7.1 #1efh (e) False. For a counterexample, consider the identity transformation on $\mathbb{R}^2$. The only eigenvalue is $\lambda = 1$, and every vector is an eigenvector. $\{e_1\}$ and $\{e_2\}$ are an example of distinct cycles. (f) False. We need to take a basis of K_λ which can be partitioned into cycles. (h) True. A previous homework problem showed that for any operator T, when $\dim(\ker(T^k)) = \dim(\ker(T^{k+1}))$, we have $\dim(\ker(T^k)) = \dim(\ker(T^\ell))$ for all $\ell \geq k$. Thus after n "steps" it must stabilize since there V is n-dimensional.

7.1 #2 (a) The characteristic polynomial is $\chi(t) = t^2 - 4t + 4$, so $t = 2$ is the only eigenvalue. $A - 2I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$ has rank 1, so in particular we have an eigenvector $(1,1)^t$. We want to comptute the cycle terminating here; we solve $(A - 2I)x = (1,1)^t$ to find $x = (0,1)^t$, so we have a basis $\beta_2 = \{(1,1)^t, (0,1)^t\}$ which is a cycle generated by $(0,1)^t$. The Jordan normal form (in this basis) is $[T]_\beta = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

(b) The charactersitic polynomial is $\chi(t) = t^2 - 3t - 4$. Thus the eigenvalues are $t = 4, -1$. $A - 4I$ has kernel generated by $(2,3)^t$ and $A + I$ has kernel generated by $(1,-1)^t$. Thus we have a basis of eigenvectors and in this basis β, we have $[T]_\beta = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$. (Note: In the case that a matrix is diagonalizable, finding its Jordan form is the same thing as diagonalizing it. The purpose of introducing Jordan form is to generalize to the case of non-diagonalizable matrices.)

(c) The characteristic polynomial is $\chi(t) = -t^3 + 3t^2 - 4$. Testing its rational roots, we find that it factors $-(t-2)^2(t+1)$, so its eigenvalues are $t = -1, 2$. For the eigenvalue 2, we find that $(A - 2I) = \begin{pmatrix} 9 & -4 & -5 \\ 21 & -10 & -11 \\ 3 & -1 & -2 \end{pmatrix}$, which we can row-reduce to $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ and so has rank 2, with corresponding eigenvector $(1,1,1)^t$. We compute the cycle terminating here: $(A - 2I)x = (1,1,1)^t$ gives $x = (1,2,0)^t$. Thus, $\beta_2 = \{(1,1,1)^t, (1,2,0)^t\}$ is a cycle of generalized eigenvectors of A corresponding to $\lambda = 2$, generated by $(1,2,0)^t$. Next, for the eigenvalue -1, we find $A + I = \begin{pmatrix} 12 & -4 & -5 \\ 21 & -7 & -11 \\ 3 & -1 & 1 \end{pmatrix}$, which row reduces to $\begin{pmatrix} 3 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ so is rank 2. We have $(A + I)^2 = 3\begin{pmatrix} 15 & -5 & -7 \\ 24 & -8 & -13 \\ 6 & -2 & -1 \end{pmatrix}$, which row reduces to $\begin{pmatrix} 3 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and has rank 2. An eigenvector for this eigenvalue is $(1,3,0)^t$. Thus, we have a basis of generalized eigenvectors $\beta = \{(1,1,1)^t, (1,2,0)^t, (1,3,0)^t\}$; so we have $[T]_\beta = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

(d) The characteristic polynomial of the matrix is $(t-2)^2(t-3)^2$, so the eigenvalues are

$t = 2, 3$. For $t = 2$, we have $A - 2I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}$. This matrix has rank 3, so we have eigenvector $(1, 0, 0, 0)^t$. To find the corresponding cycle, we look for x such that $(A - 2I)x = (1, 0, 0, 0)^t$, e.g. $(0, 1, 0, -1)^t$ will do. For $t = 3$, we have $A - 3I = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}$, which has rank 2, so it has a two-dimensional eigenspace spanned by eigenvectors $(1, 1, 1, 0)^t$ and $(0, 0, 0, 1)^t$. Define the basis $\beta = \{(1, 0, 0, 0)^t, (0, 1, 0, -1)^t, (1, 1, 1, 0)^t, 0, 0, 0, 1)^t\}$; we have $[T]_\beta = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

