HW #3; Date 9/5/2017
MATH 110 Linear Algebra
with Professor Stankova

Section 1.4: Linear Combinations

1)

(a) True. For any nonempty set $\{v_1, \ldots, v_n\}$ we have $0 \cdot v_1 + \cdots 0 \cdot v_n = 0 + 0 + \ldots 0 = 0$.

(b) False. $\mathrm{Span}(\emptyset) = \{0\}$ by the definition before Theorem 1.5. One reason for this definition is that we would like to say that the span of n linearly independent vectors is an n-dimensional space, and this definition makes the statement true for $n = 0$.

(c) True, by Theorem 1.5. As proof, let $\overline{S} = \bigcap_{S \subseteq W \subseteq V} W$ be the intersection of all subspaces of V that contain S. We know that $\mathrm{Span}(S)$ contains S, so $\overline{S} \subseteq \mathrm{Span}(S)$. But according to Theorem 1.5 any subspace of V that contains S also contains the span of S, so $\mathrm{Span}(S) \subseteq \overline{S}$. This proves that $\mathrm{Span}(S) = \overline{S}$.

NOTE: This uses the technique of proving that two sets A and B are equal by proving that $A \subseteq B$ and $B \subseteq A$.

(d) False. Multiplying an equation by zero may give a non-equivalent system of equations.

(e) True. See the explanation in Example 1 of Section 1.4.

(f) False. $x_1 = 1$; $x_1 = -1$ is a system of two linear equations in one variable with no solution.

2d)

We replace the system of equations by its matrix of coefficients.

$$\begin{bmatrix} 1 & 2 & 2 & 0 & 2 \\ 1 & 0 & 8 & 5 & -6 \\ 1 & 1 & 5 & 5 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 & 0 & 2 \\ 0 & -2 & 6 & 5 & -8 \\ 0 & -1 & 3 & 5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 & 0 & 2 \\ 0 & -1 & 3 & 5 & 1 \\ 0 & -2 & 6 & 5 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 & 0 & 2 \\ 0 & 1 & -3 & -5 & -1 \\ 0 & 0 & 0 & -5 & -10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 0 & 2 \\ 0 & 1 & -3 & -5 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 & 0 & 2 \\ 0 & 1 & -3 & 0 & 9 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 8 & 0 & -16 \\ 0 & 1 & -3 & 0 & 9 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

So that x_3 is free, and we have $x_1 = -8x_3 - 16$, $x_2 = 3x_3 + 9$, and $x_4 = 2$.

3b)

Solving the vector equation

$$x_1 \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

gets us the following linear system (in matrix form).

$$\begin{bmatrix} 1 & -3 & 2 & 0 \\ 2 & 2 & -1 & 0 \\ -3 & 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 8 & -5 & 0 \\ 0 & -8 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 8 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 1 & \frac{-5}{8} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{8} & 0 \\ 0 & 1 & \frac{-5}{8} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that there is a nontrivial solution to this equation. For example,

$$\frac{-1}{8}\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \frac{5}{8}\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So that

$$8\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} + 5\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

This shows that $x_1 = 5x_2 + 8x_3$.

4c)

If we wish to solve $-2x^3 - 11x^2 + 3x + 2 = a(x^3 - 2x^2 + 3x - 1) + b(2x^3 + x^2 + 3x - 2)$, then we need the polynomials on each side to have the same coefficient for each monomial. We get a system of equations:

$$\begin{bmatrix} 1 & 2 & -2 \\ -2 & 1 & -11 \\ 3 & 3 & 3 \\ -1 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -15 \\ 0 & -3 & 9 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From this we derive the solution

$$-2x^3 - 11x^2 + 3x + 2 = 4(x^3 - 2x^2 + 3x - 1) + (-3)(2x^3 + x^2 + 3x - 2)$$

5)

For each part, we convert the linear dependence question into the matrix form of a system of equations and use row operations to find a solution, if any.

Part	System	In Span	Relation
(a)	$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 2 & 1 & 1 \end{bmatrix}$	Yes	$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$
(b)	$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$	No	
(c)	$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$	No	
(d)	$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & -3 \end{bmatrix}$	Yes	$\begin{bmatrix} 2 \\ -1 \\ 1 \\ -3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$
(e)	$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 3 \end{bmatrix}$	Yes	$-x^3 + 2x^2 + 3x + 3$ $= (-1)(x^3 + x^2 + x + 1) + (3)(x^2 + x + 1) + (1)(x + 1)$
(f)	$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}$	No	
(g)	$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ -1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 4 \end{bmatrix}$	Yes	$\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} = 3 \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + 4 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + (-2) \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$
(h)	$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$	No	

6)

Label the given elements e_1, e_2, e_3. If F is a field of characteristic $\neq 2$, then any element $(a_1, a_2, a_3) \in F^3$ can be written as

$$(a_1, a_2, a_3) = \frac{a_1}{2}(e_1 + e_2 - e_3) + \frac{a_2}{2}(e_1 - e_2 + e_3) + \frac{a_3}{2}(-e_1 + e_2 + e_3).$$

So that e_1, e_2, e_3 generate F^3. They do not generate F^3 if F is a field of characteristic 2 because the number of ones in each triple will always be even.

