

# Homework 5

by Carlos Rubio

February 11th, 2025

## Partial Derivatives

### Problem 1

Determine the signs of the partial derivatives for the function  $f$  whose graph is shown (see worksheet). The point  $(1, 2, f(1, 2))$  is marked.

1.  $f_x(1, 2)$
2.  $f_y(1, 2)$
3.  $f_{xx}(1, 2)$
4.  $f_{yy}(1, 2)$
5.  $f_{xy}(1, 2)$
6.  $f_{yx}(1, 2)$

### Answer

To be honest, I struggled a little bit to answer this questions by inspection because of the perspective issue of projecting 3D into 2D. I also made the mistake of not realizing we wanted the derivative at *the given point*. After checking the solution sheet, I started judging the grids on the surface I was able to start answering.

Also, it helps projecting a slice going in one of the directions into its orthogonal plane (at least, that's what I mentally tried).

1. Positive
  2. Negative
  3. Positive
  4. Negative
  5. Positive
-

## Problem 2

Find the first partial derivative of the following functions:

1.  $f(x, y) = x^2 - 3y^4$
2.  $u(r, \theta) = \sin(r \cos(\theta))$

### Answer

1.

$$\begin{aligned}f_x(x, y) &= \frac{\partial(x^2 - 3y^4)}{\partial x} \\&= \frac{\partial x^2}{\partial x} - 3y^4 \frac{\partial(1)}{\partial x} \\&= 2x\end{aligned}$$

$$\begin{aligned}f_y(x, y) &= \frac{\partial}{\partial y}(x^2 - 3y^4) \\&= x^2 \frac{\partial(1)}{\partial y} - 3 \frac{\partial(y^4)}{\partial y} \\&= -12y^3\end{aligned}$$

---

2.

$$\begin{aligned}u_r(r, \theta) &= \frac{\partial \sin(r \cos \theta)}{\partial r} \\&= \frac{\partial \sin u}{\partial u} \cdot \cos \theta \frac{\partial r}{\partial r} \\&= \cos(r \cos(\theta)) \cdot \cos(\theta)\end{aligned}$$

$$\begin{aligned}u_\theta(r, \theta) &= \frac{\partial \sin(r \cos \theta)}{\partial \theta} \\&= \frac{\partial \sin u}{\partial u} \cdot r \frac{\partial(\cos \theta)}{\partial \theta} \\&= \cos(r \cos \theta) \cdot -r \sin \theta\end{aligned}$$

---

### Problem 3

Use implicit differentiation to find  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  if

$$yz + x \ln y = z^2$$

**Answer**

$$\begin{aligned}\frac{\partial(yz)}{\partial x} + \ln y \frac{\partial(x)}{\partial x} &= \frac{\partial z^2}{\partial x} \\ y \frac{\partial z}{\partial x} + \ln y &= 2z \frac{\partial z}{\partial x} \\ \ln y &= 2z \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial x} \\ &= \frac{\partial z}{\partial x} (2z - y) \\ \frac{\ln y}{2z - y} &= \frac{\partial z}{\partial x}\end{aligned}$$

---

$$\begin{aligned}\frac{\partial(yz)}{\partial y} + \frac{\partial(x \ln y)}{\partial y} &= \frac{\partial z^2}{\partial y} \\ z + y \frac{\partial z}{\partial y} + \frac{x}{y} &= 2z \frac{\partial z}{\partial y} \\ z + \frac{x}{y} &= 2z \frac{\partial z}{\partial y} - y \frac{\partial z}{\partial y} \\ &= (2z - y) \frac{\partial z}{\partial y} \\ \frac{z + \frac{x}{y}}{2z - y} &= \frac{\partial z}{\partial y} \\ \frac{z}{2z - y} + \frac{x}{y(2z - y)} &= \\ \frac{zy + x}{y(2z - y)} &= \frac{\partial z}{\partial y}\end{aligned}$$

---

**Problem 4**

Find all the second partial derivatives of the function

$$w(u, v) = \sqrt{1 + uv^2}$$

**Answer**

$$\begin{aligned}w_u(u, v) &= \frac{\partial \sqrt{1 + uv^2}}{\partial u} \\&= \frac{\partial x^{\frac{1}{2}}}{\partial x} \cdot \frac{\partial (1 + uv^2)}{\partial u} \\&= \frac{1}{2\sqrt{1 + uv^2}} \cdot v^2 = \frac{v^2}{2\sqrt{1 + uv^2}}\end{aligned}$$

