# Homework 5

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# Partial Derivatives

## Problem 1

Determine the signs of the partial derivatives for the function $f$ whose graph is shown (see worksheet). The point $(1, 2, f(1, 2))$ is marked.

1. $f_x(1, 2)$
2. $f_y(1, 2)$
3. $f_{xx}(1, 2)$
4. $f_{yy}(1, 2)$
5. $f_{xy}(1, 2)$
6. $f_{yx}(1, 2)$

### Answer

To be honest, I struggled a little bit to answer this questions by inspection because of the perspective issue of projecting 3D into 2D. I also made the mistake of not realizing we wanted the derivative at *the given point.* After checking the solution sheet, I started judging the grids on the surface I was able to start answering.

Also, it helps projecting a slice going in one of the directions into its orthogonal plane (at least, that's what I mentally tried).

1. Positive
2. Negative
3. Positive
4. Negative
5. Positive

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## Problem 2

Find the first partial derivative of the following functions:

1. $f(x, y) = x^2 - 3y^4$
2. $u(r, \theta) = \sin(r\cos(\theta))$

### Answer

1.

$$\begin{aligned} f_x(x, y) &= \frac{\partial(x^2 - 3y^4)}{\partial x} \\ &= \frac{\partial x^2}{\partial y} - 3y^4 \frac{\partial(1)}{\partial y} \\ &= 2x \end{aligned}$$

$$\begin{aligned} f_y(x, y) &= \frac{\partial}{\partial y}(x^2 - 3y^4) \\ &= x^2 \frac{\partial(1)}{\partial y} - 3\frac{\partial(y^4)}{\partial y} \\ &= -12y^3 \end{aligned}$$

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2.

$$\begin{aligned} u_r(r, \theta) &= \frac{\partial \sin(r\cos\theta)}{\partial r} \\ &= \frac{\partial \sin u}{\partial u} \cdot \cos\theta \frac{\partial r}{\partial r} \\ &= \cos(r\cos(\theta)) \cdot \cos(\theta) \end{aligned}$$

$$\begin{aligned} u_\theta(r, \theta) &= \frac{\partial \sin(r\cos\theta)}{\partial \theta} \\ &= \frac{\partial \sin u}{\partial u} \cdot r \frac{\partial(\cos\theta)}{\partial \theta} \\ &= \cos(r\cos\theta) \cdot -r\sin\theta \end{aligned}$$

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## Problem 3

Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$yz + x \ln y = z^2$$

**Answer**

$$\begin{aligned}
\frac{\partial(yz)}{\partial x} + \ln y \frac{\partial(x)}{\partial(x)} &= \frac{\partial z^2}{\partial x} \\
y\frac{\partial z}{\partial x} + \ln y &= 2z\frac{\partial z}{\partial x} \\
\ln y &= 2z\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial x} \\
&= \frac{\partial z}{\partial x}(2z - y) \\
\frac{\ln y}{2z - y} &= \frac{\partial z}{\partial x}
\end{aligned}$$

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$$\begin{aligned}
\frac{\partial(yz)}{\partial y} + \frac{\partial(x \ln y)}{\partial y} &= \frac{\partial z^2}{\partial y} \\
z + y\frac{\partial z}{\partial y} + \frac{x}{y} &= 2z\frac{\partial z}{\partial y} \\
z + \frac{x}{y} &= 2z\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial y} \\
&= (2z - y)\frac{\partial z}{\partial y} \\
\frac{z + \frac{x}{y}}{2z - y} &= \frac{\partial z}{\partial y} \\
\frac{z}{2z - y} + \frac{x}{y(2z - y)} &= \\
\frac{zy + x}{y(2z - y)} &=
\end{aligned}$$

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## Problem 4

Find all the second partial derivatives of the function

$$w(u,v) = \sqrt{1+uv^2}$$

**Answer**

$$\begin{aligned}
w_u(u,v) &= \frac{\partial \sqrt{1+uv^2}}{\partial u} \\
&= \frac{\partial x^{\frac{1}{2}}}{\partial x} \cdot \frac{\partial (1+uv^2)}{\partial u} \\
&= \frac{1}{2\sqrt{1+uv^2}} \cdot v^2 \quad = \frac{v^2}{2\sqrt{1+uv^2}}
\end{aligned}$$