7)

Any element $(a_1, a_2, \ldots, a_n) \in F^n$ can be written as

$$(a_1, a_2, \ldots, a_n) = a_1 e_1 + a_2 e_2 + \cdots a_n e_n.$$

8)

Any element $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in P_n$ can be written as

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = (a_n)x^n + (a_{n-1})x^{n-1} + \cdots + (a_0)1$$

which is a linear combination of the specified polynomials. If this proof feels silly, that's because it is. We defined P_n to be any polynomial of degree n, and wrote out an arbitrary polynomial in a notation that already implied the basis $\{x^n, x^{n-1}, \ldots, 1\}$. For this reason, we sometimes call the set $\{x^n, x^{n-1}, \ldots, 1\}$ the *standard* basis (and likewise with the basis for F^n in Problem 7).

10)

Any symmetric matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ can be written as

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

so if S is the set of all symmetric 2×2 matrices then $S \subseteq \operatorname{Span}\{M_1, M_2, M_3\}$. Conversely, M_1, M_2, M_3 are all symmetric and any linear combination of symmetric matrices must also be symmetric, so $\operatorname{Span}\{M_1, M_2, M_3\} \subseteq S$. Therefore, $S = \operatorname{Span}\{M_1, M_2, M_3\}$.

12)

If W is a subspace of V, then $\operatorname{Span} W = W$, because subspaces are closed under taking linear combinations of elements.

For the reverse direction, suppose that W is a subset of V such that $\operatorname{Span} W = W$. By Theorem 1.5, $\operatorname{Span} W$ is a subspace of V and therefore so is W.

14)

First suppose that $x \in \text{Span}(S_1 \cup S_2)$. Then we can write $x = \sum_{i=1}^{k} a_i v_i$ for some vectors $v_i \in S_1 \cup S_2$ and scalars a_i, and in particular we can split the sum according to which vectors are in S_1 and which are in S_2 but not S_1:

$$x = \sum_{i: v_i \in S_1} a_i v_i + \sum_{i: v_i \in S_2 \setminus S_1} a_i v_i$$

In particular, the first of the above sums is an element of $\text{Span}(S_1)$, and the second is an element of $\text{Span}(S_2)$, so the sum x of these two elements is in $\text{Span}(S_1) + \text{Span}(S_2)$. This shows that $\text{Span}(S_1 \cup S_2) \subseteq \text{Span}(S_1) + \text{Span}(S_2)$.

For the reverse inclusion, suppose that $y \in \text{Span}(S_1) + \text{Span}(S_2)$. Then we can write $y = w_1 + w_2$ where $w_1 \in \text{Span}(S_1)$ and $w_2 \in \text{Span}(S_2)$, and we can further write

$$w_1 = \sum_{i=1}^{k_1} a_i^{(1)} v_i^{(1)}, \quad w_2 = \sum_{j=1}^{k_2} a_j^{(2)} v_j^{(2)}$$

where the vectors $v_i^{(1)}$ are in S_1 and the vectors $v_j^{(2)}$ are in S_2. However, if we consider the sum $w_1 + w_2$ as a single linear combination, then this linear combination is of vectors that are either in S_1 or in S_2, or in other words which are in $S_1 \cup S_2$. Thus $y \in \text{Span}(S_1 \cup S_2)$, and we see that $\text{Span}(S_1) + \text{Span}(S_2) \subseteq \text{Span}(S_1 \cup S_2)$. We conclude that we have equality of the two sets: $\text{Span}(S_1 \cup S_2) = \text{Span}(S_1) + \text{Span}(S_2)$.

15)

Let $x \in \text{Span}(S_1 \cap S_2)$. Then x can be written $x = a_1 x_1 \cdots + a_n x_n$, where $x_1, \ldots, x_n \in S_1 \cap S_2$. Since $x_1, \ldots, x_n \in S_1$, we conclude that $x \in \text{Span}\, S_1$. Likewise since $x_1, \ldots, x_n \in S_2$, we conclude that $x \in S_2$. Thus $x \in \text{Span}\, S_1 \cap \text{Span}\, S_2$.
If $S_1 = S_2$, then it is easy to see that we have equality.

To see that the two spaces are not necessarily equal, consider the example $V = \mathbb{R}$ with $S_1 = \{1\}$ and $S_2 = \{-1\}$. Then $\text{Span}(S_1 \cap S_2) = \text{Span}(\emptyset) = \{0\}$, but $\text{Span}(S_1) \cap \text{Span}(S_2) = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$.

