Real field

① **Defn** (ε-δ definition) Suppose that $x_0 \in (a,b)$ and $(a,b)/\{x_0\} \subset \mathrm{dom} f$. We have $\lim_{x\to x_0} f(x) = A$ if and only if for any $\varepsilon>0$, there exists $\delta>0$, such that

$$|f(x)-A|<\varepsilon \quad \text{whenever } 0<|x-x_0|<\delta.$$

② **Defn** (x tends to ∞) Suppose that the domain of f contains $(\alpha,+\infty)$ for some $\alpha\in\mathbb{R}$. We have $\lim_{x\to\infty} f(x) = A$ if and only if for any $\varepsilon>0$, there exists $K>\alpha$, such that

$$|f(x)-A|<\varepsilon \quad \text{whenever } x>K.$$

Similarly, we can define $\lim_{x\to-\infty} f(x) = A$. $\lim_{|x|\to\infty} f(x) = A$

Thm (Sandwich theorem) Suppose that $0<|x-x_0|<\varepsilon$, if

$$f(x)\le h(x)\le g(x)$$

and $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = A$, then we have

$$\lim_{x\to x_0} h(x) = A$$

Thm (Two important limits)

$$① \lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad ② \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e$$

已知 $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$

Proof: Let $x\ge 1$, then

$$\left(1+\frac{1}{[x]+1}\right)^{[x]} \le \left(1+\frac{1}{x}\right)^x \le \left(1+\frac{1}{[x]}\right)^{[x]+1}$$

$$\Rightarrow \left(1+\frac{1}{[x]+1}\right)^{[x]+1}\left(1+\frac{1}{[x]+1}\right)^{-1} \le \left(1+\frac{1}{x}\right)^x \le \left(1+\frac{1}{[x]}\right)^{[x]}\left(1+\frac{1}{[x]}\right)$$

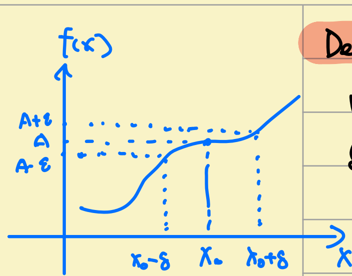
$$\lim_{[x]+1\to\infty}\left(1+\frac{1}{[x]+1}\right)^{[x]+1}\left(1+\frac{1}{[x]+1}\right)^{-1} \le \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \le \lim_{[x]\to\infty}\left(1+\frac{1}{[x]}\right)^{[x]}\left(1+\frac{1}{[x]}\right)$$

$$e\cdot 1 \le \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \le e\cdot 1$$

$$\Rightarrow \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

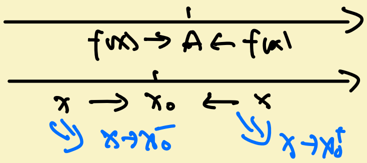
$\lim_{x \to \infty} f(x)$

$\lim_{x \to x^0} f(x)$



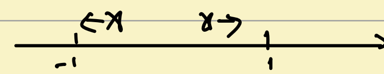
Defn (One-sided limits) Suppose that $(x_0-d, x_0) \subset \text{dom} f$ for some $d>0$. We have $\lim_{x \to x_0^-} f(x) = A$ if and only if for any $\varepsilon>0$, there exists $\delta>0$ and $\delta<d$, such that

$$|f(x)-A|<\varepsilon \quad \text{whenever} \quad x \in (x_0-\varepsilon, x_0)$$



Defn (One-sided limits) Suppose that $(x_0, x_0+d) \subset \text{dom} f$ for some $d>0$. We have $\lim_{x \to x_0^+} f(x) = A$ if and only if for any $\varepsilon>0$, there exists $\delta>0$ and $\delta<d$, such that

$$|f(x)-A|<\varepsilon \quad \text{whenever} \quad x \in (x_0, x_0+\delta)$$



E.g.: $f(x) = \sqrt{1-x^2}$ at $x = \pm 1$

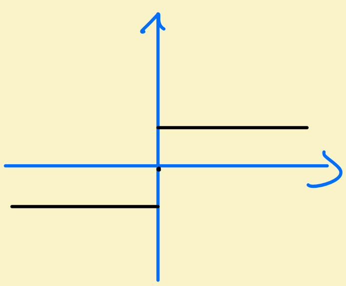
$\text{dom} f = [-1, 1]$ since $|x| \le 1$, we have $1-x^2 = (1+x)(1-x) \le 2(1-x)$

for any $0<\varepsilon<1$, when $2(1-x)<\varepsilon^2$ we have $\sqrt{1-x^2}<\varepsilon$.