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$$\begin{aligned}
w_{uu}(u,v) &= \frac{\partial}{\partial u}\left(\frac{v^2}{2\sqrt{1+uv^2}}\right) \\
&= \frac{v^2}{2} \cdot \frac{\partial}{\partial u}\left((1+uv^2)^{-\frac{1}{2}}\right) \\
&= -\frac{v^4}{4} \cdot (1+uv^2)^{-\frac{3}{2}}) \\
&= -\frac{v^4}{4\sqrt{(1+uv^2)^3}}
\end{aligned}$$

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$$\begin{aligned}
w_{uv}(u,v) &= \frac{\partial}{\partial v}\left(\frac{v^2}{2\sqrt{1+uv^2}}\right) \\
&= \frac{1}{2}\frac{\partial}{\partial v}\left(v^2(1+uv^2)^{-\frac{1}{2}}\right) \\
&= \frac{1}{2}\left(\frac{\partial v^2}{\partial v}(1+uv^2)^{-\frac{1}{2}} + v^2\frac{\partial}{\partial v}(1+uv^2)^{-\frac{1}{2}}\right) \\
&= \frac{1}{2}\left(\frac{2v}{\sqrt{1+uv^2}} - 2u\frac{v^3}{2(1+uv^2)^{\frac{3}{2}}}\right) \\
&= \frac{1}{2}\left(\frac{2v}{\sqrt{1+uv^2}} - \frac{2uv^3}{2(1+uv^2)^{\frac{3}{2}}}\right)
\end{aligned}$$

## Tangent Planes and Linear Approximation

### Problem 5

Find an equation of the tangent plane to the surface given by

$$z = \frac{x}{y^2}$$

at the point $(-4, 2, -1)$.

**Answer**

Let $(x_0, y_0, z_0)$ be the given point, and let $T$ be the tangent plane to the function $z$. Therefore, $T = z - z_0 = \frac{\partial z}{\partial x}(x - x_0) + \frac{\partial z}{\partial y}(y - y_0)$.

$$\begin{aligned}
\frac{\partial z}{\partial x} \mid_{-4,2,-1} &= \frac{\partial}{\partial x}\left(\frac{x}{y^2}\right) \\
&= \frac{\partial x}{\partial x}\left(\frac{1}{y^2}\right) + x\frac{\partial y^{-2}}{\partial x} \\
&= \frac{1}{y^2} - \frac{2x}{y^3} \cdot \frac{\partial y}{\partial x} \\
&= \frac{1}{y^2} \\
&= \frac{1}{4}
\end{aligned}$$

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$$\begin{aligned}
\frac{\partial z}{\partial y} \mid_{-4,2,-1} &= \frac{\partial}{\partial y}\left(\frac{x}{y^2}\right) \\
&= \frac{\partial x}{\partial y}\left(\frac{1}{y^2}\right) + x\frac{\partial y^{-2}}{\partial y} \\
&= \frac{\partial x}{\partial y}\left(\frac{1}{y^2}\right) - \frac{2x}{y^3} \\
&= -\frac{2x}{y^3} \\
&= -\frac{2(-4)}{(2)^3} \\
&= \frac{8}{8} \\
&= 1
\end{aligned}$$

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Therefore:

$$\begin{aligned}
z &= \frac{1}{4}(x + 4) + (y - 2) - 1 \\
&= \frac{x}{4} + y - 2
\end{aligned}$$

is the equation for the tagent plane at $(-4, 2, 1)$.

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## Problem 6

Verify the linear approximation

$$\frac{y-1}{x+1} \approx x + y - 1$$

at $(0,0)$.

