

Differentiation 微分 derivative 导数

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$(1) (f \pm g)' = f' \pm g'$$

$$(2) (c \cdot f)' = c \cdot f'$$

$$(3) (f \cdot g)' = f' \cdot g + f \cdot g'$$

$$(4) \left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2} \quad (g \neq 0)$$

**Lemma** Suppose that  $f$  is differentiable at point  $x_0$ , if and only if there exists  $F(x)$ , which is continuous at  $x_0$ , such that

$$f(x) - f(x_0) = F(x)(x - x_0).$$

$$\text{and } F(x) = \begin{cases} \frac{f(x) - f(x_0)}{x - x_0} & x \in (x_0 - \varepsilon, x_0) \cup (x_0, x_0 + \varepsilon) \\ f'(x_0) & x = x_0 \end{cases}$$

$$\text{Proof: } (\Rightarrow) \text{ Since } \lim_{x \rightarrow x_0} F(x) = f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = F\left(\lim_{x \rightarrow x_0} x\right)$$

we have that  $F(x)$  is continuous at  $x_0$ .

( $\Leftarrow$ ) Suppose that  $F(x)$  is continuous at  $x_0$ , and we have

$$f(x) - f(x_0) = F(x)(x - x_0).$$

$$\text{Since the limit } \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} F(x) = F(x_0)$$

exists, we conclude that  $f$  is differentiable at  $x_0$ .

**Thm (Chain rule)** Suppose that a function  $f$  is differentiable at  $x_0$ , and a function  $g$  is differentiable at  $y_0 = f(x_0)$ . Then

$$\begin{aligned} (g \circ f)'(x_0) &= g'(y_0) \cdot f'(x_0) \\ &= g'(f(x_0)) \cdot f'(x_0) \end{aligned}$$

Proof: Since  $g$  is differentiable at  $y_0$ , we have

$$g(y) = g(y_0) + G(y)(y - y_0)$$

and  $G$  is continuous function, and  $G(y_0) = g'(y_0)$  Let  $y = f(x_0 + \Delta x)$

$$g(f(x_0 + \Delta x)) = g(f(x_0)) + G(f(x_0 + \Delta x))(f(x_0 + \Delta x) - f(x_0)) \quad y_0 = f(x_0)$$

$$g(f(x_0 + \Delta x)) - g(f(x_0)) = G(f(x_0 + \Delta x))(f(x_0 + \Delta x) - f(x_0))$$

$$\frac{g(f(x_0 + \Delta x)) - g(f(x_0))}{\Delta x} = G(f(x_0 + \Delta x)) \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Since  $f$  is differentiable at  $x_0$ , and  $g$  is differentiable at  $y_0$ .

$$\lim_{\Delta x \rightarrow 0} \frac{g(f(x_0 + \Delta x)) - g(f(x_0))}{\Delta x} = \lim_{\Delta x \rightarrow 0} G(f(x_0 + \Delta x)) \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

$$(g \circ f)'(x_0) = G(f(x_0)) \cdot f'(x_0)$$

$$= g'(y_0) \cdot f'(x_0)$$

Note:  $(g \circ f)'(x) = \frac{dg}{dx} = \frac{df}{dx} \cdot \frac{dg}{df}$

$$(h \circ g \circ f)'(x) = \frac{dh}{dx} = \frac{df}{dx} \cdot \frac{dg}{df} \cdot \frac{dh}{dg}$$

E.g.:  $g(x) = a^x$   $g'(x) = a^x \cdot \ln a$

$$= \exp(\ln(a^x)) = \exp(x \cdot \ln a) \quad \text{Let } f(x) = x \cdot \ln a$$

$$(a^x)' = (g \circ f)'(x) = g'(y) \cdot f'(x) \quad g(y) = \exp(y)$$

$$= \exp(y) \cdot \ln a = \exp(x \cdot \ln a) \cdot \ln a$$

$$= a^x \cdot \ln a$$

E.x.:  $y = \sin x^2$

$$y = \frac{1}{3\sqrt{x^2 + x + 1}}$$

$$y = (\sin x)^2$$

$$y = (2x+1)^5 (x^3 - 2x+3)^3$$

$$\textcircled{1} \quad y = \sin x^2 \quad f(x) = x^2 \quad g(f) = \sin f \\ y' = \cos f \cdot 2x = \cos x^2 \cdot 2x$$

