Final Exam. Linear Algebra 110

Instructor: Zvezdelina Stankova

Student name: ______________________________

GSI's name: ______________________________

• **DO NOT OPEN THE EXAM UNTIL TOLD TO DO SO!**

• PREPARE YOUR STUDENT PHOTO ID TO SHOW TO YOUR PROCTOR.

• Please, do *all* problems as best as you can. The exam is 2 hours and 50 minutes long. You may **not** leave during the first 30 minutes and you may **not** during the last 30 minutes of the exam: no restroom breaks and no early submission of your exam during those times.

• Use the provided sheets to write your solutions. You may also use the back of each page for scratch work or solutions. If you need more paper, raise your hand to ask for additional paper. Submit *all* work which you wish to be graded and staple it to this packet! **Loose sheets of paper will NOT be graded!** Ask the instructor for a stapler at the end of the exam.

• The exam is closed notes and books, which means: **no class notes, no session notes, no review notes, no textbooks, and no other materials can be used during the exam.** You can use only your **cheat sheet**, this exam packet and additional blank paper provided by the instructor. You cannot use your paper to write on, even if it is only scratch work. The cheat sheet is <u>two sides</u> of <u>one</u> regular 8×11 sheet, handwritten only by you. A cheat sheet which doesn't conform to these specifications (for example, there is stuff written on both sides of the sheet) will be disqualified, and the student's exam may be annulled.

• **NO CALCULATORS ARE ALLOWED DURING THE EXAM!**

• **You cannot ask for help in any form from other students, you cannot look at their exams and copy, you cannot cheat in any way – the exam is an individual assignment and must be done only by you.** If you have a question regarding the statement of a problem, raise your hand when the instructor or a GSI is around.

• Think of this exam as an important homework. Check your reasoning and calculations very carefully. Justify *all* your answers, include all intermediate steps and calculations. If you are not sure about how to write something in mathematical notation, explain clearly *in words* what you mean and what you are doing. **Unjustified answers, even if correct, will receive no credit!** On the contrary, good justifications and good work on a problem may receive a lot of credit even if the final answer is incorrect.

• Before turning in your exam, please, sign the statement below. **exams which are not signed will not be graded.**

I, the student whose name and signature appear on this exam, have completed the exam without any outside help from people or other sources. I have used only my cheat sheet conforming to the specifications written above. I have not cheated in any way, and I have followed all instructions above. I have stapled all solutions and work wished to be graded to the present exam packet.

Student signature: ______________________________

Problem	Score
#1	
#2	
#3	
#4	
#5	
#6	
#7	
Total	

Problem 1 (10 pts). True or False? To discourage guessing, the problem will be graded as follows:

- 1 pt for each correct answer.
- 0 pts for a blank.
- -1 pts for each incorrect answer.
- If anything else but "True" or "False" is written, more than one answer is written, or the answer is hard to read, you will get -1 points.

Note: In all questions below V and W are vector spaces over a field F. They may or may not be finite-dimensional. Do not assume anything that is not given. If a statement claims that several things are true, but, in reality, even one is false, then the statement is false.

(1) Subsets of linearly *dependent* sets are linear *dependent.*

Answer: __________

(2) If $T : V \to W$ is a function with $T(\vec{v}_1 + b\vec{v}_2) = T(\vec{v}_1) + bT(\vec{v}_2)$ for any $\vec{v}_1, \vec{v}_2 \in V$ and any scalar $b \in F$, then T is *linear.*

Answer: __________

(3) Just like diagonal matrices, *Jordan canonical matrices* are easy to work with because they *commute* with each other.

Answer: __________

(4) The *Triangle Inequality* in V makes sense only if we already have a *norm* on V.

Answer: __________

(5) There is a linear operator T on V that preserves the *norm* but *not* the *inner product* on V.

Answer: __________

(6) *Hermitian* matrices are necessarily *normal* matrices but not the other way around.

Answer: __________

(7) A square matrix over $\mathbb{R}$ is *orthogonal* if and only if it is *invertible.*

Answer: __________

(8) If two *continuous real-valued functions* $f(x)$ and $g(x)$ on $[3,5]$ satisfy $\left(\int_3^5 f(x)g(x)\,dx\right)^2 = \left(\int_3^5 f^2(x)\,dx\right) \cdot \left(\int_3^5 g^2(x)\,dx\right)$, then one of them must be a *scalar multiple* of the other.

Answer: __________

(9) The *smallest angle* between a given vector $\vec{v}$ and any vector $\vec{w}$ in a subspace W of a finite-dimensional inner product space V is made between $\vec{v}$ and its *orthogonal projection* $\mathrm{proj}_W \vec{v}$.