**Answer**

Let $z = \frac{y-1}{x+1}$, and $(x_0, y_0) = (0,0)$. Then

$$z_0 = z(0,0) = \frac{0-1}{0+1} = -1$$

We know that the linear approximation of $z$ at the point $(x_0, y_0, z_0)$ can be found using the following formula:

$$z - z_0 = z_x(x - x_0) + z_y(y - y_0)$$

From computation, we know that:

$$\begin{aligned}
z_x \;|_{(0,0,-1)} &= \frac{\partial}{\partial x}\left(\frac{y-1}{x+1}\right)|_{(0,0,-1)} \\
&= \frac{\partial}{\partial x}((y-1)(x+1)^{-1})|_{(0,0,-1)} \\
&= \frac{\partial(y-1)}{\partial x}\left(\frac{1}{x+1}\right) + \frac{\partial(x+1)^{-1}}{\partial x}(y-1)|_{(0,0,-1)} \\
&= \frac{\partial(x+1)^{-1}}{\partial x}(y-1)|_{(0,0,-1)} \\
&= \frac{1-y}{(x+1)^2} \;|_{(0,0,-1)} \\
&= 1
\end{aligned}$$

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$$\begin{aligned}
z_y|_{(0,0,-1)} &= \frac{\partial}{\partial y}\left(\frac{y-1}{x+1}\right)|_{(0,0,-1)} \\
&= \frac{\partial}{\partial y}((y-1)(x+1)^{-1})|_{(0,0,-1)} \\
&= \frac{\partial(y-1)}{\partial y}\left(\frac{1}{x+1}\right) + \frac{\partial(x+1)^{-1}}{\partial y}(y-1)|_{(0,0,-1)} \\
&= \frac{1}{x+1} \\
&= 1
\end{aligned}$$

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Therefore:

$$\begin{aligned}
z + 1 &= 1x + 1y \\
z &= x + y - 1
\end{aligned}$$

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## Problem 7

Given that $f$ is a differentiable function with $f(2,5) = 6$, $f_x(2,5) = 1$ and $f_y(2,5) = -1$, use linear approximation to estimate $f(2.2, 4.9)$.

### Answer

We can approximate $f(x,y) \approx (x-2) + (y+5) + 6 = x - y + 9$. If we plug in $(2.2, 4.9)$, we get

$$\begin{aligned} f(2.2, 4.9) &= 2.2 - 4.9 + 9 \\ &= 6.3 \end{aligned}$$

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## Problem 8

Find the differential of the function $u = \sqrt{x^2 + 3y^2}$.

### Answer

We know that $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} + dy$,

$$\begin{aligned} u_x &= \frac{\partial}{\partial x} \sqrt{x^2 + 3y^2} \\ &= \frac{1}{2\sqrt{x^2 + 3y^2}} \frac{\partial}{\partial x} (x^2 + 3y^2) \\ &= \frac{x}{\sqrt{x^2 + 3y^2}} \end{aligned}$$

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$$\begin{aligned} u_y &= \frac{\partial}{\partial y} \sqrt{x^2 + 3y^2} \\ &= \frac{1}{2\sqrt{x^2 + 3y^2}} \frac{\partial}{\partial y} (x^2 + 3y^2) \\ &= \frac{3y}{\sqrt{x^2 + 3y^2}} \end{aligned}$$

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Then,

$$\begin{aligned} du &= \frac{x}{\sqrt{x^2 + 3y^2}} dx + \frac{3y}{\sqrt{x^2 + 3y^2}} dy \\ &= \frac{x dx + 3y dy}{\sqrt{x^2 + 3y^2}} \end{aligned}$$

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## Problem 9

The length and width of a rectangle are measured as 30 $cm$ and 24 $cm$, respectively, with an error in measurement of at most 0.1 $cm$ in each. Use differentials to estimate the maximum error in the calculated area of the rectangle.

### Answer

We know that the area of a rectangle $A$ is a function of its base $b$ and height $h$: $A = bh$. Therefore, its differential $dA$ is given as

$$dA = \frac{\partial A}{\partial b} db + \frac{\partial A}{\partial h} dh$$

where $db = dh = 0.1\ cm$. Then

$$\frac{\partial A}{\partial b} = \frac{\partial}{\partial b} bh = h$$

$$\frac{\partial A}{\partial h} = \frac{\partial}{\partial h} bh = b$$

Therefore,

$$\begin{aligned} dA &= 24(0.1) + 30(0.1) \\ &= 5.4 \end{aligned}$$

or our estimate of the maximum error is 5.4 $cm$.

