)$, $\mathbf{v} = \langle 1,1,1\rangle$.

We first need the unit vector in the direction of $\mathbf{v}$. Since $|\mathbf{v}| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$, the unit vector is $\hat{\mathbf{v}} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$. Then we compute the gradient vector $\nabla f = \langle f_x, f_y, f_z\rangle$. The first partials and gradient are then

$$f_x = y^2\tan^{-1} z, \qquad f_y = 2xy\tan^{-1} z, \qquad f_z = \frac{xy^2}{1+z^2}$$

$$\nabla f = \left\langle y^2\tan^{-1} z,\ 2xy\tan^{-1} z,\ \frac{xy^2}{1+z^2} \right\rangle$$

Then $D_{\mathbf{v}}f = \nabla f \cdot \hat{\mathbf{v}}$ and we get

$$D_{\mathbf{v}}f(2,1,1) = \frac{\cdot y^2\tan^{-1} z}{\sqrt{3}} + \frac{2xy\tan^{-1} z}{\sqrt{3}} + \frac{xy^2}{\sqrt{3}(1+z^2)}\Bigg|_{(x,y,z)=(2,1,1)} = \frac{5}{4\sqrt{3}}\pi + \frac{1}{\sqrt{3}}$$

2. Find the maximum rate of change of

$$f(x, y, z) = x \ln(yz)$$

at the point $\left(1, 2, \frac{1}{2}\right)$ **and the direction in which it occurs.**

The maximum rate of change is always $|\nabla f|$ and occurs in the direction $\mathbf{u} = \dfrac{\nabla f}{|\nabla f|}$.

$$\nabla f\left(1, 2, \frac{1}{2}\right) = \langle f_x, f_y, f_z \rangle = \left\langle \ln(yz),\ \frac{x}{y},\ \frac{x}{z} \right\rangle \Bigg|_{(1,2,1/2)} = \left\langle 0,\ \frac{1}{2},\ 2 \right\rangle$$

And so the maximum rate of change and the direction are

$$\left|\nabla f\left(1, 2, \frac{1}{2}\right)\right| = \sqrt{0 + \frac{1}{4} + 4} = \frac{\sqrt{17}}{2}, \qquad \frac{\nabla f}{|\nabla f|}\Bigg|_{(1,2,1/2)} = \left\langle 0,\ \frac{1}{\sqrt{17}},\ \frac{4}{\sqrt{17}} \right\rangle.$$

3. Find equations of (a) the tangent plane and (b) the normal line to the surface

$$x = y^2 + z^2 + 1$$

at the point $(3, 1, -1)$. ***Hint: Recall that the normal line is the line through the point that is perpendicular (i.e. orthogonal) to the surface.***

To find these, we write the function as a level curve $F(x, y, z) = k$ and then ∇F is a normal vector to the surface. The level curve equation here is

$$x - y^2 - z^2 = 1.$$

This makes $F(x, y, z) = x - y^2 - z^2$ and hence the gradient at $(3, 1, -1)$ is

$$\nabla F(3, 1, -1) = \langle 1,\ -2y,\ -2z \rangle|_{(3,1,-1)} = \langle 1, -2, 2 \rangle.$$

This is the normal vector to the tangent plane at $(3, 1, -1)$ and hence the equations for the tangent plane and normal line to the surface are

$$(x - 3) - 2(y - 1) + 2(z + 1) = 0$$

$$\mathbf{r}(t) = \langle 3 + t, 1 - 2t, -1 + 2t \rangle$$

Maximum and Minimum Values

4. Suppose $(0,2)$ is a critical point of a function g with continuous second deriavatives. In each case below, what can you say about g? Explain your reasoning.

(a) $g_{xx}(0,2) = -1, \quad g_{xy}(0,2) = 6, \quad g_{yy}(0,2) = 1.$

In this case,

$$D = g_{xx}g_{yy} - (g_{xy})^2 = (-1)(1) - (6)^2 = -37 < 0$$

making this point a saddle point for the curve.

(b) $g_{xx}(0,2) = -1, \quad g_{xy}(0,2) = 2, \quad g_{yy}(0,2) = -8.$

In this case,

$$D = g_{xx}g_{yy} - (g_{xy})^2 = (-1)(-8) - (2)^2 = 4 > 0$$

and $g_{xx}(0,2) = -1 < 0$. Thus by the second derivative test, this point is is a local maximum for the curve.

