HW #4; date: Feb 2, 2017
MATH 110 Linear Algebra
with Professor Stankova

1.6 Bases and Dimension

1. (a) False. The empty set is a basis for the zero vector space.

   (b) True. It follows from Theorem 1.10.

   (c) False. See Example 5.

   (d) False. The sets $\{1\}$ and $\{2\}$ are different bases of $\mathbb{R}^1$.

   (e) True. It is Corollary 1 to Theorem 1.10.

   (f) False. The dimension is $n+1$.

   (g) False. The dimension is mn.

   (h) True. It follows from Corollary 2 to Theorem 1.10.

   (i) False. In $\mathbb{R}^1$, the subset $\{1,2\}$ spans $\mathbb{R}^1$, but 2 can be expressed two different linear combinations $2\times 1+0\times 2$ and $0\times 1+1\times 2$.

   (k) True. It is Theorem 1.11.

   (l) True. It follows from Corollary 2 to Theorem 1.10.

2. (b) Since
   $$3(2,-4,1)+4(0,3,-1)-(6,0,-1)=0,$$
   the set is linearly dependent. Thus it is not a basis for $\mathbb{R}^3$.

   (e) Since
   $$4(1,3,-2)+2(-3,1,3)-(-2,-10,-2)=0,$$
   the set is linearly dependent. Thus it is not a basis for $\mathbb{R}^3$.

3. (c) The reduced row echelon form of the matrix
   $$\begin{pmatrix} 1 & -2 & -2 \\ -2 & 3 & -1 \\ 1 & -1 & 6 \end{pmatrix}$$
   is the identity matrix, so the set is a basis for $\mathbb{P}_2(\mathbb{R})$.

   (e) Since
   $$7(1+2x-x^2)-2(4-2x+x^2)-(-1+18x-9x^2)=0,$$
   the set is linearly dependent. Thus it is not a basis for $\mathbb{R}^3$.

4. No. The dimension of $\mathbb{P}_3(\mathbb{R})$ is four, and three elements cannot span four dimensional vector space.

5. No. The dimension of $\mathbb{R}^3$ is three, and four elements in three dimensional vector space cannot be linearly independent.

6. The sets
$$\{(1,0),(0,1)\}, \quad \{(1,1),(0,1)\}, \quad \{(1,0),(1,1)\}$$
are bases for F^2. Using Example 14 as a guide, the sets
$$\{E_{11},E_{12},E_{21},E_{22}\}, \quad \{E_{11},E_{12},E_{21},E_{11}+E_{22}\}, \quad \{E_{11},E_{12},E_{21},E_{12}+E_{22}\}$$
are bases for $\mathrm{M}_{2\times 2}(F)$ where E_{ij} denotes the matrix whose only nonzero entry is 1 at (i,j).

7. The set $\{u_1,u_2,u_3\}$ is a basis for $\mathbb{R}^3$ since the reduced row echelon form of the matrix
$$\begin{pmatrix} 2 & -3 & 1 \\ 1 & 37 & -17 \\ -3 & -5 & 8 \end{pmatrix}$$
is the identity matrix.

8. The reduced row echelon form of the matrix
$$\begin{pmatrix} 2 & -6 & 3 & 2 & -1 & 0 & 1 & 2 \\ -3 & 9 & -2 & -8 & 1 & -3 & 0 & -1 \\ 4 & -12 & 7 & 2 & 2 & -18 & -2 & 1 \\ -5 & 15 & -9 & -2 & 1 & 9 & 3 & -9 \\ 2 & -6 & 1 & 6 & -3 & 12 & -2 & 7 \end{pmatrix}$$
is
$$\begin{pmatrix} 1 & -3 & 0 & 4 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$
The first, third, fifth, and seventh columns are pivot columns. Thus by an algorithm in Math 54, the set $\{u_1,u_3,u_5,u_7\}$ is a basis for W.