# The Chain Rule

## Problem 10

Use the chain rule to find $\frac{dz}{dt}$ and $\frac{dw}{dt}$.

1. $z = xy^3 - x^2y$, where $x = t^2 + 1$ and $y = t^2 - 1$.
2. $w = \ln\sqrt{x^2 + y^2 + z^2}$, where $x = \sin t$, $y = \cos t$, $z = \tan t$.

### Answer

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1.

$$\frac{dx}{dt} = \frac{dy}{dt} = 2t$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(xy^3 - x^2y) = y^3 - 2xy$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(xy^3 - x^2y) = 3xy^2 - x^2$$

$$\begin{aligned}\frac{dz}{dt} &= 2t((y^3 - 2yx) + (3xy^2 - x^2)) \\ &= 2t\left[\left((t^2-1)^3 - 2(t^2-1)(t^2+1)\right) + \left(3(t^1+1)(t^2-1)^2 - (t^2+1)^2\right)\right]\end{aligned}$$

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2.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\frac{dx}{dt} = \cos t, \frac{dy}{dt} = -\sin t, \frac{dz}{dt} = \sec^2 t$$

$$\begin{aligned}\frac{\partial w}{\partial x} &= \frac{\partial}{\partial x}\ln\sqrt{x^2+y^2+z^2} \\ &= \frac{1}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} \\ &= \frac{1}{\sqrt{x^2+y^2+z^2}}\left(\frac{1}{2\sqrt{x^2+y^2+z^2}}\right)\left(\frac{\partial}{\partial x}(x^2+y^2+z^2)\right) \\ &= \left(\frac{1}{2(x^2+y^2+z^2)}\right)(2x) \\ &= \frac{x}{(x^2y^3+z^2)} \\ &= \frac{\sin t}{\sin^2 t + \cos^2 t + \tan^2 t} \\ &= \frac{\sin t}{1 + \tan^2 t}\end{aligned}$$

$$
\begin{aligned}
\frac{\partial w}{\partial y} &= \frac{\partial}{\partial y} \ln \sqrt{x^2 + y^2 + z^2} \\
&= \frac{1}{\sqrt{x^2 + y^2 + z^2}} \left( \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \right) \left( \frac{\partial}{\partial y} (x^2 + y^2 + z^2) \right) \\
&= \left( \frac{1}{2(x^2 + y^2 + z^2)} \right) (2y) \\
&= \frac{y}{(x^2 y^3 + z^2)} \\
&= \frac{\cos t}{\sin^2 t + \cos^2 t + \tan^2 t} \\
&= \frac{\cos t}{1 + \tan^2 t}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial w}{\partial z} &= \frac{\partial}{\partial z} \ln \sqrt{x^2 + y^2 + z^2} \\
&= \frac{1}{\sqrt{x^2 + y^2 + z^2}} \left( \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \right) \left( \frac{\partial}{\partial z} (x^2 + y^2 + z^2) \right) \\
&= \left( \frac{1}{2(x^2 + y^2 + z^2)} \right) (2z) \\
&= \frac{z}{(x^2 y^3 + z^2)} \\
&= \frac{\tan t}{\sin^2 t + \cos^2 t + \tan^2 t} \\
&= \frac{\tan t}{1 + \tan^2 t}
\end{aligned}
$$

Then,

$$
\begin{aligned}
\frac{dw}{dt} &= \frac{\sin t}{1 + \tan^2 t} (\cos t) + \frac{\cos t}{1 + \tan^2 t} (-\sin t) + \frac{\tan t}{1 + \tan^2 t} (\sec^2 t) \\
&= \frac{\sin t \cos t}{1 + \tan^2 t} - \frac{\sin t \cos t}{1 + \tan^2 t} + \frac{\tan t}{\cos^2 t (1 + \tan^2 t)} \\
&= \frac{\tan t}{\cos^2 t + \sin^2 t} \\
&= \tan t
\end{aligned}
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## Problem 11

