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1.Poisson Regression(6.6)

Exercise 4 of Chapter 5 describes an experiment from Haberman (1978) involving subjects reporting one stressful event. The number of events recalled i months before an interview y_i is Poisson distributed with mean λ_i, where the $\{\lambda_i\}$ satisfy the loglinear regression model

$$\log \lambda_i = \beta_0 + \beta_1 i.$$

One is interested in learning about the posterior density of the regression coefficients (β_0, β_1).

a) Using the output of `laplace`, construct a Metropolis random walk algorithm for simulating from the posterior density. Use the function `rwmetrop` to simulate 1000 iterates, and compute the posterior mean and standard deviation of β_1.
b) Construct a Metropolis independence algorithm, and use the function `rwindep` to simulate 1000 iterates from the posterior. Compute the posterior mean and standard deviation of β_1.
c) Use a table such as Table 6.2 to compare the posterior estimates using the three computational methods.

在这里，我们的数据y_i服从均值为λ_i的 poisson 分布，即：

$$p(y_i = k) = \frac{\lambda_i^k}{k!} e^{-\lambda_i}$$

其中：

$$log\lambda_i = \beta_0 + \beta_1 i$$

即是：

$$\lambda_i = e^{\beta_0 + \beta_1 i}$$

所以，当参数未知时，其条件概率为：

$$p(y_i = k|\beta_0, \beta_1) = \frac{e^{(\beta_0+\beta_1 i)k}}{k!} e^{-e^{\beta_0+\beta_1 i}}$$

似然函数（$\theta = \beta_0, \beta_1$）为：

$$L(\theta|data) = \prod_{i=1}^{18} [\frac{e^{(\beta_0+\beta_1 i)y_i}}{y_i!} e^{-e^{\beta_0+\beta_1 i}}]$$

Log 似然为：

$$lnL(\theta|data) = \sum_{i=1}^{18} [(\beta_0 + \beta_1 i)y_i - e^{(\beta_0+\beta_1 i)} - ln(y_i!)]$$

由于y_i固定，其似然函数正比于：

$$lnL(\theta|data) \propto \sum_{i=1}^{18} [(\beta_0 + \beta_1 i)y_i - e^{(\beta_0+\beta_1 i)}]$$

由题意，给参数施加一个均匀先验后，其最终的后验概率为：

$$g(\theta|data) \propto \sum_{i=1}^{18}[(\beta_0+\beta_1 i)y_i - e^{(\beta_0+\beta_1 i)}]$$

1.1 a——rwmetrop and analysis

在这里，我们仿照本章 6.7 节给出解答过程。

首先，写出所求解出的后验概率函数，随后规定模拟次数与 burn-in period。由于在实验中发现模拟次数 sim=1000 时结果的不稳定性很大，为方便比较，我们模拟 10000 次以获得更稳定的效果。

```
rm(list=ls())
library(LearnBayes)
data=c(15,11,14,17,5,11,10,4,8,10,7,9,11,3,6,1,1,4)
f=function(beta,data){
  beta0=beta[1]
  beta1=beta[2]
  i=1:18
  y=data
  sum((beta0+beta1*i)*y-exp(beta0+beta1*i))
}
sim=10000
burnin=2000
```

题目首先让我们使用 laplace 函数。我们采取一个小技巧（类似 6.7 节找初值的方法）来获取一个较好的初值。我们知道y_i服从均值为$\beta_0+\beta_1 i$的 poisson 分布，由于：

