**HW 18; 3/22/2017**
**MATH 110 Linear Algebra**
**with Professor Stankova**

7.1 #1fh (f) False. We need to take a basis of $K_\lambda$ which can be partitioned into cycles. (h) True. A previous homework problem showed that for any operator $T$, when $\dim(\ker(T^k)) = \dim(\ker(T^{k+1}))$, we have $\dim(\ker(T^k)) = \dim(\ker(T^\ell))$ for all $\ell \geq k$. Thus after $n$ "steps" it must stabilize since there $V$ is $n$-dimensional.

7.1 #2 (a) The characteristic polynomial is $\chi(t) = t^2 - 4t + 4$, so $t = 2$ is the only eigenvalue. $A - 2I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$ has rank 1, so in particular we have an eigenvector $(1,1)^t$. We want to comptute the chain terminating here; we solve $(A - 2I)x = (1,1)^t$ to find $x = (0,1)^t$, so we have a basis $\beta_2 = \{(1,1)^t, (0,1)^t\}$ which is a cycle generated by $(0,1)^t$. The Jordan normal form (in this basis) is $[T]_\beta = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

(b) The charactersitic polynomial is $\chi(t) - t^2 - 3t - 4$. Thus the eigenvalues are $t = 4, -1$. $A - 4I$ has kernel $(2,3)^t$ and $A + I$ has kernel $(1,-1)^t$. Thus we have a basis of eigenvectors and in this basis $\beta$, we have $[T]_\beta = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$.

(c) The characteristic polynomial is $\chi(t) = -t^3 + 3t^2 - 4$. Testing its rational roots, we find that it factors $-(t-2)^2(t+1)$, so its eigenvalues are $t = -1, 2$. For the eigenvalue 2, we find that $(A - 2I) = \begin{pmatrix} 9 & -4 & -5 \\ 21 & -10 & -11 \\ 3 & -1 & -2 \end{pmatrix}$, which we can row-reduce to $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ and so has rank 2, with corresponding eigenvector $(1,1,1)^t$. We compute the chain terminating here: $(A - 2I)x = (1,1,1)^t$ gives $x = (1,2,0)^t$. Thus, $\beta_2 = \{(1,1,1)^t, (1,2,0)^t\}$ is $A$-cyclic generated by $(1,2,0)^t$. Next, for the eigenvalue $-1$, we find $A + I = \begin{pmatrix} 12 & -4 & -5 \\ 21 & -7 & -11 \\ 3 & -1 & 1 \end{pmatrix}$, which row reduces to $\begin{pmatrix} 3 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ so is rank 2. We have $(A + I)^2 = 3\begin{pmatrix} 15 & -5 & -7 \\ 24 & -8 & -13 \\ 6 & -2 & -1 \end{pmatrix}$, which row reduces to $\begin{pmatrix} 3 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and has rank 2. An eigenvector for this eigenvalue is $(1,3,0)^t$. Thus, we have a basis of generalized eigenvectors $\beta = \{(1,1,1)^t, (1,2,0)^t, (1,3,0)^t\}$; so we have $[T]_\beta = 2\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

(d) The charactersitic polynomial of the matrix is $(t-2)^2(t-3)^2$, so the eigenvalues are

$t = 2, 3$. For $t = 2$, we have $A - 2I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}$. This matrix has rank 3, so we have eigenvector $(1, 0, 0, 0)^t$. To find the corresponding chain, we look for $x$ such that $(A - 2I)x = (1, 0, 0, 0)^t$, e.g. $(0, 1, 0, -1)^t$ will do. For $t = 3$, we have $A - 3I = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}$, which has rank 2, so it has a two-dimensional eigenspace spanned by eigenvectors $(1, 1, 1, 0)^t$ and $(0, 0, 0, 1)^t$. Define the basis $\beta = \{(1, 0, 0, 0)^t, (0, 1, 0, -1)^t, (1, 1, 1, 0)^t, 0, 0, 0, 1)^t\}$; we have $[T]_\beta = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

7.1 #3 (a) Let us choose the standard basis $\gamma$. Then since $T(1) = 2$, $T(x) = 2x - 1$, $T(x^2) = 2x^2 - 2x$, we have $A = [T]_\gamma = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{pmatrix}$. This matrix is almost in Jordan canonical form as written. It is upper triangular, so the only eigenvalue is 2. $A - 2I = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ has rank 2, so we have an eigenvector $(1, 0, 0)^t$. To compute the rest of the chain, we want $x$ such that $(A - 2I)x = (1, 0, 0)^t$, e.g. $x = (0, -1, 0)^t$. Finally, we want a $y$ such that $(A - 2I)y = x = (0, -1, 0)^t$, e.g. $y = (0, 0, 1/2)$. Thus, we have basis $\beta = \{1, -x, \frac{1}{2}x^2\}$ whereby $[T]_\beta = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

7.1 #3 (a) Let us forgo a choice of basis and compute the chains directly. Note that $(T - 2)(f) = -f'$, and that $(T - 2)(t^2) = -2t$, $(T - 2)(-2t) = 2$, and $(T - 2)(2) = 0$. Thus, the basis $\beta = \{t^2, -2t, 2\}$ is a chain in $K_2$ for $T$ ($K_0$ for $T - 2$), and we have $[T]_\beta = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 2 \end{pmatrix}$.

(b) Let us forgo a choice of basis and compute the chains directly. Observe that $T(\frac{1}{2}t^2) = t$, $T(t) = 1$, and $T(1) = 0$. Thus, $\{1, t, \frac{1}{2}t^2\}$ is a basis for $K_0$ which is a chain. Next. note that $(T - I)(te^t) = e^t$, and $(T - I)(e^t) = 0$ so we have $\{e^t, te^t\}$ is a chain. So, in the basis $\beta = \{1, t, \frac{1}{2}t^2, e^t, te^t\}$ we have $[T]_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$