Continuous function

$$\lim_{x \to x_0} f(x) = f(\lim_{x \to x_0} x)$$

Left continuous, Right continuous at x_0 $\Leftrightarrow$ continuous at x_0

Discontinuous function

Type I discontinuity: Removable discontinuity
Jump discontinuity.

Type II discontinuity: At least one-side limit doesn't exist

Properties of continuous function on $[a, b]$

① Boundedness

② Minimum. Maximum exist

③ Intermediate value theorem $\exists \mu$ $f(a) < \mu < f(b)$

Uniform continuity (一致连续性)

Defn (Uniform continuity) Let $f: S \to \mathbb{R}$ be a function. Suppose for any $\varepsilon > 0$, there exists $\delta > 0$, such that $x_1, x_2 \in S$, and $|x_1 - x_2| < \delta$, then

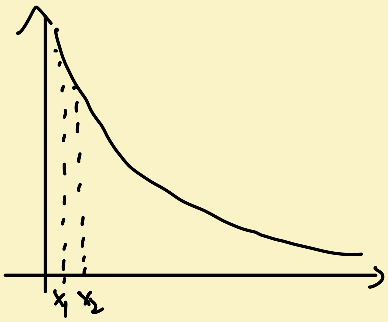
$$|f(x_1) - f(x_2)| < \varepsilon$$

We say f is uniformly continuous.

Defn (Cauchy convergence criterion) A sequence $\{x_n\}$ converges, if and only if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that $n_1, n_2 \geq N$,

$$|x_{n_1} - x_{n_2}| < \varepsilon$$

Note: Uniform continuity is stronger than continuity.



E.g. $f(x) = \frac{1}{x} : (0,1) \to \mathbb{R}$. $f(x)$ is continuous, but is not uniformly continuous.

Proof: Given $\varepsilon > 0$, then for $|f(x_1) - f(x_2)| < \varepsilon$ to hold for all $x_1, x_2 \in (0,1)$, we have

$$\varepsilon > \left| \frac{1}{x_1} - \frac{1}{x_2} \right| = \frac{|x_2 - x_1|}{x_1 x_2}$$

$$\Rightarrow \quad |x_2 - x_1| < \varepsilon \cdot x_1 \cdot x_2$$

we must have $\delta \leq \varepsilon \cdot x_1 x_2$ for all $x_1, x_2 \in (0,1)$, then $\delta \leq 0$ but $\delta > 0$ by definition of uniform continuity. Contradiction.

E.g.: $f(x) = x^2 : \mathbb{R} \to \mathbb{R}$ is not uniformly continuous.

Proof: Construct two sequences: $a_n = n + \frac{1}{n}$, $b_n = n$, then

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} (n + \tfrac{1}{n} - n) = \lim_{n\to\infty} \frac{1}{n} = 0$$

Let $x_1 = a_n$, $x_2 = b_n$, we have $|x_1 - x_2| < \delta$ for $n > N$ but

$$|f(x_1) - f(x_2)| = |(n + \tfrac{1}{n})^2 - n^2| = \left|2 + \frac{1}{n^2}\right| \geq 2$$

Then f is not uniformly continuous.

E.g.: $f(x) = x^2 : [a, b] \to \mathbb{R}$ is uniformly continuous.

Proof: Let $[a, b] = [0, 1]$ take $0 \leq x_1, x_2 \leq 1$

$$|f(x_1) - f(x_2)| = |x_1^2 - x_2^2| = |x_1 + x_2|\,|x_1 - x_2| \leq 2\,|x_1 - x_2|$$

Therefore, given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{2}$, If $|x_1 - x_2| < \delta$, then

$$|f(x_1) - f(x_2)| < \varepsilon.$$

E.g.: $f(x) = \frac{1}{x}$ is uniformly continuous on $[1, +\infty)$.

Thm. (Theorem of uniform continuity) If a function f is continuous on $[a, b]$, then f is a uniformly continuous function on $[a, b]$.

