Homework 4

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Arc Length and Curvature

Problem 1

Find the length of the curve $\boldsymbol{r}(t) = \langle 2t^{\frac{2}{3}}, \cos(2t), \sin(2t) \rangle$, $0 \leq t \leq 1$.

Answer

Derivative of $x = 2t^{\frac{2}{3}}$.

$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dt}\left(2t^{\frac{2}{3}}\right) \\ &= 2\frac{d}{dt}\left(t^{\frac{2}{3}}\right) \\ &= \frac{4}{3}t^{-\frac{1}{3}} \end{aligned}$$

Derivative of $y = \cos(2t)$.

$$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt}(\cos(2t)) \\ &= -2\sin(2t) \end{aligned}$$

Derivative of $z = \sin(2t)$.

$$\begin{aligned} \frac{dz}{dt} &= \frac{d}{dt}(\sin(2t)) \\ &= 2\cos(2t) \end{aligned}$$

$$\begin{aligned} |\boldsymbol{r}'(t)|\ dt &= \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \\ &= \left(\frac{4}{3}t^{-\frac{1}{3}}\right)^2 + (-2\sin(2t))^2 + (2\cos(2t))^2 \\ &= \frac{16}{9}t^{-\frac{2}{3}} + 4\sin^2(2t) + 4\cos^2(2t) \\ &= \frac{16}{9}t^{-\frac{2}{3}} + 4 \\ &= \frac{16}{9t^{\frac{2}{3}}} + 4 \\ &= \frac{16}{9t^{\frac{2}{3}}} + \frac{36t^{\frac{2}{3}}}{9t^{\frac{2}{3}}} \\ &= \frac{4}{9t^{\frac{2}{3}}}\left(4 + 9t^{\frac{2}{3}}\right) \end{aligned}$$

Let $u = 4 + 9t^{\frac{2}{3}} \implies du = 9\left(\frac{2}{3t^{\frac{1}{3}}}\right)dt \implies \frac{1}{9}du = \left(\frac{2}{3t^{\frac{1}{3}}}\right)dt$. Also, let L be the length of $\boldsymbol{r}(t)$. In other words, its arclength.

$$\begin{aligned}
L &= \int_0^1 |\boldsymbol{r}'(t)| \; dt \\
&= \int_0^1 \frac{2}{3t^{\frac{1}{3}}}\sqrt{4 + t^{\frac{2}{3}}} dt \\
&= \frac{1}{9}\int_4^{13} \sqrt{u} du \\
&= \frac{1}{9}\int_4^{13} u^{\frac{1}{2}} du \\
&= \frac{2}{27}u^{\frac{3}{2}} \;|_4^{13} \\
&= \frac{2}{27}\left(13^{\frac{3}{2}} - 4^{\frac{3}{2}}\right) \;|_4^{13} \\
&= \frac{2}{27}\left(13^{\frac{3}{2}} - 8\right)
\end{aligned}$$

Problem 2

Reparametrize the curve

$$\boldsymbol{r}(t) = e^t\boldsymbol{i} + e^t\sin(t)\boldsymbol{j} + e^t\cos(t)\boldsymbol{k}$$

with respect to arc length measured from the point $(1, 0, 1)$ in the direction of increasing t.

Answer

$$\begin{aligned}
L &= \int_0^t \sqrt{\left(\frac{d}{du}(e^u)\right)^2 + \left(\frac{d}{du}(e^u\sin u)\right)^2 + \left(\frac{d}{du}(e^u\cos u)\right)^2}\,du \\
&= \int_0^t \sqrt{(e^u)^2 + (e^u\sin u + e^u\cos u)^2 + (e^u\cos u - e^u\sin u)^2}\,du \\
&= \int_0^t \sqrt{e^{2u} + e^{2u}(\sin u + \cos u)^2 + e^{2u}(\cos u - \sin u)^2}\,du \\
&= \int_0^t \sqrt{e^{2u}(1 + (\sin u + \cos u)^2 + (\cos u - \sin u)^2)}\,du \\
&= \int_0^t \sqrt{e^{2u}(1 + (\sin^2 u + 2\sin u\cos u + \cos^2 u) + (\cos^2 u - 2\cos u\sin u + \sin^2 u))}\,du \\
&= \int_0^t \sqrt{e^{2u}(1 + (1 + 2\sin u\cos u) + (1 - 2\cos u\sin u))}\,du \\
&= \int_0^t \sqrt{e^{2u}(1+1+1)}\,du \\
&= \sqrt{3}\int_0^t e^u\,du \\
&= \sqrt{3}(e^t - 1) \\
&=
\end{aligned}$$

