Real Analysis

GitHub.com/styluck/ra   4次作业, Homework 20%

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Real space 实空间

数学分析

1 基础：记号，集合，区间，欧氏距离

2. 函数极限. ε-δ定义，极限是否存在，夹逼定理，极限运算，
   经典极限.

3. 连续函数：函数是否连续，有界性，最值定理，介值定理，
   一致连续定义

4. 求导：求导法则，链式法则，隐函数，
   罗尔定理，拉格朗日中值定理，洛必达法则
   泰勒定理.

Notation: $\mathbb{R}$: the set of real numbers

quotient 商 → $\mathbb{Q}$: the set of rational number   ratio

$x \in \mathbb{Q}$: $x = \frac{p}{q}$, $q \neq 0$, $p, q \in \mathbb{Z}$ 整数集

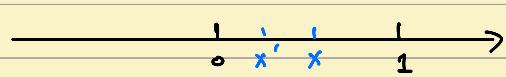
$\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$ $\mathbb{I}$: irrational number

$\sqrt{2}$, π, $1+\sqrt{2}$,

$0.999\cdots = 1$

The set of rational numbers is dense (稠密的)

The set of real numbers is continuous (连续的)



1.1 Set theory

Defn: A set is a collection of different things.

Properties: ① Well-definedness (确定性) $a \in A$ or $a \notin A$ (in / not in)

② Distinct elements (互异性) $A = \{1, 6, 3\}$ $B = \{1, 3, 6, 6\} = \{1, 3, 6\}$

③ Unorders (无序性)

Notations: $A \subset B \Leftrightarrow \{A \subseteq B$ and $A \neq B\}$

Property: if $A \subseteq B$, and $B \subseteq A$ if and only if $A = B$

Union (并集) $A \cup B = \{x \in \mathbb{R} \mid x \in A \text{ or } x \in B\}$

Intersection (交集) $A \cap B = \{x \in \mathbb{R} \mid x \in A \text{ and } x \in B\}$

Complement (补集) $A^c = \{x \in \mathbb{R} \mid x \notin A\}$

Empty set : ϕ a set that contains no elements

Index set : $A_1, A_2, \cdots A_n \rightarrow$ sets

$I = \{1, 2, \cdots, n\} \rightarrow$ Index set

$$A_1 \cup A_2 \cup A_3 \cdots \cup A_n =: \bigcup_{i \in I} A_i = \{x \in \mathbb{R} \mid x \in A_i \text{ for some } i \in I\}$$

$$\bigcap_{i \in I} A_i = \{x \in \mathbb{R} \mid x \in A_i \text{ for all } i \in I\}$$

De Morgan's laws :

① $\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c$

② $\left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^c$

1.2 The Intervals 区间

Defn: An interval is a subset of the set of real numbers $\mathbb{R}$

given $a, b \in \mathbb{R}$, $a<b$

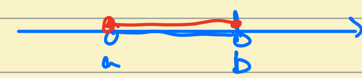
Notations:

Finite interval:

$(a, b) := \{x \in \mathbb{R} \mid a<x<b\}$ open interval

$[a, b] := \{x \in \mathbb{R} \mid a \le x \le b\}$ closed interval

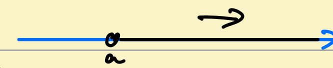
$[a, b) := \{x \in \mathbb{R} \mid a \le x < b\}$



$(a, b] := \{x \in \mathbb{R} \mid a < x \le b\}$

infinite interval:

$(a, +\infty) := \{x \in \mathbb{R} \mid x > a\}$



$[a, +\infty) := \{x \in \mathbb{R} \mid x \geq a\}$

$(-\infty, b) := \{x \in \mathbb{R} \mid x < b\}$

$(-\infty, b] := \{x \in \mathbb{R} \mid x \le b\}$

$(-\infty, +\infty) := \mathbb{R}$

Notation: $\mathbb{R}_{+} := [0, +\infty)$ $\mathbb{R}_{++} := (0, +\infty)$

E.g. Given $x, y \in \mathbb{R}$, $x<y$, then there exists a rational number r such that

$$x<r<y$$

Proof: Since $x<y$, there exists $n>0$, such that $\bar{x}_n < y_n$. let

$$r := \frac{1}{2}(\bar{x}_n + y_n)$$

$$\Rightarrow \quad x \le \bar{x}_n < r < y_n < y$$

We conclude $x<r<y$, and r is a rational number.

x: $a_0 . a_1 a_2 a_3 \cdots a_n \cdots$

$x = 11.347165$

$a_0 = 11$ $a_1 = 3$ $a_2 = 4$

$a_3 = 7 \cdots$

过剩近似: $\bar{x}_n = 11.3472$ $n=4$

不足近似: $x_n = 11.3471$

$x_n \le x \le \bar{x}_n$

Properties of $\mathbb{R}$

① . Closure property: (封闭性) Any arithmetic operation $(+, -, \times, \div)$ is performed on two elements of $\mathbb{R}$, with the answer being another element of $\mathbb{R}$.

