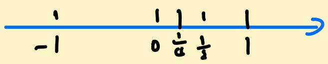
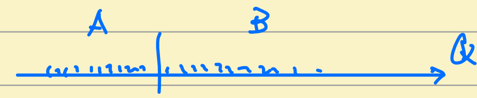
Real Analysis : Analyse the real field.



Real field : complete (完备性，或称为连续性)

$0.9999\cdots = 1$?

Dedekind's cut (戴德金分割)



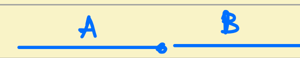
① $A \cap B = \phi$

② $A \cup B = \mathbb{Q}$

③ $\forall a \in A,\ b \in B$, we have $a < b$

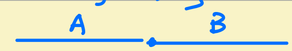
There exist three different cases:

Case 1 : $\sup A \in A$ and $\inf B \notin B$



E.g. $A = \{x \mid x \in \mathbb{Q},\ x \leq 2\}$ $B = \{x \mid x \in \mathbb{Q},\ x > 2\}$

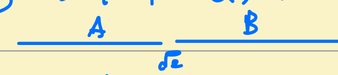
Case 2 : $\sup A \notin A$ and $\inf B \in B$



E.g. $A = \{x \mid x \in \mathbb{Q},\ x < 2\}$ $B = \{x \mid x \in \mathbb{Q},\ x \geq 2\}$

Case 3 : $\sup A \notin A$ and $\inf B \notin B$

E.g. $A = \{x \mid x \in \mathbb{Q}\ \ x < 0 \text{ or } x^2 \leq 2\}$ $B = \{x \mid x \in \mathbb{Q},\ x > 0 \text{ and } x^2 > 2\}$



Case 4 : $\sup A \in A$ and $\inf B \in B$ (Impossible)

Consider a cut Case 1 : $A \mid B \rightarrow 0.999\cdots$

Case 2 : $C \mid D \rightarrow 1$

$A = \{x \mid x \in \mathbb{Q},\ x < 0.999\cdots\}$

$C = \{x \mid x \in \mathbb{Q},\ x < 1\}$

We only need to prove $A = C \iff A \subseteq C$ and $C \subseteq A$

Proof: ① $A \subseteq C$ take $x \in A$, then $x < 0.99\cdots \Rightarrow x < 1 \Rightarrow x \in C$

② $C \subseteq A$ take $x \in C$, then $x < 1$, since $x \in \mathbb{Q}$,

we have $x = \frac{p}{q} < 1$, p, q are integers . $p < q$,

$$1-x = 1-\frac{p}{q} = \frac{q-p}{q} \geqslant \frac{1}{q}$$

There exists $n > N$ such that $10^n > q \Rightarrow \frac{1}{10^n} < \frac{1}{q}$

$$1-x = \frac{q-p}{q} \geqslant \frac{1}{q} > \frac{1}{10^n} \Rightarrow 1-x > \frac{1}{10^n}$$

$$x < 1-\frac{1}{10^n} = \overbrace{0.999\cdots}^{n个} < 0.999\cdots \qquad x \in A$$

Hence $A = C \Rightarrow 0.999\cdots = 1$

Limits of functions

Defn: the limit of function f at x_0

$$\lim_{x \to x_0} f(x) \qquad\qquad \lim_{n \to \infty} f(n) \quad n \in \mathbb{N}_+$$

the limit of function f as x_0 goes-to infinity.

$$\lim_{x \to +\infty} f(x) \qquad\qquad \lim_{x \to -\infty} f(x)$$

Defn (ε-δ-definition) Let $f: \mathbb{R} \to \mathbb{R}$ is a real-valued function. Suppose that $x_0 \in (a, b)$, and f is well-defined on (a, b) except possibly (可是可不是) at x_0, then the limit of f at x_0 is A if and only if:

> for any $\varepsilon > 0$, there exists a $\delta > 0$, such that
> $$|f(x) - A| < \varepsilon$$
> whenever $0 < |x - x_0| < \delta$