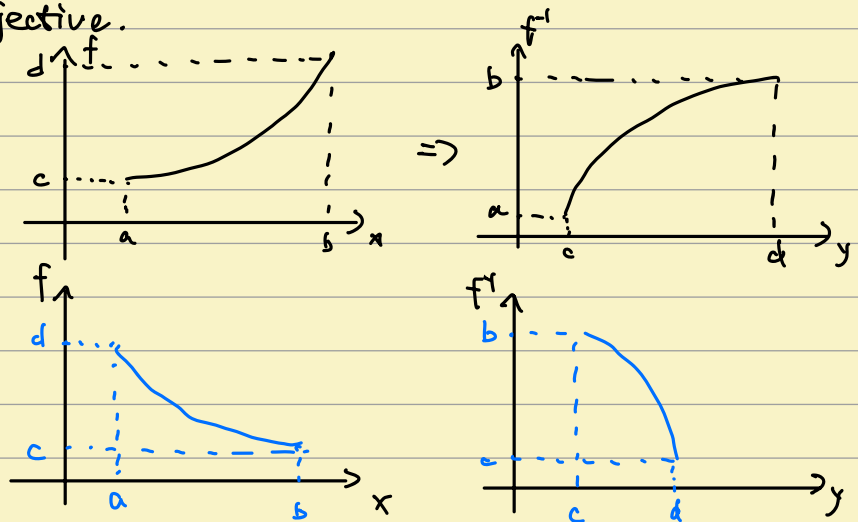
$$\textcircled{2} \quad y = \sin^2 x \quad f(x) = \sin x \quad g(f) = f^2 \\ y' = 2f \cdot \cos x = 2 \sin x \cdot \cos x$$

$$\textcircled{3} \quad y = \frac{1}{\sqrt[3]{x^2+x+1}} \quad f(x) = x^2+x+1 \quad g(f) = \frac{1}{\sqrt[3]{f}} = f^{-\frac{1}{3}} \\ y' = -\frac{1}{3} f^{-\frac{4}{3}} \cdot (2x+1) = -\frac{1}{3} (x^2+x+1)^{-\frac{4}{3}} (2x+1)$$

$$\textcircled{4} \quad y = (2x+1)^5 (x^3-2x+3)^3 \quad f_1(x) = 2x+1 \quad f_2(x) = x^3-2x+3 \\ g_1(f) = f^5 \quad g_2(f) = f^3 \\ y_1 = (2x+1)^5 \quad y_1' = 5f_1^4 \cdot 2 = 10(2x+1)^4 \\ y_2 = (x^3-2x+3)^3 \quad y_2' = 3f_2^2 \cdot (3x^2-2) = 3(x^3-2x+3)^2 (3x^2-2)$$

$$y' = y_1' \cdot y_2 + y_1 \cdot y_2' = 10(2x+1)^4 \cdot (x^3-2x+3)^3 + 3(2x+1)^5 (x^3-2x+3)^2 (3x^2-2)$$

**Thm.** Suppose that  $f: [a, b] \rightarrow [c, d]$  is continuous and strictly <sup>decreasing</sup> increasing, so that  $f(a) = c$ ,  $f(b) = d$ . Then the inverse function  $f^{-1}: [c, d] \rightarrow [a, b]$  exists, and is continuous, strictly <sup>decreasing</sup> increasing, and surjective.



