6.1 #1 (a) True (though it must satisfy some properties). (b) True. In this text it's the case that you assume the field is $\mathbb{R}$ or $\mathbb{C}$. (c) False. It is conjugate-linear in the second variable. (d) False. You can always scale an inner product by nonzero factor. (e) False. (f) False. (g) False. Take the dot product, $x = e_1$, $y = e_1 + e_2$, $z = e_1 + 2e_2$. (h) True. In particular, $\langle y, y\rangle = 0$ so $y = 0$.

6.1 #2 We have $\langle x, y\rangle = 8 + 5i$, $||x|| = \sqrt{7}$, $||y|| = \sqrt{14}$, $||x + y|| = \sqrt{37}$. We verify that $|8 + 5i| = \sqrt{89} < \sqrt{7}\sqrt{14} = \sqrt{98}$ and $\sqrt{7} + \sqrt{14} > \sqrt{37}$.

6.1 #3 We have we have $\langle f, g\rangle = \int_0^1 te^t = (te^t - e^t)|_0^1 = 1$, $||f|| = \sqrt{\int t^2} = \frac{1}{\sqrt{3}}$ and $||g|| = \sqrt{\int e^{2t}} = \sqrt{\frac{e^2-1}{2}}$, and, $||f + g|| = \sqrt{\int t^2 + 2te^t + e^{2t}} = \sqrt{1/3 + 2 + e^2/2 - 1/2} = \sqrt{11/6 + e^2/2}$. We verify that $|1| < \sqrt{\frac{e^2-1}{6}}$ and $\sqrt{1/3} + \sqrt{(e^2 - 1)/2} > \sqrt{4/3 + e^2}$.

6.1 #4 (a) We see that $\langle cx, y\rangle = \mathrm{tr}(y^*cx) = c\mathrm{tr}(y^*x) = c\langle x, y\rangle$ and that $\overline{\langle x, y\rangle} = \overline{\mathrm{tr}(y^*x)} = \mathrm{tr}(\overline{\overline{y^t}x}) = \mathrm{tr}(y^t\overline{x}) = \mathrm{tr}(\overline{x}^ty) = \langle y, x\rangle$. Here we use that the trace of the transpose of a matrix is equal to the trace of the matrix (which is obvious). (b) $||A|| = 4$, $||B|| = \sqrt{3}$, and $\langle A, B\rangle = 1 - 3i$.

6.1 #5 Since $A^* = A$, this inner product is conjugate-symmetric, i.e. $\langle x, y\rangle = xAy^* = (xAy^*)^t = \bar{y}A^tx^t = \overline{yA^*x^*} = \langle y, x\rangle$. It is clearly linear in the first variable. We only need to show it is positive definite. To this end, we compute $||(a, b)|| = ||a||^2 - \bar{a}bi + a\bar{b}i + 2||b||^2 = ||a||^2 + 2||b||^2 + i(\langle a, b\rangle - \overline{\langle a, b\rangle}) = ||a||^2 + 2||b||^2 - 2\Im(\langle a, b\rangle) > ||a||^2 + ||b||^2 - 2||a||||b|| + ||b||^2 = (||a|| - ||b||)^2 + ||b||^2 > 0$ unless $a = b = 0$ (the second to last inequality is Cauchy-Schwartz, and also using that $|\Im(\langle a, b\rangle| \leq |\langle a, b\rangle|)$. We compute $\langle(1 - i, 2 + 3i), (2 + i, 3 - 2i)\rangle = 16 - 7i$.

6.1 #8 (a) If $(a, b) = (c, d) = (1, 1)$, we have $\langle(a, c), (b, d)\rangle = 0$ but $(a, b) \neq 0$ (b) Not linear in each variable, e.g. $\langle cA, B\rangle \neq c\langle A, B\rangle$. (c) If $f(x) = g(x) = 1$, then $\langle f, f\rangle = 0$.

6.1 #9 (a) If Let $z = a_1b_1 + \cdots a_nb_n$ be arbitrary. Then $\langle x, z\rangle = \overline{a_1}\langle x, b_1\rangle + \cdots + \overline{a_n}\langle x, b_n\rangle = 0$, so $x = 0$. (b) If $\langle x, z\rangle = \langle y, z\rangle$, then $\langle x - y, z\rangle = 0$ for every z, so $x - y = 0$, so $x = y$.

6.1 #10 $||x + y||^2 = \langle x + y, x + y\rangle = ||x||^2 + \langle x, y\rangle + \langle y, x\rangle + ||y||^2 = ||x||^2 + ||y||^2$. Let x, y be vectors that give two sides of a triangle meeting at a right angle (i.e. they are orthogonal). Then $||x||^2$ is the square of the length of one side, $||y||^2$ the other, and $||x + y||^2$ is the square of the length of the hypotenuse.

6.1 #11 $||x+y||^2+||x-y||^2 = ||x||^2+\langle x, y\rangle+\langle y, x\rangle+||y||^2+||x||^2-\langle x, y\rangle-\langle y, x\rangle+|y||^2 = 2||x||^2+2||y||^2$. It says that the squares of the lengths of the diagonals is equal to the sum of the squares of the sides.

