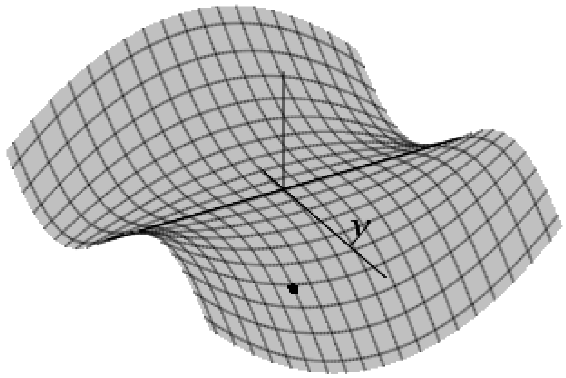
MATH 2210 HOMEWORK WORKSHEET 2

Name: KEY

Partial Derivatives

1. Determine the signs of the partial derviatves for the function f whose graph is shown. The point $\big(1, 2, f(1, 2)\big)$ is marked.



(a) $f_x(1, 2)$

Positive, since as we move in the positive x direction from the point $(1, 2, f(1, 2))$, the z coordinate increases.

(b) $f_y(1, 2)$

Negative, since as we move in the positive y direction from the point $(1, 2, f(1, 2))$, the z coordinate decreases.

(c) $f_{xx}(1, 2)$

Positive, since as we move along the x-direction, that is along a $y =$ constant slice, from the point $(1, 2, f(1, 2))$, the curve cut out is concave up.

(d) $f_{yy}(1, 2)$

Negative, since as we move along the y-direction, that is along an $x =$ constant slice, from the point $(1, 2, f(1, 2))$, the curve cut out is concave down.

(e) $f_{xy}(1, 2)$

Positive. This one is a little difficult to see. However, as we move along the positive y-direction, that is along an$x =$ constant slice, from the point $(1, 2, f(1, 2))$, the slope of the tangent line parallel to the x-axis gets steeper, that is, it increases.

2. Find the first partial derivatives of the following functions.

(a) $f(x, y) = x^2y - 3y^4$

$f_x(x, y) = 2xy$

To compute the partial derivative with respect to x, treat y as a constant and differentiate with respect to x.

$f_y(x, y) = x^2 - 12y^3$

To compute the partial derivative with respect to y, treat x as a constant and differentiate with respect to y.

(b) $u(r, \theta) = \sin(r \cos \theta)$

$u_r(r, \theta) = \cos(r \cos \theta) \cos \theta$

To compute the partial derivative with respect to r, treat θ as a constant and differentiate with respect to r. Don't forget the chain rule.

$u_\theta(r, \theta) = \cos(r \cos \theta)(-r \sin \theta)$

To compute the partial derivative with respect to θ, treat r as a constant and differentiate with respect to θ. Don't forget the chain rule.

3. Use implicit differentiation to find $\partial z/\partial x$ and $\partial z/\partial y$ if

$$yz + x \ln y = z^2.$$

$$\begin{aligned}
\frac{\partial}{\partial x}(yz + x \ln y) &= \frac{\partial}{\partial x}(z^2) \\
y\frac{\partial z}{\partial x} + \ln y &= 2z\frac{\partial z}{\partial x} \\
\ln y &= 2z\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial x} \\
\ln y &= (2z - y)\frac{\partial z}{\partial x}
\end{aligned}$$

$$\frac{\partial z}{\partial x} = \frac{\ln y}{2z - y}$$

$$\begin{aligned}
\frac{\partial}{\partial y}(yz + x \ln y) &= \frac{\partial}{\partial y}(z^2) \\
y\frac{\partial z}{\partial y} + z + \frac{x}{y} &= 2z\frac{\partial z}{\partial y} \\
z + \frac{x}{y} &= 2z\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial y} \\
\frac{zy + x}{y} &= (2z - y)\frac{\partial z}{\partial y}
\end{aligned}$$

