Section 4.1 (6 Exercises)

In exercises 5-13, determine whether each set equipped with the given operations is a vector space. For those that are not veector spaces, identify the vector space axioms that fail.

Exercise 4.1.5

The set of all pairs of real numbers of the form (x, y) where $x \geq 0$,with the standard operations of $\mathbb{R}^2$.

Answer

No, as it fails Theorem (6). Proof by contradiction:

Let S be the formentioned set, and assume S is a vector space. Let $\boldsymbol{u}$ be a vector in S, such that $u_1 > 0$. Therefore, $\forall k \in \mathbb{R}, k\boldsymbol{v} \in S$. Let $k < 0$. Now $k\boldsymbol{u} = (ku_1, ku_2) \implies ku_1 < 0$, which contradicts Theorem (6). Therefore, S is not a vector space. ■

Exercise 4.1.7

The set of all triples of real numbers with the standard vector addition but with scalar multiplication defined by

$$k(x, y, z) = (k^2x, k^2y, k^2z)$$

Answer

No, as it contradicts (8):

Let S be the formentioned set. Assume that it is a vector space. Therefore, $km(\boldsymbol{u}) = k\boldsymbol{u} + m\boldsymbol{v}$. Therefore:

$$\begin{aligned} km(\boldsymbol{u}) &= k\boldsymbol{u} + m\boldsymbol{u} \\ k(m^2u_1, m^2u_2, m^2u_3) &= (k^2u_1, k^2u_2, k^2u_3) + (m^2u_1, m^2u_2, m^2u_3) \\ (k^2m^2u_1, k^2m^2u_2, k^2m^2u_3) &= ((k^2 + m^2)u_1, (k^2 + m^2)u_2, (k^2 + m^2)u_3) \end{aligned}$$

which is a contradiction, as $\forall k \in \mathbb{R}, \forall m \in \mathbb{R}, k^2m^2 \neq k^2 + m^2$. ■

Exercise 4.1.9

The set of all 2×2 matrices of the form

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

with the standard matrix addition and scalar multiplication.

Answer

Yes, it is a vector space.

Exercise 4.1.11

The set of all pairs of real numbers of the form $(1, x)$ with the operations

$$(1, y) + (1, y') = (1, y + y') \quad \text{and} \quad k(1, y) = (1, ky)$$

Answer

Surprisingly, it is a vector space. A way to understand this problem intuitively thinking that the first component (the 1) has the same additive and multiplicative properties as the conventional 0.

Exercise 4.1.13

Verify Axioms 3, 7, 8 and 9 for the vector space given in Example 4.

Answer

In example for, we were given the set V consisting of all 2×2 matrices with real entries. Now, consider matrices $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ in V, and $k, m \in \mathbb{R}$.

3.
$$A + (B + C) = (A + B) + C$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \left(\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \right) = \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right) + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} + c_{11} & b_{12} + c_{12} \\ b_{21} + c_{21} & b_{22} + c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} + b_{11} + c_{11} & a_{12} + b_{12} + c_{12} \\ a_{21} + b_{21} + c_{21} & a_{22} + b_{22} + c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} + c_{11} & a_{12} + b_{12} + c_{12} \\ a_{21} + b_{21} + c_{21} & a_{22} + b_{22} + c_{22} \end{bmatrix}$$

7.
$$k(A + B) = kA + kB$$

$$k \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix} + \begin{bmatrix} kb_{11} & kb_{12} \\ kb_{21} & kb_{22} \end{bmatrix}$$

$$\begin{bmatrix} k(a_{11} + b_{11}) & k(a_{12} + b_{12}) \\ k(a_{21} + b_{21}) & k(a_{22} + b_{22}) \end{bmatrix} = \begin{bmatrix} ka_{11} + kb_{11} & ka_{12} + kb_{12} \\ ka_{21} + kb_{21} & ka_{22} + kb_{22} \end{bmatrix}$$

