HW #5; date: Feb 7, 2017
MATH 110 Linear Algebra
with Professor Stankova

2.1 Linear Transformations, Null spaces, and Ranges

1. (a) True. By definition.

   (b) False. Consider the linear transformation $T : \mathbb{C} \to \mathbb{C}$ given by $T(x) = \bar{x}$. It preserves sums, but it does not preserve scalar products because $iT(i) = i\bar{i} = 1$ but $T(i \cdot i) = T(-1) = -1$.

   (c) False. Consider the function $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = x + 1$. It is one-to-one, but $T(0) \neq 0$. It is true if one assumes that T is linear though.

   (d) True. Since $T(0) = T(0 + 0) = T(0) + T(0)$, we have $T(0) = 0$ if we cancel $T(0)$ in the both sides. Alternatively, $T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0$

   (e) False. Consider the linear transformation $T : \mathbb{R} \to \mathbb{R}^3$ given by $T(x) = (x, 0, 0)$. Then nullity$(T) = 0$ and rank$(T) = 1$, but $\dim W = 3$.

   In fact, the rank and nullity don't really depend on how large the codomain vector space is. Given any transformation $T : V \to W$, if one composes this with a one-to-one linear transformation $W \to W'$ one obtains a linear transformation $T' : V \to W'$ which has the same rank and nullity, but the dimension of W' has changed. This shows that rank and nullity really aren't dependent on the codomain vector space, just the image vector space

   (f) False. Consider the linear transformation $T : \mathbb{R} \to \{0\}$ given by $T(x) = 0$. Then $\{1\}$ is linearly independent, but $\{T(1)\}$ is linearly dependent.

   (g) True. It is the uniquness part of Theorem 2.6.

   (h) True. It follows from the existence part of Theorem 2.6.

3. The function T is a linear transformation since it preserves sums and scalar products. A basis for $\operatorname{Ker} T$ is $\emptyset$—indeed, if $T(a_1, a_2) = (0, 0, 0)$ then $a_1 = -a_2$ and $2a_1 = a_2$, which has only the solution (a_1, a_2). A basis for $\operatorname{Im} T$ is $\{(1, 0, 0), (0, 0, 1)\}$—these are clearly linearly independent vectors in the image, and since the image has dimension 2 (by Rank-Nullity) the must therefore be a basis. Thus $\dim \operatorname{Ker} T = 0$ and $\dim \operatorname{Im} T = 2$. These verify the dimension theorem since $\dim V = 2$. Moreover, T is one-to-one by Theorem 2.4, and T is not onto by Theorem 2.5.

5. The function T is a linear transformation since it preserves sums and scalar products. If $f(x) = a_0 + a_1 x + a_2 x^2$, then $T(f(x)) = a_0 x + a_1(x^2 + 1) + a_2(x^3 + 2x)$. A basis for $\operatorname{Ker} T$ is $\emptyset$, which can be seen directly from the fact that $\{1, x^2 + 1, x^3 + 2x\}$ are linearly independent vectors in $P_3(\mathbb{R})$. A basis for $\operatorname{Im} T$ is $\{1, x^2 + 1, x^3 + 2x\}$, this follows immediately from

their independent and the explicit expression for $T(a0+a_1x+a_2x^2)$ given above. Thus $\dim \operatorname{Ker} T = 0$ and $\dim \operatorname{Im} T = 3$. These verify the dimension theorem since $\dim V = 3$. Moreover, T is one-to-one by Theorem 2.4, and T is not onto by Theorem 2.5.

6. The function T is a linear transformation since trace preserves sums and scalar products. Let E_{ij} denote the $n \times n$ matrix whose only nonzero entry is 1 at (i,j). Then $T(E_{11}) = 1$, so $\{1\}$ is a basis for $\operatorname{Im} T$. If $T(M) = 0$, then the sum of the diagonal entries of M is zero, so a basis for $\operatorname{Ker} T$ is

$$\{E_{22} - E_{11}, \dots, E_{nn} - E_{11}\} \cup \{E_{ij}\}_{i \neq j}.$$

Thus $\dim \operatorname{Ker} T = n(n-1)+n-1 = n^2-1$ and $\dim \operatorname{Im} T = 1$. These verify the dimension theorem since $\dim V = n^2$. Moreover, T is not one-to-one by Theorem 2.4, and T is onto by Theorem 2.5.