$$E(y_i)=\beta_0+\beta_1 i$$

我们采用y_i替代$E(y_i)$：

$$y_i=\beta_0+\beta_1 i$$

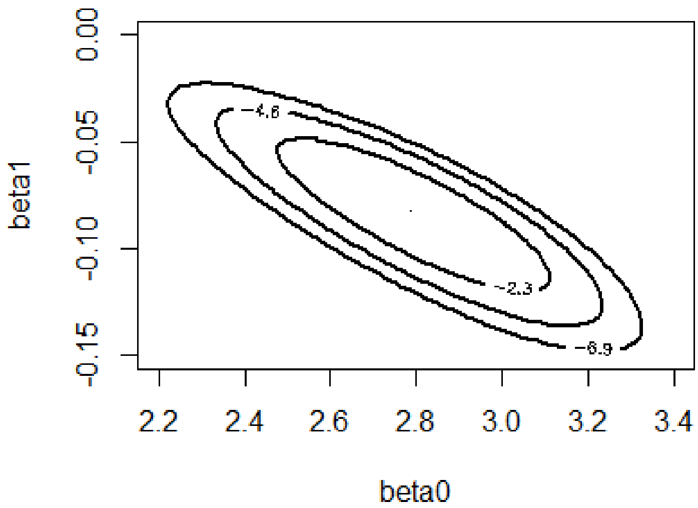
通过带入(i,y_i)的两组数据(1,15),(9,8)求得一组(β_0,β_1)=(127/8, -7/8)，我们使用这组参数值作为初值。求得 laplace 的 mode 与 var。

```
start2=c(127/8,-7/8)
fit=laplace(f,start2,data)
fit
$mode
[1]  2.79392439 -0.08230398

$var
            [,1]         [,2]
[1,]  0.02200785 -0.002066740
[2,] -0.00206674  0.000280728
```

接下来，通过 Laplace 输出的 mode 我们利用 mycontour 函数多次调试后画出区域图：

```
mycontour(f,c(2.2,3.4,-0.15,0),data,xlab="beta0",ylab="beta1")
```



接下来，我们通过 laplace 函数的输出设置随机游走模拟的 proposal 参数，进行 10000 次模拟，并计算接受率。此接受率在 25%-45%之间，我们比较满意。

proposal=list(var=fit$var,scale=2)

fit2=rwmetrop(f,proposal,start1,sim,data)

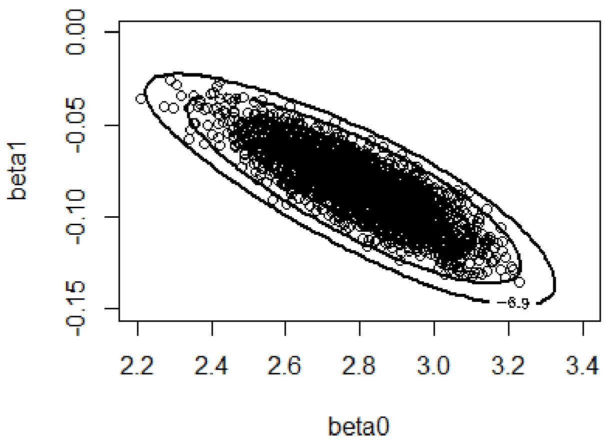
fit2$accept

[1] 0.2873

随后，我们将模拟的点除去 burn-in period 后（如果不去除 burn-in preiod 会有一些离群点），在图上展示：

mycontour(f,c(2.2,3.4,-0.15,0),data,xlab="beta0",ylab="beta1")

points(fit2$par[burnin:sim, 1], fit2$par[burnin:sim, 2])



接下来，我们仿照 6.7 节给出随机游走模拟与 laplace 近似的对比，可以看出，无论是 β_0,β_1的均值与方差，两种方法差别都很小。

post.means=apply(fit2$par,2,mean)

post.sds=apply(fit2$par,2,sd)

modal.sds=sqrt(diag(fit$var))

> cbind(c(fit$mode), modal.sds)