8.
$$(k + m)A = kA + mA$$

$$(k + m) \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix} + \begin{bmatrix} ma_{11} & ma_{12} \\ ma_{21} & ma_{22} \end{bmatrix}$$

$$\begin{bmatrix} (k + m)a_{11} & (k + m)a_{12} \\ (k + m)a_{21} & (k + m)a_{22} \end{bmatrix} = \begin{bmatrix} ka_{11} + ma_{11} & ka_{12} + ma_{12} \\ ka_{21} + ma_{21} & ka_{22} + ma_{22} \end{bmatrix}$$

9.
$$(km)A = k(mA)$$

$$(km) \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = k \begin{bmatrix} ma_{11} & ma_{12} \\ ma_{21} & ma_{22} \end{bmatrix}$$

$$\begin{bmatrix} (km)a_{11} & (km)a_{12} \\ (km)a_{21} & (km)a_{22} \end{bmatrix} = \begin{bmatrix} kma_{11} & kma_{12} \\ kma_{21} & kma_{22} \end{bmatrix}$$

Exercise 4.1.15

With the addition and scalar multiplication operations defined in Example 7, show that $V = \mathbb{R}^2$ satisfies Axioms 1-9.

Answer

Axioms 1 through 5 hold, since V holds the same definition for addition operation as $\mathbb{R}^2$, and the set of vectors in V is the set of vectors in $\mathbb{R}^2$.

6.

$$V = \mathbb{R}^2 \wedge \boldsymbol{u} \in \mathbb{R}^2 \iff \boldsymbol{u} \in V$$

7.

$$\begin{aligned} k(\boldsymbol{u} + \boldsymbol{v}) &= k\boldsymbol{u} + k\boldsymbol{v} \\ k((u_1, u_2) + (v_1, v_2)) &= k(u_1, u_2) + k(v_1, v_2) \\ k(u_1 + v_1, u_2 + v_2) &= (ku_1, 0) + (kv_1, 0) \\ (k(u_1 + v_1), 0) &= (ku_1 + kv_1, 0) \end{aligned}$$

8. Let $k + m = a$.

$$\begin{aligned} (k + m)\boldsymbol{u} &= k\boldsymbol{u} + m\boldsymbol{u} \\ a(u_1, u_2) &= k(u_1, u_2) + m(u_1, u_2) \\ (au_1, 0) &= (ku_1, 0) + (mu_1, 0) \\ ((k + m)u_1, 0) &= (ku_1 + mu_1, 0) \end{aligned}$$

9. Let $km = a$.

$$\begin{aligned} (km)\boldsymbol{u} &= k(m\boldsymbol{u}) \\ a(u_1, u_2) &= k(m(u_1, u_2)) \\ (au_1, 0) &= (k(mu_1, 0)) \\ (kmu_1, 0) &= (kmu_1, 0) \end{aligned}$$

Section 4.2 (6 Exercises)

Exercise 4.2.1

Use the *Theorem 4.2.1* to determine which of the followig are subspaces of $\mathbb{R}^3$.

1. All vectors of the form $(a, 0, 0)$.
2. All vectors of the form $(a, 1, 1)$.
3. All vectors of the form (a, b, c), where $b = a + c$.
4. All vectors of the form (a, b, c), where $b = a + c + 1$.
5. All vectors of the form $(a, b, 0)$.

Answer

1. Yes, as $k\boldsymbol{u} = (ku_1, 0, 0)$ and $\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1, 0, 0)$.
2. No, as $k\boldsymbol{u} = (ku_1, k, k)$ and $\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1, 2, 2)$.
3. Yes, as $k\boldsymbol{u} = (ku_1, k(u_1 + u_3), u_3)$ and $\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1, (u_1 + u_3) + (v_1 + v_3), u_3 + v_3)$.
4. No, as $k\boldsymbol{u} = (ku_1, k(u_1 + u_3 + 1), u_3) \neq (ku_1, k(u_1 + u_3), u_3)$.
5. Yes, as $k\boldsymbol{u} = (ku_1, ku_2, 0)$ and $\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1, u_2 + u_2, 0)$.

Exercise 4.2.3

Use the *Theorem 4.2.1* to determine which of the following are subspaces of P_3.

