Homework 2

1. (a) Show $(1+x)^n = 1 + nx + o(x)$ as $x \to 0$

taylor expansion of $(1+x)^n$

$f(x) = (1+x)^n$, $a=0$

$f(0) = 1$

$f'(x) = n(1+x)^{n-1}$

$f'(0) = n$

$f''(x) = n(n-1)(1+x)^{n-2}$

$f''(0) = n(n-1)$

$f'''(x) = n(n-1)(n-2)(1+x)^{n-3}$

$f'''(0) = n(n-1)(n-2)$

$$f(x) = 1 + nx + \overbrace{\frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots}^{= o(x)?}$$

$$\lim_{x \to 0} \left| \frac{\frac{1}{2}n(n-1)x^2 + \frac{1}{6}n(n-1)(n-2)x^3 + \ldots}{x} \right|$$

$$= \lim_{x \to 0} \left| \frac{n(n-1)x}{2} + \frac{n(n-1)(n-2)x^2}{6} + \ldots \right|$$

$= 0$, since exponent of x is larger in num.

thus, $f(x) = (1+x)^n = 1 + nx + o(x)$

(b) Show $x \sin\sqrt{x} = O(x^{3/2})$ as $x \to 0$

$$\lim_{x \to 0} \left| \frac{x \sin\sqrt{x}}{x^{3/2}} \right| = \lim_{x \to 0} \left| \frac{\sin\sqrt{x}}{\sqrt{x}} \right|$$

$$= \lim_{x \to 0} \left| \frac{\frac{1}{2}\cos\sqrt{x} \cdot x^{-1/2}}{\frac{1}{2} x^{-1/2}} \right| \leftarrow \text{l'hopitals}$$

$$= \lim_{x \to 0} |\cos\sqrt{x}|$$

$$= 1 > 0$$

thus, $x\sin\sqrt{x} = O(x^{3/2})$

(c) show $e^{-t} = O\left(\frac{1}{t^2}\right)$ as $t \to \infty$

$$\lim_{t \to \infty} \left| \frac{e^{-t}}{1/t^2} \right| = \lim_{t \to \infty} \left| \frac{t^2}{e^t} \right|$$

$$= \lim_{t \to \infty} \left| \frac{2t}{e^t} \right| \leftarrow \text{l'hopitals}$$

$$= \lim_{t \to \infty} \left| \frac{2}{e^t} \right| \leftarrow \text{l'hopitals}$$

$$= 0$$

thus, $e^{-t} = O\left(\frac{1}{t^2}\right)$

(d) show $\int_0^\varepsilon e^{-x^2} dx = O(\varepsilon)$ as $\varepsilon \to 0$

$$\lim_{\varepsilon \to 0} \left| \frac{\int_0^\varepsilon e^{-x^2} dx}{\varepsilon} \right| = \lim_{\varepsilon \to 0} |e^{-\varepsilon^2}| \leftarrow \text{l'hopitals}$$

$$= \lim \left| \frac{1}{e^{\varepsilon^2}} \right|$$

$$= 1 > 0$$

thus, $\int_0^\varepsilon e^{-x^2}dx = O(\varepsilon)$

2. $Ax = b$

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1+10^{-10} & 1-10^{-10} \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1-10^{10} & 10^{10} \\ 1+10^{10} & -10^{10} \end{bmatrix}, \quad \text{perturbation in } b = \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \end{bmatrix}$$

(a) find the exact formula for change in the soln

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Delta x = A^{-1}\Delta b = \begin{bmatrix} 1-10^{10} & 10^{10} \\ 1+10^{10} & -10^{10} \end{bmatrix}\begin{bmatrix} 1+\Delta b_1 \\ 1+\Delta b_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\cancel{10^{10}} + \Delta b_1(1-10^{10}) + \cancel{10^{10}} + \Delta b_2 10^{10} \\ 1+\cancel{10^{10}} + \Delta b_1(1+10^{10}) - \cancel{10^{10}} - \Delta b_2 10^{10} \end{bmatrix}$$

$$= \begin{bmatrix} 1+\Delta b_1(1-10^{10}) + \Delta b_2 10^{10} \\ 1+\Delta b_1(1+10^{10}) - \Delta b_2 10^{10} \end{bmatrix}$$

$$|\Delta x - x| = \left| \begin{bmatrix} 1+\Delta b_1(1-10^{10}) + \Delta b_2 10^{10} \\ 1+\Delta b_1(1+10^{10}) - \Delta b_2 10^{10} \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} \Delta b_1(1-10^{10}) + \Delta b_2 10^{10} \\ \Delta b_1(1+10^{10}) - \Delta b_2 10^{10} \end{bmatrix} \right|$$

$$= \begin{bmatrix} \Delta b_1(1-10^{10}) + \Delta b_2 10^{10} \\ \Delta b_1(1+10^{10}) - \Delta b_2 10^{10} \end{bmatrix}$$

(b) whats the condition # of A?

$$\kappa(A) = \|A^{-1}\|_2 \|A\|_2$$
$$= 2 \times 10^{10} \cdot 1$$
$$= 2 \times 10^{10}$$

[code in repo]

(c) let Δb_1 and Δb_2 be of magnitude 10^{-5}

$$\text{relative error} = |\Delta x - x|$$
$$= \begin{bmatrix} 10^{-5}(1-10^{10}) + 10^{-5}10^{10} \\ 10^{-5}(1+10^{10}) - 10^{-5}10^{10} \end{bmatrix}$$
$$= \begin{bmatrix} 10^{-5} - \cancel{10^{5}} + \cancel{10^{5}} \\ 10^{-5} + \cancel{10^{5}} - \cancel{10^{5}} \end{bmatrix}$$
$$= \begin{bmatrix} 10^{-5} \\ 10^{-5} \end{bmatrix}$$

relative error = perturbation

same value of perturbation is more realistic

3. $f(x) = e^x - 1$

(a) what is $\kappa(f(x))$?

$$\kappa = \frac{|f'(c)||x|}{|f(x)|}, \quad c \in (x, \Delta x)$$
$$= \frac{|e^c||x|}{|e^x - 1|}$$
$$= \frac{e^c|x|}{|e^x - 1|}$$

(b) consider computing f(x) using this algo:

1. y= math.e^x
2. return y-1

this algorithm is not stable because it includes subtraction of values that could be close to equal if $x \sim 0$

(c) let $x = 9.999999995000000 \times 10^{-10}$, thus $f(x) = 10^{-9}$

algorith returns $1.000000082740371 \times 10^{-9}$
returns 8 correct digits
this is expected because the algorithm is unstable

(d) find a polynomial approx. for f(x)

taylor series

$f(x) = e^x - 1, \quad a = 0$

$f(0) = 0$

$f'(x) = e^x = f''(x) = f'''(x) = \ldots$

$f'(0) = 1$

$f(x) = x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

(e) verify (d) [code in repo]