① Implicit differentiation $\quad x^3+y^3=6xy \Rightarrow x^3+y(x)^3=6x\cdot y(x)$

logarithmic differentiation $\quad y=x^{\sqrt{x}} \Rightarrow \ln y=\sqrt{x}\cdot\ln x$

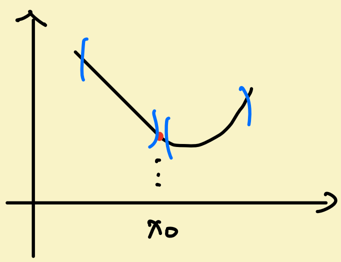
② Rolle's theorem : ① continuous on $[a,b]$ ② differentiable on (a,b) ③ $f(a)=f(b)$ $\Rightarrow$ $\exists\, c\in(a,b)$ $f'(c)=0$

$\downarrow$

Lagrangian MVT : ① continuous on $[a,b]$ ② differentiable on (a,b) $\Rightarrow$ $\exists\, c\in(a,b)$ $f'(c)=\frac{f(b)-f(a)}{b-a}$

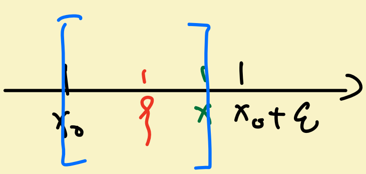
$\downarrow$

Cauchy's mean value theorem

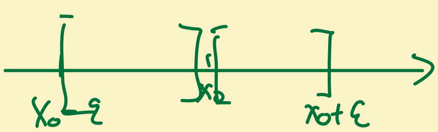


Corollary : Suppose f is ① continuous on $(x_0-\varepsilon,\ x_0+\varepsilon)$, and is ② differentiable on $(x_0-\varepsilon, x_0)\cup(x_0, x_0+\varepsilon)$, and ③ $\lim\limits_{x\to x_0} f'(x)$ exists then f is differentiable at x_0, and

$$f'(x_0)=\lim_{x\to x_0} f'(x)$$



$x_0 \Leftarrow x$



Proof : Take $x\in(x_0, x_0+\varepsilon)$. then the mean value theorem holds on $[x_0, x]$. There exists $\xi\in(x_0, x) \Rightarrow f'(\xi)=\frac{f(x)-f(x_0)}{x-x_0}$

When $x\to x_0^+$, we have $\xi\to x_0^+$. Then

$$\lim_{x\to x_0^+}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{\xi\to x_0^+} f'(\xi)=f'(x_0+0)$$

Similarly, we have $\lim\limits_{\xi\to x_0^-} f(\xi)=f'(x_0-0)$

Since $\lim\limits_{x\to x_0} f'(x)$ exists, we have $f'(x_0+0)=f'(x_0-0)=f'(x_0)$

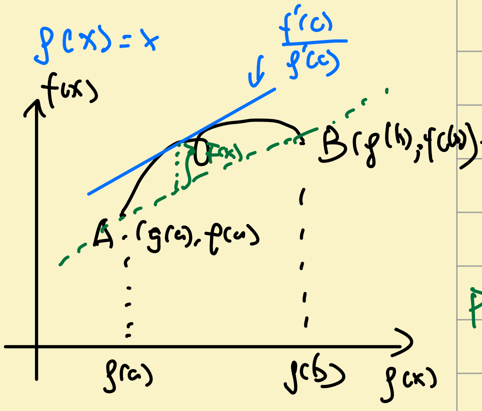
E.g.: $f(x)=\begin{cases} x+\sin x^2 & x\le 0\\ \ln(x+1) & x>0\end{cases}$ $\quad f'(x)\begin{cases} 1+2x\cdot\cos x^2 & x\le 0\\ \frac{1}{x+1} & x>0\end{cases}$

$f'(0^-)=\lim\limits_{x\to 0^-}(1+2\cdot 0\cdot\cos 0^2)=1$

$f'(0^+)=\lim\limits_{x\to 0^+}\left(\frac{1}{1+0}\right)=1$ $\quad\Rightarrow f'(0)=1$

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$g(x) = x$



It is more general than Lagrangian MVT.

