HW #2; Date 9/5/2017
MATH 110 Linear Algebra
with Professor Stankova

Section 1.3: Subspaces

1)

(a) False. Take as V the vector space $\mathbb{R}^2$ with the usual definitions of vector addition and scalar multiplication, and let W be the line $\{(a_1, 1) \mid a_1 \in \mathbb{R}\} \subseteq \mathbb{R}^2$ with vector addition and scalar multiplication defined by

$$(a_1, 1) + (a_2, 1) = (a_1 + a_2, 1), \quad c(a_1, 1) = (ca_1, 1)$$

Then W is a vector space which is also a subset of V, but it is not necessarily a subspace since it does not use the same vector addition and scalar multiplication as V. Indeed, we can tell that W is not a subspace of V because the zero element of V is not contained in W.

(b) False. The empty set is a *subset* of every vector space, but it is not a subspace since $0 \notin \emptyset$.

(c) True. If V is a subspace that is not trivial, then it contains the trivial subspace $W = \{0\}$, which in particular is not equal to V.

(d) False. The sets $\{1\}$ and $\{2\}$ are subsets of $\mathbb{R}$, but $\{1\} \cap \{2\} = \emptyset$ is not a subspace of $\mathbb{R}$. It is true that the intersection of any two *subspaces* of V is a subspace of V.

(e) True. By definition, all of the off-diagonal entries are zero. There are only n diagonal entries in an $n \times n$ matrix, so there can never be more than n nonzero entries.

(f) False. For example, $\operatorname{tr}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 + 1 = 2 \neq 1$. The trace is always the *sum* of the diagonal entries.

(g) False. $\mathbb{R}^2$ consists of ordered pairs of real numbers, whereas W is a triple of numbers with the last being zero, so in particular these two vector spaces are not equal as sets. It is, however, true that W is *isomorphic* to $\mathbb{R}^2$.

5)

Recall that a square matrix is called symmetric if it is equal to its own transpose. Using two basic facts about transposes of matrices, that for any matrix A,

$$(A^t)^t = A$$

and that for any matrices A and B,

$$(A + B)^t = A^t + B^t$$

we see that

$$\begin{aligned}(A + A^t)^t &= A^t + (A^t)^t \\ &= A^t + A \\ &= A + A^t\end{aligned}$$

This proves that $A + A^t$ is always equal to its own transpose, and thus is symmetric.

8b)

W_2 is not a subspace because it does not containt the zero element. It is also not closed under either addition or scalar multiplication; any sum of two vectors or any vector and scalar other than 1 will demonstrate this.

8d)

ANSWER 1: Let (a_1, a_2, a_3) and (b_1, b_2, b_3) be elements of W_4, and let $c \in \mathbb{R}$. Then

- $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$ satisfies

$$(a_1 + b_1) - 4(a_2 + b_2) - (a_3 + b_3) = (a_1 - 4a_2 - a_3) + (b_1 - 4b_2 - b_3) = 0$$

- $c \cdot (a_1, a_2, a_3) = (ca_1, ca_2, ca_3)$ satisfies

$$ca_1 - 4ca_2 - ca_3 = c(a_1 - 4a_2 - a_3) = 0$$

- $(0, 0, 0)$ satisfies $0 - 4(0) - 0 = 0$.
- $(-a_1, -a_2, -a_3)$ is in W_4.

So W_4 satisfies all of the properties of a subspace.

ANSWER 2: Similar to the result from problem 18, it is sufficient to prove that W_4 is nonempty and $ax + y \in W_4$ whenever $a \in \mathbb{R}$ and $x, y \in W_4$. W_4 is nonempty since $(0, 0, 0) \in W_4$, and if $x, y \in W_4$ then

$$(ax_1 + y_1) - 4(ax_2 + y_2) - (ax_3 + y_3) = a(x_1 - 4x_2 - x_3) + (y_1 - 4y_2 - y_3) = a \cdot 0 + 0 = 0$$

for any $a \in \mathbb{R}$. Thus W_4 is a subspace.

8f)

$(\sqrt{3}, \sqrt{5}, 0)$ and $(0, \sqrt{6}, \sqrt{3})$ are in W_6, but $(\sqrt{3}, \sqrt{5}, 0) + (0, \sqrt{6}, \sqrt{3})$ is not in W_6. This means that W_6 is not closed under addition, so it is not a subspace.