Proof: Prove by contradiction. Suppose f is continuous but not uniformly continuous on $[a, b]$. By the definition of uniform continuity:

$\exists \varepsilon_0 > 0$ such that $\forall \delta > 0$, and $\exists x, y$ $|x - y| < \delta$, $\Rightarrow |f(x) - f(y)| \geq \varepsilon_0$

Since δ is arbitrary,

$\exists \varepsilon_0 > 0$ such that $\delta = 1$ and $\exists x_1, y_1$, $|x_1 - y_1| < 1 \Rightarrow |f(x_1) - f(y_1)| \geq \varepsilon_0$

$\exists \varepsilon_0 > 0$ such that $\delta = \frac{1}{2}$ and $\exists x_2, y_2$, $|x_2 - y_2| < \frac{1}{2} \Rightarrow |f(x_2) - f(y_2)| \geq \varepsilon_0$

$\vdots$

$\exists \varepsilon_0 > 0$ such that $\delta = \frac{1}{n}$ and $\exists x_n, y_n$, $|x_n - y_n| < \frac{1}{n} \Rightarrow |f(x_n) - f(y_n)| \geq \varepsilon_0$

By above, we constructed bounded sequences $\{x_n\}, \{y_n\} \subset [a, b]$.

By Bolzano-Weierstrass theorem, there exists a convergent subsequence $\{x_{n_k}\}$ such that $\lim_{k \to +\infty} x_{n_k} = c \in [a, b]$. Then

$$|y_{n_k} - c| = |y_{n_k} - x_{n_k} + x_{n_k} - c| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - c|$$

$$\leq \frac{1}{n_k} + \varepsilon_0$$

As both $\frac{1}{n_k}$ and $|x_{n_k} - c|$ converge to 0 as $k \to +\infty$, we have $\{y_{n_k}\}$ also converges to c as $k \to +\infty$. But

$$|f(x_{n_k}) - f(c)| = |f(x_{n_k}) - f(y_{n_k}) + f(y_{n_k}) - f(c)|$$

$$\geq |f(x_{n_k}) - f(y_{n_k})| - |f(y_{n_k}) - f(c)|$$

$$\geq \varepsilon_0 - |f(y_{n_k}) - f(c)|$$

$$\Rightarrow |f(x_{n_k}) - f(c)| + |f(y_{n_k}) - f(c)| \geq \varepsilon_0$$

At least one of $\{x_{n_k}\}$ or $\{y_{n_k}\}$ does not converge. Thus f is not continuous. Contradiction. Hence, f is uniformly continuous.

Let $f(x): S \to \mathbb{R}$ be a function, then $f(x)$ is uniformly continuous if and only if for any sequences $\{x_n\}, \{y_n\} \subset S$,

$$\lim_{n\to\infty}(x_n - y_n) = 0 \Rightarrow \lim_{n\to\infty}(f(x_n) - f(y_n)) = 0 .$$

Proof: ($\Rightarrow$) By the uniform continuity of f, $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $\forall x_n, y_n \in S$, $|x_n - y_n| < \delta \Rightarrow |f(x_n) - f(y_n)| < \varepsilon$.

That is, if $\lim_{n\to\infty}(x_n - y_n) = 0 \Rightarrow \lim_{n\to\infty}(f(x_n) - f(y_n)) = 0$

($\Leftarrow$) Prove by contradiction. Suppose that $f(x)$ is not uniformly continuous. $\exists \varepsilon > 0$, $\forall \delta > 0$, $\exists x, y \in S$, such that $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$.

Let $\delta_1 = 1$ $\exists |x_1 - y_1| < 1$ but $|f(x_1) - f(y_1)| \geq \varepsilon$.

Let $\delta_2 = \frac{1}{2}$, $\exists |x_2 - y_2| < \frac{1}{2}$ but $|f(x_2) - f(y_2)| \geq \varepsilon$.

