Midterm I. MATH 110 Linear Algebra

Instructor: Zvezdelina Stankova

STUDENT NAME: ______________________ STUDENT ID: ______________

GSI'S NAME: ______________________________________

• DO NOT OPEN THE MIDTERM UNTIL TOLD TO DO SO!

• Please, do *all* Problems 1 through 5, as best as you can. The midterm about 70 minutes long. You may **not** leave early.

• Use the provided sheets to write your solutions. You may use the back of each page for the remainder of your solutions; in such a case, put an arrow at the bottom of the page and indicate that the solution continues on the back page. Before the test starts, pull out and use several of your own sheets for scratch work. **No extra sheets of paper can be submitted with this exam due to the grading computer software!** Thus, use the front and the back of the sheets in the exam for the actual solutions, and not for scratch work.

• The exam is closed notes and books, which means: **no classnotes, no session notes, no review notes, no textbooks, and no other materials can be used during the midterm.** You can use only your **cheat sheet**, this exam packet and the additional blank pages that you pulled out before the exam started. The cheat sheet is one side of a regular 8×11 sheet, handwritten only by you. A cheat sheet that doesn't conform to these specifications (for example, there is stuff written on both sides of the sheet) will be disqualified, and the student's midterm may be annulled.

• **NO CALCULATORS ARE ALLOWED DURING THE MIDTERM!**

• **You cannot ask for help in any form from other students, you cannot look at their midterms and copy, you cannot cheat in any way – the midterm is an individual assignment and must be done only by you.** If you have a question regarding the statement of a problem, raise your hand when the instructor or a GSI is around.

• Think of this midterm as an important homework. Check your reasoning and calculations carefully. Justify *all* your answers, include all intermediate steps and calculations. If you are not sure about how to write something in math notation, explain clearly *in words* what you mean and what you are doing. **Unjustified answers, even if correct, will receive no credit!** On the contrary, good justifications and good work on a problem may receive a lot of credit even if the final answer is incorrect.

• Before turning in your midterm, please, sign the statement below. **Midterms that are not signed will not be graded.**

I, the student whose name and signature appear on this midterm, have completed the midterm without any outside help from people or other sources. I have used only my cheat sheet conforming to the specifications written above. I have not cheated in any way, and I have followed all instructions above.

STUDENT SIGNATURE: ______________________________________

Problem 1. (10 pts) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a function. Explain why each T below is *not* a linear transformation.

(a) $T(a, b) = (1, b)$.

- $T(\vec{0}) = T(0,0) = (1,0) \neq \vec{0}$

  But a linear transformation sends $\vec{0} \to \vec{0}$

  Thus, T is not linear.

(b) $T(a, b) = (|a - b|, 0)$.

- Let $\vec{v} = (1,0)$. A linear transformation T would send $T(-\vec{v}) = -T(\vec{v})$, but here we have:
  - $T(-\vec{v}) = T(-1,0) = (1,0)$
  - $-T(\vec{v}) = -T(1,0) = -(1,0)$

$\Rightarrow T(-\vec{v}) \neq -T(\vec{v})$ and hence T is not linear.

Problem 2. (20 pts) Answer briefly each question in the allotted space below.

(a) If V and W are finite-dimensional spaces over F, why does $\dim V = \dim W$ imply that $V \cong W$? Explain. (Do not quote that that this is a theorem from class or somewhere else.)

Since $\dim V = \dim W$, we can take bases $\beta = \{\vec{v}_1, \ldots, \vec{v}_n\}$ of V and $\gamma = \{\vec{w}_1, \ldots, \vec{w}_n\}$ of W of $\underline{n}$ elements each.

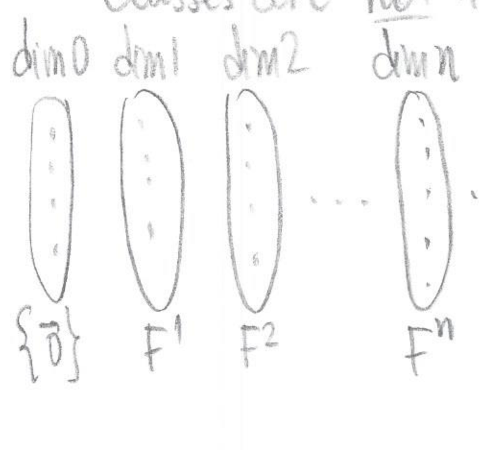
- Define $T(\vec{v}_i) = \vec{w}_i \quad \forall\, i = 1, 2, \ldots, n$
- Extend T linearly to V; i.e., define

$$T(c_1\vec{v}_1 + \ldots + c_n\vec{v}_n) = c_1\vec{w}_1 + \ldots + c_n\vec{w}_n \quad \forall\, c_i \in F.$$

- Since T is linear and sends a basis to a basis, then T is 1-1 and onto, hence a bijection, and finally, an isomorphism. Thus, $V \xrightarrow[\cong]{T} W$.

(b) Why do we care that the relation "is isomorphic to" ($\cong$) on the set of vector spaces over F is an *equivalence relation*? What is this useful for?