Therefore,

$$\begin{aligned}
L &= \sqrt{3}(e^t - 1) \\
\frac{L}{\sqrt{3}} &= e^t - 1 \\
\frac{L}{\sqrt{3}} + 1 &= e^t \\
\ln\left(\frac{L}{\sqrt{3}} + 1\right) &= t
\end{aligned}$$

Therefore, we get this ugly reparametrization:

$$\boldsymbol{r}(t) = \left(\frac{L}{\sqrt{3}} + 1\right)\boldsymbol{i} + \left(\frac{L}{\sqrt{3}} + 1\right)\sin\left(\ln\left(\frac{L}{\sqrt{3}} + 1\right)\right)\boldsymbol{j} + \left(\frac{L}{\sqrt{3}} + 1\right)\cos\left(\ln\left(\frac{L}{\sqrt{3}} + 1\right)\right)\boldsymbol{k}$$

Problem 3

Consider the curve given by $\boldsymbol{r}(t) = \langle \sin^3 t, \cos^3 t, \sin^2 t \rangle$, $0 \leq t \leq \frac{\pi}{2}$.

1. Find the unit tangent vector. *Note: This question was asked on the previous homework as well.*
2. Find the unit normal vector.
3. Find the unit binormal vector.
4. Find the curvature.

Answer

The derivative of $\boldsymbol{r}'(t)$:

$$\begin{aligned}\boldsymbol{r}'(t) &= \langle \frac{d}{dt}\sin^3(t), \frac{d}{dt}\cos^3(t), \frac{d}{dt}\sin^2(t) \rangle \\ &= \langle 3\sin^2(t)\cos(t), -3\cos^2(t)\sin(t), 2\sin(t)\cos(t) \rangle\end{aligned}$$

The norm of $|\boldsymbol{r}'(t)|$:

$$\begin{aligned}|\boldsymbol{r}'(t)| &= \sqrt{\left(3\sin^2(t)\cos(t)\right)^2 + \left(3\cos^2(t)\sin(t)\right)^2 + (2\sin(t)\cos(t))^2} \\ &= \sqrt{9\sin^4(t)\cos^2(t) + 9\cos^4(t)\sin^2(t) + 4\sin^2(t)\cos^2(t)} \\ &= \sqrt{\sin^2(t)\cos^2(t)(9\sin^2(t) + 9\cos^2(t) + 4)} \\ &= \sqrt{\sin^2(t)\cos^2(t)(9+4)} \\ &= \sqrt{13}\sin(t)\cos(t)\end{aligned}$$

a) The unit tangent vector $\boldsymbol{T}(t)$:

$$\begin{aligned}T(t) &= \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|} \\ &= \frac{1}{\sqrt{13}\sin(t)\cos(t)}\langle 3\sin^2(t)\cos(t), -3\cos^2(t)\sin(t), 2\sin(t)\cos(t) \rangle \\ &= \langle \frac{3}{\sqrt{13}}\sin(t), -\frac{3}{\sqrt{13}}\cos(t), \frac{2}{\sqrt{13}} \rangle\end{aligned}$$

The derivative of $\boldsymbol{T}'(t)$:

$$\begin{aligned}T'(t) &= \langle \frac{3}{\sqrt{13}}\frac{d}{dt}\sin(t), -\frac{3}{\sqrt{13}}\frac{d}{dt}\cos(t), \frac{d}{dt}\frac{2}{\sqrt{13}} \rangle \\ &= \langle \frac{3}{\sqrt{13}}\cos(t), \frac{3}{\sqrt{13}}\sin(t), 0 \rangle\end{aligned}$$

The norm of $|\boldsymbol{T}'(t)|$:

$$\begin{aligned}
|T'(t)| &= \sqrt{\left(\frac{3}{\sqrt{13}}\cos(t)\right)^2 + \left(\frac{3}{\sqrt{13}}\sin(t)\right)^2 + 0} \\
&= \sqrt{\frac{9}{13}\cos^2(t) + \frac{9}{13}\sin^2(t)} \\
&= \sqrt{\frac{9}{13}} \\
&= \frac{3}{\sqrt{13}}
\end{aligned}$$

b) The unit normal vector $\boldsymbol{N}(t)$.