9. (a) The function T is not linear since $-2T(1,0) \neq T(-2,0)$.

(b) The function T is not linear since $-2T(1,0) \neq T(-2,0)$. Alternatively, $T(0) \neq 0$

(c) The function T is not linear since $-2T(1,0) \neq T(-2,0)$.

(d) The function T is not linear since $-2T(1,0) \neq T(-2,0)$.

(e) The function T is not linear since $-2T(1,0) \neq T(-2,0)$.

10. We have $T(2,3) = 3T(1,1) - T(1,0) = (5,11)$. If $T(a,b) = 0$, then

$$T(a,b) = T(a-b,0) + T(b,b) = (a-b)(1,4) + b(2,5) = (a+b, 4a+b),$$

so $a+b = 4a+b = 0$. Then $a = b = 0$, so $\operatorname{Ker} T = \{0\}$. Thus T is one-to-one. Alternatively, one easily sees that $(1,4)$ and $(2,5)$ form a basis for $\mathbb{R}^2$ so that the image of T is all of $\mathbb{R}^2$. By the Rank-Nullity Theorem we deduce that $\dim \ker T = 0$ so that T is one-to-one as desired.

11. The existence follows from Theorem 2.6. We have

$$T(8,11) = 2T(1,1) + 3T(2,3) = (5,-3,16).$$

12. If T is a linear transformation satisfying the condition, then

$$T(0,0,0) = 2T(1,0,3) + T(-2,0,-6) = (2,2) + (2,1) \neq (0,0),$$

which is a contradiction. Thus there are no linear transformations satisfying the conditions.

13. If $c_1v_1 + \cdots + c_kv_k = 0$ where $c_1, \dots, c_k$ are scalars, then $T(c_1v_1 + \cdots + c_kv_k) = 0$, so $c_1T(v_1) + \cdots + c_kT(v_k) = 0$. Since $\{T(v_1), \dots, T(v_k)\}$ is linearly independent, we have $c_1 = \cdots = c_n = 0$. This shows that $\{v_1, \dots, v_k\}$ is linearly independent.

Alternatively, if $\{v_1, \ldots, v_k\}$ are linearly dependent then $U := \operatorname{span}\{v_1, \ldots, v_k\}$ has dimension less than k. Thus, evidently the image U under T must have dimension less than k. But, the image of T contains $\operatorname{span}\{w_1, \ldots, w_k\}$ which as dimension k (since the vectors are independent)—this is a contradiction.

14. (a) Assume that T is one-to-one, and let $\{v_1, \ldots, v_n\}$ be a linearly independent set in V. If $c_1T(v_1) + \cdots + c_nT(v_n) = 0$ where $c_1, \ldots, c_n$ are scalars, then $T(c_1v_1 + \cdots + c_nv_n) = 0$, so $c_1v_1 + \cdots + c_nv_n = 0$ since T is one-to-one. Then $c_1 = \cdots = c_n = 0$ since $\{v_1, \ldots, v_n\}$ is linearly independent. This shows that $\{T(v_1), \ldots, T(v_n)\}$ is linearly independent. On the other hand, assume that T preserves linear independecy. If $T(v) = 0$, then $\{v\}$ should be linearly dependent since otherwise T does not preserve linear independency, so $v = 0$. By Theorem 2.4, we conclude that T is one-to-one.

    Alternatively, note that T is one-to-one if and only if $\ker T = \{0\}$. This is clearly equivalent to the fact that for all subspaces $U \subseteq V$ one has that $\ker(T\mid_U) = \{0\}$ (where $T\mid_U$ is the restriction of T to U). But, by the Rank-Nullity theorem, this is equivalent to the claim that for all $U \subseteq V$ a subspace one has that $\dim U = \dim \operatorname{im}(T\mid_U)$. But, evidently this is equivalent to the claim that T sends linearly independent subsets to linearly independent subsets since, for $\{v_1, \ldots, v_k\}$ linearly independent, one has that $\{T(v_1), \ldots, T(v_k)\}$ is linearly independent if nd only if its span has dimension k.

    (b) If S is linearly independent, then $T(S)$ is linearly independent by (a). If S is linearly dependent, then $T(S)$ is linearly dependent since T preserves linear dependency. Explicitly, if $S = \{v_1, \ldots, v_n\}$ is linearly independent, then we can write $c_1v_1 + \cdots + c_nv_n = 0$ for scalars not all zero; then $T(S) = \{T(v_1), \ldots, T(v_n)\}$ and then $0 = T(0) = T(c_1v_1 + \cdots + c_nv_n) = c_1T(v_1) + \cdots + c_nT(v_n) = 0$.

