HW #8; date: Sept. 26, 2017
MATH 110 Linear Algebra
with Professor Stankova

Section 2.4

Exercise 2.4.1:

(a) False: it should be $([T]_\alpha^\beta)^{-1} = [T^{-1}]_\beta^\alpha$.

(b) True: this follows easily from the definition.

(c) False; $T : V \to W$ while $L_A : \mathbb{F}^{\dim(V)} \to \mathbb{F}^{\dim(W)}$.

(d) False: $\dim(M_{2\times 3}(F)) = 6 \neq 5 = \dim(F^5)$.

(e) True: two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.

(f) False: this could happen when A and B are not square matrices.

(g) True: by definition.

(h) True: by Theorem 2.18 of the textbook.

(i) True: otherwise it will fails to be one-to-one or onto.

Exercise 2.4.2:

(b) T cannot be invertible as $\dim(\mathbb{R}^2) = 2 \neq 3 = \dim(\mathbb{R}^3)$.

(d) T cannot be invertible as $\dim(P_3(\mathbb{R})) = 4 \neq 3 = \dim(P_2(\mathbb{R}))$.

(f) T is invertible as it is one-to-one; as the domain and co-domain spaces have the same dimension, it follows that it is invertible. To see it is one-to-one, suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \ker(T)$; then, $\begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow a = 0,\ a+b = 0 \Rightarrow b = 0,\ c = 0,\ c+d = 0 \Rightarrow d = 0$. Thus, T has a trivial kernel.

Exercise 2.4.3:
Recall that two finite-dimensional vector spaces are isomorphic if and only if they have the same dimension. By checking dimensions, we deduce that:

(a) is not isomorphic: $\dim(F^3) = 3 \neq 4 = \dim(P_3(F))$.

(b) is isomorphic: $\dim(F^4) = 4 = \dim(P_3(F))$.

(c) is isomorphic: $\dim(M_{2\times 2(\mathbb{R})}) = 4 = \dim(P_3(\mathbb{R}))$.

(d) is not isomorphic: $\dim(V) = 3 \neq 4 = \dim(\mathbb{R}^4)$.

Exercise 2.4.4: *Let A and B be $n \times n$ invertible matrices. Prove that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.*

Proof. We first note that the matrices AB and $B^{-1}A^{-1}$ are $n \times n$ matrices so we only need to prove that

$$(AB)(B^{-1}A^{-1}) = I,$$

where I denotes the $n \times n$ identity matrix. We can readily check this by using the associativity of matrix multiplication (Theorem 2.16) and the fact that $IC = CI = C$ for all $C \in \mathrm{M}_{n\times n}(\mathbb{R})$ (Theorem 2.12(c)):

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

□

Exercise 2.4.5: *Let A be invertible. Prove that A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.*

Proof. We know that $A \cdot A^{-1} = I$, where I is the identity matrix. Taking transpose on both sides, we get $(A \cdot A^{-1})^t = (A^{-1})^t \cdot A^t = I^t = I$. As A^t is a square matrix, we conclude that A^t is invertible with inverse $(A^{-1})^t$.

□

Exercise 2.4.6: *Prove that if A is invertible and $AB = O$, then $B = O$.*

Proof. Multiplying both sides of the equation $AB = O$ on the left by the inverse A^{-1} of A, we obtain

$$A^{-1}AB = A^{-1}O \implies IB = B = O$$

as desired; here I denotes the identity matrix of the same dimensions as A.

□

Exercise 2.4.7:

(a) *Suppose that $A^2 = O$. Prove that A is not invertible.*

Proof. Suppose A is invertible, then by Exercise 6 (with $B = A$), we get $A = O$. This is a contradiction, since O is not invertible. Therefore, A cannot be invertible.

□

(b) *Suppose that $AB = O$ for some non-zero $n \times n$ matrix B. Could A be invertible? Explain.*

Proof. If A were invertible, then, by Exercise 6, we would have $B = O$. This is a contradiction and therefore A cannot be invertible.

