HW #10; Date 10/3/2017
MATH 110 Linear Algebra
with Professor Stankova

Section 3.1: Elementary Matrix Operations and Elem. Matrices

1)

(a) True. By definition, an elementary matrix is obtained by performing an elementary operation on I_n, the $n \times n$ identity matrix.

(b) False. For example, one is allowed to multiple a row/column by any nonzero scalar, which introduces an entry in the elementary matrix which contains that scalar.

(c) True. For example, multiply a row of the identity matrix by the scalar 1.

(d) False. This would mean that any two successive elementary operations could be performed in a single elementary operation, which is not true. For example,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix},$$

the latter of which is not elementary.

(e) True. This is part of the content of Theorem 3.2.

(f) False. For example,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

the latter of which is not elementary.

(g) True. The associated row operation becomes a column operation or vice versa.

(h) False. For example, take $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot A = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$. Then there is *no* matrix C (let alone an elementary column operation) such that $B = AC$. This is because the columns of AC must be linear combinations of the columns of A.

(i) True. Since elementary row operations are invertible, one can simply perform the inverse operation of that performed on A to obtain B.

2)

To get from A to B, subtract two times the first column of A from the second column of A. To get from B to C, subtract the first row from the second row. One way to get from C to I_3 is the following sequence of elementary operations:

- Subtract the first row from the third row

- Multiply the second row by $-\frac{1}{2}$
- Add 3 times the second row to the third row
- Subtract the third row from the second row
- Subtract 3 times the third row from the first row

Transforming C to I_3 then progresses as follows:

$$\begin{aligned}
\begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 1 & -3 & 1 \end{pmatrix} &\underset{R_3 \to R_3 - R_1}{\sim} \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 0 & -3 & -2 \end{pmatrix} \\
&\underset{R_2 \to -\frac{1}{2}R_2}{\sim} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & -3 & -2 \end{pmatrix} \\
&\underset{R_3 \to R_3 + 3 \cdot R_2}{\sim} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\
&\underset{R_2 \to R_2 - R_3}{\sim} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&\underset{R_1 \to R_1 - 3 \cdot R_3}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

3)

(a) The matrix is obtained from the identity matrix by swapping the first row and the third row. The inverse would then swap these columns back, and the elementary matrix that does this is

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

(b) The matrix is obtained from the identity matrix by multiplying the second row by three. The inverse would be to multiply the second row by $\frac{1}{3}$, and the elementary matrix that does this is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(c) The matrix is obtained from the identity matrix by adding -2 times the first row to the third row. The inverse would be to add two times the first row to the third row, and the elementary matrix that does this is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

12)

We will prove this for any number of rows m by induction on n, the number of columns.

Base case: $n = 1$. If all entries of A are zero, then we are done. Otherwise, say A has a non-zero entry in the i-th row. Swap rows 1 and i (elementary operation 1), then subtract multiples of row 1 to zero out all other rows:

$$\begin{bmatrix} * \\ * \\ \vdots \\ \alpha_i \\ \vdots \\ * \end{bmatrix} \mapsto \begin{bmatrix} \alpha_i \\ * \\ \vdots \\ * \\ \vdots \\ * \end{bmatrix} \mapsto \begin{bmatrix} \alpha_i \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Inductive step: Suppose as an inductive hypothesis that we can reduce an $m \times n$ matrix to upper triangular form by a series of row operations, and consider an $m \times (n+1)$ matrix A. Then we can write

$$A = \begin{bmatrix} A_0 & a \end{bmatrix},$$

where A_0 is an $m \times n$ matrix, and a is the final column of A. By the inductive assumption, we can find elementary row operations which reduce A_0 to an upper triangular form U; applying these operations to A results in a new matrix

$$A' = \begin{bmatrix} U & a' \end{bmatrix}.$$

If $n + 1 > m$, then the condition of being upper triangular does not impose any extra restriction on the column a', since the last column of the matrix is after the initial $m \times m$ square submatrix which is required to have zeros below the main diagonal. Thus in this case, we already have that A' is upper triangular.

