Real Analysis: Exercise 2

Due date: 2025-11-18

Question 1. 1. Give an example that the intersection of infinitely many open sets is not an open set.

2. Give an example that the union of infinitely many closed sets is not a closed set.

Question 2. What can you say about x and a if you are told that, for every $\epsilon > 0$, $|x - a| < \epsilon$? What if you are told that, for every positive integer n, $|x - a| < 1/n$?

Question 3. Determine the following limits and prove that your answer is correct by directly appealing to the definition of the limit.

1. $\lim_{x\to\infty} \dfrac{3x^3 - 5x^2 + 7x - 13}{2x^3 - \pi}$.

2. $\lim_{x\to 1} 2x^2 - 3x + 5$.

3. $\lim_{x\to 2} 1/(1 - x)$.

4. $\lim_{x\to\infty} (1/x)\sin x$.

5. $\lim_{x\to 0} x\sin(1/x)$.

Question 4. Let f and g be real-valued functions of a real variable. For each of the following statements, either prove it is true using the definition of the limit or give a counterexample to show that it is false.

1. Suppose that $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to\infty} g(x) = b$. If $f(x) < g(x)$ for all $x \in \mathbb{R}$, then $a < b$.

2. If $\lim_{x\to a} f(x) = \ell$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} f(x)g(x) = \infty$.

3. If $\lim_{x\to a_-} f(x) = \lim_{x\to a_+} f(x) = \ell$, then $f(a) = \ell$.

4. If $\lim_{x\to b} f(x) = c$ and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f\big(g(x)\big) = c$.

Question 5. Suppose now that $b < a < c$ and that f is a real valued function defined for all $x \in (b, c)$. Suppose also that ℓ is a real number. What can you say about f if the following hold? (It might help to think about the following property; a function is said to be *bounded* on the interval $[c, d]$ if there is a number $K > 0$ (called a bound) such that $-K \leq f(x) \leq K$ for all $c \leq x \leq d$.)

1. for all $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - \ell| < \epsilon$ whenever $0 < |x - a| < \delta$.

2. for all $\delta > 0$, there exists $\epsilon > 0$ such that $|f(x) - \ell| < \epsilon$ whenever $0 < |x - a| < \delta$.

3. there exists $\epsilon > 0$, such that for all $\delta > 0$, $|f(x) - \ell| < \epsilon$ whenever $0 < |x - a| < \delta$.

4. there exists $\delta > 0$ such that for all $\epsilon > 0$, $|f(x) - \ell| < \epsilon$