Homework 4

1. $$\frac{T(x,t) - T_s}{T_i - T_s} = \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

$T(x,t)$ - temp. at dist x (m), and time t (sec)

T_s = constant temp. during cold period

T_i = initial soil temp. before cold

α = thermal conductivity (m^2/sec)

$T_i = 20°C$

$T_s = -15°C$

$a = 0.138 \cdot 10^{-6}\ m^2/s$

(a) How deep should a water main be buried so that it will only freeze after 60 days?

when t is given:

$$\frac{T(x) - T_s}{T_i - T_s} = \text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

$$T(x) - T_s = (T_i - T_s)\,\text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

$$\left.\begin{aligned} T(x) &= (T_i - T_s)\,\text{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right) + T_s \\ T'(x) &= \frac{2}{\sqrt{\pi}}(T_i - T_s)\,e^{-x^2/4\alpha t} \end{aligned}\right]$$

f and f' for root finding prob.

$60 \text{ days}\left[\frac{24 \text{ hr}}{\text{day}}\right]\left[\frac{60 \text{ min}}{\text{hr}}\right]\left[\frac{60 \text{ sec}}{\text{min}}\right] = 5.184 \cdot 10^6 \text{ s}$

plot in github

(b) compute approximate depth using bisection with $a=0$, $b=\bar{x}$

$\bar{x} = 10$

approximate root $= -3.926 \times 10^{-13}$ m

code in github

(c) compute approximate depth using Newton's method with $x_0 = 0.01$

approximate root $= 0.67696$ m

if I try $\bar{x}=10$ as the initial value, I get a runtime warning and nan returns as root and the error message reads fail

code in github

2. let $f(x)$ denote a function with root α of multipitity m

(a) mathematical definition of a function with root α of mult. m

$f(x) = (x-\alpha)^m q(x)$

$f(\alpha) = f'(\alpha) = \ldots = f^{(m-1)}(\alpha) = 0, \; f^{(m)}(\alpha) \neq 0$

(b) show that newtons method on f(x) only converges linearly to α

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$g'(x) = 1 - \frac{[f'(x)f'(x) - f(x)f''(x)]}{[f'(x)]^2}$$

$$= 1 + \frac{f(x)f''(x)}{[f'(x)]^2}$$

$$f'(x) = (x-\alpha)^m q'(x) + m(x-\alpha)^{m-1} q(x)$$

$$f''(x) = (x-\alpha)^m q''(x) + m(x-\alpha)^{m-1} q'(x) + m(x-\alpha)^{m-1} q'(x) + m(m-1)(x-\alpha)^{m-2} q(x)$$

$$= (x-\alpha)^m q''(x) + 2m(x-\alpha)^{m-1} q'(x) + m(m-1)(x-\alpha)^{m-2} q(x)$$

$$g'(\alpha) = \frac{1}{m} \leq 1$$

thus f(x) converges linearly to α

(c) Show that fixed point iteration applied to $g(x) = x - m\frac{f(x)}{f'(x)}$ is second order convergent

$$g'(x) = 1 - m\left[\frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2}\right]$$

$$g'(\alpha) = 1 - m\left(\frac{1}{m}\right) = 0$$

$$g''(x) = -m\left[\frac{[f'(x)]^2[f(x)f^{(3)}(x) + f'(x)f''(x)] - 2[f'(x)][f''(x)]^2}{[f'(x)]^4}\right]$$

$$g''(x) = -m\frac{\begin{aligned}&[[q''(x)(x-\alpha) + q'(x) + mq''(x)][(x-\alpha)q'(x) + q''(x)] + [(x-\alpha)q''(x) + 2q'(x)][(x-\alpha)q'(x) \\ &+ mq(x)] - [[(x-\alpha)q(x)][(x-\alpha)q^{(3)}(x) + q''(x) + q''(x) + mq''(x)] + [(x-a)q''(x) + q'(x) \\ &+ mq'(x)][(x-\alpha)q'(x) + q(x)] - [mq(x)] [2(x-a)q'(x) + mq(x)][(x-\alpha)q''(x) + q'(x) \\ &+ mq'(x)]\end{aligned}}{[(x-a)^m q(x) + m(x-a)^{m-1}q(x)]^4}$$

$$g''(\alpha) = -m\frac{[q(\alpha)q'(\alpha) + mq''(\alpha)q(\alpha) + 2mq'(\alpha)q(\alpha) - q'(\alpha)q(\alpha) + mq'(\alpha)q(\alpha)]}{m^4 q(\alpha)^4}$$

$$= -\left[\frac{m^4 q(\alpha)^3 q'(\alpha) + m^4 q''(\alpha)q(\alpha)^3 + 2m^3 q(\alpha)^3 q'(\alpha)}{m^4 q(\alpha)^4}\right]$$

$$= -\left[\frac{mq'(\alpha) + mq''(\alpha) + 2q'(\alpha)}{mq(\alpha)}\right]$$

$$= \frac{-q'(\alpha) - q''(\alpha)}{q(\alpha)} - \frac{2q'(\alpha)}{mq(\alpha)}$$

$$\neq 0$$

thus, $g(x)$ is second order convergent

[some scratch work I did not include]

3. sequence $\{x_k\}_{k=1}^{\infty}$ that converges to α, derive a relationship between $\log(|x_{k+1}-\alpha|)$ and $\log(|x_k-\alpha|)$

$$\lim_{k\to\infty} \frac{|x_{k+1}-\alpha|}{|x_k-\alpha|^p} = \lambda$$

when k is large,

$$|x_{k+1}-\alpha| = \lambda|x_k-\alpha|^p$$

$$\log(|x_{k+1}-\alpha|) = \log(\lambda|x_k-\alpha|^p)$$

$$\log(|x_{k+1}-\alpha|) = \log(\lambda)\cdot p\log(|x_k-\alpha|)$$

$$\log(|x_{k+1}-\alpha|) = p\log(\lambda)\log(|x_k-\alpha|)$$

4. $f(x) = e^{3x} - 27x^6 + 27x^4e^x - 9x^2e^{2x} \in (3,5)$

$f(x) = (e^x - 3x^2)^3$

the modified method from #2 since $f(x)$ has a root with higher order than 1

5. Newton and secant method to approximate the largest root of $f(x) = x^6 - x - 1$

$f'(x) = 6x^5 - 1$

actual root of $f(x)$ is 1.1347

(a) Newton error:

iteration	error
1	0.8653
2	0.5459
3	0.2960
4	0.1203
5	0.0268
6	0.0017
7	3.05×10^{-5}
8	2.41×10^{-5}
9	2.41×10^{-5}
10	2.41×10^{-5}

secant error:

iteration	error
1	0.1186
2	0.0559
3	0.0170
4	2.1684×10^{-3}
5	1.1681×10^{-4}
6	2.3646×10^{-5}
7	2.4138×10^{-5}
8	2.4138×10^{-5}
9	2.4138×10^{-5}

the error does decrease as I would expect; nonlinear convergence