If $n + 1 \leq m$, then we can further divide the matrix A' into the block form

$$A' = \begin{bmatrix} U_0 & a_1' \\ 0 & a_2' \end{bmatrix},$$

where U_0 is an $n \times n$ diagonal matrix and a_1' and a_2' are the first n and last $m - n$ entries respectively of a'. By the argument for the $n = 1$ case, we can find a sequence of elementary row operations which make a_2' upper triangular, where all coordinates are zero except possibly the first. Applying these row operations to the rows in $\begin{bmatrix} 0 & a_2' \end{bmatrix}$, we transform the lower submatrix of A' into one which is zero in all coordinates except possibly the upper right corner. This is sufficient to make A' upper triangular, and this completes the inductive argument.

NOTE: Proofs that a matrix can be reduced to a particular form (upper triangluar, diagonal, etc.) often rely on induction on the size of the matrix. This is a good technique to become familiar with. For another example, read through the proof of Theorem 3.6 in the book.

Section 3.3: Systems of Linear Equations—Theoretical Aspects

1)

(a) False. A system of equations can be inconsistent if it is nonhomogeneous.

(b) False. A homogenuous system of equations with more unknowns than equations will have more than one solution, for instance.

(c) True. The zero vector is always a solution.

(d) False. If the matrix A in a system $Ax = b$ is not full rank (i.e. $\operatorname{rank}(A) = n$), then the system may have more than one solution. As a silly example, any $x \in \mathbb{R}^n$ solves the system $0_n x = 0$, where 0_n is the $(n \times n)$ zero matrix.

(e) False. If $\operatorname{rank}(A) < n$ and the system is nonhomogeneous, then it could be inconsistent.

(f) False. The homogeneous system *always* has a solution, so this gives us no reason to believe that the nonhomogeneous system would also have a solution. Theorem 3.11 gives a useful criterion for determining when a system is consistent.

(g) True. The solution has to be unique, and we know the zero vector is a solution, so it follows that the zero vector is the only solution.

(h) False. This is true only in the homogenuous case.

2)

(a) We see that $x_1 = -3x_2$ in both cases and so the dimension of the solution set is 1 and a basis is $\{(3, -1)\}$.

(c) Adding -2 times the first row of the coefficient matrix to the second row, we see that $-3x_2 + 3x_3 = 0$ and so $x_2 = x_3$. We then substitue this back into the first row to get that $x_1 + x_2 = 0$ and so $x_1 = -x_2$. It follows that the dimension is 1 and a basis is $\{(-1, 1, 1)\}$.

(d) Adding the second row to the first row, we get $3x_1 = 0$ and so $x_1 = 0$. Plugging this into the second row, we see that $-x_2 + x_3 = 0$ and so $x_2 = x_3$. Plugging $x_1 = 0$ and $x_2 = x_3$ into the third row, we see that the result is just $0 = 0$, which means that the third equation is redundant. It follows that the dimension is 1 and a basis is $\{(0, 1, 1)\}$.

(g) We see that we have freedom to choose x_3 and x_4, in which case x_2 is constrained to be $x_2 = x_3 - x_4$ by the second row and x_1 is constrained to be $x_1 = -3x_3 + x_4$ by the first row. It follows that the dimension is 2 and a basis is $\{(-3, 1, 1, 0), (1, -1, 0, 1)\}$.

3)

We make use of Theorem 3.9 to find the set of all solutions based on a single nonhomogeneous solution and the set of all homogeneous solutions.

(a) One solution is $(5, 0)$. The solution set is then $(5, 0) + \mathrm{Span}(\{(3, -1)\})$.

(c) One solution is $(3, 0, 0)$. The solution set is then $(3, 0, 0) + \mathrm{Span}(\{(-1, 1, 1)\})$.

(d) Adding the first and second rows, we see that a solution must have $x_1 = 2$. The remaining equations then become $x_2 - x_3 = 1$, so a solution is given by $(2, 2, 1)$. The solution set is then $(2, 2, 1) + \mathrm{Span}(\{(0, 1, 1)\})$.

