HW 1; 8/29/2017
MATH 110 Linear Algebra
with Professor Stankova

Section 1.1

6. We think of the midpoint of (a,b) and (c,d) as the point reached by starting at (a,b) and walking halfway to (c,d). We know that the vector $(c,d)-(a,b)=(c-a,d-b)$ points this direction, and has magnitude equal to the distance between the points. We compute

$$(a,b)+\frac{1}{2}(c-a,d-b)=\left(\frac{a+c}{2},\frac{b+d}{2}\right).$$

7. Let $\vec{x}=(a,b)$ and $\vec{y}=(c,d)$ be non-collinear vectors in the plane. Then $\vec{x}$ and $\vec{y}$ define a parallelogram with endpoints $\vec{0}$, $\vec{x}$, $\vec{y}$, and $\vec{x}+\vec{y}$, and every parallelogram may be defined in such a manner. (This is because an arbitary parallelogram may be translated so that one of its vertices is the origin. Labelling the adjacent edges $\vec{x}$ and $\vec{y}$, it is clear by the paralleogram law for addition that the opposite vertex is $\vec{x}+\vec{y}$.) The diagonal joining $\vec{x}$ and $\vec{y}$ has midpoint $\left(\frac{a+c}{2},\frac{b+d}{2}\right)$, and the diagonal joining $\vec{0}$ and $\vec{x}+\vec{y}=(a+c,b+d)$ also has midpoint $\left(\frac{a+c}{2},\frac{b+d}{2}\right)$. It follows that the two diagonals intersect at their midpoints, i.e., that they bisect each other.

Section 1.2

1. True/False

(a) True, by definition (VS 3).
(b) False, the zero vector of a vector space is unique (Corollary 1 on page 11, proved in the first lecture).
(c) False, if $x=\vec{0}$, then $ax=bx=\vec{0}$ for any $a,b\in F$.
(d) False, if $a=0$, then $ax=ay=\vec{0}$ for any $x,y\in V$.
(e) True, a vector in F^n, written as a column vector, can be considered as an $n\times 1$ matrix with entries in F.
(f) False, the convention is #(rows) $\times$ #(columns).
(g) False, any two polynomials may be added in $P(F)$ (see Example 4 on page 10).
(h) False, if f and g are polynomials of degree n, then $f+g$ is a polynomial of degree less than or equal to n. To see that $f+g$ can possibly be of lower degree, consider $f(x)=x+1$ and $g(x)=-x+1$. In this case, f and g each have degree 1, while $f+g$ has degree 0.
(i) True, if f is a polynomial of degree n, then we may write $f(x)=a_nx^n+...+a_1x+a_0$, where $a_n\neq 0$. Then, $cf=(ca_n)x^n+...+(ca_1)x+ca_0$ is a polynomial of degree n since $c\neq 0$ and $a_n\neq 0$ implies $ca_n\neq 0$.
(j) True, let $f(x)=c$.
(k) True, by definition (see Example 3 on page 9).

7. By definition, we need to show that

$$f(0)=g(0),\quad f(1)=g(1),$$

and

$$f(0)+g(0)=h(0),\quad f(1)+g(1)+h(1).$$

We compute:

$$f(0) = g(0) = 1, \quad f(1) = g(1) = 3,$$

and

$$f(0) + g(0) = h(0) = 2, \quad f(1) + g(1) = h(1) = 6.$$

10. Let $f, g, h \in V$ and $c, r \in \mathbb{R}$. Then $f+g$ (resp., cf) is again a differentiable real-valued function on the line with derivative $f' + g'$ (resp., cf'). Addition is clearly commutative and associative (as these properties hold for any functions, not just differentiable ones):

$$f(r) + g(r) = g(r) + f(r), \qquad (f(r) + g(r)) + h(r) = f(r) + (g(r) + h(r)),$$

and the function that is identically 0 serves as the zero vector. The additive inverse of a function f is given by the function $-f$ where $(-f)(r) = -f(r)$ for $r \in \mathbb{R}$. Multiplying by the scalar $c = 1$ clearly returns the original vector. If $d \in \mathbb{R}$, then

$$((cd)f)(r) = cdf(r) = c(df)(r) = (c(df))(r).$$

and

$$((c+d)f)(r) = (c+d)f(r) = cf(r) + df(r) = (cf)(r) + (df)(r).$$

Lastly, we observe that

$$(c(f+g))(r) = c(f+g)(r) = cf(r) + cg(r) = (cf)(r) + (cg)(r).$$

11. The closure properties are stipulated to be satisfied:

$$0 + 0 = 0, \quad c0 = 0.$$

Addition is clearly both commutative and associative since the result can only be the single vector 0; for the same reason, the vector 0 is necessarily both the identity element and its own inverse. Clearly, $1 \cdot 0 = 0$. The remaining properties are likewise true (both sides of the required equalities are necessarily equal to the sole vector 0) in all 3 cases.