$$\frac{\partial z}{\partial y} = \frac{zy + x}{y(2z - y)}$$

4. Find all the second partial derivatives of the function

$$w(u,v) = \sqrt{1+uv^2}.$$

<u>First Derivatives</u>

$$w_u(u,v) = \frac{1}{2\sqrt{1+uv^2}}(v^2) = \frac{v^2}{2\sqrt{1+uv^2}}$$

$$w_v(u,v) = \frac{1}{2\sqrt{1+uv^2}}(2uv) = \frac{uv}{\sqrt{1+uv^2}}$$

<u>Second Derivatives</u>

$$w_{uu}(u,v) = \frac{-v^2\dfrac{1}{\sqrt{1+uv^2}}(v^2)}{4(1+uv^2)} = -\frac{v^4}{4(1+uv^2)^{3/2}}$$

$$w_{uv}(u,v) = \frac{4v\sqrt{1+uv^2} - v^2\dfrac{1}{\sqrt{1+uv^2}}(2uv)}{4(1+uv^2)} = 2v\left(\frac{2\sqrt{1+uv^2} - \dfrac{uv^2}{\sqrt{1+uv^2}}}{4(1+uv^2)}\right)$$

$$= v\left(\frac{\dfrac{2(1+uv^2)-uv^2}{\sqrt{1+uv^2}}}{2(1+uv^2)}\right) = \frac{v(2+uv^2)}{2(1+uv^2)^{3/2}}$$

$$w_{vv}(u,v) = \frac{u\sqrt{1+uv^2} - uv\dfrac{1}{2\sqrt{1+uv^2}}(2uv)}{1+uv^2} = u\left(\frac{\dfrac{1+uv^2-uv^2}{\sqrt{1+uv^2}}}{1+uv^2}\right) = \frac{u}{(1+uv^2)^{3/2}}$$

$$w_{vu}(u,v) = w_{uv}(u,v) = \frac{v(2+uv^2)}{2(1+uv^2)^{3/2}} \qquad \text{by Clairaut's Theorem}$$

Tangent Planes and Linear Approximations

5. Find an equation of the tangent plane to the surface given by

$$z = \frac{x}{y^2}$$

at the point $(-4, 2, -1)$**.**

$$\left.\frac{\partial z}{\partial x}\right|_{(-4,2,-1)} = \left.\frac{1}{y^2}\right|_{(-4,2,-1)} = \frac{1}{4}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(-4,2,-1)} = -\left.\frac{2x}{y^3}\right|_{(-4,2,-1)} = -\frac{-8}{8} = 1$$

$$\begin{aligned} z - z_0 &= \left.\frac{\partial z}{\partial x}\right|_{(-4,2,-1)}(x - x_0) + \left.\frac{\partial z}{\partial y}\right|_{(-4,2,-1)}(y - y_0) \\ z - (-1) &= \frac{1}{4}(x - (-4)) + (y - 2) \\ z &= -1 + \frac{1}{4}x + 1 + y - 2 \\ z &= \frac{1}{4}x + y - 2 \end{aligned}$$

To compute the coefficients in the tangent plane equation, we need to compute both first partials of z at the point $(-4, 2, -1)$. From there, it is simply plugging all the appropriate values into the formula for the tangent plane and solving for z.

6. Verify the linear approximation

$$\frac{y-1}{x+1} \approx x + y - 1$$

at $(0, 0)$**.**

Let $z = \dfrac{y-1}{x+1}$.

$$\left.\frac{\partial z}{\partial x}\right|_{(0,0)} = -\left.\frac{y-1}{(x+1)^2}\right|_{(0,0)} = \frac{1}{1} = 1$$

$$\left.\frac{\partial z}{\partial y}\right|_{(0,0)} = \left.\frac{1}{x+1}\right|_{(0,0)} = \frac{1}{1} = 1$$

$$\begin{aligned} L(x,y) &= z(0,0) + z_x(0,0)(x-0) + z_y(0,0)(y-0) \\ &= -1 + x + y \\ L(x,y) &= x + y - 1 \end{aligned}$$

Thus $\dfrac{y-1}{x+1} \approx x + y - 1$ near $(0, 0)$.