The $\cong$ relation partitions all vector spaces$/_F$ into non-overlapping set (called <u>classes</u>) of mutually isomorphic spaces within each class. Two vector spaces from different classes are <u>not</u> isomorphic. The vector spaces within each class have the same dimension. Standard representatives of these classes are $\{\vec{0}\}$, $F^1, F^2, \ldots, F^n, \ldots$ We can concentrate on proving theorems only for F^n, as they will carry over to all $V \cong F^n$.



Problem 3. **(25 pts)** Let $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be defined by $T(f(x)) = 2f'(x) + \int_0^x 3f(t)\,dt$.

(a) Prove that T is a linear transformation.

(*Note:* You may assume here well-known properties of derivatives and integrals.)

- $T(f(x)+g(x)) = 2(f(x)+g(x))' + \int_0^x 3(f(t)+g(t))\,dt =$

($\forall f, g \in P_2(\mathbb{R})$, $\forall c \in F$)

$= [2f'(x) + \int_0^x 3f(t)dt] + [2g'(x) + \int_0^x 3g(t)dt] = T(f(x)) + T(g(x))$

- $T(cf(x)) = 2(cf(x))' + \int_0^x 3cf(t)dt = c[2f'(x) + \int_0^x 3f(t)dt] = cT(f(x))$

(b) In the standard bases β of $P_2(\mathbb{R})$ and γ of $P_3(\mathbb{R})$, find the matrix $[T]_\beta^\gamma$. (Start β and γ with the *lowest* powers of x.) Show all calculations.

- $T(1) = 2\cdot 1' + \int_0^x 3\cdot 1\,dt = 0 + 3x = 3x$
- $T(x) = 2\cdot x' + \int_0^x 3\cdot t\,dt = 2 + \frac{3x^2}{2}$
- $T(x^2) = 2\cdot(x^2)' + \int_0^x 3\cdot t^2 dt = 4x + \frac{\cancel{3}x^3}{\cancel{3}}$
$= 4x + x^3$

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & 2 & 0 \\ 3 & 0 & 4 \\ 0 & 3/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{4\times 3}$$

(c) Explain why T is not onto, but it is 1-1.

- <u>T cannot be onto</u> since $\dim \mathrm{Im}\,T \leq \dim P_2(\mathbb{R}) = 3 < 4 = \dim P_3(\mathbb{R})$ (by Dim. thm)

so $\mathrm{Im}\,T \subsetneq P_3(\mathbb{R})$.

- <u>T is 1-1</u> b/c the columns of $[T]_\beta^\gamma$ are lin. independent (if $\mathrm{Ker}\,T \neq \{\vec{0}\}$ then these columns would have participated in a non-trivial lin. relation, but they do not.)

Problem 4. **(25pts)** Let $T : V \to V$ be a linear transformation on a vector space V over $\mathbb{C}$ such that $T^2 = I_V$. Let $W_1 = \{\vec{v} \in V \mid T(\vec{v}) = \vec{v}\}$ and $W_2 = \{\vec{v} \in V \mid T(\vec{v}) = -\vec{v}\}$.

(a) Prove that $V = W_1 \oplus W_2$. (*Note*: You may assume that W_1 and W_2 are subspaces of V. Writing $\vec{v}$ as a sum $\frac{1}{2}(\vec{v} + \text{something}) + \frac{1}{2}(\vec{v} - \text{something})$ may be useful.)

(1) $W_1 \cap W_2 = \{\vec{0}\}$. Indeed, if $\vec{v} \in W_1 \cap W_2$ then $T(\vec{v}) = \vec{v} = -\vec{v}$

$\Rightarrow 2\vec{v} = \vec{0} \underset{F=\mathbb{C}}{\Rightarrow} \vec{v} = \vec{0} \Rightarrow W_1 \cap W_2 = \{\vec{0}\}$

(2) $V = W_1 + W_2$. Indeed, let $\vec{v} \in V$. We write it as:

$$\vec{v} = \frac{1}{2}(\underbrace{\vec{v} + T(\vec{v})}_{\vec{w_1}}) + \frac{1}{2}(\underbrace{\vec{v} - T(\vec{v})}_{\vec{w_2}}) = \frac{1}{2}\vec{w_1} + \frac{1}{2}\vec{w_2}$$

- To show $\vec{w_1} \in W_1$: $T(\vec{w_1}) = T(\vec{v} + T(\vec{v})) = T(\vec{v}) + \overset{I_V}{T^2}(\vec{v})$
  $= T(\vec{v}) + I_V(\vec{v}) = T(\vec{v}) + \vec{v} = \vec{w_1} \in W_1$
- To show $\vec{w_2} \in W_2$: $T(\vec{w_2}) = T(\vec{v} - T(\vec{v})) = T(\vec{v}) - T^2(\vec{v})$
  $= T(\vec{v}) - \vec{v} = -(\vec{v} - T(\vec{v})) = -\vec{w_2} \in W_2$
- Since $\frac{1}{2}\vec{w_1} \in W_1$ & $\frac{1}{2}\vec{w_2} \in W_2$, we have written $\vec{v}$ as a sum of elt's of W_1 & W_2

(1) & (2) $\Rightarrow V = W_1 \oplus W_2$ ■

(b) If $\dim V = 2$, list all possible matrices $[T]_\alpha^\alpha$ in a basis α of V made from a basis of W_1 and a basis of W_2, including the cases when one of the subspaces W_1 and W_2 is $\{\vec{0}\}$.

