HW #7; date: September 19, 2017
MATH 110 Linear Algebra
with Professor Stankova

2.3, #1 (a) False. The order should be preserved. (b) True. This is by definition. (c) False. The basis are wrong; should read $[U]_\beta^\gamma$. (d) True. One can check that $I[v]_\beta = [I_V(v)] = [v]_\beta$. (e) False. This is only true (the expression only makes sense) if $V = W$. (f) False. Take $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$. (g) False. L_A is a function from $\mathbb{R}^n \to \mathbb{R}^m$. (h) False. Take $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. (i) True. This is because $(A+B)v = Av + Bv$. (j) True, by definition of I.

2.3, #2(b) $A^t = \begin{pmatrix} 2 & -3 & 4 \\ 5 & 1 & 2 \end{pmatrix}$, $A^tB = \begin{pmatrix} 23 & 19 & 0 \\ 26 & -1 & 10 \end{pmatrix}$ $BC^t = \begin{pmatrix} 12 \\ 16 \\ 29 \end{pmatrix}$ $CB = \begin{pmatrix} 27 & 7 & 9 \end{pmatrix}$ $CA = \begin{pmatrix} 20 & 26 \end{pmatrix}$

2.3, #3 $[U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$, $[T]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}$, and $UT(a + bx + cx^2) = U((3b + 2a) + (3b + 6c)x + (4c)x^2) = (2a + 6b + 6c, 4c, 2a - 6c)$, so that $[UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}$.

2.3, #4 (a) $(1, -1, 4, 6)$, (b) $(-6, 2, 0, 6)$, (c) (5) (d) (12).

2.3, #9 Take $U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $A = U$, $B = T$.

2.3, #11 Suppose that $T^2 = 0$. Then, if $x \in R(T)$, this means that $x = Ty$ for some $y \in V$. Then, $Tx = T^2y = 0y = 0$, so $x \in N(T)$. Conversely, suppose that $R(T) \subseteq N(T)$. Then, for any $x \in V$, $T^2x = T(Tx)$. Since $Tx \in R(T) \subseteq N(T)$, $T(Tx) = 0$ as desired.

2.3, #12 (a) Let $x \in V$ with $T(x) = 0$. Then $UT(x) = U(0) = 0$. Since UT is one-to-one, this implies $x = 0$. So T is one-to-one. However, U does not ahve to be one-to-one. For example, take $T : 0 \to V$ and $U : V \to 0$; for any nonzero V this provides an example. (b) Since UT is onto, for every $z \in Z$ there is $v \in V$ such that $UT(v) = z$. Then, for every $z \in Z$, there is a $w = T(v)$ such that $Uw = UT(v) = z$. T does not have to be onto: for example, take the same example as in part (a). (c) Suppose that U, T are one-to-one and onto. First we prove UT is one-to-one. Suppose that $x \neq y$. Since T is one-to-one, $T(x) \neq T(y)$. Since U is one-to-one, $U(Tx) \neq U(Ty)$, proving the claim. Now we prove that UT is onto. Take $z \in Z$. Since U is onto, tere is a $w \in W$ such that $U(w) = z$. Since T is onto, there is a v such that $T(v) = w$. Then, $UT(v) = U(w) = z$.

2.3, #13 This is a direct calculation. Note that $AB = (\sum_{k=0}^n a_{ik}b_{kj})_{ij}$ and $BA = (\sum_{k=0}^n b_{ik}a_{kj})_{ij}$. Then, $\mathrm{tr}(AB) = \sum_{i=0}^n \sum_{k=0}^n a_{ik}b_{ki}$, and $\mathrm{tr}(BA) = \sum_{i=0}^n \sum_{k=0}^n b_{ik}a_{ki}$. Swapping the labeling on the indices i and k equate the two. Note that $A^t = (a_{ji})_{ij}$. Then, $\mathrm{tr}(A^t) = \sum_{i=0}^n a_{ii} = \mathrm{tr}(A)$.

2.3, #16 (a) Consider the restricted map $T' : R(T) \to R(T^2)$. It is onto by definition of range, and since $\dim(R(T^2)) = \dim(R(T)) = d$, the null space $N(T') = 0$. By definition, $N(T') = N(T) \cap R(T)$. Furthermore, since $\dim(R(T)+N(T)) = \dim(R(T)) + \dim(N(T)) - \dim(R(T) \cap N(T)) = n - 0$, we have that $R(T) + N(T) = V$, so $V = R(T) \oplus N(T)$.

(b) Since $R(T^{k+1}) \subseteq R(T^k)$ for any integer k, we have $\dim(R(T^{k+1})) \leq \dim(R(T^k))$. Since a sequence of positive integers cannot strictly decrease forever, there must be a k such that $\dim(R(T^{k+1})) = \dim(R(T^k))$, so $R(T^{k+1}) = R(T^k)$. The restricted map $T' : R(T^k) \to R(T^k)$ is onto, and therefore also one-to-one. It follows that $N(T'^k) = \{0\}$, so $N(T^k) \cap R(T^k) = \{0\}$. Then, since $\dim(N(T^k)) + \dim(R(T^k)) = n$, we have $V = N(T^k) \oplus R(T^k)$.

2.3, #17 The linear transformations such that $T^2 = T$ are the projection maps, i.e. we have a decomposition $V = V_0 \oplus V_1$ where $V_0 = N(T)$ and $V_1 = \{x \in V \mid T(x) = x\}$. For any $x \in V$ we can write $x = (x - T(x)) + T(x)$; here, $x - T(x) \in V_0$ since $T(x - T(x)) = T(x) - T^2(x) = 0$, and $T(x) \in V_1$ since $T(Tx) = T^2x = Tx$. To show that $V_1 \cap V_0 = \{0\}$, note that if $T(x) = x$ and $T(x) = 0$, then $x = 0$.