MATH 2210 HOMEWORK WORKSHEET 5

Name: KEY

Functions of Several Variables

1. A company makes three sizes of cardboard boxes: small, medium, and large. It costs $\$2.50$ to make a small box, $\$4.00$ for a medium box, and $\$4.50$ for a large box. Fixed costs are $\$8,000$.

(a) Express the cost of making x small boxes, y medium boxes, and z large boxes as a function of three variables: $C = f(x, y, z)$.

$C = 2.5x + 4y + 4.5z + 8000$

The cost is equal to the fixed costs plus \$2.50 each for the x small boxes, \$4.00 each for the y medium boxes, and \$4.50 each for the z large boxes.

(b) Find $f(3000, 5000, 4000)$ and interpret it.

$$\begin{aligned} &f(3000, 5000, 4000) \\ &\quad = 2.5(3000) + 4(5000) + 4.5(4000) + 8000 \\ &\quad = 7500 + 20000 + 18000 + 8000 \\ &\quad = 53500 \end{aligned}$$

To compute this, simply plug in $x = 3000$, $y = 5000$, and $z = 4000$ into f and simplify.

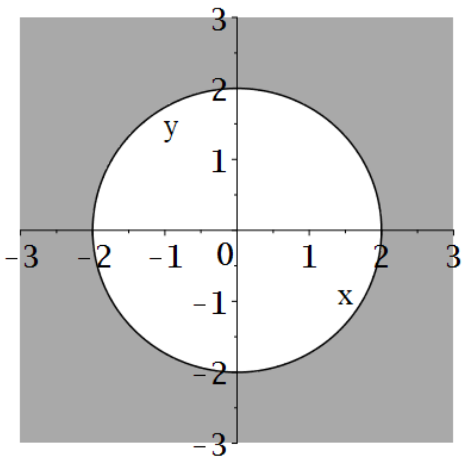
It costs \$53,500.00 to make 3,000 small boxes, 5,000 medium boxes, and 4,000 large boxes.

(c) What is the domain of f?

Mathematically, the domain is all of $\mathbb{R}^3$ because the function is a multivariate polynomial. However, as a real life application, the domain would be restricted by the amount of material or money the company could acquire, but there is not enough information to determine this.

2. Find and sketch the domain of the function.

$$f(x,y) = \sqrt{x^2+y^2-4}$$



The function's domain is only restricted by the square root; whatever is under the square root must be nonnegative. Thus the domain is

$$x^2+y^2-4 \geq 0$$
$$x^2+y^2 \geq 4$$

or the outside and edge of the circle centered at the origin of radius 2.

3. Match the function with its graph (labeled I-IV and listed on the following page). Give reasons for your choices.

(a) $f(x,y) = \dfrac{1}{1+x^2+y^2}$

II. We have that $f(0,0)=1$ and along any line $y=mx$ through the origin, the function becomes $z = 1/(1+(m^2+1)x^2)$ which decays to 0 as $x \to \pm\infty$ (i.e. as we move away from the origin).

(b) $f(x,y) = \dfrac{1}{1+x^2y^2}$

I. We have the $f=1$ along the lines $y=0$ and $x=0$. Along any other line $y=mx$ through the origin, the function becomes $z = 1/(1+m^2x^4)$ which decays to 0 as $x \to \pm\infty$ (i.e. as we move away from the origin).

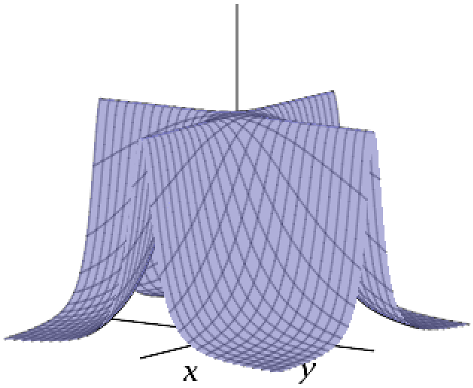
(c) $f(x,y) = \ln(x^2+y^2)$

III. The natural log goes to $-\infty$ as its argument goes to 0. The expression x^2+y^2 only equals zero at the origin $(0,0)$ and increases along circles centered at the origin as the radius increases. Thus we essentially get the typical $\ln x$ graph rotated in a circle centered around the z-axis.

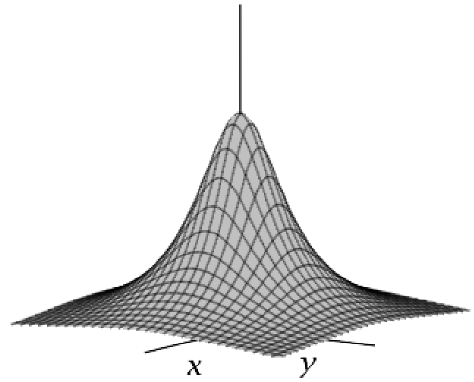
(d) $f(x,y) = \cos\left(\sqrt{x^2+y^2}\right)$

IV. This is the only graph that oscillates and so must match with the only function to have an oscillating function. However, note that the function oscillates out like ripples in a pond, that is, along concentric circles and that the argument in the cos is in fact the distance from the point $(0,0)$ and so would be the same aroung concentric circles.