□

Exercise 2.4.9: *Let A and B be $n \times n$ matrices such that AB is invertible. Prove that A and B are invertible. Give an example to show that arbitrary matrices A and B need not be invertible if AB is invertible.*

Proof. We first prove that A and B are invertible under the original hypotheses. We shall do this by considering the corresponding left-multiplication transformations $L_A,\ L_B : F^n \to F^n$ and using Corollary 2 on page 102 of the text: A and B are invertible matrices if and only if L_A and L_B are invertible linear transformations.

By the same Corollary, our assumption that AB is invertible tells us that $L_{AB} = L_A L_B$ is an invertible linear transformation. It follows from Exercise 2.3.12 that L_A is onto and L_B is one-one. As the dimensions of the domain and co-domain are the same, we infer that both L_A and L_B are invertible. By the discussion above, we conclude that A and B are invertible matrices.

□

To wrap up this exercise, we construct an example to show that for arbitrary (*i.e.*, not necessarily square) matrices A and B with AB invertible, A and B need not be invertible. Take

$$A = \begin{pmatrix} 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then A and B are not square matrices, so they cannot be invertible; on the other hand, $AB = \begin{pmatrix} 1 \end{pmatrix} = I_1$, the 1×1 identity matrix, which is invertible.

Exercise 2.4.13: *Let $\sim$ mean "is isomorphic to". Prove that $\sim$ is an equivalence relation on the class of vector spaces over F.*

Proof. We need to show that the isomorphism relation $\sim$ between vector spaces is reflexive, symmetric, and transitive:

- First, we have that $V \sim V$ by identity map.
- Next, if $V \sim W$, then there exists a linear isomorphism $T : V \to W$. As T^{-1} exists and is an isomorphism map from W to V, we have $W \sim V$.
- Finally, if $V \sim W$ and $W \sim Z$, there exist linear isomorphism $T : V \to W$ and $U : W \to Z$. Then $U \circ T : V \to Z$ is a linear isomorphism so that $V \sim Z$.

□

Exercise 2.4.14: *Let $V = \left\{ \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} : a, b, c \in F \right\}$. Construct an isomorphism from V to F^3.*

To construct an isomorphism between two vector spaces of the same finite dimension, it suffices to construct a bijective map between their bases.

Note that V is three-dimensional and $\alpha = \left\{ v_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is a basis of V. Let $\beta = \{e_1, e_2, e_3\}$ denote the standard basis of $\mathbb{F}^3$.

Let $T : V \to \mathbb{F}^3$ be the map $T(v_i) = e_i, i = 1, 2, 3$. For any $v \in V$, v is a linear combination of the basis elements in α, i.e. $v = a_1v_1 + a_2v_2 + a_3v_3$ for some scalars a_1, a_2, a_3. Then $T(v) = a_1e_1 + a_2e_2 + a_3e_3$. This linear map is one-to-one and onto.

Exercise 2.4.15: *Let V and W be n-dimensional vector spaces, and let $T : V \to W$ be a linear transformation. Suppose that β is a basis for V. Prove that T is an isomorphism if and only if $T(\beta)$ is a basis for W.*

Proof. We first prove the "only if" implication. So assume that $T : V \to W$ is an isomorphism; we first claim that then $T(\beta)$ must be a linearly independent set of n vectors in W. To that end, write $\beta = \{v_1, \ldots, v_n\}$. Because T is one-to-one, the vectors $T(v_k)$ are *distinct* for $1 \leq k \leq n$, and thus $T(\beta) = \{T(v_1), \ldots, T(v_n)\}$ is a set of n vectors in W. Now suppose that some linear combination of these vectors is equal to the zero vector of W:

$$a_1T(v_1) + \ldots + a_nT(v_n) = \mathit{0}$$

for some scalars $a_1, \ldots, a_n \in \mathbb{R}$. It suffices to prove that $a_k = 0$ for all k. Now, again because T is a one-to-one linear transformation, we have $\ker(T) = \{\mathit{0}\}$; thus,

$$a_1T(v_1) + \ldots + a_nT(v_n) = T(a_1v_1 + \ldots + a_nv_n) = \mathit{0} \implies a_1v_1 + \ldots + a_nv_n = \mathit{0}.$$

Since β is assumed to be a basis, in particular the v_k are mutually linearly independent; thus, we must have $a_k = 0$ for all k as desired. So $T(\beta)$ is a linearly independent set of n vectors in W; because $\dim(W) = n$, $T(\beta)$ is a basis for W.