(g) From the homogeneous case we know that we can set x_3 and x_4 freely, so set $x_3 = 0, x_4 = 1$. Solving for x_1 and x_2 leads to the solution $(0, 0, 0, 1)$. The solution set is then

$$(0, 0, 0, 1) + \mathrm{Span}\left(\{(-3, 1, 1, 0), (1, -1, 0, 1)\}\right).$$

4)

(a) We have coefficient matrix given by

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix},$$

with inverse matrix

$$A^{-1} = -\begin{pmatrix} 5 & -3 \\ -2 & 1 \end{pmatrix}.$$

In particular, we can find the solution to the system by computing

$$A^{-1}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -11 \\ 5 \end{pmatrix}.$$

(b) Simiarly, we have coefficient matrix given by

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & -2 & 1 \end{pmatrix},$$

with inverse matrix

$$A^{-1} = \frac{1}{9}\begin{pmatrix} 3 & 0 & 3 \\ 1 & 3 & -2 \\ -4 & 6 & -1 \end{pmatrix}.$$

Thus the solution to the system is given by the computation

$$A^{-1}\begin{pmatrix}5\\1\\4\end{pmatrix} = \begin{pmatrix}3\\0\\-2\end{pmatrix}.$$

6)

This amounts to solving $a + b = 1$ and $2a - c = 11$. Solving for b and c in terms of a, we have that $b = 1 - a$ and $c = 2a - 11$. Thus the solution set is

$$\left\{ \begin{pmatrix}a\\1-a\\2a-11\end{pmatrix} : a \in \mathbb{R} \right\}$$

WARNING: In this context, $T^{-1}(1, 11)$ does not mean that T is invertible! The problem is asking for the set $\{v \in \mathbb{R}^3 : T(v) = (1, 11)\}$, the set of all vectors that get mapped to $(1, 11)$ under T. This is known as the *preimage* or *inverse image* of $(1, 11)$ under the map T.

7)

A system of linear equations $Ax = b$ has solutions iff the vector b is in the span of the columns of A, as follows from Theorem 3.11.

(a) For coefficient matrix A given by

$$A = \begin{bmatrix}1 & 1 & -1 & 2\\1 & 1 & 2 & 0\\2 & 2 & 1 & 2\end{bmatrix}$$

we can express the column span in various forms by applying elementary column operations and eliminating unnecessary columns.

$$\begin{aligned}
&\mathrm{Span}\left\{ \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\0\\2\end{bmatrix} \right\}\\
=&\mathrm{Span}\left\{ \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\0\\2\end{bmatrix} \right\}\\
=&\mathrm{Span}\left\{ \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}0\\3\\3\end{bmatrix}, \begin{bmatrix}0\\-1\\-1\end{bmatrix} \right\}\\
=&\mathrm{Span}\left\{ \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}.
\end{aligned}$$

Since $b = (2, 1, 4)$ is not in this span, the system of equations does not have a solution.

(d) We work with the span of the columns of the coefficient matrix

$$A = \begin{bmatrix} 1 & 1 & 3 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & 8 & -1 \end{bmatrix}.$$

In particular we have

$$\begin{aligned}
&\operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 8 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\} \\
&= \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 3 \end{bmatrix} \right\} \\
&= \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 3 \end{bmatrix} \right\} \\
&= \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ -1 \end{bmatrix} \right\} \\
&= \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.
\end{aligned}$$

This set is linearly independent and so (by the Dimension Theorem) spans all of $\mathbb{R}^4$. We conclude that $b = (0, 1, 1, 0)$ is in the span, so the system of equations has a solution.

(e) We work with the span of the columns of the coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ 1 & -4 & 7 \end{bmatrix}$$

In particular we have

$$
\begin{aligned}
&\operatorname{Span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}2\\1\\-4\end{bmatrix},\begin{bmatrix}-1\\2\\7\end{bmatrix}\right\}\\
=&\operatorname{Span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}0\\-3\\-6\end{bmatrix},\begin{bmatrix}0\\4\\8\end{bmatrix}\right\}\\
=&\operatorname{Span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}0\\1\\2\end{bmatrix}\right\}.
\end{aligned}
$$

Since $b = (1, 3, 4)$ is not in this span, the system of equations does not have a solution.

