HW #9; date: Sept. 26, 2017
MATH 110 Linear Algebra
with Professor Stankova

Section 2.5

<u>**Exercise 2.5.1**</u>**:**

(a) False: it should be $[x'_j]_\beta$.

(b) True: this is Theorem 2.22.

(c) True: this is Theorem 2.23.

(d) False: it should be $B = Q^{-1}AQ$.

(e) True: this follows from the definition of similarity and Theorem 2.23.

<u>**Exercise 2.5.2**</u>**:** *For each of the following pairs of ordered bases β and β' for $\mathbb{R}^2$, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.*

(b) $\beta = \{(-1,3),(2,-1)\}$ *and* $\beta' = \{(0,10)\,,\,(5,0)\}$.

Note that $\left[\begin{pmatrix}0\\10\end{pmatrix}\right]_\beta = \begin{pmatrix}4\\2\end{pmatrix}$ and $\left[\begin{pmatrix}5\\0\end{pmatrix}\right]_\beta = \begin{pmatrix}1\\3\end{pmatrix}$ so we have $[I_{\mathbb{R}^2}]_{\beta'}^{\beta} = \begin{pmatrix}4&1\\2&3\end{pmatrix}$.

(c) $\beta = \{(2,5),(-1,-3)\}$ *and* $\beta' = \{(e_1,e_2\}$.

As $[I_{\mathbb{R}^2}]_{\beta}^{\beta'} = \begin{pmatrix}2&-1\\5&-3\end{pmatrix}$, we have

$$[I_{\mathbb{R}^2}]_{\beta'}^{\beta} = \left([I_{\mathbb{R}^2}]_{\beta}^{\beta'}\right)^{-1} = \frac{1}{(2)(-3)-(-1)(5)}\begin{pmatrix}-3&1\\-5&2\end{pmatrix} = \begin{pmatrix}3&-1\\5&-2\end{pmatrix}.$$

<u>**Exercise 2.5.3**</u>**:** *For each of the following pairs of ordered bases β and β' for $\mathrm{P}_2(\mathbb{R})$, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.*

(b) $\beta = \{1,x,x^2\}$ *and* $\beta' = \{a_2x^2+a_1x+a_0\,,\,b_2x^2+b_1x+b_0\,,\,c_2x^2+c_1x+c_0\}$.

Solution: We follow the same basic approach as in Exercise 2. The β-coordinate vectors of the basis vectors in β' can easily be computed by inspection in this case:

$$[a_2x^2+a_1x+a_0]_\beta = \begin{pmatrix}a_0\\a_1\\a_2\end{pmatrix},\ [b_2x^2+b_1x+b_0]_\beta = \begin{pmatrix}b_0\\b_1\\b_2\end{pmatrix},\ [c_2x^2+c_1x+c_0]_\beta = \begin{pmatrix}c_0\\c_1\\c_2\end{pmatrix}.$$

So the change-of-coordinates matrix is

$$Q = [I_{P_2(\mathbb{R})}]_{\beta'}^{\beta} = \begin{pmatrix}a_0&b_0&c_0\\a_1&b_1&c_1\\a_2&b_2&c_2\end{pmatrix}.$$

(d) $\beta = \{x^2 - x + 1\,,\, x + 1\,,\, x^2 + 1\}$ *and* $\beta' = \{x^2 + x + 4\,,\, 4x^2 - 3x + 2\,,\, 2x^2 + 3\}$.

Solution: For convenience, write $\beta' = \{f_1, f_2, f_3\}$ and $\beta = \{g_1, g_2, g_3\}$ (as ordered bases). Then we compute the coordinate vectors:

$$f_1 = a_1g_1 + a_2g_2 + a_3g_3 \iff \begin{cases} a_1 + a_3 &= 1 \\ -a_1 + a_2 &= 1 \\ a_1 + a_2 + a_3 &= 4 \end{cases} \iff [f_1]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix};$$

$$f_2 = b_1g_1 + b_2g_2 + b_3g_3 \iff \begin{cases} b_1 + b_3 &= 4 \\ -b_1 + b_2 &= -3 \\ b_1 + b_2 + b_3 &= 2 \end{cases} \iff [f_2]_\beta = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix};$$

$$f_3 = c_1g_1 + c_2g_2 + c_3g_3 \iff \begin{cases} c_1 + c_3 &= 2 \\ -c_1 + c_2 &= 0 \\ c_1 + c_2 + c_3 &= 3 \end{cases} \iff [f_2]_\beta = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

So the change-of-coordinates matrix is

$$Q = [\mathrm{I}_{\mathrm{P}_2(\mathbb{R})}]_{\beta'}^{\beta} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & 3 & 1 \end{pmatrix}.$$

(f) $\beta = \{2x^2 - x + 1\,,\, x^2 + 3x - 2\,,\, -x^2 + 2x + 1\}$ *and* $\beta' = \{9x - 9\,,\, x^2 + 21x - 2\,,\, 3x^2 + 5x + 2\}$.

