Differentiation : A funtion f is differentiable at point x_0, iff

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x_0)}{\Delta x}$$

exists.

Formulas :

① $(f \pm g)' = f' \pm g'$ ② $(c \cdot f)' = c \cdot f'$

③ $(f \cdot g)' = f' \cdot g + f \cdot g'$ ④ $\left(\frac{f}{g}\right)' = \frac{1}{g^2}(f' \cdot g - f \cdot g')$ $g \neq 0$

⑤ $(g \circ f)'(x) = g'(f) \cdot f'(x)$ ⑥ $g'(y) = \frac{1}{f'(x)}$

Leibniz formula

$$(f \pm g)' = f' \pm g' \quad \Rightarrow \quad (f \pm g)^{(n)} = f^{(n)} \pm g^{(n)}$$

$$(f \cdot g)' = f' \cdot g + f \cdot g' \quad \Rightarrow \quad (f \cdot g)^{(n)} \ ?$$

Let $y = f \cdot g$

$y' = f' \cdot g + f \cdot g'$

$y'' = (f' \cdot g + f \cdot g')' = (f'' \cdot g + f' \cdot g') + (f' \cdot g' + f \cdot g'')$

$= f^{(2)} \cdot g^{(0)} + 2 f^{(1)} \cdot g^{(1)} + f^{(0)} \cdot g^{(2)}$

$y''' = (f''' \cdot g + f'' \cdot g') + 2(f'' \cdot g' + f' \cdot g'') + (f' \cdot g'' + f \cdot g''')$

$= f^{(3)} \cdot g^{(0)} + 3 f^{(2)} \cdot g^{(1)} + 3 f^{(1)} \cdot g^{(2)} \cdot f^{(0)} \cdot g^{(3)}$

$(x+y)^n$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

⋮

The Leibniz's formula

$$(f \cdot g)^{(n)} = f^{(n)} \cdot g^{(0)} + C_n^1 f^{(n-1)} \cdot g^{(1)} + C_n^2 f^{(n-2)} \cdot g^{(2)} + \cdots + C_n^k f^{(n-k)} \cdot g^{(k)} + \cdots + f^{(0)} \cdot g^{(n)}$$

$$= \sum_{k=0}^{n} C_n^k f^{(n-k)} \cdot g^{(k)}$$

$$(f\cdot g)^{(n)} = \sum_{k=0}^{n} C_n^k f^{(n-k)} \cdot g^{(k)}$$

E.x $y = x^2 \cdot e^x$ find $y^{(n)}$

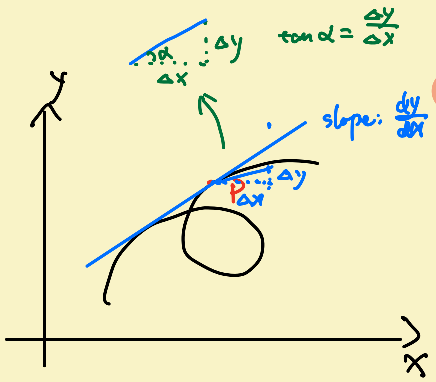
Let $f = x^2$ $g = e^x$

$$y^{(n)} = \sum_{k=0}^{n} C_n^k f^{(n-k)} \cdot g^{(k)} = \sum_{k=0}^{n} C_n^k f^{(k)} \cdot g^{(n-k)}$$

$$= \sum_{k=0}^{n} C_n^k f^{(k)} \cdot e^x$$

$$= x^2 \cdot e^x + n \cdot 2x \cdot e^x + n(n-1) e^x \qquad f^{(0)} \cdot g^{(n)} + C_n^1 f^{(1)} \cdot g^{(n-1)} + C_n^2 f^{(2)} \cdot g^{(n-2)}$$

$$= e^x [x^2 + 2nx + n(n-1)]$$



Parametric derivative

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}$$ t: the parameter

$$\frac{\Delta y}{\Delta x} = \frac{\psi(t_0+\Delta t) - \psi(t_0)}{\varphi(t_0+\Delta t) - \varphi(t_0)}$$

$$\tan\alpha = \lim_{\Delta t\to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta t\to 0} \frac{\psi(t_0+\Delta t) - \psi(t_0)}{\varphi(t_0+\Delta t) - \varphi(t_0)}$$

$$= \lim_{\Delta t\to 0} \frac{\frac{\psi(t_0+\Delta t) - \psi(t_0)}{\Delta t}}{\frac{\varphi(t_0+\Delta t) - \varphi(t_0)}{\Delta t}} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\psi'(t)}{\varphi'(t)}$$

