6.8 #18 Let $A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 3 \end{pmatrix}$ and $f = (2\sqrt{2}, 0, 2\sqrt{2})$. We want to solve the equation $x^t Ax + f \cdot x + 1 = 0$. Let us find an orthonormal basis that diagonalizes A. The eigenvalues are $2, 3, 4$ with corresponding eigenvectors $(1, 0, 1)$, $(0, 1, 0)$, and $(1, 0, -1)$. We can normalize them to $\frac{1}{\sqrt{2}}(1, 0, 1)$, $(0, 1, 0)$ and $\frac{1}{\sqrt{2}}(1, 0, -1)$. In this basis, the equation looks like $x^t \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} x + (4, 0, 0)x + 1 = 0$. If $x = (x_1, x_2, x_3)^t$, this is the equation $2x_1^2 + 3x_2^2 + 4x_3^2 + 4x_1 + 1 = 2(x_1 + 1)^2 + 3x_2^2 + 4x_3^2 - 1 = 0$. The shape is an ellipsoid centered at $(-1, 0, 0)$.

In-class challenge 1: Find the determinant of the matrix whose diagonal entries are p and off-diagonal entries are all q. Solution: Let the $n \times n$ matrix be A. Note that $A - (p - q)I$ is a matrix whose entries are all q. This matrix is rank 1, so the algebraic multiplicity of the eigenvalue $p - q$ is at least $n - 1$. Further, note that $A - (p + (n - 1)q)$ is a matrix whose entries are q off the diagonal, and $-(n - 1)q$ on the diagonal. Notice that the sum of the entries in each row (or each column) is zero, so they live in a subspace of F^n cut out by the equation $x_1 + x_2 + \cdots + x_n = 0$. So the columns cannot be a basis for F^n. So this matrix is not invertible, and $p + (n - 1)q$ is an eigenvalue. Its rank must be 1 since the sum of algebraic multiplicities cannot exceed n. Thus, the determinant is the product of all the roots of the characteristic polynomial, so it is $(p - q)^{n-1}(p + (n - 1)q)$.

In-class challenge 2: What conditions on p, q must be imposed so that A is positive definite? Solution: We need all the eigenvalues to be positive (and real), so we need $p - q > 0$ and $p + (n - 1)q > 0$. In other words, $p > q$, and $p > (1 - n)q$.

In-class challenge 3: For which angle $\theta \in (0, \pi)$ is there a basis of $\mathbb{R}^n$ such that any two basis vectors form an angle of θ? Solution: Suppose it is possible. normalize the basis vectors and put them in as the columns of A. We have AA^t is a matrix of the above form, where $p = 1$ and $q = \cos(\theta)$. By assumption, $p = 1 > q$. We also need $p = 1 > (1 - n)q$, i.e. $q < \frac{1}{1-n}$, i.e. $\theta < \arccos(\frac{1}{1-n})$.