1. All polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$.
2. All polyomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 + a_1 + a_2 + a_3 = 0$.
3. All polynomials of the form $a_0 + a_1x + a_2x^2 + a_3x^3$ in which a_0, a_1, a_2, a_3 are rational numbers.
4. All polynomials of the form $a_0 + a_1x_1$, where a_0 and a_1 are real numbers.

Answer

1. Yes. Consider that any polynomial in P_3 can be expressed as the dot product of $\boldsymbol{a} \cdot \boldsymbol{x} = (a_0, a_1, a_2, a_3) \cdot (1, x, x^2, x^3)$. Then, the sum of any polynomial will always be the the dot product of the sum of their coefficient vectors and the literal vector.
2. Yes.
3. No, as $k \in \mathbb{R} \nRightarrow k \in \mathbb{Q}$.
4. Yes.

Exercise 4.2.7

For which of the following are linear combinations of $\boldsymbol{u} = (0, -2, 2)$ and $\boldsymbol{v} = (1, 3, -1)$?

1. $(2, 2, 2)$
2. $(0, 4, 5)$
3. $(0, 0, 0)$

Answer

1. $2\boldsymbol{u} + 2\boldsymbol{v} = (0, -4, 4) + (2, 6, -2) = (2, 6 - 4, 4 - 2) = (2, 2, 2)$
3. $0\boldsymbol{u} + 0\boldsymbol{v} = (0, 0, 0)$

Exercise 4.2.9

Which of the following are linear combinations of

$$A = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$$

(a) $A = \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$

(b) $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(c) $C = \begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$

Answer

(a) Note that

$$aA + bB + cC = \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$$

$$\begin{bmatrix} 4a+b & -b+2c \\ -2a+2b+c & -2a+3b+4c \end{bmatrix} = \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$$

maps to the following system of equations:

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & -1 & 2 \\ -2 & 2 & 1 \\ -2 & 3 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -1 \\ -8 \end{bmatrix}$$

which is a system with infinite solutions. Therefore, the given matrix is a linear combination for matrices A, B, and C.

(b) The given matrix is the 0 vector of its vector space. Therefore, it's a linear combination of A, B, and C.

(a)

Exercise 4.2.11

In each part, determine whether the vectors span $\mathbb{R}^3$.

(a) $\boldsymbol{v}_1 = (2,2,2), \boldsymbol{v}_2 = (0,0,3), \boldsymbol{v}_3 = (0,1,1)$

(b) $\boldsymbol{v}_1 = (2,-1,3), \boldsymbol{v}_2 = (4,1,2), \boldsymbol{v}_3 = (8,-1,8)$

Answer

(a) Yes.

(b) No. $\boldsymbol{v}_1 = \frac{1}{2}(\boldsymbol{v}_3 - \boldsymbol{v}_1)$. As one of the vectors is a linear combination of the others, and $\mathbb{R}^3$ requires a basis of 3 vectors, this basis does not span $\mathbb{R}^3$.

Exercise 4.2.19

In each part, let $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ be multiplication by A, and let $\boldsymbol{u}_1 = (1,2)$ and $\boldsymbol{u}_2 = (-1,1)$. Determine whether the set $\left\{T_{A(\boldsymbol{u}_1)}, T_{A(\boldsymbol{u}_2)}\right\}$ spans $\mathbb{R}^2$.

(a)
$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$

(b) $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$

Answer

(a) $A\boldsymbol{u}_1 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, $A\boldsymbol{u}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. The set spans $\mathbb{R}^2$. See that $\det(A)\det([\boldsymbol{u}_1 \ \ \boldsymbol{u}_2]) \neq 0$.

(b) $A\boldsymbol{u}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $A\boldsymbol{u}_2 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$. The set does not span $\mathbb{R}^2$. See that $\det(A)\det([\boldsymbol{u}_1 \ \ \boldsymbol{u}_2]) = 0$.

Section 4.3 (8 Exercises)

Exercise 4.3.3

In each part, determine whether the vectors are linearly independent or are linearly dependent in $\mathbb{R}^4$.

(a) $(3, 8, 7, -3), (1, 5, 3, -1), (2, -1, 2, 6), (4, 2, 6, 4)$

(b) $(3, 0, -3, 6), (0, 2, 3, 1), (0, -2, -2, 0), (-2, 1, 2, 1)$

Answer

(a) Linearly dependent.

