Final Exam. Linear Algebra 110

Instructor: Zvezdelina Stankova

STUDENT NAME: ______________________________ STUDENT ID: ____________________

GSI'S NAME: ____

• **DO NOT OPEN THE EXAM UNTIL TOLD TO DO SO!**

• PREPARE YOUR STUDENT PHOTO ID TO SHOW TO YOUR PROCTOR.

• Do *all* problems as best as you can. The exam is 2 hrs and 50 min long. For security reasons, you may leave **neither** during the first 30 minutes **nor** during the last 30 minutes of the exam: no restroom breaks and no early submission of exams during those times. No exceptions.

• Use the provided sheets to write your solutions. You may use the back of each page for the remainder of your solutions; in such a case, put an arrow at the bottom of the page and indicate that the solution continues on the back page. Before the test starts, pull out and use several of your own sheets for scratch work. **No extra sheets of paper can be submitted with this exam due to the grading computer software!** Thus, use the front and the back of the sheets in the exam for the actual solutions, and not for scratch work.

• The exam is closed notes and books, which means: **no classnotes, no session notes, no review notes, no textbooks, and no other materials can be used during the exam.** You can use only your **cheat sheet**, this exam packet and additional blank paper provided by the instructor. You cannot use your paper to write on, even if it is only scratch work. The cheat sheet is two sides of one regular 8×11 sheet, handwritten only by you. A cheat sheet which doesn't conform to these specifications will be disqualified, and the student's exam may be annulled.

• **NO CALCULATORS ARE ALLOWED DURING THE EXAM!**

• **You cannot ask for help in any form from other students, you cannot look at their exams and copy, you cannot cheat in any way – the exam is an individual assignment and must be done only by you.** If you have a question regarding the statement of a problem, raise your hand when the instructor or a GSI is around.

• Think of this exam as an important homework. Check your reasoning and calculations very carefully. Justify *all* your answers, include all intermediate steps and calculations. If you are not sure about how to write something in mathematical notation, explain clearly *in words* what you mean and what you are doing. **Unjustified answers, even if correct, will receive no credit!** On the contrary, good justifications and good work on a problem may receive a lot of credit even if the final answer is incorrect.

• Before turning in your exam, please, sign the statement below. **Exams that are not signed will not be graded.**

I, the student whose name and signature appear on this exam, have completed the exam without any outside help from people or other sources. I have used only my cheat sheet conforming to the specifications written above. I have not cheated in any way, and I have followed all instructions above. I have submitted all solutions and work wished to be graded in the present exam packet.

STUDENT SIGNATURE: ____

Problem 1. (10pts) Let $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $T(p(x)) = p'(x) - p''(x)$.

(a) Prove that T is a linear transformation.

(*Note:* You may assume here well-known properties of derivatives.)

(b) In the standard bases β of $P_3(\mathbb{R})$ and γ of $P_2(\mathbb{R})$ (start β and γ with the highest powers of x), find the matrix $[T]_\beta^\gamma$. Show all calculations.

(c) Describe $\operatorname{Ker} T$. Exhibit a basis for $\operatorname{Ker} T$. What is $\operatorname{rank}(T)$? Explain.

$\operatorname{rank}(T) =$

Problem 2 (15pts) Let V be a finite-dimensional space and T be a linear operator on V.

(a) Define what a generalized eigenspace $K_\lambda(T)$ is.

(b) If $U = T - \lambda I$ for some eigenvalue λ of T, what is the stabilizing constant l for λ? Why does it exist? Explain.

(c) What are the dot diagram and the Jordan Canonical Form (JCF) for T if $V = K_6(T)$ and $r_1 = 5$, $r_2 = 9$, $r_3 = 13$, and $r_4 = 14 = r_5$, where $r_j = \dim \operatorname{Ker} U^j$ for $U = T - 6I$ and $j \geq 1$? (*Note:* You may simplify JCF by appropriate notation $J_k(\lambda)$ for Jordan blocks.)

Problem 3. (15pts) Let V be a vector space over $\mathbb{C}$.

(a) Define what an inner product $\langle\ ,\ \rangle$ on V is.

(b) For some two vectors $\vec{w}_1$ and $\vec{w}_2$ in V we know that $\langle \vec{x}, \vec{w}_1 \rangle = \langle \vec{x}, \vec{w}_2 \rangle$ for all $\vec{x} \in V$, where $\langle\ ,\ \rangle$ is some inner product on V. Prove that $\vec{w}_1 = \vec{w}_2$.

