① Infinitesimal quantity

$x_0 \to 0$

$\sin x \sim x$ $\arcsin x \sim x$

$\tan x \sim x$ $\arctan x \sim x$

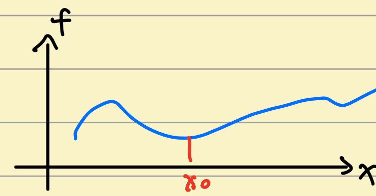
$1-\cos x \sim \frac{1}{2}x^2$

$\ln(x+1) \sim x$ $e^x - 1 \sim x$ $a^x - 1 \sim x \ln a$

$\sqrt[n]{1+x} - 1 \sim \frac{1}{n}x$

② Squeeze theorem $f(x) \le h(x) \le g(x)$, $\lim f(x) = \lim g(x) = A$ $\Rightarrow h(x) = A$

③ $\lim_{x\to 0} \frac{\sin x}{x} = 1$ $\lim_{x\to\infty} (1+\frac{1}{x})^x = e$



Continuous function

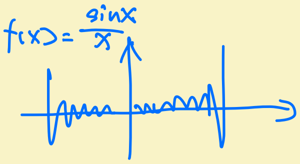
dom $f \supseteq (x_0-\delta, x_0+\delta)$

Defn: (continuous function) Suppose that $f(x)$ is defined on $(x_0-\delta, x_0+\delta)$ with $\delta > 0$. If

$$\lim_{x\to x_0} f(x) = f(x_0)$$

then we say that f is continuous at x_0.

$f(x) = \frac{\sin x}{x}$



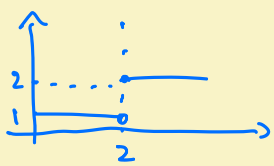
Note: We have 3 requirements for the continuity definition.

① f is defined at x_0

② $\lim_{x\to x_0} f(x)$ exists.

③ $\lim_{x\to x_0} f(x) = f(x_0)$

$\lim_{x\to x_0} f(x) = f(\lim_{x\to x_0} x)$



$f(2) = 2$

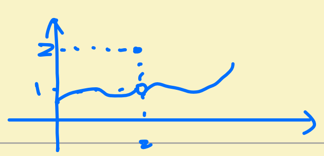
Defn ($\varepsilon-\delta$ definition for continuous function) For any $\varepsilon > 0$, there exists $\delta > 0$, such that $|x - x_0| < \delta$, we have

$$|f(x) - f(x_0)| < \varepsilon$$

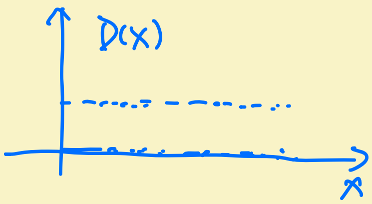
① f is defined at x_0

② $\lim_{x\to x_0} f(x)$ exists

③ $\lim_{x\to x_0} f(x) = f(x_0)$



$f(2) = 2$ $\lim_{x\to 2} f(x) = 1$



E.g. $D(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

define $f(x) = x \cdot D(x)$ is continuous at $x = 0$

for any $\varepsilon > 0$, take $\delta = \varepsilon$, then we have

$$|f(x) - f(0)| = |x \cdot D(x) - 0| \le |x - 0| < \varepsilon$$

$f(0) = 0 \cdot 1 = 0$ $\quad |x \cdot D(x)| \le |x| \cdot |D(x)| \le |x| \cdot 1 \quad |D(x)| \le 1$

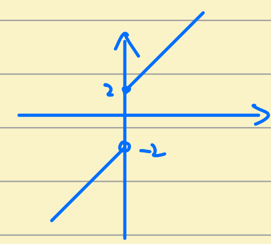
Defn: (Left and right continuity) If a function f is defined on $(x_0 - \delta, x_0]$, we say f is left-continuous if

$$\lim_{x \to x_0^-} f(x) = f(x_0)$$

$[x_0, x_0 + \delta)$, we say f is right-continuous if

$$\lim_{x \to x_0^+} f(x) = f(x_0)$$

E.g. $f(x) = \begin{cases} x+2 & x \ge 0 \\ x-2 & x < 0 \end{cases}$



Thm. A function f is continuous at x_0 if and only if f is both left continuous and right continuous at x_0.

E.g. $\lim_{x \to 0^+} f(x) = 2 = f(0)$

$\lim_{x \to 0^-} f(x) = 0 \ne f(0)$ $\quad$ f is not continuous

Discontinuous function

If a function f satisfies one of the following:

① f is not defined at x_0.