9. We have
$$(a_1,a_2,a_3,a_4) = (a_1,a_2,a_3,a_4) = a_1u_1+(a_2-a_1)u_2+(a_3-a_2)u_3+(a_4-a_3)u_4.$$

11. Both of the reduced row echelon forms of the matrices
$$\begin{pmatrix} 1 & 1 \\ a & 0 \end{pmatrix}, \quad \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$
are the identity matrix. Thus the sets are bases for V.

12. The reduced row echelon form of the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

is the identity matrix. Thus the set is a basis for V.

13. The solution of the system is $x_1 = x_2 = x_3$. Thus the set $\{(1,1,1)\}$ is a basis for this subspace of $\mathbb{R}^3$.

14. First consider W_1. Here a_2 and a_5 may be arbitrary, so $(0,1,0,0,0)$ and $(0,0,0,0,1)$ satisfy the equation trivially, and by inspection $(1,0,1,0,0)$ and $(1,0,0,1,0)$ also solve the equation. This set is linearly independent, and any other solution of W_1 may be written in terms of this set, hence it is a basis, and thus W_1 has dimension 4.

    Now consider W_2. The two equations are independent, and may be solved uniquely and independently: we see that $(0,1,1,1,0)$ and $(1,0,0,0,-1)$ solve the equations, and any other solution of the equations is a linearly combination of these solutions. Moreover, they are linearly independent and thus form a basis. We conclude W_2 has dimension 2.

22. If $W_1 \cap W_2 \neq W_1$, choose a basis S for $W_1 \cap W_2$, and extend it to a basis T for W_1. Since S does not span W_1, $S \neq T$. Thus $\dim(W_1 \cap W_2) \neq \dim W_1$. On the other hand, if $\dim(W_1 \cap W_2) \neq \dim W_1$, then $W_1 \cap W_2 \neq W_1$. From these, we conclude that the necessary and sufficient condition for $\dim(W_1 \cap W_2) = \dim W_1$ is $W_1 \cap W_2 = W_1$, which is equivalent to $W_1 \subset W_2$. In other words, v needs to be dependent on the vectors $\{v_1, \ldots, v_k\}$.

23. (a) Since $W_1 \subset W_2$, by the solution of the above problem, $\dim W_1 = \dim W_2$ if and only if $W_1 = W_2$. This is equivalent to $v \in W_1$.

    (b) Let S be a basis for W_1. Since $S \cup \{v\}$ spans W_2, $\dim W_2$ is less than or equal to the number of elements of $S \cup \{v\}$, which is $\dim W_1 + 1$. Thus the relation is $\dim W_2 = \dim W_1 + 1$ since we have assumed $\dim W_2 \neq \dim W_1$.

24. The polynomials $f, f', f'', \ldots, f^{(n)}$ all have different degrees, so they are linearly independent. This means that they form a basis for $\mathbb{P}_n(\mathbb{R})$ since its dimension is $n+1$. The statement follows from this.

26. Consider the linear transformation $T : \mathbb{P}_n(\mathbb{R}) \to \mathbb{R}$ given by

$$T(f) = f(a).$$

    Then it is surjective since $T(c) = c$ for constants c, and its kernel is the subspace in the problem. By Theorem 2.3, the dimension of the subspace is $\dim \mathbb{P}_n - \dim \mathbb{R} = n$.

    Alternatively, one sees (using the Taylor expansion of a polynomial at a) that $1, x-a, (x-a)^2, \ldots, (x-a)^n$ is a basis of $P_n(\mathbb{R})$. The condition that $f(a) = 0$ then states merely that the coefficient of 1 is zero, so that basis of this subspaces is $\{x-a, (x-a)^2, \ldots, (x-a)^n\}$.

28. If $\{v_1, \ldots, v_n\}$ is a basis for V when V is regarded as a vector space over $\mathbb{C}$, then $\{v_1, \ldots, v_n, iv_1, \ldots, iv_n\}$ is a basis for V when V is regarded as a vector space over $\mathbb{R}$. Thus the dimension becomes $2n$.

