HW #22; date: Nov. 21, 2017
MATH 110 Linear Algebra
with Professor Stankova

6.2 #1 (a) False. They must be linearly independent. (b) True. Use Gram-Schmidt on any basis. (c) True. Say $W = S^\perp$. If $x \in W$ and $y \in W$, this means that $x \cdot s = 0$ and $y \cdot s = 0$ for any $s \in S$. Then, $(ax + by) \cdot s = a(x \cdot s) + b(y \cdot s) = 0 + 0 = 0$, so $ax + by \in W$. (d) False. It should be an orthonormal basis. (e) True. (f) False. In $\mathbb{R}^{\not{}}$, $\{(0,0),(1,1)\}$ is an orthogonal set, but linearly dependent. (g) True. If we have a linear dependence $\sum a_i x_i = 0$, then we have $0 = \langle \sum a_i x_i, \sum a_i x_i \rangle = \sum |a_i|^2$ (since $\langle x_i, x_j \rangle = 0$ if $i \neq j$ and $\langle x_i, x_i \rangle$.) So this expression can only be zero if all of the $a_i = 0$.

6.2 #2bcdgij (b) $v_1 = (1,1,1)$. $v_2 = (0,1,1) - \frac{(1,1,1)\cdot(0,1,1)}{(1,1,1)\cdot(1,1,1)}(1,1,1) = \frac{1}{3}(-2,1,1)$. $v_3 = (0,0,1) - \frac{(0,0,1)\cdot(1,1,1)}{3}(1,1,1) - \frac{(0,0,1)\cdot(-2,1,1)}{6}(-2,1,1) = \frac{1}{2}(0,-1,1)$ (note that I ignored the constant $\frac{1}{3}$ in the calcuation here – it's because the numerator an denominator will cancel out). The basis is $(1,1,1), \frac{1}{3}(-2,1,1), \frac{1}{2}(0,-1,1)$. The orthonormal basis is $u_1 = \frac{1}{\sqrt{3}}(1,1,1)$, $u_2 = \frac{1}{\sqrt{6}}(-2,1,1)$, $u_3 = \frac{1}{\sqrt{2}}(0,-1,1)$. The Fourier coefficients of $(1,1,2)$ are $\frac{4}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}$. We verify: $\frac{4}{\sqrt{3}}\frac{1}{\sqrt{3}}(1,1,1) + \frac{1}{\sqrt{6}}\frac{1}{\sqrt{6}}(-2,1,1) + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(0,-1,1) = \frac{4}{3}(1,1,1) + \frac{1}{6}(-2,1,1) + \frac{1}{2}(0,-1,1) = (1,1,2)$.

(c) $v_1 = 1$. $v_2 = x - \frac{\int_0^1 x\,dx}{\int_0^1 1\,dx}1 = x - \frac{1}{2}$, $v_3 = x^2 - \frac{\int_0^1 x\,dx}{1}1 - \frac{\int_0^1 x^2(x-1/2)\,dx}{\int_0^1 (x-1/2)^2\,dx}x = x^2 - \frac{1}{3} - (x - \frac{1}{2}) = x^2 - x + \frac{1}{6}$. The orthonormal basis is $u_1 = 1$, $u_2 = \frac{1}{\sqrt{\int_0^1 (x-1/2)^2\,dx}}(x - \frac{1}{2}) = \sqrt{3}(2x-1)$, $u_3 = \frac{1}{\int_0^1 (x^2-x+1/6)^2\,dx}(x^2 - x - \frac{1}{6}) = \sqrt{180}(x^2 - x + \frac{1}{6}) = \sqrt{5}(6x^2 - 6x + 1)$. The Fourier coefficients of $h(x) = 1 + x$ are $\frac{3}{2}$, $\frac{\sqrt{3}}{6}$ and 0. We verify: $\frac{3}{2} + \frac{\sqrt{3}}{6}\sqrt{3}(2x-1) = \frac{3}{2} + \frac{1}{2}(2x-1) = x + 1$.