Thm: Suppose  $f: [a, b] \rightarrow [c, d]$  is continuous and strictly increasing. Let  $y_0 = f(x_0)$ . If  $f$  is differentiable at  $x_0$ , and  $f'(x_0) \neq 0$ , then  $f^{-1}$  is differentiable at  $y_0$ , and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} \quad \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

Proof: Since  $f$  is continuous and strictly increasing, then  $f^{-1}$  is also continuous and strictly increasing. Hence, we can write

$$y = f(x), \quad y_0 = f(x_0) \quad \text{and then} \quad x = f^{-1}(y) \quad \text{and} \quad x_0 = f^{-1}(y_0)$$

Since  $f(x)$  is differentiable at  $x_0$ ,

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

$$\Rightarrow \lim_{x \rightarrow x_0} \frac{y - y_0}{f^{-1}(y) - f^{-1}(y_0)} = f'(x_0)$$

$$\Rightarrow \lim_{y \rightarrow y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{f'(x_0)}$$

Since  $f$  is continuous at  $x_0$ ,  $f(x) \rightarrow f(x_0)$  as  $x \rightarrow x_0$  that is  $y \rightarrow y_0$  as  $x \rightarrow x_0$ . Hence

$$\lim_{y \rightarrow y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{f'(x_0)}$$

$$\Rightarrow [f^{-1}(y_0)]' = \frac{1}{f'(x_0)}$$

E.g.:  $y = \sin x \quad x = \arcsin y$

let  $y = f(x) = \sin x \quad x = g(y) = \arcsin y$ .

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{\cos x} = \frac{1}{\sqrt{\cos^2 x}} = \frac{1}{\sqrt{1 - \sin^2 x}} = \frac{1}{\sqrt{1 - y^2}}$$

E.x.:  $x = \arctan y \quad y = \tan x \quad (\tan x)' = \sec^2 x = \frac{1}{\cos^2 x}$

$$= \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \tan^2 x + 1$$

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{\sec^2 x} = \cos^2 x = \frac{\cos^2 x}{1}$$

$$= \frac{\cos^2 x}{\sin^2 x + \cos^2 x} = \frac{1}{\tan^2 x + 1} = \frac{1}{y^2 + 1}$$

$$\textcircled{1} (f \pm g)' = f' \pm g'$$

$$\textcircled{2} (c \cdot f)' = c \cdot f'$$

$$\textcircled{3} (f \cdot g)' = f' \cdot g + f \cdot g' \quad \textcircled{4} \left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2} \quad (g \neq 0)$$

$$\textcircled{5} (g \circ f)'(x) = g'(f) \cdot f'(x) \quad \textcircled{6} g'(y) = \frac{1}{f'(x)}$$

E.g.: Let  $y = \frac{(x+5)^2 (x-4)^{\frac{1}{3}}}{(x+2)^5 (x+4)^{\frac{1}{2}}}$  ( $x > 4$ ), find  $y'$

$$\ln y = \ln \frac{(x+5)^2 (x-4)^{\frac{1}{3}}}{(x+2)^5 (x+4)^{\frac{1}{2}}} = 2 \ln(x+5) + \frac{1}{3} \ln(x-4) - 5 \ln(x+2) - \frac{1}{2} \ln(x+4)$$

$$\frac{d \ln y}{d x} = \frac{d}{d x} \left( 2 \ln(x+5) + \frac{1}{3} \ln(x-4) - 5 \ln(x+2) - \frac{1}{2} \ln(x+4) \right)$$

$$\frac{1}{y} y' = \frac{2}{x+5} + \frac{1}{3(x-4)} - \frac{5}{x+2} - \frac{1}{2(x+4)}$$

$$y' = y \left[ \frac{2}{x+5} + \frac{1}{3(x-4)} - \frac{5}{x+2} - \frac{1}{2(x+4)} \right]$$

$$= \frac{(x+5)^2 (x-4)^{\frac{1}{3}}}{(x+2)^5 (x+4)^{\frac{1}{2}}} \left[ \frac{2}{x+5} + \frac{1}{3(x-4)} - \frac{5}{x+2} - \frac{1}{2(x+4)} \right]$$