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$ (if $W_2 = \{\vec{0}\}$, $W_1 = V$)

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ = reflection across a line (if $\dim W_1 = \dim W_2 = 1$)

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2$ (if $W_1 = \{\vec{0}\}$, $W_2 = V$)

$\Rightarrow [T]_\alpha^\alpha$ is diagonal w/ ± 1 along the diagonal

Problem 5 (20pts). True or False? You must completely bubble (fill in) one of the circles or leave blank. To discourage guessing, the problem will be graded as follows:

- 2 pt for each correct answer. • 0 pts for a blank. • -2 pt for each incorrect answer.
- If anything is written other than the bubbling, more than one circle is bubbled (or partially bubbled), or it is hard to tell which answer is bubbled, you will get -2 points.

Notes: A statement is TRUE if it is true in *all* cases that satisfy its hypothesis. A statement is FALSE if it is false even in just *one* case that satisfies its hypothesis, regardless of what happens in all other cases. If a statement claims that several things are true, but in reality even one of them is false, then the whole statement is FALSE.

(1) $\mathbb{C}$ is a vector space over $\mathbb{Q}$.

● Ⓕ $\mathbb{Q} \subseteq \mathbb{C}$ - subfield

(2) Unions of linearly *independent* sets are linearly *independent*.

Ⓣ ● If $\vec{v} \neq \vec{0}$ in $\mathbb{R}^2$, then $\{\vec{v}\}$ & $\{2\vec{v}\}$ are each L.I., but $\{\vec{v}, 2\vec{v}\}$ is Lin. Dep.

(3) If $T : V \to W$ is a function with $T(c\vec{v}_1 + \vec{v}_2) = cT(\vec{v}_1) + T(\vec{v}_2)$ for any $\vec{v}_1, \vec{v}_2 \in V$ and any scalar $c \in F$, then T must be *linear.*

● Ⓕ If $\vec{v}_2 = \vec{0}$, $T(c\vec{v}_1) = cT(\vec{v}_1)\ \forall \vec{v}_1 \in V$; if $c=1$, $T(\vec{v}_1+\vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$.

(4) The set of functions $f : \mathbb{R} \to \mathbb{R}$ that are twice-differentiable (i.e., have a second derivative) forms a vector subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

● Ⓕ If $f''(x)$ & $g''(x)$ $\exists$, then $(c_1 f(x) + c_2 g(x))'' = c_1 f''(x) + c_2 g''(x)$ also $\exists$.

(5) One axiom of vector spaces states that if $\vec{x} = \vec{0}$ and c is a scalar, then $c\vec{x} = \vec{0}$.

Ⓣ ● $c\vec{0} = \vec{0}\ \forall c \in F$ is a property of vector spaces (which we proved).

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(6) If $V = V_1 \oplus V_2$ for two subspaces V_1 and V_2 of a vector space V, then the union of any basis β of V_1 and any basis γ of V_2 is a basis for V. (Compare w/ Problem 4b.)

● Ⓕ Yes, $\beta \cup \gamma$ is a basis for V, including when $\beta = \emptyset$ or $\gamma = \emptyset$

(7) Although an isomorphism always carries linearly dependent vectors to linearly dependent vectors, a linear transformation that is *not onto* may fail to do that.

Ⓣ ● Any lin. transformation takes lin. dep. to lin. dep. vectors.

(8) There exists a linear transformation $T : S_{3\times3}(\mathbb{R}) \to A_{3\times3}(\mathbb{R})$ (from symmetric to skew-symmetric matrices) whose kernel has dimension 2.

Ⓣ ● $n(T) = \dim S_{3\times3} - r(T) = 6 - r(T) \geq 6 - 3 = 3 > 2$ b/c $r(T) \leq \dim A_{3\times3} = 3$.

(9) If $W_1 \cap W_2 = \{\vec{0}\}$ for two subspaces of a finite-dimensional vector space V, then $\dim(\mathrm{Span}(W_1 \cup W_2)) = \dim W_1 + \dim W_2$.

● Ⓕ $\mathrm{Span}(W_1 \cup W_2) = W_1 + W_2$ and $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \underbrace{\dim W_1 \cap W_2}_{0}$

(10) Matrix multiplication is defined in such a way that for any vector space V over F of dimension n, the standard isomorphisms $\xi : \mathcal{L}(V, V) \to M_{n\times n}(F)$ (corresponding to various fixed bases of V) respect not only the vector space operations but also turn composition of functions into multiplication of matrices.

● Ⓕ $V_\beta \xrightarrow{T} V_\gamma \xrightarrow{U} V_\alpha$ $[U \circ T]_\beta^\alpha = [U]_\gamma^\alpha \cdot [T]_\beta^\gamma$