---

$$\begin{aligned}w_{uu}(u, v) &= \frac{\partial}{\partial u} \left( \frac{v^2}{2\sqrt{1 + uv^2}} \right) \\&= \frac{v^2}{2} \cdot \frac{\partial}{\partial u} \left( (1 + uv^2)^{-\frac{1}{2}} \right) \\&= -\frac{v^4}{4} \cdot (1 + uv^2)^{-\frac{3}{2}} \\&= -\frac{v^4}{4\sqrt{(1 + uv^2)^3}}\end{aligned}$$

---

$$\begin{aligned}w_{uv}(u, v) &= \frac{\partial}{\partial v} \left( \frac{v^2}{2\sqrt{1 + uv^2}} \right) \\&= \frac{1}{2} \frac{\partial}{\partial v} \left( v^2 (1 + uv^2)^{-\frac{1}{2}} \right) \\&= \frac{1}{2} \left( \frac{\partial v^2}{\partial v} (1 + uv^2)^{-\frac{1}{2}} + v^2 \frac{\partial}{\partial v} (1 + uv^2)^{-\frac{1}{2}} \right) \\&= \frac{1}{2} \left( \frac{2v}{\sqrt{1 + uv^2}} - 2u \frac{v^3}{2(1 + uv^2)^{\frac{3}{2}}} \right) \\&= \frac{1}{2} \left( \frac{2v}{\sqrt{1 + uv^2}} - \frac{2uv^3}{2(1 + uv^2)^{\frac{3}{2}}} \right)\end{aligned}$$

## Tangent Planes and Linear Approximation

### Problem 5

Find an equation of the tangent plane to the surface given by

$$z = \frac{x}{y^2}$$

at the point  $(-4, 2, -1)$ .

### Answer

Let  $(x_0, y_0, z_0)$  be the given point, and let  $T$  be the tangent plane to the function  $z$ . Therefore,  $T = z - z_0 = \frac{\partial z}{\partial x}(x - x_0) + \frac{\partial z}{\partial y}(y - y_0)$ .

$$\begin{aligned}\frac{\partial z}{\partial x} \big|_{-4, 2, -1} &= \frac{\partial}{\partial x} \left( \frac{x}{y^2} \right) \\ &= \frac{\partial x}{\partial x} \left( \frac{1}{y^2} \right) + x \frac{\partial y^{-2}}{\partial x} \\ &= \frac{1}{y^2} - \frac{2x}{y^3} \cdot \frac{\partial y}{\partial x} \\ &= \frac{1}{y^2} \\ &= \frac{1}{4}\end{aligned}$$

---

$$\begin{aligned}\frac{\partial z}{\partial y} \big|_{-4, 2, -1} &= \frac{\partial}{\partial y} \left( \frac{x}{y^2} \right) \\ &= \frac{\partial x}{\partial y} \left( \frac{1}{y^2} \right) + x \frac{\partial y^{-2}}{\partial y} \\ &= \frac{\partial x}{\partial y} \left( \frac{1}{y^2} \right) - \frac{2x}{y^3} \\ &= -\frac{2x}{y^3} \\ &= -\frac{2(-4)}{(2)^3} \\ &= \frac{8}{8} \\ &= 1\end{aligned}$$

---

Therefore:

$$\begin{aligned}z &= \frac{1}{4}(x + 4) + (y - 2) - 1 \\ &= \frac{x}{4} + y - 2\end{aligned}$$

is the equation for the tangent plane at  $(-4, 2, 1)$ .

---

## Problem 6

Verify the linear approximation

$$\frac{y-1}{x+1} \approx x + y - 1$$

at  $(0, 0)$ .

