HW #12; date: March 2, 2017
MATH 110 Linear Algebra
with Professor Stankova

4.2 Determinants of Order n

1. (a) False. See Exercise 4.2.1(a).

   (b) True. It follows from Theorem 4.4.

   (c) True. It follows from Corollary to Theorem 4.4.

   (d) True. It follows from Theorem 4.5.

   (e) False. We should have $\det(B) = k\det(A)$.

   (f) False. We should have $\det(B) = \det(A)$.

   (g) False. The identity 2×2 matrix has rank 2 with nonzero determinant.

   (h) True. Let A be an upper triangular $n \times n$ matrix. The statement holds for $n = 1$. Let us use an induction on n. Assume $n > 1$. Consider the $(n-1) \times (n-1)$ matrix B obtained by removing the first row and the first column of A, and let $a_{11}, \ldots, a_{nn}$ denote the diagonal entries of A. Then B is an upper triangular matrix, so by induction

   $$\det B = a_{22} \cdots a_{nn}.$$

   By the cofactor expansion formula, we have

   $$\det A = a_{11} \det B = a_{11} \cdots a_{nn},$$

   which completes the induction process.

3. Add $-\frac{5}{7}$ of the third row to the second row in the matrix in the left hand side, and multiply $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{7}$ to the first, second and third row respectively. Then we obtain the matrix in the right hand side. Thus k should be $2 \times 3 \times 7 = 42$.

7. The determinant is

   $$-(-1)\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} - (-3)\begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} = -12.$$

10. The determinant is

   $$-(-1)\begin{vmatrix} 2+i & 0 \\ -1 & 1-i \end{vmatrix} + 3\begin{vmatrix} i & 0 \\ 0 & 1-i \end{vmatrix} - 2i\begin{vmatrix} i & 2+i \\ 0 & -1 \end{vmatrix} = 4 + 2i.$$

11. The determinant is

   $$-(-1)\begin{vmatrix} 2 & 1 & 3 \\ 0 & -2 & 2 \\ -1 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 3 \\ 1 & -2 & 2 \\ 3 & 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 & 3 \\ 1 & 0 & 2 \\ 3 & -1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} -2 & 2 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ -2 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ -2 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} - 6\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = -3.$$

17. Adding $6\times$(row 2) to (row 3), we get

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 5 \\ 0 & -16 & 33 \end{pmatrix},$$

so the determinant is $-(1 \times 33 - 1 \times (-16)) = -49$.

22. To simplify the first column, apply elementary operations of type 3 to the matrix, and we get

$$\begin{pmatrix} 1 & * & * & * \\ 0 & 2 & 1 & -41 \\ 0 & 4 & 7 & -77 \\ 0 & 1 & -2 & -33 \end{pmatrix}.$$

Then to simplify the second column, apply elementary operations of type 3 to the above matrix, and we get

$$\begin{pmatrix} 1 & * & * & * \\ 0 & 2 & 1 & -41 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & -\frac{5}{2} & -\frac{25}{2} \end{pmatrix}.$$

Thus the determinant is $1 \times 2 \times (5(-\frac{25}{2}) - 5(-\frac{5}{2})) = -100$.

25. Applying Theorem 4.3 to each row, we get $\det B = k^n \det A$ since there are n rows.

26. By Exercise 25, we should have $(-1)^n = 1$. This is true if one of the following conditions are satisfied.

   (1) n is even.

   (2) The characteristic of F is 2.

4.3 Properties of Determinants

1. (a) False. The elementary matrices of type 2 may not have determinant 1.

   (b) True. It follows from Theorem 4.7.

   (c) False. A square matrix is invertible if and only if the determinant is not 0.

   (d) True. An $n \times n$ matrix M has rank n if and only if M is invertible, which is equivalent to $\det M \neq 0$.

(e) False. We have $\det A = \det A^t$, which may be different from $-\det A^t$.

(f) True. See p. 224.

(g) False. $0x = 0$ cannot be solved by Cramer's rule.

(h) False. The matrix M_k should be the $n \times n$ matrix from A by replacing column k of A by b,

7. We have

$$\det M_1 = 0, \quad \det M_2 = 48, \quad \det M_3 = -64, \quad \det A = -4.$$

Thus the solution is $(x_1, x_2, x_3) = (0, -12, 16)$.