9)

For the first two intersections, we can use a sequence of substitutions to simplify the conditions describing the intersection. We write

$$W_1 \cap W_3 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid a_1 = 3a_2, a_3 = -a_2, 2a_1 - 7a_2 + a_3 = 0\}$$

In particular, we use the first two conditions to rewrite the third as $6a_2 - 7a_2 - a_2 = 0$, which is just $a_2 = 0$. But in this case, $a_1 = 3a_2 = 0$, and $a_3 = -a_2 = 0$, so we conclude that $W_1 \cap W_3 = \{(0, 0, 0)\}$ is the trivial subspace.

Similarly, we have

$$W_1 \cap W_4 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid a_1 = 3a_2, a_3 = -a_2, a_1 - 4a_2 - a_3 = 0\}$$

In this case however, when we use the first two conditions to rewrite the third, we get $3a_2 - 4a_2 + a_2 = 0$, which just simplifies to $0 = 0$. This condition is thus redundant in the presence of the first two, and we find that $W_1 \cap W_4 = W_1$. In particular, this means that W_1 is actually a subspace of W_4.

For the last intersection, we can use row reduction to find the solutions of the system of equations given by

$$W_3 \cap W_4 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid 2a_1 - 7a_2 + a_3 = 0, a_1 - 4a_2 - a_3 = 0\}$$

which can be expressed as $W_3 \cap W_4 = \{(-11a_3, -3a_3, a_3) \in \mathbb{R}^3 \mid a_3 \in \mathbb{R}\}$.

13)

Fix $s_0 \in S$, and let $W = \{f \in \mathcal{F}(S, F) \mid f(s_0) = 0\}$. We show that W is a subspace of $\mathcal{F}(S, F)$ by checking the necessary properties. If $f, g \in W$ and $c \in F$, we have $f(s_0) = g(s_0) = 0$. Then

- $\hat{0}(s_0) = 0$, where $\hat{0}$ denotes the zero function, so $\hat{0} \in W$
- $(f + g)(s_0) = f(s_0) + g(s_0) = 0 + 0 = 0$, so $f + g \in W$
- $(cf)(s_0) = c \cdot f(s_0) = c \cdot 0 = 0$, so $cf \in W$

14)

Let $f, g \in \mathcal{C}(S, F)$. Then f is nonzero on some finite set S_f and g is nonzero on some finite set S_g. Let $c \in F$.

- The zero function is nonzero only on the empty set, which in particular should be considered finite.
- $(f + g)$ is nonzero on a subset of $S_f \cup S_g$, so in particular this is a finite set.
- cf is nonzero on S_f, unless $c = 0$. If $c = 0$, then we have the zero function, which is also a member of our space, as noted above.

15)

It is in fact a subspace of $C(\mathbb{R})$. To see this, first note that since every differentiable function is continuous, the set of differentiable functions is at least a subset of $C(\mathbb{R})$. Now from standard calculus properties we show that the differentiable functions satisfies the necessary properties to form a subspace. If f and g are differentiable, and $c \in \mathbb{R}$, then

- The zero function is differentiable
- $(f+g)$ is differentiable with derivative $f' + g'$
- (cf) is differentiable with derivative cf'

17)

We must show both directions to prove the if and only if statement. If W is a subspace of V, then $W \neq \emptyset$, since it must contain the zero vector. Further, if $x, y \in W$ and $a \in F$, then $ax \in W$ by closure under scalar multiplication, and $x + y \in W$ by closure under addition.

On the other hand, if W is a nonempty subset of V such that $ax \in W$ and $x + y \in W$ for any $x, y \in W$ and any $a \in F$, then

- W is clearly closed under addition.
- W is clearly closed under multiplication by scalars.
- Since W is nonempty, it contains some element x. In particular, since it is closed under multiplication by scalars, this means that $0x = 0 \in W$, so W contains the zero element.

By Theorem 1.3 we conclude that W is a subspace, and this completes the argument.

18)

We must show both directions to prove the if and only if statement. If W is a subspace of V, then $0 \in W$. Also if $x, y \in W$ and $a \in F$, then $ax \in W$ by closure under multiplication and $ax + y \in W$ by closure under addition.