$$\begin{aligned}
N(t) &= \frac{\boldsymbol{T}'(t)}{|\boldsymbol{T}'(t)|} \\
&= \frac{\sqrt{13}}{3}\langle\frac{3}{\sqrt{13}}\cos(t), \frac{3}{\sqrt{13}}\sin(t), 0\rangle \\
&= \langle\cos(t), \sin(t), 0\rangle
\end{aligned}$$

c) The binormal vector $\boldsymbol{B}(t)$.

$$\begin{aligned}
\boldsymbol{B}(t) &= \boldsymbol{T}(t) \times \boldsymbol{N}(t) \\
&= \langle\frac{3}{\sqrt{13}}\sin(t), -\frac{3}{\sqrt{13}}\cos(t), \frac{2}{\sqrt{13}}\rangle \times \langle\cos(t), \sin(t), 0\rangle \\
&= \langle-\frac{2}{\sqrt{13}}\sin(t), \frac{2}{\sqrt{13}}\cos(t), \frac{3}{\sqrt{13}}(\sin^2 t + \cos^2 t)\rangle \\
&= \langle-\frac{2}{\sqrt{13}}\sin(t), \frac{2}{\sqrt{13}}\cos(t), \frac{3}{\sqrt{13}}\rangle
\end{aligned}$$

d) The curvature κ:

$$\begin{aligned}
\kappa &= \frac{|\boldsymbol{T}'(t)|}{|\boldsymbol{r}'(t)|} \\
&= \left(\frac{3}{\sqrt{13}}\right)\left(\frac{1}{\sqrt{13}\sin(t)\cos(t)}\right) \\
&= \frac{3}{13\sin(t)\cos(t)} \\
&= \frac{6}{13\sin(2t)} \\
&= \frac{6}{13}\csc(2t)
\end{aligned}$$

Motion in Space: Velocity and Acceleration

Problem 4

Find the velocity, speed and acceleration of a particle moving with position function:

$$\boldsymbol{r}(t) = (2t^2 - 3)\boldsymbol{i} + 2t\boldsymbol{j}$$

Sketch the path the particle on the axes below and draw the position, velocity and acceleration vectors for $t = 1$.

Answer

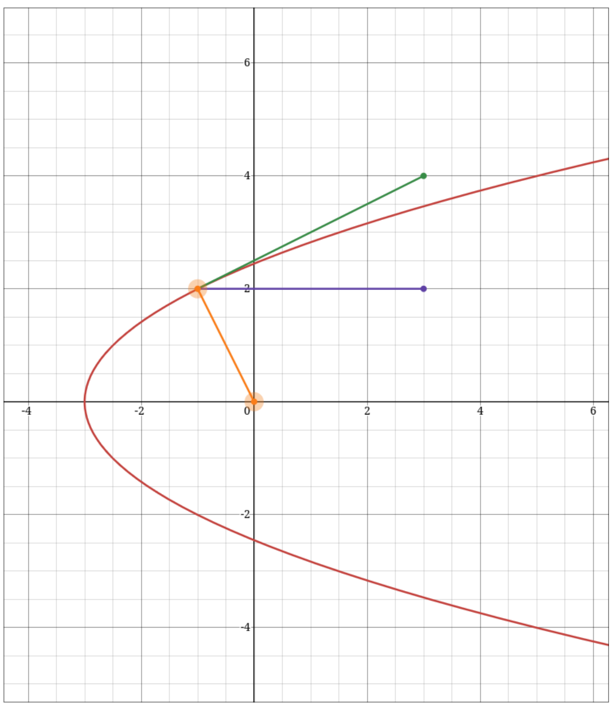
$$\begin{aligned}\boldsymbol{v}(t) &= \boldsymbol{r}'(t) \\ &= 4t\boldsymbol{i} + 2\boldsymbol{j}\end{aligned}$$

$$\begin{aligned}|\boldsymbol{v}(t)| &= \sqrt{(4t)^2 + 2^2} \\ &= \sqrt{16t^2 + 4} \\ &= \sqrt{4(4t^2 + 1)} \\ &= 2\sqrt{4t^2 + 1}\end{aligned}$$

$$\begin{aligned}\boldsymbol{a}(t) &= \boldsymbol{v}'(t) \\ &= 4\boldsymbol{i}\end{aligned}$$

At $t = 1$, this functions have values $\boldsymbol{r}(t) = -\boldsymbol{i} + 2\boldsymbol{j}$, $\boldsymbol{v}(t) = 4\boldsymbol{i} + 2\boldsymbol{j}$, $|\boldsymbol{v}(t)| = 2\sqrt{5}$ and $\boldsymbol{a}(t) = 4\boldsymbol{i}$.