Now let us prove the "if" implication. To that end, assume that $T(\beta) = \{T(v_1), \ldots, T(v_n)\}$ is a basis for W. Let $y \in W$ be arbitrary. Using the assumed linearity of T, we can find scalars $b_1, \ldots, b_n \in \mathbb{R}$ such that

$$y = b_1T(v_1) + \ldots + b_nT(v_n) = T(b_1v_1 + \ldots + b_nv_n) = T(x),$$

where $x = b_1v_1 + \ldots + b_nv_n \in V$. So we've shown that for all $y \in W$, $y \in \operatorname{Im}(T)$. In other words, T is onto. By the Dimension Theorem, because $\dim(V) = \dim(W)$, T must also be one-to-one; thus, T is an isomorphism.

□

Exercise 2.4.16: *Let B be an $n \times n$ invertible matrix. Define $\Phi : \mathrm{M}_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ by $\Phi(A) = B^{-1}AB$. Prove that Φ is an isomorphism.*

Proof. We first show that Φ is a *linear* transformation. Let $A_1, A_2 \in M_{n\times n}(\mathbb{R})$ and $c \in \mathbb{R}$ be arbitrary. Then

$$\Phi(cA_1 + A_2) = B^{-1}(cA_1 + A_2)B = (cB^{-1}A_1 + B^{-1}A_2)B = cB^{-1}A_1B + B^{-1}A_2B = c\Phi(A_1) + \Phi(A_2),$$

so Φ is linear. Next, let $C \in M_{n\times n}(\mathbb{R})$ be any $n \times n$ matrix. Then

$$\Phi(BCB^{-1}) = B^{-1}(BCB^{-1})B = (B^{-1}B)\,C\,(B^{-1}B) = I_n\,C\,I_n = C,$$

where I_n denotes the $n \times n$ identity matrix. In particular, $C \in \operatorname{Im}(\Phi)$; as C was arbitrary, Φ is onto. Since Φ is a linear transformation from the finite-dimensional vector space $M_{n\times n}(\mathbb{R})$ to itself, by the Dimension Theorem Φ must also be one-to-one. Thus, Φ is an isomorphism.

□

<u>Exercise 2.4.17</u>: *Let V and W be finite-dimensional vector spaces and $T : V \to W$ be an isomorphism. Let V_0 be a subspace of V.*

(a) *Prove that $T(V_0)$ is a subspace of W.*

Proof. We need only use the fact that T is a linear transformation here; *i.e.*, we do not need to use the full fact that T is an *isomorphism*. Because V_0 is a subspace of V, the zero vector $\mathit{0}_V$ of V is an element of V_0. So $\mathit{0}_W = T(\mathit{0}_V) \in T(V_0)$. To see that $T(V_0)$ is closed under addition and scalar multiplication in W, let $w_1, w_2 \in T(V_0)$ and $c \in \mathbb{R}$ be arbitrary. Then there exist $v_1, v_2 \in V_0$ such that $T(v_1) = w_1$ and $T(v_2) = w_2$. Furthermore, because V_0 is a subspace of V, we have $cv_1 + v_2 \in V_0$. Thus,

$$cw_1 + w_2 = cT(v_1) + T(v_2) = T(cv_1 + v_2) \in T(V_0).$$

As c, w_1, and w_2 were arbitrary, $T(V_0)$ is closed under addition and scalar multiplication in W. Thus, $T(V_0)$ is a subspace of W.

□

(b) *Prove that* $\dim(V_0) = \dim\big(T(V_0)\big)$.

Proof. Here we need to use the full fact that T is an isomorphism. Consider the *restriction* T_R of T to the subspace V_0. The restriction of a linear transformation to a subspace of its domain was defined in a previous homework; in this case we may view it as the linear transformation

$$T_R : V_0 \longrightarrow T(V_0)$$

defined by $T_R(x) := T(x)$ for all $x \in V_0$. By definition, $\mathrm{Im}\big(T_R\big) = T(V_0)$, so T_R is onto. Moreover, if $x \in \ker(T_R)$, then

$$\mathit{0}_W = T_R(x) = T(x) \implies x = \mathit{0}_V,$$

because T is an isomorphism, hence one-to-one. Thus, T_R is one-to-one; since we also noted that it's onto, $T_R : V_0 \to T(V_0)$ is an isomorphism. In other words, V_0 and $T(V_0)$ are isomorphic vector spaces and, by Theorem 2.19, $\dim(V_0) = \dim\big(T(V_0)\big)$.