### Answer

Let  $z = \frac{y-1}{x+1}$ , and  $(x_0, y_0) = (0, 0)$ . Then

$$z_0 = z(0, 0) = \frac{0-1}{0+1} = -1$$

We know that the linear approximation of  $z$  at the point  $(x_0, y_0, z_0)$  can be found using the following formula:

$$z - z_0 = z_x(x - x_0) + z_y(y - y_0)$$

From computation, we know that:

$$\begin{aligned} z_x|_{(0,0,-1)} &= \frac{\partial}{\partial x} \left( \frac{y-1}{x+1} \right) \Big|_{(0,0,-1)} \\ &= \frac{\partial}{\partial x} ((y-1)(x+1)^{-1}) \Big|_{(0,0,-1)} \\ &= \frac{\partial(y-1)}{\partial x} \left( \frac{1}{x+1} \right) + \frac{\partial(x+1)^{-1}}{\partial x} (y-1) \Big|_{(0,0,-1)} \\ &= \frac{\partial(x+1)^{-1}}{\partial x} (y-1) \Big|_{(0,0,-1)} \\ &= \frac{1-y}{(x+1)^2} \Big|_{(0,0,-1)} \\ &= 1 \end{aligned}$$

---

$$\begin{aligned} z_y|_{(0,0,-1)} &= \frac{\partial}{\partial y} \left( \frac{y-1}{x+1} \right) \Big|_{(0,0,-1)} \\ &= \frac{\partial}{\partial y} ((y-1)(x+1)^{-1}) \Big|_{(0,0,-1)} \\ &= \frac{\partial(y-1)}{\partial y} \left( \frac{1}{x+1} \right) + \frac{\partial(x+1)^{-1}}{\partial y} (y-1) \Big|_{(0,0,-1)} \\ &= \frac{1}{x+1} \\ &= 1 \end{aligned}$$

---

Therefore:

$$\begin{aligned} z + 1 &= 1x + 1y \\ z &= x + y - 1 \end{aligned}$$

---

**Problem 7**

Given that  $f$  is a differentiable function with  $f(2, 5) = 6$ ,  $f_x(2, 5) = 1$  and  $f_y(2, 5) = -1$ , use linear approximation to estimate  $f(2.2, 4.9)$ .

**Answer**

We can approximate  $f(x, y) \approx (x - 2) + (y + 5) + 6 = x - y + 9$ . If we plug in  $(2.2, 4.9)$ , we get

$$\begin{aligned} f(2.2, 4.9) &= 2.2 - 4.9 + 9 \\ &= 6.3 \end{aligned}$$

---

**Problem 8**

Find the differential of the function  $u = \sqrt{x^2 + 3y^2}$ .

**Answer**

We know that  $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$ ,

$$\begin{aligned} u_x &= \frac{\partial}{\partial x} \sqrt{x^2 + 3y^2} \\ &= \frac{1}{2\sqrt{x^2 + 3y^2}} \frac{\partial}{\partial x} (x^2 + 3y^2) \\ &= \frac{x}{\sqrt{x^2 + 3y^2}} \end{aligned}$$

---

$$\begin{aligned} u_y &= \frac{\partial}{\partial y} \sqrt{x^2 + 3y^2} \\ &= \frac{1}{2\sqrt{x^2 + 3y^2}} \frac{\partial}{\partial y} (x^2 + 3y^2) \\ &= \frac{3y}{\sqrt{x^2 + 3y^2}} \end{aligned}$$

---

Then,

$$\begin{aligned} du &= \frac{x}{\sqrt{x^2 + 3y^2}} dx + \frac{3y}{\sqrt{x^2 + 3y^2}} dy \\ &= \frac{xdx + 3ydy}{\sqrt{x^2 + 3y^2}} \end{aligned}$$

---



**Problem 9**

The length and width of a rectangle are measured as 30 *cm* and 24 *cm*, respectively, with an error in measurement of at most 0.1 *cm* in each. Use differentials to estimate the maximum error in the calculated area of the rectangle.

**Answer**

We know that the area of a rectangle  $A$  is a function of its base  $b$  and height  $h$ :  $A = bh$ . Therefore, its differential  $dA$  is given as

$$dA = \frac{\partial A}{\partial b}db + \frac{\partial A}{\partial h}dh$$

where  $db = dh = 0.1$  *cm*. Then

$$\frac{\partial A}{\partial b} = \frac{\partial}{\partial b}bh = h$$

$$\frac{\partial A}{\partial h} = \frac{\partial}{\partial h}bh = b$$

Therefore,

$$\begin{aligned}dA &= 24(0.1) + 30(0.1) \\ &= 5.4\end{aligned}$$

or our estimate of the maximum error is 5.4 *cm*.

