HW #15; date: Oct. 24, 2017
MATH 110 Linear Algebra
with Professor Stankova

5.2.2 (e) The characteristic polynomial is $-\lambda(-\lambda(1-\lambda)+1)+1 = -\lambda^3+\lambda^2-\lambda+1 = -\lambda(\lambda^2+1)+(\lambda^2+1) = (1-\lambda)(\lambda^2+1)$. This polynomial does not split over $\mathbb{R}$ so it's not diagonalizable.

(g) The characteristic polynomial is $(3-\lambda)((4-\lambda)(1-\lambda)+2) - (2(1-\lambda)+2) + (-2+(4-\lambda)) = (3-\lambda)(4-\lambda)(1-\lambda) + (4-\lambda) = (4-\lambda)(\lambda^2-4\lambda+4) = (4-\lambda)(2-\lambda)^2$, so the roots are $\lambda = 2, 4$. Note that $A - 2I = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{pmatrix}$ so $\{(1,0,-1)^t, (0,1,-1)^t\}$ is a basis for the eigenspace E_2. In addition, $A - 4I = \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ -1 & -1 & -3 \end{pmatrix}$; row operations yield $\begin{pmatrix} -1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & -2 & -4 \end{pmatrix}$ so $\{(-1,-2,1)^t\}$ is a basis for E_4. Thus, the matrix is diagonalizable with $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{pmatrix}$ yielding $A = QDQ^{-1}$.

5.2.3 (c) With respect to the standard basis e, we have $[T]_e = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, which gives the characteristic polynomial $(2-\lambda)(1+\lambda^2)$. As this does not split over $\mathbb{R}$, we conclude that T is not diagonalizable.

(d) Using the standard basis the matrix is $[T]_e = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ which has characteristic polynomial $\lambda(1-\lambda)(\lambda-2)$; the eigenvalues are $\lambda = 0, 1, 2$ which are distinct so T is diagonalizable.

(f) This transformation is diagonalizable. Its eigenspaces are three 3-dimensional subspace of symmetric matrices (which has eigenvalue 1 under T since $T(A) = A$ if A is symmetric) and the one-dimensional subspace of skew-symmetric matrices (which has eigenvalue -1).

5.2.4 Let b_i be the eigenvector corresponding to eigenvalue λ_i, $i = 1, \ldots, n$. The set $\beta = \{b_1, \ldots, b_n\}$ then must be linearly independent in F^n by Theorem 5.5 so the matrix Q with b_i as the ith column must be invertible. Note then that the ith column of AQ is $\lambda_i b_i$ so that $AQ = QD$, where D is the diagonal matrix with λ_i's on the diagonal. It follows that $A = QDQ^{-1}$.

5.2.5 Claim: If A is diagonalizable then $\det(A - tI)$ splits as a polynomial over t. To see this, note that if A is diagonalizable, say $A = QDQ^{-1}$, then $\det(A - tI) = \det(QDQ^{-1} - tI) = \det(Q(D-tI)Q^{-1}) = \det(D - tI) = \prod_{i=1}^{n}(\lambda_i - t)$.

5.2.6 (a) A linear transformation T is diagonalizable if and only if it has a basis of eigenvectors. Let E_λ denote the λ-eigenspace; this is equivalent to asking that $\sum_\lambda \dim(E_\lambda) = \dim(V)$. By Theorem

5.7, $\dim(E_\lambda)$ is at most equal to the multiplicity of the root λ in the characteristic polynomial. Since eigenspaces for different eigenvalues have zero intersection, if any $\dim(E_\lambda)$ is strictly less than the algebraic multiplicity, it would be impossible for the dimension of the eigenspaces to be $\dim(V)$. Thus, we need equalities across the board, and we need for every root of the characteristic polynomial to be in the field F.

(b) Replace T by L_A.

Find a direct formula for the terms of the sequence: $a_{n+1} = 4a_n - 3a_{n-1}$, $a_0 = 0$, $a_1 = 1$.

Solution: We have

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} a_1 \\ a_0 \end{pmatrix}$$

Denote the matrix A. We will diagonalize A; it has characteristic polynomial $t^2-4t+3 = (t-3)(t-1)$. Its eigenvalues are $1, 3$, with eigenvectors $(1,1)^t$ and $(3,1)^t$ respectively. So, $A = QDQ^{-1}$ where $Q = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$. We can compute $Q^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 3 \\ 1 & -1 \end{pmatrix}$. Since $A^n = QD^nQ^{-1}$, we have

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_0 \end{pmatrix}$$

Plugging in the initial values and multiplying out, we find that

$$a_n = \frac{3^n - 1}{2}$$

5.2.18 (a) Let the basis be $b_1, \ldots, b_n$; by assumption, $T(b_i) = \lambda_i b_i$ and $U(b_i) = \mu_i b_i$. Then $TU(b_i) = T(\mu_i b_i) = \mu_i T(b_i) = \mu_i \lambda_i b_i$. Similarly, $UT(b_i) = \lambda_i \mu_i b_i$. The two are equal, and two linear transformations which are equal on a basis are equal.

(b) Suppose that $A = QDQ^{-1}$ and $B = QD'Q^{-1}$. It's easy to check that $DD' = D'D$. Then $AB = QDQ^{-1}QD'Q^{-1} = QDD'Q^{-1}$, and $BA = QD'Q^{-1}QDQ^{-1} = QD'DQ^{-1} = QDD'Q^{-1}$.

5.2.19 Let $b_1, \ldots, b_n$ be a basis of eigenvectors for T with eigenvalues λ_i. Then $T^n(b_i) = \lambda^n b_i$. Then this basis is also a basis of eigenvectors for T^n, so $[T^n]_\beta$ is diagonal ($[T]_\beta$ is diagonal by assumption).

Challenge from class: Find a direct formula for the nth term of the Fibonacci sequence $F_{n+1} = F_n + F_{n-1}$, $F_0 = 0$, $F_1 = 1$.

Solution: We have

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$$

Denote the matrix A. We will diagonalize A; it has characteristic polynomial $t^2 - t - 1$. Let us introduce some notation: let $\phi = \frac{1+\sqrt{5}}{2}$, so that $\phi^{-1} = \frac{1+\sqrt{5}}{2}\frac{1-\sqrt{5}}{1-\sqrt{5}} = \frac{1-\sqrt{5}}{2}$. Note that $\phi + \phi^{-1} = 1$. Then the eigenvalues are ϕ, ϕ^{-1} and the eigenvectors $(\phi, 1)^t, (\phi^{-1}, 1)^t$ respectively. So, $A = QDQ^{-1}$ where $Q = \begin{pmatrix} \phi & \phi^{-1} \\ 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} \phi & 0 \\ 0 & \phi^{-1} \end{pmatrix}$. We can compute $Q^{-1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -\phi^{-1} \\ -1 & \phi \end{pmatrix}$. Since $A^n = QD^nQ^{-1}$, we have

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \phi & \phi^{-1} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \phi^n & 0 \\ 0 & \phi^{-n} \end{pmatrix} \begin{pmatrix} 1 & -\phi^{-1} \\ -1 & \phi \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Plugging in the initial values and multiplying out, we find that

$$a_n = \frac{\phi^n - \phi^{-n}}{\sqrt{5}}$$