(b) Linearly independent.

Exercise 4.3.5

In each part, determine whether the matrices are linearly independent or dependent.

(a) $$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Answer

(a) Independent.

(b) Independent.

Exercise 4.3.7

In each part, determine whether the three vectors lie in a plane in $\mathbb{R}^3$.

(a) $\boldsymbol{v}_1 = (2, -2, 0), \boldsymbol{v}_2 = (6, 1, 4), \boldsymbol{v}_3 = (2, 0, -4)$

(b) $\boldsymbol{v}_1 = (-6, 7, 2), \boldsymbol{v}_2 = (3, 2, 4), \boldsymbol{v}_3 = (4, -1, 2)$

Answer

(a) They don't lie in a plane.

(b) They lie in a plane.

Exercise 4.3.13

In each part, let $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ be a multiplication by A, and let $\boldsymbol{u}_1 = (1, 2)$ and $\boldsymbol{u}_2 = (-1, 1)$. Determine whether the set $\left\{T_{A(\boldsymbol{u}_1)}, T_{A(\boldsymbol{u}_2)}\right\}$ is linearly independent in $\mathbb{R}^2$.

(a) $$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$

(b) $$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$$

Answer

(a) $A\boldsymbol{u}_1 = \left[\begin{smallmatrix} -1 \\ 2 \end{smallmatrix}\right], A\boldsymbol{u}_2 = \left[\begin{smallmatrix} -2 \\ 2 \end{smallmatrix}\right]$. The set is linearly independent in $\mathbb{R}^2$.

(b) $A\boldsymbol{u}_1 = \left[\begin{smallmatrix} -1 \\ 2 \end{smallmatrix}\right], A\boldsymbol{u}_2 = \left[\begin{smallmatrix} -2 \\ 4 \end{smallmatrix}\right]$. The set is dependent in $\mathbb{R}^2$.

Exercise 4.3.17

(Calculus required) The functions

$$f_1(x) = x \quad \text{and} \quad f_2(x) = \cos(x)$$

are linearly independent in $F(-\infty, \infty)$ because neither function is a scalar multiple of the other. Confirm the linear independence using the Wronskian.

Answer

$$f_1{}' = 1, f_1{}'' = 0, f_2{}' = -\sin(x), f_2{}'' = -\cos(x)$$

$$\begin{aligned} W(x) &= \begin{vmatrix} x & \cos(x) \\ 1 & -\sin(x) \end{vmatrix} \\ &= -x\sin(x) - \cos(x) \end{aligned}$$

And $\forall x \in \mathbb{R}$ $-(x\sin(x) + \cos(x))$ is not necesarily 0. Therefore, the functions are linearly independent.

Exercise 4.3.19

(Calculus required) Use the Wronskian to show that the following sets of vectors are linearly independent.

(a) $1, x, e^x$
(b) $1, x, x^2$

Answer

(a)

$$\begin{aligned} W(x) &= \begin{vmatrix} 1 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{vmatrix} \\ &= e^x \end{aligned}$$

which is nonzero for $x \in \mathbb{R}$.

(a)

$$\begin{aligned} W(x) &= \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} \\ &= 2 - 2x \end{aligned}$$

which is nonzero for $x \in \mathbb{R}$.

Exercise 4.3.21

(Calculus required) Use the Wronskian to show that the functions $f_1(x) = \sin x$, $f_2(x) = \cos x$, and $f_3(x) = x\cos x$ are linearly independent vectors in $C^\infty(-\infty, \infty)$

Answer

$$W(x) = \begin{vmatrix} \sin x & \cos x & x\cos x \\ \cos x & -\sin x & \cos x - x\sin x \\ -\sin x & -\cos x & -2\sin x - x\cos x \end{vmatrix}$$

$$= \sin(x)\big(2\sin^2 x + 2x\cos(x)\sin(x) - \cos^2 x\big)$$

$$-\cos(x)\big(-\sin(x)\cos(x) - x\cos^2(x) - x\sin^2 x\big)$$

$$+x\cos(x)\big(-\cos^2 x - \sin^2 x\big)$$