Thm (Cauchy's mean value theorem) Suppose there are two functions f and g, satisfying

① f and g are continuous on $[a,b]$,

② f and g are differentiable on (a,b). $g'(x) \neq 0 \ \forall x \in (a,b)$

Then there exists $c \in (a,b)$, such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Proof: Define $F(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}(g(x)-g(a))$

then we have $F(x)$ is continuous on $[a,b]$, and

$F(x)$ is differentiable on (a,b)

Moreover, $F(a) = F(b) = 0$. By Rolle's theorem $\exists \xi \in (a,b)$

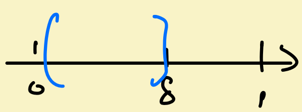
$$F'(\xi) = f'(\xi) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(\xi) = 0$$

$$\Rightarrow \frac{f'(\xi)}{g'(\xi)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

E.x. Suppose that f is differentiable on $(0,1]$, $\lim_{x\to 0^+} \sqrt{x} f'(x) = A$.
Prove that f is uniformly continuous on $(0,1]$.

$g(x) = \sqrt{x}$

$g'(x) = \frac{1}{2\sqrt{x}}$



Proof: Since $\lim_{x\to 0^+} \sqrt{x} \cdot f'(x) = A$, then $\exists\, 0<\delta<1$, if $0<x<\delta$, we have

$$|\sqrt{x} f'(x) - 0| < |A| + 1 =: M$$

For any $x, y \in (0, \delta]$ and $x<y$. By Cauchy's mean value theorem, there exists $c \in (x,y)$ such that

$$\left|\frac{f(x)-f(y)}{\sqrt{x}-\sqrt{y}}\right| = \left|\frac{f'(c)}{g'(c)}\right| = 2\sqrt{x}\cdot|f'(c)| < 2M$$

$$\Rightarrow |f(x)-f(y)| \leq 2M|\sqrt{x}-\sqrt{y}| \leq 2M\cdot\delta$$

Then $f(x)$ is uniformly continuous on $(0,\delta]$.
Since $f(x)$ is differentiable on $[\delta,1]$, then it is contionuous on $[\delta,1]$ which is a closed interval. Hence f is uniformly continuous on $[\delta,1]$

Rolle's theorem

$\downarrow$

Lagrangian MVT

$\downarrow$

Cauchy's mean value theorem

$\downarrow$

L'Hospital's rule

$\boxed{\frac{0}{0} \quad \frac{\infty}{\infty}}$

Corollary (L'Hospital's rule) Suppose that f and g are differentiable and $g'(x) \neq 0$ on $(x_0-\varepsilon, x_0) \cup (x_0, x_0+\varepsilon)$. Suppose that

① $\lim_{x\to x_0} f(x) = 0$, and $\lim_{x\to x_0} g(x) = 0$, or

② $\lim_{x\to x_0} f(x) = \infty$, and $\lim_{x\to x_0} g(x) = \infty$,

and $\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = A$ $(A \in \mathbb{R} \cup \{-\infty, +\infty\})$. Then we have

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)} = A$$

Proof: Since $f(x_0) = g(x_0) = 0$

$$\lim_{x\to x_0^+} \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \lim_{c\to x_0^+} \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \lim_{x\to x_0^+} \frac{f(x)}{g(x)} = \lim_{c\to x_0^+} \frac{f'(c)}{g'(c)}$$

$x \to x_0 \leftarrow x$

Similarly, we have $\lim_{x\to x_0^-} \frac{f(x)}{g(x)} = \lim_{x\to x_0^-} \frac{f'(c)}{g'(c)}$

Since $\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = A$ exists, we have $\lim_{x\to x_0} \frac{f(x)}{g(x)} = A$

