Real Analysis: Answer to Exercise 1

Due date: 2024-11-04

Question 1.
- (a) Yes, $l \in A$.
- (b) No, $10 \notin A$.
- (c) Yes since $A \subseteq B$ and $B \subseteq A$.
- (d) No since $apple \in A$ but $apple \notin C$.
- (e) Yes since $C = \{1, 2\} \subseteq \{1, 2, apple\} = A$.
- (f) Yes since $C \subseteq A$ and $apple \in A \setminus C$.
- (g) No since $apple \in A$ but $apple \notin C$.
- (h) No since $pear \in D$ but $pear \notin B$.

Question 2.
- (a) $A \cap (B \cup C) = \{1, 2, 3\} \cap \{2, 3, fish, fowl, 7, 8\} = \{2, 3\}$.
- (b) $(A \cap B) \cup (A \cap C) = \{2, 3\} \cup \{2\} = \{2, 3\}$.
- (c) $A \cup (B \cap C) = \{1, 2, 3\} \cup \{2\} = \{1, 2, 3\}$.
- (d) $(A \cup B) \cap (A \cup C) = \{1, 2, 3, fish\} \cap \{1, 2, 3, fowl, 7, 8\} = \{1, 2, 3\}$.
- (e) $C^c = U \backslash C = \{1, 3, 4, 5, 6, fish\}$.
- (f) $(C^c)^c = C = \{2, fowl, 7, 8\}$.
- (g) $(A \cap C)^c = \{2\}^c = \{1, 3, 4, 5, 6, 7, 8, fish, fowl\}$.
- (h) $A^c \cup C^c = \{4, 5, 6, 7, 8, fish, fowl\} \cup \{1, 3, 4, 5, 6, fish\} = \{1, 3, 4, 5, 6, 7, 8, fish, fowl\}$.
- (i) $(A \cup B)^c = \{1, 2, 3, fish\}^c = \{4, 5, 6, 7, 8, fowl\}$.
- (j) $A^c \cap B^c = \{4, 5, 6, 7, 8, fish, fowl\} \cap \{1, 4, 5, 6, 7, 8, fowl\} = \{4, 5, 6, 7, 8, fowl\}$.

Question 3.
- (a) $(1, 3) \cup (2, 15) = (1, 15)$.
- (b) $[1, 8] \cap [4, 16] = [4, 8]$.
- (c) $[66, 76] - [72, 100] = [66, 72)$.
- (d) $[0, \infty) \cup (-10, 10) = (-10, \infty)$.
- (e) $[27, 29] - (26, 28) = [28, 29]$.

Question 4. To denote the number of elements in a finite set e.g. A, we use the notation $|A|$. It follows that $|X \times Y| = |X| \cdot |Y| = mn$ i.e. there are mn elements in $X \times Y$.

Question 5. (a) There are infinitely many valid answers to this question... here is one of them. Consider splitting the integers into the even integers, and, the odd integers. By setting $A = \{x \in \mathbb{Z} : x/2 \in \mathbb{Z}\} \cup \{0\}$ and $B = \{x \in \mathbb{Z} : (x+1)/2 \in \mathbb{Z}\}$, it follows that:

- A and B have infinitely many elements;
- $A \cap B = \{0\}$; and
- $A \cup B = \mathbb{Z}$, as required.

(b) Assume sets C and D exist. Then since $s \in C \cap D$ it follows that $s \in C$. Since $s \in C$, it follows that $s \in C \cup D$, which is a contradiction (since $s \notin C \cup D = \{b, i, g\}$). Therefore, such sets do not exist.

Question 6. Let $f, g : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = x^2, g(x) = \sin x \quad \forall x \in \mathbb{R}.$$

Then

$$f \circ g(x) = \sin^2(x) \neq \sin(x^2) = g \circ f(x).$$

The functions f and g meet the criteria of the question, as required.

Question 7. A function $f : X \to Y$ is injective if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for all $x_1, x_2 \in X$. Let $f(x_1) = f(x_2)$ for $x_1, x_2 \in X$. Via (1) it follows that $x_1 = g(f(x_1)) = g(f(x_2)) = x_2$. We conclude that f is injective, as required.

A function $g : Y \to X$ is surjective if for all $x \in X$ there exists $y \in Y$ such that $g(y) = x$. Let $x \in X$. Via (1), it follows that $g(f(x)) = x$. Setting $y = f(x)$ it follows that $g(y) = x$. Therefore g is surjective, as required.

Question 8. It is possible. The following example highlights the general idea. Let $f : \{1, 2\} \to \{1, 2, 3\}$ and $g : \{1, 2, 3\} \to \{1, 2\}$ be given by

$$f(x) = \begin{cases} 1, & x = 1, \\ 2, & x = 2, \end{cases} \quad g(x) = \begin{cases} 1, & x = 1, \\ 2, & x = 2, \\ 2, & x = 3. \end{cases}$$

Thus, f is not surjective since $f(\{1, 2\}) = \{1, 2\} \neq \{1, 2, 3\}$. Moreover, g is not injective since $g(2) = g(3) = 2$. Therefore neither f or g is bijective. However, $g \circ f : \{1, 2\} \to \{1, 2\}$ is given by

$$g \circ f(x) = \begin{cases} 1, & x = 1, \\ 2, & x = 2. \end{cases}$$

We observe that $g \circ f$ is bijective, as required.