Midterm I. Linear Algebra 110

Instructor: Zvezdelina Stankova

STUDENT NAME: ____________________

GSI'S NAME: ____________________

• **DO NOT OPEN THE MIDTERM UNTIL TOLD TO DO SO!**

• Please, do *all* Problems 1 through 5, as best as you can. The midterm about 70 minutes long. You may **not** leave early.

• Use the provided sheets to write your solutions. You may also use the back of each page for scratch work or solutions. If you need more paper, raise your hand to ask for additional paper. Submit *all* work which you wish to be graded and staple it to this packet! **Loose sheets of paper will NOT be graded!** Ask the instructor for a stapler at the end of the exam.

• The exam is closed notes and books, which means: **no classnotes, no session notes, no review notes, no textbooks, and no other materials can be used during the midterm.** You can use only your **cheat sheet**, this exam packet and additional blank paper provided by the instructor. You cannot use your paper to write on, even if it is only scratch work. The cheat sheet is one side of a regular 8×11 sheet, handwritten only by you. A cheat sheet which doesn't conform to these specifications (for example, there is stuff written on both sides of the sheet) will be disqualified, and the student's midterm may be annulled.

• **NO CALCULATORS ARE ALLOWED DURING THE MIDTERM!**

• **You cannot ask for help in any form from other students, you cannot look at their midterms and copy, you cannot cheat in any way – the midterm is an individual assignment and must be done only by you.** If you have a question regarding the statement of a problem, raise your hand when the instructor or a GSI is around.

• Think of this midterm as an important homework. Check your reasoning and calculations very carefully. Justify *all* your answers, include all intermediate steps and calculations. If you are not sure about how to write something in mathematical notation, explain clearly *in words* what you mean and what you are doing. **Unjustified answers, even if correct, will receive no credit!** On the contrary, good justifications and good work on a problem may receive a lot of credit even if the final answer is incorrect.

• Before turning in your midterm, please, sign the statement below. **Midterms which are not signed will not be graded.**

I, the student whose name and signature appear on this midterm, have completed the midterm without any outside help from people or other sources. I have used only my cheat sheet conforming to the specifications written above. I have not cheated in any way, and I have followed all instructions above. I have stapled all solutions and work wished to be graded to the present exam packet.

STUDENT SIGNATURE: ____________________

Problem	Score
#1	
#2	
#3	
#4	
#5	
Total	/100

Problem 1 (20 pts). Consider the linear transformation $T : \mathbb{R}^3 \to P_3(\mathbb{R})$ given by

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a + bx + (a+b)x^3.$$

(a) Find the matrix $[T]_\beta^\gamma$ of T relative to the standard bases β and γ of $\mathbb{R}^3$ and $P_3(\mathbb{R})$.

(b) Explain why T is neither 1-1 nor onto.

Problem 2 (30 pts). Let V be the set of matrices defined as follows:

$$V = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}.$$

(a) Show that V is a vector space over $\mathbb{R}$.

(b) Give a basis for V and find $\dim V$.

(c) Consider $\mathbb{C}$, the set of complex numbers, as a vector space over $\mathbb{R}$.
Give a basis for $\mathbb{C}$ and find its dimension.

(d) Let $f : V \to \mathbb{C}$ be a function defined by $f\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a + ib$ for all $a, b \in \mathbb{R}$.
Prove that f is an isomorphism of vectors spaces over $\mathbb{R}$.

Problem 3 (20 pts). Let V be a vector space and let $\mathcal{L}(V, V)$ be the vector space of all linear transformations from V to V.

(a) State what it means for several linear transformations $T_1, T_2, \ldots, T_k$ from V to V to be linearly independent in $\mathcal{L}(V, V)$.

(b) Let $V = P_5(\mathbb{R})$. Let T and U be linear transformations from V to V defined by

$$T(f(x)) = f(2) + f'(x) \text{ and } U(f(x)) = f''(x) \text{ for all polynomials } f(x) \in V.$$

Prove that T and U are linearly independent in $\mathcal{L}(V, V)$.

Problem 4 (10 pts). Let V be a vector space of dimension 2017. Let $T : V \to V$ be a linear transformation and let $\vec{v}$ be some vector in V. Prove that for a large enough positive integer m the vectors

$$\vec{v}, T(\vec{v}), T^2(\vec{v}), T^3(\vec{v}), \ldots, T^m(\vec{v})$$

will be linearly *dependent* in V.

Problem 5 (20 pts). True or False?

To discourage guessing, the problem will be graded as follows:

- 2 pts for each correct answer.
- 0 pts for a blank.
- -2 pts for each incorrect answer.
- If anything else but "True" or "False" is written, more than one answer is written, or the answer is hard to read, you will get -2 point.

Note: In all questions below V and W are vector spaces over a field F. They may or may not be finite-dimensional. Thus, read the questions carefully and do not assume anything that is not given.

(1) The intersection of any two *subsets* of V is a *subspace* of V.

Answer: ___________

(2) Subsets of linearly *independent* sets are linear *independent*.

Answer: ___________

(3) If $T : V \to W$ is a function with $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$ for any $\vec{v}_1, \vec{v}_2 \in V$, then T is *linear*.

Answer: ___________

(4) If $T : V \to W$ is a linear transformation, $\dim V = m$, $\dim W = n$, and β and γ are ordered bases of V and W, respectively, then $[T]_\beta^\gamma$ is an $m \times n$ matrix.

Answer: ___________

(5) For square matrices, to be a *left-sided* inverse of a matrix A is the same as to be a *double-sided* inverse of A.

Answer: ___________

(6) The *Replacement Theorem* is used, among other places, in the proof that the *dimension* of a finite-dimensional vector space is a well-defined concept.

Answer: ___________

(7) $V \cong W$ if and only if $\dim V = \dim W$.

Answer: ___________

(8) There are at least *three* different linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T^2 = T$.

Answer: ___________

(9) If $\dim V = \dim W = n$ and $T : V \to W$ is any linear transformation, then there are bases of V and W with respect to which the matrix of T is diagonal with only 1's and 0's along the diagonal.

Answer: ___________

(10) If V and W are finite-dimensional, then $\dim \mathcal{L}(V, W) = \dim \mathcal{L}(W, V)$.

Answer: ___________

GOOD LUCK!