In the figure, the orange segment represents the position vector, the green segment represents the velocity, and the purple represents the acceleration at $t = 1$. The red line is $\boldsymbol{r}(t)$.



Problem 5

Find the tangential and normal components of the acceleration vector of the curve

$$\boldsymbol{r}(t) = t\boldsymbol{i} + 2e^t\boldsymbol{j} + e^{2t}\boldsymbol{k}$$

Answer

Computing the second derivative of $\boldsymbol{r}(t)$:

$$\begin{aligned}\boldsymbol{v}(t) &= \boldsymbol{i} + 2e^t\boldsymbol{j} + 2e^{2t}\boldsymbol{k}\\ \boldsymbol{a}(t) &= 2e^t\boldsymbol{j} + 4e^{2t}\boldsymbol{k}\end{aligned}$$

The speed $|\boldsymbol{v}(t)|$:

$$\begin{aligned}|\boldsymbol{v}(t)| &= \sqrt{1^2 + (2e^t)^2 + (2e^{2t})^2}\\ &= \sqrt{1^2 + 2(1)(2e^{2t}) + (2e^{2t})^2}\\ &= \sqrt{(1 + 2e^{2t})^2}\\ &= 1 + 2e^{2t}\end{aligned}$$

The cross product of $\boldsymbol{v}$ and $\boldsymbol{a}$:

$$\begin{aligned}\boldsymbol{v} \times \boldsymbol{a} &= ((2e^t)(4e^{2t}) - (2e^t)(2e^{2t}))\boldsymbol{i} - (4e^{2t})\boldsymbol{j} + (2e^t)\boldsymbol{k}\\ &= ((2e^t)(4e^{2t} - 2e^{2t}))\boldsymbol{i} - (4e^{2t})\boldsymbol{j} + (2e^t)\boldsymbol{k}\\ &= (2e^t)(2e^{2t})\boldsymbol{i} - 4e^{2t}\boldsymbol{j} + 2e^t\boldsymbol{k}\\ &= 4e^{3t}\boldsymbol{i} - 4e^{2t}\boldsymbol{j} + 2e^t\boldsymbol{k}\end{aligned}$$

The tangential component:

$$\begin{aligned}a_{\boldsymbol{T}} &= \frac{\boldsymbol{v} \cdot \boldsymbol{a}}{|\boldsymbol{v}|}\\ &= \frac{(1)(0) + (2e^t)(2e^t) + (2e^{2t})(4e^{2t})}{1 + 2e^{2t}}\\ &= \frac{4e^{2t} + 8e^{4t}}{1 + 2e^{2t}}\\ &= (4e^{2t})\frac{1 + 2e^{2t}}{1 + 2e^{2t}}\\ &= 4e^{2t}\end{aligned}$$

The normal component:

$$\begin{aligned}
a_{\boldsymbol{N}} &= \frac{|\boldsymbol{v}(t) \times \boldsymbol{a}(t)|}{|\boldsymbol{v}(t)|} \\
&= \frac{\sqrt{\left((4e^{3t})^2 + (4e^{2t})^2 + (2e^t)^2\right)}}{1 + 2e^{2t}} \\
&= \frac{\sqrt{16e^{6t} + 16e^{4t} + 4e^{2t}}}{\sqrt{1 + 4e^{2t} + 4e^{4t}}} \\
&= \frac{\sqrt{(4e^{2t})(4e^{4t} + 4e^{2t} + 1)}}{\sqrt{1 + 4e^{2t} + 4e^{4t}}} \\
&= 2e^t \frac{\sqrt{(4e^{4t} + 4e^{2t} + 1)}}{\sqrt{1 + 4e^{2t} + 4e^{4t}}} \\
&= 2e^t
\end{aligned}$$

Therefore, the acceleration $\boldsymbol{a}(t)$ in terms of its components:

$$\boldsymbol{a}(t) = 4e^{2t}\boldsymbol{T} + 2e^t\boldsymbol{N}$$