7.1 #3 (a) Let us choose the standard basis γ. Then since $T(1) = 2$, $T(x) = 2x - 1$, $T(x^2) = 2x^2 - 2x$, we have $A = [T]_\gamma = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{pmatrix}$. This matrix is almost in Jordan canonical form as written. It is upper triangular, so the only eigenvalue is 2. $A - 2I = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ has rank 2, so we have an eigenvector $(1, 0, 0)^t$. To compute the rest of the cycle, we want x such that $(A - 2I)x = (1, 0, 0)^t$, e.g. $x = (0, -1, 0)^t$. Finally, we want a y such that $(A - 2I)y = x = (0, -1, 0)^t$, e.g. $y = (0, 0, 1/2)$. Thus, we have basis $\beta = \{1, -x, \frac{1}{2}x^2\}$ whereby $[T]_\beta = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

7.1 #3 (a) Let us forgo a choice of basis and compute the cycles directly. Note that $(T - 2I)(f) = -f'$, and that $(T - 2I)(t^2) = -2t$, $(T - 2I)(-2t) = 2$, and $(T - 2I)(2) = 0$. Thus, the basis $\beta = \{t^2, -2t, 2\}$ is a cycle in K_2 for T (K_0 for $T - 2I$), and we have $[T]_\beta = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

(b) Let us forgo a choice of basis and compute the cycles directly. Alternatively, one could compute the matrix representation of T with respect to the given basis and find the eigenvalues through this matrix as in our first solution for part a). However, our knowledge of calculus suggests that looking at the generalized eigenspaces for $\lambda = 0, 1$ will be enough and this is justified below by the size of the cycles.

Observe that $T(\frac{1}{2}t^2) = t$, $T(t) = 1$, and $T(1) = 0$. Thus, $\{1, t, \frac{1}{2}t^2\}$ is a basis for K_0 which is a cycle. Next. note that $(T - I)(te^t) = e^t$, and $(T - I)(e^t) = 0$ so we have $\{e^t, te^t\}$ is a cycle.

So, in the basis $\beta = \{1, t, \frac{1}{2}t^2, e^t, te^t\}$ we have $[T]_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

(c) First, note that for a matrix A, and a matrix B with columns u, v, AB is the matrix with columns Au, Av. Thus, it makes sense to treat each column of B separately. Furthermore, note that $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has eigenvector 1 with a generalized eigenbasis equal to the standard basis. Thus, we have a basis

$$\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$$

and in this basis, $[T]_\beta = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(d) First, note that $(T - 2I)(A) = A^t$, which we know is a diagonalizable transformation with eigenvalues ± 1. Thus, T is also diagonalizable with eigenvalues $1, 3$ and the same eigenbasis. So take, for example,

$$\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$$

which has $[T]_\beta = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

7.2 #1abcdef (a) false/true; This point is controversial. The problem is that there is no **the** Jordan canonical form. We could reorder the eigenvalues and get another matrix in Jordan normal form, which makes the statement false. However, every diagonal matrix is already in Jordan normal form, which is why the book says this is true. (b) true; a Jordan basis is a Jordan basis regardless of how it is expressed. (c) false; for example the zero matrix and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ each have the characteristic polynomial λ^2 , yet are dissimilar. (The zero matrix is only similar to the zero matrix.) (d) true; if $A = PJP^{-1}$ and $B = QJQ^{-1}$, then $A = PQ^{-1}BQP^{-1} = (PQ^{-1})B(PQ^{-1})^{-1}$ (Alternatively, one may use the fact that similarity of matrices is transitive.) (e) true; the Jordan normal form is the representation of the matrix in a particular basis (f) false; see example for (c)

7.2 #2

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{pmatrix}$$

7.2 #3 (a) $\chi(t) = -(t-2)^5(t-3)^2$

(b) For $\lambda = 2$,

$$\begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \end{array}$$

For $\lambda = 3$,

$$\begin{array}{cc} \bullet & \bullet \end{array}$$

(c) $\lambda = 3$

(d) for $\lambda = 2$, min is 3; for $\lambda = 3$, min is 1

(e) (i) 3, 0 (ii) 1, 0 (iii) 2, 2 (iv) 4, 2. (Values are for $\lambda = 2$ first, then $\lambda = 3$ next.)

7.2 #4abc (a) By example 5, the characteristic polynomial is $-(t-1)(t-2)^2$. Then for $\lambda = 1$, $A - I = \begin{pmatrix} -4 & 3 & -2 \\ -7 & 5 & -3 \\ 1 & -1 & 1 \end{pmatrix}$, and we have corresponding eigenvector $(1,2,1)^t$. For $\lambda = 2$, we have $A - 2I = \begin{pmatrix} -5 & 3 & -2 \\ -7 & 4 & -3 \\ 1 & -1 & 0 \end{pmatrix}$, which has rank 2; we have eigenvalue $(1,1,-1)^t$. Next, we solve the equation $\begin{pmatrix} -5 & 3 & -2 \\ -7 & 4 & -3 \\ 1 & -1 & 0 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, e.g. $(0,1,1)^t$. Thus, we have $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$.