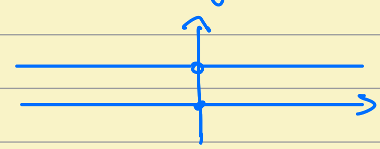
② f has no limit at x_0

③ $\lim_{x\to x_0} f(x) \neq f(x_0)$

Removable discontinuity (可去间断点)

E.g. $f(x) = |\operatorname{sgn}(x)|$

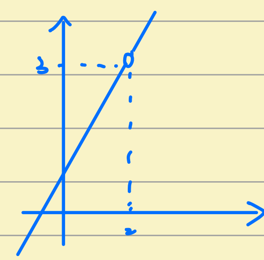
$$\operatorname{sgn}(x) = \begin{cases} -1 & x<0 \\ 0 & x=0 \\ 1 & x>0 \end{cases}$$



$$g(x) = \begin{cases} |\operatorname{sgn}(x)| & x\neq 0 \\ 1 & x=0 \end{cases}$$

E.g. $f(x) = \dfrac{x^2-x-2}{x-2}$

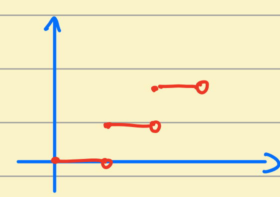
$$g(x) = \begin{cases} \dfrac{x^2-x-2}{x-2} & x\neq 2 \\ 3 & x=2 \end{cases}$$



Jump discontinuity (跳跃间断点)

E.g. $f(x) = [x]$

$$\lim_{x\to x_0^-} f(x) \neq \lim_{x\to x_0^+} f(x)$$



Type I discontinuity: Removable discontinuity & Jump discontinuity

Both left and right limits exist

Type II discontinuity: At least one-side limit does not exist

E.g.: $f(x) = \frac{1}{x}$ $\quad f(x) = \frac{1}{x^2}$ $\quad D(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}$

Continuous function on an Interval

Defn: A function f is said to be continuous on (a,b), if f is continuous at every point on (a,b).

Defn: A function f is said to be continuous on $[a,b]$, if f is continuous at every point on (a,b), and f is right/left continuous at a/b.

Properties of continuous function

Thm (Boundedness) Suppose that f is continuous on $[a,b]$, then f is bounded.

E.g.: $f(x) = \frac{1}{x}$ is not bounded at $(0,1)$

Thm (Boundedness) Suppose that f is continuous at x_0, then there exists $\delta > 0$ such that f is bounded on $(x_0-\delta, x_0+\delta)$.

Thm (保号性) Suppose that f is continuous at x_0. if $f(x_0) > 0$, then there exists $\delta > 0$, such that $f(x) > 0$, for any $x \in (x_0-\delta, x_0+\delta)$

Thm (Arithmetic operation) Suppose that f, g are continuous at x_0, then

$$\left.\begin{array}{l} ① \ f \pm g \\ ② \ f \cdot g \\ ③ \ f/g, \quad g(x_0) \neq 0 \end{array}\right\} \text{ are continuous at } x_0.$$

$g(f(x)) = g \circ f(x)$

Thm (Composite function) Suppose f is continuous at x_0, and g is continuous at u_0, with $u_0 = f(x_0)$. then $g(f(x))$ is continuous at x_0.

Proof: Since g is continuous at u_0.

$\forall \varepsilon > 0$, $\exists \delta_1 > 0$, such that $|u-u_0| < \delta_1 \Rightarrow |g(u) - g(u_0)| < \varepsilon$.

Since f is continuous at x_0.

$\forall \delta_1 > 0$, $\exists \delta > 0$, such that $|x-x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \delta_1$

$\Rightarrow \forall \varepsilon > 0$, $\exists \delta > 0$, such that $|x - x_0| < \delta \Rightarrow |g\circ f(x) - g \circ f(x_0)| < \varepsilon$.

Note: $\lim\limits_{x\to x_0} g(f(x)) = g(\lim\limits_{x\to x_0} f(x)) = g(f(\lim\limits_{x\to x_0} x))$

E.g. $\lim\limits_{x\to 1} \sin(1-x^2) = \sin(\lim\limits_{x\to 1}(1-x^2)) = \sin 0 = 0$

Defn (Maximum, minimum) Suppose that f is defined on an interval I, if there exists $x_0 \in I$, such that for any $x \in I$, we have

$$f(x_0) \geq f(x) \qquad \text{(最大值)}$$

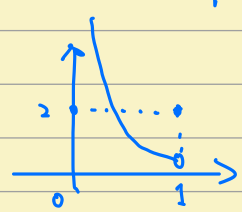
then we say f has maximum on I, and the maximum of f on I is $f(x_0)$

$$f(x_0) \leq f(x)$$

then we say f has minimum on I, and the minimum of f on I is $f(x_0)$

E.g.: $f(x) = x$ has no maximum or minimum on $(0,1)$

E.g.: $f(x) = \begin{cases} \frac{1}{x} & x \in (0,1) \\ 2 & x = 0 \text{ or } x = 1 \end{cases}$ $\quad f(x)$ is defined on $[0,1]$



$[a,b]$ + continuity $\Leftrightarrow$ bounded

Thm (Boundedness) Suppose f is continuous on $[a,b]$, then f is bounded on $[a,b]$.

