HW #20; date: April 6, 2017
MATH 110 Linear Algebra
with Professor Stankova

Find the Jordan normal form of the following matrices.

$$\begin{pmatrix} 3 & 1 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 7 & 1 & 2 & 1 \\ -17 & -6 & -1 & 0 \end{pmatrix}$$

The characteristic polynomial of this matrix A is the characteristic polynomial of $\begin{pmatrix} 3 & 1 \\ -4 & 1 \end{pmatrix}$ multiplied with the characteristic polynomial of $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$, i.e. $(t^2 - 4t + 7)(t-1)^2$. The only repeated root is $t = 1$, so we will investigate this. $A - I$ has rank 3, so the eigenspace is one-dimensional. The roots of $t^2 - 4t + 7$ are $t = 2 \pm i\sqrt{3}$. Thus, the Jordan normal form is $\begin{pmatrix} 2+i\sqrt{3} & 0 & 0 & 0 \\ 0 & 2-i\sqrt{3} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

It is clear that the only eigenvalue is 1. The rank of $A - I = \begin{pmatrix} 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is 3. The rank of $(A-I)^2 = \begin{pmatrix} 0 & 0 & 4 & 12 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is 2. The rank of $(A-I)^3$ is 1s. Thus, the Jordan normal form is $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This matrix permutes the standard basis vectors in a cycle. In particular, $A^5 = I$. Thus any eigenvector λ must satisfy $\lambda^5 = 1$. Let ω be a fifth root of unity, i.e. a complex number such that $\omega^5 = 1$ and $\omega \neq 1$. Then, we have the following eigenvalue-eigenvector pairs:

$$\lambda = 1, (1, 1, 1, 1, 1)^t \qquad \lambda = \omega, (1, \omega, \omega^2, \omega^3, \omega^4) \qquad \lambda = \omega^2, (1, \omega^2, \omega^4, \omega, \omega^3)$$

$$\lambda = \omega^3, (1, \omega^3, \omega, \omega^4, \omega^2) \qquad \lambda = \omega^4, (1, \omega^4, \omega^3, \omega^2)$$

So the Jordan normal form is $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 & 0 \\ 0 & 0 & 0 & \omega^3 & 0 \\ 0 & 0 & 0 & 0 & \omega^4 \end{pmatrix}$

Find a matrix Q such that $Q^{-1}AQ$ is a Jordan matrix where $A = \begin{pmatrix} 2 & -1 & 2 \\ 2 & 2 & 1 \\ -1 & 2 & 2 \end{pmatrix}$. The characteristic polynomial is $(t-3)(t^2-3t+9)$. If ω is a third root of unity, then the roots of the quadratic factor are $3(\omega+1) = -3\omega^2$ and $3(\omega^2+1) = -3\omega$ (recall that $1+\omega+\omega^2 = 0$). The eigenvector for $t = 3$ is $(1, 1, 1)^t$. For the quadratic factor, we can compute it directly or try to find a shortcut by playing around. Let's try, just for kicks, to compute $A(1, \omega, \omega^2)^t = -3(\omega, \omega^2, 1)^t$, and $A(1, \omega^2, \omega)^t = -3(\omega^2, \omega, 1)$. Thus, we have

$$J = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -3\omega^2 & 0 \\ 0 & 0 & -3\omega \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$$

Find the Jordan normal form of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & -2 & 2 & 1 \end{pmatrix}$$

The characteristic polynomial is $t^5 - t^4 - 2t^3 + 2t^2 + t - 1 = (t-1)^3(t+1)^2$. We have that $A + I = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & -1 & -2 & 2 & 2 \end{pmatrix}$ has rank 4. We know that $(A+I)^2$ must have rank 3, since $(A+I)^k$ eventually stabilizes at rank 3 (i.e. $(A+I)^2$ must have at most rank 4; it cannot have rank less than 3 because the rank of $(A+I)^k$ decreases as k increases and stabilizes at rank

3, and it cannot have rank 4 because it does cannot stailize at rank 4). Next, we have that $A-I = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & -1 & -2 & 2 & 0 \end{pmatrix}$ has rank 4. $(A-I)^2 = \begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 1 & -1 & -2 & 3 & -1 \\ -1 & 2 & 1 & -4 & 2 \end{pmatrix}$ has rank 3. Thus $(A-I)^3$ must have rank 2. Thus, the Jordan normal form is $\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

7.2 #1gh (g) False; unique up to ordering. (h) True by construction (ordering is determined).

Challenge Find the Jordan canonical form of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Similar to earlier problem. Let ω be an nth root of unity. This matrix is diagonalizable. Its diagonal form has diagonal $1, \omega, \omega^2, \dots, \omega^{n-1}$, and the eigenvectors are $(1, 1, 1, \dots, 1)$, $(1, \omega, \omega^2, \cdots, \omega^{n-1})$, $(1, \omega^2, \omega^4, \cdots, \omega^{2(n-1)})$, et cetera.

For the matrix

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 0 & 1 & 2 & \cdots & n-2 & n-1 \\ 0 & 0 & 1 & \cdots & n-3 & n-2 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & 2 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

We claim that $d_j - d_{j-1} = 1$ for every j (until it is zero) (note $\lambda = 1$ is the only eigenvalue); in particular, the Jordan normal form is $\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$ To see this, note that $(A-I)e_i = 2e_{i-1} + 3e_{i-2} + \cdots$. So, $(A-I)^k = ae_{i-k} + be_{i-k-1} + \cdots$ where a, b are some positive constants that I don't need to figure out. In particular, $(A-I)^k$ has zeroes below the diagonal shifted up by k and positive values on and above it. The claims follows.