(d) Note: in this problem you have to be really careful about the order of the Hermitian product, since the entries are complex. $v_1 = (1, i, 0)$ and $v_2 = (1-i, 2, 4i) - \frac{(1-i)\cdot 1 + 2\cdot -i + 4i\cdot 0}{2}(1,i,0) = (1-i, 2, 4i) - (\frac{1-3i}{2}, \frac{3+i}{2}, 0) = (\frac{1+i}{2}, \frac{1-i}{2}, 4i)$. Orthonormal: $u_1 = \frac{1}{\sqrt{2}}(1,i,0)$ and $u_2 = \frac{1}{2\sqrt{17}}(1+i, 1-i, 8i)$. Fourier coefficients of $(3+i, 4i, -4)$ are $\frac{1}{\sqrt{2}}3 + i + 4i\cdot(-i) = \frac{1}{\sqrt{2}}(7+i)$ and $\frac{1}{2\sqrt{17}}(34i) = \sqrt{17}i$. Verify: $\frac{1}{2}(7+i)(1,i,0) + \frac{1}{2}i(1+i, 1-i, 8i) = (3+i, 4i, -4)$

(g) Note that the Frobenius product of a matrix with itself is the square of its entries, and the Frobenius product of is the "dot product" of the two matrix "component-wise." This makes the calcuations a bit easier. $v_1 = \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix} - \frac{-3-5+45-1}{9+1+25+1}\begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix}$. $v_3 = \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix} - \frac{-72}{36}\begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix} - \frac{-72}{72}\begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix} = \begin{pmatrix} 9 & -3 \\ 6 & -6 \end{pmatrix}$. Orthonormal: $u_1 = \frac{1}{6}\begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}$, $u_2 = \frac{1}{6\sqrt{2}}\begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix}$, $u_3 = \frac{1}{9\sqrt{2}}\begin{pmatrix} 9 & -3 \\ 6 & -6 \end{pmatrix}$. Finally, the Fourier coefficients of $\begin{pmatrix} -1 & 27 \\ -4 & 8 \end{pmatrix}$ are $24, 6\sqrt{2}, -9\sqrt{2}$. Verify: $\frac{1}{6}\begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}24 + \frac{1}{6\sqrt{2}}\begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix}6\sqrt{2} + \frac{1}{9\sqrt{2}}\begin{pmatrix} 9 & -3 \\ 6 & -6 \end{pmatrix}(-9\sqrt{2}) = \begin{pmatrix} -1 & 27 \\ -4 & 8 \end{pmatrix}$.

(i) $v_1 = \sin(t)$, $v_2 = \cos(t) - \frac{\int_0^\pi \sin(t)\cos(t)\,dt}{\int_0^\pi \sin(t)^2\,dt}\sin(t) = \cos(t)$, $v_3 = 1 - \frac{\int_0^\pi \sin(t)\,dt}{\int_0^\pi \sin(t)^2\,dt}\sin(t) - \frac{\int_0^\pi \cos(t)\,dt}{\int_0^\pi \cos(t)^2\,dt}\cos(t) = 1 - \frac{2}{\pi/2}\sin(t) = 1 - \frac{4}{\pi}\sin(t)$ and $v_4 = t - \frac{\int_0^\pi t\sin(t)\,dt}{\int_0^\pi \sin(t)^2\,dt}\sin(t) - \frac{\int_0^\pi t\cos(t)\,dt}{\int_0^\pi \cos(t)^2\,dt}\cos(t) - \frac{\int_0^\pi t - \frac{4}{\pi}t\sin(t)\,dt}{\int_0^\pi (1-\frac{4}{\pi}\sin(t))^2\,dt}(1 -$

$\frac{4}{\pi}\sin(t)) = t - \frac{\pi}{\pi/2}\sin(t) - \frac{-2}{\pi/2}\cos(t) - \frac{\frac{1}{2}(\pi^2-8)}{\pi-\frac{8}{\pi}}(1 - \frac{4}{\pi}\sin(t)) = t - 2\sin(t) + \frac{4}{\pi}\cos(t) - \frac{\pi}{2} + 2\sin(t) = t + \frac{4}{\pi}\cos(t) - \frac{\pi}{2}$. Orthonormal: $u_1 = \sqrt{\frac{2}{\pi}}\sin(t)$, $u_2 = \sqrt{\frac{2}{\pi}}\cos(t)$, $u_3 = \sqrt{\frac{\pi}{\pi^2-8}}(1 - \frac{4}{\pi}\sin(t))$, and $u_4 = \sqrt{\frac{12\pi}{\pi^4-96}}(t + \frac{4}{\pi}\cos(t) - \frac{\pi}{2})$. The Fourier coefficients of $g(t) = 2t + 1$ are $\sqrt{\frac{2}{\pi}}(2 + 4\pi)$, $\sqrt{\frac{2}{\pi}}8$, $\sqrt{\frac{\pi}{\pi^2-8}}\frac{(2\pi+1)(\pi^2-8)}{\pi}$, $\sqrt{\frac{12\pi}{\pi^4-96}}\frac{\pi^4-96}{3\pi}$. The final verification is left to the reader.