12. Let f, g be even functions and $c \in \mathbb{R}$. Then cf and $f+g$ are again even functions:

$$(cf)(-r) = cf(-r) = cf(r) = (cf)(r), \quad (f+g)(-r) = f(-r)+g(-r) = f(r)+g(r) = (f+g)(r).$$

The function that is identically 0 is clearly an even function, and the inverse $-f$ is also even:

$$(-f)(-r) = -f(-r) = -f(r) = (-f)(r).$$

The remaining properties follow as in **10.**

14. Yes: the real numbers are a subset of the complex numbers. Having established properties (VS 1-8) for the field of complex numbers (and noting that $\mathbb{R}$ and $\mathbb{C}$ have the same unit), it follows that (VS 1-8) hold for the smaller field of real numbers. Similarly, if a space is closed under scalar multiplication by complex numbers, it is also closed under scalar multiplication by the smaller set of real numbers. Addition doesn't involve scalars so closure is immediate from V being a vector space over complex numbers.

Note: We didn't use any specific properties of V, just the fact that it is a vector space over complex numbers. By the above discussion, we have actually shown that any complex vector space may be viewed as a real vector space.

15. No: for example, V lacks closure under scalar multiplication by complex numbers. In particular, let $n = 2$. Note that $(1, 1) \in V$, but that $i \cdot (1, 1) = (i, i) \notin V$.

17. No: for example, (VS 5) (i.e. $1x = x$) does not hold for any element (a_1, a_2) for which $a_2 \neq 0$.

18. No: for example, addition is not commutative as soon as $a_1 \neq b_1$. In particular $(1, 0) + (2, 0) = (5, 0)$, but $(2, 0) + (1, 0) = (4, 0)$.

19. No: for example, (VS 8) does not hold,

$$(0,1) = 1 \cdot (0,1) = \left(\frac{1}{2} + \frac{1}{2}\right)(0,1) \neq (0,4) = \frac{1}{2}(0,1) + \frac{1}{2}(0,1)$$

21. Let $(v_1, w_1), (v_2, w_2) \in Z$ and $c \in F$. Then

$$v_1 + v_2 \in V, \quad w_1 + w_2 \in W, \quad cv_1 \in V, \quad cw_1 \in W.$$

Thus,

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \in Z, \quad c(v_1, w_1) = (cv_1, cw_1) \in Z.$$

Addition is commutative (resp., associative) since addition is defined componentwise and the addition in each component is commutative (resp., associative). Similarly, if 0_V is the zero vector of V and 0_W is the zero vector of W, then $(0_V, 0_W)$ acts as a zero vector for Z. If $(v, w) \in Z$ and $(-v)$ is the additive inverse of v in V and $(-w)$ is the additive inverse of w in W, then $(-v, -w) + (v, w) = (0_V, 0_W)$ so $(-v, -w)$ is an additive inverse for (v, w).

If $d \in F$ is another scalar, then

$$\begin{aligned}
&\text{(VS 5)} & 1(v_1, w_1) &= (1v_1, 1w_1) = (v_1, w_1) \\
&\text{(VS 6)} & (cd)(v_1, w_1) &= (cdv_1, cdw_1) \\
& & &= (c(dv_1), c(dw_1)) = c(dv_1, dw_1) = c(d(v_1, w_1)) \\
&\text{(VS 7)} & c((v_1, w_1) + (v_2, w_2)) &= c(v_1 + v_2, w_1 + w_2) \\
& & &= (c(v_1 + v_2), c(w_1 + w_2)) = (cv_1 + cv_2, cw_1 + cw_2) \\
& & &= (cv_1, cw_1) + (cv_2, cw_2) = c(v_1, w_1) + c(v_2, w_2) \\
&\text{(VS 8)} & (c+d)(v_1, w_1) &= ((c+d)v_1, (c+d)w_1) = (cv_1 + dv_1, cw_1 + dw_1) \\
& & &= (cv_1, cw_1) + (dv_1, dw_1) = c(v_1, w_1) + d(v_1, w_1).
\end{aligned}$$

22. Recall that two $m \times n$ matrices A and B are equal if and only if

$$A_{ij} = B_{ij} \text{ for all } 1 \leq i \leq m \text{ and } 1 \leq j \leq n.$$

Since there are mn entries in an $m \times n$ matrix, and Z_2 has two elements, it follows that $\mathbf{M}_{m \times n}(Z_2)$ has 2^{mn} vectors. (If this is confusing, consider that there are two possible entries in the first position, two in the next, etc.)