modal.sds

[1,] 2.79392439 0.14835042

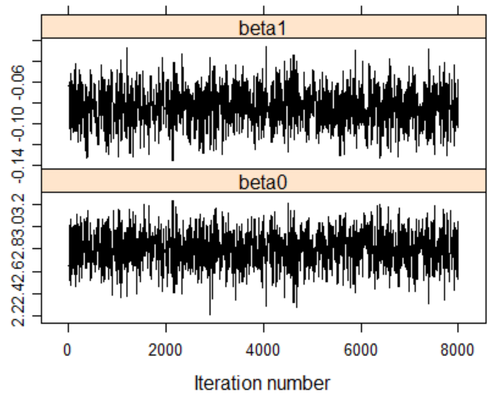
[2,] -0.08230398 0.01675494

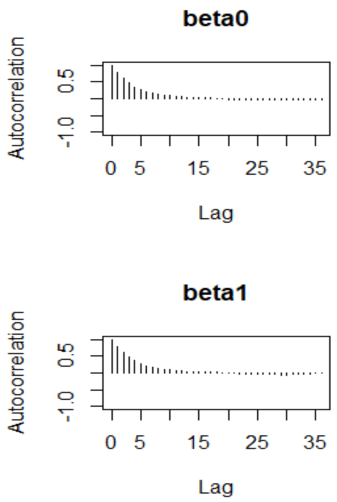
> cbind(post.means, post.sds)

```
         post.means     post.sds
[1,]   2.80483591 0.17222863
[2,] -0.08355152 0.01775821
```

接下来，我们仿照 6.8 节给出 MCMC 模拟的输出分析：（依旧丢弃了 burn-in period），从模拟的轨迹图和自相关图我们都能发现模拟的效果很好。

```
library(coda)
library(lattice)
dimnames(fit2$par)[[2]]=c("beta0", "beta1")
xyplot(mcmc(fit2$par[-c(1:burnin), ]), col = "black")
par(mfrow = c(2, 1))
autocorr.plot(mcmc(fit2$par[-c(1:burnin), ]), auto.layout = FALSE)
```





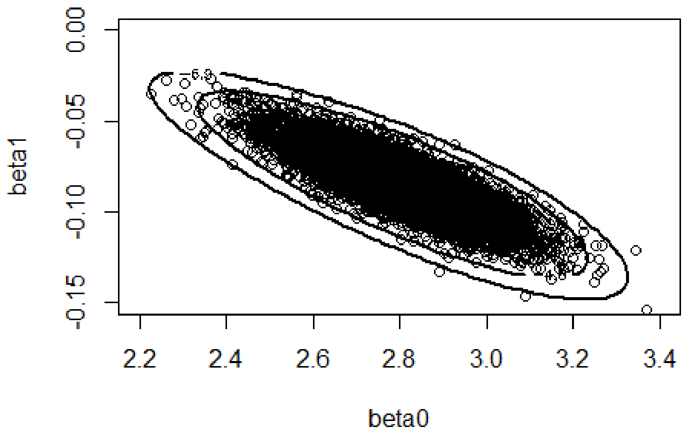
最后，我们输出题目要求的β_1的后验均值和方差

```
> mean(fit2$par[burnin:sim,2])
[1] -0.08398493
> sd(fit2$par[burnin:sim,2])
[1] 0.01683423
```

1.2 b——indepmetrop

接下来，我们使用 MCMC 算法的独立模型，首先依旧从 laplace 的输出中获得 proposal 的参数，依旧进行 10000 次模拟，burn-in period 为 2000。接受率为 0.9118。之后画出模拟图。

```
proposal2=list(var = fit$var, mu = t(fit$mode))
fit3=indepmetrop(f, proposal2, start2, sim, data)
fit3$accept
[1,] 0.9118
mycontour(f,c(2.2,3.4,-0.15,0),data,xlab="beta0",ylab="beta1")
points(fit3$par[burnin:sim, 1], fit3$par[burnin:sim, 2])
```



最后，我们输出题目要求的β_1的后验均值和方差

> mean(fit3$par[burnin:sim,2])

[1] -0.08432691

> sd(fit3$par[burnin:sim,2])

[1] 0.016868

1.3 c——Gibbs

接下来，我们利用 gibbs 采样，通过调整参数得到了一组可以接受的接受率。之后画出模拟图。

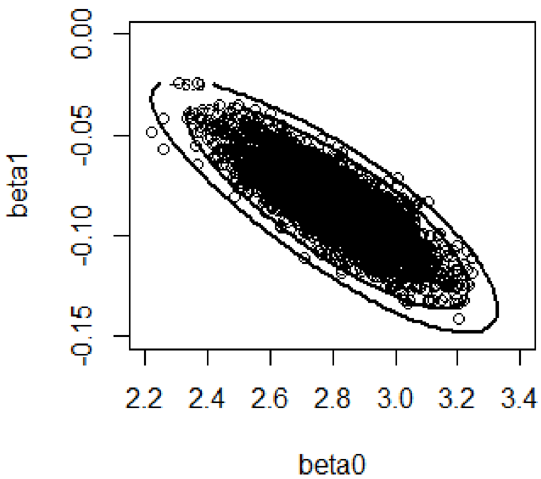
> fit4=gibbs(f, start2, sim, c(0.3, 0.05), data)