8)

The range is spanned by $\{T(e_1), T(e_2), T(e_3)\} = \{(1, 0, 1), (1, 1, 0), (0, -2, 2)\}$. Notice that $T(e_3) = 2(T(e_1) - T(e_2))$, and so the range is also spanned by $\{(1, 0, 1), (1, 1, 0)\}$. In particular:

(a) We see that $(1, 3, -2) = 3T(e_2) - 2T(e_1)$.

(b) We see that $(2, 1, 1) = T(e_1) + T(e_2)$.

9)

This follows largely from Theorem 2.13. Assume that A is an $m \times n$ matrix. Note that

$$Ax = A\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i A e_i = \sum_{i=1}^{n} x_i u_i,$$

where u_i is the ith column of A. But $R(L_A) = \operatorname{Span}(\{u_1, \ldots, u_n\})$. Thus, the equation $Ax = b$ has a solution iff b is in the span of the columns of A iff b is in the range of L_A.

Section 2.6: Dual Bases

1)

(a) False. By definition, a linear functional is a linear transformation from a vector space to its field of scalars.

(b) True. In general, a linear functional on a vector space of dimension n may be represented as a $1 \times n$ matrix.

(c) True. The instructions said to assume that all vector spaces are finite-dimensional. In the infinite-dimensional case, this is in general false.

(d) True. This follows from the identification of V and its double dual V^{**}, and so V can be regarded as the dual of V^*.

(e) False. The statement depends on the choice of isomorphism T and is not true in general.

(f) True. By definition, $\mathrm{dom}((T^t)^t) = \mathrm{codom}(T^t)^* = (\mathrm{dom}(T)^*)^* = V^{**}$.

(g) True. Since V and W are finite-dimensional, we can conclude that $\dim(V^*) = \dim(V) = \dim(W) = \dim(W^*)$.

(h) False. The derivative of a function is not necessarily a scalar. This map is, however, a linear transformation.

2)

(a) f is a linear functional since it is a linear transformation (it is the sum of compositions of linear transformations) and maps into the field of scalars $\mathbb{R}$.

(b) f is not a linear functional since it does not map into the field of scalars $\mathbb{R}$.

(c) f is a linear functional since it is a linear transformation (see Exercise 6 in Section 2.1) and maps into the field of scalars F.

(d) f is not a linear map since, for instance, $f(2,0,0) \neq f(1,0,0)+f(1,0,0)$, so in particular it is not a linear functional.

(e) f is a linear functional since it is a linear transformation (see Exercise 15 in Section 2.1) and maps into the field of scalars $\mathbb{R}$.

(f) f is a linear functional since it is a linear transformation (specifically a coordinate projection) and maps into the field of scalars F.

3)

(a) Let $\beta^* = \{f_1, f_2, f_3\}$. Then

$$\begin{aligned}
f_1(1,0,1) &= 1 = 1\cdot f_1(e_1) + 0\cdot f_1(e_2) + 1\cdot f_1(e_3)\\
f_1(1,2,1) &= 0 = 1\cdot f_1(e_1) + 2\cdot f_1(e_2) + 1\cdot f_1(e_3)\\
f_1(0,0,1) &= 0 = 0\cdot f_1(e_1) + 0\cdot f_1(e_2) + 1\cdot f_1(e_3).
\end{aligned}$$

We see that $f_1(e_3) = 0$ from the last equation, which implies that $f_1(e_1) = 1$ in the first equation, and so from the second equation $f_1(e_2) = -1/2$. Thus,

$$f_1(x,y,z) = x - \frac{y}{2}.$$

Similarly,

$$\begin{aligned}
f_2(1,0,1) &= 0 = 1\cdot f_2(e_1) + 0\cdot f_2(e_2) + 1\cdot f_2(e_3)\\
f_2(1,2,1) &= 1 = 1\cdot f_2(e_1) + 2\cdot f_2(e_2) + 1\cdot f_2(e_3)\\
f_2(0,0,1) &= 0 = 0\cdot f_2(e_1) + 0\cdot f_2(e_2) + 1\cdot f_2(e_3).
\end{aligned}$$