(j) $v_1 = (1, i, 2 - i, -1)$, $v_2 = (2 + 3i, 3i, 1 - i, 2i) - \frac{(2+3i,3i,1-i,2i)\cdot(1,-i,2+i,-1)}{8}(1, i, 2 - i, -1) = (1 + 3i, 2i, -1, 1 + 2i)$, $v_3 = (-1 + 7i, 6 + 10i, 11 - 4i, 3 + 4i) - \frac{(-1+7i,6+10i,11-4i,3+4i)\cdot(1,-i,2+i,-1)}{8}(1, i, 2 - i, -1) - \frac{(-1+7i,6+10i,11-4i,3+4i)\cdot(1-3i,-2i,-1,1-2i)}{20}(1+3i, 2i, -1, 1+2i) = (-7+i, 6+2i, 5, 5)$. Normalized: $u_1 = \frac{1}{\sqrt{8}}(1, i, 2 - i, -1)$, $u_2 = \frac{1}{\sqrt{20}}(1 + 3i, 2i, -1, 1 + 2i)$, $u_3 = \frac{1}{\sqrt{140}}(-7 + i, 6 + 2i, 5, 5)$. Fourier coefficients of $(-2 + 7i, 6 + 9i, 9 - 3i, 4 + 4i)$: $\frac{24+8i}{\sqrt{8}} = \sqrt{8}(3 + i)$, $\frac{44-12i}{\sqrt{20}}$, $\frac{112-4i}{\sqrt{140}}$. The final verification is left to the reader.

6.2 #3 β is already orthonormal. Then, we have that the Fourier coefficients are $\frac{7}{\sqrt{2}}$ and $\frac{-1}{\sqrt{2}}$.

6.2 #4 We want (x, y, z) satisfying the equations $x - iz = 0$ and $x + 2y + z = 0$. These are just matrix equations, so we can solve $\begin{pmatrix} 1 & 0 & i \\ 1 & -2 & 1 \end{pmatrix} x = 0$. This row reduces to $\begin{pmatrix} 1 & 0 & -i \\ 0 & 2 & 1+i \end{pmatrix}$, so we have that $S^\perp$ is spanned by the vector $(i, -(1 + i)/2, 1)$.

6.2 #5 $S_0^\perp$ is the plane normal to the vector x_0. $S^\perp$ is the line normal to the plane spanned by x_1, x_2.

6.2 #6 Using Theorem 6.6, write $x = y + z$ where $y \in W^\perp$ and $z \in W$. Then, we have $\langle x, y\rangle = \langle y + z, y\rangle = \langle y, y\rangle + \langle z, y\rangle = ||y||^2$, since z and y are orthogonal by construction. This is nonzero if and only if y is nonzero; however, we assumed that $x \notin W$, and if $y = 0$ then $x = z \in W$, so y was nonzero by assumption.

6.2 #7 If $z \in W^\perp$, then $\langle z, v\rangle = 0$ for every $v \in W$, in particular if $v \in \beta$. Conversely, suppose that $\langle z, v\rangle = 0$ for every $v \in \beta$. Since β is a basis, every $w \in W$ can be written $w = a_1v_1 + \dots a_rv_r$ for $v_i \in \beta$. Then, $\langle z, w\rangle = \langle z, \sum a_iv_i\rangle = \sum \overline{a_i}\langle z, v_i\rangle = 0$.

6.2 #8 We induct on the indices i. For $i = 1$ the statement is obvious, since Gram-Schmidt does not do anything to the first vector. Suppose that the statement is true up for $i = 1, \dots, k$. Then Gram-Schmidt says that $v_{k+1} = w_{k+1} - \sum_{i=1}^k \frac{\langle w_{k+1}, v_i\rangle}{\langle v_i, v_i\rangle} v_i = w_{k+1} - \sum_{i=1}^k \frac{\langle w_{k+1}, w_i\rangle}{\langle w_i, w_i\rangle} w_i = w_{k+1}$, since the w_i are orthogonal, completing the proof.