    These establish the if any only if statement.

    (c) By (a), $T(\beta)$ is linearly independent. Since β spans V and T is onto, $T(\beta)$ spans W. Thus $T(\beta)$ is a basis for W.

15. If $T(f) = 0$, then $f = 0$ since the derivative of $T(f)$ is f. Thus T is one-to-one. If $T(f) = 1$, then $f = 0$ since the derivative of $T(f)$ is f, but it is impossible since $T(0) = 0$. Thus 1 is not in $\operatorname{Im} T$, so T is not onto.

18. If $T(x, y) = (y, 0)$, then $\operatorname{Ker} T = \operatorname{Im} T = \operatorname{span}\{(1, 0)\}$.

20. Becase T preserves sums and scalar products, and V_1 is closed under sums and scalar products, $T(V_1)$ is also closed under sums and scalar products. Since $T(0) = 0$, $T(V_1)$ contains the zero element, so $T(V_1)$ is a subspace of W. Let c_1 and c_2 be scalars, and let v_1 and v_2 be elements of V. If $T(v_1)$ and $T(v_2)$ are in W_1, then $T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2)$ is

also in W_1 since W_1 is a subspace of W. Since $T(0) = 0$ is in W_1, the set $\{v \in V : T(v) \in W_1\}$ is a subspace of V.

22. Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation. By Section 2.2, T is given by a matrix multiplication. Thus if $m = 1$, then

$$T(x_1, \ldots, x_n) = a_1 x_1 + \cdots + a_n x_n$$

for some $a_1, \ldots, a_n \in \mathbb{F}$.

To understand precisely what this has to do with the generalization of $T : \mathbb{F}^n \to \mathbb{F}^m$, consider that any such T looks like $T(x_1, \ldots, x_n) = (T_1(x_1, \ldots, x_n), \ldots, T_m(x_1, \ldots, x_n))$ where $T_i : \mathbb{F}^n \to \mathbb{F}$ are linear maps. Thus, we see that there are scalars $a_{i,j}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$ such that $T(x_1, \ldots, x_n) = (a_{1,1}x_1 + \cdots + a_{1,n}x_n, \ldots, a_{m,1}x_1 + \cdots + a_{m,n}x_n)$ by using the expression for the linear maps $T_i : \mathbb{F}^n \to \mathbb{F}$ mentioned above. Of course, this is no different than the matrix interpretation of a linear map between the standard powers of $\mathbb{F}$.

26. (a) Let x and y be elements of V. If $x = x_1 + x_2$ and $y = y_1 + y_2$ with $x_1, y_1 \in W_1$ and $x_2, y_2 \in W_2$, then for $c, d \in F$, we have $cx + dy = (cx_1 + dy_1) + (cx_2 + dy_2)$ with $cx_1 + dy_1 \in W_1$ and $cx_2 + dy_2 \in W_2$. By the definition of T, $T(cx + dy) = cx_1 + dy_1$. Thus T is linear. If $T(x) = x$, then x is in W_1 since $T(x)$ is in W_1 by definition. On the other hand, if x is in W_1, then $x = x + 0$ with $x \in W_1$ and $0 \in W_2$, so $T(x) = x$ by the definition of T. Thus $W_1 = \{x \in V : T(x) = x\}$.

(b) By (a), $W_1 \subset \operatorname{Im} T$. On the other hand, $\operatorname{Im} T \subset W_1$ by the definition of T. Thus $W_1 = \operatorname{Im} T$. If $T(x) = 0$, then $x = x_2$, so x is in W_2. On the other hand, if x is in W_2, then $x = 0 + x$ with $0 \in W_1$ and $x \in W_2$, so $T(x) = 0$. Thus $W_2 = \operatorname{Ker} T$.

(c) If $W_1 = V$, then $W_2 = \{0\}$, and $x = x + 0$ with $x \in W_1$ and $0 \in W_2$. Thus $T(x) = x$.

(d) If $W_1 = \{0\}$, then $W_2 = V$, and $x = 0 + x$ with $0 \in W_1$ and $x \in W_2$. Thus $T(x) = 0$.