

Differentiation $\{ \}$ derivative \exists

$$f(x) \quad f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\textcircled{1} (f \pm g)' = f' + g' \quad \textcircled{2} (c \cdot f)' = c \cdot f'$$

$$\textcircled{3} (f \cdot g)' = f \cdot g' + f' \cdot g \quad \textcircled{4} \left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2} \quad (g \neq 0)$$

Lemma Suppose that f is differentiable at point x_0 . if and only if there exists $F(x)$, which is continuous at x_0 , such that

$$f(x) - f(x_0) = F(x)(x - x_0)$$

$$\text{and } F(x) = \begin{cases} \frac{f(x) - f(x_0)}{x - x_0} & x \in (x_0 - \epsilon, x_0) \cup (x_0, x_0 + \epsilon) \\ f'(x_0) & x = x_0 \end{cases}$$

$$\text{Proof: } \textcircled{\Rightarrow} \text{ Since } \lim_{x \rightarrow x_0} F(x) = f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = F(\lim_{x \rightarrow x_0} x)$$

we have that $F(x)$ is continuous at x_0 .

$\textcircled{\Leftarrow}$ Suppose that $F(x)$ is continuous at x_0 , and we have

$$f(x) - f(x_0) = F(x)(x - x_0)$$

$$\text{Since the limit } \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} F(x) = F(x_0)$$

exists, we conclude that f is differentiable at x_0 .

Then (Chain rule) Suppose that a function f is differentiable at x_0 , and a function g is differentiable at $y_0 = f(x_0)$. Then

$$(g \circ f)'(x_0) = g'(y_0) \cdot f'(x_0)$$

$$= g'(f(x_0)) \cdot f'(x_0)$$

Proof: Since g is differentiable at y_0 , we have

$$g(y) = g(y_0) + G(y)(y - y_0)$$

and G is continuous function, and $G(y_0) = g'(y_0)$ let $y = f(x_0 + \Delta x)$

$$g(f(x_0 + \Delta x)) = g(f(x_0)) + G(f(x_0 + \Delta x))(f(x_0 + \Delta x) - f(x_0)) \quad y = f(x_0)$$

$$g(f(x_0 + \Delta x)) - g(f(x_0)) = G(f(x_0 + \Delta x))(f(x_0 + \Delta x) - f(x_0))$$

$$\frac{g(f(x_0 + \Delta x)) - g(f(x_0))}{\Delta x} = G(f(x_0 + \Delta x)) \frac{(f(x_0 + \Delta x) - f(x_0))}{\Delta x}$$

Since f is differentiable at x_0 , and g is differentiable at y_0

$$\lim_{\Delta x \rightarrow 0} \frac{g(f(x_0 + \Delta x)) - g(f(x_0))}{\Delta x} = \lim_{\Delta x \rightarrow 0} G(f(x_0 + \Delta x)) \frac{(f(x_0 + \Delta x) - f(x_0))}{\Delta x}$$

$$(g \circ f)'(x_0) = G(f(x_0)) \cdot f'(x_0)$$

$$= g'(y_0) \cdot f'(x_0)$$

Note: $(g \circ f)'(x) = \frac{dg}{dx} = \frac{df}{dx} \cdot \frac{dg}{df}$

$$(h \circ g \circ f)'(x) = \frac{dh}{dx} = \frac{df}{dx} \cdot \frac{dg}{df} \cdot \frac{dh}{dg}$$

E.g.: $g(x) = a^x \quad g'(x) = a^x \cdot \ln a$

$$= \exp(\ln(a^x)) = \exp(x \cdot \ln a) \quad \text{Let } f(x) = x \cdot \ln a$$