E.g.: The derivative $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$

$f(x) = x^2$ Let $x = x_0 + h$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$= \lim_{h \to 0} \frac{(x_0+h)^2 - x_0^2}{h} = \lim_{h \to 0} \frac{\cancel{x_0^2} + 2x_0 \cdot h + h^2 - \cancel{x_0^2}}{h}$$

$$= \lim_{h \to 0} \frac{2x_0 \cdot \cancel{h} + h^{\cancel{2}}}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x_0 + h = 2x_0$$

任取ε，能找到δ，
⟺ 建立ε与δ之间的关系，
得证

E.g.: Show that $\lim_{x \to 8} f(x) = x = 8$

Proof: For any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x - 8| < \delta$ we have $|x - 8| < \varepsilon$. Then we can take $\delta = \varepsilon$.

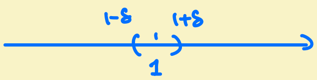
E.g.: Show that $\lim_{x \to 8} f(x) = 10x - 2 = 78$

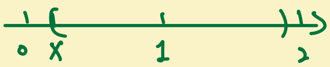
Proof: For any $\varepsilon > 0$, there exists a $\delta > 0$ such that if
$|x - 8| < \delta \Rightarrow |10x - 2 - 78| = |10x - 80| < \varepsilon \Rightarrow |x - 8| < \frac{\varepsilon}{10}$
Then we can take $\delta = \frac{\varepsilon}{10}$. Then by definition $\lim_{x \to 8} f(x) = 78$

E.g. $f(x) = \frac{x^2 - 4}{x - 2}$ show that $\lim_{x \to 2} f(x) = 4$

Proof: For any $\varepsilon > 0$, there exists a $\delta > 0$ such that if
$|x - 2| < \delta$
$|f(x) - 4| = \left| \frac{x^2 - 4}{x - 2} - 4 \right|$ suppose $x \neq 2$, then
$\left| \frac{x^2 - 4}{x - 2} - 4 \right| = |x + 2 - 4| = |x - 2| < \varepsilon$

Then we can take $\delta = \varepsilon$, then by definition $\lim_{x \to 2} f(x) = 4$





$|x-1|<\delta$ $\quad |2x+1|$

$|x-1|<\dfrac{3\varepsilon}{|2x+1|}<3\varepsilon$

E.g. Show that $\displaystyle\lim_{x\to 1}\frac{x^2-1}{2x^2-x-1}=\frac{2}{3}$

Proof: Suppose $x\neq 1$, then $\left|\dfrac{x^2-1}{2x^2-x-1}-\dfrac{2}{3}\right|=\left|\dfrac{(x+1)(x-1)}{(2x+1)(x-1)}-\dfrac{2}{3}\right|$

$$=\left|\frac{x+1}{2x+1}-\frac{2}{3}\right|=\left|\frac{3(x+1)-2(2x+1)}{3(2x+1)}\right|=\frac{|x-1|}{3|2x+1|}<\varepsilon$$

Let $|x-1|<1 \Rightarrow x>0 \Rightarrow |2x+1|>1$

Then we can take $\delta=\min\{3\varepsilon,1\}$, then we have

$\dfrac{|x-1|}{3|2x+1|}<\dfrac{|x-1|}{3}<\varepsilon$ by definition, $\displaystyle\lim_{x\to 1}\frac{x^2-1}{2x^2-x-1}=\frac{2}{3}$

Note:

① δ依赖于ε，ε越小，则δ越小，但δ取更小也无妨.

