HW #23; date: Nov. 21, 2017
MATH 110 Linear Algebra
with Professor Stankova

6.3 #1 (a) True. (b) False. The dimensions don't match up; there are n^2 linear operators but only n vectors y. What's true is that all linear functionals are of that form. (c) False. The basis needs to be orthonormal. (d) True. Theorem. (e) False. The coefficients must be conjugated. (f) True. (g) True.

6.3 #2abc (a) $y = (1, -2, 4)$. (b) $y = (1, -2)$ (c) Say $h(x) = a + bx + cx^2$. We want $g(1) = 1$, $g(x) = 1$ and $g(x^2) = 2$. This tells us that $g(1) = \int_0^1 a + bx + cx^2 \; dx = a + \frac{1}{2}b + \frac{1}{3}c = 1$. Next, $g(x) = \int_0^1 ax + bx^2 + cx^3 \; dx = \frac{1}{2}a + \frac{1}{3}b + \frac{1}{4}c = 1$. Next, $\int_0^1 ax^2 + bx^3 + cx^4 \; dx = \frac{1}{3}a + \frac{1}{4}b + \frac{1}{5}c = 2$. Putting these together we have a system

$$\begin{pmatrix} 6 & 3 & 2 \\ 6 & 4 & 3 \\ 20 & 15 & 12 \end{pmatrix} x = \begin{pmatrix} 6 \\ 12 \\ 60 \end{pmatrix}$$

Solving it, we find

$$\begin{pmatrix} 6 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 6 \\ 6 \\ 30 \end{pmatrix}$$

i.e. $c = 30$, $b = -24$, $a = 3$, i.e. $h(x) = 3 - 24x + 30x^3$.

6.3 #3abc (a) $\langle (x, y), T^*(3, 5) \rangle = \langle T(x, y), (3, 5) \rangle = \langle (2x + y, x - 3y), (3, 5) \rangle = 6x + 3y + 5x - 15y = 11x - 12y$. Thus, $T^*(3, 5) = (11, -12)$.

(b) $\langle (x, y), T^*(3 - i, 1 + 2i) \rangle = \langle T(x, y), (3 - i, 1 + 2i) \rangle = \langle (2x + iy, (1 - i)x), (3 - i, 1 + 2i) \rangle = (2x + iy)(3 + i) + ((1 - i)x)(1 - 2i) = x(6 + 2i + (1 - i)(1 - 2i)) + y(-1 + 3i) = x(5 - i) + y(-1 + 3i)$. So, $\langle T^*(3 - i, 1 + 2i), (x, y) \rangle = \overline{x}(5 + i) + \overline{y}(-1 - 3i)$, so $T^*(3 - i, 1 + 2i) = (5 + i, -1 - 3i)$.

(c) We have $\langle a + bt, T^*(4 - 2t) \rangle = \langle T(a + bt), 4 - 2t \rangle = \langle (b + 3a) + 3bt, 4 - 2t \rangle = \int_{-1}^1 ((b + 3a) + 3bt)(4 - 2t) \; dt = \int_{-1}^1 4(b + 3a) - 6bt^2 \; dt = 8(b + 3a) - 4b = 24a + 4b$. So, $\langle T^*(4 - 2t), a + bt \rangle = 24a + 4b$. Say $T^*(4 - 2t) = c + dt$. Then, $\langle c + dt, 1 \rangle = 24$, i.e. $\int_{-1}^1 c + dt \; dt = 2c = 24$, i.e. $c = 12$. Also, $\langle c + dt, t \rangle = \int_{-1}^1 ct + dt^2 \; dt = \frac{2}{3}d = 4$, i.e. $d = 6$. Thus, $T^*(4 - 2t) = 12 + 6t$.

6.3 #6 $U_1^* = (T + T^*)^* = T^* + (T^*)^* = T^* + T = U_1$ and $U_2^* = (TT^*)^* = (T^*)^* T^* = TT^* = U_2$.

6.3 #8 Let U be the inverse of T. Then $TU = I$. Taking the adjoint of this equation, we have $U^*T^* = I^* = I$, so U^* is the inverse of T^*. The converse follows by reciprocity.

6.3 #9 We need to show that $\langle Tx, y \rangle = \langle x, Ty \rangle$. To this end, write $x = x' + x''$ and $y = y' + y''$ where $x', y' \in W$ and $x'', y'' \in W^\perp$. Then, $\langle Tx, y \rangle = \langle x', y' + y'' \rangle = \langle x', y' \rangle$. On the other hand, $\langle x, Ty \rangle = \langle x' + x'', y' \rangle = \langle x', y' \rangle$, completing the proof.

