Proof

A method for establishing a truth

Proposition

A statement that can be either true or false. Most sentences are propositions, but not all of them.

Ex $\forall n \in \mathbb{N}\; n^2 + n + 41$ is prime

multiple variables $\forall\ \exists$ quantifiers

This is a special proposition called a

Predicate

- A predicate's truth depends on its variables values

Axiom

An axiom is something you believe to be true because it is reasonable

L. Deductions

Logical deductions are used to combine axioms and true propositions such that more true propositions are formed

EX
$$\frac{P \quad P \to Q}{Q}$$

if P is T and P→Q is T then Q must be true

$(P \land (P \to Q)) \to Q$ same

Induction

1 $P(0)$ is true
1 $\forall n \in \mathbb{N},\ P(n) \to P(n+1)$
1 $P(n)$ is true for $\forall n \in \mathbb{N}$

Induction is used to prove that a statement, in coding, holds for every value of that variable.

How to use induction
1 State that the proof uses induction
2 Define an appropriate predicate
3 Prove that $P(0)$ is true
4 Prove that $P(n) \to P(n+1)$ for all $\mathbb{N}$
5 Invoke induction

Number Theory

is the study of integers

Graphs

A bunch of dots connected by lines.
Formally, a graph is a pair of sets V(nodes) and E(edges), where V is a non empty set and E is a set of 2-item subsets of V called edges

x_1, x_2
- are adjacent
- the edge is incident to x_1 and x_2
- two vertices connected means degree two

$V = \{x_1, x_2, x_3\}$ $|V| = 6$

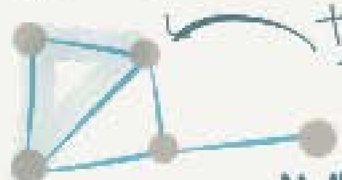
$E = \{\{x_1, x_2\} \{x_3, x_4\} \dots\}$

cardinality

Graph colouring problem

Given a graph G with K colors assign each color to each node such that adjacent nodes have different colors.

- the minimum number of K and that you can do that is called **chromatic number of G** $\chi(G)$



triangle forces χ to be 3.

This is not efficient computationally!!

the is an NP complete. can't be done under $O(n^2)$. NP complete are connected by common ... good solutions

However if the basic has low degree $\leq d$ **it will use d+1 colors at value**

Basic colouring Alg

1 Order nodes $V_1 V_2 \ldots n$
2 Order colors
3 Color nodes with lowest legal color

num colors used depends on # 1 ordering. So How to order effectively?

Stable marriage problem

- N boys and N girls
- each boy has his ranked list for all girls
- each girl has a ranked list of all boys

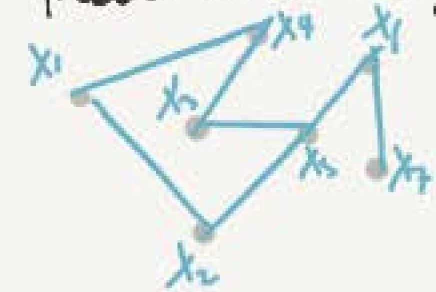
Find a perfect matching without rogues

Stable marriage algorithm?

Greedy

Matching problem

Given a graph G a matching is a subgraph where every node has degree one



$\{X_1 - X_2, X_5 - X_6, X_3 - X_4\}$

matching of size 3

- A matching is perfect if it has size |V| half

The weight of a matching M is the sum of the weight on the edges M.

A min weight matching is a perfect matching with min weight

$\{A-B, C-D\}$ weight 20

Given a matching M, X and Y form a rogue couple if they prefer each other over their matches (A-D)

A match is stable if there are no rogue couple

Walks and Paths

A walk is a sequence of vertices connected by edges $v_0 - v_1 - v_2 \dots v_k$ } length k

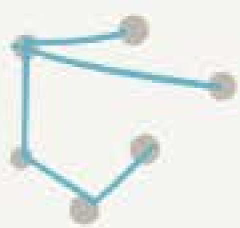
A path is a special types of walks in which each V is different

Two vertices are **connected** if there's a path between

A graph is **connected** if every pair of vertices in the graph is connected

A **closed walk** starts and ends at the same vertex

If $k \geq 3$ and if all vertices are different then it's a **cycle**



A tree is both connected and without cycles

A spanning tree is a subgraph where a graph that smaller spans all the vertices within a graph

You can create a spanning tree from any graph that is connected

How to make a min weight tree

NEXT COLUMN!