② 只需求$f(x)$在$(x_0-\delta,x_0)\cup(x_0,x_0+\delta)$有定义即可，不需求在$x=x_0$处有定义

E.x. show that $\displaystyle\lim_{x\to 8}x^2=64$

Proof ①: $|x^2-64|=|x-8||x+8|<\varepsilon$ $\qquad |x-8|<\dfrac{\varepsilon}{|x+8|}$ $\qquad |x+8|<?$

Let $|x-8|<1 \Rightarrow |x|<9 \quad |x+8|<17$

Then we have $|x-8||x+8|<|x-8|\cdot 17<\varepsilon$

take $\delta=\min(1,\frac{\varepsilon}{17})$. Then by definition, $\displaystyle\lim_{x\to 8}x^2=64$

Proof ②: Suppose $|x-8|<\delta$, $\Rightarrow -\delta<x-8<\delta \Rightarrow |x|<\delta+8$

$\Rightarrow |x+8|<\delta+16 \Rightarrow |x+8||x-8|<|\delta+16|\cdot\delta<\varepsilon$

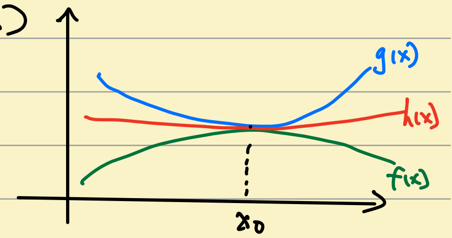
$(\delta+16)\cdot\delta<\varepsilon \Rightarrow$ we take $\delta<-8+\sqrt{64+\varepsilon}$

Then by definition, we have $\displaystyle\lim_{x\to 8}x^2=64$.

Thm (Squeeze theorem, or Sandwich theorem 夹逼定理)

Suppose $\alpha > 0$, for all $x \in (x_0-\alpha, x_0) \cup (x_0, x_0+\alpha)$, such that

$$f(x) \le h(x) \le g(x)$$



and $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = A$

then $\lim\limits_{x\to x_0} h(x) = A$

Proof: For any $\varepsilon > 0$, by definition, there exists $\delta_1 > 0$, such that

$|x-x_0| < \delta_1 \Rightarrow |f(x)-A| < \varepsilon \iff A-\varepsilon < f(x) < A+\varepsilon$

For any $\varepsilon > 0$, there exists $\delta_2 > 0$, such that

$|x-x_0| < \delta_2 \Rightarrow |g(x)-A| < \varepsilon \iff A-\varepsilon < g(x) < A+\varepsilon$

Take $\delta = \min(\delta_1, \delta_2)$, we have

$$|x-x_0| < \delta \Rightarrow \begin{cases} A-\varepsilon < f(x) < A+\varepsilon \\ A-\varepsilon < g(x) < A+\varepsilon \end{cases}$$

Since we have $f(x) \le h(x) \le g(x)$

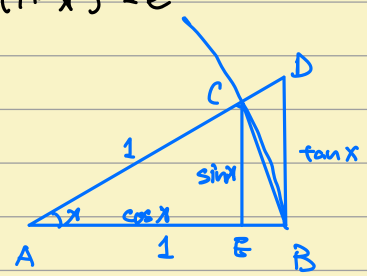
$$\Rightarrow A-\varepsilon < f(x) \le h(x) \le g(x) < A+\varepsilon$$

$$\Rightarrow A-\varepsilon \le h(x) \le A+\varepsilon \iff |h(x)-A| < \varepsilon$$

By definition, we conclude $\lim\limits_{x\to x_0} h(x) = A$

Thm 2.7 (Two important limits)

① $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ ② $\lim\limits_{x\to +\infty} \left(1+\dfrac{1}{x}\right)^x = e$



Proof the area of $\triangle ABC = \dfrac{\sin x}{2}$

the area of $\triangle ABD = \dfrac{\tan x}{2}$

the area of segment $\overset{\frown}{ABC} = \dfrac{x}{2}$

$$\triangle ABC < \overset{\frown}{ABC} < \triangle ABD$$

$\Rightarrow \frac{\sin x}{2} < \frac{x}{2} < \frac{\tan x}{2}$

$\Rightarrow \sin x < x < \tan x = \frac{\sin x}{\cos x}$

$\Rightarrow 1 < \frac{x}{\sin x} < \frac{1}{\cos x} \Rightarrow \cos x < \frac{\sin x}{x} < 1$

Since $\lim_{x \to 0} \cos x = 1$

By the squeeze theorem we have $\lim_{x\to 0} \frac{\sin x}{x} = 1$