9. An upper triangular matrix is invertible if and only if its determinant is nonzero. It is equivalent to the statement that the product of all the diagonal entries is nonzero by Exercise 4.2.1(h), and it is equivalent to the statement that all the diagonal entries are nonzero.

10. If $M^k = O$, then $\det(M^k) = 0$, so $(\det M)^k = 0$ by Theorem 4.7. Thus $\det M = 0$.

11. Assume that M is skew-symmetric. Then

$$\det M = \det(-M^t) = \det(-M) = (-1)^n \det M$$

by Theorem 4.8 and Exercise 4.2.25. If n is odd, then $\det M = -\det M$, so $\det M = 0$. If n is even, we do not have anything from the above argument, and actually there is an invertible 2×2 matrix that is skew-symmetric. Here is an example.

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

12. If Q is orthogonal, then

$$\det I = \det(QQ^t) = \det Q \det(Q^t) = (\det Q)^2$$

by Theorem 4.7 and 4.8. Since $\det I = 1$, we get $\det Q = \pm 1$.

13. (a) Let us use an induction on n. The statement is trivial for $n = 1$. Assume $n > 1$. By induction, $\det \overline{\tilde{M}_{ij}} = \overline{\det \tilde{M}_{ij}}$ for each i and j. Thus by the cofactor expansion formula,

$$\det \overline{M} = \sum_{j=1}^{n} (-1)^{1+j} \overline{M_{1j}} \det \overline{\tilde{M}_{ij}}$$

$$= \overline{\sum_{j=1}^{n} (-1)^{1+j} M_{1j} \det \tilde{M}_{ij}} = \overline{\det M}.$$

(b) If Q is unitary, then

$$1 = \det I = \det Q \det Q^* = \det Q \det \overline{Q} = \det Q \overline{\det Q} = |\det Q|^2.$$

Thus $|\det Q| = 1$.

15. If $A = QBQ^{-1}$ where Q is an invertible matrix, then

$$\det A = \det Q \det B \det(Q^{-1}) = \det Q \det B (\det Q)^{-1} = \det B.$$

17. If $AB = -BA$, then

$$\det A \det B = \det(AB) = \det(-BA) = (-1)^n \det B \det A.$$

Thus if n is odd and the characteristic of the field is not 2, $\det A \det B = 0$, so at least one of A and B is not invertible.

22. (c) Put

$$f(c_0, \ldots, c_n) = \det M.$$

Then f is a polynomial. For each $i \neq j$, if $c_i - c_j = 0$, then f becomes 0 since two rows of M become equal, and this means that the polynomial $(c_i - c_j)$ should divide f. Thus the product of all the $(c_i - c_j)$ with $i \neq j$ should divide f, so

$$f = g \prod_{i<j} (c_j - c_i) \tag{*}$$

for some polynomial g of $c_0, \ldots, c_n$. Let us compare the degrees. The polynomial f has degree $0 + 1 + \cdots + n = \frac{n(n+1)}{2}$, and the polynomial $\prod_{i<j}(c_j - c_i)$ has degree $\binom{n}{2} = \frac{n(n+1)}{2}$. This means that the degree of g is 0, so g is constant. Now, let us compare the coefficients of $c_1 c_2^2 \cdots c_n^n$ in the both sides of (*). The coefficient of $c_1 c_2^2 \cdots c_n^n$ of f is 1 (the reason is that in the formula of $\det M$, the coefficient of $M_{11} M_{22} \cdots M_{nn}$ is 1, and we have $c_1 c_2^2 \cdots c_n^n = M_{11} M_{22} \cdots M_{nn}$). On the other hand, In the expression

$$(c_1 - c_0)(c_2 - c_0)(c_2 - c_1)(c_3 - c_0)(c_3 - c_1)(c_3 - c_2) \cdots (c_n - c_{n-1}),$$

to obtain $c_1 c_2^2 \cdots c_n^n$, we should select the first variables

$$c_1 c_2 c_2 c_3 c_3 c_3 \cdots c_n.$$

Thus the coefficient of $c_1 c_2^2 \cdots c_n^n$ for $\prod_{i<j}(c_j - c_i)$ is also 1. In (*), the coefficient of $c_1 c_2^2 \cdots c_n^n$ for f and $\prod_{i<j}(c_j - c_i)$ are both 1, so g should be 1. This completes the proof.