Solution: Again, write $\beta' = \{f_1, f_2, f_3\}$ and $\beta = \{g_1, g_2, g_3\}$. Compute the coordinate vectors (in the interest of saving space I will not show my work in solving the systems of equations, but you should):

$$f_1 = a_1g_1 + a_2g_2 + a_3g_3 \iff \begin{cases} 2a_1 + a_2 - a_3 &= 0 \\ -a_1 + 3a_2 + 2a_3 &= 9 \\ a_1 - 2a_2 + a_3 &= -9 \end{cases} \iff [f_1]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix};$$

$$f_2 = b_1g_1 + b_2g_2 + b_3g_3 \iff \begin{cases} 2b_1 + b_2 - b_3 &= 1 \\ -b_1 + 3b_2 + 2b_3 &= 21 \\ b_1 - 2b_2 + b_3 &= -2 \end{cases} \iff [f_2]_\beta = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix};$$

$$f_3 = c_1g_1 + c_2g_2 + c_3g_3 \iff \begin{cases} 2c_1 + c_2 - c_3 &= 3 \\ -c_1 + 3c_2 + 2c_3 &= 5 \\ c_1 - 2c_2 + c_3 &= 2 \end{cases} \iff [f_3]_\beta = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

So the change-of-coordinates matrix is

$$Q = [\mathrm{I}_{\mathrm{P}_2(\mathbb{R})}]_{\beta'}^{\beta} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 2 \\ 3 & 4 & 1 \\ -1 & 5 & 2 \end{pmatrix}.$$

Exercise 2.5.4:

We know $[T]_\beta = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$. Thus, $[T]_{\beta'} = [I]_\beta^{\beta'}[T]_\beta[I]_{\beta'}^\beta = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 13 \\ -5 & 9 \end{pmatrix}$.

Exercise 2.5.5:

We know $[T]_\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Thus, $[T]_{\beta'} = [I]_\beta^{\beta'}[T]_\beta[I]_{\beta'}^\beta = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$.

Exercise 2.5.6: *For each matrix A and ordered basis β, find $[\mathrm{L}_A]_\beta$. Also, find an invertible matrix Q such that $[\mathrm{L}_A]_\beta = Q^{-1}AQ$.*

(b) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ *and* $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$.

Solution: For convenience, write $\beta = \{v_1, v_2\}$ (as an ordered basis). Then the j-th column of $[\mathrm{L}_A]_\beta$ is the β-coordinate vector $[\mathrm{L}_A(v_j)]_\beta$. So we compute

$$\mathrm{L}_A(v_1) = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3v_1 + 0v_2 \implies [\mathrm{L}_A(v_1)]_\beta = \begin{pmatrix} 3 \\ 0 \end{pmatrix};$$

$$\mathrm{L}_A(v_2) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0v_1 + (-1)v_2 \implies [\mathrm{L}_A(v_2)]_\beta = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

So

$$[\mathrm{L}_A]_\beta = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}.$$

By the Corollary on page 115 of the text, the change-of-coordinates matrix Q that changes β-coordinates into standard-basis-coordinates, given by

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

satisfies $[\mathrm{L}_A]_\beta = Q^{-1}AQ$.

(d) $A = \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix}$ *and* $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Solution: Again, for convenience write $\beta = \{v_1, v_2, v_3\}$. Compute the relevant β-coordinate vectors:

$$\mathrm{L}_A(v_1) = \begin{pmatrix} 6 \\ 6 \\ -12 \end{pmatrix} = 6v_1 + 0v_2 + 0v_3 \implies [\mathrm{L}_A(v_1)]_\beta = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix};$$

$$\mathrm{L}_A(v_2) = \begin{pmatrix} 12 \\ -12 \\ 0 \end{pmatrix} = 0v_1 + 12v_2 + 0v_3 \implies [\mathrm{L}_A(v_2)]_\beta = \begin{pmatrix} 0 \\ 12 \\ 0 \end{pmatrix};$$

$$\mathrm{L}_A(v_3) = \begin{pmatrix} 18 \\ 18 \\ 18 \end{pmatrix} = 0v_1 + 0v_2 + 18v_3 \implies [\mathrm{L}_A(v_3)]_\beta = \begin{pmatrix} 0 \\ 0 \\ 18 \end{pmatrix}.$$

So we have

$$[\mathrm{L}_A]_\beta = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 18 \end{pmatrix}.$$

Again using the Corollary on page 115 of the text, the change-of-coordinates matrix

$$Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{bmatrix}$$

satisfies $[\mathrm{L}_A]_\beta = Q^{-1}AQ$.

Remark: In both parts of this exercise, it was easy enough to solve (by inspection) the systems of linear equations required to obtain the relevant coordinate vectors to construct $[\mathrm{L}_A]_\beta$. On the other hand, if the vectors of the basis β are explicitly presented to you in standard coordinates, the matrix Q is *always* easy to construct: its columns are simply the standard coordinate vectors of the basis vectors from β, arranged in their proper order. So in general, you have a choice of which computation you'd rather do:

(1) solve the linear systems needed to compute the relevant coordinate vectors, or

(2) invert the matrix Q, by whatever technique you desire (probably either by row operations or by Cramer's rule).