(c) $g_{xx}(0,2) = 4, \quad g_{xy}(0,2) = 6, \quad g_{yy}(0,2) = 9.$

In this case,

$$D = g_{xx}g_{yy} - (g_{xy})^2 = (4)(9) - (6)^2 = 0$$

and so the second derivative test is inconclusive. Thus without more information, we cannot determine whether this is a local minimum, maximum, or saddle point.

5. Find the local maximum and minimum values and saddle point(s) of the function.

(a) $f(x, y) = y(\mathrm{e}^x - 1)$

First we find the critical points where $f_x = f_y = 0$. That is,

$$f_x = y\mathrm{e}^x = 0, \qquad f_y = \mathrm{e}^x - 1 = 0$$

Since $\mathrm{e}^x \neq 0$, $f_x = 0$ only when $y = 0$ and $f_y = 0$, when $\mathrm{e}^x = 1$ or $x = 0$. Thus the only critical point is $(0, 0)$.
We next compute the second derivatives and compute D.

$$f_{xx} = y\mathrm{e}^x, \qquad f_{xy} = \mathrm{e}^x, \qquad f_{yy} = 0$$

$$D = f_{xx}f_{yy} - f_{xy}^2 = y\mathrm{e}^x \cdot 0 - \mathrm{e}^{2x} = -\mathrm{e}^{2x}.$$

Thus $D = -1$ at the critical point $(0, 0)$ and thus $(0, 0)$ is a saddle point.

(b) $f(x, y) = xy\mathrm{e}^{-(x^2+y^2)/2}$

First we find the critical points where $f_x = f_y = 0$. That is,

$$\begin{aligned} f_x &= -x^2y\mathrm{e}^{-(x^2+y^2)/2} + y\mathrm{e}^{-(x^2+y^2)/2} = \mathrm{e}^{-(x^2+y^2)/2}y(1 - x^2) = 0, \\ f_y &= -xy^2\mathrm{e}^{-(x^2+y^2)/2} + x\mathrm{e}^{-(x^2+y^2)/2} = \mathrm{e}^{-(x^2+y^2)/2}x(1 - y^2) = 0 \end{aligned}$$

Thus since an exponential is never 0, $f_x = 0$ when either $y = 0$ or $x = \pm 1$ and $f_y = 0$ when either $x = 0$ or $y = \pm 1$. But if $y = 0$, then $y \neq \pm 1$ and so for both f_x and f_y to be 0, it must be that $x = 0$. Similarly if $x = \pm 1$, then $x \neq 0$ and so $y = \pm 1$. Thus we have 5 critical points $(0, 0)$, $(1, 1)$, $(1, -1)$, $(-1, 1)$, and $(-1, 1)$.
We next compute the second derivatives and D.

$$\begin{aligned} f_{xx} &= -xy(1 - x^2)\mathrm{e}^{-(x^2+y^2)/2} - 2xy\mathrm{e}^{-(x^2+y^2)/2} = \mathrm{e}^{-(x^2+y^2)/2}xy(x^2 - 3) \\ f_{xy} &= -y^2(1 - x^2)\mathrm{e}^{-(x^2+y^2)/2} + (1 - x^2)\mathrm{e}^{-(x^2+y^2)/2} = \mathrm{e}^{-(x^2+y^2)/2}(1 - x^2)(1 - y^2) \\ f_{yy} &= -xy(1 - y^2)\mathrm{e}^{-(x^2+y^2)/2} - 2xy\mathrm{e}^{-(x^2+y^2)/2} = \mathrm{e}^{-(x^2+y^2)/2}xy(y^2 - 3) \\ D &= f_{xx}f_{yy} - f_{xy}^2 = \mathrm{e}^{-(x^2+y^2)}\left[x^2y^2(x^2 - 3)(y^2 - 3) - (1 - x^2)^2(1 - y^2)^2\right] \end{aligned}$$

Then we have

$$@(0,0),\ D = -1 \qquad @(1,1),\ D = \frac{4}{\mathrm{e}^2} \text{ and } f_{xx} = -\frac{2}{\mathrm{e}} \qquad @(1,-1),\ D = \frac{4}{\mathrm{e}^2} \text{ and } f_{xx} = \frac{2}{\mathrm{e}}$$

$$@(-1,1),\ D = \frac{4}{\mathrm{e}^2} \text{ and } f_{xx} = \frac{2}{\mathrm{e}} \qquad @(-1,-1),\ D = \frac{4}{\mathrm{e}^2} \text{ and } f_{xx} = -\frac{2}{\mathrm{e}}$$

Thus by the second derivative test, $(0, 0)$ is a saddle point, $(1, 1)$ and $(-1, -1)$ are local maximums, and $(1, -1)$ and $(-1, 1)$ are local minimums.