24. Put

$$f(a_0, \ldots, a_{n-1}) = \det(tI + A).$$

Let us use an induction on n to prove

$$f(a_0, \ldots, a_{n-1}) = a_0 + a_1 t + \cdots + a_{n-1}t^{n-1} + t^n.$$

This is trivial for $n = 1$, so assume $n > 1$. By the cofactor expansion using the first row, we get

$$f(a_0, \ldots, a_{n-1}) = tf(a_1, \ldots, a_{n-1}) + a_0.$$

Using the induction hypotheses, we get

$$f(a_0, \ldots, a_{n-1}) = t(a_1+a_2t+\cdots+a_{n-1}t^{n-2}+t^{n-1})+a_0 = a_0+a_1t+\cdots+a_{n-1}t^{n-1}+t^n.$$

This completes the induction process.

26. (g) The classical adjoint is

$$\begin{pmatrix} 18 & 28 & -6 \\ -20 & -21 & 37 \\ 48 & 14 & -16 \end{pmatrix}.$$

4.4 Summary - Important Facts about Determinants

1. (a) True. The cofactor expansion formula can be used for columns and rows.

   (b) "Wise" is not a mathematical terminology. Nevertheless, the answer is True.

   (c) True. The determinant of a not invertible matrix is 0.

   (d) False. We should have $\det B = -\det A$.

   (e) False. If the scalar is r, then $\det B = r\det A$.

   (f) True. See p. 217.

   (g) True. See 4.2.1(h).

   (h) False. We should have $\det A^t = \det A$.

   (i) True. See Theorem 4.7.

   (j) True. See Corollary to Theorem 4.7.

   (k) True. See Corollary to Theorem 4.7.

5. If A is an $m \times m$ matrix, use the cofactor expansion formula for the last $(n - m)$ rows of M. Then we get

$$\det M = \det A.$$

6. (Challenge) Consider the two matrices

$$N = \begin{pmatrix} A & B \\ O & I \end{pmatrix}, \quad N' = \begin{pmatrix} I & O \\ O & C \end{pmatrix}.$$

Then we have $\det N = \det A$ by Exercise 5, and similarly, we have $\det N' = \det C$. Since $N'N = M$, we have

$$\det M = \det N' \det N = \det C \det A.$$

Challenges from Class

The first matrix has determinant

$$(\omega - 1)(\omega^2 - 1)(\omega^2 - \omega)$$

by Exercise 4.3.22. The second matrix is a special case of the third matrix. If the size of the third matrix is $n \times n$, let $f_n(a, b)$ denote the determinant of the third matrix. We will show

$$f_n(a, b) = (a + (n-1)b)(a - b)^n$$

by induction on n. It is trivial for $n = 1$, so assume $n > 1$. We will first consider the case when a is nonzero. Applying elementary row operations of type 3 to the first column of the third matrix, we get

$$\begin{pmatrix} a & b & \cdots & b & b \\ 0 & a - \frac{b^2}{a} & \cdots & b - \frac{b^2}{a} & b - \frac{b^2}{a} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & b - \frac{b^2}{a} & \cdots & a - \frac{b^2}{a} & b - \frac{b^2}{a} \\ 0 & b - \frac{b^2}{a} & \cdots & b - \frac{b^2}{a} & a - \frac{b^2}{a} \end{pmatrix}.$$

Thus

$$f_n(a, b) = a f_{n-1}\left(a - \frac{b^2}{a}, b - \frac{b^2}{a}\right),$$

which is equal to

$$a\left(a - \frac{b^2}{a} + (n-2)\left(b - \frac{b^2}{a}\right)\right)\left(a - \frac{b^2}{a} - \left(b - \frac{b^2}{a}\right)\right)^{n-2} = (a+(n-1)b)(a-b)^{n-1}$$

by the induction hypothesis. Thus

$$f_n(a, b) - (a + (n-1)b)(a - b)^n \qquad (*)$$

is 0 if $a \neq 0$. Consider (*) as a polynomial of a. Then (*) has infinitely many roots since (*) is 0 for $a \neq 0$, so (*) should be the zero polynomial. This completes the induction process.