Now let W be a subset of V such that $0 \in W$ and $ax+y \in W$ whenever $a \in F$ and $x, y \in W$. Then,

- $0 \in W$ by assumption
- Taking $a = 1$, we see that $x + y \in W$ for any $x, y \in W$
- Taking $y = 0$, we see that $ax \in W$ for any $x \in W$, $a \in F$

By Theorem 1.3 we conclude that W is a subspace, and this completes the argument.

19)

We must show both directions to prove the if and only if statement. First, if $W_1 \subseteq W_2$, then $W_1 \cup W_2 = W_2$, so it's a subspace, and similarly if $W_2 \subseteq W_1$. For the converse statement, we will give a proof of the contrapositive: if $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$, then $W_1 \cup W_2$ is not a subspace.

Consider elements $x_1 \in W_1 \setminus W_2$, and $x_2 \in W_2 \setminus W_1$, and look at the sum $x_1 + x_2$. If $x_1 + x_2$ were in W_1 then $(x_1 + x_2) - x_1 = x_2$ would also be in W_1 because W_1 is a subspace, but this is false. Similarly, if $x_1 + x_2$ were in W_2 then $(x_1 + x_2) - x_2 = x_1$ would also be in W_2, but this is false. Therefore $(x_1 + x_2) \notin W_1 \cup W_2$, so $W_1 \cup W_2$ is not closed under addition and is therefore not a subspace.

As an example, consider the x-axis and y-axis in $\mathbb{R}^2$. Each of these axes on their own are subspaces of $\mathbb{R}^2$, but their union is not, since it is not closed under addition. As problem (23) shows, the smallest subspace that contains both W_1 and W_2 is the direct sum $W_1 + W_2$, which in the case of (x-axis) + (y-axis) is all of $\mathbb{R}^2$, a 2-dimensional plane.

20)

We will proceed using a proof by induction. If you don't really remember how to do proofs by induction, now is an excellent time to review it. Consult your local GSI today!

We will prove for each $k \geq 1$ the proposition $P(k)$, that for any vectors $w_1, \ldots, w_k \in W$ and scalars $a_1, \ldots, a_k$, we have that the linear combination $a_1 w_1 + \cdots + a_k w_k \in W$.

BASE CASE: $n = 1$. If $w_1 \in W$ and $a_1 \in F$, then $a_1 w_1 \in W$ by closure of W under scalar multiplication. This proves the proposition $P(1)$.

INDUCTIVE STEP: Suppose as an inductive hypothesis that $P(k)$ is true. We want to show that under this assumption, $P(k+1)$ is also true. So suppose that $w_1, \ldots, w_k, w_{k+1} \in W$ and $a_1, \ldots, a_k, a_{k+1}$ are scalars. Then $a_1 w_1 + \cdots + a_k w_k \in W$ by our inductive hypothesis, and $a_{k+1} w_{k+1} \in W$ by closure of W under scalar multiplication. In particular, we can now conclude

$$a_1 w_1 + \cdots + a_k w_k + a_{k+1} w_{k+1} = (a_1 w_1 + \cdots + a_k w_k) + (a_{k+1} w_{k+1}) \in W$$

because of closure of W under vector addition. This proves $P(k+1)$, and so the inductive step is complete. By mathematical induction we can conclude that $P(n)$ holds (unconditionally) for any $n \geq 1$.

22)

Odd functions: Let f, g be odd functions in $\mathcal{F}(F_1, F_2)$, and let $c \in F_2$. Then

- $(af + g)(-x) = (af)(-x) + g(-x) = a \cdot (f(-x)) - g(x) = -(a \cdot f(x) + g(x)) = -(af + g)(x)$. So $af + g$ is odd.
- Let $z \in \mathcal{F}(F_1, F_2)$ be the zero function. Then $z(-x) = 0 = -0 = -z(x)$. So z is odd.

By Exercise 18, the set of odd functions is a vector subspace.

Even functions: Let f, g be even functions in $\mathcal{F}(F_1, F_2)$, and let $c \in F_2$. Then

- $(af+g)(-x) = (af)(-x) + g(-x) = a \cdot (f(-x)) - g(x) = a \cdot f(x) + g(x) = (af+g)(x)$. So $af + g$ is even.
- Let $z \in \mathcal{F}(F_1, F_2)$ be the zero function. Then $z(-x) = 0 = z(x)$. So z is even.

Again using Exercise 18, the set of even functions is a vector subspace.

