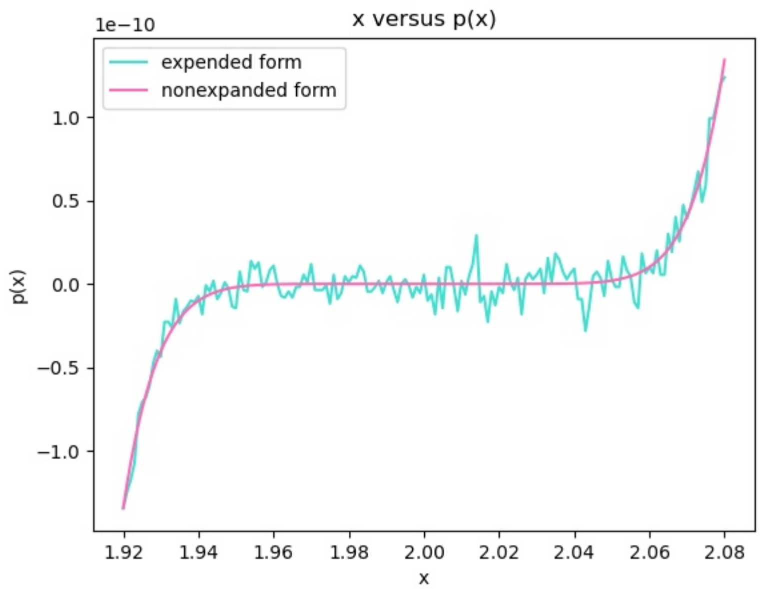
Homework #1

1. $p(x) = (x-2)^9$

(i) plot $p(x)$ for $x = 1.920, 1.921, \ldots, 2.080$ evaluating p via coefficients

(ii) plot $p(x)$ via $p(x) = (x-2)^9$

```python
import matplotlib.pyplot as plt
import numpy as np
import numpy.linalg as la
import math

def driver():
        x = np.arange(1.920, 2.080, 0.001)

        p1 = lambda x: x**9 - 18*x**8 + 144*x**7 - 672*x**6 + 2016*x**5 - 4032*x**4 + 5376*x**3 - 4608*x**2 + 2304*x - 512

        p2 = lambda x: (x-2)**9

        y = p1(x)
        g = p2(x)

        plt.plot(x, y, color='turquoise', label='expended form')
        plt.plot(x, g, color='hotpink', label='nonexpanded form')
        plt.xlabel('x')
        plt.ylabel('p(x)')
        plt.title('x versus p(x)')
        plt.legend()
        plt.savefig('Downloads/appm4600/Problem1')

        return
driver()
```



(iii) what is the difference?

The nonexpanded form is smooth while the coefficient form is not. The discrepancy is caused by catastrophic cancellation.

The nonexpanded form is the correct plot.

2. how would you preform the calculations to avoid cancellation?

(i) $\sqrt{x+1}-1$ for $x \approx 0$

$$\sqrt{x+1}-1\left[\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right] = \frac{(x+1)-1}{\sqrt{x+1}+1}$$

$$= \frac{x}{\sqrt{x+1}+1}$$

adding one in the denominator has less risk of losing digits than subtracting one in the numerator with the original for

(ii) $\sin(x)-\sin(y)$ for $x \approx y$

$$\sin(x)-\sin(y)\left[\frac{\sin(x)+\sin(y)}{\sin(x)+\sin(y)}\right]$$

$$= \frac{\sin^2(x)-\sin^2(y)}{\sin(x)+\sin(y)}$$

$$= \frac{\sin(x-y)\sin(x+y)}{\sin(x)+\sin(y)}$$

I'm not too certain about this one because $\sin(x-y) \approx 0$ in the numerator... but hopefully no significant digits would be lost in this form since we multiply by $\sin(x+y) > 0$

(iii) $\frac{1-\cos(x)}{\sin(x)}$ for $x \approx 0$

$$\frac{1-\cos(x)}{\sin(x)}\left[\frac{1+\cos(x)}{1+\cos(x)}\right] = \frac{1-\cos^2(x)}{\sin(x)(1+\cos(x))}$$
$$= \frac{\sin(x)}{1+\cos(x)}$$

in this case, rather than having $1-\cos(x) \approx 0$ in the numerator, we have addition in the denom with $1+\cos(x) > 0$

3. Find $P_2(x)$ for $f(x) = (1+x+x^3)\cos(x)$ about $x_0 = 0$

$f(x) = \cos(x)(1+x+x^3)$

$f(0) = \cos(0)(1) = 1$

$f'(x) = \cos(x)(1+3x^2) - \sin(x)(1+x+x^3)$

$f'(0) = \cos(0) - \sin(0) = 1$

$f''(x) = \cos(x)(6x) - \sin(x)(1+3x^2)$
$- \sin(x)(1+3x^2) - \cos(x)(1+x+x^3)$

$= 6x\cos(x) - 2\sin(x)(1+3x^2) - \cos(x)(1+x+x^3)$

$f''(0) = -\cos(0) = -1$

$P_2(x) = 1 + x - \frac{x^2}{2}$

(a) use $P_2(0.5)$ to approx. $f(0.5)$, consider error.