Take $\delta = \frac{\varepsilon^2}{2}$, then $0<1-x<\delta \iff x \in (1-\delta, 1)$ we have $\sqrt{1-x^2}<\varepsilon$

that is $\lim_{x \to 1^-} \sqrt{1-x^2} = 0$.

Thm $\lim_{x \to x_0} f(x) = A$ if and only if $\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = A$



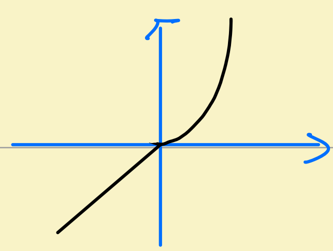
E.g. $f(x) = \text{sgn}(x) = \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$

$\lim_{x \to 0^-} f(x) = -1 \quad x \to 0^-$

$\iff x \in (-\varepsilon, 0)$

$\lim_{x \to 0^+} f(x) = 1 \quad x \to 0^+$

since $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$ the limit $\lim_{x \to 0} f(x)$ does not exist.



E.g. $f(x) = \begin{cases} x^2 & x \ge 0 \\ x & x<0 \end{cases}$ verify $\lim_{x \to 0} f(x)$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x = 0 \qquad \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 = 0$

Since $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 0$ we have $\lim_{x \to 0} f(x) = 0$

Properties of limits.

六种极限的类型：

$\lim_{x\to x_0} f(x)$ $\lim_{x\to\infty} f(x)$ $\lim_{x\to-\infty} f(x)$ $\lim_{|x|\to\infty} f(x)$

$\lim_{x\to x_0^-} f(x)$ $\lim_{x\to x_0^+} f(x)$

① Uniqueness (唯一性) If $\lim f(x)=a$ and $\lim f(x)=b$, then $a=b$

② Boundedness (有界性) If $\lim_{x\to x_0} f(x)$ exists then $f(x)$ is bounded at $0<|x-x_0|$

$\lim_{x\to\infty} f(x)$ exists then $f(x)$ is bounded at $x>M$

⋮

③ 保号性 If $\lim_{x\to x_0} f(x)=A>0$, then for any $0<r<A$, we have $f(x)>r>0$ whenever $0<|x-x_0|<\delta$.

④ 保不等式性 If $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} g(x)$ and $f(x)\le g(x)$ whenever $0<|x-x_0|<\delta$, then

$$\lim_{x\to x_0} f(x) \le \lim_{x\to x_0} g(x)$$

Proof: ① Suppose $\lim f(x)=a$, $\lim f(x)=b$

$\forall \varepsilon>0$, $\exists\ \delta_1>0$, such that $0<|x-x_0|<\delta \Rightarrow |f(x)-a|<\frac{\varepsilon}{2}$

$\forall \varepsilon>0$, $\exists\ \delta_2>0$, such that $0<|x-x_0|<\delta \Rightarrow |f(x)-b|<\frac{\varepsilon}{2}$

$$|a-b| = |(f(x)-a)-(f(x)-b)|$$
$$\le |f(x)-a|+|f(x)-b| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

then we have $a=b$.

② $0<|x-x_0|<\varepsilon \Rightarrow |f(x)-a|<\varepsilon$

$\Rightarrow |f(x)| = |f(x)-a+a| \le |f(x)-a|+|a| < |a|+\varepsilon$.

Hence $f(x)$ is bounded on $(x_0-\varepsilon, x_0)\cup(x_0, x_0+\varepsilon)$

③ Suppose $A>0$, for any $0<r<A$ take $\varepsilon=A-r$, then there exists $\delta>0$, $0<|x-x_0|<\delta \Rightarrow |f(x)-A|<\varepsilon \Rightarrow f(x)>A-\varepsilon=r$

④. Suppose $f(x) \le g(x)$ holds on $(x_0-\delta, x_0)\cup(x_0, x_0+\delta)$

$\forall \varepsilon>0$, $\exists \delta_1>0$, such that $0<|x-x_0|<\delta_1$

$\Rightarrow |f(x)-A|<\varepsilon \Rightarrow f(x)>A-\varepsilon$

$\forall \varepsilon>0$, $\exists \delta_2>0$, such that $0<|x-x_0|<\delta_2$

$\Rightarrow |g(x)-B|<\varepsilon \Rightarrow g(x)<B+\varepsilon$

Take $\delta=\min(\delta_1,\delta_2)$, when $0<|x-x_0|<\delta$

$$A-\varepsilon<f(x)\le g(x)<B+\varepsilon$$

$$\Rightarrow A<B+2\varepsilon \Rightarrow A\le B$$

Prop: a) $\lim(f(x))^n = (\lim f(x))^n$

b) $\lim c = c$

c) $\lim\limits_{x\to a} x = a$

d) $\lim\limits_{x\to a} x^n = a^n$

e) $\lim\limits_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$

f) $\lim \sqrt[n]{f(x)} = \sqrt[n]{\lim f(x)}$

Prop: (Arithmetic operations)

a) $\lim cf(x) = c\cdot\lim f(x)$

b) $\lim(f(x)\pm g(x)) = \lim f(x) \pm \lim g(x)$

c) $\lim f(x)\cdot g(x) = \lim f(x)\cdot\lim g(x)$

d) $\lim \dfrac{f(x)}{g(x)} = \dfrac{\lim f(x)}{\lim f(x)}$ given $g(x)\ne 0$ and $\lim g(x)\ne 0$.