> fit4$accept

```
      [,1]   [,2]
[1,] 0.3161 0.2386
```

mycontour(f,c(2.2,3.4,-0.15,0),data,xlab="beta0",ylab="beta1")

points(fit4$par[burnin:sim, 1], fit4$par[burnin:sim, 2])



Gibbs 采样的β_1的后验均值和方差

> mean(fit4$par[burnin:sim,2])

[1] -0.08436472

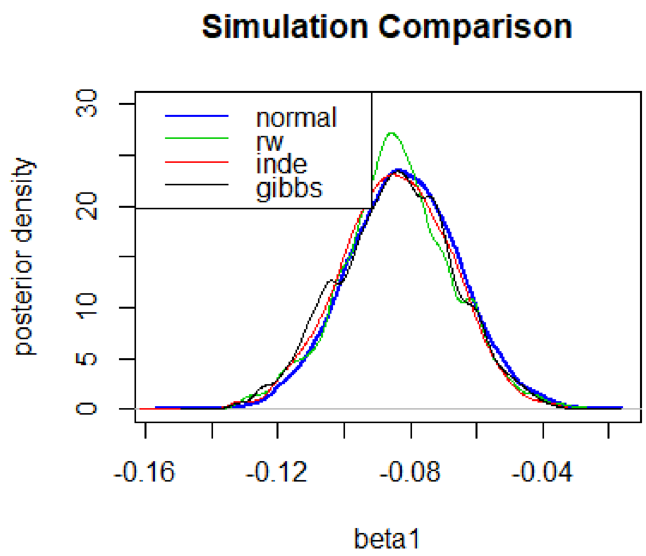
> sd(fit4$par[burnin:sim,2])

[1] 0.01737734

1.4 Comparison

我们画出β_1的四种方法模拟出的后验密度图，可以看到除了随机游走方法，其他方法都非常近似于正态分布。

```
plot(density(rnorm(sim,fit$mode[2],sqrt(fit$var[2,2]))),
     type="l",col=4,lwd=2,lty=1,main="Simulation Comparison",
     xlab="beta1",ylab="posterior density",ylim=c(0,30))
lines(density(fit2$par[burnin:sim,2]),type="l",col=3,lwd=1)
lines(density(fit3$par[burnin:sim,2]),type="l",col=2,lwd=1)
lines(density(fit4$par[burnin:sim,2]),type="l",col=1,lwd=1)
legend("topleft",c("normal","rw","inde","gibbs"),
       lty=1,col=c(4,3,2,1))
```



接下来，我们仿照 table6.2 做出四种算法的对比，首先就β_1的 0.05/0.5/0.95 分位数而言，随机游走，独立模型和 gibbs 采样三种方法区别不大，都要好于正态。

```
> r12=rnorm(sim,fit$mode[2],sqrt(fit$var[2,2]))#normal
> r22=fit2$par[burnin:sim,2]#rw
> r32=fit3$par[burnin:sim,2]#inde
> r42=fit4$par[burnin:sim,2]#gibbs
>
> t12=c(quantile(r12,0.05),quantile(r12,0.5),quantile(r12,0.95))
> t22=c(quantile(r22,0.05),quantile(r22,0.5),quantile(r22,0.95))
> t32=c(quantile(r32,0.05),quantile(r32,0.5),quantile(r32,0.95))
> t42=c(quantile(r42,0.05),quantile(r42,0.5),quantile(r42,0.95))
> t12 #normal
         5%         50%         95%
-0.10950724 -0.08203023 -0.05477066
> t22 #rw
```

```
          5%          50%          95%
-0.11316273 -0.08419039 -0.05656357
> t32 #inde
          5%          50%          95%
-0.11261074 -0.08422269 -0.05726191
> t42#gibbs
          5%          50%          95%
-0.11345120 -0.08387958 -0.05673468
```

