HW #13; date: Oct. 17, 2017
MATH 110 Linear Algebra
with Professor Stankova

5.1 # 1 (a) False. Take $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. It only has one distinct eigenvalue, which is 0. (b) True. Every nonzero scalar multiple of an eigenvector is an eigenvector. (c) True. Take $\begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}$ over the field $\mathbb{Z}/3\mathbb{Z}$. The characteristic polynomial is $(2-t)t-2 = -(t^2+t+2) = 0$. One can check that $t = 0, 1, 2$ are not solutions to this equation (modulo 3). For a less exotic example, also the rotation by 90 degrees matrix in $\mathbb{R}^2$ has no eigenvalues $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. (d) False. They can certainly be zero. (e) False. See answer to part b. (f) False. Take $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. 1 and 2 are eigenvectors but 3 is not. (g) False. Take $T : P(\mathbb{R}) \to P(\mathbb{R})$ ($P(\mathbb{R})$ means real-valued polynomials) defined by $T(f) = 5f$. 5 is an eigenvalue. (h) True. If we put the basis of eigenvectors into the columns of a matrix Q, then QAQ^{-1} is diagonal. (i) True. Say$P^{-1}AP = B$. Then if λ is an eigenvalue of A with eigenvector v, then $P^{-1}v$ is an eigenvector of B with eigenvalue λ. (j) False. See previous sentence. (k) False. Take $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. e_1 and e_2 are eigenvectors but $e_1 + e_2$ is not.

5.1 # 2b,d,f (b) Since $T(3+4x) = -6-8x = -2(3+4x)$ and $T(2+3x) = -6-9x = -3(2+3x)$, we have $[T]_\beta = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}$ and since this is diagonal, the basis is a basis of eigenvectors. (d) $T(x-x^2) = 4+4x+4x^2 = 2(x-x^2) - 2(-1+x^2) + (-1-x+x^2)$, $T(-1+x^2) = 2-2x^2 = -2(-1+x^2)$, and $T(-1-x+x^2) = -3x-3x^2 = -(x-x^2)+2(-1+x^2)-2(-1-x+x^2)$. So, $[T]_\beta = \begin{pmatrix} 2 & 0 & -1 \\ -2 & -2 & 2 \\ 1 & 0 & -2 \end{pmatrix}$, not diagonal, so not a basis of eigenvectors. (f) Since $T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -3\begin{pmatrix} -3 & 0 \\ -3 & 0 \end{pmatrix}$, $T\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}$, $T\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$, $T(\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$, we have $[T]_\beta = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, so it is a basis of eigenvectors.

5.1 # 3b,c (b) $\det(tI-A) = (t-1)(t-2)(t-3)$, so the eigenvalues are $1, 2, 3$. The eigenvectors are bases for $N(A-I), N(A-2I), N(A-3I)$, so $(1,1,-1)^t$, $(1,-1,0)^t, (1,0,-1)^t$ respectively. Since there are three such vectors for different eigenvalues, they must form a basis. The matrix is $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$. (c) The characteristic polynomial $\det(tI-A) = (t-i)(t+i)-2 = t^2-1$, so the eigenbalues are $t = 1, -1$. The corresponding eigenvectors are $(-1, i-1)^t$ and $(-1, i+1)^t$ respectively. These form a basis and $Q = \begin{pmatrix} -1 & -1 \\ i-1 & i+1 \end{pmatrix}$.

5.1 # 4a,e,j (a) The eigenvalues are $3, 4$. The eigenvectors are $(3, 5)$ and $(1, 2)$ respectively. (e) Written in the standard basis, this gives a matrix $\begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix}$. The characteristic polynomial is $(2-\lambda)((1-\lambda)(3-\lambda)-3) = -\lambda(2-\lambda)(4-\lambda)$, so the eigenvectors are $\lambda = 0, 2, 4$. The corresponding eigenvalues are $(3, -1, 0)^t, (3, 13, -4)^t, (1, 1, 0)^t$, so the corresponding eigenbasis is $\beta = \{3-x, 3+13x-4x^2, 1+x\}$. (j) In the standard basis, $\begin{pmatrix} 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. The characteristic polynomial is $(3-t)(-t^3+t) = t(3-t)(1-t)(1+t)$, so the eigenvalues are $-1, 0, 1, 3$. The corresponding eigenvectors are $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 5 \\ 0 & -3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 3 & 0 \end{pmatrix}$.

5.1 #8 (a) Suppose that T is not invertible. Then there is a nonzero vector $x \in N(T)$. Then $T(x) = 0x$, so it is a eigenvector with eigenvalue zero. Conversely, suppose that $T(x) = 0x$ for nonzero x. Then $x \in N(T)$ is nonzero, so T is not invertible. (b) Suppose that $T(x) = \lambda x$, so that $\lambda^{-1}T(x) = x$. Then $T^{-1}(x) = T^{-1}(\lambda^{-1}T(x)) = \lambda^{-1}T^{-1}(T(x)) = \lambda^{-1}(x)$. By reciprocity of inverses, the converse follows. (c) Take $T = L_A$ and the same proofs go through.

5.1 #9 The determinant of an upper triangular matrix is the product of the entries on the diagonal. If A is upper triangular with diagonal entires $d_1, \dots, d_n$, then $A - tI$ is also upper triangular with diagonal entries $d_1 - t, \dots, d_1 - t$, so the determinant $\det(A - tI) = (d_1 - t)\cdots(d_n - t)$, so the eigenvalues are $d_1, \dots, d_n$.