E.x. ① $\lim_{x\to\pi} \frac{1+\cos x}{\tan^2 x} = \lim_{x\to\pi} \frac{-\sin x}{2\tan x \sec^2 x} = -\lim_{x\to\pi} \frac{\sin x}{2\frac{\sin x}{\cos x}\cdot\frac{1}{\cos^2 x}}$

$= -\lim_{x\to\pi} \frac{\cos^3 x}{2} = \frac{1}{2}$

$\ln(1+x) \sim x$

② $\lim_{x\to 0} \frac{e^x - (1+2x)^{\frac{1}{2}}}{\ln(1+x^2)} = \lim_{x\to 0} \frac{e^x - (1+2x)^{\frac{1}{2}}}{x^2} = \lim_{x\to 0} \frac{e^x - (1+2x)^{-\frac{1}{2}}}{2x}$

$= \lim_{x\to 0} \frac{e^x + (1+2x)^{-\frac{3}{2}}}{2} = 1$

③ $\lim_{x\to 0^+} \frac{\sqrt{x}}{1-e^{\sqrt{x}}}$ let $t = \sqrt{x}$ $x\to 0^+ \Rightarrow t\to 0^+$

$\Rightarrow \lim_{t\to 0^+} \frac{t}{1-e^t} = \lim_{t\to 0^+} \frac{1}{-e^t} = -1$

④ $\lim\limits_{x\to+\infty}\dfrac{x+\sin x}{x} = \lim\limits_{x\to\infty}\overset{\to 1}{\dfrac{x}{x}} + \lim\limits_{x\to\infty}\overset{\to 0}{\dfrac{\sin x}{x}} = 1$

But $\lim\limits_{x\to+\infty}\dfrac{1+\cos x}{1}$ does not exist! Therefore L'Hospital's rule fails.

⑤ $\lim\limits_{x\to0^+} x\cdot\ln x$ $\qquad 0\cdot\infty \Rightarrow \dfrac{\infty}{\infty}$

$= \lim\limits_{x\to0^+}\dfrac{\ln x}{\frac{1}{x}} = \lim\limits_{x\to0^+}\dfrac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim\limits_{x\to0^+}(-x) = 0$

⑥ $\lim\limits_{x\to0}(\cos x)^{\frac{1}{x^2}}$ $\qquad 1^\infty \Rightarrow e^{\frac{0}{0}}$ $\qquad e^{\ln x} = x$

$= \lim\limits_{x\to0}\exp\left(\dfrac{1}{x^2}\ln\cos x\right) = \lim\limits_{x\to0}\exp\left(\dfrac{\ln\cos x}{x^2}\right) = \exp\left(\lim\limits_{x\to0}\dfrac{-\tan x}{2x}\right)$

$= \exp\left(\lim\limits_{x\to0}\dfrac{-\sec x}{2}\right) = e^{-\frac{1}{2}}$

⑦ $\lim\limits_{x\to0^+}(\sin x)^{\frac{a}{1+\ln x}}$ $\quad a\in\mathbb{R}$ $\qquad 0^0 \Rightarrow \exp\left(\dfrac{\infty}{\infty}\right)$

$= \exp\left(\lim\limits_{x\to0^+} a\dfrac{\ln\sin x}{1+\ln x}\right) = \exp(a)\exp\left(\lim\limits_{x\to0^+}\dfrac{\frac{\cos x}{\sin x}}{\frac{1}{x}}\right)$

$= \exp(a)\cdot\exp\left(\lim\limits_{x\to0^+}\cos x\cdot\dfrac{x}{\sin x}\right) = \exp(a)$

⑧ $\lim\limits_{x\to\infty}\left(x+\sqrt{1+x^2}\right)^{\frac{1}{\ln x}}$ $\qquad \infty^0 \Rightarrow e^{\frac{\infty}{\infty}}$