Min spanning tree

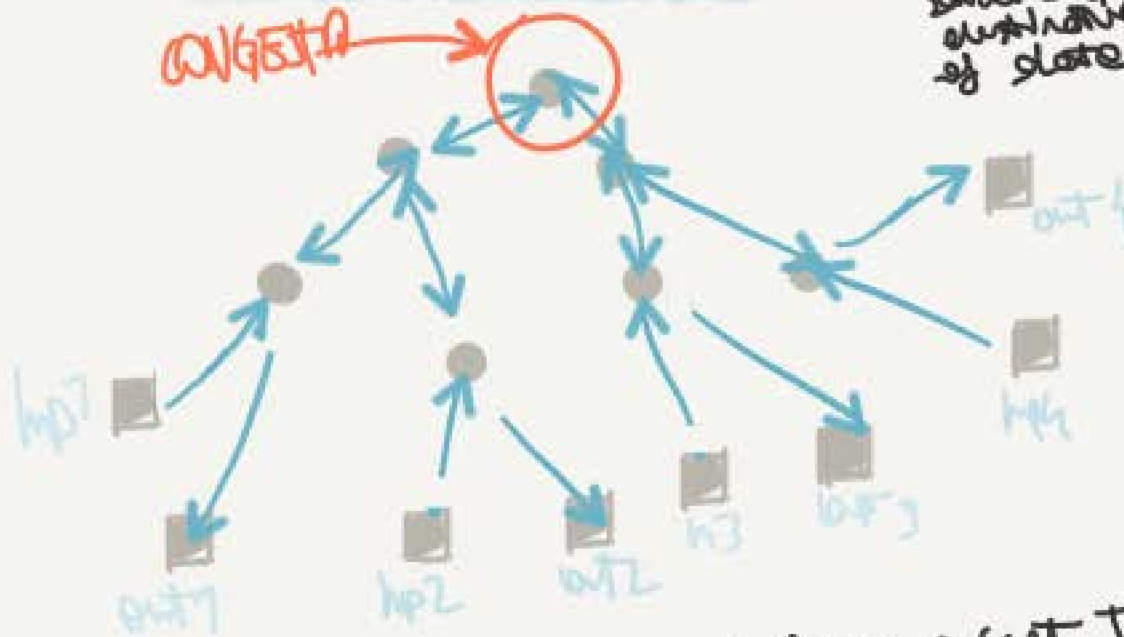
Algorithm, grow the subgraph one edge at a time such that at each step we add the min weighted edge that keeps a subgraph acyclic

Communication networks

- complete binary tree

= switch, used to direct packets

= terminal, source and destination of data



Latency, time required for a packet to travel from an input to an output

Diameter, shortest path between the input and output farthest apart

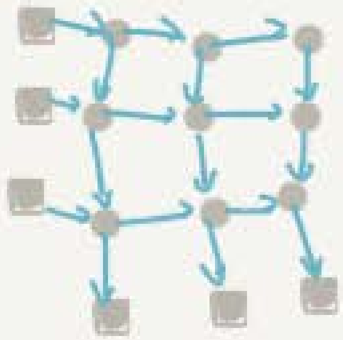
Switch size, # of input and outputs to a switch (BT is 3x3)

switches, ... (BT is 2n-1)

Congestion, in the worst case (BT is N)

2D ARRAY →

- 2D array



Diameter is 2N
sw. size is 2X2
switches is N^2
congestion is 2

- Butterfly structure

"BEATIFUL
GRAPH HERE"

Rows are numbered from 0 to n,
levels here are numbered in 01 notation

When passing data from a level to another, for every bit that is different you cross, for every equal bit you go straight

Diameter is 2+logN
sw. size is 2X2
switches is (1+LogN)
congestion is $\sqrt{N}$ or $\sqrt{\frac{N}{2}}$

- Benes

Is a butterfly network attached to another butterfly network

"ANOTHER
GRAPH"

the outer subgraphs (upper and lower) forms other Benes networks

WE GET CONGESTION 1

An Euler tour is a walk that traverses every edge exactly once and starts and finishes at the same vertex

A connected graph has an Euler tour iff every vertex has even degree

The degree of a node is given by how many times appears in the tour, doubled

Directed graph in which each edge goes from a vertex to another.
Each vertex has a tail and a head
The total # of tails represents the outdegree while the # of heads is the indegree

A digraph (directed graph) is strongly connected if for all vertices there exists a directed path that connect each other

A digraph is acyclic if it does not contain any directed cycle (DAGs)

Tournament graph
Vertices represents teams, directed edge pointing represents a won game

A directed Hamiltonian path is a directed walk that has every vertex exactly once. Is present in every tournament graph

Relations

A relation from a set A to a set B is a subset of the cross product of the two

$R = \{(a,b) : \text{student } a \text{ takes class } b\}$

A relation on A is a subset $R \subseteq A \times A$

$A = \mathbb{N}$ $\quad xRy$ iff $x \mid y$

relation symbol

Set A together with R is a directed graph

$G = (V, E)$ with $V = A$, $E = R$

PROPERTIES

- A relation is **reflexive** if xRx for all $x \in A$ (not the case in the graph)
- **symmetric** if $xRy \Rightarrow yRx$ for all $x, y \in A$
- **antisymmetric** if $xRy \wedge yRx \Rightarrow x = y$
- **transitive** if $xRy \wedge yRz \Rightarrow xRz$

