

Teoria informacji kolokwium 4 grudnia 2024

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1 Problem 2

1.1 Task 1

We observe that for binary input and perfect channel combined in series with Z-channel we have:

$$\begin{cases} P_A(\alpha \rightarrow \alpha) = 1 \\ P_A(\beta \rightarrow \beta) = 1 \\ P_B(\alpha \rightarrow \alpha) = 1 - p \\ P_B(\alpha \rightarrow \beta) = p \\ P_B(\beta \rightarrow \beta) = 1 \end{cases} \quad (1)$$

$Pr(B | A, C)$ won't be well defined for a channel matrix, because:

$$\begin{cases} Pr(B = \alpha | A = \alpha, C = \beta) = 1 \\ Pr(B = \alpha | A = \alpha, C = \alpha) = 1 \end{cases} \quad (2)$$

Hence the requirement that there are no zero elements in matrices of both channels will come into play. We consider the probability to be well defined if:

1. $\exists_{f \perp P(A)} \forall_{\substack{a \in A \\ c \in C}} Pr(B = b | A = a, C = c) = f(...)$
2. $\forall_{\substack{a \in A \\ c \in C}} 0 < Pr(B = b | A = a, C = c) < 1$
3. $\forall_{\substack{a \in A \\ c \in C}} \sum_{b \in B} Pr(B = b | A = a, C = c) = 1$

Now let's start by expanding the probability using Bayes' theorem:

$$Pr(b | a, c) = \frac{Pr(a, c | b) * Pr(b)}{Pr(a, c)} \quad (3)$$

For numerator we use the fact that $Pr(x, y | z) = Pr(x | y, z) * Pr(y | z)$ and for enumerator we use $Pr(x, y) = Pr(x | y) * Pr(y)$:

$$Pr(b | a, c) = \frac{Pr(a | b, c) * Pr(b) * Pr(c | b)}{Pr(a | c) * Pr(c)} \quad (4)$$

Now we use conditional independence constraint so $Pr(a | b, c) = Pr(a | b)$:

$$Pr(b | a, c) = \frac{Pr(a | b) * Pr(b) * Pr(c | b)}{Pr(a | c) * Pr(c)} \quad (5)$$

We use the fact that $Pr(x | y) = Pr(y | x) * \frac{Pr(x)}{Pr(y)}$:

$$Pr(b | a, c) = \frac{Pr(b | a) * Pr(a) * Pr(b) * Pr(c | b)}{Pr(a | c) * Pr(c) * Pr(b)} \quad (6)$$

After reorganising terms:

$$Pr(b | a, c) = (Pr(b | a) * Pr(c | b)) * \frac{Pr(a)}{Pr(a | c)} * \frac{1}{Pr(c)} \quad (7)$$

Now let's observe that $\frac{Pr(x)}{Pr(x | y)} = \frac{Pr(y)}{Pr(y | x)}$:

$$Pr(b | a, c) = (Pr(b | a) * Pr(c | b)) * \frac{Pr(c)}{Pr(c | a)} * \frac{1}{Pr(c)} \quad (8)$$

This can be written as:

$$Pr(b | a, c) = P_{\Gamma_1}(a \rightarrow b) * P_{\Gamma_2}(b \rightarrow c) * \frac{1}{Pr(c | a)} \quad (9)$$

We observe that $Pr(c | a)$ in fact does not depend on the distribution of A , which proves first point. In all our transformations we implicitly use the facts that $Pr(a), Pr(b), Pr(c) \neq 0$ Second point is trivial to show, because $0 < P_{\Gamma_1}(a \rightarrow b), P_{\Gamma_2}(b \rightarrow c), Pr(c | a) < 1$ Third point is direct consequence of conditional independence.

1.2 Task 2

Let's assume we take channel Γ_E and squeeze the symbol space using mapping $s_1 \in (A \times C)^A := s_1(a, c) = a$

In such scenario:

$$\begin{aligned} P_{\Gamma_E}(s_1(a, c) \rightarrow b) &= \sum_c P_{\Gamma_E}(a, c \rightarrow b) \\ &= \sum_c P_{\Gamma_1}(a \rightarrow b) * P_{\Gamma_2}(b \rightarrow c) * \frac{1}{Pr(c | a)} \\ P_{\Gamma_F}(a \rightarrow b) &= P_{\Gamma_1}(a \rightarrow b) * \sum_c \frac{P_{\Gamma_2}(b \rightarrow c)}{Pr(c | a)} \end{aligned}$$

We know that $\sum_c Pr(c | a) = 1$

If we compare $P_{\Gamma_F}(a \rightarrow b)$ and $P_{\Gamma_1}(a \rightarrow b)$ then in $I(A, B)$ we will notice that in worst case we can select such a coefficients that Γ_F is as bad as using Γ_1 The original matrix Γ_F has more rows than Γ_E (such that rows of Γ_F are linear combinations of Γ_E) meaning that the capacity of original Γ_E cannot be lower.

Hence we prove that capacity of Γ_E is no lower than Γ_F

1.3 Task 3

My best guess is that we use some prime factors to make Γ_E have all rows independent, then that will work. However I was unable to showcase some concrete example.