Real Analysis: Exercise 4

Due date: 2024-12-16

Question 1. Let f be a real-valued function, continuous on $[0, 1]$, with the following property: For every real y, either there is no x in $[0, 1]$ for which $f(x) = y$ or there is exactly one such x. Prove that f is strictly monotonic (i.e. strictly increasing or strictly decreasing) on $[0, 1]$.

Question 2. A number x such that $f(x) = x$ is called a fixed point of f. Suppose $f : [0, 1] \to [0, 1]$ is continuous. Show that f has a fixed point.

Question 3. Suppose that the function $f : [0, 1] \to [0, 1]$ is continuous. Use the intermediate Value Theorem to prove that there exists $c \in [0, 1]$ such that

$$f(c) = c(2 - c^2)\,.$$

Question 4. Suppose for $f : [0,1] \to \mathbb{R}$ we have $|f(x) - f(y)| \leq K|x-y|$ for all $x, y \in [0,1]$, and $f(0) = f(1) = 0$. Prove that $|f(x)| \leq \frac{K}{2}$ for all $x \in [0,1]$.

Note: A function $f : X \to \mathbb{R}$ is called Lipschitz continuous if there exists a $K > 0$ such that

$$|f(x) - f(y)| \leq K|x-y| \quad \text{for all } x \text{ and } y \text{ in } X\,.$$

Question 5. Suppose that f is continuous on $[a,b]$, and for each $x \in [a,b]$, there is a $y \in [a,b]$ such that $|f(y)| \leq \frac{1}{2}|f(x)|$. Show that there is a $c \in [a,b]$ such that $f(c) = 0$.

Question 6. Assume that f is uniformly continuous on a bounded interval (a,b). Prove that f must be bounded on (a,b). Give an example that for unbounded interval, uniformly continuous function may not be bounded.

Question 7. Assume that f is continuous on $(-\infty, \infty)$ and

$$\lim_{x\to\infty} f(x) = A \quad \text{and} \quad \lim_{x\to-\infty} f(x) = B\,,$$

here A, B are finite numbers. Show that f is uniformly continuous on $(-\infty, \infty)$.