6.5 #1 (a) True. $A^* = A^{-1}$, and $A^{-1}A = AA^{-1}$. (b) False (over $\mathbb{R}$); for example, take a rotation operator in $\mathbb{R}^2$. True over $\mathbb{C}$ by theorem. (c) False. Not every invertible matrix is unitary. For example, again take a rotation operator in $\mathbb{R}^2$. (d) True. Being unitarily equivalent is a special kind of similarity (e) False. Take the identity matrix, summed with itself. (f) True; the adjoint is the inverse. (g) False. β must be orthonormal. (h) False. It could be a Jordan block. (i) False. It was proven in class that preservation of the norm implies preservation of the inner product. A generalization of this result may be found in Theorem 6.22 in your textbook.

6.5 #2 (a) The characteristic polynomial is $t^2 - 2t - 3 = (t-3)(t+1) = 0$, so the eigenvalues are $3, -1$. For 3, the eigenvector is $(1,1)$ and for -1, it is $(1,-1)$. We normalize these to get $P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$.

(b) This matrix is normal so we expect to find a unitary matrix. It is a 90 degree rotation matrix, so the eigenvalues are $\pm i$. The eigenvectors are $(\pm i, 1)$. So, $P = \frac{1}{\sqrt{2}}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

(c) The characteristic polynomial is $t^2 - 7t - 8 = (t-8)(t+1)$. So the eigenvalues are $8, -1$. The corresponding eigenvectors are $(3-3i, 6)$ and $(3-3i, -3)$. So we have $U = \begin{pmatrix} \frac{1-i}{\sqrt{6}} & \frac{1-i}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$ and $D = \begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}$.

(d) By the last challenge problem, we know that the eigenvalues are -2 with eigenvectors $(1,0,-1)$ and $(1,-1,0)$ and 4 with eigenvector $(1,1,1)$. We need to apply Gram-Schmidt to the first eigenspace: we get vectors $(1,0,-1)$ and $(1,-2,1)$. Normalizing, we find $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$ and $D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

(e) The eigenvalues are 1 and 4 with the same eigenvectors as (d). So we have the same $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

6.5 #3 By a theorem from class, it is sufficient to show that the composition of norm-preserving linear operators is norm-preserving. Suppose that $||Tx|| = ||x||$ and $||Sx|| = ||x||$. Then $||T(Sx)|| = ||Sx|| = ||x||$.

6.5 #4 In this problem, $(T_z)^* = T_{\overline{z}}$, the operator which takes $u \mapsto \overline{z}u$. Thus, T_z is normal for all z because $z\overline{z} = \overline{z}z$ for all $z \in \mathbb{C}$. It is self-adjoint for real z because that is when $z = \overline{z}$. It is unitary when $||z|| = 1$ because that is when $z\overline{z} = 1$.

6.5 #5 (a) no; they don't even have the same eigenvalues. (1 with multiplicity 2, and 1, -1) (b) no; the second matrix has eigenvalues which are half the first one (c) no; the second matrix is not invertible but the first is, so they can't even be similar. (d) yes; (d) is normal (it is orthogonal) and its eigenvalues are the same as the diagonal matrix (that is, you get the matrix on the right by orthogonally diagonalizing the matrix on the left) (e) no; the first matrix is not orthogonally diagonalizable (because it is not symmetric) while the second one is (because it is symmetric).

6.5 #6 We have $\langle Tf, Tf \rangle = \int_0^1 |f(t)|^2 |h(t)|^2 \, dt$. For this to be equal to $\int_0^1 |f(t)|^2 dt$ for *all* continuous f, integrating dt has to be the same as integrating $|h(t)|^2 dt$. Thus, $|h(t)|^2 = 1$ for all t. (See Math 104 for a rigorous proof.)

6.5 #7 There is an orthonormal basis β such that $[T]_\beta$ is diagonal. Let U be the matrix such that $[U]_\beta$ is the diagonal matrix whose entries are square roots (choice a square root for each diagonal entry) of the diagonal entries of $[T]_\beta$. It is unitary by construction and satisfies the desired condition. (Recall that square roots are well-defined over the complex numbers, though not unique.)

6.5 #9 Yes. Let $x = a_1 b_1 + \cdots + a_n b_n$ where b_i is the basis. Then $||U(x)|| = \sum_{i,j} a_i a_j \langle U(b_i), U(b_j) \rangle = \sum_{i,j} a_i a_j \langle b_i, b_j \rangle = ||x||$.

6.5 #10 A is diagonalizable. Since $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, it suffices to prove the result for the diagonal matrix. The first statement follows easily. For the second, note that if $A = PDP^*$, then $A^* = PD^*P^*$, so $A^*A = PD^*DP^*$, and the diagonal entries of DD^* are $|\lambda_i|^2$.

6.5 #11 Take for example $A = \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}$. Note: One may find such a matrix by using the Gram-Schmidt process to extend $(1/3, 2/3, 2/3)$ to an orthonormal basis for $\mathbb{R}^3$.

6.5 #15 (a) If U is unitary, it is invertible. If $U(W) \neq W$. then by rank-nullity there is a vector in w which is killed by U, contradicting that is invertible. (b) Let $x \in W^\perp$. Then for any $y \in W$, since $U(W) = W$, we have $y' \in W$ such that $U(y') = y$, and so we have $\langle U(x), y \rangle = \langle U(x), U(y') \rangle = \langle x, y' \rangle = 0$ for every $y \in W$, so $U(x) \in W^\perp$.

6.5 #17 Let U be unitary and upper triangular. The columns are orthogonal under the standard Hermitian inner product. In particular, the first column is e_1, so this means that the first row must be zero except for the diagonal entry. Repeating this argumuent for the second column (which we now know is e_2), et cetera, we find that U must be diagonal.

6.5 #21 (a) Suppose that $A = PBP^*$. Then $A^*A = PB^*P^*PBP^* = PB^*BP^*$, so they have the same trace by the same argument is problem 10. (b) The trace is the sum of the diagonal entries. The first diagonal entry of A^*A is $|A_{11}|^2 + \cdots + |A_{n1}|^2$. The ith diagonal entry is $|A_{1i}|^2 + \cdots + |A_{ni}|^2$. Summing them all up gives the left hand side. Now do the same for B. (c) The left hand side is $1 + 4 + 4 + 1 = 10$, and the right hand side is $1 + 16 + 1 + 1 = 19$.