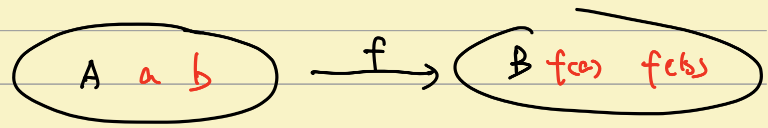
② Ordered: (有序性) any two elements of $\mathbb{R}$, one of the following holds: $a<b$, $b<a$, $a=b$

③ Transitivity: (传递性) if $a<b$, $b<c \Leftrightarrow a<c$

④ Density: (稠密性) if $a<b$, then there exist $c \in \mathbb{R}$, $a<c<b$

single-valued function $\rightarrow$

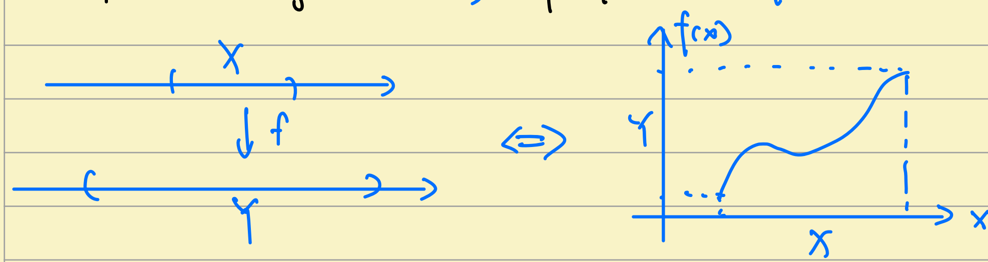
1.3. Functions



Real function $f: X \to Y$ with $X, Y \subseteq \mathbb{R}$

X: the domain (定义域) of f : $\mathrm{Dom}(f)$

Y: the range (值域) of f : $\mathrm{Im}(f)$ codomain, image

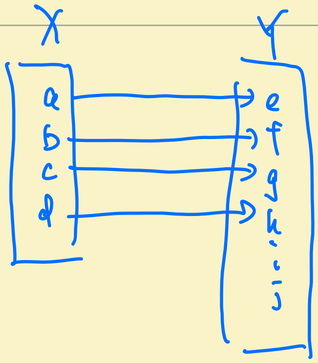


Graph (图) of f: $\{(x, y) \in \mathbb{R}^2 : y = f(x), x \in X, y \in Y\}$

E.g.: $f(x) = 1$ $x \in \mathbb{R}$ $\mathrm{Dom}(f): \mathbb{R}$

$\mathrm{Im}(f): \{1\}$

E.g.: $\varphi(x) = |x|$, $x \in \mathbb{R}$ $\psi(x) = \sqrt{x^2}$ $x \in \mathbb{R}$

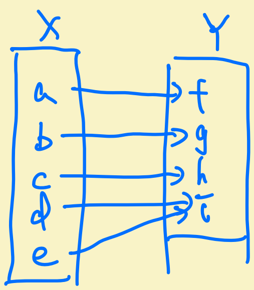


Defn: Let $f: X \to Y$ be a real function

① The function is <u>injective</u> (单射) (or one-to-one) if for any $a, b \in X$, $f(a) = f(b)$ implies $a = b$

Equivalently, if $a \neq b$ then $f(a) \neq f(b)$.

The function is called <u>injection</u> (单射函数)



② The function is <u>surjective</u> if $f(X) = Y$

Then the function is called <u>surjection</u> (满射函数)

E.g. $f(x) = e^x$ $\quad f: \mathbb{R} \to \mathbb{R}$ $\quad$ injective but not surjective

E.g. $f(x) = x^2$ $\quad f: \mathbb{R} \to \mathbb{R}_+$ $\quad$ surjective but not injective

E.g. $f(x) = x^3$ $\quad f: \mathbb{R} \to \mathbb{R}$ $\quad$ both injective and surjective

3. The function f is bijective if it is both injective and surjective. Then the function is called bijection.

<u>Operations of the functions</u>

Given real functions $f_1: X_1 \to Y_1$, $f_2: X_2 \to Y_2$

define $X := X_1 \cap X_2$ if $X \neq \phi$

$$F(x) = f_1(x) + f_2(x)$$
$$G(x) = f_1(x) - f_2(x)$$
$$H(x) = f_1(x) \cdot f_2(x)$$

Let $X^* = X_1 \cap \{ x \mid f_2(x) \neq 0, \text{ and } x \in X_2 \} \neq \phi$

$$\lfloor (x) = \frac{f_1(x)}{f_2(x)}$$

E.g. $f_1(x) = \sqrt{1-x^2}$ $\quad f_2(x) = \sqrt{x^2-4}$

$F(x) = f_1(x) + f_2(x)$ $\quad F: X \to Y$ $\quad X = \phi$

$\therefore F(x)$ is not well-defined.