## The Chain Rule

### Problem 10

Use the chain rule to find  $\frac{dz}{dt}$  and  $\frac{dw}{dt}$ .

1.  $z = xy^3 - x^2y$ , where  $x = t^2 + 1$  and  $y = t^2 - 1$ .
2.  $w = \ln \sqrt{x^2 + y^2 + z^2}$ , where  $x = \sin t$ ,  $y = \cos t$ ,  $z = \tan t$ .

### Answer

---

1.

$$\frac{dx}{dt} = \frac{dy}{dt} = 2t$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(xy^3 - x^2y) = y^3 - 2xy$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(xy^3 - x^2y) = 3xy^2 - x^2$$

$$\begin{aligned}\frac{dz}{dt} &= 2t((y^3 - 2xy) + (3xy^2 - x^2)) \\ &= 2t\left[\left((t^2 - 1)^3 - 2(t^2 - 1)(t^2 + 1)\right) + \left(3(t^2 + 1)(t^2 - 1)^2 - (t^2 + 1)^2\right)\right]\end{aligned}$$

---

2.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$$

$$\frac{dx}{dt} = \cos t, \frac{dy}{dt} = -\sin t, \frac{dz}{dt} = \sec^2 t$$

$$\begin{aligned}\frac{\partial w}{\partial x} &= \frac{\partial}{\partial x} \ln \sqrt{x^2 + y^2 + z^2} \\ &= \frac{1}{\sqrt{x^2 + y^2 + z^2}} \frac{\partial}{\partial x} \sqrt{x^2 + y^2 + z^2} \\ &= \frac{1}{\sqrt{x^2 + y^2 + z^2}} \left( \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \right) \left( \frac{\partial}{\partial x} (x^2 + y^2 + z^2) \right) \\ &= \left( \frac{1}{2(x^2 + y^2 + z^2)} \right) (2x) \\ &= \frac{x}{(x^2 + y^2 + z^2)} \\ &= \frac{\sin t}{\sin^2 t + \cos^2 t + \tan^2 t} \\ &= \frac{\sin t}{1 + \tan^2 t}\end{aligned}$$

$$\begin{aligned}
\frac{\partial w}{\partial y} &= \frac{\partial}{\partial y} \ln \sqrt{x^2 + y^2 + z^2} \\
&= \frac{1}{\sqrt{x^2 + y^2 + z^2}} \left( \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \right) \left( \frac{\partial}{\partial y} (x^2 + y^2 + z^2) \right) \\
&= \left( \frac{1}{2(x^2 + y^2 + z^2)} \right) (2y) \\
&= \frac{y}{(x^2 y^3 + z^2)} \\
&= \frac{\cos t}{\sin^2 t + \cos^2 t + \tan^2 t} \\
&= \frac{\cos t}{1 + \tan^2 t} \\
\frac{\partial w}{\partial z} &= \frac{\partial}{\partial z} \ln \sqrt{x^2 + y^2 + z^2} \\
&= \frac{1}{\sqrt{x^2 + y^2 + z^2}} \left( \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \right) \left( \frac{\partial}{\partial z} (x^2 + y^2 + z^2) \right) \\
&= \left( \frac{1}{2(x^2 + y^2 + z^2)} \right) (2z) \\
&= \frac{z}{(x^2 y^3 + z^2)} \\
&= \frac{\tan t}{\sin^2 t + \cos^2 t + \tan^2 t} \\
&= \frac{\tan t}{1 + \tan^2 t}
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\sin t}{1 + \tan^2 t} (\cos t) + \frac{\cos t}{1 + \tan^2 t} (-\sin t) + \frac{\tan t}{1 + \tan^2 t} (\sec^2 t) \\
&= \frac{\sin t \cos t}{1 + \tan^2 t} - \frac{\sin t \cos t}{1 + \tan^2 t} + \frac{\tan t}{\cos^2 t (1 + \tan^2 t)} \\
&= \frac{\tan t}{\cos^2 t + \sin^2 t} \\
&= \tan t
\end{aligned}$$