(b) By example 5, the characteristic polynomial is $-(t-1)(t-2)^2$. Then for $\lambda = 1$, $A - I = \begin{pmatrix} -1 & 1 & -1 \\ -4 & 3 & -2 \\ -2 & 1 & 0 \end{pmatrix}$, and we have corresponding eigenvector $(1,2,1)^t$. For $\lambda = 2$, we have

$A-2I = \begin{pmatrix} -2 & 1 & -1 \\ -4 & 2 & -2 \\ -2 & 1 & -1 \end{pmatrix}$, which has rank 1; we have basis of eigenvalues $(1,0,-2)^t, (1,2,0)^t$.

Thus, we have $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & -2 & 0 \end{pmatrix}$.

(c) By example 5, the characteristic polynomial is $-(t-1)(t-2)^2$. Then for $\lambda = 1$, $A - I = \begin{pmatrix} -1 & -1 & -1 \\ -3 & -2 & -2 \\ 7 & 5 & 5 \end{pmatrix}$, and we have corresponding eigenvector $(0,1,-1)^t$. For $\lambda = 2$, we have $A-2I = \begin{pmatrix} -2 & -1 & -1 \\ -3 & -3 & -2 \\ 7 & 5 & 4 \end{pmatrix}$, which has rank 1; we have basis of eigenvalues $(1,1,-3)^t$.

Next, we solve the equation $\begin{pmatrix} -2 & -1 & -1 \\ -3 & -3 & -2 \\ 7 & 5 & 4 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$, e.g. $(-1,0,1)^t$. Thus, we have $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & -3 & 1 \end{pmatrix}$.

7.2 #4d d) After much computation, one may find that $\det(A-tI) = t^2(t-2)^2$, hence the eigenvalues are $\lambda = 0, 2$, each with multiplicity 2. To find generalized eigenspaces, we look first at

$$A - 2I = \begin{pmatrix} -2 & -3 & 1 & 2 \\ -2 & -1 & -1 & 2 \\ -2 & 1 & -3 & 2 \\ -2 & -3 & 1 & 2 \end{pmatrix}$$

By inspection, $N(A-2I) = Span\{(1,0,0,1)^t, (-1,1,1,0)^t\}$. (The first should be obvious, the second maybe not so much. If you're not sure, just row reduce to find the null space as usual. But be aware there are many different bases for the same null space.) We found two linearly independent eigenvectors of A with eigenvalue 2, so we are done here and $\lambda = 2$ may be treated much as with diagonalization.

The matrix $A - 0I$ is just A, and we may see from inspection (or row reduction), that $(1,1,1,1)^t \in N(A)$. However, $rk(A) = 3$, so $nullity(A) = 1$, and we do not have enough eigenvectors to form a basis. However, solving the equation $Ax = (1,1,1,1)^t$, we find that (one particular solution of) x is $(0,-1,-2,0)$. Thus, $A = QJQ^{-1}$, where

$$Q = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

In-class challenge: How does the existence of Jordan normal form (over $\mathbb{C}$) imply Cayley-Hamilton Theorem for a matrix over $\mathbb{R}$? **Solution:** Let A be a matrix with real entries; we can consider it over the larger field $\mathbb{C}$ (the algebraic closure of $\mathbb{R}$). In particular, we can write $A = PJP^{-1}$ where P, J and P^{-1} may have imaginary (non-real) entries. Let f be a polynomial; we claim that $f(A) = 0$ if and only if $f(J) = 0$. This is because $(PJP^{-1})^k = PJ^kP^{-1}$ for any k, and so $f(PJP^{-1}) = Pf(J)P^{-1}$ for any polynomial f, so $f(A) = Pf(J)P^{-1}$, proving the claim. Furthermore, the characteristic polynomial of J is the same as the characteristic polynomial of A, so it suffices to show that the complex matrix J satisfies the Cayley-Hamilton theorem. Let χ be the characteristic polynomial. Equivalently, we know that V is the direct sum of the K_λ, which are T-invariant, so it suffices to show that $\chi(T|_{K_\lambda}) = 0$. Suppose we have the factor $(t - \lambda)^{r_\lambda}$ in $\chi(t)$, where r_λ is as large as possible. By Theorem 7.9, $(T - \lambda)^{r_\lambda}$ is zero on K_λ. Thus, $\chi(T)$ is zero on K_λ and the result follows.