Review Topics for Midterm 1 in MATH 110 Linear Algebra

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1. Definitions

Have a thorough understanding of the following definitions and concepts. What is/are

(1) *vector space*? What are the vector space operations and the vector space defining properties? What special elements does every vector space have?

(2) Why do we need a *field* to be associated to a vector space? What is a *field*? Give examples of finite and classic infinite fields.

(3) the *prototype* of a vector space?

(4) the classic examples of vector spaces, such as $\mathbb{R}^n$, $\mathcal{F}(\mathbb{R},\mathbb{R})$, $\mathcal{D}(\mathbb{R},\mathbb{R})$, $\mathcal{C}(\mathbb{R},\mathbb{R})$, $P(\mathbb{F})$, $P_n(\mathbb{F})$, $M_{m\times n}(F)$, sequences $\{a_n\}$ with elements in a field F? Describe each space.

(5) the difference between $\mathcal{C}/_{\mathbb{R}}$ and $\mathcal{C}/_{\mathbb{C}}$?

(6) basic properties of vector spaces that follow from the definition of vector space? e.g., cancellation law, uniqueness of $\vec{0}$, uniqueness of additive inverses, multiplying by 0-scalar, multiplying by $\vec{0}$, inverse of $\vec{x}$ written in two ways;

(7) vector subspaces? Definition and shortcut for confirming that a set is a vector subspace?

(8) intersection, sum and direct sum of two subspaces? is the union of two subspaces a subspace?

(9) classic types of matrices: upper-triangular, lower triangular, diagonal, symmetric, skew-symmetric?

(10) transpose of a matrix? transpose of a sum? transpose of a product of a scalar with a matirx?

(11) a *matrix*? the *size* of a matrix?

(12) a *linear combination* of vectors? What is the connection between the vector form of a linear system and a linear combination of vectors?

(13) the *identity* matrix?

(14) the *sum* of two matrices? Is it defined for any two matrices?

(15) the *product* of two matrices? What are the restrictions on the size of the two matrices?

(16) the *product* of a matrix and a scalar? Is it defined for any matrix and any scalar?

(17) a *linear transformation*? How do we decide on the size of a matrix A defining a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$? What is the *domain* and the *codomain* of T? What are the *input* and the *output* vectors for T? How do we compute $T(\vec{x})$?

(18) the *unit* vectors in $\mathbb{R}^n$? How do we identify the columns of a matrix A in terms of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$?

(19) the *fundamental linear properties* of linear transformations T? Why do they work? How do they relate to corresponding properties of matrices? What does it mean that T respects linear combinations?

(20) a non-linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$? Give an example. Is translation in the plane a linear transformation? Why?

(21) the *composition* of two linear transformations T_1 and T_2? If T_1 is given by a matrix A and T_2 – by a matrix B, what is the matrix of $T_2 \circ T_1$? of $T_1 \circ T_2$?

(22) If A and B are invertible square $n \times n$ matrices, are AB and BA invertible? If yes, what are their inverses?

(23) the *image* of a linear transformation T? Where does it live: in the domain or the codomain? If T is given by a matrix A, how do we find $\operatorname{Im} A$?

(24) the *span* of $\vec{v}_1, ..., \vec{v}v_m \in \mathbb{R}^n$? What is the connection between images of linear transformations and spans of vectors?

(25) the *kernel* of a linear transformation? How do we find it using linear systems?

(26) a *subspace* of $\mathbb{R}^n$? How do we verify that a subset of $\mathbb{R}^n$ is a subspace? Examples? Non-examples of subspaces?

(27) *linearly independent* vectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_k$ in a subspace V? *linearly dependent* vectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_k$?

(28) a *basis* of V formed by $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k\}$?

(29) a *linear relation* among vectors? a *trivial* and a *non–trivial* relation?

(30) the *dimension* of a subspace V?

(31) the *nullity* of a matrix A?

2. Theorems

Have a thorough understanding of each of the following theorems (laws, propositions, corollaries, etc.) Know how to **apply** each theorem appropriately in problems.

(1) **Columns of a Matrix.** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with matrix $A_{m\times n}$. Then the columns of A are the images of the unit vectors $\vec{e}_1, \vec{e}_2, ..., \vec{e}_n$ in $\mathbb{R}^n$ under T:

$$A = \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \\ | & | & & | \end{bmatrix}$$

(2) **Linear Properties of Linear Transformations.** Linear transformations respect addition and scalar multiplication, i.e. they respect linear combinations. In other words, if $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, $\vec{v}, \vec{w} \in \mathbb{R}^n$ and $k, k_1, k_2 \in \mathbb{R}$, then

(a) $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$;

(b) $T(k\vec{v}) = kT(\vec{v})$;

(c) $T(k_1\vec{v} + k_2\vec{w}) = k_1T(\vec{v}) + k_2T(\vec{w})$.