Composed functions (复合函数)

Given $y = f(u)$ $\quad u \in D$, $u = g(x)$, $x \in X$
Let $X^* = X \cap \{ x \mid g(x) \in D \} \neq \phi$, then the composed function is denoted as

$$y = f(g(x)) = f \circ g(x)$$

E.g.:

$y = f(u) = \sqrt{u}$ $\quad u = g(x) = 1 - x^2$ $\quad X = \mathbb{R}$

$y = f \circ g(x) = \sqrt{1-x^2}$

$X^*: X \cap \{ x \mid g(x) \in D \}$
$= \mathbb{R} \cap [-1, 1]$
$= [-1, 1]$

Inversed function (反函数)

Given a real function : $f: X \to Y$, its inversed function is

$$\varphi^{-1}: f(X) \to X$$

E.g. $f: \mathbb{R} \to \mathbb{R}$ $\quad y = f(x) = x^2$

$f^{-1}: \mathbb{R}_+ \to \mathbb{R}$ $\quad x = \sqrt{y}$
$x = -\sqrt{y}$

Monotone function (单调函数)

Defn: $f: X \to Y \quad \forall x_1, x_2 \in X, \ x_1 < x_2$, if

① $f(x_1) \le f(x_2)$, then f is monotonically increasing

② $f(x_1) < f(x_2)$, then f is strictly monotonically increasing

③ $f(x_1) \ge f(x_2)$, then f is monotonically decreasing

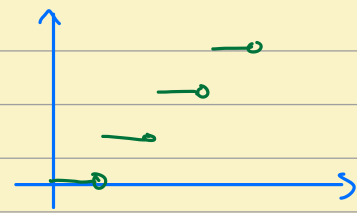
④ $f(x_1) > f(x_2)$, then f is strictly monotonically decreasing

E.g. $f(x) = x^3 \quad \forall x_1, x_2 \in \mathbb{R}, \quad x_1 < x_2$

$$x_2^3 - x_1^3 = (x_2 - x_1)\left[\left(x_2 + \frac{x_1}{2}\right)^2 + \frac{3}{4}x_1^2\right] > 0$$

$\Rightarrow x_2^3 > x_1^3 \quad \Rightarrow f$ is strictly monotonically increasing.

E.g. $f(x) = \lceil x \rceil$



Thm: Let $y = f(x)$. $f: X \to Y$. If f is strictly monotonically increasing then its inversed function f^{-1} is also strictly monotonically increasing.

Thm: Let $y = f(x)$. $f: X \to Y$. If f is strictly monotonically decreasing then its inversed function f^{-1} is also strictly monotonically decreasing.

1.4. Euclidean distance on the real space (欧式距离)

Absolute value $\quad |a| = \begin{cases} a, & a \ge 0 \\ -a, & a < 0 \end{cases}$

Properties of absolute value

1. $|a| = |-a| \geq 0$ iff $a=0$ $|a|=0$
2. $-|a| \leq a \leq |a|$
3. $|a| \leq h \iff -h \leq a \leq h$
4. $|a| - |b| \leq |a \pm b| \leq |a| + |b|$ $\quad |a+b| \lessgtr |a-b|$
5. $|a \cdot b| = |a| \cdot |b|$
6. $\dfrac{|a|}{|b|} = \left|\dfrac{a}{b}\right|$ $(b \neq 0)$

$\mathbb{R}\times\mathbb{R}$ ↪ 卡笛尔积

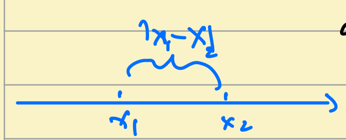
$\mathbb{R}^2$ Cartesian product

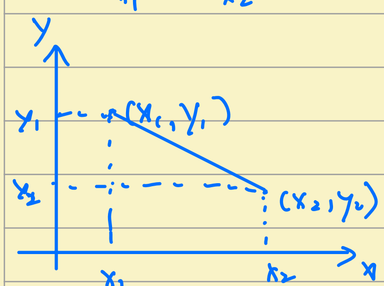
Defn. On $\mathbb{R}$, the Euclidean distance is the function.