其次，对于β_0的 0.05/0.5/0.95 分位数而言，四种方法相差极小，说明其有较好的正态性。

```
> r12=rnorm(sim,fit$mode[1],sqrt(fit$var[1,1])) #normal
> r22=fit2$par[burnin:sim,1] #rw
> r32=fit3$par[burnin:sim,1] #inde
> r42=fit4$par[burnin:sim,1] #gibbs
>
> t11=c(quantile(r12,0.05),quantile(r12,0.5),quantile(r12,0.95))
> t21=c(quantile(r22,0.05),quantile(r22,0.5),quantile(r22,0.95))
> t31=c(quantile(r32,0.05),quantile(r32,0.5),quantile(r32,0.95))
> t41=c(quantile(r42,0.05),quantile(r42,0.5),quantile(r42,0.95))
> t11#normal
      5%      50%      95%
2.550118 2.795169 3.041423
> t21#rw
      5%      50%      95%
2.550140 2.802765 3.035306
> t31 #inde
      5%      50%      95%
2.555045 2.799454 3.041913
> t41#gibbs
      5%      50%      95%
2.549682 2.805797 3.057397
```

1.5 MAP

在本题中，由于后验概率形式比较简单，我们其实可以直接采用极大化后验概率(MAP)的方法求出最优的参数。即：

$$\beta_0,\beta_1=\underset{\beta_0,\beta_1}{argmax}\sum_{i=1}^{18}[(\beta_0+\beta_1 i)y_i-e^{(\beta_0+\beta_1 i)}]$$

我们调用了 GA 包里的遗传算法对后验概率函数进行极大化求解。

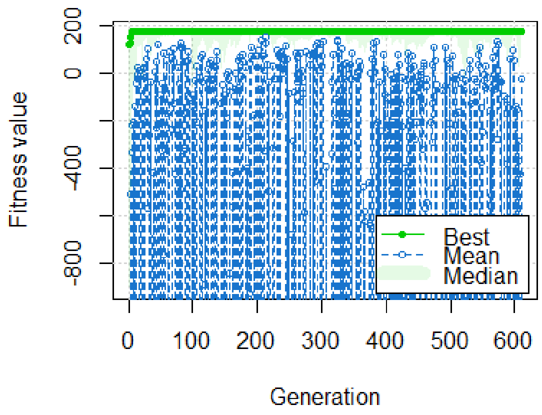
```
library(GA)
f_ga=function(beta0,beta1){
  data=c(15,11,14,17,5,11,10,4,8,10,7,9,11,3,6,1,1,4)
  i=1:18
```

```
  y=data
  sum((beta0+beta1*i)*y-exp(beta0+beta1*i))
}

GA=ga(type = "real-valued",
        fitness =   function(x) f_ga(x[1], x[2]),
        lower = c(1, -2), upper = c(8, 1),
        popSize = 50, maxiter = 1000, run = 100)
summary(GA)
```

经过 611 次迭代后得到最优解，β_0 =2.82466，$\beta_1 = -0.8636$ 这个结果与我们模拟得到的结果基本相同，相互佐证，说明的模拟方法与 MAP 的方法都具有很好的效果。

```
GA results:
Iterations             = 611
Fitness function value = 174.8327
Solution =
           x1          x2
[1,] 2.82466 -0.08636227
plot(GA)
```



2.Inference about the Box-Cox transformation(6.10)

Suppose one observes the positive values $y_1, ..., y_n$ that exhibit some right-skewness. Box and Cox (1964) suggested using the power transformation

$$w_i = \frac{y_i^\lambda - 1}{\lambda}, i, = 1, ..., n,$$

such that $w_1, ..., w_n$ represent a random sample from a normal distribution with mean μ and standard deviation σ. Suppose that the vector of parameters (λ, μ, σ) is assigned the noninformative prior proportional to $1/\sigma$. Then the posterior density of θ is given, up to a proportionality constant, by

$$g(\theta|y) \propto \frac{1}{\sigma} \prod_{i=1}^{n} \left[\phi\left(\frac{y_i^\lambda - 1}{\lambda}; \mu, \sigma \right) y_i^{\lambda-1} \right].$$

