HW #16; date: Oct. 24, 2017
MATH 110 Linear Algebra
with Professor Stankova

5.2.14 (b) Let $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$. Then the system is $x' = Ax$, where $A = \begin{pmatrix} 8 & 10 \\ -5 & -7 \end{pmatrix}$. We diagonalize A: Its characteristic polynomial is $\lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$. Corresponding eigenvectors to $\lambda = 3$ and $\lambda = -2$ are $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, respectively, so we get $A = QDQ^{-1}$ with $D = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$ and $Q = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}$. If $y = Q^{-1}x$, we get $y' = Dy$, hence $y = \begin{pmatrix} c_1 e^{3t} \\ c_2 e^{-2t} \end{pmatrix}$. Finally, $x = Qy = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{3t} \\ c_2 e^{-2t} \end{pmatrix} = \begin{pmatrix} -2c_1 c^{3t} - c_2 e^{-2t} \\ c_1 e^{3t} + c_2 e^{-2t} \end{pmatrix}$.

(c) Let $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$. Then the system is $x' = Ax$ where $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. The eigenvalues of A are 1 and 2, with corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, the first two corresponding to $\lambda = 1$ and the last corresponding to $\lambda = 2$. Then $A = QDQ^{-1}$, with $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Letting $t = Q^{-1}x$, we get $y' = Dy$, hence $y = \begin{pmatrix} c_1 e^t \\ c_2 e^t \\ c_3 e^{2t} \end{pmatrix}$. Then $x = Qy = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^t \\ c_2 e^t \\ c_3 e^{2t} \end{pmatrix} = \begin{pmatrix} c_1 e^t + c_3 e^{2t} \\ c_2 e^t + c_3 e^{2t} \\ c_3 e^{2t} \end{pmatrix}$.

5.4.1 (a) False: V is always $T-$invariant.
(b) True: Theorem 5.21.
(c) False: for example if $v = \lambda w$.
(d) False: for example if $T(v) = 0$.
(e) True: Cayley-Hamilton.
(f) True: Use exercise 4.3.24.
(g) True: Theorem 5.25.

5.4.2 (b) No. Take $f(x) = x^2 \in P_2(\mathbb{R})$; then $T(f)(x) = x^3$, which has degree $3 > 2$.

(e) No. W consists of symmetric matrices. The questions is whether $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A$ must be symmetric if A is symmetric. $T(A)$ is obtained from A by swapping the rows; taking $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ provides a counterexample.

5.4.4 Let $w \in W$. We claim that $T^k(w) \in W$ for any k by induction. It's true by assumption for $k = 1$ (and tautologically for $k = 0$). Suppose it is true for $k = m$, i.e. $T^m(w) \in W$. Then $T(T^m(w)) \in W$ since $T^m(w) \in W$ and W is T-invariant, concluding the proof of the claim. Furthermore, since W is a subspace, and linear combination of $T^k(w)$ for various k are also elements of W, so $g(T)(w) \in W$.

5.4.5 Suppose that $\{W_\alpha\}_{\alpha\in S}$ is a collection of T-invariant subspaces. If $w \in \cap_{\alpha\in S} W_\alpha$ then $w \in W_\alpha$ for all $\alpha \in S$ so that $T(w) \in W_\alpha$ by invariance. Hence, $T(w) \in \cap_{\alpha\in S} W_\alpha$, proving that $\cap_{\alpha\in S} W_\alpha$ is T-invariant.

5.4.6 (b) $z = x^3$, $T(z) = 6x$, and $T^2(z) = 0$. So, the cyclic subspace generated by z has basis $\{x^2, 6x\}$.

(d) $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Note that $T(z) = Az$ is A with its columns swapped, so $T(z) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$. $T^2(z) = A^2 = \begin{pmatrix} 3 & 3 \\ 6 & 6 \end{pmatrix}$, which is a scalar multiple of $T(z)$. Thus $\{z, T(z)\} = \{z, A\}$ is a basis for the subspace generated by z.