23a)

For $w \in W_1$, we have that $w = w + 0 \in W_1 + W_2$, so $W_1 \subseteq W_1 + W_2$. Similarly $W_2 \subseteq W_1 + W_2$.

Now we use Exercise 18 to prove that $W_1 + W_2$ is a vector subspace. Since W_1 and W_2 both contain 0, we know that $0 = 0 + 0 \in W_1 + W_2$. Let $x, y \in W_1 + W_2$, and let $a \in F$. We can write $x = x_1 + x_2$ and $y = y_1 + y_2$, where $x_1, y_1 \in W_1$ and $x_2, y_2 \in W_2$. Then we have

$$ax + y = a(x_1 + x_2) + (y_1 + y_2) = (ax_1 + y_1) + (ax_2 + y_2) \in W_1 + W_2$$

Thus by Exercise 18 we see that $W_1 + W_2$ is a vector subspace.

23b)

Let X be some subspace of V containing both W_1 and W_2, and let $w \in W_1 + W_2$. Then $w = w_1 + w_2$, where $w_1 \in W_1$ and $w_2 \in W_2$. In particular since $w_1, w_2 \in X$, we have that $w = w_1 + w_2 \in X$ as well by closure under vector addition. Since w was arbitrary, we conclude that all elements of $W_1 + W_2$ are elements of X, and so $W_1 + W_2 \subseteq X$.

Note that this result can be interpreted as meaning that $W_1 + W_2$ is the *smallest possible* subspace that contains both W_1 and W_2.

24)

Since the problem asserts that W_1 and W_2 are subspaces of F^n, we will take this fact for granted (although you should check why this is the case if you are unsure!).

First we check that the intersection of W_1 and W_2 is just the zero vector:

$$W_1 \cap W_2 = \{(a_1, \dots, a_n) \mid a_n = 0 \text{ and } a_1 = a_2 = \cdots = a_{n-1} = 0\} = \{(0, 0, \dots, 0)\}$$

Next we show that $F^n = W_1 + W_2$, which we will do by showing that $W_1 + W_2 \subseteq F^n$ and that $F^n \subseteq W_1 + W_2$. The first containment $W_1 + W_2 \subseteq F^n$ is true since both W_1 and W_2 are subspaces of F^n (which is closed under addition).

For the second containment, for any element $\vec{a} \in F^n$ we can say

$$\vec{a} = (a_1, a_2, \dots, a_n) = (a_1, \dots, a_{n-1}, 0) + (0, \dots, 0, a_n) \in W_1 + W_2.$$

Thus $F^n \subseteq W_1 + W_2$. We have shown that each set contains the other, and so $F^n = W_1 + W_2$. Therefore $F^n = W_1 \oplus W_2$.

NOTE: If $F^n = W_1 \oplus W_2$, then the decomposition $\vec{a} = \vec{w_1} + \vec{w_2}$ is also *unique*. A later problem proves this fact.

25)

We will skip the proof that W_1 and W_2 are subspaces of $P(F)$... you should check this on your own if you are unsure why it is true. First observe that

$$W_1 \cap W_2 = \{a_n x^n + \cdots + a_1 x + a_0 \mid a_i = 0 \text{ for } i \text{ odd, and } a_i = 0 \text{ for } i \text{ even}\} = \{0\}$$

Second, note that any polynomial $p = \sum_{i=1}^{n} a_i x^i$ can be written as

$$p = \sum_{i=1}^{n} a_i x^i = \sum_{i \text{ odd}} a_i x^i + \sum_{i \text{ even}} a_i x^i \in W_1 + W_2$$

so $P(F) \subseteq W_1 + W_2$. Since $W_1 + W_2 \subseteq P(F)$ by the same logic as in Problem (24), the two sets are equal. Therefore, $P(F) = W_1 \oplus W_2$.

26)

Again skip the proof that W_1 and W_2 are subspaces of $M_{m\times n}(F)$. First observe that

$$W_1 \cap W_2 = \{A \in M_{m\times n}(F) \mid A_{ij} = 0 \text{ for } i > j \text{ and } A_{ij} = 0 \text{ for } i \leq j\} = \{\vec{0}\}$$

Second, we need to show that $M_{m\times n} = W_1 + W_2$. Take an arbitrary $A \in M_{m\times n}$, and consider the matrices $B \in W_1$ and $C \in W_2$ such that

$$B_{ij} = \begin{cases} A_{ij}, & i \leq j \\ 0, & \text{otherwise} \end{cases} \qquad C_{ij} = \begin{cases} A_{ij}, & i > j \\ 0, & \text{otherwise} \end{cases}$$

Then $A = B + C \in W_1 + W_2$. The conclusions that $M_{m\times n} = W_1 + W_2$ and that $M_{m\times n} = W_1 \oplus W_2$ follow using the same logic as in the previous two problems.