$$(a^x)' = (g \circ f)'(x) = g'(y) \cdot f'(x) \quad g(y) = \exp(y)$$

$$= \exp(y) \cdot \ln a = \exp(x \cdot \ln a) \cdot \ln a$$

$$= a^x \cdot \ln a$$

E.g.: $y = \sin x^2$

$$y = \frac{1}{3\sqrt{x^2+x+1}}$$

$$y = (\sin x)^2$$

$$y = (2x+1)^5 (x^3 - 2x+3)^3$$

$$\textcircled{1} \quad y = \sin x^2 \quad f(x) = x^2 \quad g(f) = \sin f$$

$$y' = \cos f \cdot 2x = \cos x^2 \cdot 2x$$

$$\textcircled{2} \quad y = \sin^2 x \quad f(x) = \sin x \quad g(f) = f^2$$

$$y' = 2f \cdot \cos x = 2 \sin x \cdot \cos x$$

$$\textcircled{3} \quad y = \frac{1}{\sqrt[3]{x^2+x+1}} \quad f(x) = x^2+x+1 \quad g(f) = \frac{1}{\sqrt[3]{f^2}} = f^{-\frac{1}{3}}$$

$$y' = -\frac{1}{3} f^{-\frac{4}{3}} \cdot (2x+1) = -\frac{1}{3} (x^2+x+1)^{-\frac{4}{3}} (2x+1)$$

$$\textcircled{4} \quad y = (2x+1)^5 (x^3-2x+3)^3 \quad f_1(x) = 2x+1 \quad f_2(x) = x^3-2x+3$$

$$g_1(f) = f^5 \quad g_2(f) = f^3$$

$$y_1 = (2x+1)^5$$

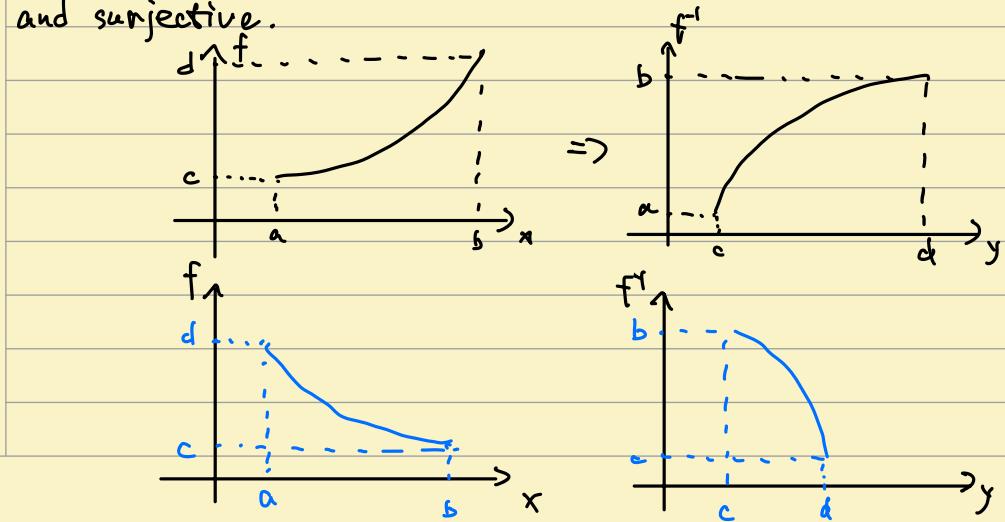
$$y'_1 = 5f_1^4 \cdot 2 = 10(2x+1)^4$$

$$y_2 = (x^3-2x+3)^3$$

$$y'_2 = 3f_2^2 \cdot (3x^2-2) = 3(x^3-2x+3)^2 (3x^2-2)$$

$$y' = y'_1 \cdot y_2 + y_1 \cdot y'_2 = 10(2x+1)^4 \cdot (x^3-2x+3)^3 + 3(2x+1)^5 (x^3-2x+3)^2 (3x^2-2)$$

Thm. Suppose that $f : [a, b] \rightarrow [c, d]$ is continuous and strictly increasing, so that $f(a) = c$, $f(b) = d$. Then the inverse function $f^{-1} : [c, d] \rightarrow [a, b]$ exists, and is continuous, strictly increasing, and surjective.