□

<u>Exercise 2.4.19(a)</u>:
We first show that T is a linear isomorphism. T is linear because of the properties of the transpose operation. T is an isomorphism as T is one-to-one and onto.
Next we compute (a) the matrix $[T]_\beta$. $T(E^{11}) = E^{11}, T(E^{12}) = E^{21}, T(E^{21}) = E^{12}, T(E^{22}) = E^{22}$.
Therefore, we get $[T]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

<u>Exercise 2.4.23</u>: *Let V denote the vector space defined in Example 5 of Section 1.2, and let $W = \mathrm{P}(\mathbb{R})$. Define*

$$T : V \to W \quad \textit{by} \quad T(\sigma) = \sum_{i=0}^{n} \sigma(i) x^i,$$

where n is the largest integer such that $\sigma(n) \neq 0$. Prove that T is an isomorphism.

Remark 1: To be utterly pedantic, the statement of the exercise does not define $T(0_V)$. (The zero vector 0_V of V is the sequence σ_0 defined by $\sigma_0(m) = 0$ for all m, so it does not make sense to say "n is the largest integer such that $\sigma_0(n) \neq 0$.") So if we want to be completely precise, we should further define $T(0_V) = 0_W$, where 0_W is the zero polynomial in $\mathrm{P}(\mathbb{R})$. This exercise is also confusing as stated because Example 5 in Section 1.2 defines V to be the space of functions from the *positive* integers to $\mathbb{R}$; thus, "$\sigma(0)$" does not make sense for $\sigma \in V$ under that definition. The natural thing to do is to modify the definition of V slightly, taking V to be the vector space of functions from the *nonnegative* integers to $\mathbb{R}$. (N.B.: The two definitions yield *isomorphic* vector spaces. To check your understanding, find a reasonably obvious isomorphism from "the new V" to "the old V.") We'll use that modified definition of V for the following proof.

Remark 2: Note that V and W here are *infinite-dimensional* vector spaces; accordingly, we cannot hope to make any use of the Dimension Theorem or other finite-dimensional techniques.

Proof. We first show that T is a linear transformation. Let $c \in \mathbb{R}$ and $\sigma, \eta \in V$ be arbitrary. Let N be the largest integer such that *either* $\sigma(N) \neq 0$ or $\eta(N) \neq 0$; if $\sigma = \eta = 0_V$, then set $N = 0$. Then

$$T(c\sigma + \eta) = \sum_{i=0}^{N} (c\sigma + \eta)(i)x^i = c\sum_{i=0}^{N} \sigma(i)x^i + \sum_{i=0}^{N} \eta(i)x^i = cT(\sigma) + T(\eta),$$

so T is linear.

Next, we show T is one-to-one. Indeed, suppose $T(\sigma) = 0_W$, the zero polynomial. Then by definition of the zero polynomial and the map T, we know that:

- $\sigma(n) = 0$ for all integers $n > 0$, and
- $\sigma(0) = 0$.

In other words, $\sigma(n) = 0$ for all nonnegative integers n, and thus $\sigma = 0_V$. So $\ker(T) = \{0_V\}$, and T is one-to-one.

Finally, let $f \in W = \mathrm{P}(\mathbb{R})$ be an arbitrary polynomial, and write

$$f(x) = \sum_{i=0}^{n} a_i x^i,$$

so that $\deg(f) = n$. Define a sequence $\sigma \in V$ by

$$\sigma(k) = \begin{cases} a_k & \text{for } k \leq n \\ 0 & \text{for } k > n. \end{cases}$$

Then $T(\sigma) = f$; since $f \in W$ was arbitrary, T is onto.

Thus, we've shown that T is a linear transformation that is both one-to-one and onto; in other words, T is an isomorphism.

□