$\vdots$

Let $\delta_n = \frac{1}{n}$, $\exists |x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \varepsilon$.

That is $\lim_{n\to\infty}|x_n - y_n| = 0$ but $\lim_{n\to\infty}(f(x_n) - f(y_n)) \neq 0$

Contradiction. Hence, f is uniformly continuous.

Continuity of Elementary functions (初等函数的连续性)

Thm: Every basic elementary function is continuous on its domain.

E.g.: basic elementary functions.

① $f(x)=c$, ② $f(x)=x$ ③ $f(x)=x^{a}$ $a\in\mathbb{R}$.

④ $f(x)=\log_a x$ $a>0$.

⑤ $f(x)=\sin x$, $f(x)=\cos x$ $f(x)=\tan x$ $f(x)=\cot x$

$f(x)=\arcsin x$ $f(x)=\arccos x$ $f(x)=\arctan x$ $f(x)=\operatorname{arccot} x$

Thm. Every function that is obtained by algebra from basic elementary functions is continuous on its domain.

E.x. ① Expand the following functions to be continuous on $\mathbb{R}$

$$f(x)=\frac{x^3-8}{x-2} \Rightarrow f(x)=\begin{cases}\frac{x^3-8}{x-2} & x\neq 2\\ 12 & x=2\end{cases}$$

$$f(x)=\frac{1-\cos x}{x^2} \Rightarrow f(x)=\begin{cases}\frac{1-\cos x}{x^2} & x\neq 0\\ \frac{1}{2} & x=0\end{cases}$$

$$f(x)=x\cdot\cos\frac{1}{x} \Rightarrow f(x)=\begin{cases}x\cdot\cos\frac{1}{x} & x\neq 0\\ 0 & x=0\end{cases}$$

② Prove that if f and g are uniformly continuous functions on $[a,b]$, then $f-g$ is also uniformly continuous on $[a,b]$.

Proof: By the definition of uniform continuity.

$\forall \varepsilon>0$, $\exists \delta_1>0$, $\exists x_1, x_2\in[a,b]$, such that $|x_1-x_2|<\delta_1$

$\Rightarrow |f(x_1)-f(x_2)|<\frac{\varepsilon}{2}$

$\forall \varepsilon > 0$, $\exists \delta_2 > 0$, $\exists x_1, x_2 \in [a,b]$, such that $|x_1 - x_2| < \delta_2$

$\Rightarrow |g(x_1) - g(x_2)| < \frac{\varepsilon}{2}$

Let $\delta = \min(\delta_1, \delta_2)$ then $|x_1 - x_2| < \delta \Rightarrow$

$|(f-g)(x_1) - (f-g)(x_2)| = |(f(x_1) - g(x_1)) - (f(x_2) - g(x_2))|$

$= |(f(x_1) - f(x_2)) - (g(x_1) - g(x_2))| \leq |f(x_1) - f(x_2)| + |g(x_1) - g(x_2)| < \varepsilon$

By the definition, $f-g$ is uniformly continuous.

③ Suppose that function f satisfies the Lipschitz continuity:

$$|f(x_1) - f(x_2)| \leq L|x_1 - x_2|$$

for all $x_1, x_2 \in [a,b]$ with $L > 0$. Prove that f is uniformly continuous on $[a,b]$.

Proof: By the definition of uniform continuity.

$\forall \varepsilon > 0$, $\exists \delta > 0$, $\exists x_1, x_2 \in [a,b]$, such that $|x_1 - x_2| < \delta$

$\Rightarrow |f(x_1) - f(x_2)| < \varepsilon$.

Since $|f(x_1) - f(x_2)| \leq L|x_1 - x_2| < L \cdot \delta$.

Let $\delta = \frac{\varepsilon}{L}$. if $|x_1 - x_2| < \delta$, we have

$$|f(x_1) - f(x_2)| < L \cdot \frac{\varepsilon}{L} = \varepsilon,$$

Hence f is uniformly continuous.