$= \exp\left(\lim\limits_{x\to\infty}\dfrac{\ln(x+\sqrt{1+x^2})}{\ln x}\right) = \exp\left(\lim\limits_{x\to\infty}\dfrac{\frac{1}{x+\sqrt{1+x^2}}\cdot\left(1+\frac{x}{\sqrt{1+x^2}}\right)}{\frac{1}{x}}\right)$

$= \exp\left(\lim\limits_{x\to\infty}\dfrac{\frac{1}{\cancel{x+\sqrt{1+x^2}}}\cdot\frac{\cancel{\sqrt{1+x^2}+x}}{\sqrt{1+x^2}}}{\frac{1}{x}}\right) = \exp\left(\lim\limits_{x\to\infty}\dfrac{x}{\sqrt{1+x^2}}\right)$

$= \exp\left(\lim\limits_{x\to\infty}\dfrac{1}{\sqrt{\frac{1}{x^2}+1}}\right) = e$

⑨ $\lim\limits_{x\to1}\left(\dfrac{1}{x-1} - \dfrac{1}{\ln x}\right)$ $\qquad \infty-\infty \Rightarrow \dfrac{0}{0}$

$= \lim\limits_{x\to1}\dfrac{\ln x - x + 1}{(x-1)\ln x} = \lim\limits_{x\to1}\dfrac{\frac{1}{x}-1}{\ln x+\frac{x-1}{x}} = \lim\limits_{x\to1}\dfrac{1-x}{x\ln x+x-1} = \lim\limits_{x\to1}\dfrac{-1}{\ln x+2} = -\dfrac{1}{2}$

Exercise 5. $\lim_{x\to c} f'(x) = f'(c)$

By definition: $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $|x-c| < \delta \Rightarrow |f'(x) - f'(c)| < \varepsilon$.

$$|f'(x) - f'(c)| \le \left| f'(x) - \frac{f(x+h) - f(x)}{h} \right| + \left| \frac{f(x+h) - f(x)}{h} - \frac{f(c+h) - f(c)}{h} \right| + \left| \frac{f(c+h) - f(c)}{h} - f'(c) \right|$$

①, ②, ③

For ①, since $f'(x)$ exists $\exists -\delta < h < \delta$ such that $\left| f'(x) - \frac{f(x+h) - f(x)}{h} \right| < \frac{\varepsilon}{3}$

For ③, since $f'(c)$ exists $\exists -\delta < h' < \delta$ such that $\left| f'(c) - \frac{f(c+h) - f(c)}{h} \right| < \frac{\varepsilon}{3}$

For ②, by MVT

$$\left| \frac{f(c+h) - f(c)}{h} - \frac{f(x+h) - f(x)}{h} \right| = |f'(\xi_1) - f'(\xi_2)| < \frac{\varepsilon}{3}$$

3. $f(x) = \begin{cases} \dfrac{\tan x - x}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

① Prove $f(x)$ differentiable at $x = 0$

② Justify $f(x)$ is continuous at $x = 0$

$$\lim_{h\to 0} \frac{f(0+h) - \cancel{f(0)}^{\,0}}{h} = \lim_{h\to 0} \frac{\frac{\tanh - h}{h^2} - 0}{h} = \lim_{h\to 0} \frac{\tanh - h}{h^3}$$

$1-\cos^2 x \sim \frac{1}{2}x^2$

$$= \lim_{h\to 0} \frac{\sec^2 h - 1}{3h^2} = \lim_{h\to 0} \frac{1 - \cos^2 x}{3h^2\cos^2 x} = \frac{1}{6}$$

$\sim \frac{1}{2}x^2$

$$\lim_{x\to 0} \frac{\tan x - x}{x^2} = \lim_{x\to 0} \frac{\sec^2 x - 1}{2x} = \lim_{x\to 0} \frac{1 - \cos^2 x}{2x\cos^2 x} = 0$$