Teoria informacji kolokwium 4 grudnia 2024

Piotr Styczyński[1]

[1]MIM UW (ps386038)

2024-12-08 (updated: 2024-12-08)

1 Problem 1

Intuitivelly choosing F gives us the values for each X while X only says about some of the values. We can construct such set L that $f_i(x) = x \iff x \leq k$ for $k = \inf\{k : (n-k)! \geq n\}$ For such a contruct $X = x \wedge x \leq k \implies Y = y = x$ If $k = n$ then we have just $X = x \implies Y = y$ That is discussed later in examples section.

Now let's take a look at the fact that:

$$\begin{cases} I(Y;X) = H(Y) - H(Y \mid X) \\ I(Y;F) = H(Y) - H(Y \mid F) \end{cases} \tag{1}$$

$$\begin{gathered} I(Y;X) \leq I(Y;F) \\ -H(Y \mid X) \leq -H(Y \mid F) \end{gathered}$$

$$\begin{aligned} &\sum_x Pr(X) * \sum_y Pr(Y = y \mid X = x) * \log Pr(Y = y \mid X = x) \leq \\ &\leq \sum_f Pr(F) * \sum_y Pr(Y = y \mid F = f) * \log Pr(Y = y \mid F = f) \end{aligned}$$

We can get rid of any of the $Pr(X)$ and $Pr(F)$:

$$\begin{aligned} &\sum_x \sum_y Pr(Y = y \mid X = x) * \log Pr(Y = y \mid X = x) \leq \\ &\leq \sum_f \sum_y Pr(Y = y \mid F = f) * \log Pr(Y = y \mid F = f) \end{aligned}$$

We expand the definition using $Pr(b) = \sum_a Pr(a) * Pr(b \mid a)$:

$$\begin{aligned} &\sum_x \sum_y \{(\sum_f Pr(F) * Pr(Y = y \mid X = x, F = f)) * \log \sum_f Pr(F) * Pr(Y = y \mid X = x, F = f)\} \leq \\ &\leq \sum_f \sum_y \{(\sum_x Pr(X) * Pr(Y = y \mid F = f, X = x)) * \log \sum_x Pr(X) * Pr(Y = y \mid F = f, X = x)\} \end{aligned}$$

We can get rid of $Pr(F)$ and $Pr(X)$:

$$\sum_x \sum_y \{(\sum_f Pr(Y=y \mid X=x, F=f)) * \log \sum_f Pr(F) * Pr(Y=y \mid X=x, F=f)\} \leq$$
$$\leq \sum_f \sum_y \{(\sum_x Pr(Y=y \mid F=f, X=x)) * \log \sum_x Pr(X) * Pr(Y=y \mid F=f, X=x)\}$$

If we define $a_{x,y,z} := Pr(Y=y \mid X=x, F=f)$ and $k = 1/n$ rename z as follows: $z := f$, we get:

$$\sum_x \sum_y \{(\sum_z a_{x,y,z}) * \log \sum_z k * a_{x,y,z}\} \leq$$
$$\leq \sum_z \sum_y \{(\sum_x a_{x,y,z}) * \log \sum_x k * a_{x,y,z}\}$$

$$\sum_x \sum_y \{k * (\sum_z a_{x,y,z}) + (\sum_z a_{x,y,z}) * \log \sum_z a_{x,y,z}\} \leq$$
$$\leq \sum_z \sum_y \{k * (\sum_x a_{x,y,z}) + (\sum_x a_{x,y,z}) * \log \sum_x a_{x,y,z}\}$$

The first term in inner sum dissappear:

$$\sum_x \sum_y \{(\sum_z a_{x,y,z}) * \log \sum_z a_{x,y,z}\} \leq$$
$$\leq \sum_z \sum_y \{(\sum_x a_{x,y,z}) * \log \sum_x a_{x,y,z}\}$$

$$\sum_x \sum_y \{(\sum_z a_{x,y,z}) * \log \sum_z a_{x,y,z}\} \leq$$
$$\leq \sum_z \sum_y \{(\sum_z a_{x,y,z}) * \log \sum_x a_{x,y,z}\}$$

We use Golden lemma here for property that (for iteration over x) each function is permutation so each value will occur once and (for iteration over z) we will have multiple non-zero elements.

We can observe that Golden Lemma says that euality holds if respective values are equal. In that case it can be implied by having symmetrical matrix (special cases are discussed below. Such case can correspond to $|S_n| = |L|$ with $X \perp F$)

1.1 Examples

We observe that for $X = F$ we alsways have equality in all of the cases.

Now, if we assume $X \perp F$ and $|S_n| = |L|$ then there are $(n-1)!$ permutations for each pair that sends x into y:

$$Pr(Y=y \mid X=x) = Pr(F(x)=y \mid X=x) = Pr(F(x)=y) = 1/n \tag{2}$$

That is not dependent on x which shows that $Y \perp X$ and we can also show that $Y \perp F$ In that case we have equality, because the mutual information is zeroed. The first condition implies that this happens for $n = 1$ and $n = 2$

In more general case for any n and $F = g(X)$ we have equality, because $Y = g(X)(X)$ and both terms are zeroed.