Problem 4. (20pts) Let $V_{\langle\ ,\ \rangle}$ be a non-zero finite-dimensional inner-product space over $\mathbb{C}$.

(a) Does V always have an orthonormal basis? Explain and cite all relevant theorems.

(b) Let $f : V \to \mathbb{C}$ be a linear transformation. By following the steps below, prove that there is a unique $\vec{y} \in V$ for which $f(\vec{x}) = \langle \vec{x}, \vec{y} \rangle$ for all $\vec{x} \in V$.

Start with an orthonormal basis $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for V.

- If $\vec{y}$ is such a vector, write $\vec{y}$ in basis β. What is $f(\vec{v}_i)$?

  $f(\vec{v}_i) =$

- If the desired $\vec{y}$ exists, then $\vec{y} =$

Your formula must **not** contain the coordinates of $\vec{y}$. It must depend on f and β.

- Did you just prove that $\vec{y}$ exists or that it is unique? Answer: ________________
- Finally, show that the $\vec{y}$ you found above actually works for *all* $\vec{x} \in V$.

- Did you just prove that $\vec{y}$ exists or that it is unique? Answer: ________________

Problem 5. (20pts) Let $V_{\langle\ ,\ \rangle}$ be an inner product vector space over $\mathbb{C}$. Let T be a linear operator. If $U : V \to V$ is a function that satisfies:

$$\langle T(\vec{x}), \vec{y}\rangle = \langle \vec{x}, U(\vec{y})\rangle \text{ for all } \vec{x}, \vec{y} \in V,$$

prove that U is also a linear transformation.

(*Note:* You cannot use that U may be equal to the adjoint T^*, assuming the latter exists at all, or that we may have proven this in class, etc. You need to assume only what is given in this problem, and if helpful, also result(s) of other problem(s) on this final exam.)

Problem 6 (20pts). True or False? You must completely bubble (fill in) one of the circles or leave blank. To discourage guessing, the problem will be graded as follows:

- 2 pt for each correct answer. • 0 pts for a blank. • -2 pt for each incorrect answer.
- If anything is written other than the bubbling, more than one circle is bubbled (or partially bubbled), or it is hard to tell which answer is bubbled, you will get -2 points.

Notes: A statement is TRUE if it is true in *all* cases that satisfy its hypothesis. A statement is FALSE if it is false even in just *one* case that satisfies its hypothesis, regardless of what happens in all other cases. If a statement claims that several things are true, but in reality even one of them is false, then the whole statement is FALSE.

(1) There is a linear operator T on some vector space V_F with *no* T-invariant subspaces.
Ⓣ Ⓕ

(2) Any linear operator on an n-dimensional vector space that has fewer than n *distinct* eigenvalues is *not* diagonalizable.
Ⓣ Ⓕ

(3) If B is a matrix obtained by interchanging two rows or two columns of a square matrix A, then $\det A = \det B$.
Ⓣ Ⓕ

(4) If the *homogenous* system corresponding to a given system of linear equations has a solution, then the given system *has* a solution.
Ⓣ Ⓕ

(5) Just like diagonal matrices, *Jordan matrices* of same sizes are relatively easy to work with because their determinants, traces, and characteristic polynomials are easy compute, even if they do *not* necessarily *commute* with each other and even if their powers may require some more computation to be found.
Ⓣ Ⓕ

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(6) If V is a finite-dimensional vector space over $\mathbb{C}$, then the map $^* : \mathcal{L}(V,V) \to \mathcal{L}(V,V)$ sending any linear transformation T to its adjoint T^* is itself *not* linear.
Ⓣ Ⓕ

(7) If an upper-triangular matrix A is *unitary*, then its *diagonal* entries must satisfy $|a_{ii}| = 1$ for all i.
Ⓣ Ⓕ

(8) For a real 3×4 matrix A, the function $q(\vec{x}) = ||A\vec{x}||^2$ cannot define a *positive semi-definite quadratic form* on $\mathbb{R}^4$ because A is *not* square and hence it is *non-invertible.*
Ⓣ Ⓕ

(9) If $\mathbb{R}^n$ is considered as an inner product space with respect to the standard dot product, the *Gramian* of any n vectors in $\mathbb{R}^n$ can always be represented as a product of a some matrix and its *transpose*, even in the case when the vectors are linearly *dependent.*
Ⓣ Ⓕ

(10) There are square matrices over $\mathbb{C}$ that are *symmetric* but *not* diagonalizable.
Ⓣ Ⓕ