6.2 #10 For $x \in V$, write $x = w + z$ where $w \in W$ and $z \in W^\perp$, which can be done uniquely by Theorem 6.6. Define $T(x) = w$. This is a well-defined function, but we need to show that it is linear. To see this, write $x' = w' + z'$. Then, the $ax + bx' = (aw + bw') + (az + bz')$. Since W and $W^\perp$ are subspaces, $aw+bw' \in W$ and $az+bz' \in W^\perp$. Since the decomposition in Theorem 6.6 was unique, it must be the case that $T(ax + bx') = aw + bw' = aT(x) + bT(x')$. Further, since w and z are orthogonal, we have $||x||^2 = ||w||^2 + ||z||^2$ by the Pythagorean Theorem. Thus, $||T(x)|| = ||w|| = \sqrt{||x||^2 - ||z||^2} \leq ||x||$.

6.2 #11 The ijth entry of AB is the dot product between the ith row of A and the jth column of B. Thus the ijth entry of AA^* is the dot product between the ith row of A and the jth column of A^*, which is the conjugate of the jth row of A. Thus, $AA^* = I$ if and only if $\langle a_i, a_j\rangle$ is 0 when $i \neq j$ and 1 when $i = j$, where the brackets indicate the standard inner product on $\mathbb{C}^n$.

6.2 #13c First, we show that $W \subset (W^\perp)^\perp$. Suppose that $w \in W$. We claim that $w \in (W^\perp)^\perp$; that is, for any x such that $\langle x, y\rangle = 0$ for very $y \in W$, we have $\langle w, x\rangle = 0$. But $\langle w, x\rangle = \overline{\langle x, w\rangle} = 0$, just taking $y = w$. Next, we show that $(W^\perp)^\perp \subset W$. Suppose that $x \in (W^\perp)^\perp$. Write $x = w + w'$, where $w \in W$ and $w' \in W^\perp$ using Theorem 6.6. We want to show that $x \in W$. Suppose not; using Exercise 6, we should be able to choose $y \in W^\perp$ such that $\langle x, y\rangle \neq 0$. But in fact we can't, since $x \in (W^\perp)^\perp$, i.e. $\langle x, y\rangle = 0$ for every $y \in W^\perp$. Thus, $x \in W$.

6.2 #14 First, we will show that $(W_1 + W_2)^\perp \subset W_1^\perp \cap W_2^\perp$. If $x \in (W_1 + W_2)^\perp$, then in particular it is in $(W_1 \cup W_2)^\perp$ since $W_1 \cup W_2 \subset W_1 + W_2$. That is, $\langle x, w\rangle = 0$ if $w \in W_1$ or $w \in W_2$. Thus, $x \in W_1^\perp$ and $x \in W_2^\perp$, so $x \in W_1^\perp \cap W_2^\perp$. Now, we will show that $(W_1 + W_2)^\perp \supset W_1^\perp \cap W_2^\perp$. Suppose that $x \in W_1^\perp \cap W_2^\perp$. This means that $\langle x, w_1\rangle = 0$ for all $w_1 \in W_1$, and the same is true for all $w_2 \in W_2$. Thus, for any $w \in W$, written as $w = w_1 + w_2$, we have $\langle x, w_1 + w_2\rangle = 0$, proving the first equation. For the second equation, apply Exercise 13c.

6.2 #19bc (b) First, we find a basis for W. $\beta = \{(-3, 1, 0), (2, 0, 1)\}$ will do. We then make it orthogonal via Gram-Schmidt: $v_1 = (-3, 1, 0)$ and $v_2 = (2, 0, 1) - \frac{-6}{10}(-3, 1, 0) = (2, 0, 1) + (-9/5, -3/5, 0) = \frac{1}{5}(1, -3, 5)$. Let's instead set $v_2 = (1, -3, 5)$ since the constant doesn't matter. The projection is $\frac{(2,1,3)\cdot(-3,1,0)}{10}(-3, 1, 0) + \frac{(2,1,3)\cdot(1,-3,5)}{35}(1, -3, 5) = (13/6, -17/10, 2)$. (c) We computed the orthogonal basis in Problem 2c, which is $\{1, x - \frac{1}{2}\}$. Then, the projection is $\frac{\int_0^1 4+3x-2x^2\,dx}{\int_0^1 1\,dx} 1 + \frac{\int_0^1 (x-\frac{1}{2})(4+3x-2x^2)\,dx}{\int_0^1 (x-\frac{1}{2})^2\,dx}(x - \frac{1}{2}) = \frac{6}{29} + \frac{1/12}{1/12}(x - \frac{1}{2}) = x - \frac{17}{58}$.