Chain rule: $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

Inverse function: $\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

E.x: $\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$ find $y'(x)$

$$\frac{dy}{dx} = \frac{(b\sin t)'}{(a\cos t)'} = -\frac{b\cos t}{a\sin t} = -\frac{b}{a}\cot t$$

E.x. $\begin{cases} x = a(t-\sin t) \\ y = a(1-\cos t) \end{cases}$ find $y''(x)$

$$\frac{dy}{dx} = \frac{[a(1-\cos t)]'}{[a(t-\sin t)]'} = \frac{\sin t}{1-\cos t} = \cot\frac{t}{2}$$

$$\boxed{\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\cdot\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}} = \frac{\left(\cot\frac{t}{2}\right)'}{[a(t-\sin t)]'}$$

$$= \frac{-\frac{1}{2}\csc^2\frac{t}{2}}{a(1-\cos t)} = -\frac{1}{4a}\csc^4\frac{t}{2}$$

Implicit differentiation 隐含数求导

E.g.: $y = \sin^2 x$ $\quad y = \sqrt[3]{x^3+1}$

E.g.: $x^2+y^2=25 \Rightarrow y = \pm\sqrt{25-x^2}$

$x^2+2xy+y^2=10$

E.g.: $x^3+y^3=6xy$ find $y'(3)$

$\Rightarrow x^3+[y(x)]^3 = 6x\cdot[y(x)]$

Let $g(x) = x^3+[y(x)]^3$ $\quad h(x) = 6x\cdot y(x)$

$$\boxed{\text{If } g(x) = h(x) \Rightarrow g'(x) = h'(x)}$$

$g'(x) = 3x^2+3y^2\cdot y'$ $\quad h'(x) = 6\cdot y + 6x\cdot y'$

$g'(x) = h'(x) \Rightarrow 3x^2+3y^2\cdot y' = 6\cdot y + 6x\cdot y'$

$$\Rightarrow y' = \frac{6y-3x^2}{3y^2-6x} = \frac{2y-x^2}{y^2-2x}$$

$x^3+y^3=6xy$ $\quad x=3 \Rightarrow y=3$ $\quad \therefore y'(3) = -1$

E.x. $x^2+2xy+y^2=10$ find $y'(x)$

$$(x+y)^2=10$$
$$x+y=\pm\sqrt{10}$$
$$y=\pm\sqrt{10}-x$$
$$y'=-1$$

$$g(x)=x^2+2xy+y^2$$
$$g'(x)=2x+2y+2xy'+2y\cdot y'=0$$

$$y'=-\frac{2x+2y}{2x+2y}=-1$$

E.x. $\sin(x+y)=y^2\cos x$ find $y'(x)$

$$g(x)=\sin(x+y) \qquad g'(x)=(1+y')\cdot\cos(x+y)$$
$$h(x)=y^2\cos x \qquad h'(x)=2y\cdot y'\cos x-y^2\sin x$$

$$\Rightarrow \cos(x+y)+y'\cdot\cos(x+y)=y'\cdot 2y\cos x-y^2\sin x$$
$$y'(\cos(x+y)-2y\cos x)=-\cos(x+y)-y^2\sin x$$

$$\Rightarrow y'=\frac{\cos(x+y)+y^2\sin x}{2y\cos x-\cos(x+y)}$$

Logarithmic differentiation

E.g.: $y=\dfrac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$ find $y'(x)$

$$\ln y=\frac{3}{4}\ln x+\frac{1}{2}\ln(x^2+1)-5\ln(3x+2)$$

$$(\ln y)'=\left(\frac{3}{4}\ln x+\frac{1}{2}\ln(x^2+1)-5\ln(3x+2)\right)'$$

$$\Rightarrow \frac{y'}{y}=\frac{3}{4x}+\frac{1}{2}\frac{2x}{x^2+1}-\frac{5\cdot 3}{3x+2}$$

$$=y'=\frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x}+\frac{x}{x^2+1}-\frac{15}{3x+2}\right)$$