---

**Problem 11**

Use the chain rule to find  $\frac{\partial z}{\partial s}$  and  $\frac{\partial z}{\partial t}$  if

$$z = \sqrt{x}e^{xy}, \text{ where } x = 1 + st \text{ and } y = s^2 - t^2$$

**Answer**

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

$$\frac{\partial x}{\partial t} = s, \frac{\partial y}{\partial t} = -2t$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} \sqrt{x}e^{xy} = \frac{e^{xy}}{2\sqrt{x}} + \sqrt{xy}e^{xy} = e^{xy} \left( \frac{1}{2\sqrt{x}} + \sqrt{xy} \right)$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} \sqrt{x}e^{xy} = \sqrt{x^3}e^{xy}$$

$$\begin{aligned} \frac{\partial z}{\partial t} &= s \left( \frac{e^{xy}}{2\sqrt{x}} + \sqrt{xy}e^{xy} \right) - 2t \left( \sqrt{x^3}e^{xy} \right) \\ &= e^{\sqrt{1-st}(s^2+t^2)} \left( s \left( \frac{1}{2\sqrt{1-st}} + \sqrt{1-st}(s^2+t^2) \right) - 2t\sqrt{(1-st)^3} \right) \end{aligned}$$


---

Note that  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  have been computed.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

$$\frac{\partial x}{\partial s} = t, \frac{\partial y}{\partial s} = 2s$$

$$\begin{aligned} \frac{\partial z}{\partial s} &= t \left( \frac{e^{xy}}{2\sqrt{x}} + \sqrt{xy}e^{xy} \right) + 2s \left( \sqrt{x^3}e^{xy} \right) \\ &= e^{\sqrt{1-st}(s^2+t^2)} \left( t \left( \frac{1}{2\sqrt{1-st}} + \sqrt{1-st}(s^2+t^2) \right) + 2s\sqrt{(1-st)^3} \right) \end{aligned}$$


---

**Problem 12**

Use the chain rule to find  $\frac{\partial w}{\partial r}$  and  $\frac{\partial w}{\partial \theta}$  if

$$w = xy + yz + zx \text{ where } x = r \cos \theta, y = r \sin \theta \text{ and } z = r\theta$$

when  $r = 2$  and  $\theta = \frac{\pi}{2}$ .

**Answer**

We know that for  $(r, \theta) = (2, \frac{\pi}{2})$ ,  $x = 0$ ,  $y = 2$ ,  $z = \pi$ . Also:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r}$$

$$\frac{\partial x}{\partial r} = \cos \theta, \frac{\partial y}{\partial r} = \sin \theta, \frac{\partial z}{\partial r} = \theta$$

$$\begin{aligned} \frac{\partial w}{\partial x} &= \frac{\partial xy}{\partial x} + \frac{\partial yz}{\partial x} + \frac{\partial xz}{\partial x} \\ &= y + z \end{aligned}$$

$$\begin{aligned} \frac{\partial w}{\partial y} &= \frac{\partial xy}{\partial y} + \frac{\partial yz}{\partial y} + \frac{\partial xz}{\partial y} \\ &= x + z \end{aligned}$$

$$\begin{aligned} \frac{\partial w}{\partial z} &= \frac{\partial xy}{\partial z} + \frac{\partial yz}{\partial z} + \frac{\partial xz}{\partial z} \\ &= x + y \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial w}{\partial r} \Big|_{(2, \frac{\pi}{2})} &= (2 + \pi)(0) + (\pi)(1) + (\pi)\left(\frac{\pi}{2}\right) \\ &= 2\pi \end{aligned}$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial \theta}$$

$$\frac{\partial x}{\partial \theta} = -\sin \theta, \frac{\partial y}{\partial \theta} = \cos \theta, \frac{\partial z}{\partial \theta} = r$$

$$\begin{aligned} \frac{\partial w}{\partial \theta} \Big|_{(2, \frac{\pi}{2})} &= (2 + \pi)(-2) + (\pi + 0)(0) + (2)(0 + 2) \\ &= -4 - 2\pi + 4 \\ &= -2\pi \end{aligned}$$

**Problem 13**

Use the equations

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

to compute  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$ , where

$$yz + x \ln y = z^2$$

### Answer

Let the given equation be  $F(x, y, z)$ . Then

$$\frac{\partial F}{\partial x} = \ln y$$

$$\frac{\partial F}{\partial y} = z + \frac{x}{y}$$

$$\frac{\partial F}{\partial z} = y - 2z$$

Therefore,

$$\frac{\partial z}{\partial x} = -\frac{\ln y}{y - 2z}$$

$$\frac{\partial z}{\partial y} = -\frac{z + \frac{x}{y}}{y - 2z} = -\frac{zy + x}{y(y - 2z)}$$

### Problem 14

The radius of a right circular cone is increasing at a rate of 1.8 *in/s* while its height is decreasing at a rate of 2.5 *in/s*. At what rate is the volume of the cone changing when the radius is 120 *in* and the height is 140 *in*?