Suppose this transformation model is fit to the following survival times (from Collett, 1994) of patients in a study on multiple myeloma.

```
13 52  6 40 10  7 66 10 10 14 16  4
65  5 11 10 15  5 76 56 88 24 51  4
40  8 18  5 16 50 40  1 36  5 10 91
18  1 18  6  1 23 15 18 12 12 17  3
```

a) Write an R function to compute the logarithm of the posterior distribution of $(\lambda, \mu, \log\sigma)$.
b) Use `laplace` to find the posterior mode of $(\lambda, \mu, \log\sigma)$ using an initial starting value of (0.1, 3, 0.5).
c) Use an MCMC algorithm such as random walk Metropolis, independent Metropolis, or Gibbs sampling to simulate 10,000 values from the posterior distribution.
d) Construct 90% interval estimates of λ, μ, and σ.
e) For these data, use the result from part (d) to decide whether a log or square root transformation is more appropriate for these data.

本题，我们基于一组数据，利用 Box-Cox 方法对其做正态性转化。首先，题目给出了后验概率，我们需要写出 log 后验，并进行变量代换。

$$lng(\theta|data) \propto \sum_{i=1}^{n}[ln\phi(\frac{{y_i}^\lambda - 1}{\lambda}; \mu, e^{\theta_3}) + (\lambda - 1)lny_i]$$

2.1 a

根据题意，我们写出函数计算上述 log 后验概率：

```
#6.10a
rm(list=ls())
set.seed(11)
library(LearnBayes)
data=c(13,52,6,40,10,7,66,10,10,14,16,4,65,5,11,10,15,5,76,
       56,88,24,51,4,40,8,18,5,16,50,40,1,36,5,10,91,18,1,
       18,6,1,23,15,18,12,12,17,3)
g=function(theta,data){
  lambda=theta[1]
  mu=theta[2]
  sigma=exp(theta[3])
  y=data
  sum(log(dnorm((y^lambda-1)/lambda,mu,sigma))+(lambda-1)*log(y))
}
```

2.2 b

接下来，使用 laplace 函数找到后验 mode：

```
> start=c(0.1,3,0.5)
> laplacefit=laplace(g,start,data)
> laplacefit$mode
[1] 0.1371790 3.2757185 0.4565021
```

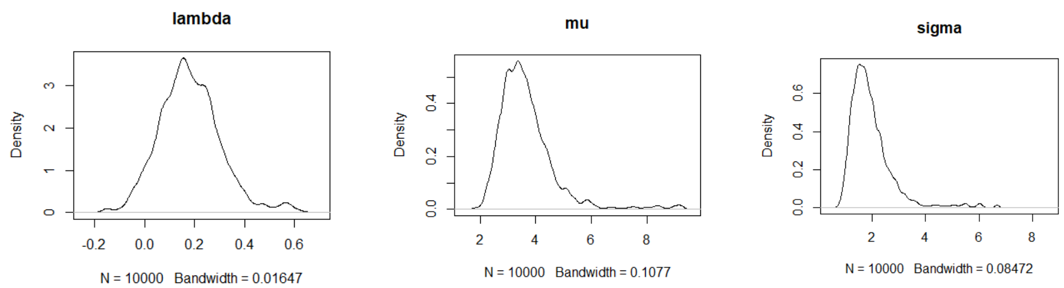
2.3 c

在此，我们使用三种 MCMC 算法进行模拟。此处由于估计的第三个参数为$\log\sigma$，为了得到真实值，我们需要取其指数。其接受率我们都控制在合理的范围内。其中 gibbs 方法的调参很麻烦，不太容易找到好的参数。

```
#rw
> proposal1=list(var=laplacefit$var,scale=1.7)
> fitrw=rwmetrop(g,proposal1,start,10000,data)
> fitrw$accept
[1] 0.2341
> pararw=fitrw$par
> pararw[,3]=exp(pararw[,3])
#inde
proposal2=list(var = laplacefit$var, mu = t(laplacefit$mode))
> fitinde=indepmetrop(g, proposal2, start, 10000, data)
```

```
> fitinde$accept
          [,1]
[1,] 0.681
> parainde=fitinde$par
> parainde[,3]=exp(parainde[,3])
#gibbs
> fitgibbs=gibbs(g,start,10000,scale =c(0.15,1.5,0.55), data)
> fitgibbs$accept
          [,1]     [,2]     [,3]
[1,] 0.221 0.2424 0.2272
> paragibbs=fitgibbs$par
> paragibbs[,3]=exp(paragibbs[,3])
```