We see that $f_2(e_3) = 0$ from the last equation, which implies that $f_1(e_1) = 0$ in the first equation, and so $f_2(e_2) = 1/2$. Thus,

$$f_2(x, y, z) = \frac{y}{2}.$$

Finally,

$$\begin{aligned}
f_3(1, 0, 1) &= 0 = 1 \cdot f_3(e_1) + 0 \cdot f_3(e_2) + 1 \cdot f_3(e_3) \\
f_3(1, 2, 1) &= 0 = 1 \cdot f_3(e_1) + 2 \cdot f_3(e_2) + 1 \cdot f_3(e_3) \\
f_3(0, 0, 1) &= 1 = 0 \cdot f_3(e_1) + 0 \cdot f_3(e_2) + 1 \cdot f_3(e_3).
\end{aligned}$$

We see that $f_3(e_3) = 1$ from the last equation, which implies that $f_3(e_1) = -1$ in the first equation, and so $f_3(e_2) = 0$. Thus,

$$f_3(x, y, z) = -x + z.$$

4)

Suppose a linear combination of the functionals f_i gives the zero functional, so that for all $x, y, z \in \mathbb{R}$,

$$(af_1 + bf_2 + cf_3)(x, y, z) = (a + b)x + (-2a + b + c)y + (b - 3c)z = 0.$$

Then evaluating at e_1, we have the relation $a + b = 0$. Evaluating at e_2, we have the relation $-2a + b + c = 0$. And evaluating at e_3, we have the relation $b - 3c = 0$. Solving for a and c in terms of b and using the middle relation, we get that $b = 0$ and so $a = c = 0$. Thus f was the trivial linear combination to begin with, and this implies that $\{f_1, f_2, f_3\}$ is a linearly independent set of size $3 = \dim(V^*)$. It follows that $\{f_1, f_2, f_3\}$ is a basis for V^*.

For the second part of the problem, we want vectors v_1, v_2, and v_3 in $\mathbb{R}^3$ such that

$$f_i(v_j) = \delta_{ij}.$$

For $v_1 = (a_1, b_1, c_1)$, this amounts to

$$a_1 - 2b_1 = 1, \quad a_1 + b_1 + c_1 = 0, \quad b_1 - 3c_1 = 0.$$

Thus,

$$1 + 2b_1 + b_1 + \frac{b_1}{3} = 0$$

and so

$$v_1 = (2/5, -3/10, -1/10).$$

Similarly, for $v_2 = (a_2, b_2, c_2)$, this amounts to

$$a_2 - 2b_2 = 0, \quad a_2 + b_2 + c_2 = 1, \quad b_2 - 3c_2 = 0.$$

Thus,

$$2b_2 + b_2 + \frac{b_2}{3} = 1$$

and so

$$v_2 = (3/5, 3/10, 1/10).$$

Finally, for $v_3 = (a_3, b_3, c_3)$, this amounts to

$$a_3 - 2b_3 = 0, \quad a_3 + b_3 + c_3 = 0, \quad b_3 - 3c_3 = 1.$$

Thus,

$$2b_3 + b_3 + \frac{b_3 - 1}{3} = 0$$

and so

$$v_3 = (1/5, 1/10, -3/10).$$

5)

Suppose $af_1(p(x)) + bf_2(p(x)) = 0$ for all $p(x) \in P_1(\mathbb{R})$. Taking $p(x) = 1$, we have the relation $a + 2b = 0$. Taking $p(x) = x$, we have the relation $\frac{a}{2} + 2b = 0$. This implies that $a = b = 0$, and so $\{f_1, f_2\}$ is a linearly independent set of size $2 = \dim(V^*)$. It follows that $\{f_1, f_2\}$ is a basis for V^*.

For the second part of the problem, we need to find polynomials p_1 and p_2 such that

$$f_i(p_j(x)) = \delta_{ij}.$$

Solving this problem algebraically, we can get that

$$\int_0^1 ax + b = \frac{a}{2} + b, \quad \text{and} \quad \int_0^2 ax + b = 2a + 2b,$$

which leads to the two systems of equations

$$\begin{pmatrix} \frac{1}{2} & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{2} & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which are solved by the tuples

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}.$$

Thus letting $p_1(x) = -2x + 2$ and $p_2(x) = x - \frac{1}{2}$, we have that $\{f_1, f_2\}$ is the dual basis of the ordered basis $\{p_1, p_2\}$.