Homework 7 Solutions in Math 110 Fall 2016

with Professor Stankova

3.4 System of Linear Equations–Computational Aspects

1. (a) False. Elementary column operations change the solution set.

   (b) True. It follows from Theorem 3.13.

   (c) True. It follows from Theorem 3.16.

   (d) True. It follows from Theorem 3.14.

   (e) False. The equation $0x = 1$ has no solutions.

   (f) True. It follows from Theorem 3.15.

   (g) True. It follows from Theorem 3.16.

2. (b) The reduced row echelon form of the augmented matrix is

$$\begin{pmatrix} 1 & 0 & 5 & 9 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

   so the solution set is $\{(9, 4, 0) + x_3(-5, -3, 1)\}$.

   (d) The reduced row echelon form of the augmented matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \frac{7}{13} \\ 0 & 1 & 0 & 0 & \frac{16}{13} \\ 0 & 0 & 1 & 0 & \frac{14}{13} \\ 0 & 0 & 0 & 1 & -\frac{18}{13} \end{pmatrix},$$

   so the solution set is $\{(\frac{7}{13}, \frac{16}{13}, \frac{14}{13}, -\frac{18}{13})\}$.

   (f) The reduced row echelon form of the augmented matrix is

$$\begin{pmatrix} 1 & 0 & 0 & -1 & -3 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix},$$

   so the solution set is $\{(-3, 3, 1, 0) + x_4(1, -2, 0, 1)\}$.

3. (a) The augmented matrix $(A'|b')$ contains a row in which the only nonzero entry lies in the last column if and only if the system of linear equations corresponding to $(A'|b')$ is inconsistent, which is equivalent to $\operatorname{rank}(A') \neq \operatorname{rank}(A'|b')$ by Theorem 3.11.

   (b) The equation $Ax = b$ is consistent if and only if $A'x = b'$ is consistent by Corollary to Theorem 3.13, which is equivalent to the statement that the augmented matrix $(A'|b')$ does not contain a row in which the only nonzero entry lies in the last column by the proof of (a).

4. (b) The reduced row echelon form of the augmented matrix is

$$\begin{pmatrix} 1 & 1 & 0 & -\frac{1}{2} & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

   so the solution set is $\{(1, 0, 1, 0) + x_2(-1, 1, 0, 0) + x_4(\frac{1}{2}, 0, \frac{1}{2}, 1)\}$. A basis for the solution set of the corresponding homogeneous system is $\{(-1, 1, 0, 0), (\frac{1}{2}, 0, \frac{1}{2}, 1)\}$.

5. We have

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & -1 & -2 \\ 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & 1 & 4 \\ -1 & -1 & 3 & -2 & -7 \\ 3 & 1 & 1 & 0 & 9 \end{pmatrix},$$

and this should be A since this is row equivalent to the original 3×5 matrix.

6. By the same trick as before, we get that

$$\begin{aligned} A &= \begin{pmatrix} 1 & -1 & 3 & * \\ -2 & 1 & -9 & * \\ -1 & 2 & 2 & * \\ 3 & -4 & 5 & * \end{pmatrix} \begin{pmatrix} 1 & -3 & 0 & 4 & 0 & 5 \\ 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 & 3 \\ -2 & 1 & -9 \\ -1 & 2 & 2 \\ 3 & -4 & 5 \end{pmatrix} \begin{pmatrix} 1 & -3 & 0 & 4 & 0 & 5 \\ 0 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -3 & -1 & 1 & 0 & 3 \\ -2 & 6 & 1 & -5 & 1 & -9 \\ -1 & 3 & 2 & 2 & -3 & 2 \\ 3 & -9 & 4 & 0 & 2 & 5 \end{pmatrix} \end{aligned}$$

Note that it does not matter what the entries labeled $(*)$ are, since the final row of the reduced echelon form of A is zero. If this row were not zero, our answer would not be unique.

7. The reduced row echelon form of the matrix

$$A = \begin{pmatrix} 2 & 1 & -8 & 1 & -3 \\ -3 & 4 & 12 & 37 & -5 \\ 1 & -2 & -4 & -17 & 8 \end{pmatrix}$$

is

$$A' = \begin{pmatrix} 1 & 0 & * & * & 0 \\ 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where the entries labeled $(*)$ are real numbers. Let $C'_1, \ldots, C'_5$ denote the columns of A'. Then C'_3 and C'_4 are in the span of $\{C'_1, C'_2, C'_5\}$, and this relation is preserved by elementary row operations, so u_3 and u_4 are in the span of $\{u_1, u_2, u_5\}$. Moreover, $\{C'_1, C'_2, C'_5\}$ is linearly independent, and so the same is true for $\{u_1, u_2, u_5\}$. Thus $\{u_1, u_2, u_5\}$ is a basis for $\mathbb{R}^3$.