Conversely, if $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function which respects sums and scalar multiplication, then f is a linear transformation (i.e. f is given by some matrix $A_{m\times n}$ so that $f(\vec{v}) = A\vec{v}$.

(3) **Inverse of a 2×2 matrix.** A 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible iff its determinant $\det A = ad - bc \neq 0$. In such a case, the inverse matrix $A^{-1} = \dfrac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

(4) **Composition of Linear Transformations.** Let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ and $T_B : \mathbb{R}^m \to \mathbb{R}^k$ be two linear transformations given by matrices $A_{m\times n}$ and $B_{k\times m}$, respectively. Then the composition function $T = T_B \circ T_A : \mathbb{R}^n \to \mathbb{R}^k$ is also a linear transformation give by the matrix BA.

(5) **Properties of Matrix Products.**

(a) In general, matrices do **not** commute: $AB \neq BA$ for "most" choices of A and B.

(b) Matrices commute in special cases, e.g. an invertible matrix A commutes with its inverse A^{-1}: $AA^{-1} = A^{-1}A = I_n$.

(c) Matrix products are associative: $(AB)C = A(BC)$.

(d) I_n acts as the number "1" in the set of matrices: $A_{m\times n}\cdot I_n = A_{m\times n}$, $I_m\cdot A_{m\times n} = A_{m\times n}$.

(e) If A and B are invertible $n\times n$ matrices, then their product is also invertible and given by $(AB)^{-1} = B^{-1}A^{-1}$.

(f) If A and B are $n \times n$ such that $AB = I_n$, then both A and B are invertible, they are inverses of each other: $A^{-1} = B$ and $B^{-1} = A$, and also $BA = I_n$.

(g) Distributivity: $A \cdot (B + C) = AB + AC$.

(6) **Images and Span.** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation given by $A_{m\times n}$. Then $\mathrm{Im}T = \mathrm{Im}A$ is the span of the column-vectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$ of A: $\mathrm{Im}T = \mathrm{span}\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$.

(7) **Image of an Invertible Transformation.** If $A_{n\times n}$ is invertible, then the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by A has "full–image": $\mathrm{Im}T = \mathbb{R}^n$.

(8) **Properties of Images and Kernels.** If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation given by $A_{m\times n}$, then its kernel $\mathrm{Ker}T$ is a subspace of the domain $\mathbb{R}^n$, and its image $\mathrm{Im}T$ is a subspace of the codomain $\mathbb{R}^m$. In other words, images and kernels are closed under addition and scalar multiplication (and hence, under linear combinations), and contain $\vec{0}$.

(9) **Non-trivial Kernel.** If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation given by $A_{m\times n}$ such that $n > m$, then $\mathrm{Ker}T$ contains more than just $\vec{0}$: the transformation T is collapsing some of the dimensions of the domain $\mathbb{R}^n$ by mapping it into a smaller space $\mathbb{R}^m$.

(10) **Smallest Kernel.** Let $A_{m\times n}$. Then $\mathrm{Ker}A = \{\vec{0}\}$ iff $m \geq n$ and $\mathrm{rk}A = n$. In other words, a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ has trivial kernel $\mathrm{Ker}T = \{\vec{0}\}$ iff $n \leq m$ (the domain is a smaller space than the codomain) and A has the largest possible rank (n), which in this case is the number of columns of A. In particular, if $A_{n\times n}$ is a square matrix, then $\mathrm{Ker}A = \{\vec{0}\}$ iff $RREF(A) = I_n$, i.e. iff A is invertible.

(11) **Invertibility Equivalences.** Let $A_{n\times n}$. TFAE: A is invertible iff $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$ for every $\vec{b} \in \mathbb{R}^n$ iff $RREF(A) = I_n$ iff $\mathrm{rk}A = n$ iff $\mathrm{Ker}A = \{\vec{0}\}$ iff $\mathrm{im}A = \mathbb{R}^n$.

(12) **Subspaces of $\mathbb{R}^2$.** The only subspaces of $\mathbb{R}^2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1, x_2 \in \mathbb{R} \right\}$ are

(a) $\{\vec{0}\}$, the zero (trivial) subspace;

(b) lines through the origin;

(c) $\mathbb{R}^2$ (everything).