An **equivalence relation** is reflexive, symmetric and transitive

Ex $x = y$, $x \equiv y \pmod 5$

An equivalent class of $x \in A$ is the set of all the elements in A related to x by R: denoted $[x]$

A **partition** of A is a collection of disjoint, non empty sets $A_1 \dots A_n$ whose union is A

A relation is a weak partial order if is reflexive, antisymmetric and transitive

NOTATION $\preccurlyeq$ instead of R

A poset is a graph with vertex set A and edge set $\preccurlyeq$

"REPRESENTED WITH HASSE DIAGRAM"

A and B are **incompatible** if $a \prec b$ or neither $b \prec a$

A and B are compatible otherwise

A **total order** is a partial order in which every pair is compatible

A total order consistent with a partial order is a **topological sort**.

A top. sort of a poset $(A, \preccurlyeq)$ is a total order $(A, \preccurlyeq_T)$ such that $\preccurlyeq \subseteq \preccurlyeq_T$

Factorial

$n! = \prod_{i=1}^{n} i$; where $\prod$ denotes the product

$\ln(n!) = \ln(1 \cdot 2 \cdot 3 \dots n)$
$= \sum_{i=1}^{n} \ln(i)$

Approximated is

$\frac{n^n}{e^{n-1}} \le n! \le \frac{n^{n+1}}{e^{n+1}}$

tighter bounds for that are given by the Stirling's formula

$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\, e^{\varepsilon(n)}$

Asymptotic notation

How a function grows in the limit

tilde $f(x) \sim g(x)$ if $\frac{f(x)}{g(x)} = 1$

oh, big oh $f(x) = O(g(x))$

if $\lim_{x \to \infty} \left| \frac{f(x)}{g(x)} \right| < \infty$ (finite)

How $f(x)$ grows at the same rate or slower than $g(x)$

Thm: let $f(x) = x$, $g(x) = x^2$, then $f(x) = O(g(x))$

Pf $\lim_{x \to \infty} \frac{x}{x^2} = 0 < \infty$

$f(x) = O(g(x))$, $f(x)$ is $O(g)$, $f(x) \in O(g(x))$

omega $f(x) = \Omega(g(x))$

if $\lim_{x \to \infty} \left| \frac{f(x)}{g(x)} \right| > 0$

for a lower bound version: this is the opposite of "oh"

theta $f(x) = \Theta(g(x))$

if $\lim_{x \to \infty} \left| \frac{f(x)}{g(x)} \right| > 0$ & $< \infty$

$f(x)$ is $\Theta(g(x))$ iff $f(x)$ is $O(g(x))$ and $\Omega(g(x))$

little oh

$f(x) = o(g(x))$ if $\lim_{x \to \infty} \left| \frac{f(x)}{g(x)} \right| = 0$

little omega

$f(x) = \omega(g(x))$ if $\lim_{x \to \infty} \left| \frac{f(x)}{g(x)} \right| = \infty$

Sums and asymptotics

Sums arises often in computer science, although e.g. $1+2+3\ldots N$ can be represented with sigma, it is a lot easier to use a closed form

$$\sum_{i=0}^{\log N} 2^i \rightarrow 2N-1$$

This chapter is about to find closed forms

For a geometric series like

$$\sum_{j=0}^{n-1} x^j = 1+x+x^2+x^3\ldots x^{n-1}$$

can be simplified to $\dfrac{1-x^n}{1-x}$

Perturbation model

Given a sum with a defined structure, it is useful to perturb and combine the sum with the perturbation such that we get something simpler

$$S = 1+x+x^2+x^3\ldots x^n$$
$$xS = x+x^2+x^3\ldots x^n$$

PERTURBATION

the subtraction gives

$$S = \frac{1-x^n}{1-x}$$

What if our series is infinite? for $|x|<1$

$$\sum_{i=0}^{\infty} x^i = \frac{1-\cancel{x^n}}{1-x}$$

x is less than 1.00 n grows, it goes to 0

Stability

Let's use sums to find out how many blocks I can stack in a certain way before they fall

Ex

Given n blocks of length 1

r_j is the amount by which the beam extends beyond the table

Stability constraint, the center of mass C_k of the top k blocks must lie on the $k+1$ blocks

Using greedy we want to know where is r_1. with, $C_k = r_{k+1}$ (edge of block underneath)

$$\underbrace{r_k - r_{k+1}}_{\text{how much further the } k \text{ block sticks out}} = \frac{1}{2k}$$

Greedy here follows Harmonic sum pattern

Harmonic sum

$$r_1 = \frac{1}{2}\sum_{i=n}^{n}\frac{1}{i}$$