Use the chain rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ if

$$z = \sqrt{x}e^{xy}, \text{where } x = 1 + st \text{ and } y = s^2 - t^2$$

**Answer**

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial x}{\partial t} = s, \frac{\partial y}{\partial t} = -2t$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\sqrt{x}e^{xy} = \frac{e^{xy}}{2\sqrt{x}} + \sqrt{x}ye^{xy} = e^{xy}\left(\frac{1}{2\sqrt{x}} + \sqrt{x}y\right)$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\sqrt{x}e^{xy} = \sqrt{x^3}e^{xy}$$

$$\begin{aligned}\frac{\partial z}{\partial t} &= s\left(\frac{e^{xy}}{2\sqrt{x}} + \sqrt{x}ye^{xy}\right) - 2t\left(\sqrt{x^3}e^{xy}\right) \\ &= e^{\sqrt{1-st}(s^2+t^2)}\left(s\left(\frac{1}{2\sqrt{1-st}} + \sqrt{1-st}(s^2+t^2)\right) - 2t\sqrt{(1-st)^3}\right)\end{aligned}$$

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Note that $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ have been computed.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial x}{\partial s} = t, \frac{\partial y}{\partial s} = 2s$$

$$\begin{aligned}\frac{\partial z}{\partial s} &= t\left(\frac{e^{xy}}{2\sqrt{x}} + \sqrt{x}ye^{xy}\right) + 2s\left(\sqrt{x^3}e^{xy}\right) \\ &= e^{\sqrt{1-st}(s^2+t^2)}\left(t\left(\frac{1}{2\sqrt{1-st}} + \sqrt{1-st}(s^2+t^2)\right) + 2s\sqrt{(1-st)^3}\right)\end{aligned}$$

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## Problem 12

Use the chain rule to find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$ if

$$w = xy + yz + zx \text{ where } x = r\cos\theta, y = r\sin\theta \text{ and } z = r\theta$$

when $r = 2$ and $\theta = \frac{\pi}{2}$.

### Answer

We know that for $(r, \theta) = \left(2, \frac{\pi}{2}\right)$, $x = 0, y = 2, z = \pi$. Also:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial x}{\partial r} = \cos\theta =, \frac{\partial y}{\partial r} = \sin\theta, \frac{\partial z}{\partial r} = \theta$$

$$\begin{aligned}\frac{\partial w}{\partial x} &= \frac{\partial xy}{\partial x} + \frac{\partial yz}{\partial x} + \frac{\partial xz}{\partial x}\\ &= y + z\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial y} &= \frac{\partial xy}{\partial y} + \frac{\partial yz}{\partial y} + \frac{\partial xz}{\partial y}\\ &= x + z\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial z} &= \frac{\partial xy}{\partial z} + \frac{\partial yz}{\partial z} + \frac{\partial xz}{\partial z}\\ &= x + y\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial w}{\partial r}\ |_{(2,\frac{\pi}{2})} &= (2+\pi)(0) + (\pi)(1) + (\pi)\left(\frac{\pi}{2}\right)\\ &= 2\pi\end{aligned}$$

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$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \theta}$$

$$\frac{\partial x}{\partial r} = -\sin\theta, \frac{\partial y}{\partial r} = \cos\theta, \frac{\partial z}{\partial r} = r$$

$$\begin{aligned}\frac{\partial w}{\partial r}\ |_{(2,\frac{\pi}{2})} &= (2+\pi)(-2) + (\pi+0)(0) + (2)(0+2)\\ &= -4 - 2\pi + 4\\ &= -2\pi\end{aligned}$$

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## Problem 13

Use the equations

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$
$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

to compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, where

$$yz + x \ln y = z^2$$

**Answer**

Let the given equation be $F(x, y, z)$. Then

$$\frac{\partial F}{\partial x} = \ln y$$
$$\frac{\partial F}{\partial y} = z + \frac{x}{y}$$
$$\frac{\partial F}{\partial z} = y - 2z$$