E.x. $y = x^{\sqrt{x}}$ find $y'(x)$

$\ln y = \sqrt{x} \cdot \ln x$ $\frac{y'}{y} = \frac{1}{2}\frac{1}{\sqrt{x}} \cdot \ln x + \frac{\sqrt{x}}{x}$

$\Rightarrow y' = x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right)$

Mean value theorem

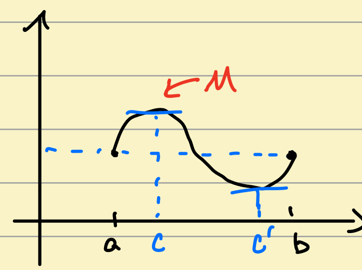
To find the properties of a function based on the properties of its derivative

Rolle's theorem

Thm: (Rolle's theorem) Suppose that a function f satisfies:

1. f is continuous on the closed interval $[a, b]$
2. f is differentiable on the open interval (a, b)
3. $f(a) = f(b)$

Then, there is at least a number $c \in (a, b)$, such that $f'(c) = 0$.



Proof: By the minimum-maximum theorem if f is continuous on $[a, b]$, then there exist $M = \max f(x)$ and $m = \min f(x)$

Case 1: $f(a) = f(b) < M$,

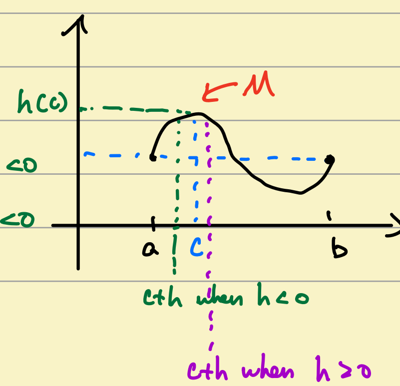
Case 2: $m < f(a) = f(b)$, or

Case 3: $m = f(a) = f(b) = M$

There is $h \in \mathbb{R}$,

when $h < 0$ and $h \to 0$, then $f(c+h) - f(c) < 0$

when $h > 0$ and $h \to 0$, then $f(c+h) - f(c) < 0$



$$\lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0$$

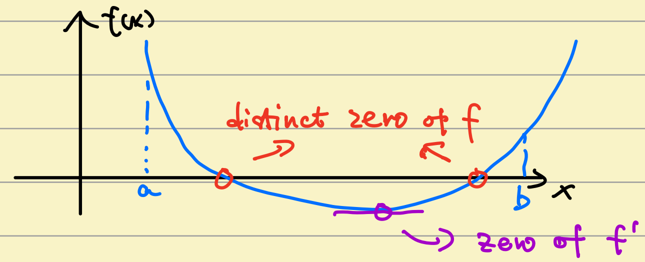
$$\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \leq 0$$

since f is differentiable at $c \in (a,b)$

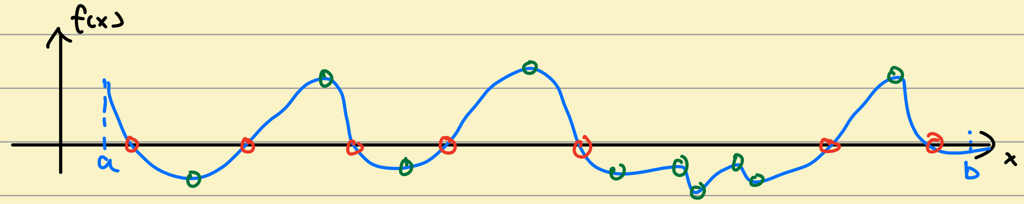
$$f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} = 0$$

Hence $f'(c) = 0$

Corollary: Suppose that a function f is differentiable on (a,b), then between any 2 distinct zeros of f, there is at least a zero of f'.



Corollary: Suppose that a function f is differentiable on (a,b). and it has n distinct zeros, then f' has at least $n-1$ different zeros.