Answer: __________

(10) Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ matrices. If $\sum_{i,j=1}^{n} |a_{ij}|^2 = \sum_{i,j=1}^{n} |b_{ij}|^2$ then A and B are *unitarily equivalent.*

Answer: __________

Problem 2 (15 pts) Let A be a square matrix over $\mathbb{C}$ with two eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -5$. It is known that the two dot diagrams corresponding to λ_1 and λ_2, respectively, are:

$$\begin{array}{lllll} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & & & \end{array} \qquad\qquad \begin{array}{ll} \bullet & \bullet \\ \bullet & \bullet \end{array}$$

(a) Find the following numbers, with brief one-line explanations of how you found them:

- the algebraic multiplicities $m_a(\lambda_i)$ of the two eigenvalues:
  $m_a(2) =$
  $m_a(-5) =$
- the geometric multiplicities $m_g(\lambda_i)$ of the two eigenvalues:
  $m_g(2) =$
  $m_g(-5) =$
- the stabilizing constants l_i for the two eigenvalues:
  $l_1 =$
  $l_2 =$
- the dimensions d_i of the generalized eigenspaces $K_{\lambda_i}(A)$:
  $\dim K_2(A) =$
  $\dim K_{-5}(A) =$
- the number n_i of Jordan blocks for each of the two eigenvalues:
  $n_1 =$
  $n_2 =$
- the dimensions of $\mathrm{Ker}(A - 2I)^3$ and $\mathrm{Ker}(A + 5I)^3$:
  $\dim \mathrm{Ker}(A - 2I)^3 =$
  $\dim \mathrm{Ker}(A + 5I)^3 =$

(b) What is the Jordan canonical form of A? (You may simplify via Jordan block notation $J_n(\lambda)$.)

Problem 3 (15 pts) Consider the following matrix:

$$A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -2 & -2 \\ -2 & 5 & 4 \end{pmatrix}.$$

It is known that $\text{char}_A(\lambda) = (1-\lambda)^3$.

(a) Find the Jordan canonical form for A. (No need to find Jordan canonical bases!)

(b) If two students, Vera and Wendy, both work on this problem and obtain different Jordan bases $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$, how many of Vera's vectors will be for sure scalar multiples of Wendy's vectors? Why?

Problem 4 (15 pts) In this problem we consider $\mathbb{C}$ as a vector space over the field $\mathbb{R}$. Let $\alpha = a+bi$ be a complex number, where $a, b \in \mathbb{R}$.

(a) Consider the function $f_\alpha : \mathbb{C} \to \mathbb{C}$ that multiplies every complex number z by α:

$$f_\alpha(z) = \alpha z \ \forall\, z \in \mathbb{C}.$$

Prove that f_α is a linear transformation of $\mathbb{C}$ over $\mathbb{R}$.

(b) Find the matrix of f_α in the standard basis $\{1, i\}$ of $\mathbb{C}$ over $\mathbb{R}$. What is the characteristic polynomial of f_α?

(c) For what α is f_α diagonalizable over $\mathbb{R}$? Why?

Problem 5 (15 pts)

(a) State Cauchy-Buniakovski-Schwartz Inequality on an inner product space V. Don't forget to say when equality is attained.

(b) Prove that

$$(1 \cdot 2016 + 2 \cdot 2017 + 3 \cdot 2018 + \cdots + 2015 \cdot 4030)^2 < (1^2 + 2^2 + \cdots + 2015^2) \cdot (2016^2 + 2017^2 + \cdots + 4030^2).$$

Explain why this is *not* an equality.

Problem 6 (15 pts) Let T be a self-adjoint (hermitian) operator on a finite-dimensional complex inner product space V.

(a) Prove that for all $\vec{x} \in V$ we have $||T(\vec{x}) - i\vec{x}||^2 = ||T(\vec{x})||^2 + ||\vec{x}||^2$.

(b) Prove that $T - iI$ is invertible. (*Hint:* Argue by contradiction. What vector could possibly be in the kernel of $T - iI$? Use part (a).)

Problem 7 (15 pts) Let A be an 5×5 real or complex matrix such that $A^{23} = O$ (the zero-matrix).

(a) What are the eigenvalues of A? Explain. (*Hint:* Apply A and A^{23} to an eigenvector $\vec{v}$ of A.)

(b) Prove that $A^5 = O$. (*Hint:* What is $\mathrm{char}_A(\lambda)$? Do we know some theorem related to $\mathrm{char}_A(\lambda)$?)

(c) Is A diagonalizable? Prove or give a counterexample.