用随机游走的模拟结果做展示：



2.4 d

三种方法构建的 90%置信区间：

```
> apply(pararw,2,quantile,c(.05,.95))
                  [,1]        [,2]        [,3]
5%    -0.00628621 2.536748 1.109721
95%    0.39289331 5.383972 3.390467
> apply(parainde,2,quantile,c(.05,.95))
                  [,1]        [,2]        [,3]
5%    -0.01250405 2.504812 1.070034
95%    0.35470770 4.801247 3.039811
> apply(paragibbs,2,quantile,c(.05,.95))
               [,1]        [,2]        [,3]
5%    0.0010025 2.563410 1.109809
95% 0.4288908 5.704552 3.821411
```

2.5 e

经过查阅资料我们发现当 lambda=0 时 Box-Cox 方法为 log 变换，通过随机游走和独立性采样得到的后验置信区间里都包含有 0，所以我们认为，对于这组数据做 log 变换更加合适。

3.Is a basketball player streaky? (8.4)

Is a basketball player streaky?
Kobe Bryant is one of the most famous players in professional basketball. Shooting data were obtained for Bryant for the first 15 games in the 2006 season. For game i, one records the number of field goal attempts n_i and the number of successful field goals y_i; the data are displayed in Table 8.6. If p_i denotes the probability that Kobe makes a shot during the ith game, it is of interest to compare the nonstreaky hypothesis

$$M_0 : p_1 = ... = p_{15} = p, \ p \sim \text{uniform}(0, 1)$$

against the streaky hypothesis that the p_i vary according to a beta distribution

$$M_K : p_1, ..., p_{15} \text{ random sample from beta}(K\eta, K(1-\eta)), \eta \sim \text{uniform}(0, 1).$$

Use the function `laplace` together with the function `bfexch` to compute the logarithm of the Bayes factor in support of the streaky hypothesis M_K. Compute the log of the Bayes factors for values of K = 10, 20, 50, and 100. Based on your work, is there much evidence that Bryant displayed true streakiness in his shooting performance in these 15 games?

Table 8.6. Shooting data for Kobe Bryant for the first 15 games during the 2006 basketball season.

Game	(y, n)	Game	(y, n)
1	(8, 15)	9	(12, 23)
2	(4, 10)	10	(9, 18)
3	(5, 7)	11	(8, 24)
4	(12, 19)	12	(7, 23)
5	(5, 11)	13	(19, 26)
6	(7, 17)	14	(11, 23)
7	(10, 19)	15	(7, 16)
8	(5, 14)		

本题目，我们来判断 Kobe 投球的命中率是否遵循非条纹/条纹假设，我们仿照 8.7 节对 baseball 的做法。由于两个题目的基本问题是一致的，我们采取 8.7 节推导出的结果。
我们先构建了 bfexch 函数来计算贝叶斯因子B_k：

```
bfexch=function (theta, datapar)
  {
    y = datapar$data[, 1]
    n = datapar$data[, 2]
    K = datapar$K
    eta = exp(theta)/(1 + exp(theta))
    logf = function(K, eta, y, n)
```

```
      lbeta(K * eta + y, K * (1 - eta) + n - y) -
      lbeta(K * eta, K * (1 - eta))
    sum(logf(K, eta, y, n)) + log(eta * (1 - eta)) -
      lbeta(sum(y) + 1, sum(n - y) + 1)
}
```

导入数据，设置本题的精度参数 K:

```
data=cbind(c(8,4,5,12,5,7,10,5,12,9,8,7,19,11,7),
           c(15,10,7,19,11,17,19,14,23,18,24,23,26,23,16))
k=c(10,20,50,100)
```