Equivalently, if f' has m distinct zeros, then f has at most $m+1$ distinct zeros,

$0 \in \partial f(c) \Leftrightarrow 0 = f'(c)$

g is subderivative if

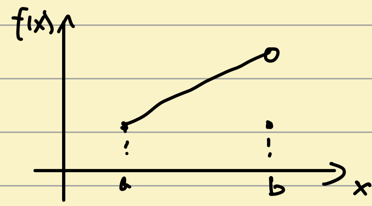
$f(c) + g(x-c) < f(x)$

Assumptions of Rolle's theorem:

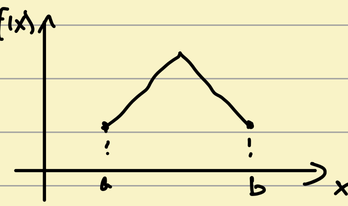
① f is continuous on the closed interval $[a, b]$

② f is differentiable on the open interval (a, b)

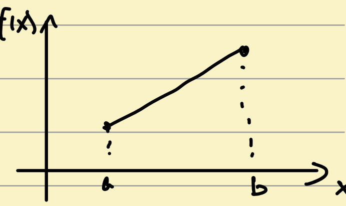
③ $f(a) = f(b)$



lack ①



lack ②



lack ③

Lagrangian Mean value theorem

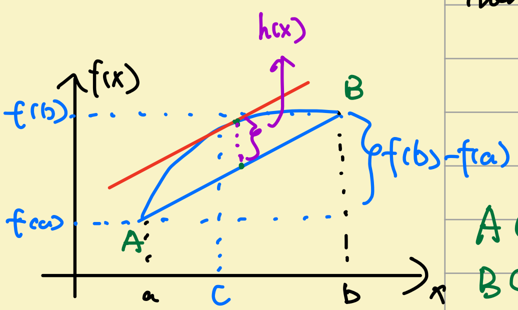
Thm (Lagrangian MVT) Suppose that a function f satisfies:

① f is continuous on the closed interval $[a, b]$

② f is differentiable on the open interval (a, b).

Then, there is at least a $c \in (a, b)$, such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$



Proof: The equation of the secant line AB can be written as

$A(a, f(a))$ $\qquad y(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$

$B(b, f(b))$

Define $h(x) = f(x) - y(x)$

$$= f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

Then we have $h(x)$ is ① continuous on $[a, b]$

② differentiable on (a, b), and

③ $h(a) = f(a) - f(a) - \frac{f(b)-f(a)}{b-a}(a-a) = 0$

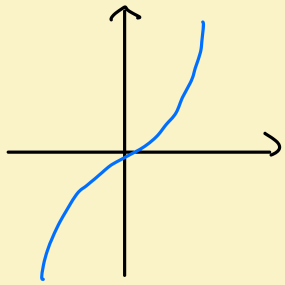
$h(b) = f(b) - f(a) - \frac{f(b)-f(a)}{b-a}(b-a) = 0$

By Rolle's theorem, there is $c \in (a, b)$ such that $h'(c) = 0$

$$\Rightarrow f'(c) - \frac{f(b)-f(a)}{b-a} = 0$$

$\equiv$：恆等于

constant：常數



Corollary: Suppose that a function f is

① continuous on $[a,b]$, and ② differentiable on (a,b), then

(i) If $f'(x) \equiv 0 \quad \forall x \in (a,b)$, then f is a constant on $[a,b]$

(ii) If $f'(x) > 0 \quad \forall x \in (a,b)$, then f is strictly increasing on $[a,b]$

(iii) If $f'(x) < 0 \quad \forall x \in (a,b)$, then f is strictly decreasing on $[a,b]$

Proof: By MVT applied to $f(x)$ on $[a,x]$

Ex: Show that if $-\frac{\pi}{4} < a < b < \frac{\pi}{4}$, then

$$|\tan b - \tan a| < 2|b-a|$$

Let $f(x) = \tan x$ then $f'(x) = \sec^2 x = \frac{1}{\cos^2 x}$

By Mean value theorem, there is $c \in (a, b)$, such that

$$\sec^2 c = \frac{\tan b - \tan a}{b-a}$$

Since $\cos^2 x > \frac{1}{2} \quad \forall x \in (-\frac{\pi}{4}, \frac{\pi}{4})$ we have $\sec^2 c < 2$

$$\Rightarrow \frac{\tan b - \tan a}{b-a} < 2$$

$$\Rightarrow |\tan b - \tan a| < 2|b-a|$$