6.3 #10 Suppose that $\langle Tx, Ty\rangle = \langle x, y\rangle$. Taking $x = y$, we find that $||Tx||^2 = ||x||^2$. Since $||Tx||$ and $||x||$ are positive, we have $||Tx|| = ||x||$. Conversely, suppose that $||Tx|| = ||x||$ for all $x \in V$. Then, $||T(x+y)||^2 = \langle T(x+y), T(x+y)\rangle = ||Tx||^2 + ||Ty||^2 + \langle T(x), T(y)\rangle + \langle T(y), T(x)\rangle$ and $||x+y||^2 = ||x||^2 + \langle x, y\rangle + \langle y, x\rangle + ||y||^2$. These two are equal, so in particular $\Re(\langle T(x), T(y)\rangle) = \Re(\langle x, y\rangle)$. To find that the imaginary parts are equal, apply the same argument to $x + iy$: we have on one hand

$$\begin{aligned} ||x||^2 + ||y||^2 + \langle x, iy\rangle + \langle iy, x\rangle &= ||x||^2 + ||y||^2 - i\langle x, y\rangle + i\overline{\langle x, y\rangle} \\ &= ||x||^2 + ||y||^2 + 2\Im(\langle x, y\rangle) \end{aligned}$$

while $||T(x+iy)|| = ||x||^2 + ||y||^2 + \Im(\langle Tx, Ty\rangle)$, so $\Im(\langle x, y\rangle) = \Im(\langle Tx, Ty\rangle)$.

6.3 #11 Suppose that $T^*T = 0$. Then, $\langle T^*Tx, y\rangle = \langle Tx, Ty\rangle = 0$ for all x, y. In particular, taking $y = x$ we find that $||Tx||^2 = 0$, so in particular $T(x) = 0$ for all x, so $T = 0$. The result is still true if we assume $TT^* = 0$: take $\langle x, TT^*y\rangle = 0$, and applying the same argument, we find that $T^* = 0$. This means that $\langle Tx, y\rangle = \langle x, T^*y\rangle = 0$ for all x, y. In particular, taking $y = Tx$, we find that $||Tx||^2 = 0$ for all x, so $T = 0$.

6.3 #14 T is linear because inner products are linear in the first variable (not that they are not linear in the second variable – conjugate linear). We claim that $S(x) = \langle x, z\rangle y$ is the adjoint. (The intuition here is to think of matrices; T has range z, i.e. the columns are multiples of z, so when we take the transpose, the rows become z, i.e. the nullspace consists of vectors that dot with z to zero). To see this, note that $\langle T(x), w\rangle = \langle\langle x, y\rangle z, w\rangle = \langle x, y\rangle\langle z, w\rangle$. On the other hand, $\langle x, S(w)\rangle = \langle x, \langle w, z\rangle y\rangle = \overline{\langle w, z\rangle}\langle x, y\rangle = \langle z, w\rangle\langle x, y\rangle$.

6.3 #18 Since the determinant is a polynomial in the entries of a matrix with real coefficients, $\det(\overline{A}) = \overline{\det(A)}$. Furthermore, $\det(A^t) = \det(A)$. Combining these two yields the desired statement.

6.3 #20a We want to minimize the error $|| \begin{pmatrix} -3 & 1 \\ -2 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} x - \begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix} ||$. The least squares solution is a solution to the system

$$\begin{pmatrix} 14 & -4 \\ -4 & 4 \end{pmatrix} x = \begin{pmatrix} -38 \\ 18 \end{pmatrix}$$

i.e. $x = (-2, 2.5)$, i.e. the linear least squares solution is $-2t + 2.5$. The error is $||(-0.5, 0.5, 0.5, -0.5)|| = 1$.

The quadratic least squares: We want to minimize the error $|| \begin{pmatrix} 9 & -3 & 1 \\ 4 & -2 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} x - \begin{pmatrix} 9 \\ 6 \\ 2 \\ 1 \end{pmatrix} ||$. The least squares solution is a solution to the system

$$\begin{pmatrix} 98 & -34 & 14 \\ -34 & 14 & -4 \\ 14 & -4 & 4 \end{pmatrix} x = \begin{pmatrix} 106 \\ -38 \\ 18 \end{pmatrix}$$

i.e. $x = (1/3, -4/3, 2)$, i.e. the quadratic least squares solution is $\frac{1}{3}t^2 - \frac{4}{3}t + 2$. The error is $||(1, 0, 0, 0)|| = 1$.

6.3 #24 (a) We check $T(a\sigma + b\tau)(k) = \sum_{i=k}^{\infty}(a\sigma + b\tau)(i) = a\sum_{i=k}\sigma(i) + b\sum_{i=k}^{\infty}\tau(i) = aT(\sigma) + bT(\tau)$. (b) It suffices to check that both sides agree after evaluating at any k. Indeed, $T(e_n)(k) = \sum_{i=k}^{\infty} e_n(k)$ is 1 if $k \leq n$ and 0 otherwise. On the other side, $\sum_{i=1}^{n} e_i(k)$ is 1 if $k \leq n$ and 0 otherwise. (c) Suppose an adjoint S existed. Then, we have $\langle T(e_n), e_m\rangle = \langle e_n, S(e_m)\rangle$. The left hand side is $\langle \sum_{i=1}^{n} e_i, e_m\rangle$ which is 1 if $m \leq n$ and 0 otherwise. In particular, $\langle e_n, S(e_m)\rangle = 1$ if $n \geq m$. Take, for example, $m = 1$, and write $S(e_1) = \sum a_k e_k$; then $\langle e_n, S(e_1)\rangle = \sum_k \overline{a_k}\langle e_n, e_k\rangle = 1$ for $n \geq 1$, so in particular, $a_n = 1$ for all $n \geq 1$, which contradicts the requirement that we only allow finite sums.