HW #24; Date 11/28/2017
MATH 110 Linear Algebra
with Professor Stankova

Section 6.4: Normal and Self-Adjoint Operators

1. (a) True. A normal operator is one which commutes with its adjoint. A self-adjoint operator is its own adjoint, and in particular, any operator commutes with itself.

(e) True. This is true from the lemma on page 373, part (a).

(f) True. Their matrices with respect to any orthonormal basis are the identity matrix and the zero matrix, which are equal to their own conjugate transpose.

(h) True. This follows from Theorem 6.17, since a self-adjoint linear operator has an (orthonormal) basis of eigenvectors.

2. (a) Let us do this one directly. We want to compare the inner products

$$\langle T(a,b),(x,y)\rangle = \langle (2a-2b,-2a+5b),(x,y)\rangle = 2ax-2bx-2ay+5by$$

and

$$\langle (a,b),T(x,y)\rangle = \langle (a,b),(2x-2y,-2x+5y)\rangle = 2ax-2bx-2ay+5by$$

The expressions are equal for arbitrary points (a,b) and (x,y), so T is self-adjoint, and thus normal. The eigenvalues of A are 1 and 6 with eigenvectors $(2,1)$ and $(1,-2)$. These are already orthogonal but not orthonormal; to make them orthonormal, we just rescale them by their norms to obtain the orthonormal eigenbasis $\frac{1}{\sqrt{5}}(2,1)$, $\frac{1}{\sqrt{5}}(1,-2)$.

(b) We can try another method. The matrix for this linear transformation with respect to the standard basis on $\mathbb{R}^3$ is given by

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 5 & 0 \\ 4 & -2 & 5 \end{pmatrix}$$

Over a real vector space, the conjugate transpose of a matrix is just the transpose. In particular, $A^* = A^t \neq A$, so because A is not symmetric, we see that A is not self-adjoint. To check if it is normal, we compute that the a_{11} entry of AA^t is 2 and of A^tA is 17, so $AA^t \neq A^tA$, and the transformation is not normal.

Since the inner product space is over $\mathbb{R}$, the linear transformation has an orthonormal basis if and only if it is self-adjoint by Theorem 6.17, so we conclude that there is no orthonormal eigenbasis for T.

(c) The matrix for this operator is

$$A = \begin{pmatrix} 2 & i \\ 1 & 2 \end{pmatrix}$$

This matrix is not equal to its conjugate transpose, so the operator is not self-adjoint. However, we can compute

$$AA^* = A^*A = \begin{pmatrix} 5 & 2+2i \\ 2-2i & 5 \end{pmatrix}$$

Thus we see that the operator is normal. Since the inner product space is complex, the fact that T is normal implies that it has an orthonormal eigenbasis. Its eigenvalues are $2+\sqrt{i}$ and $2-\sqrt{i}$, where $\sqrt{i}$ denotes the complex number $\sqrt{2}/2+\sqrt{2}/2i$. The corresponding eigenvectors are $(\sqrt{i}, 1)$ and $(\sqrt{i}, -1)$, which are orthogonal, but not orthonormal. Both vectors have norm $\sqrt{2}$, so the corresponding orthonormal vectors are obtained by dividing by this norm, giving $(\sqrt{i}/2, 1/\sqrt{2})$ and $(\sqrt{i}/2, -1/\sqrt{2})$.

(d) We can show that T is not a normal operator (and thus that it is not self-adjoint), using a proof by contradiction. Suppose that T is normal. In this case, by Theorem 6.15 part (a), we have for any polynomial p that $\|T(p)\| = \|T^*(p)\|$. For the particular polynomial $p = 1$, we note that since $T(1) = 0$, this implies that $\|T^*(1)\| = \|T(1)\| = \|0\| = 0$, and so we must also have $T^*(1) = 0$.

However, if we try to compute $\langle T(x), 1\rangle$, then we find that

$$\langle T(x), 1\rangle = \int_0^1 T(x) \cdot 1\, dx = \int_0^1 1\, dx = 1$$

but on the other hand

$$\langle T(x), 1\rangle = \langle x, T^*(1)\rangle = \int_0^1 x \cdot T^*(1)\, dx = \int_0^1 0\, dx = 0$$

This is a contradiction, so our assumption that T was normal must have been false. In particular, a linear operator in a complex vector space has an orthonormal eigenbasis if and only if it is normal by Theorem 6.16, so no such eigenbasis exists for T.

(e) The standard basis $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ for $M_{2\times 2}(\mathbb{R})$ is orthonormal, and the transpose operator T represented in this basis has matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is a symmetric, real matrix, so T is self-adjoint and thus normal. An orthogonal eigenbasis for this matrix is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

To obtain an orthonormal eigenbasis, we normalize them:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(f) Again in the orthonormal standard basis, we can represent T with the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

This is a symmetric, real matrix, so the operator is self adjoint and thus normal. This operator has eigenvectors

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

These are orthogonal, so we just have to normalize:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

4. Suppose that TU is self-adjoint. This means that $TU = (TU)^* = U^*T^* = UT$ (for the last step, use the fact that U and T are self adjoint). Conversely, if $UT = TU$, the $(TU)^* = U^*T^* = UT = TU$, so TU is self-adjoint.