16)

By definition, we know that every element in $\text{Span}\, S$ can be written in at least one way, so we just have to prove that this way is unique. Suppose that $v = a_1 v_1 + \cdots + a_n v_n = b_1 v_1 + \cdots + b_n v_n$ are two ways of writing the same element $v \in \text{Span}\, S$. Then

$$0 = v - v = (a_1 v_1 + \cdots a_n v_n) - (b_1 v_1 + \cdots b_n v_n) = (a_1 - b_1) v_1 + \cdots (a_n - b_n) v_n.$$

By the stated property for S it follows that $a_1 - b_1 = 0, \ldots, a_n - b_n = 0$ and therefore that $a_1 = b_1, \ldots, a_n = b_n$. So the two representations are the same, implying that every element in $\text{Span}\, S$ can be uniquely written as a linear combination of elements of S.

NOTE: This is a standard way to prove that there is a *unique* element x in a set S with some property P. Namely, let $x, y \in S$ both have property P, and then to show that $x = y$.

17)

If W is the trivial subspace, then there are only two generating subsets, $\{0\}$ and $\emptyset$, so this is a trivial example where W has only finitely many generating subsets.

More generally, if our vector space V is over an infinite field, and W is a nontrivial subspace, then let S be a generating set of W with some nonzero element x. We may replace S with $S' = S \setminus \mathrm{Span}(\{x\}) \cup \{x\}$ to get a set which still generates W but contains no scalar multiples of x. Then for each nonzero $a \in F$, the set $S' \setminus \{x\} \cup \{ax\}$ is also a generating set for W, and so there are infinitely many such sets.

If the field F is finite (in particular this means that the characteristic of F is nonzero), then the situation is more complicated. In this case, a nontrivial subspace W may be finite, in which case W has only finitely many subsets, and thus only finitely many generating subsets.

If W is infinite in this setting, it is somewhat more involved to argue that there are infinitely many generating subsets. One way is to take a minimal generating set B, also called a basis (why does such a set exist in general?), and remove a single element x. Since W is infinite but F is finite, a basis B must be infinite, and thus likewise $B \setminus \{x\}$. Then if C is any finite subset of $B \setminus \{x\}$, it can be shown that

$$B' = B \setminus \{x\} \cup \{x + \sum_{y \in C} y\}$$

is a basis of W which is distinct for each distinct subset C. This gives an infinite collection of generating sets.

To summarize, W has only finitely many generatings sets if and only if W is finite. This can occur if W is the trivial subspace, or if V is over a finite field (in particular with nonzero characteristic) and W is nontrivial but finite.

Section 1.5: Linear Dependence and Independence

1)

(a) False: consider the set $\{(1,0),(2,0),(0,1)\}$. The set is linearly dependent, but the third element cannot be expressed as a linear combination of the first two.

(b) True, because $1 \cdot 0 + 0 \cdot v_1 + 0 \cdot v_2 + \cdots 0 \cdot v_n = 0$ is a linear dependence relation.

(c) False. Linearly dependent sets must be nonempty.

(d) False, in general. Consider the subset $\{(1,0),(0,1)\} \subseteq \{(1,0),(2,0),(0,1)\}$.

(e) True. If there were some linear combination $a_1 v_1 + \ldots + a_n v_n = 0$ with elements of the subset, then this would also be a linear dependence relation for the original set.

(Note that this is using the contrapositive of the original statment: If a subset of V is linearly dependent then so is V.)

(f) If this notation is supposed to mean that the *set* $\{x_1, \ldots, x_n\}$ is linearly independent, then the conclusion is true, because it is the definition of linear independence.

2b)

If $a \begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix} + b \begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix} = 0$, then $a - b = 0$, $-2a + b = 0$, $-a + 2b = 0$, and $4a - 4b = 0$. Using just the first and second equations, we see that the only solution is $a = b = 0$. So this set is linearly independent.

2d)

There is a linear dependence relation between these polynomials:

$$4(x^3 - x) + (-3)(2x^2 + 4) + 2(-2x^3 + 3x^2 + 2x + 6) = 0$$

This shows the set is linearly dependent.

2f)

We are looking for solutions to $x_1 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. So we look at the matrix of coefficients and use row reduction to solve.

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 0 & 2 & 0 \\ 2 & 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -3 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & \frac{5}{2} & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \frac{5}{2} & 0 \end{bmatrix}$$

We conclude that the only solution is $x_1 = x_2 = x_3 = 0$. The set is therefore linearly independent.

5)

Suppose that $(a_n)x^n + \cdots + (a_1)x + (a_0)1 = 0$. The left side of the equation is just the polynomial $p(x) = a_n x^n + \cdots + a_1 x + a_0$. Since polynomials are equal if and only if their coefficients are equal, the equation implies that $a_i = 0$ for each $i = 0, 1, \ldots, n$, which shows that the polynomials $1, x, \ldots, x^n$ are linearly independent.