(13) **Subspaces of $\mathbb{R}^3$.** The only subspaces of $\mathbb{R}^3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x_1, x_2, x_3 \in \mathbb{R} \right\}$ are

(a) $\{\vec{0}\}$, the zero (trivial) subspace;

(b) lines through the origin;

(c) planes through the origin;

(d) $\mathbb{R}^3$ (everything).

(14) **Linear Independence and Relations.** $\vec{v}_1, \vec{v}_2, ..., \vec{v}_m$ in $\mathbb{R}^n$ are linearly independent iff none of them can be expressed as a linear combination of the others iff there is no non–trivial linear relation among them iff the only linear relation among them is the trivial one. In other words, to verify that $\vec{v}_1, \vec{v}_2, ..., \vec{v}_m$ in $\mathbb{R}^n$ are linearly independent, one has to show that

$$\text{if } c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m = \vec{0} \text{ then } c_1 = c_2 = \cdots = c_m = 0.$$

(15) **Linear Independence, Kernel and Rank.** $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$ in $\mathbb{R}^m$ are linearly independent iff

$$\operatorname{Ker} \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} = \{\vec{0}\} \text{ iff } \operatorname{rk} \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} = n.$$

This can only happen if $m \geq n$, i.e. more columns than rows in the above matrix.

(16) **Bases and Unique Linear Combinations.** Let V be a subspace of $\mathbb{R}^n$. Then $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ form a basis for V iff

(a) the $\vec{v}_i$'s span V and are linearly independent, or equivalently,

(b) every vector $\vec{v} \in V$ can be written as a unique linear combination of the $\vec{v}_i$'s, i.e.

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m$$

for some unique $c_1, c_2, ..., c_m \in \mathbb{R}$.

(17) **Dimensions of Kernels and Images.** Let A be an $m \times n$ matrix. Then

(a) $\dim(\operatorname{Ker}A)$ equals to the number of non–leading variables in $RREF(A)$, i.e. $\dim(\operatorname{Ker}A) = n - \operatorname{rk}A$. Further, a basis for $\operatorname{Ker}A$ is given by the "basis" solutions of the system $A\vec{x} = \vec{0}$ (after assigning letters to the non–leading variables in $RREF(A)$, solving the system and splitting the solutions into a linear combination of the "basis" vectors with coefficients equal to the assigned letters.)

(b) $\dim(\operatorname{Im}A)$ equals to the number of leading variables in $RREF(A)$, i.e. $\dim(\operatorname{Im}A) = \operatorname{rk}A$. Further, a basis for $\operatorname{Im}(A)$ is given by the columns of A corresponding to the leading variables in $RREF(A)$.

(18) **Fundamental Theorem of Linear Algebra.** Let A be an $m \times n$ matrix. Then

$$\dim(\operatorname{Ker}A) + \dim(\operatorname{Im}A) = n,$$

since the number of non–leading variables plus the number of leading variables equals the number of all variables, n. In other words, the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ given by A contracts as many dimensions of the domain $\mathbb{R}^n$ as there are in the kernel of T and results in the image of T inside the codomain. An equivalent formulation is

$$\text{null}(A) + \text{rk}(A) = n.$$

(19) **Bases of $\mathbb{R}^n$.** The standard basis of $\mathbb{R}^n$ is given by the n unit vectors $\{\vec{e}_1, \vec{e}_2, ..., \vec{e}_n\}$, and hence $\dim \mathbb{R}^n = n$. Given n vectors $\vec{v}_1, \vec{v_2}, ..., \vec{v}_n$ of $\mathbb{R}^n$, they form a basis for $\mathbb{R}^n$ iff for $A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$: Ker(A) $= \{\vec{0}\}$ i.e. $\text{rk}(A) = n$, i.e. $RREF(A) = I_n$, i.e. A is invertible.

(20) **Number of vectors in a basis.** Let V be a subspace of $\mathbb{R}^n$.

(a) If $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k\}$ is a basis for V, then any set of more that k vectors in V is linearly dependent: if $\{\vec{w}_1, \vec{w}_2, ..., \vec{w}_m\}$ is a set of m vectors in V with $m > k$, then the $\vec{w}_i$'s are linearly dependent.

(b) Any two bases of V have the same number of vectors, hence we can talk about dimension of V equal to this common number of vectors in each basis of V.

(21) **Linear Independent Sets, Basis Sets and Spanning Sets.** Let V be a subspace of $\mathbb{R}^n$ with $\dim V = k$.

(a) Any basis set of V has k vectors.

(b) Any spanning set of V has *at least* k vectors.

(c) Any linearly independent set in V has *at most* k vectors.

(d) If a linearly independent set in V has *exactly* k vectors, then it is a basis for V.

