We define $Q_c(z) = z^2 + c$ and

$$Q_c^m(z) = \overbrace{(Q_c \circ \cdots \circ Q_c)}^{m \text{ times}}(z)$$

Note that

$$Q_c^{n+m}(z) = Q_c^n(Q_c^m(z)) \tag{1}$$

Theorem 1 (Triangle Inequalities)**.** *For $z, w \in C$,*

$$|z + w| \leq |z| + |w| \tag{2}$$

$$|z + w| \geq |z| - |w| \tag{3}$$

Lemma 1. *Let $z, c \in \mathbf{C}$ and $k = \max(|c|, 2)$. Then for any m*

$$|z| > k \implies |Q_c^m(z)| > k$$

Proof. If $|z| > k$ then $|z| > |c|$ and $|z| > 2$. This implies that there is some $\varepsilon > 0$ such that,

$$|z| - 1 = 1 + \varepsilon > 1$$

By Theorem 1,

$$|Q_c(z)| = |z^2 + c| \geq |z^2| - |c|$$

Since $|z| > |c|$, we have,

$$\begin{aligned} |Q_c(z)| &> |z^2| - |z| \\ &= (|z| - 1)|z| \\ &= (1 + \varepsilon)|z| > z > k \end{aligned} \tag{4}$$

Since $Q_c^n(z) = Q_c(Q_c^{n-1}(z))$, we have,

$$Q_c^m(z) > Q_c^{m-1}(z) > \cdots > Q_c^2(z) > Q_c^1(z) > z > k \quad \square \tag{5}$$

**Is this already sufficient to prove Theorem 2?*

Theorem 2 (Escape Criteria)**.** *Let $z, c \in \mathbf{C}$ and $k = \max(|c|, 2)$. Then if $|Q_c^m(z)| > k$ for any m, then $|Q_c^n(z)|$ is unbounded as $n \to \infty$.*