To compute the coefficients in the linear approximation equation, we need to compute both first partials of z at the point $(0, 0)$. From there, it is simply plugging all the appropriate values into the formula for the linear approximation.

7. Given that f is a differentiable function with $f(2,5) = 6$, $f_x(2,5) = 1$, and $f_y(2,5) = -1$, use a linear approximation to estimate $f(2.2, 4.9)$.

By the linear approximation formula, we have

$$L(x,y) = f(2,5) + f_x(2,5)(x-2) + f_y(2,5)(y-5) = 6 + (x-2) - (y-5) = x - y + 9$$

Then $f(2.2, 4.9) \approx L(2.2, 4.9) = 2.2 - 4.9 + 9 = 6.3$.

8. Find the differential of the function $u = \sqrt{x^2 + 3y^2}$.

To compute the differential, we use the differential formula $du = \dfrac{\partial u}{\partial x}\,dx + \dfrac{\partial u}{\partial y}\,dy$. Since

$$\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{x^2+3y^2}}(2x) = \frac{x}{\sqrt{x^2+3y^2}} \qquad \frac{\partial u}{\partial y} = \frac{1}{2\sqrt{x^2+3y^2}}(6y) = \frac{3y}{\sqrt{x^2+3y^2}}$$

we have

$$du = \frac{x}{\sqrt{x^2+3y^2}}\,dx + \frac{3y}{\sqrt{x^2+3y^2}}\,dy = \frac{1}{\sqrt{x^2+3y^2}}(x\,dx + 3y\,dy).$$

9. The length and width of a rectangle are measured as 30 cm and 24 cm, respectively, with an error in measurement of at most 0.1 cm in each. Use differentials to estimate the maximum error in the calculated area of the rectangle.

The area formula for a rectangle of length ℓ and width w is $A = \ell w$. The differential of this formula is

$$dA = \ell\,dw + w\,d\ell.$$

Since the max measurement error of both the length and width is 0.1 cm, we have that

$$d\ell = \Delta\ell = 0.1 \text{ cm} \qquad \text{and} \qquad dw = \Delta w = 0.1 \text{ cm}.$$

We know that the max error of the area ΔA is approximately dA, that is $\Delta A \approx dA$, then to estimate the max error of the rectangle, we need to plug in the above errors on ℓ and w along with the measurements of $\ell = 30$ cm and $w = 24$ cm into the equation for dA to obtain the approximate maximum error in the area

$$\Delta A \approx dA = (30 \text{ cm})(0.1 \text{ cm}) + (24 \text{ cm})(0.1 \text{ cm}) = 5.4 \text{ cm}^2.$$

Thus the approximate maximum error in the calculated area of the rectangle is 5.4 cm^2.

The Chain Rule

10. Use the chain rule to find $\frac{dz}{dt}$ and $\frac{dw}{dt}$.

(a) $z = xy^3 - x^2y$, **where** $x = t^2 + 1$, **and** $y = t^2 - 1$.

By the chain rule, $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$. Then we have

$$\frac{\partial z}{\partial x} = y^3 - 2xy \qquad \frac{dx}{dt} = 2t$$
$$\frac{\partial z}{\partial y} = 3xy^2 - x^2 \qquad \frac{dy}{dt} = 2t$$

Plugging this into the formula above yields

$$\begin{aligned}\frac{dz}{dt} &= (y^3 - 2xy)(2t) + (3xy^2 - x^2)(2t)\\ &= \left[(t^2-1)^3 - 2(t^2+1)(t^2-1)\right](2t) + \left[3(t^2+1)(t^2-1)^2 - (t^2+1)^2\right](2t)\\ &= 8t^7 - 18t^5 - 4t^3 + 6t\end{aligned}$$

You do not have to simplify to get the last line. The previous line is sufficient.

(b) $w = \ln\sqrt{x^2+y^2+z^2}$, **where** $x = \sin t$, $y = \cos t$, **and** $z = \tan t$.