5.4.7 Linearity comes for free, since it is linear on the larger subspace. The only thing to do is prove it is well-defined on the invariant subspace, i.e. that $T(w) \in W$ for $w \in W$; this is true by definition of invariance.

5.4.8 If $v \in W$, and $T(v) = \lambda v$, then this is still true once we consider v as an element of the larger V.

5.4.10 (b) Using the standard basis, T has matrix representation $\begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. This is upper triangular, so the characteristic polynomial is $f(t) = t^4$. The matrix for T_W is $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, which has characteristic polynomial $g(t) = t^2$. Clearly, $g(t)$ divides $f(t)$.

(d) The transformation has matrix using the standard basis $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix}$. This matrix has rank 2, so $\lambda = 0$ is an eigenvector with dimension two eigenspace. By inspection, $(1, 0, 2, 0)^t$ and $(0, 1, 0, 2)^t$ are eigenvectors with eigenvalue 3. Thus the characteristic polynomial is $f(t) = t^2(t-3)^2$. For T_W, we have matrix $\begin{pmatrix} 0 & 0 \\ 1 & 3 \end{pmatrix}$, which has characteristic polynomial $t(t-3)$. Again, $g(t)$ divides $f(t)$.

5.4.13 It's clear that $g(T)(v) \in W$, since W is T-invariant by construction. Now, suppose that $w \in W$. As W is a T-cyclic subspace generated by v, it has a basis of the form $\{v, T(v), \ldots, T^{k-1}(v)\}$ by Theorem 5.22, where $k = \dim(W)$. Hence, we can write

$$w = a_0 v + a_1 T(v) + \ldots + a_{k-1} T^{k-1}(v).$$

Setting $g(t) = a_0 + a_1 t + \ldots + a_{k-1} t^{k-1}$ gives $w = g(T)(v)$.

5.4.14 The construction already satisfies this.

5.4.15 Let A be a matrix, and apply the Cayley-Hamilton theorem to $T = L_A : F^n \to F^n$. Note that $T^n = L_{A^n}$.

5.4.17 By the Cayley-Hamilton theorem, if $\chi(t)$ is the characteristic polynomial for A, then $\chi(A) = 0$. Thus, if A is $n \times n$, and since the coefficient of t^n in $\chi(t)$ is nonzero, we have $A^n = c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I$ (I'm being sloppy about signs but they do not matter). So, $A^n \in \operatorname{span}(I, A, A^2, \ldots, A^{n-1})$. Applying A^k to the previous equation also tells us that A^{n+k} is also in this span, and the dimension of the span of a finite set of vectors is less than or equal to the number of vectors, which is n.

5.4.18 (a) $f(t) = \det(A - \lambda I)$; so $f(0) = a_0 = \det(A - 0) = \det(A)$. Thus, A is invertible if and only if $a_0 \neq 0$.

(b) It suffices to check that $A(\frac{-1}{a_0}((-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n)) = \frac{-1}{a_0}((-1)^n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A = \frac{-1}{a_0}(f(A) - a_0 I) = I$. The penultimate equality follows from Cayley-Hamilton.

(c) The characteristic polynomial of A is $f(t) = (1-t)(2-t)(-1-t) = -t^3 + 2t^2 + t - 2$ so, by (b),

$$A^{-1} = \frac{1}{2}(-A^2 + 2A + I) = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1/2 & 3/2 \\ 0 & 0 & -1 \end{pmatrix}.$$

5.4.19 Let us use induction on k. For $k = 1$, the matrix is (a_0), which has characteristic polynomial $a_0 - t$, so this establishes the base case. Now assume it is true for $k = m - 1$. Let us expand by cofactors on $A - \lambda I$, for the $m \times m$ matrix, along the top row. The only nonzero entries are in the first and the last column, so we have $(-\lambda)g(\lambda) + (-1)^m(a_0)$; the first term is obtained via the inductive hypothesis, where $g(t) = (-1)^{m-1}(a_1 + a_2 t + \cdots + a_{m-1}t^{m-2} + t^{m-1})$, and for the second summand, by noting that the matrix resulting from deleting the last column and first row is upper diagonal with 1's on the diagonal. Evaluating out this expression, we find $(-1)^m(a_1\lambda + a_2\lambda^2 + \cdots + a_{m-1}\lambda^{m-1} + \lambda^m) + (-1)^m a_0$, which gives the desired expression after regrouping.