### Answer

Let the volume of the cone be a function  $V(r, h) = \frac{1}{3}\pi r^2 h$ . From chain rule, we know that

$$\frac{dV}{dt} = \frac{\partial V}{\partial r} \frac{dr}{dt} + \frac{\partial V}{\partial h} \frac{dh}{dt}$$

where  $\frac{dr}{dt}$  and  $\frac{dh}{dt}$  are the rate of change of the radius and height of the cone with respect of time, respectively. Then,

$$\frac{\partial V}{\partial r} = \frac{2}{3}\pi r h, \text{ and } \frac{\partial V}{\partial h} = \frac{1}{3}\pi r^2$$

Therefore, for  $r = 120$ ,  $h = 140$ ,  $\frac{dr}{dt} = 1.8$ ,  $\frac{dh}{dt} = 2.5$ :

$$\begin{aligned} \frac{dV}{dt} &= \pi \left( \frac{10}{18} \right) \left( \frac{2}{3} \right) (140)(100) + \pi \left( \frac{25}{10} \right) \left( \frac{1}{3} \right) (120)^2 \\ &= 40 \left( \frac{271400}{9} \right) \pi \\ &\approx 3789458.872 \end{aligned}$$

### Problem 15

Show that any function of the form

$$z = f(x + at) + g(x - at)$$

is a solution of the wave equation

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

*Hint: Let  $u = x + at$  and  $v = x - at$ .*

### Answer

Note that  $\frac{\partial z}{\partial t} = \frac{\partial f}{\partial t} + \frac{\partial g}{\partial t}$  and  $\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}$ . From computation, we know that

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{df}{du} \frac{\partial u}{\partial t} \\ &= a \frac{df}{du} \\ &= a f'(u)\end{aligned}$$

$$\begin{aligned}\frac{\partial g}{\partial t} &= \frac{dg}{dv} \frac{\partial v}{\partial t} \\ &= -a \frac{dg}{dv} \\ &= -a g'(v)\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{df}{du} \frac{\partial u}{\partial x} \\ &= \frac{df}{du} \\ &= f'(u)\end{aligned}$$

$$\begin{aligned}\frac{\partial g}{\partial x} &= \frac{dg}{dv} \frac{\partial v}{\partial x} \\ &= \frac{dg}{dv} = g'(v)\end{aligned}$$

Then  $\frac{\partial z}{\partial t} = a f'(u) - a g'(v)$ , and  $\frac{\partial z}{\partial x} = f'(u) + g'(v)$ . Now,

$$\begin{aligned}\frac{\partial^2 z}{\partial t^2} &= \frac{\partial}{\partial t} \frac{\partial z}{\partial t} \\ &= a \frac{\partial}{\partial t} (f'(u) - g'(v)) \\ &= a \left( \frac{\partial}{\partial t} f'(u) - \frac{\partial}{\partial t} g'(v) \right) \\ &= a (a f''(u) - (-a) g'(v)) \\ &= a^2 (f''(u) + g''(v))\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x} \frac{\partial z}{\partial x} \\
&= \frac{\partial}{\partial x} (f'(u) + g'(v)) \\
&= \left( \frac{\partial}{\partial x} f'(u) + \frac{\partial}{\partial t} g'(v) \right) \\
&= f''(u) + g''(v)
\end{aligned}$$

Therefore  $a^2 \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2}$ . ■

Honestly, while I was able to solve this problem with one of the tutors, I'm still a little uncertain of the way I'm manipulating the composition of these functions. I see that we have a succession of chain rules: we simplify the input in the first part in order to find some total derivatives, then we find the derivative of those totals by looking at their partials... It might take a bit...