First note that $\ln\sqrt{x^2+y^2+z^2} = \frac{1}{2}\ln(x^2+y^2+z^2)$ and if we plug in the above,

$$x^2 + y^2 + z^2 = \sin^2 t + \cos^2 t + \tan^2 t = 1 + \tan^2 t = \sec^2 t.$$

By the chain rule $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$. Then we have

$$\frac{\partial w}{\partial x} = \frac{x}{x^2+y^2+z^2} \qquad \frac{\partial w}{\partial y} = \frac{y}{x^2+y^2+z^2} \qquad \frac{\partial w}{\partial z} = \frac{z}{x^2+y^2+z^2}$$
$$\frac{dx}{dt} = \cos t \qquad \frac{dy}{dt} = -\sin t \qquad \frac{dz}{dt} = \sec^2 t$$

Plugging these into the formula above yields

$$\begin{aligned}\frac{dw}{dt} &= \frac{x}{x^2+y^2+z^2}(\cos t) + \frac{y}{x^2+y^2+z^2}(-\sin t) + \frac{z}{x^2+y^2+z^2}(\sec^2 t)\\ &= (\sin t)(\cos^3 t) - (\sin t)(\cos^3 t) + \tan t\\ &= \tan t\end{aligned}$$

11. Use the chain rule to find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ if

$$z = \sqrt{x}\mathrm{e}^{xy}, \quad \textbf{where} \quad x = 1 + st, \quad \textbf{and} \quad y = s^2 - t^2.$$

We have that $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$ and $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$. Since

$$\begin{aligned}
\frac{\partial z}{\partial x} &= y\sqrt{x}\mathrm{e}^{xy} + \frac{\mathrm{e}^{xy}}{2\sqrt{x}} & \frac{\partial x}{\partial s} &= t & \frac{\partial y}{\partial s} &= 2s \\
\frac{\partial z}{\partial y} &= x^{3/2}\mathrm{e}^{xy} & \frac{\partial x}{\partial t} &= s & \frac{\partial y}{\partial t} &= -2t,
\end{aligned}$$

we have that

$$\begin{aligned}
\frac{\partial z}{\partial s} &= \left(y\sqrt{x}\mathrm{e}^{xy} + \frac{\mathrm{e}^{xy}}{2\sqrt{x}}\right)t + \left(x^{3/2}\mathrm{e}^{xy}\right)(2s) \\
&= \mathrm{e}^{(1+st)(s^2-t^2)}\left(t(s^2-t^2)\sqrt{1+st} + \frac{t}{2\sqrt{1+st}} + 2s(1+st)^{3/2}\right) \\
\frac{\partial z}{\partial t} &= \left(y\sqrt{x}\mathrm{e}^{xy} + \frac{\mathrm{e}^{xy}}{2\sqrt{x}}\right)s + \left(x^{3/2}\mathrm{e}^{xy}\right)(-2t) \\
&= \mathrm{e}^{(1+st)(s^2-t^2)}\left(s(s^2-t^2)\sqrt{1+st} + \frac{s}{2\sqrt{1+st}} - 2t(1+st)^{3/2}\right)
\end{aligned}$$

12. Use the chain rule to find $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial \theta}$ if

$$w = xy + yz + zx, \quad \textbf{where} \quad x = r\cos\theta, \quad y = r\sin\theta, \quad \textbf{and} \quad z = r\theta,$$

when $r = 2$ and $\theta = \dfrac{\pi}{2}$.

We know that $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$ and $\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \theta}$ and that when $r = 2$ and $\theta = \dfrac{\pi}{2}$, $x = 0$, $y = 2$, and $z = \pi$. Since

$$\begin{aligned}
\frac{\partial w}{\partial x} &= y + z & \frac{\partial x}{\partial r} &= \cos\theta & \frac{\partial y}{\partial r} &= \sin\theta & \frac{\partial z}{\partial r} &= \theta \\
\frac{\partial w}{\partial y} &= x + z & \frac{\partial x}{\partial \theta} &= -r\sin\theta & \frac{\partial y}{\partial \theta} &= r\cos\theta & \frac{\partial z}{\partial \theta} &= r \\
\frac{\partial w}{\partial z} &= y + x
\end{aligned}$$