Then: Suppose $f: [a, b] \rightarrow [c, d]$ is continuous and strictly increasing. Let $y_0 = f(x_0)$. If f is differentiable at x_0 , and $f'(x_0) \neq 0$, then f^{-1} is differentiable at y_0 , and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} \quad \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

Proof: Since f is continuous and strictly increasing, then f^{-1} is also continuous and strictly increasing. Hence, we can write

$y = f(x)$, $y_0 = f(x_0)$ and then $x = f^{-1}(y)$ and $x_0 = f^{-1}(y_0)$. Since $f(x)$ is differentiable at x_0 .

$$\lim_{y \rightarrow y_0} \frac{f(x) - f(x_0)}{y - y_0} = f'(x_0)$$

$$\Rightarrow \lim_{x \rightarrow x_0} \frac{y - y_0}{f^{-1}(y) - f^{-1}(y_0)} = f'(x_0)$$

$$\Rightarrow \lim_{x \rightarrow x_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{f'(x_0)}$$

Since f is continuous at x_0 , $f(x) \rightarrow f(x_0)$ as $x \rightarrow x_0$. That is $y \rightarrow y_0$ as $x \rightarrow x_0$. Hence

$$\lim_{y \rightarrow y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{f'(x_0)}$$

$$\Rightarrow [f^{-1}(y_0)]' = \frac{1}{f'(x_0)}$$

E.g.: $y = \sin x \quad x = \arcsin y$

Let $y = f(x) = \sin x \quad x = g(y) = \arcsin y$.

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-y^2}}$$

E.g.: $x = \arctan y \quad y = \tan x \quad (\tan x)' = \sec^2 x = \frac{1}{\cos^2 x}$

$$= \frac{\sin^2 x + \cos^2 x}{\cos^2 x}$$

$$= \tan^2 x + 1$$

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{\sec^2 x} = \cos^2 x = \frac{\cos^2 x}{1}$$

$$= \frac{\cos^2 x}{\sin^2 x + \cos^2 x} = \frac{1}{\tan^2 x + 1} = \frac{1}{y^2 + 1}$$

$$\textcircled{1} (f \pm g)' = f' + g' \quad \textcircled{2} (c \cdot f)' = c \cdot f'$$

$$\textcircled{3} (f \cdot g)' = f \cdot g' + f' \cdot g \quad \textcircled{4} \left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2} \quad (g \neq 0)$$

$$\textcircled{5} (g \circ f)'(x) = g'(f(x)) \cdot f'(x) \quad \textcircled{6} g'(y) = \frac{1}{f'(x)}$$

E.g.: Let $y = \frac{(x+5)^2 (x-4)^{\frac{1}{3}}}{(x+2)^5 (x+4)^{\frac{1}{2}}} \quad (x > 4)$, find y'

$$\ln y = \ln \frac{(x+5)^2 (x-4)^{\frac{1}{3}}}{(x+2)^5 (x+4)^{\frac{1}{2}}} = 2 \ln(x+5) + \frac{1}{3} \ln(x-4) - 5 \ln(x+2) - \frac{1}{2} \ln(x+4)$$

$$\frac{d \ln y}{dx} = \frac{d}{dx} \left(2 \ln(x+5) + \frac{1}{3} \ln(x-4) - 5 \ln(x+2) - \frac{1}{2} \ln(x+4) \right)$$

$$\frac{1}{y} y' = \frac{2}{x+5} + \frac{1}{3(x-4)} - \frac{5}{x+2} - \frac{1}{2(x+4)}$$

$$y' = y \left[\frac{2}{x+5} + \frac{1}{3(x-4)} - \frac{5}{x+2} - \frac{1}{2(x+4)} \right]$$

$$= \frac{(x+5)^2 (x-4)^{\frac{1}{3}}}{(x+2)^5 (x+4)^{\frac{1}{2}}} \left[\frac{2}{x+5} + \frac{1}{3(x-4)} - \frac{5}{x+2} - \frac{1}{2(x+4)} \right]$$