E.g.:  $y = x^{x^x}$  Let  $u = x^x$   $y = x^u$   $\ln y = u \ln x$

$$\frac{d}{dx} \ln y = \frac{d}{dx} (u \cdot \ln x) \Rightarrow \frac{y'}{y} = u' \ln x + \frac{u}{x}$$

$$\Rightarrow y' = y \cdot \left( u' \ln x + \frac{u}{x} \right)$$

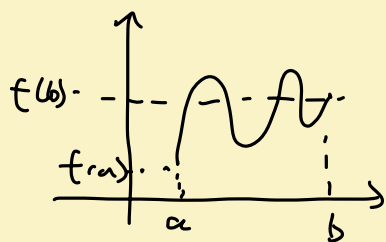
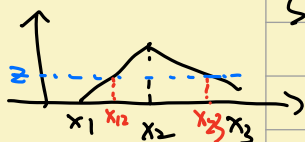
$$u = x^x \quad \ln u = x \cdot \ln x$$

$$\frac{d}{dx} \ln u = \frac{d}{dx} (x \cdot \ln x) \Rightarrow \frac{u'}{u} = \ln x + 1$$

$$\Rightarrow u' = u(\ln x + 1)$$

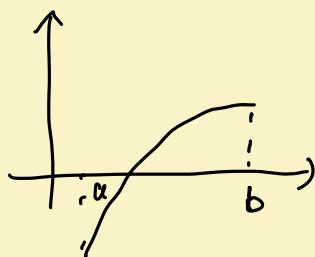
$$y' = x^{x^x} \cdot \left( x^x (\ln x + 1) \cdot \left( \ln x + \frac{x^x}{x} \right) \right)$$

$$= x^{x^x} \cdot x^x \left( \ln^2 x + \ln x + \frac{1}{x} \right)$$



$$\exists x \in [a, b]$$

$$\Rightarrow f(x) = \frac{f(b) - f(a)}{2}$$



$$f(a) < 0 \quad f(b) > 0$$

$$\exists x \in [a, b] \quad f(x) = 0$$

## Exercise 4.

1. Since there is at most one  $x$  for  $f(x) = y$ ,  $f$  is **injective**.

Suppose that  $f$  is not strictly monotonic.  $x_1 < x_2 < x_3$

$$f(x_1) < f(x_2), \quad f(x_2) > f(x_3)$$

we can find a value  $z \in \mathbb{R}$  such that

$$f(x_1) < z < f(x_2) \quad f(x_2) > z > f(x_3)$$

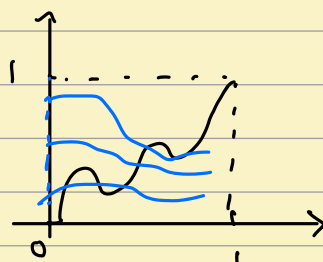
There exist  $x_{12} \in (x_1, x_2)$  such that  $f(x_{12}) = z$

$x_{23} \in (x_2, x_3)$  such that  $f(x_{23}) = z$

a contradiction with  $f$  being an injection

2.  $f(x) = x$   $f$  is continuous.  $f: [0, 1] \rightarrow [0, 1]$

$$\text{Let } g(x) = f(x) - x$$



① If  $f(0) = 0$  <sup>and/or</sup>  $f(1) = 1$ . It is done

② If  $f(0) > 0$ ,  $f(1) < 1$

$$\Rightarrow g(0) = f(0) > 0$$

$$g(1) = f(1) - 1 < 0$$

By the intermediate value theorem

$$\exists x \in (0, 1) \text{ such that } g(x) = 0$$

$$\Rightarrow g(x) = f(x) - x = 0 \Rightarrow f(x) = x$$

6. Assume  $f$  is not bounded on  $(a, b)$ , that is  $\exists x \in (a, b)$  such that  $|f(x)| \rightarrow +\infty$ , there exist  $\{a_n\} \subset (a, b)$

$|f(a_n)| \rightarrow \infty$  that is  $\{b_n\} \subset (a, b)$ , such that

$$f(b_n) = 2^n \rightarrow +\infty$$

Since  $\{b_n\}$  is bounded, then there exists a convergent subsequence  $b_{n_k}$   $|b_{n_k} - b_{n_{k+1}}| < \delta \quad \forall k \geq N \in \mathbb{N}$  for any  $\delta > 0$ .

but  $|f(b_{n_k}) - f(b_{n_{k+1}})| = |2^{n_k} - 2^{n_{k+1}}| \geq 2$  then

$f$  is not uniformly continuous. contradiction.