Midterm II. MATH 110 Linear Algebra

Instructor: Zvezdelina Stankova

STUDENT NAME: ________________________________ STUDENT ID: ____________________

GSI'S NAME: ____

• DO NOT OPEN THE MIDTERM UNTIL TOLD TO DO SO!

• Please, do *all* Problems 1 through 5, as best as you can. The midterm about 70 minutes long. You may **not** leave early.

• Use the provided sheets to write your solutions. You may use the back of each page for the remainder of your solutions; in such a case, put an arrow at the bottom of the page and indicate that the solution continues on the back page. Before the test starts, pull out and use several of your own sheets for scratch work. **No extra sheets of paper can be submitted with this exam due to the grading computer software!** Thus, use the front and the back of the sheets in the exam for the actual solutions, and not for scratch work.

• The exam is closed notes and books, which means: **no classnotes, no session notes, no review notes, no textbooks, and no other materials can be used during the midterm.** You can use only your **cheat sheet**, this exam packet and the additional blank pages that you pulled out before the exam started. The cheat sheet is one side of a regular 8×11 sheet, handwritten only by you. A cheat sheet that doesn't conform to these specifications (for example, there is stuff written on both sides of the sheet) will be disqualified, and the student's midterm may be annulled.

• **NO CALCULATORS ARE ALLOWED DURING THE MIDTERM!**

• **You cannot ask for help in any form from other students, you cannot look at their midterms and copy, you cannot cheat in any way – the midterm is an individual assignment and must be done only by you.** If you have a question regarding the statement of a problem, raise your hand when the instructor or a GSI is around.

• Think of this midterm as an important homework. Check your reasoning and calculations carefully. Justify *all* your answers, include all intermediate steps and calculations. If you are not sure about how to write something in math notation, explain clearly *in words* what you mean and what you are doing. **Unjustified answers, even if correct, will receive no credit!** On the contrary, good justifications and good work on a problem may receive a lot of credit even if the final answer is incorrect.

• Before turning in your midterm, please, sign the statement below. **Midterms that are not signed will not be graded.**

I, the student whose name and signature appear on this midterm, have completed the midterm without any outside help from people or other sources. I have used only my cheat sheet conforming to the specifications written above. I have not cheated in any way, and I have followed all instructions above.

STUDENT SIGNATURE: ____

Problem 1. **(20pts)** Calculate the following determinant. Include all calculations. Be clear and organized. (*Hint:* One way to start is by adding all columns to some column.)

$$\det \begin{pmatrix} 7 & 1 & 1 & 1 \\ 1 & 7 & 1 & 1 \\ 1 & 1 & 7 & 1 \\ 1 & 1 & 1 & 7 \end{pmatrix} =$$

Problem 2 (20 pts) It is known that $\text{char}_A(\lambda) = -\lambda^3$ for the following matrix:

$$A = \begin{pmatrix} 1 & 1 & -1 \\ -3 & -3 & 3 \\ -2 & -2 & 2 \end{pmatrix}.$$

(a) Find the dot diagram for the Jordan canonical form for A. (No need to find Jordan canonical bases!! The calculations are minimal here.)

(b) The Jordan canonical form for A is: $J =$

(c) Is A diagonalizable? Why or why not?

Problem 3. (20pts) Let A be a square matrix over $\mathbb{C}$ with two eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 7$. It is known that the two dot diagrams corresponding to λ_1 and λ_2, respectively, are:

$$\begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & & & \end{array} \qquad\qquad \begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \end{array}$$

Find the following numbers, with brief **one-line** explanations of how you found them:

- the algebraic multiplicities $m_a(\lambda_i)$ of the two eigenvalues:

  $m_a(-2) =$

  $m_a(7) =$

- the geometric multiplicities $m_g(\lambda_i)$ of the two eigenvalues:

  $m_g(-2) =$

  $m_g(7) =$

- the stabilizing constants l_i for the two eigenvalues:

  $l_1 =$

  $l_2 =$

- the dimensions d_i of the generalized eigenspaces $K_{\lambda_i}(A)$:

  $\dim K_{-2}(A) =$

  $\dim K_7(A) =$

- the number n_i of Jordan blocks for each of the two eigenvalues:

  $k_1 =$

  $k_2 =$

Problem 4. (20pts) What is the *smallest* n for which the set

$$\{I_3, A, A^2, \cdots, A^n\}$$

is linearly *dependent* in the space of matrices $M_{3\times 3}(\mathbb{R})$ for *any* 3×3 real matrix A? Explain. (*Hint:* $\mathrm{char}_A(A) =$? Which theorem am I hinting at?)

- Answer: $n =$ ________
- This n works for any 3×3 real matrix A because:

- A *smaller* n does **not** work for some 3×3 real matrix A since there is a counterexample:

Problem 5 (20 pts). True or False? You must completely bubble one of the circles or leave blank. To discourage guessing, the problem will be graded as follows:

- 2 pt for each correct answer. • 0 pts for a blank. • -2 pt for each incorrect answer.
- If anything is written, other than the bubbling, more than one circle is bubbled (or partially bubbled), or it is hard to tell which answer is bubbled, you will get -2 point.

Notes: All matrices below are **square**. T is a linear operator on a vector space V.

A statement is TRUE if it is true in *all* cases that satisfy its hypothesis. A statement is FALSE if it is false even in just *one* case that satisfies its hypothesis, regardless of what happens in all other cases. If a statement claims that several things are true, but in reality even one of them is false, then the whole statement is FALSE.

(1) If an $n \times n$ matrix has n *distinct* eigenvalues in the base field F, then the matrix is diagonalizable over F, but the converse statement is **not** necessarily true.

Ⓣ Ⓕ

(2) The sum of all *algebraic* multiplicities for a matrix A **cannot** exceed the sum of all *geometric* multiplicities for a matrix A.

Ⓣ Ⓕ

(3) Solving any *non-homogenous* system of linear equations can be interpreted as finding a *translate* of the kernel of some linear transformation by a certain vector.

Ⓣ Ⓕ

(4) Any *non-zero T-invariant* subspace of V contains a (non-zero) T-*cyclic* subspace of V.

Ⓣ Ⓕ

(5) To prove that $\operatorname{rk} A = \operatorname{rk} A^t$ for any matrix A, in class we went through $RREF(A)$ and used its properties.

Ⓣ Ⓕ

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(6) Just like diagonal matrices, *Jordan blocks* of same sizes are easy to work with because they *commute* with each other.

Ⓣ Ⓕ

(7) If the characteristic polynomials of a matrix A with real entries does **not** split over $\mathbb{R}$, then A does **not** have a Jordan form over $\mathbb{C}$.

Ⓣ Ⓕ

(8) If we plug a real matrix A into its own characteristic polynomial and we do **not** obtain the zero matrix O, then we must have made a mistake.

Ⓣ Ⓕ

(9) To reduce the proof of Cayley-Hamilton theorem from any matrix A to its Jordan Canonical Form, we used *both* the relationship between *similar* matrices and the relationship between their characteristic polynomials.

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(10) $\det(A + B) = \det A + \det B$ is *impossible* for 3×3 matrices, unless at least one of them is the *zero* matrix.

Ⓣ Ⓕ