Thm Suppose $\lim\limits_{y\to y_0} f(y)=A$, $\lim\limits_{x\to x_0} g(x)=y_0$, then $\lim\limits_{x\to x_0} f(g(x))=A$

Proof: $\forall \varepsilon_1>0$, $\exists \delta_1>0$, such that $0<|y-y_0|<\delta_1 \Rightarrow |f(y)-A|<\varepsilon_1$

$\forall \varepsilon_2>0$, $\exists \delta_2>0$, such that $0<|x-x_0|<\delta_2 \Rightarrow |g(x)-y_0|<\varepsilon_2$

Take $\delta_1=\varepsilon_2$

$\Rightarrow \forall \varepsilon_1>0$, $\exists \delta_2>0$, such that $0<|x-x_0|<\delta_2$

$\Rightarrow |f(g(x))-A|<\varepsilon_1$

E.g.: $\lim\limits_{x\to -2}\dfrac{x^3+2x^2-1}{5-3x} = \dfrac{(-2)^3+2(-2)^2-1}{5-3(-2)} = -\dfrac{1}{11}$

E.g.: $\lim\limits_{x\to 1}\frac{x^2-1}{x-1} = \lim\limits_{x\to 1}\frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = \lim\limits_{x\to 1} x+1 = 2$

E.g.: $\lim\limits_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2} = \lim\limits_{x\to 0}\frac{(\sqrt{x^2+9}-3)(\sqrt{x^2+9}+3)}{x^2(\sqrt{x^2+9}+3)} = \lim\limits_{x\to 0}\frac{x^2}{x^2}\frac{1}{\sqrt{x^2+9}+3} = \frac{1}{6}$

E.g.: $\lim\limits_{x\to\infty}\frac{3x^2+2x-1}{x^2-2} = \lim\limits_{x\to\infty}\frac{3+\cancel{\frac{2}{x}}-\cancel{\frac{1}{x^2}}}{1-\cancel{\frac{2}{x^2}}} = 3$

E.g.: Suppose $\lim\limits_{x\to x_0} f(x)=A$, prove $\lim\limits_{h\to 0} f(x_0+h)=A$

Let $x=x_0+h$, then $h=x-x_0$, $x\to x_0$ as $h\to 0$

$\lim\limits_{x\to x_0} f(x)=A \iff \lim\limits_{h\to 0} f(x_0+h)=A$

$\lim\limits_{x\to\infty}(1+\frac{1}{x})^x=e$

E.g.: Prove $\lim\limits_{x\to-\infty}(1+\frac{1}{x})^x=e$

Let $y=-(x+1)$, then $x=-y-1$, $x\to-\infty$ as $y\to\infty$

$\lim\limits_{x\to-\infty}(1+\frac{1}{x})^x = \lim\limits_{y\to\infty}\left(1-\frac{1}{y+1}\right)^{-(y+1)} = \lim\limits_{y\to\infty}\left(\left(\frac{y}{y+1}\right)\right)^{-(y+1)}$

$= \lim\limits_{y\to\infty}\left(\frac{y+1}{y}\right)^{y+1} = \lim\limits_{y\to\infty}\left(1+\frac{1}{y}\right)^{y+1} = \lim\limits_{y\to\infty}(1+\frac{1}{y})^y(1+\frac{1}{y}) = e$

E.g. Prove $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e$

Let $y=\frac{1}{x}$, then $x=\frac{1}{y}$, $x\to 0$ as $y\to\pm\infty$

$\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = \lim\limits_{|y|\to\infty}(1+\frac{1}{y})^y = e$

Linkage b/w limit of function and limit of sequence →

Thm (Heine's theorem) $\lim\limits_{x\to x_0} f(x)=A$ if and only if for any sequence $\{x_n\}\subset \operatorname{dom} f$ satisfying $x_n\neq x_0$, $x_n\to x_0$ as $n\to\infty$, we have $\lim\limits_{n\to\infty} f(x_n)=A$

Proof: (⇒) Suppose $\lim\limits_{x\to x_0} f(x)=A$, then for any $\varepsilon>0$, $\exists\ \delta>0$ such that $0<|x-x_0|<\delta$, we have $|f(x)-A|<\varepsilon$.

