HW #14; date: Oct. 17, 2017
MATH 110 Linear Algebra
with Professor Stankova

5.2 #1a-g (a) False. Take the identity matrix. (b) False. They can be scalar multiples of each other. (c) True by definition. (d) True. If v had two eigenvalues, λ and μ, then $Av = \lambda v = \mu v$, so $(\lambda - \mu)v = 0$, so $\lambda - \mu = 0$. (e) True. (f) True. It means there is a basis of eigenvectors. (g) True. If $A = PDP^{-1}$, then Pe_1 is an eigenvector with eigenvalue the first diagonal entry of D.

5.2 #2bdf (b) The characteristic polynomial is $(1-t)^2 - 9 = t^2 - 2t - 8 = (t-4)(t+2)$. The matrix is diagonalizable. The eigenvalues are $4, -2$ with eigenvectors $(1,1)^t, (1,-1)^t$ respectively. Q is the matrix whose columns are the eigenvectors, so $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. (d) The characteristic polynomial is $(7-t)(-5-t)(3-t)+4(8(3-t)) = (3-t)((7-t)(-5-t)+32) = (3-t)(t^2-2t+3)$. The polynomial t^2-2t+3 has no real roots, so the matrix is not diagonalizble. (f) The matrix is not diagonalizable since for the eigenvalue 1, which has multiplicity two, $A - I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, and the nullspace of this matrix has dimension 1.

5.2 #3abe (a) In the standard basis, this linear transformation is strictly upper triangular with zeroes on the diagonal (since derivatives lower the degree of polynomials). Thus the only eigenvalue is zero. However this is not the zero transformation, so it is not diagonalizable. (b) By inspection, we have $T(x^1 + 1) = x^2 + 1$, $T(x) = x$, and $T(x^2 - 1) = -(x^2 - 1)$. Thus it is diagonalizable, with basis $\beta = \{x^2 + 1, x, x^2 - 1\}$. (e) The matrix for this transformation under the standard basis is $\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$. The characteristic polynomial is $(1-t)^2+1 = t^2-2t+2$. The roots are $t = 1 \pm i$. The corresponding eigenvectors are $(1,1)^t$ and $(1,-1)^t$, so this gives a basis β.

5.2 #7 First diagonalize A. Suppose we did, $A = PDP^{-1}$, then $A^n = (PDP^{-1})^n = PD^nP^{-1}$. It has characteristic polynomial $(1-t)(3-t)-8 = t^2-4t-5 = (t-5)(t+1)$, so eigenvalues $5, -1$, with corresponding eigenvectors $(1,1)^t$ and $(2,-1)^t$. So, we have $P = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, D = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$, and $P^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$. So,

$$A^n = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 5^n & 0 \\ 0 & (-1)^n \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 5^n + 2(-1)^n & 2(5)^n - 2(-1)^n \\ 5^n - (-1)^n & 2(5)^n + (-1)^n \end{pmatrix}$$

5.2 #8 A is diagonalizable if and only if it has a basis of eigenvectors. Since $\dim(E_{\lambda_1}) = n-1$, there are $n-1$ linearly independent eigenvectors for the eigenvalue λ_1, say $v_1, \ldots, v_{n-1}$. Since λ_2

is an eigenvalue, it has an eigenvector v_n, which, and since it has a different eigenvalue it is linearly independent to the other v_i, provides us with the desired basis.

5.2 #9 (a) Let $A = [T]_\beta$. Then $A - tI$ is still upper triangular, with diagonal entries $d_i - t$, so $\det(A - tI) = (d_1 - t) \cdots (d_n - t)$, which is split. (b) Take $T = L_A$.

5.2 #10 Let d_i be the diagonal entries. By the previous problem, the characteristic polynomial is $(d_1 - t) \cdots (d_n - t)$. However since the eigenvalues are λ_i with multiplicity m_i, the λ_i must appear as roots in the characteristic polynomial with multiplicity m_i, so m_i of the d_j are equal to λ_i.

5.2 #11 (a) Note that $\mathrm{tr}(A) = \mathrm{tr}(PAP^{-1})$; this is because $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, so $\mathrm{tr}(PAP^{-1}) = \mathrm{tr}(AP^{-1}P) = \mathrm{tr}(A)$. Thus, the trace of A is the trace of the upper triangular matrix it is similar to. By the previous problem, we just sum up the diagonal entries, which we know to be the eigenvalues with proper multiplicities, i.e. $\sum m_i \lambda_i$. (b) Since $\det(A) = \det(PAP^{-1})$, and for an upper triangular matrix the determinant is the product of the diagonal entries, and the diagonal entries of the upper triangular PAP^{-1} are the eigenvalues with multiplicities, the result follows.

5.2 #12 (a) Suppose that $Tx = \lambda x$. Then $T^{-1}Tx = T^{-1}\lambda x$, so $x = \lambda T^{-1}x$, so $T^{-1}x = \lambda^{-1}x$. Thus if x is an eigenvector of T with eigenvalue λ, then x is an eigenvector of T^{-1} with eigenvalue λ^{-1}. The converse follows by reciprocity. (b) If we have a basis of eigenvectors of T, by part (a) that same set is a basis of eigenvectors of T^{-1}.

5.2 #13 (a) Take $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. For eigenvalue $\lambda = 0$, the eigenvector for A is the span of $(1, -1)^t$ and for A^t is the span of $(0, 1)^t$. (b) We want to show that the dimensions of $N(A - \lambda I)$ and $N(A^t - \lambda I)$ are equal. Equivalently, by rank nullity, we can show that they have the same rank. However, $(A - \lambda I)^t = A^t - \lambda I$, and the rank of a matrix is equal to the rank of its transpose. (c) If A is diagonalizable, with eigenvalues $\lambda_1, \ldots, \lambda_k$, then $\sum \dim(E_{\lambda_i}) = \dim(V)$. By (b) the same is true for E'_{λ_i}; in particular, there is a basis of eigenvectors for A^t. and so A^t is diagonalizable.

In-class 2c Diagonalize A so that $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix} \frac{-1}{2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$. Then

$$A^{2016} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 5^{2016} & 0 \\ 0 & (-2)^{2016} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 5^{2016} + 2^{2016} & 2^{2016} - 2^{2016} \\ 5^{2016} - 2^{2016} & 5^{2016} + 2^{2016} \end{pmatrix}$$