6.2 #22 (a) Take $v_1 = \sqrt{t}$ and $v_2 = t - \frac{\int_0^1 t\sqrt{t}\,dt}{\int_0^1 t\,dt}\sqrt{t} = \sqrt{t} - \frac{5/2}{1/2}\sqrt{t} = t - 5\sqrt{t}$. We then normalize $u_1 = \sqrt{2t}$ and $u_2 = \frac{\sqrt{6}}{\sqrt{53}}(t - 5\sqrt{t})$. (b) We project to W: $(\int_0^1 t^2\sqrt{t}\,dt)\sqrt{t} + \sqrt{6/53}(\int_0^1 t^2(t - 5\sqrt{t})\,dt)(t - 5\sqrt{t}) = \frac{7}{2}\sqrt{t} - \sqrt{6/53}(69/4)(t - 5\sqrt{t})$.

6.2 #23 (a) We need to check the properties for inner products on V. However, note that $F^n \subset V$ as the "first n components." Any sentence we have to check involves vectors $v \in V$ which are in F^n for large enough n (i.e. since only finitely many coefficients can be nonzero, take n to be the maximum index of such coefficient). This means that for each such sentence, our verification of the inner product properties happens in F^n, in which we know they hold, so the result follows. (b) It's not hard to check that these vectors are orthogonal, and also normal (just check that $\sum_n e_i(n)e_j(n) = e_i(i)e_j(i) + e_i(j) + e_j(j) = \delta_{ij}$). We also need to show it is a basis; let $\sigma \in V$. It is nonzero at finitely many indices, say indexed by the finite set S. Then, $\sigma = \sum_{s\in S} \sigma(s)e_s$, so the set spans. It is also linearly independent since it is orthogonal, so it is a basis. (c) (i) Note that $\sigma_n(k) = 1$ if $k = 1, n$ and zero otherwise. Suppose that $e_1 = \sum_{s\in S} a_s\sigma_s$ (finite sum). Choose some $k \in S$. Then, $e_1(k) = 0$ but $\sum_{s\in S} a_s\sigma_s(k) = a_k$, we have that $a_k = 0$. Since k was chosen arbitrarily, we have $a_k = 0$ for all $k \in S$, i.e. that the right hand is zero. But, $e_1(1) = 1$, i.e. $e_1 \neq 0$, so we have a contradiction. (ii) Let $w \in W^\perp$. Let us write $w = \sum_{s\in S} a_s e_s$ where S is some finite set containing 1. Note that for $k \geq 2$, we have $0 = \langle w, \sigma_k\rangle = \sum_{s\in S} a_s\langle e_s, \sigma_k\rangle = \sum_{s\in S} a_s\langle e_s, e_1 + e_k\rangle$. This expression is $a_1 + a_k$ if $k \in S$ and a_1 if $k \notin S$. This shows that $a_1 = 0$ and hence $a_k = 0$ for every $k \in S$, $k \geq 2$ so $w = 0$. Thus, $(W^\perp)^\perp = V$, which is not W.

Challenge: Prove that if the rows of a square matrix are orthonormal (under the dot product), then the columns are also orthonormal. **Solution**: Suppose that the rows are orthonormal. In general, the ijth entry of AB (where A, B are matrices) is the dot product between the ith row of A and the jth column of B. Thus, the ijth entry of AA^t is the dot product of the ith row of A and the jth row of A^t. The jth row of A^t is the jth column of A, so this is 1 if $i = j$ and 0 if $i \neq j$. In other words, $AA^t = I$. Since A is square, this means that $A^{-1} = A^t$. In particular, a right inverse is a left inverse, so $A^tA = I$. The ijth entry of A^tA is the dot product between the ith column of A and the jth column of A. This says that the columns of A are orthogonal.