Homework 5

1. $f(x,y) = 3x^2 - y^2 = 0$
   $g(x,y) = 3xy^2 - x^3 - 1 = 0$

(a) $x_0 = y_0 = 1$, iterate using the following,

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \begin{bmatrix} 1/6 & 1/18 \\ 0 & 1/6 \end{bmatrix} \begin{bmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{bmatrix}, \quad n = 0, 1, 2, \ldots$$

this method converges to [0.5, 0.8660254] in 33 iterations

(b) provide some motivation for the choice of the 2x2 matrix in (a)

$$J(x,y) = \begin{bmatrix} 6x & -2y \\ 3y^2 - 3x^2 & 6xy \end{bmatrix}$$

$$J(1,1) = \begin{bmatrix} 6 & -2 \\ 0 & 6 \end{bmatrix}$$

$$J^{-1}(1,1) = \frac{1}{36} \begin{bmatrix} 6 & 2 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 1/6 & 1/18 \\ 0 & 1/6 \end{bmatrix}$$

this iteration is the same as 'Lazy Newton' where the jacobian of the initial guess is used for each iteration.

(c) use Newtons method and compare

Newton converges to the same root in 5 iterations, much less than lazy Newton from part (a)

(d) from your numerical result, find the exact solution

$f(x,y)-g(x,y)=0$, should be true at $(0.5, 0.866..)$

$3x^2-y^2-3xy^2+x^3+1=0$

$3(0.5)^2-(0.86603)^2-3(0.5)(0.866003)^2+(0.5)^3+1=0$

$1.63871\times10^{-8}\approx 0$

thus, the results are accurate

2. find a region D in the xy plane where the fixed point iteration

$$\begin{cases} x_{n+1}=\frac{1}{\sqrt{2}}\sqrt{1+(x_n+y_n)^2}-\frac{2}{3} \\ y_{n+1}=\frac{1}{\sqrt{2}}\sqrt{1+(x_n-y_n)^2}-\frac{2}{3} \end{cases}$$

will converge for any $(x_0,y_0)\in D$

$$G(x,y) = \begin{cases} g_1(x,y) = \frac{1}{\sqrt{2}}\sqrt{1+(x+y)^2} - \frac{2}{3} \\ g_2(x,y) = \frac{1}{\sqrt{2}}\sqrt{1+(x-y)^2} - \frac{2}{3} \end{cases}$$

$G(x,y) \in \mathbb{R}^2$ and $G(x,y)$ continuous

want: $D \subset \mathbb{R}^2$ st $G(x,y) \in D$ when $(x,y) \in D$

fix $(x,y) \in [0,1]$

$$0.040 \approx \frac{1}{\sqrt{2}} - \frac{2}{3} \leq g_1(x,y) \leq \frac{\sqrt{5}}{\sqrt{2}} - \frac{2}{3} \approx 0.914$$

$$0.040 \approx \frac{1}{\sqrt{2}} - \frac{2}{3} \leq g_2(x,y) \leq \frac{\sqrt{2}}{\sqrt{2}} - \frac{2}{3} \approx 0.333$$

$D = [0,1] \times [0,1]$

$\exists\, k \leq 1$ st $\left|\frac{dg_i(x,y)}{dx_j}\right| \leq \frac{k}{2}$, where $i = 1,2$, $j = 1,2$

$$\left|\frac{dg_1}{dx}\right| = \left|\frac{2(x+y)}{2\sqrt{2}\sqrt{1+(x+y)^2}}\right| = \left|\frac{x+y}{\sqrt{2(1+(x+y)^2)}}\right| \leq \frac{2}{\sqrt{10}}$$

$x=1, y=1$

$$\left|\frac{dg_1}{dy}\right| \leq \frac{2}{\sqrt{10}}$$

$$\left|\frac{dg_2}{dx}\right| \leq \frac{2}{\sqrt{10}} \approx 0.6325$$

$x=0, y=1$
$x=1, y=0$

$$\left|\frac{dg_2}{dy}\right| = \left|\frac{-2(x-y)}{2\sqrt{2}\sqrt{1+(x-y)^2}}\right| = \left|\frac{-(x-y)}{\sqrt{2(1+(x-y)^2)}}\right| \leq \frac{1}{2}$$

thus $\left|\frac{dg_i}{dx_j}\right| \leq \frac{1}{2}$, $k = 1 \leq 1$

3. let $f(x,y)$ be a smooth function st $f(x,y)=0$ defines a smooth curve in the xy-plane

(a) derive the iteration scheme

$$\begin{cases} x_{n+1} = x_n - d f_x, \\ y_{n+1} = y_n - d f_y \end{cases} \quad \text{where } d = \frac{f}{f_x^2 + f_y^2}$$

x_n, y_n, f_x, f_y are constants

let $g(x,y) = \frac{x - x_n}{f_x} - \frac{y - y_n}{f_y} = 0$

$$g_x = \frac{1}{f_x}, \quad g_y = \frac{-1}{f_y}$$

$$J = \begin{bmatrix} f_x & f_y \\ 1/f_x & -1/f_y \end{bmatrix}$$

$$J^{-1} = \frac{1}{-f_x/f_y - f_y/f_x} \begin{bmatrix} -1/f_y & -f_y \\ -1/f_x & f_x \end{bmatrix}$$

$$= \frac{-f_x f_y}{f_x^2 + f_y^2} \begin{bmatrix} -1/f_y & -f_y \\ -1/f_x & f_x \end{bmatrix}$$

$$= \frac{f_x f_y}{f_x^2 + f_y^2} \begin{bmatrix} 1/f_y & f_y \\ 1/f_x & -f_x \end{bmatrix}$$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \frac{f_x f_y}{f_x^2 + f_y^2} \begin{bmatrix} 1/f_y & f_y \\ 1/f_x & -f_x \end{bmatrix} \begin{bmatrix} f(x,y) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \frac{f_x f_y}{f_x^2 + f_y^2} \begin{bmatrix} f(x,y)/f_y \\ f(x,y)/f_x \end{bmatrix}$$

$$= \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \frac{1}{f_x^2 + f_y^2} \begin{bmatrix} f_x f(x,y) \\ f_y f(x,y) \end{bmatrix}$$

$$= \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \frac{f(x,y)}{f_x^2 + f_y^2} \begin{bmatrix} f_x \\ f_y \end{bmatrix}$$

which is equal to $\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - d \begin{bmatrix} f_x \\ f_y \end{bmatrix}$

(b) use this iteration on $x^2 + 4y^2 + 4z^2 = 10$ with $x_0 = y_0 = z_0 = 1$