5.4.21 Choose any nonzero $v \in V$. Either $T(v) = cv$ (for some scalar c) or not. If not, then $\{v, T(v)\}$ is linearly independent, so they span V, and V is a T-cyclic subspace. If $T(v) = cv$, then take $w \notin \operatorname{span}(v)$. Then either $T(w) = dw$ or not. If not, again we have that V is a T-cyclic subspace, since $\{w, T(w)\}$ are linearly independent. If $T(w) = dw$, then $c = d$ implies that $T = cI$ as $\{v, w\}$ is a basis for V. If $c \neq d$, then $T(v+w) = cv + dw$, which is not a multiple of $v + w$ so $\{v+w, T(v+w)\}$ forms a basis for V and so V is again a T-cyclic subspace. This covers all the cases.

5.4.23 We perform induction on k. The $k = 1$ case is easy. Now assume that the statement is true for $k = m - 1$, and let λ_i be the eigenvalue for v_i. We have that $T(v_1 + \cdots + v_m) = \lambda_1 v_1 + \cdots + \lambda_m v_m \in W$ by invariance of W. Since W is a subspace, take $\lambda_1(v_1 + \cdots + v_n) - T(v_1 + \cdots + v_n) = (\lambda_1 - \lambda_2)v_2 + \cdots + (\lambda_1 - \lambda_m)v_m \in W$. None of these coefficients are zero since the eigenvalues are distinct. As $(\lambda_1 - \lambda_i)v_i \in W$ for all i, by the inductive hypothesis $v_i \in W$ for every i, and hence $v_1 = (v_1 + \cdots + v_n) - (v_2 + \cdots v_n) \in W$.

5.4.24 Let W be an invariant subspace. Let $V = E_1 \oplus \cdots \oplus E_k$ be a decomposition into eigenspaces (which exists since T is diagonalizable) with corresponding eigenvalues $\lambda_1, \ldots, \lambda_k$. We can write any $w \in W$ as $w = v_1 + \cdots + v_k$ where some of the v_i are allowed to be zero. By the previous exercise, each $v_i \in W$ (recall again that some of these might be zero). Thus every $w \in W$ can be written as a sum of eigenvectors, i.e. there is an eigenbasis for W, and the claim follows.

5.4.26 Let the eigenvectors be $v_1, \dots, v_n$. We claim for $v = v_1 + \dots + v_n$, that $\{v, T(v), \dots, T^{n-1}(v)\}$ is linearly independent. Equivalently, we want to show that the cyclic subspace generated by v has dimension n. Let W be the cyclic subspace generated by v. Then $v \in W$, and W is T-invariant, so by problem 23, each $v_i \in W$. Thus W has dimension n.

Challenge from class: if $T : V \to V$ is a linear operator on a n-dimensional space, and v is a vector in V, prove that $\{v, T(v), T^2(v), \dots, T^m(v)\}$ is linearly dependent for large enough m, and $\{I, T, T^2, \dots, T^\ell\}$ is linearly dependent for large enough ℓ.

Proof: When $m > n$, the first set has more vectors than the dimension of the vector space, and must be linearly dependent by the replacement theorem. When $\ell \geq n^2$, the set has more vectors than the dimension of $L(V, V)$, and must be linearly dependent. However, by Cayley-Hamilton, we know that this set must be linearly dependent for $\ell \geq n$. For an example of T such that $\{I, T, T^2, \dots, T^\ell\}$ is linearly independent for any $\ell < n$, take $T = \text{diag}(1, 2, 3, \dots, n)$ (any distinct elements on the diagonal will do).