Midterm II. Linear Algebra 110

Instructor: Zvezdelina Stankova

STUDENT NAME: ______________________________

GSI'S NAME: ______________________________

• **DO NOT OPEN THE MIDTERM UNTIL TOLD TO DO SO!**

• Please, do *all* Problems 1 through 5, as best as you can. The midterm about 70 minutes long. You may **not** leave early.

• Use the provided sheets to write your solutions. You may also use the back of each page for scratch work or solutions. If you need more paper, raise your hand to ask for additional paper. Submit *all* work which you wish to be graded and staple it to this packet! **Loose sheets of paper will NOT be graded!** Ask the instructor for a stapler at the end of the exam.

• The exam is closed notes and books, which means: **no classnotes, no session notes, no review notes, no textbooks, and no other materials can be used during the midterm.** You can use only your **cheat sheet**, this exam packet and additional blank paper provided by the instructor. You cannot use your paper to write on, even if it is only scratch work. The cheat sheet is one side of a regular 8×11 sheet, handwritten only by you. A cheat sheet which doesn't conform to these specifications (for example, there is stuff written on both sides of the sheet) will be disqualified, and the student's midterm may be annulled.

• **NO CALCULATORS ARE ALLOWED DURING THE MIDTERM!**

• **You cannot ask for help in any form from other students, you cannot look at their midterms and copy, you cannot cheat in any way – the midterm is an individual assignment and must be done only by you.** If you have a question regarding the statement of a problem, raise your hand when the instructor or a GSI is around.

• Think of this midterm as an important homework. Check your reasoning and calculations very carefully. Justify *all* your answers, include all intermediate steps and calculations. If you are not sure about how to write something in mathematical notation, explain clearly *in words* what you mean and what you are doing. **Unjustified answers, even if correct, will receive no credit!** On the contrary, good justifications and good work on a problem may receive a lot of credit even if the final answer is incorrect.

• Before turning in your midterm, please, sign the statement below. **Midterms which are not signed will not be graded.**

I, the student whose name and signature appear on this midterm, have completed the midterm without any outside help from people or other sources. I have used only my cheat sheet conforming to the specifications written above. I have not cheated in any way, and I have followed all instructions above. I have stapled all solutions and work wished to be graded to the present exam packet.

STUDENT SIGNATURE: ______________________________

Problem	Score
#1	
#2	
#3	
#4	
#5	
Total	/100

Problem 1 (15 pts). Let A be an $n \times n$ matrix with complex entries ($n \geq 2$) for which the following is true:

$$\det(A - \lambda I) = (7 - \lambda)^n.$$

(a) Is A necessarily diagonalizable? Prove or give a counterexample.

(b) Prove that A is invertible.

Problem 2 (25 pts). Consider the linear operator $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ given by:

$$T(f(x)) = f(x) + f(1) \cdot (1 + x^3) \text{ for all } f(x) \in P_3(\mathbb{R}).$$

(a) Find $\mathrm{char}_T(\lambda)$. Factor and simplify the characteristic polynomial as much as possible.
(*Hint:* Make sure that first you correctly find a matrix of T in some basis. Which basis?)

(b) Diagonalize T; i.e., find a diagonal matrix D and a basis β for $P_3(\mathbb{R})$ such that the matrix of T in β is D: $[T]_\beta = D$.

(*Note:* In the end, make sure that you write the diagonal matrix D and the basis β consisting of polynomials, NOT vectors in $\mathbb{R}^4$!)

Problem 3 (20 pts). Let T and U be two linear operators on a finite-dimensional space V. Suppose that T and U commute with each other; i.e., $TU = UT$.

(a) What is a U-invariant subspace of V? Define it.

(b) If $E_5(T)$ is the eigenspace of T corresponding to eigenvalue $\lambda = 5$, prove that $E_5(T)$ is a U-invariant subspace of V.

(*Hint:* Use the definition from part (b) and the given relation between U and T.)

Problem 4 (20 pts). Let $T : V \to V$ be a linear operator on V over F with $\dim V = n$.

(a) Write down one criterion for T to be *diagonalizable*. Your answer must start like this:

T is diagonalizable if and only if

(b) Why do we want to *diagonalize* matrices in Linear Algebra?

Where can we *apply* diagonalizability of matrices? Briefly describe one application of diagonalizability of matrices.

(c) What is the *dual space* V^* of V? Define it.

Problem 5 (20 pts). True or False?
To discourage guessing, the problem will be graded as follows:

- 2 pts for each correct answer.
- 0 pts for a blank.
- -2 pts for each incorrect answer.
- If anything else but "True" or "False" is written, more than one answer is written, or the answer is hard to read, you will get -2 point.

Note: Do not assume anything that is not given. If a statement claims that several things are true, but, in reality, even one is false, then the statement is false.

(1) Characteristic polynomials can be defined for *any* linear transformation $T : V \to W$ where V and W are finite-dimensional spaces.

Answer: __________

(2) The determinant of a lower-triangular matrix does not depend on the entries *under* the diagonal.

Answer: __________

(3) Even though square $n \times n$ matrices A and B do *not* commute in general, we always have $\det(AB) = \det(BA)$.

Answer: __________

(4) Every square matrix has at least one eigenvalue but not necessarily an eigenvector.

Answer: __________

(5) The sum of two eigenvectors of an operator T is always an eigenvector of T.

Answer: __________

(6) Similar matrices always have the same characteristic polynomials, eigenvalues, traces, and determinants, but not necessarily the same eigenvectors.

Answer: __________

(7) If a system $A\vec{x} = \vec{0}$ over $\mathbb{R}$ of n linear equations with n unknowns has at least one solution, then for any $b \in \mathbb{R}^n$ the system $A\vec{x} = \vec{b}$ also has at least one solution.

Answer: __________

(8) $E_{\lambda_1} \cap E_{\lambda_2}$ is *empty* for any two distinct eigenvalues λ_1 and λ_2 of a linear operator T.

Answer: __________

(9) Every matrix over $\mathbb{C}$ whose characteristic polynomial splits over $\mathbb{C}$ is either *diagonalizable* or is similar to a *Jordan block*.

Answer: __________

(10) The transposition operator $T : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ (i.e., $T(A) = A^t$) is *diagonalizable* with diagonal entries only $+1$ and -1, and there are *more* $+1$'s than -1's along the diagonal.

Answer: __________

GOOD LUCK!