E.g.: $y = x^{x^x}$ Let $u = x^x \quad y = x^u \quad \ln y = u \ln x$

$$\frac{d}{dx} \ln y = \frac{d}{dx} (u \ln x) \Rightarrow \frac{y'}{y} = u' \ln x + \frac{u}{x}$$

$$\Rightarrow y' = y \left(u' \ln x + \frac{u}{x} \right)$$

$$u = x^x \quad \ln u = x \ln x$$

$$\frac{d}{dx} \ln u = \frac{d}{dx} (x \ln x) \Rightarrow \frac{u'}{u} = \ln x + 1$$

$$\Rightarrow u' = u (\ln x + 1)$$

$$y' = x^{x^x} \cdot \left(x^x (\ln x + 1) \cdot \ln x + \frac{x^x}{x} \right)$$

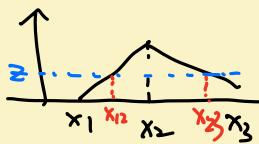
$$= x^{x^x} \cdot x^x \left(\ln^2 x + \ln x + \frac{1}{x} \right)$$

Exercise 4.

1. Since there is at most one x for $f(x) = y$, f is injective.

Suppose that f is not strictly monotonic. $x_1 < x_2 < x_3$

$$f(x_1) < f(x_2), \quad f(x_2) > f(x_3)$$



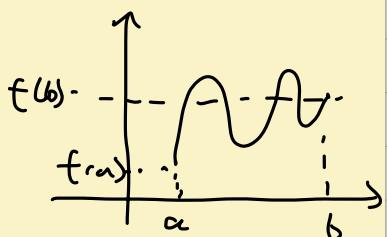
we can find a value $z \in \mathbb{R}$ such that

$$f(x_1) < z < f(x_2) \quad f(x_2) > z > f(x_3)$$

there exist $x_{12} \in (x_1, x_2)$ such that $f(x_{12}) = z$

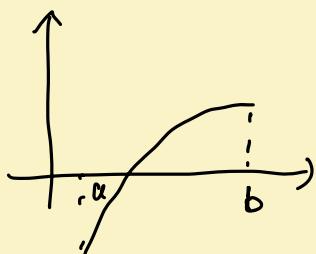
$x_{23} \in (x_2, x_3)$ such that $f(x_{23}) = z$

a contradiction with f being an injection



$\exists x \in [a, b]$

$$\Rightarrow f(x) = \frac{f(b) - f(a)}{2}$$

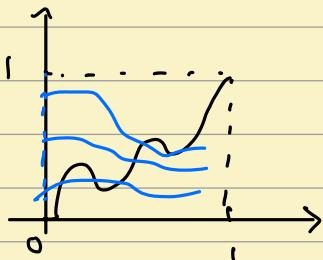


$$f(a) < 0 \quad f(b) > 0$$

$$\exists x \in [a, b] \quad f(x) = 0$$

2. $f(x) = x$ f is continuous. $f: [0, 1] \rightarrow [0, 1]$

$$\text{Let } g(x) = f(x) - x$$



- ① If $f(0) = 0 \checkmark f(1) = 1$. It is done
- ② If $f(0) > 0, f(1) < 1$
 $\Rightarrow g(0) = f(0) > 0$
 $g(1) = f(1) - 1 < 0$

By the intermediate value theorem

$\exists x \in (0, 1)$ such that $g(x) = 0$

$$\Rightarrow g(x) = f(x) - x = 0 \Rightarrow f(x) = x$$

3. Assume f is not bounded on (a, b) , that is $\exists x \in (a, b)$

such that $|f(x)| \rightarrow +\infty$, there exist $\{a_n\} \subset (a, b)$

$|f(a_n)| \rightarrow \infty$ that is $\{b_n\} \subset (a, b)$, such that

$$f(b_n) = 2^n \rightarrow +\infty$$

Since $\{b_n\}$ is bounded, then there exists a convergent subsequence b_{n_k} $|t_{n_k} - b_{n_{k+1}}| < \delta \quad \forall k \geq N \in \mathbb{N}$ for any $\delta > 0$.

$$\text{but } |f(b_{n_k}) - f(b_{n_{k+1}})| = |2^{n_k} - 2^{n_{k+1}}| \geq 2 \text{ then}$$

f is not uniformly continuous. contradiction.