    Explicitly, if $v \in V$ has unique $\mathbb{C}$-linear combination of vectors in $\{v_1, \ldots, v_n\}$ as $\sum_{j=1}^{n}(a_j + b_j i)v_j$ then, expanding out the summands, gives the unique $\mathbb{R}$-linear combination of v in the $\mathbb{R}$-basis $\{v_1, \ldots, v_n, iv_1, \ldots, iv_n\}$ as $v = \sum_{j=1}^{n}(a_j v_j + b_j(iv_j))$.

29. (b) By Exercise 29(a), $\dim W_1 + \dim W_2 = \dim V$ if and only if $\dim(W_1 \cap W_2) = 0$. This is equivalent to $W_1 \cap W_2 = \{0\}$, which is equivalent to $V = W_1 \oplus W_2$ since we have assumed $V = W_1 + W_2$.

30. We have $\dim W_1 = 3$ and $\dim W_2 = 2$. Indeed, the number of independent linear conditions imposed on V to obtain W_1 is 1 (that $a_{1,1} = a_{1,2}$) and the number of conditions on W_2 imposed is 2 (that $a_{1,1} = 0$ and that $a_{1,2} = -a_{2,1}$). Since imposing n independent linear conditions on a space reduces the dimension by n, the claim follows.

    The intersection $W_1 \cap W_2$ is

$$\left\{ \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} : a \in F \right\},$$

    so its dimension is 1—an obvious basis being given by $\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$. Thus $\dim(W_1 + W_2) = 3 + 2 - 1 = 4$ by Exercise 29(a).

31. (a) It follows from $\dim(W_1 \cap W_2) \le \dim W_2$.

    (b) By Exercise 29(a),

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) \le \dim W_1 + \dim W_2 = m + n.$$

    Alternatively, it's clear that if $\{v_1, \ldots, v_k\}$ spans W_1 and $\{u_1, \ldots, u_\ell\}$ spans W_2 then $\{v_1, \ldots, v_k, u_1, \ldots, u_\ell\}$ spans $W_1 + W_2$. Thus, taking theses spanning sets to be bases shows that there is a spanning set of $W_1 + W_2$ of size $\dim(W_1) + \dim(W_2)$ so that $\dim(W_1 + W_2) \leqslant \dim(W_1) + \dim(W_2)$.

34. (a) Let $S = \{v_1, \ldots, v_m\}$ be a basis of W_1, and extend it to a basis $T = \{v_1, \ldots, v_m, w_1, \ldots, w_n\}$ of V. Consider the subspace $W_2 = \operatorname{span}\{w_1, \ldots, w_n\}$. Then $W_1 + W_2 = V$ since T spans V, and $W_1 \cap W_2 = \{0\}$ since T is linearly independent. Thus $V = W_1 \oplus W_2$.

    (b) Consider the subspaces

$$W_2 = \operatorname{span}\{(0,1)\}, \quad W_2' = \operatorname{span}\{(1,1)\}.$$

Then the sets $\{(1,0),(0,1)\}$ and $\{(1,0),(1,1)\}$ are bases for $\mathbb{R}^2$, so $W_1 + W_2 = V$, and $W_1 + W_2' = V$. Moreover, $W_1 \cap W_2 = \{0\}$, $W_1 \cap W_2' = \{0\}$. Thus $V = W_1 \oplus W_2 = W_1 \oplus W_2'$.

In fact, one can see that if v is any vector not in in W_1 then $\text{span}\{v\}$ will be a subspace of V whose direct sum with W_1 is V. Indeed, the intersection $\text{span}\{v\} \cap W_1 = \{0\}$ since if $\alpha v \in W_1$ and $\alpha \neq 0$ then, by the vector space property for W_1, we'd have that $\alpha^{-1}(\alpha v) = v \in W_1$ which is a contradiction. Thus, we see that $W_1 \cap \text{span}\{v\} = \{0\}$. But, we then get from Exercise 29 (a) that $\dim(W_1 + \text{span}\{v\}) = 1 + 1 - 0 = 2$. Since $\dim \mathbb{R}^2 = 2$ this implies that $W_1 + \text{span}\{v\} = \mathbb{R}^2$ so that $W_1 \oplus \text{span}(\{v\}) = \mathbb{R}^2$ as desired.