6. (a) We compute

$$T_1^* = \frac{1}{2}(T + T^*)^* = \frac{1}{2}(T^* + T) = T_1$$

so T_1 is self-adjoint. Similarly

$$T_2^* = \frac{1}{\overline{(2i)}}(T - T^*) = -\frac{1}{2i}(T^* - T) = \frac{1}{2i}(T - T^*) = T_2$$

Finally, we check that

$$T_1 + iT_2 = \frac{1}{2}(T + T^*) + i\frac{1}{2i}(T - T^*) = \frac{T + T}{2} + \frac{T^* - T^*}{2} = T$$

(b) Suppose that $T = U_1 + iU_2$. Then

$$T + T^* = (U_1 + iU_2) + (U_1^* - iU_2^*) = (U_1 + iU_2) + (U_1 - iU_2) = 2U_1$$

This shows that $U_1 = T_1$. Similarly,

$$T - T^* = (U_1 + iU_2) - (U_1 - iU_2) = 2iU_2$$

so $U_2 = T_2$.

(c) If T is normal, then $T^*T = TT^*$, or

$$(T_1^* - iT_2^*)(T_1 + iT_2) = (T_1 + iT_2)(T_1^* - iT_2)$$

Using the fact that T_1 and T_2 are self adjoint, this is equivalent to

$$T_1^2 - iT_2T_1 + iT_1T_2 + T_2^2 = T_1^2 + iT_2T_1 - iT_1T_2 + T_2^2$$

Subtracting common terms and moving to one side, we have

$$2i(T_2T_1 - T_1T_2) = 0$$

This shows that $T_1T_2 = T_2T_1$. On the other hand, all of the steps in this sequence of equations are equivalences, so if we start with $T_1T_2 = T_2T_1$, we can reverse all of the above computations to show that $T^*T = TT^*$.

10. Starting from the left hand side,

$$\begin{aligned}
&\langle T(x) \pm ix, T(x) \pm ix\rangle \\
&= \langle T(x), T(x)\rangle \pm \langle T(x), ix\rangle \pm \langle ix, T(x)\rangle + \langle ix, ix\rangle \\
&= \|Tx\|^2 \mp i\,\langle T(x), x\rangle \pm i\,\langle T(x), x\rangle + \|x\|^2 \\
&= \|Tx\|^2 + \|x\|^2
\end{aligned}$$

In particular, $(T - iI)x = 0$ if and only if $\|(T - iI)x\|^2 = \|T(x)\|^2 + \|x\|^2 = 0$, and this can only happen if $\|x\|^2 = 0$, i.e. $x = 0$. Thus $T - iI$ is invertible. Furthermore, let C be its inverse; then $C^*(T + iI) = ((T^* - iI)C)^* = ((T - iI)C)^* = I^* = I$, so C^* is the inverse to $T + iI$.

11. (a) We have

$$\langle T(x), x\rangle = \langle x, T^*(x)\rangle = \langle x, T(x)\rangle = \overline{\langle T(x), x\rangle}$$

so the expression is equal to its own complex conjugate, and thus is real.

(b) We have

$$\begin{aligned}
&\langle T(x + y), x + y\rangle \\
&= \langle T(x) + T(y), x + y\rangle \\
&= \langle T(x), x\rangle + \langle T(x), y\rangle + \langle T(y), x)\rangle + \langle T(y), y\rangle \\
&= \langle T(x), y\rangle + \langle T(y), x\rangle
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\langle T(x + iy), x + iy\rangle \\
&= \langle T(x) + iT(y), x + iy\rangle \\
&= \langle T(x), x\rangle + i\,\langle T(y), x\rangle - i\,\langle T(x), y\rangle + \langle T(y), y\rangle \\
&= -i\,\langle T(x), y\rangle + i\,\langle T(y), x\rangle
\end{aligned}$$

Thus we have that $\langle T(x), y\rangle + \langle T(y), x\rangle = 0$ and $\langle T(x), y\rangle - \langle T(y), x\rangle = 0$ so $\langle T(x), y\rangle = \langle T(y), x\rangle = 0$ for all x, y. Taking $y = T(x)$, we find that $\langle T(x), T(x)\rangle = 0$, so $T(x) = 0$. However, since x is an arbitrary vector, this implies that T is the zero operator.

(c) If $\langle T(x), x\rangle$ is real for all $x \in V$, then

$$\langle T(x), x\rangle = \overline{\langle T(x), x\rangle} = \overline{\langle x, T^*(x)\rangle} = \langle T^*(x), x\rangle$$

In particular, this implies that $0 = \langle T(x), x\rangle - \langle T^*(x), x\rangle = \langle (T - T^*)(x), x\rangle$ for all $x \in V$. By part (b), this means that $T - T^*$ is the zero operator, or $T = T^*$.

In-class Challenge. The given matrix is symmetric, and has real entries, so $A^* = A^t = A$, and it is self-adjoint. An operator in a finite-dimensional real vector space is diagonalizable with an orthonormal eigenbasis if and only if it is self-adjoint, by Theorem 6.17.