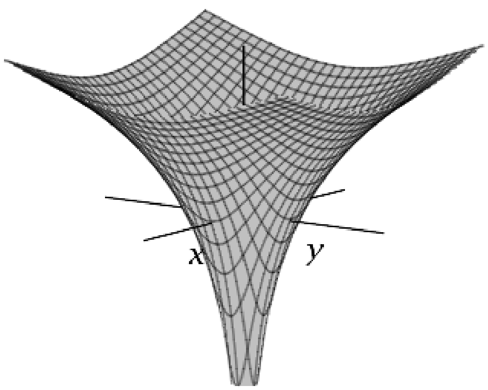
I.



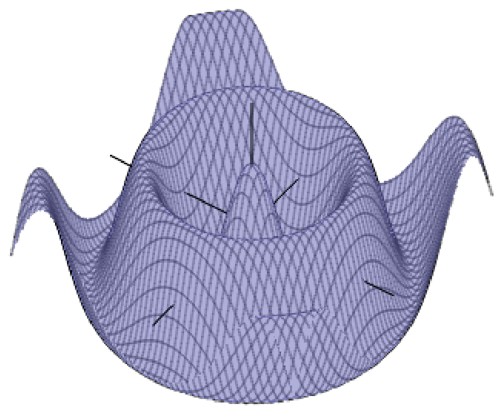
II.



III.



IV.



Limits and Continuity

4. Find the limit if it exists, or show that the limit does not exist.

(a) $\lim_{(x,y)\to(3,2)} (x^2y^3 - 4y^2)$

$$\begin{aligned}\lim_{(x,y)\to(3,2)} (x^2y^3 - 4y^2) &= 3^2 2^3 - 4(2^2) \\ &= 56\end{aligned}$$

This function is a multivariate polynomial and hence continuous everywhere, thus any limit can be computed by evaluating the function at that point.

(b) $\lim_{(x,y)\to(0,0)} \dfrac{xy^4}{x^4+y^4}$

We suspect that the limit is equal to 0, as this is the limit along the curves $y = 0$, $x = 0$, and $y = x$. We use a Squeeze Theorem argument to show that

$$\lim_{(x,y)\to(0,0)} \left|\frac{xy^4}{x^4+y^4}\right| = 0,$$

as this lets us conclude that the original limit is zero.
Note that $x^4 \geq 0$, hence $y^4 \leq x^4 + y^4$. Thus

$$0 \leq \left|\frac{xy^4}{x^4+y^4}\right| \leq |x|.$$

As $x \to 0$, we have that $|x| \to 0$. So by the Squeeze Theorem, $\left|\dfrac{xy^4}{x^4+y^4}\right| \to 0$.

(c) $\lim_{(x,y)\to(0,0)} \dfrac{xy^3}{x^2+y^6}$

We suspect that the limit does not exist due to the mismatch of powers. Along the curve $x = y$, we have that

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} = \lim_{y\to0} \frac{y^4}{y^2+y^6} = \lim_{y\to0} \frac{y^2}{1+y^4} = 0.$$

We try the curve $x = y^3$. We compute:

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} = \lim_{(y^3,y)\to(0,0)} \frac{y^6}{y^6+y^6} = \lim_{(y^3,y)\to(0,0)} \frac{y^6}{2y^6} = \frac{1}{2}.$$

Since the limit is different along two different curves, it follows that the original limit does not exist.

5. Determine the set of points at which the function is continuous.

$$F(x, y) = \frac{1 + x^2 + y^2}{1 - x^2 - y^2}.$$

This is a multivariate rational function and hence is continuous in its domain. The domain of this function is everywhere the denominator is nonzero. That is the denominator is defined by

$$1 - x^2 - y^2 \neq 0 \qquad \text{or, equivalently,} \qquad x^2 + y^2 \neq 1.$$

Thus this function is continuous on all of $\mathbb{R}^2$ except for the circle centered at the origin of radius 1.

6. Use polar coordinates to find the limit. [If (r, θ) are polar coordinatess of the point (x, y) with $r \geq 0$, note that $r \to 0^+$ as $(x, y) \to (0, 0)$.]

$$\lim_{(x,y)\to(0,0)} (x^2 + y^2) \ln(x^2 + y^2).$$

We make the substitution $r^2 = x^2 + y^2$ and exchange $(x, y) \to (0, 0)$ with $r \to 0^+$. This yields that

$$\lim_{(x,y)\to(0,0)} (x^2 + y^2) \ln(x^2 + y^2) = \lim_{r\to 0^+} r^2 \ln r^2 = \lim_{r\to 0^+} \frac{2 \ln r}{1/r^2}.$$

This limit is now of indeterminate form $\frac{\infty}{\infty}$ and so L'Hospital's rule applies. Thus we compute this limit as follows

$$\lim_{r\to 0^+} \frac{2 \ln r}{1/r^2} = \lim_{r\to 0^+} \frac{2/r}{-2/r^3} = \lim_{r\to 0^+} -\frac{r^3}{r} = \lim_{r\to 0^+} -r^2 = -0^2 = 0.$$

We can finally evaluate the function at $r = 0$ in the last equation because the function there is a polynomial which is continuous everywhere.