9. Identify W with $\mathbb{R}^3$ by the transformation

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \mapsto (a, b, c),$$

and say $S = \{C_1, C_2, C_3, C_4, C_5\}$. The reduced row echelon form of the matrix whose columns are $C_1, \ldots, C_5$ is

$$A' = \begin{pmatrix} 1 & 0 & * & 0 & * \\ 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \end{pmatrix}$$

where $*$ means a real number. By the same reasoning as the previous problem, the set

$$\left\{ \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \right\}$$

therefore forms a basis for W.

10. (a) The vector $u = (0, 1, 1, 1, 0)$ satisfies the equation, so it is an element of V. Thus S is a linearly independent subset of V since the vector is nonzero.

(b) The subspace V has a basis

$$\{u_1 = (-2, 0, 0, 0, 1), u_2 = (0, 1, 0, 0, 1), u_3 = (0, 0, -2, 0, 3), u_4 = (0, 0, 0, 2, 1)\}.$$

We need to find a basis of V that is a subset of $\{u, u_1, u_2, u_3, u_4\}$ and contains u. The reduced row echelon form of the matrix whose columns are $u, u_1, \ldots, u_4$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \end{pmatrix},$$

so as in Exercise 9, $\{u, u_1, u_2, u_3\}$ is a basis for V.

12. (a) The elements of $S = \{v_1 = (0, -1, 0, 1, 1, 0), v_2 = (1, 0, 1, 1, 1, 0)\}$ satisfy the equations, so S is a subset of V. Since v_1 and v_2 are not a multiple of the other, S is linearly independent.

(b) The reduced row echelon form of the augmented matrix is;

$$\begin{pmatrix} 1 & 0 & -1 & 1 & -1 & 3 & 0 \\ 0 & 1 & -1 & -1 & 2 & 2 & 0 \end{pmatrix},$$

the solution set has a basis

$$\{u_1 = (-3, -2, 0, 0, 0, 1), u_2 = (1, -2, 0, 0, 1, 0), u_3 = (-1, 1, 0, 1, 0, 0), u_4 = (1, 1, 1, 0, 0, 0)\}.$$

The reduced row echelon form of the matrix whose columns are $v_1, v_2, u_1, u_2, u_3, u_4$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & * & * \\ 0 & 1 & 0 & 0 & * & * \\ 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \end{pmatrix},$$

so as in Exercise 9, $\{v_1, v_2, u_1, u_2\}$ is a basis for V.

14. If the axioms (a)–(c) of reduced row echelon forms in p. 185 satisfied for $(A|b)$, then the same is true for A. Thus A is a reduced row echelon form.

4.1 Determinants of Order 2

1. (a) False. We have $\det(rI_2) = r^2$ and $r \cdot \det(I) = r$, and these are different if $r \neq 0, 1$.

(b) True. Fix c and d. Then the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(a, b) = ad - bc$ is linear. The same is true if we fix instead a and b.

(c) False. A is invertible if and only if $\det(A)$ is *not* zero.

(d) False, since determinants may have negative value. However, the area is equal to the absolute value of the determinant.

(e) True, by definition. See p. 203.

3. (b) The determinant is $(5 - 2i)7i - (6 + 4i)(-3 + i) = -8 + 41i$.

4. (c) The determinant is $4(-2) - (-1)(-6) = -14$, and the area is its absolute value, which is 14.

5. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

then $\det A = ad - bc$ and $\det B = cb - da$, so $\det B = -\det A$.

6. If
$$A = \begin{pmatrix} a & b \\ a & b \end{pmatrix},$$
then $\det A = ab - ba = 0$.

7. If
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, A^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$
then $\det A = ad - bc$ and $\det A^t = ad - cb$, so they are equal.

8. If
$$A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$
then $\det A = ad$, which is the product of the diagonal entries of A.

9. If
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix},$$
then
$$AB = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix},$$
so $\det(AB) = (ae+bg)(cf+dh) - (af+bh)(ce+dg) = (ad-bc)(eh-fg) = \det A \det B$.

11. By the properties of δ, we have
$$\begin{aligned}\delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= a\delta\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} + b\delta\begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix} \\ &= ac\delta\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + ad\delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + bc\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + bd\delta\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\ &= ad\delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + bc\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = ad + bc\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.\end{aligned}$$
When $a = b = c = d = 1$, we get
$$\delta\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
which is 0 by (ii). Thus
$$\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1,$$
and we have
$$\delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad + bc\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = ad - bc.$$