This implies that at $r = 2$ and $\theta = \dfrac{\pi}{2}$, we have

$$\begin{aligned}
\left.\frac{\partial w}{\partial r}\right|_{r=2,\theta=\pi/2} &= (y+z)\cos\theta + (x+z)\sin\theta + (y+x)\theta = 2\pi \\
\left.\frac{\partial w}{\partial \theta}\right|_{r=2,\theta=\pi/2} &= (y+z)(-r\sin\theta) + (x+z)(r\cos\theta) + (y+x)r = -2(2+\pi) + 4 = -2\pi
\end{aligned}$$

13. Use the equations

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}, \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

to compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ where

$$yz + x\ln y = z^2.$$

The above equation can be written as

$$F(x,y,z) = yz + x\ln y - z^2 = 0.$$

Then we have that

$$\frac{\partial F}{\partial x} = \ln y, \qquad \frac{\partial F}{\partial y} = z + \frac{x}{y}, \qquad \frac{\partial F}{\partial z} = y - 2z.$$

Then by the above equations, we have that

$$\frac{\partial z}{\partial x} = -\frac{\ln y}{y-2z} \qquad \frac{\partial z}{\partial y} = -\frac{z+\frac{x}{y}}{y-2z} = -\frac{zy+x}{y^2-2yz}$$

14. The radius of a right circular cone is increasing at a rate of 1.8 in/s while its height is decreasing at a rate of 2.5 in/s. At what rate is the volume of the cone changing when the radius is 120 in and the height is 140 in?

The volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$ and hence

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = \left(\frac{2}{3}\pi r h\right)\frac{dr}{dt} + \left(\frac{1}{3}\pi r^2\right)\frac{dh}{dt}.$$

Since $\frac{dr}{dt} = 1.8$ and $\frac{dh}{dt} = -2.5$, when $r = 120$ and $h = 140$, we have that

$$\frac{dV}{dt} = \frac{2}{3}\pi(120)(140)(1.8) + \frac{1}{3}\pi(120)^2(-2.5) \approx 25635.4$$

That is, the volume is increasing at a rate of 25635.4 in^3/s.

15. Show that any function of the form

$$z = f(x + at) + g(x - at)$$

is a solution of the wave equation

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}.$$

Hint: Let $u = x + at$ ***and*** $v = x - at$***.***

Notice that

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial t} + \frac{\partial g}{\partial t} \qquad \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}$$

Let $u = x + at$ and $v = x - at$. Then by the chain rule

$$\frac{\partial f}{\partial t} = \frac{df}{du}\frac{\partial u}{\partial t} \qquad \frac{\partial g}{\partial t} = \frac{dg}{dv}\frac{\partial v}{\partial t}$$
$$\frac{\partial f}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x} \qquad \frac{\partial g}{\partial x} = \frac{dg}{dv}\frac{\partial v}{\partial x}$$

Since

$$\frac{df}{du} = f'(u) = f'(x + at) \qquad \frac{\partial u}{\partial t} = a \qquad \frac{\partial u}{\partial x} = 1$$
$$\frac{dg}{dv} = g'(v) = g'(x - at) \qquad \frac{\partial v}{\partial t} = -a \qquad \frac{\partial v}{\partial x} = 1$$

Then

$$\frac{\partial z}{\partial t} = \frac{df}{du}\frac{\partial u}{\partial t} + \frac{dg}{dv}\frac{\partial v}{\partial t} = af'(x + at) - ag'(x - at)$$
$$\frac{\partial z}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x} + \frac{dg}{dv}\frac{\partial v}{\partial x} = f'(x + at) + g'(x - at)$$

Similarly

$$\frac{\partial^2 z}{\partial t^2} = a^2 f''(x + at) + a^2 g''(x - at) = a^2\left(f''(x + at) + g''(x - at)\right)$$
$$\frac{\partial^2 z}{\partial x^2} = f''(x + at) + g''(x - at)$$

Hence for $z = f(x + at) + g(x - at)$,

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}.$$