$f(0.5) \approx P_2(0.5) = 1 + \frac{1}{2} - \frac{1}{2}\left(\frac{1}{4}\right)$

$= \frac{3}{2} - \frac{1}{8} = \frac{11}{8}$

upper bound on error $= \frac{M}{(n+1)!} |x - x_0|^{n+1}$

$f'''(x) = x^3 \sin x - 9x^2 \cos x - 17x \sin x + 3\cos x + \sin x$

max at $x=0$, $f'''(0) = 3 = M$

upper bound on error $= \frac{3}{6}\left(\frac{1}{2}\right)^3 = 0.0625$

exact error $= |f(0.5) - P_2(0.5)|$

$= |1.42607 - 1.375|$

$= 0.05107$

$0.05107 \leq 0.0625$

(b) find an error bound for $P_2(x)$

error bound $= R_2(x) \leq \frac{M |x - x_0|^{n+1}}{(n+1)!}$

$R_2(x) \leq \frac{f'''(t)}{3!} x^3, \quad t \in (0, x)$

$= \frac{x^3}{6}(t^3 \sin t - 9t^2 \cos t - 17t \sin t + 3\cos t + \sin t)$

(c) approximate $\int_0^1 f(x)\,dx$ using $\int_0^1 P_2(x)\,dx$

$$\int_0^1 f(x)\,dx \approx \int_0^1 P_2(x)\,dx$$
$$= \int_0^1 1 + x - \frac{x^2}{2}\,dx$$
$$= \left[x + \frac{x^2}{2} - \frac{x^3}{6}\right]_0^1$$
$$= 1 + \frac{1}{2} - \frac{1}{6}$$
$$= \frac{4}{3}$$

(d) estimate the error in (c)

error in (c) $\approx \int_0^1 |R_2(x)|\,dx$

$\int_0^1 |R_2(x)|\,dx =$

4. $ax^2 + bx + c = 0, \quad a=1, b=-56, c=1$

(a) $r_{1,2} = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$r_{1,2} = \frac{56 \pm \sqrt{(-56)^2-4}}{2}$$

$$= \frac{56 \pm \sqrt{3132}}{2}$$

$$= 28 \pm \frac{6}{2}\sqrt{87}$$

$$= 28 \pm 3(9.327)$$

$r_1 = 28 + 27.981$
$= 55.981$

$r_2 = 28 - 27.981$
$= 0.019$

(b) manipulate $(x-r_1)(x-r_2)=0$ so that r_1 & r_2 can be related to a, b, & c

$$(x-r_1)(x-r_2)=0$$

$$(x-r_1)(x-r_2) = x^2 - r_2x - r_1x + r_1r_2$$

$$= x^2 - (r_1+r_2)x + r_1r_2$$

$r_1r_2 = 1, \quad r_1 + r_2 = 56$

$r_1 = \dfrac{1}{r_2} \qquad \dfrac{1}{r_2} + r_2 = 56$

5. $y = x_1 - x_2$, $\tilde{x}_1 = x_1 + \Delta x_1$, $\hat{x}_2 = x_2 + \Delta x_2$

$$\hat{y} = y + \underbrace{(\Delta x_1 - \Delta x_2)}_{\Delta y}$$

(a) find upper bounds on absolute and relative error

$$y = x_1 + \Delta x_1 - x_2 - \Delta x_2$$
$$=$$

(b) manipulate $\cos(x+\delta) - \cos(x)$ and plot

$$\cos(x+\delta) - \cos(x) = -2\sin\left[\frac{1}{2}(x+\delta+x)\right]\sin\left[\frac{x+\delta-x}{2}\right]$$
$$= -2\sin\left(x+\frac{\delta}{2}\right)\sin\left(\frac{\delta}{2}\right)$$

(c) $f(x+\delta) - f(x) = \delta f'(x) + \frac{\delta^2}{2} f''(\varepsilon)$, $\varepsilon \in [x, x+\delta]$

$$\cos(x+\delta) - \cos(x) = -\delta\sin(x) - \frac{\delta^2}{2}\cos(\varepsilon),$$
$$\varepsilon \in [x, x+\delta]$$