5.1 #10 (a) Let $\beta = \{b_1, \dots, b_n\}$. Then $\lambda I_V(b_i) = \lambda b_i$, so $[\lambda I_V]_\beta e_i = [\lambda I_V]_\beta [b_i]_\beta = \lambda [b_i]_\beta = \lambda e_i$, proving the claim. (b) $\chi(t) = (\lambda - t)^n$, where $n = \dim(V)$ (c) It is already diagonal, and the diagonal entries are all λ.

5.1 #11 (a) Suppose $PAP^{-1} = \lambda I$. Then $A = P^{-1}\lambda IP = \lambda P^{-1}IP = \lambda I$. (b) Since a diagonalizable matrix is similar to a diagonal matrix with its eigenvalues on the diagonal, the result follows from part (a). (c) The characteristic polynomial is $(1-t)^2$, so the matrix has only one eigenvalue: 1. By part (b), if this matrix were diagonalizable it would be a scalar matrix, since it is not a scalar matrix it is not diagonalizabl.

5.1 #14 Since $\det(B) = \det(B^t)$ for any B, $\det(A - tI) = \det((A-tI)^t) = \det(A^t - tI^t) = \det(A^t - tI)$.

5.1 #15 (a) We proceed by induction. For $m = 1$ the result is immediate, since $T^1x = Tx = \lambda x = \lambda^1 x$. Assume that $T^{m-1}x = \lambda^{m-1}x$. Then $T^m x = T(T^{m-1}x) = T(\lambda^{m-1}x) = \lambda^{m-1}Tx = \lambda^{m-1}\lambda = \lambda^m$. (b) Take $T = L_A$.

5.1 #17 (a) Take $A = (a_{ij})$; if $T(A) = A^t = \lambda A$, then $a_{ji} = \lambda a_{ij}$. Swapping i, j we have $a_{ij} = \lambda a_{ji}$, so $a_{ji} = \lambda^2 a_{ji}$. This must be true for all a_{ij}, so $\lambda^2 = 1$, so $\lambda = \pm 1$. (b) The symmetric matrices have eigenvalue 1 (since $T(A) = A^t = A$ if A is symmetric), and the skew-symmetric matrices have eigenvalue -1 (since $T(A) = A^t = -A$). (c) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (d) The first n basis vectors are diagonal matrices with a 1 on the ith diagonal entry. The next $\frac{n(n-1)}{2}$ basis vectors are symmetric, with a 1 in the i, jth entry and a 1 in the j, ith entry and zeroes elsewhere. The next $\frac{n(n-1)}{2}$ basis vectors are skew symmetric, with a 1 in the i, jth entry and a -1 in the j, ith entry and zeroes elsewhere.

5.1 #18 (a) We want to find c such that $\det(A+cB) = \det(AB^{-1}+cI)\det(B) = 0$. $\det(AB^{-1}+cI) = \mathrm{char}_{AB^{-1}}(-c)$. Since the characteristic polynomial of AB^{-1} is a polynomial of degree n (Theorem 5.3), it has a complex root, i.e. a value c such that $\mathrm{char}_{AB^{-1}}(-c) = 0$.

(b) $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

5.1 #20 Since $\det(A - tI)|_{t=0} = \det(A)$, and $f(0) = a_0$, it follows that $\det(A) = a_0$. Then apply problem 8.

5.1 #21 (a) We use mathematical induction.

Base case, $n = 1$: The statement is obviously true for a 1×1 matrix.

Inductive Step: To induct, we expand by cofactors the first row: $(a_{11} - t)f_1(t) - a_{12}f_2(t) \pm a_{1n}f_n(t)$. By induction, $f_1(t) = (a_{22} - t)\cdots(a_{nn} - t) + q(t)$ where $\deg(q) = n - 3$. Thus, $(a_{11} - t)f_1(t) = (a_{11} - t)\cdots(a_{nn} - t) + (a_{11}q(t) - tq(t))$, and $\deg(a_{11}q(t) - tq(t)) \leq n - 2$. To complete the proof, it suffices to show that the $f_i(t)$ for $i \neq 1$ are degree $\leq n - 2$. However, this is clear; when we delete the first row and any column that is not the first column, we delete two entries which are linear in t. The resulting cofactor has only $n - 2$ entries involving t, which are linear in t. The monomials in the cofactor expansion are products of distinct entries of A, so in particular this cofactor must have degree $\leq n - 2$, and we are finished. (b) Using (a), since $\deg(q) \leq n - 2$, the t^{n-1}-coefficient of $\det(A - tI)$ is the same as the t^{n-1}-coefficient of $(a_{11} - t)\cdots(a_{nn} - t)$. Using Vieta's formula, we find that $(-1)^{n-1}a_{n-1} = a_{11} + \cdots + a_{nn} = \mathrm{tr}(A)$.

5.1 #22 (a) It suffices to show that if $T(x) = \lambda x$ and $S(x) = \mu x$, then $(T + S)(x) = (\lambda + \mu)x$, and also to show that if $T(x) = \lambda x$ then $T^m(x) = \lambda^m x$, since every polynomial in T is the sum of powers of T. The first statement simply follows from linearity, and the second we've done earlier. (b) Take $T = L_A$. (c) $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$, so $g(T) = 2A^2 - A + 1 = \begin{pmatrix} 7 & 4 \\ 6 & 9 \end{pmatrix}$, so $g(T)(2,3)^t = (26,39)^t$. On the other hand, $g(4) = 13$, and $13(2,3)^t = (26,39)^t$.