Which option is the less tedious varies on a case-by-case basis, of course.

Challenge Question Exercise 2.5.7:

(a) The reflection matrix has a very simple form with respect to the basis $\alpha = \{(1, m), (-m, 1)\}$. The first vector in α is parallel to the line L, and the second vector is perpendicular to L. Therefore $[T]_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We will find the matrix for T with respect to the standard basis β of $\mathbb{R}^2$ so as to find an expression for $T(x, y)$.

$$[T]_\beta = [I]_\alpha^\beta [T]_\alpha [I]_\beta^\alpha = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}^{-1} = \frac{1}{m^2+1} \begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}.$$

And we compute:

$$[T]_\beta \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{m^2+1} \begin{pmatrix} (1-m^2)x + 2my \\ 2mx + (m^2-1)y \end{pmatrix} \implies \mathrm{T}(x, y) = \left(\frac{(1-m^2)x + 2my}{m^2+1}, \frac{2mx + (m^2-1)y}{m^2+1} \right).$$

(b) We can use the same basis α as in part (a). The projection will collapse a vector perpendicular to L to the zero vector, hence we have $[T]_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then:

$$[T]_\beta = [I]^\beta_\alpha [T]_\alpha [I]^\alpha_\beta = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}^{-1} = \frac{1}{m^2+1} \begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix}.$$

So we just compute

$$[T]_\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{m^2+1} \begin{pmatrix} x+my \\ mx+m^2y \end{pmatrix} \implies \mathrm{T}(x,y) = \left(\frac{x+my}{m^2+1}, \frac{mx+m^2y}{m^2+1} \right).$$

Remark: If you have learned elsewhere how to take orthogonal projections using "dot products," you can verify that $\mathrm{T}(x, y)$ is indeed the same orthogonal projection of (x, y) to L as computed using the dot-product method.

Exercise 2.5.9: *Prove that similarity is an equivalence relation on $M_{n\times n}(\mathbb{F})$.*

Proof. We may denote that "A is similar to B" by $A \sim B$. First A is similar to itself, since $A = I^{-1}AI$. Second, if $A \sim B$, then $A = Q^{-1}BQ$ for some $Q \in M_{n\times n}(\mathbb{F})$. This gives $QAQ^{-1} = B$. Letting $P = Q^{-1}$, we have $P^{-1} = Q$, so $B = P^{-1}AP$, and so $B \sim A$. Third, if $A \sim B$, $B \sim C$, then $A = Q^{-1}BQ$ and $B = P^{-1}CP$ for some invertible matrices P, Q. Combining these two expressions, we get

$$A = Q^{-1}(P^{-1}CP)Q = (Q^{-1}P^{-1})C(PQ) = (PQ)^{-1}C(PQ).$$

Therefore $A \sim C$.

□

Exercise 2.5.10: *Prove that if A and B are similar $n \times n$ matrices, then* $\mathrm{tr}(A) = \mathrm{tr}(B)$. Hint: *Use Exercise 13 of Section 2.3.*

Proof. Exercise 13 of Section 2.3 gives the following "commutativity" property of the trace: For any $C, D \in \mathrm{M}_{n\times n}(\mathbb{R})$, we have $\mathrm{tr}(CD) = \mathrm{tr}(DC)$. Since A and B are similar matrices, there exists some invertible $n\times n$ matrix Q such $B = Q^{-1}AQ$, by definition. So we just compute, using the "commutativity" property:

$$\mathrm{tr}(B) = \mathrm{tr}\left(Q^{-1}(AQ)\right) = \mathrm{tr}\left((AQ)Q^{-1}\right) = \mathrm{tr}\left(A(QQ^{-1})\right) = \mathrm{tr}(AI) = \mathrm{tr}(A),$$

where I denotes the $n \times n$ identity matrix, as desired.

□

Challenge: *Prove that if A and B are similar $n \times n$ matrices, then there is a linear operator T on F^n such that A and B are matrices of T with respect to some bases in F^n.*

Proof. Observe first that A is the matrix representation of L_A in the standard basis $e = \{e_1, \dots, e_n\}$. We are also given that $B = Q^{-1}AQ$ for some invertible $n \times n$ matrix Q. Define the vectors $\beta_1, \dots, \beta_n$ by

$$\beta_i = Qe_i \text{ for } i = 1, \dots, n.$$

As Q is invertible, it preserves linear independence so $\beta = \{\beta_1, \dots, \beta_n\}$ is another basis for F^n. Moreover, note that the change of coordinates matrix from β to e is given by

$$[I]_\beta^e = \begin{pmatrix} [\beta_1]_e & [\beta_2]_e & \dots & [\beta_n]_e \end{pmatrix} = Q$$

where we used the fact that $[\beta_i]_e$ is simply the ith column of Q. It follows from Theorem 2.23 that

$$[L_A]_\beta = [I]_e^\beta [L_A]_e [I]_\beta^e = Q^{-1}AQ = B.$$

Thus, A and B are merely two representations of the same linear transformation.

□