# Midterm I. Linear Algebra 110

Instructor: Zvezdelina Stankova

STUDENT NAME: ______________________________

GSI'S NAME: ______________________________

• **DO NOT OPEN THE MIDTERM UNTIL TOLD TO DO SO!**

• Please, do *all* Problems 1 through 5, as best as you can. The midterm about 70 minutes long. You may **not** leave early.

• Use the provided sheets to write your solutions. You may also use the back of each page for scratch work or solutions. If you need more paper, raise your hand to ask for additional paper. Submit *all* work which you wish to be graded and staple it to this packet! **Loose sheets of paper will NOT be graded!** Ask the instructor for a stapler at the end of the exam.

• The exam is closed notes and books, which means: **no classnotes, no session notes, no review notes, no textbooks, and no other materials can be used during the midterm.** You can use only your **cheat sheet**, this exam packet and additional blank paper provided by the instructor. You cannot use your paper to write on, even if it is only scratch work. The cheat sheet is one side of a regular $8 \times 11$ sheet, handwritten only by you. A cheat sheet which doesn't conform to these specifications (for example, there is stuff written on both sides of the sheet) will be disqualified, and the student's midterm may be annulled.

• **NO CALCULATORS ARE ALLOWED DURING THE MIDTERM!**

• **You cannot ask for help in any form from other students, you cannot look at their midterms and copy, you cannot cheat in any way – the midterm is an individual assignment and must be done only by you.** If you have a question regarding the statement of a problem, raise your hand when the instructor or a GSI is around.

• Think of this midterm as an important homework. Check your reasoning and calculations very carefully. Justify *all* your answers, include all intermediate steps and calculations. If you are not sure about how to write something in mathematical notation, explain clearly *in words* what you mean and what you are doing. **Unjustified answers, even if correct, will receive no credit!** On the contrary, good justifications and good work on a problem may receive a lot of credit even if the final answer is incorrect.

• Before turning in your midterm, please, sign the statement below. **Midterms which are not signed will not be graded.**

*I, the student whose name and signature appear on this midterm, have completed the midterm without any outside help from people or other sources. I have used only my cheat sheet conforming to the specifications written above. I have not cheated in any way, and I have followed all instructions above. I have stapled all solutions and work wished to be graded to the present exam packet.*

STUDENT SIGNATURE: ______________________________

Student name: ______________________________

GSI's name: ______________________________

| Problem | Score |
|---|---|
| #1 | |
| #2 | |
| #3 | |
| #4 | |
| #5 | |
| Total | /100 |

**Problem 1 (20 pts).** Consider the following two subspaces $W_1$ and $W_2$ of $F^5$:

$$W_1 = \{(a_1, a_2, a_3, a_4, a_5\} \in F^5 : a_1 - a_3 - a_4 = 0\}$$

and

$$W_2 = \{(a_1, a_2, a_3, a_4, a_5\} \in F^5 : a_2 = a_3 = a_4 \text{ and } a_1 + a_5 = 0\}.$$

(a) Find a basis for $W_1$. What is $\dim W_1$? Explain. Show all relevant calculations.

$\dim W_1 =$

(b) Find a basis for $W_2$. What is $\dim W_2$? Explain. Show all relevant calculations.

$\dim W_2 =$

*GOOD LUCK!*

**Problem 2 (20 pts).** Let $T : V \to W$ be a linear transformation between two vector spaces $V$ and $W$ over a field $F$.

(a) What is the *image* of $T$? Define it.

(b) What is a *subspace* $U$ of $W$? Define it.

(c) Prove that the image of $T$ is a subspace of $W$.

(d) If $\dim V < \dim W$, for what type of linear transformations $T$ is the image of $T$ of maximal possible dimension? Justify your answer.

**Problem 3 (20 pts).** Let $V$ be a vector space, and let $T : V \to V$ be a linear transformation. Prove that $T^2 = \mathcal{O}$ if and only if $\operatorname{Im} T \subseteq \operatorname{Ker} T$. Draw a picture to represent the composition $T^2$. (*Note:* The phrase "if and only if" usually signifies *two* directions in the proof. In this case you may get by with "one" proof' that shows both directions simultaneously.)

Picture:

**Problem 4 (20 pts).** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection across the line $y = \frac{4}{3}x$.

(a) Draw a good picture. Find a convenient basis $\beta$ of $\mathbb{R}^2$ in which matrix of $T$ is as simple as possible. What is this matrix $[T]_\beta$? Explain. Show all relevant calculations.

Picture:

Answer: $\beta =$ $[T]_\beta =$

(b) What is the matrix of $T$ in the standard basis of $\mathbb{R}^2$? Explain. Show all relevant calculations.

**Problem 5 (20 pts). True or False?**
To discourage guessing, the problem will be graded as follows:

- 2 pts for each correct answer.
- 0 pts for a blank.
- -2 pts for each incorrect answer.
- If anything else but "True" or "False" is written, more than one answer is written, or the answer is hard to read, you will get -2 point.

*Note:* In all questions below $V$ and $W$ are vector spaces over a field $F$. They may or may not be finite-dimensional. Thus, read the questions carefully and do not assume anything that is not given. A statement is TRUE if it is true in *all* cases that satisfy its hypothesis. A statement is FALSE if it is false even in just *one* case that satisfies its hypothesis, regardless of what happens in all other cases. If a statement claims that several things are true, but in reality even one of them is false, then the whole statement is FALSE.

(1) One *axiom* of vector spaces states that for any vector $\vec{x}$ we have $0 \cdot \vec{x} = \vec{0}$.

Answer: __________

(2) The *union* of any two subspaces of $V$ is *never* a subspace of $V$.

Answer: __________

(3) A one-to-one linear transformation $T : V \to W$ preserves both linear independence and linear dependence; i.e., $T$ carries any linearly *independent* set in $V$ to a linearly *independent* set in $W$, and any linearly *dependent* set in $V$ to a linearly *dependent* set in $W$.

Answer: __________

(4) If $T : V \to W$ and $U : W \to X$ are linear transformations of vector spaces, and the composition $UT$ is onto, then $T$ must be onto too.

Answer: __________

(5) Given any four vectors $\vec{v}_1, \vec{v}_2, \vec{w}_1, \vec{w}_2$ in $\mathbb{R}^2$, there is some linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that sends $\vec{v}_1$ to $\vec{w}_1$ and $\vec{v}_2$ to $\vec{w}_2$.

Answer: __________

(6) If $A$ and $B$ are *similar* matrices and $B$ is invertible, then $A$ must be invertible too.

Answer: __________

(7) The base case in the proof of the *Replacement Theorem* by induction consisted in showing that any non-zero vector in a finite-dimensional space $V$ *can* be extended to a basis of $V$.

Answer: __________

(8) For any linear transformation $T : V \to W$ between finite-dimensional spaces we can find bases of $V$ and $W$ with respect to which all entries of the matrix of $T$ are 0's except for, possibly, some 1's in positions $(i, i)$ ($i^{\text{th}}$ row and $i^{\text{th}}$ column).

Answer: __________

(9) If $V$ and $W$ have dimensions $m$ and $n$, respectively, then there is an isomorphism between $\mathcal{L}(V, W)$ and $M_{m \times n}(F)$.

Answer: __________

(10) A *canonical* isomorphism between a space $V$ and it double-dual $V^{**}$ means that its construction does not depend on a choice of basis for either space.

Answer: __________