27)

By definition, W_1 and W_2 are subspaces of V. Observe that

$$W_1 \cap W_2 = \{A \in V \mid A_{ij} = 0 \text{ for } i \neq j \text{ and } A_{ij} = 0 \text{ for } i \geq j\} = \{0\}$$

Let $A \in V$. Now define $B \in W_1$ and $C \in W_2$ such that

$$B_{ij} = \begin{cases} A_{ij}, & i = j \\ 0, & \text{otherwise} \end{cases} \qquad C_{ij} = \begin{cases} A_{ij}, & i < j \\ 0, & \text{otherwise} \end{cases}$$

so that $A = B + C \in W_1 + W_2$. As before, this shows that $A \in W_1 + W_2$, and so the direct sum $V = W_1 \oplus W_2$ follows.

NOTE: A standard name for W_2 is the set of *strictly upper triangular* matrices.

28)

The zero matrix is skew-symmetric, so $0 \in W_1$. Now let $A, B \in W_1$, and let $c \in F$. Then $(cA+B)^t = (cA)^t + B^t = cA^t + B^t = -cA - B = -(cA+B)$. So $cA+B$ is skew-symmetric, as desired. Thus W_1 is a vector subspace by Exercise 18.

Now assume that our field F of scalars is not of characteristic 2, so that $1+1 =: 2 \neq 0$. If $A \in W_1 \cap W_2$, then $A^t = A$ and $A^t = -A$, so that $A = -A$, and in particular $A + A = (1+1)A = 2A = 0$. Since $2 \neq 0$ in F, this implies that A is the zero matrix, so $W_1 \cap W_2 = \{0\}$.

Given an arbitrary matrix $A \in M_{n\times n}$, observe that $(A - A^t)/2$ is skew-symmetric since

$$\left(\frac{A - A^t}{2}\right)^t = \frac{A^t - A}{2} = -\frac{A - A^t}{2}$$

Similarly, $(A + A^t)/2$ is symmetric, since

$$\left(\frac{A + A^t}{2}\right)^t = \frac{A^t + A}{2} = \frac{A + A^t}{2}$$

In particular, given any matrix $A \in M_{n\times n}$ we can write

$$A = \frac{A - A^t}{2} + \frac{A + A^t}{2} \in W_1 + W_2$$

Thus $A \in W_1 + W_2$, so as usual we can conclude that $M_{n\times n} = W_1 \oplus W_2$.

NOTE: In a field of characteristic 2, every symmetric matrix is also skew-symmetric. The proof that $M_{n\times n} = W_1 + W_2$ also depends on our ability to divide by 2, which requires that $2 \neq 0$. This is why the proof would not work in a field of characteristic 2.

BONUS: Can you relate this to the earlier problem about even and odd functions?

30)

We must prove both directions of this if and only if statement. First we show that if $V = W_1 \oplus W_2$, then elements of V are uniquely representable as a sum of elements, one from each of W_1 and W_2. Then we tackle the converse.

For the forward direction, we already know that if $V = W_1 \oplus W_2$ then each vector $v \in V$ can be written in the form $v = w_1 + w_2$ for some $w_1 \in W_1, w_2 \in W_2$, so we need to show that this representation is unique.

Suppose that there is another representation $v = w_1' + w_2'$. Since $w_1 + w_2 = v = w_1' + w_2'$, we can rearrange the equation to get $(w_1 - w_1') = (w_2' - w_2)$. But the left hand side is in W_1 and the right hand side is in W_2, and $W_1 \cap W_2 = \{0\}$. Therefore, $w_1 - w_1' = 0$ and $w_2 - w_2' = 0$, so we conclude that $w_1 = w_1'$ and $w_2 = w_2'$. This proves that the (arbitrary) second representation is equal to the first, and so the representation must unique.