为了计算特定 K 值的贝叶斯因子B_k，我们使用函数 laplace，输入函数 bfexch，起始值 η=0。为了方便使用将其封装为函数。

```
log.marg=function(k){
  laplace(bfexch,0,list(data=data,K=k))$int
}
```

使用 sapply 函数，我们计算不同 K 值下的 log.BF，之后计算出贝叶斯因子的值，最后输出展示:

```
> log.BF=sapply(k,log.marg)
> BF=exp(log.BF)
> round(data.frame(k,log.BF,BF),2)
    k log.BF   BF
1  10  -1.42 0.24
2  20  -0.09 0.91
3  50   0.39 1.48
4 100   0.34 1.40
```

从输出结果我们可以看出，K=50 时贝叶斯因子值最大$B_k = 1.48$，并且该条纹模型几乎是一致性模型的 1.5 倍。这表明了 Kobe 确实表现出了一种条纹模型的特性。

4.Test of independence (example from Agresti and Franklin (2005)) (8.5)

The 2002 General Social Survey asked the question "Taken all together, would you say that you are very happy, pretty happy, or not too happy?" The survey also asked "Compared with American families in general, would you say that your family income is below average, average, or above average?" Table 8.7 cross-tabulates the answers to these two questions.

Table 8.7. Happiness and family income from 2002 General Social Survey.

	Happiness		
Income	Not Too Happy	Pretty Happy	Very Happy
Above Average	17	90	51
Average	45	265	143
Below Average	31	139	71

a) Using the Pearson chi-square statistic, use the function `chisq.test` to test the hypothesis that happiness and family income are independent. Based on the p-value, is there evidence to suggest that the level of happiness is dependent on the family income?

b) Consider two models, a "dependence model" where the underlying multinomial probability vector is uniformly distributed and an "independence model" where the cell probabilities satisfy an independence configuration and the marginal probability vectors have uniform distributions. Using the R function `ctable`, compute the Bayes factor in support of the dependence hypothesis.

c) Instead of the analysis in part (b), suppose that one wishes to compare the independence model with the "close to independence" model M_K described in Section 8.8. Using the function `bfindep`, compute the Bayes factor in support of the model M_K for values of $\log K = 2$, 3, 4, 5, 6, and 7.

d) Compare the frequentist measure of evidence against independence with the Bayesian measures of evidence computed in parts (b) and (c). Which type of measure, frequentist or Bayesian, indicates more evidence against independence?

4.1 a

第一问让我们对该列表做列联表独立性检验。我们仿照 8.8 节，先构造列联表，之后使用函数 chisq.test 求出相关性系数。P 值为 0.808，可以看出幸福与收入的相关性不显著。

```
> rm(list=ls())
> data=matrix(c(17,90,51,45,265,143,31,139,71),c(3,3),byrow = T)
> data
     [,1] [,2] [,3]
[1,]   17   90   51
[2,]   45  265  143
[3,]   31  139   71
> chisq.test(data)

	Pearson's Chi-squared test

data:  data
X-squared = 1.6047, df = 4, p-value = 0.808
```

4.2 b

仿照 8.8 节，我们可以看出 bayes 因子的值仅为 0.00066，说明 dependence model 更加符合实际。

```
> a=matrix(rep(1,9),c(3,3))
> a
     [,1] [,2] [,3]
[1,]    1    1    1
[2,]    1    1    1
[3,]    1    1    1
> ctable(data, a)
[1] 0.0006591673
```

4.3 c

同样，仿照 8.8 节，我们根据不同的迪利克雷精度参数 K 计算相应的贝叶斯因子。可以看到在本例中最大的 bayes 因子为 0.43，当 logk=7 时，说明此取值下，“close to independence”模型最符合实际。

```
> log.K=seq(2,7)
> compute.log.BF=function(log.K)log(bfindep(data,exp(log.K),100000)$bf)
> log.BF=sapply(log.K,compute.log.BF)
> BF=exp(log.BF)
> round(data.frame(log.K,log.BF,BF),2)
  log.K log.BF   BF
1     2 -15.06 0.00
2     3 -10.61 0.00
3     4  -6.49 0.00
4     5  -3.41 0.03
5     6  -1.83 0.16
```

6	7	-0.83	0.43