$$d: \mathbb{R}\times\mathbb{R} \to [0, +\infty)$$

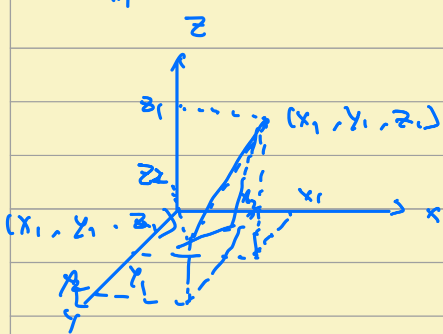
$$d(x,y) = |x-y| \quad \forall x, y \in \mathbb{R}$$

$$= \sqrt{(x-y)^2}$$



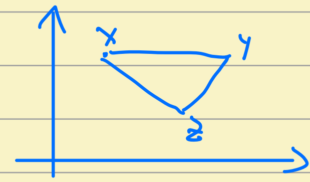


$\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

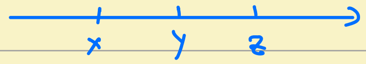


$\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$

Properties of distance function

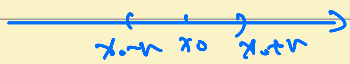


1. $|x-y| \geq 0$, $|x-y| = 0$ iff $x = y$
2. (Symmetric, 对称的) $|x-y| = |y-x|$
3. (Triangle inequality) $|x-y| \leq |x-z| + |y-z|$



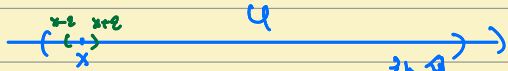
1.5 Open, closed and bounded set

Defn: Given $x_0 \in \mathbb{R}$, an open interval centered at x_0 is a set of $\mathbb{R}$ of the form $(x_0 - r, x_0 + r)$ for some $r > 0$



$(x_0 - r, x_0 + r) = \{x \in \mathbb{R} \mid |x - x_0| < r\}$
$= \{x \in \mathbb{R} \mid d(x, x_0) < r\}$

Defn: (open set) A set $U \subseteq \mathbb{R}$ is an open set if for any $x \in U$, there exists $\varepsilon > 0$, such that $(x-\varepsilon, x+\varepsilon) \subseteq U$



补集

Defn: (closed set) A set $F \subseteq \mathbb{R}$ is closed set, if the complement set F^c is an open set

Fact: A set $F \subseteq \mathbb{R}$ is a closed set if it contains all its boundary points

E.g. $\mathbb{R}$ is an open set? Yes
$\emptyset$ is an open set? Yes

$\mathbb{R}$ and $\emptyset$ are both open and closed sets

Defn: (Bounded set) A set $Y \subseteq \mathbb{R}$ is a bounded set if there exists $k \in \mathbb{R}$, such that

$$|x| \le k \quad \forall x \in Y$$

Thm:

① The union of an arbitrary (任意的) collection of open sets is an open set

② The intersection of a finite number of open sets is an open set

infinite: counter example $U_i = (\frac{1}{i}, 2)$ $i \to +\infty$ $\bigcap U_i = [0,2)$

Proof: ① Let $\{A_i\}_{i\in I}$ be a collection of open sets. Given $x \in \bigcup_{i\in I} A_i$

$\Rightarrow$ there exists $i_0 \in I$ such that $x \in A_{i_0}$ Since A_{i_0} is open,

there exists $\varepsilon > 0$, such that $(x-\varepsilon, x+\varepsilon) \subseteq A_{i_0} \subseteq \bigcup_{i\in I} A_i$

$\therefore \bigcup_{i\in I} A_i$ is open

②. $x \in \bigcap_{i\in I} A_i$ then $x \in A_i$ for all $i \in I$.

$\Rightarrow$ there exists $i \in I$ such that $x \in A_i$, for all $i \in I$

$\exists\, \varepsilon_1 > 0 \quad (x-\varepsilon_1, x+\varepsilon_1) \subseteq A_1$

$\exists\, \varepsilon_2 > 0 \quad (x-\varepsilon_2, x+\varepsilon_2) \subseteq A_2$

$\vdots$

$\exists\, \varepsilon_N > 0 \quad (x-\varepsilon_N, x+\varepsilon_N) \subseteq A_2$

$\varepsilon := \min(\varepsilon_1, \dots \varepsilon_N) > 0$

$(x-\varepsilon, x+\varepsilon) \subseteq A_i \quad \forall i \in I$

$$\Rightarrow (x-\varepsilon, x+\varepsilon) \subseteq \bigcap_{i\in I} A_i$$

$\therefore \bigcap_{i\in I} A_i$ is an open set

Thm: ① The intersection of an arbitrary collection of closed sets is a closed set.

② The union of a finite number of closed sets is a closed set.