7)

The set $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is linearly independent and generates the set of all (2×2) diagonal matrices. It may help to see that for any diagonal matrix

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

8a)

We are looking for solutions to $x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. So we look at the matrix of coefficients.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

So the only solution is $x_1 = x_2 = x_3 = 0$, and this shows the set is linearly independent as vectors in $\mathbb{R}^3$.

8b)

In a field of characteristic 2, we have the identity $1 + 1 = 0$. This gives rise to a linear dependence relation among the three vectors

$$(1) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (1) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+1 \\ 1+1 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

9)

Say $\{u, v\}$ is linearly dependent. Then there exist a, b such that $au + bv = 0$. If $a = 0$ then $bv = 0$ but $b \neq 0$, implying that $v = 0 = 0 \cdot u$. If $a \neq 0$ then $u = -\frac{b}{a}v$, so either way one vector is a scalar multiple of the other.

NOTE: This is an example of a *proof by cases*: we proved that some desired proposition P is true when $a = 0$ and also that P is true when $a \neq 0$. Since it is always true that either $a = 0$ or $a \neq 0$, this showed that P is always true.

11)

There are at most 2^n such vectors because all elements in S can be written in the form $a_1u_1 + \ldots + a_nu_n$, and since the field is Z_2 each a_i must be either 0 or 1. There are 2 options for each a_i and n such choices to make, so at most 2^n linear combinations in total.

Then because S is a linearly independent set, all of these 2^n linear combinations are distinct. If there were two that were equal then their difference would make a linear dependence relation, which would mean that S was linearly dependent. Therefore, $\text{Span}(S)$ has exactly 2^n elements.

12)

Let $\{s_1, \ldots s_n\}$ be a finite set of linearly dependent vectors in S_1. This must exist, because S_1 is linearly dependent. There exists some linear dependence relation among the vectors:

$a_1s_1 + \cdots + a_ns_n = 0$, with not all of $a_1, \ldots, a_m = 0$. Since $\{s_1, \ldots s_n\} \subseteq S_1 \subseteq S_2$, we know that S_2 has a finite set of vectors that satisfy a linear dependence relation. Thus S_2 is linearly dependent.

The corollary is precisely the contrapositive of Theorem 1.6, and if any statement is true then its contrapositive must also be true.

NOTE: *Any* if-then statement is equivalent to its contrapositive! This is an incredibly useful tool and you would do well to commit it to memory.

14)

If S is linearly dependent, then it contains a set of vectors satisfying a linear dependence relation: $a_1v_1 + \cdots + a_nv_n = 0$, where not all of the coefficients $a_1, \ldots, a_n = 0$. Let $a_i \neq 0$. Then

$$v_i = -\frac{1}{a_i}(a_1v_1 + \cdots + a_{i-1}v_{i-1} + a_{i+1}v_{i+1} + \cdots + a_nv_n) = 0$$

Labeling this first vector as v, we conclude one direction of the proof (if our set had only had one vector, the it would be the zero vector).

If S can be written as $\{v, x_1, \ldots x_n\}$ such that v is a linear combination of the other vectors, then this set satisfies a linear dependence relation, and thus S is linearly dependent. Likewise, if $S = \{0\}$, then it is linearly dependent.

19)

We will prove the contrapositive: if $\{A_1^t, \ldots, A_k^t\}$ is linearly dependent then there is some nontrivial combination $c_1A_1^t + \ldots + c_kA_k^t = 0$. But this implies that

$$0 = 0^t = (c_1A_1^t + \ldots + c_kA_k^t)^t = c_1A_1 + \ldots + c_kA_k,$$

which shows that $\{A_1, \ldots, A_k\}$ is also linearly dependent. We can therefore conclude that if $\{A_1, \ldots, A_k\}$ is a linearly independent set then so is $\{A_1^t, \ldots, A_k^t\}$.

20)

We will again proceed by showing the contrapositive: if $e^{rt} = e^{st}$, then $r = s$.

By Problem (9), if e^{rt} and e^{st} are linearly dependent then one is a scalar multiple of the other. . . without loss of generality, say $e^{rt} = ce^{st}$. Dividing both sides by e^{st} gives that $e^{(r-s)t} = c$, a constant function. Taking the derivative of both sides then gives that $(r-s)e^{(r-s)t} = 0$, and since $e^{(r-s)t}$ is never zero we conclude that $r - s = 0$, or $r = s$. This is a contradiction, however, so we conclude that these two exponentials are linearly independent.

NOTE: This is a common trick when working with solutions to differential equations: to show that two functions must be linearly dependent, divide one by the other and try to show that the derivative of the quotient is zero.