HW #6; date: September 19, 2017
MATH 110 Linear Algebra
with Professor Stankova

2.2 #1 (a) True. Check the condition for a linear transformation: $(aT+U)(c_1v_1+c_2v_2) = aT(c_1v_1+c_2v_2) + U(c_1v_1+c_2v_2) = a(c_1T(v_1)+c_2T(v_2)) + c_1U(v_1)+c_2U(v_2) = ac_1T(v_1)+c_1U(v_1)+ac_2T(v_2)+c_2U(v_2) = c_1(aT+U)(v_1)+c_2(aT+U)(v_2)$. (b) True, because every vector $x \in V$ can be written as a linear combination of basis vectors by definition. (c) False. It is a $n \times m$ matrix. (d) True. For any linear transformation T, $[T]_\beta^\gamma$ is determined uniquely by the property that $[T]_\beta^\gamma([x]_\beta) = [T(x)]_\gamma$. One can check that $[T+U]_\beta^\gamma([x]_\beta) = [(T+U)(x)]_\beta^\gamma = [T(x)+U(x)]_\beta^\gamma = [T(x)]_\beta^\gamma + [U(x)]_\beta^\gamma = ([T]_\beta^\gamma + [U]_\beta^\gamma)([x]_\beta)$. (e) True by part a. (f) False by definition.

2.2 #2(b)(f) (b) $\begin{pmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \cdots \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$.

2.2 #3 We compute $[T(b_1)]_\gamma = [(1,1,2)]_\gamma = (-1/3, 0, 2/3)$, and $[T(b_2)]_\gamma = [(-1,0,1)]_\gamma = (-1,1,0)$. Rearranging in columns, we find $[T]_\beta^\gamma = \begin{pmatrix} -1/3 & -1 \\ 0 & 1 \\ 2/3 & 0 \end{pmatrix}$. Similaly, we compute $[T(a_1)]_\gamma = [T(1,2)]_\gamma = [(-1,1,4)]_\gamma = (-7/3, 2, 2/3)$ and $[T(a_2)]_\gamma = [T(2,3)]_\gamma = [(-1,2,7)]_\gamma = (-11/3, 3, 4/3)$. So $[T]_\alpha^\gamma = \begin{pmatrix} -7/2 & -11/3 \\ 2 & 3 \\ 2/3 & 4/3 \end{pmatrix}$.

2.2, #4 If $M \in M_{2\times 2}(\mathbb{R})$ satisfies $[M]_\beta = (a,b,c,d)$, then $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $[T(M)]_\gamma = [(a+b) + 2dx + bx^2]_\gamma = (a+b, 2d, b)$. So we have the equation $[T]_\beta^\gamma \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a+b \\ 2d \\ b \end{pmatrix}$, meaning $[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

2.2, #5 (a) The transformation swaps the second and third basis vectors, fixing the first and fourth: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (b) We find $T(1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, $T(x) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$, $T(x^2) = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$. Thus, $[T]_\beta^\alpha =$

$\begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ (c) $\operatorname{tr}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = a+d$, so $[T]_\alpha^\gamma = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$. (d) $T(a+bx+cx^2) = a+2b+4c$. Thus, $[T]_\beta^\gamma = \begin{pmatrix} 1 & 2 & 4 \end{pmatrix}$. (e) $[A]_\alpha = (1, -2, 0, 4)$ (f) $[f(x)]_\beta = (3, -6, 1)$ (g) $[a]_\gamma = a$.

2.2, #8 Suppose that the vector x has coordinates $T(x) = (x_1, \dots, x_n)$ and the vector y has coordinates $T(y) = (y_1, \dots, y_n)$. It suffices to show that $cx + dy$ has coordinates $(cx_1 + dy_1, \dots, cx_n + dy_n)$. To see this note that $x = x_1 b_1 + \dots x_n b_n$ by definition of coordinates, and $y = y_1 b_1 + \cdots y_n b_n$ as well. Then, $cx + dy = (cx_1 + dy_1)b_1 + \cdots (cx_n + dy_n)b_n$.

2.2, #9 $T(a + bi) = a - bi$ by definition. Take $x = a + bi$ and $y = c + di$, and e, f real scalars. Then, $T(ex + fy) = T(ea + ebi + fc + fdi) = T((ea + fc) + (eb + fd)i) = ea + fc - (eb + fd)i = ea - ebi + fc - fdi = eT(x) + fT(y)$ as desired. We have $[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

2.2, #10 We find that $[T(v_j)]_\beta = [v_j + v_{j-1}]_\beta = e_j + e_{j-1}$, where e_k is the column vector with a 1 in the kth component. Thus, $[T]_\beta = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$.

2.2, #13 In order for $\{T, U\}$ to be linearly dependent, we need constants a, b such that $aT + bU = 0$. Since neither T nor U are zero, we know that $a \neq 0$ and $b \neq 0$. Then, write $T = (-b/a)U$. Choose $x \in V$ such that $T(x) \neq 0$ (it exists since T is nonzero). Then, $T(x) \in R(T)$, but also $U(-b/a \cdot x) = -b/a \cdot U(x) = T(x)$, so $T(x) \in R(U)$. This contradicts that $R(T) \cap R(U) = \{0\}$.

2.2, #14 Suppose that we had a linear relation $a_0 T_0 + \cdots a_n T_n = 0$. This means that there are constants a_i (not all zero) such that for all polynomials f, we have $(a_0 f + a_1 f' + \cdots + a_n f^{(n)})(x) = 0$. Let d be the minimal i such that $a_i \neq 0$, and take $f(x) = x^d$. Then, $a_i f^{(i)} = 0$ for $i < d$ by assumption, but also for $i > d$ since the derivatives of f of order higher than d vanish. Thus, $(a_0 f + a_1 f' + \cdots + a_n f^{(n)})(x) = d!$, a constant function, contradicting the claim that it is zero on all f.

2.2, #16 Choose the basis $\beta = \{b_1, \dots, b_r, b_{r+1}, \dots, b_n\}$ such that $b_1, \dots, b_r$ is a basis for the nullspace of T, and $b_{r+1}, \dots, b_n$ complete this to a basis of V in any way. Choose $\gamma = \{c_1, \dots, c_r, c_{r+1}, \dots, c_n\}$ such that $c_i = T(b_i)$ for $i \geq r+1$, and choose $c_1, \dots, c_r$ to complete this to a basis of W in any way. We have to check, however, that $T(b_i)$ for $i \geq r+1$ are linearly independent. Suppose not; then there is a relation $\sum s_i T(b_i) = T(\sum s_i b_i) = 0$. Thus, $\sum_{i=r+1}^n s_i b_i \in N(T)$, but we assumed that $b_1, \dots, b_r$ was a basis for the nullspace, so this can only be true if the coefficients are all zero. Given this claim we can verify that $[T(b_i)]_\gamma = 0$ for $i \leq r$, and $[T(b_i)]_\gamma = e_i$ for $i \geq r+1$. Thus, $[T]_\beta^\gamma = \operatorname{diag}(0, 0, 0, 0, \dots, 1, 1, 1)$, i.e. the diagonal matrix whose diagonal has r zeroes followed by $n - r$ ones.