Therefore,

$$\frac{\partial z}{\partial x} = -\frac{\ln y}{y - 2z}$$
$$\frac{\partial z}{\partial y} = -\frac{z + \frac{x}{y}}{y - 2z} = -\frac{zy + x}{y(y - 2z)}$$

## Problem 14

The radius of a right circular cone is increassing at a rate of 1.8 *in/s* while its height is decreasing at a rate of 2.5 *in/s*. At what rate is the volume of the cone changin when the radius is 120 *in* and the height is *140* in?

**Answer**

Let the volume of the cone be a function $V(r, h) = \frac{1}{3}\pi r^2 h$. From chain rule, we know that

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

where $\frac{dr}{dt}$ and $\frac{dh}{dt}$ are the rate of change of the radius and height of the cone with respect of time, respectively. Then,

$$\frac{\partial V}{\partial r} = \frac{2}{3}\pi r h, \text{ and } \frac{\partial V}{\partial h} = \frac{1}{3}\pi r^2$$

Therefore, for $r = 120, h = 140, \frac{dr}{dt} = 1.8, \frac{dh}{dt} = 2.5$:

$$\begin{aligned}\frac{dV}{dt} &= \pi\left(\frac{10}{18}\right)\left(\frac{2}{3}\right)(140)(100) + \pi\left(\frac{25}{10}\right)\left(\frac{1}{3}\right)(120)^2 \\ &= 40\left(\frac{271400}{9}\right)\pi \\ &\approx 3789458.872\end{aligned}$$

## Problem 15

Show that any function of the form

$$z = f(x + at) + g(x - at)$$

is a solution of the wave equation

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

*Hint: Let $u = x + at$ and $v = x - at$.*

### Answer

Note that $\frac{\partial z}{\partial t} = \frac{\partial f}{\partial t} + \frac{\partial g}{\partial t}$ and $\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}$. From computation, we know that

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{df}{du} \frac{\partial u}{\partial t} \\ &= a \frac{df}{du} \\ &= a f'(u) \end{aligned}$$

$$\begin{aligned} \frac{\partial g}{\partial t} &= \frac{dg}{dv} \frac{\partial v}{\partial t} \\ &= -a \frac{dg}{dv} \\ &= -a g'(v) \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{df}{du} \frac{\partial u}{\partial t} \\ &= \frac{df}{du} \\ &= f'(u) \end{aligned}$$

$$\begin{aligned} \frac{\partial g}{\partial x} &= \frac{dg}{dv} \frac{\partial v}{\partial t} \\ &= \frac{dg}{dv} &= g'(v) \end{aligned}$$

Then $\frac{\partial z}{\partial t} = af'(u) - ag'(v)$, and $\frac{\partial z}{\partial x} = f'(u) + g'(v)$. Now,

$$\begin{aligned} \frac{\partial^2 z}{\partial t^2} &= \frac{\partial}{\partial t} \frac{\partial z}{\partial t} \\ &= a \frac{\partial}{\partial t} (f'(u) - g'(v)) \\ &= a \left( \frac{\partial}{\partial t} f'(u) - \frac{\partial}{\partial t} g'(v) \right) \\ &= a(af''(u) - (-a)g'(v)) \\ &= a^2 (f''(u) + g''(v)) \end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial z}{\partial x} \\
&= \frac{\partial}{\partial x}(f'(u) + g'(v)) \\
&= \left(\frac{\partial}{\partial x}f'(u) + \frac{\partial}{\partial t}g'(v)\right) \\
&= f''(u) + g''(v)
\end{aligned}
$$

Therefore $a^2 \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2}$. ■

Honestly, while I was able to solve this problem with one of the tutors, I'm still a little uncertain of the way I'm manipulating the composition of this functions. I see that we have a succession of chain rules: we simplify the input in the first part in order to find some total derivatives, then we find the derivative of those totals by looking at their partials… It might take a bit…