For the converse, suppose that each $v \in V$ can be uniquely written in the form $w_1 + w_2$ with $w_1 \in W_1, w_2 \in W_2$. The fact that it can be written in this form at all shows that $V \subseteq W_1 + W_2$ (and therefore that $V = W_1 + W_2$, since both are subspaces of V).

To prove that $V = W_1 \oplus W_2$, we now just have to show that $W_1 \cap W_2 = \{0\}$. Since both W_1 and W_2 are subspaces, $0 \in W_1 \cap W_2$. Then suppose that $w \in W_1 \cap W_2$ for some nonzero w. Note that $w = 0 + w = w + 0$, so w (which is also in V) has a non-unique representation as a sum of elements in W_1 and W_2. This contradicts our assumptions, so no such w exists. Therefore $W_1 \cap W_2 = \{0\}$, and so $V = W_1 \oplus W_2$.

31)

(a) If $v \in W$, then $v + W = W$, since every element of $w \in W$ can be written $w = w' - v$.

If $v \notin W$, then $-v \notin W$, so $0 \notin v + W$, and W is not a subspace.

(b) If $v_1+W = v_2+W$, then in particular, $v_1 = v_1+0 \in v_2+W$, so we can write $v_1 = v_2+w$ for some $w \in W$. Rewriting this we see that $v_1 - v_2 = w \in W$.

For the converse, suppose that $v_1 - v_2 = w \in W$. Then for $x \in v_1 + W$, we can write $x = v_1 + w_1$, and so

$$x = v_1 + w_1 = v_2 + (w + w_1) \in v_2 + W$$

This shows that $v_1 + W \subseteq v_2 + W$, and a similar argument shows that the reverse inclusion also holds. We conclude that we have equality: $v_1 + W = v_2 + W$.

(c) By the above, if $v_1 + W = v_1' + W$, then $v_1 - v_1' \in W$, and if $v_2 + W = v_2' + W$, then $v_2 - v_2' \in W$. For the addition operation, notice that in particular, this implies that

$$(v_1 + v_2) - (v_1' + v_2') = (v_1 - v_1') + (v_2 - v_2') \in W$$

so again using the previous part,

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W = (v_1' + v_2') + W = (v_1' + W) + (v_2' + W)$$

For scalar multiplication, note that since $v_1 - v_1' \in W$, we also have $a(v_1 - v_1') = av_1 - av_1' \in W$ for any scalar $a \in F$. Thus we have

$$a(v_1 + W) = (av_1) + W = (av_1') + W = a(v_1' + W)$$

Thus we see that the vector addition and scalar multiplication are well-defined on the cosets, independent of the choice of representative.

(d) We show that each of the eight properties of a vector space holds. Let $x, y, z \in V$, and let $a, b \in F$. Then

(VS 1) We have $(x+W)+(y+W) = (x+y)+W = (y+x)+W = (y+W)+(x+W)$. The middle step follows from commutativity in V.

(VS 2) We have

$$\begin{aligned}
&(x+W)+\big((y+W)+(z+W)\big)\\
&=(x+W)+(y+z+W)\\
&=x+(y+z)+W\\
&=(x+y)+z+W\\
&=(x+y+W)+(z+W)\\
&=\big((x+W)+(y+W)\big)+(z+W)
\end{aligned}$$

The third equality follows from associativity in V.

(VS 3) $0+W$ is the zero vector. In particular, $(x+W)+(0+W)=(x+0)+W=x+W$.

(VS 4) $(-x)+W$ is the additive inverse of $x+W$. In particular, $(x+W)+((-x)+W)=(x-x)+W=0+W$.

(VS 5) We have $1(x+W)=(1x)+W=x+W$.

(VS 6) We have

$$\begin{aligned}
&(ab)(x+W)\\
&=(ab)x+W\\
&=a(bx)+W\\
&=a(bx+W)\\
&=a(b(x+W))
\end{aligned}$$

(VS 7) We have

$$\begin{aligned}
&a\big((x+W)+(y+W)\big)\\
&=a\big((x+y)+W\big)\\
&=a(x+y)+W\\
&=(ax+ay)+W\\
&=(ax+W)+(ay+W)\\
&=a(x+W)+a(y+W)
\end{aligned}$$

(VS 8) We have

$$\begin{aligned}
&(a+b)(x+W)\\
&=(a+b)x+W\\
&=(ax+bx)+W\\
&=(ax+W)+(bx+W)\\
&=a(x+W)+b(x+W)
\end{aligned}$$