6.1 #14 That $(A + B)^* = A^* + B^*$ follows from the statement for transposes and for conjugation commuting with addition. That $(cA)^* = \bar{c}A^*$ follows from the fact that conjugation commutes with multiplication.

6.1 #16 (a) We will ignore the constant for convenience. Linearity follows from linearity of the integral. Conjugate symmetry follows since $\int f(x)\overline{g(x)}\, dx = \overline{\int g(x)\overline{f(x)}\, dx}$. For positive definite-ness, note that $\langle f, f\rangle = \int_0^1 f(x)^2\, dx$. We need a theorem from analysis that says that any continuous function such that $f(x) \geq 0$ on $[0,1]$ and which attains a nonzero value on the interval, has nonzero integral. (b) No. For example, take $f(x) = \begin{cases} 0 & x < \frac{1}{2} \\ x - \frac{1}{2} & x \geq \frac{1}{2} \end{cases}$, and $g(x) = f(x)$. One can check that $\langle f, g \rangle = 0$, but $f \neq 0$.

6.1 #17 We know that $||x|| = 0$ if and only if $x = 0$. Thus the kernel of T is zero.

6.1 #18 First, it's easy to see that $\langle -, - \rangle'$ is linear in the first variable since $\langle -, - \rangle$ is and T is. Furthermore, $\langle y, x \rangle' = \langle T(y), T(x) \rangle = \overline{\langle T(x), T(y) \rangle} = \overline{\langle x, y \rangle'}$, so it is conjugate-symmetric. To show the final property, note that $\langle x, x \rangle' = \langle T(x), T(x) \rangle = 0$ if and only if $T(x) \neq 0$, so it is an inner product if and only if T has trivial kernel.

6.1 #19 (a) $||x \pm y||^2 = \langle x \pm y, x \pm y \rangle = \langle x, x \rangle \pm \langle x, y \rangle \pm \langle y, x \rangle + \langle y, y \rangle = ||x||^2 + ||y||^2 \pm (\langle x, y \rangle + \overline{\langle x, y \rangle})$. Now, recall that $z + \overline{z} = 2\Re(z)$, and the claim follows. (b) It suffices to assume that $||x|| > ||y||$; otherwise, swap the roles of x and y. In this case, using the triangle inequality, we have $||x - y + y|| \leq ||x - y|| + ||y||$, so $||x|| + ||y|| \leq ||x - y||$ as desired.

6.1 #20 (a) We have $\frac{1}{4}||x + y||^2 - \frac{1}{4}||x - y|| = \frac{1}{4}(\langle x, y \rangle + \langle y, x \rangle + \langle x, y \rangle + \langle y, x \rangle = \langle x, y \rangle$. Note that it is important that $F = \mathbb{R}$, so that $\langle x, y \rangle = \langle y, x \rangle$. (b) Expanding the RHS,

$$\frac{1}{4}(\sum i^k \langle x, x \rangle + i^k \langle x, i^k y \rangle + i^k \langle i^k y, x \rangle + i^k \langle i^k y, i^k y \rangle) = \frac{1}{4}(\sum i^k ||x||^2 + \langle x, y \rangle + i^{2k} \langle y, x \rangle - i^{3k} ||y||)$$

$$= \frac{1}{4}(\sum i^k ||x||^2 + i^k ||y||^2 + (-1)^k \langle y, x \rangle - (-1)^k \langle x, y \rangle) = \frac{1 + i - 1 - i}{4}(||x||^2 + ||y||^2) + \frac{2 - 2}{2}(\langle y, x \rangle) + \langle x, y$$

$$= \langle x, y \rangle.$$

Prove the following properties of complex numbers: $z + \bar{z} = 2\Re(z)$ – write $z = a + bi$, then $a + bi + a - bi = 2a$; $z - \bar{z} = 2i\Im(z)$ – write $z = a + bi$, then $a + bi - a + bi = 2bi$; $z \cdot \bar{z} = \Re(z)^2 + \Im(z)^2 = ||z||^2$ – write $z = a + bi$, then $(a + bi)(a - bi) = a^2 + b^2$; $\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$ – write $z_1 = a_1 + b_1 i, z_2 = a_2 + b_2 i$, then $\overline{a_1 + b_1 i + a_2 + b_2 i} = a_1 + a_2 - b_1 i - b_2 i = a_1 - b_1 i + a_2 - b_2 i$; $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ – let $z_1 = a + bi$ and $z_2 = c + di$, then $\overline{(a + bi)(c + di)} = \overline{ac - bd + adi + bci} = ac - bd - adi - bci = (a - bi)(c - di)$

In class challenges: Prove (a) $(1 \cdot 2016 + 2 \cdot 2017 + \cdots + 2015 \cdot 4020)^2 < (1^2 + 2^2 + \cdots + 2015^2) \cdot (2016^2 + 2017^2 + \cdots + 4020^2)$. Use Cauchy-Schwartz on the vectors $(1, 2, 3, 4, \ldots, 2015)$ and $(2016, 2017, \ldots, 4020)$. To show that the inequality is strict, we need to show that the vectors are not multiples of each other. However, this is more or less obvious. For part (b), apply the triangle inequality to these vectors.