(e) If a spanning set in V has *exactly* k vectors, then it is a basis for V.

3. Problem Solving Techniques. Algorithms

(1) *Images of Unit Vectors.* Given properties of a linear transformation T, construct its matrix A by finding and recording the images of the unit vectors $T(\vec{e}_i)$'s as A's columns. Conversely, given A, identify the images of the unit vectors $T(\vec{e}_i)$'s as the columns of A, and thus reconstruct the linear transformation T: $T(\vec{x}) = A\vec{x}$.

(2) *Compositions of Transformation.* Using appropriate matrix products, find the matrix of the composition of two transformations T_1 and T_2 given the matrices A and B of T_1 and T_2, respectively.

(3) *Image of T.* Given the matrix A of a linear transformation T, find its image as the span of the columns of A. Given properties of T, find the images of the unit vectors $T(\vec{e}_i)$'s and use their span to find $\text{Im}\, T$.

(4) *Subspaces of $\mathbb{R}^n$.* Given a set V in $\mathbb{R}^n$, show that V is a subspace of $\mathbb{R}^n$ by following one of the possible algorithms:

(a) Check that the 3 basic properties of subspaces are satisfied by V; or

(b) Identify V as the image or kernel of some linear transformation and automatically conclude that it is a subspace; or

(c) Identify V as the span of several vectors and automatically conclude that it is a subspace (this is equivalent to identifying V as the image of the matrix whose columns are the spanning vectors); or

(d) Identify V as the set of solutions of a linear system $A\vec{x} = \vec{0}$ and automatically conclude that it is a subspace. (This is equivalent to identifying V as the kernel of A.)

(5) *Non–subspaces of* $\mathbb{R}^n$. Given a set V in $\mathbb{R}^n$, show that V is **not** a subspace of $\mathbb{R}^n$ by showing that one of the 3 basic properties of subspaces is violated for V: one specific example of this violation suffices.

(6) *Linear Independence.* Given vectors $\vec{v}_1,\ \vec{v}_2, ..., \vec{v}_k$ in $\mathbb{R}^n$, show that they are linearly **independent** by setting A to be the matrix with columns the $\vec{v}_i$'s, and verifying that the system $A\vec{x} = \vec{0}$ has a unique solution $\vec{0}$. However, if $A\vec{x} = \vec{0}$ has a non–trivial solution $\vec{x} = [x_1, x_2, ..., x_k]$, (i.e. a solution in which not all x_i's are 0), then conclude that the $\vec{v}_i$'s are linearly **dependent** because they satisfy a non–trivial linear relation $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_k\vec{v}_k = \vec{0}$.

(7) *Bases.* Given vectors $\vec{v}_1,\ \vec{v}_2, ..., \vec{v}_k$ in a subspace V of $\mathbb{R}^n$, be able to decide if they form a basis for V by following one of the algorithms:

(a) Verify that the $\vec{v}_i$'s span V and that they are linearly independent; or

(b) Verify that any $\vec{x} \in V$ can be written as a **unique** linear combination of the $\vec{v}_i$'s; or

(c) If we know that $\dim V = k$ (the number of given vectors $\vec{v}_i$'s), then show either than the $\vec{v}_i$'s span V or that the $\vec{v}_i$'s are linearly independent, and automatically conclude that they are also a basis for V.

(8) *Dimension of* V. Find a basis for V and then its dimension equals the number of basis vectors.

(9) *Basis and Dimension of Kernel.* Given matrix A, solve $A\vec{x} = \vec{0}$ via reducing A to $RREF(A)$, split the solution $\vec{x}$ as a linear combination of vectors $\vec{v}_i$'s with coefficients equal to the letters assigned to the non–leading variables, and automatically conclude that those vectors $\vec{v}_i$'s form a basis for $\operatorname{Ker} A$. Also, $\dim(\operatorname{Ker} A)$ equals the number of non–leading variables found above.

(10) *Basis and Dimension of Image.* Given matrix A, find its $RREF(A)$. The columns of A corresponding to the leading variables in $RREF(A)$ form a basis for $\operatorname{Im} A$. Also, $\dim(\operatorname{Im} A)$ equals the number of leading variables found above.

4. Problems for Review

Review **all** homework problems, and all your class notes. Such a thorough review should be enough to do well on the midterm. While doing the midterm, compare the given problems with something we have done before (or on HW): quite often you will see similarities. However, you have to review the whole material **before** starting the midterm: it will take you less time this way compared to... not reviewing, getting frustrated with a problem on the midterm, not remembering where you have seen a similar one, and then ending up reviewing most stuff for every single problem.