Let $\{x_n\}\subset(x_0-\delta,x_0)\cup(x_0,x_0+\delta)$ and $\lim\limits_{n\to\infty}x_n=x_0$, then for the $\delta>0$

mention above, there exist $N \in \mathbb{N}$ such that $0<|x_n - x_0| < \delta$, therefore

$|f(x_n) - A| < \varepsilon$ that is $\lim_{n\to\infty} f(x_n) = A$.

($\Leftarrow$) Suppose $\{x_n\} \subset (x_0-\delta, x_0) \cup (x_0, x_0+\delta)$ and $\lim_{n\to\infty} x_n = x_0$ we have

$\lim_{n\to\infty} f(x_n) = A$. Prove by contradiction. suppose that $\lim_{x\to x_0} f(x) \neq A$.

then $\exists\, \varepsilon_0 > 0$, $\forall\, \delta > 0$, $|f(x) - A| \geq \varepsilon_0$ whenever $0 < |x - x_0| < \delta$.

Take $\delta' = \delta, \frac{\delta}{2}, \frac{\delta}{3} \cdots \frac{\delta}{n}$ then there exists corresponding $x_1, x_2, x_3, \cdots x_n$

such that $0 < |x_n - x| < \frac{\delta}{n}$ but $|f(x_n) - A| \geq \varepsilon_0$.

But $\{x_n\} \subset (x_0 - \delta, x_0) \cup (x_0, x_0+\delta)$ and $\lim_{n\to\infty} x_n = x_0$ but $|f(x_n) - A| \geq \varepsilon_0$

contradiction. therefore $\lim_{x\to x_0} f(x) = A$.

Thm (Heine's theorem, alternative)

$$\lim_{x\to x_0} f(x) = A \iff \forall\, x_n \to x_0 \text{ we have } \lim_{n\to\infty} f(x_n) = A.$$

E.g. Prove $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist.

Proof: Let $x_n = \frac{1}{n\pi}$, $y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$ then we have

$x_n \to 0$ as $n \to +\infty$, $y_n \to 0$ as $n \to \infty$.

$\sin\frac{1}{x_n} = \sin n\pi = 0$ but $\sin\frac{1}{y_n} = \sin\left(2n\pi + \frac{\pi}{2}\right) = 1$ as $n \to \infty$.

Therefore $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist.

Infinitesimal quantity (无穷小量)

Defn If $\lim_{x\to x_0} f(x) = 0$, then we say f is an infinitesimal quantity.
as $x \to x_0$.

E.g.: $\lim_{x\to 0} x^2 = 0$ $\quad \lim_{x\to 0} \sin x = 0$ $\quad \lim_{x\to\infty} \frac{1}{x} = 0$

Properties of infinitesimal quantity

① if $\lim_{x\to x_0} f(x) = 0$, $\lim_{x\to x_0} g(x) = 0$ $\Rightarrow$ $\lim_{x\to x_0} (f(x) \pm g(x)) = 0$

$\Rightarrow$ $\lim_{x\to x_0} f(x)\cdot g(x) = 0$

② if $\lim_{x\to x_0} f(x) = 0$, $\lim_{x\to x_0} g(x) = A$, $|A| < +\infty$, $\Rightarrow$ $\lim_{x\to x_0} f(x)\cdot g(x) = 0$

E.g. $\lim_{x\to 0} x \sin\frac{1}{x} = 0$

③ If $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0$, and $\lim_{x\to x_0} f(x) = 0$, $\lim_{x\to x_0} g(x) = 0$, then we say f is a higher order infinitesemal quantity than g. or, say.

$$f(x) = o(g(x)) \quad \text{as } x\to x_0.$$

E.g. $\lim_{x\to 0} \frac{x^2}{x} = 0$ $\qquad$ $\lim_{x\to 0} \frac{1-\cos x}{\sin x} = \lim_{x\to 0} \tan\frac{x}{2} = 0$

④ If $\lim_{x\to x_0} f(x) = 0$, $\lim_{x\to x_0} g(x) = 0$, and $\lim_{x\to x_0} \frac{f(x)}{g(x)} = c \neq 0$. then we say f and g have the same order of infinitesemal quantity, or, say

$$f(x) = O(g(x)) \quad \text{as } x\to x_0.$$

E.g. $\lim_{x\to 0} \frac{\sin x}{x} = 1$ $\qquad$ $\lim_{x\to 0} \frac{x^2}{1-\cos x} = 2$

等价无穷小，

In particular if $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$, we say f and g are equivalent infinitesemal quantity. or, say.

$$f(x) \sim g(x) \quad \text{as } x\to x_0$$