Proof: Suppose that f is continuous, but is not bounded $[a,b]$. there exists $\{x_n\} \subset [a,b]$, $\lim_{n\to\infty} f(x_n) = +\infty$. since $[a,b]$ is closed, we have that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to\infty} x_{n_k} = x_0$$

since $x_{n_k} \in [a,b]$, we have $x_0 \in [a,b]$, since $f(x)$ is continuous on x_0, $\lim_{n\to\infty} f(x_n) \neq +\infty$. contradiction. Hence $f(x)$ is bounded.

$f(x) = x \quad (0,1)$

Thm (Minimum - Maximum theorem) Suppose that f is continuous on $[a,b]$, then f has maxima and minima on $[a,b]$.

Proof: Write $\sup_{x\in[a,b]} f(x) = M$

To prove $\xi \in [a,b]$ such that $f(\xi) = M$. Suppose by contradiction that $f(x) < M$ for all $x \in [a,b]$. Let

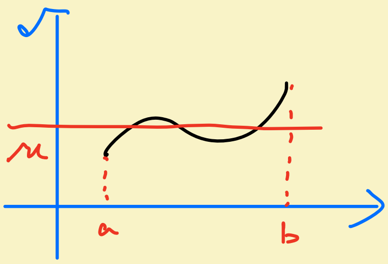
$$g(x) = \frac{1}{M - f(x)} \qquad x \in [a,b]$$

by then continuity of $g(x)$, we have $g(x)$ is bounded above.

$$0 < g(x) = \frac{1}{M - f(x)} \le G$$

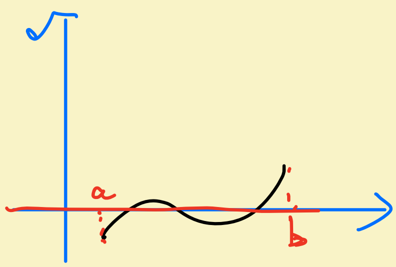
$$\Rightarrow \frac{1}{M - f(x)} \le G \Rightarrow f(x) \le M - \frac{1}{G}$$

Since M is the supremum of f on $[a,b]$, there is a contradiction as we obtained $f(x) \le M - \frac{1}{G}$ on $[a,b]$. Then there exists $\xi \in [a,b]$ such that $f(\xi) = M$.



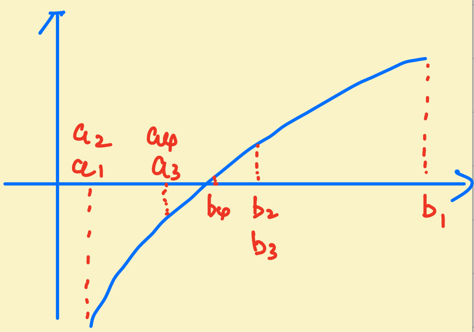
Thm (Intermediate value theorem, 介值定理) Suppose f is continuous on $[a,b]$, and $f(a) \neq f(b)$. If μ is a value such that $f(a) < \mu < f(b)$ ($f(b) < \mu < f(a)$), then there exists $x_0 \in (a,b)$ such that

$$f(x_0) = \mu.$$



Corollary (Existance of a root, 根的存在性) Suppose f is continuous on $[a,b]$, such that $f(a) < 0$ and $f(b) > 0$ (or, say, $f(a) > 0$ and $f(b) < 0$), then there exists $x_0 \in (a,b)$ such that

$$f(x_0) = 0$$



Proof: We define two sequences : $\{a_n\}$ and $\{b_n\}$.

① $a_1 := a$ and $b_1 := b$

② If $f\left(\frac{a_n+b_n}{2}\right) \geq 0$, let $a_{n+1} = a_n$ and $b_{n+1} = \frac{a_n+b_n}{2}$ $\Rightarrow f(b_{n+1}) \geq 0$

③ If $f\left(\frac{a_n+b_n}{2}\right) < 0$, let $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = b_n$ $\Rightarrow f(a_{n+1}) \leq 0$

Fact: ① if $a_n < b_n$, then $a_{n+1} < b_{n+1}$. that is $a_n < b_n \ \forall n \in \mathbb{N}$

② $\{a_n\}$ is monotonically increasing, and $\{b_n\}$ is monotonically decreasing.

③ Both $\{a_n\}$ and $\{b_n\}$ are bounded.

By the monotone convergence theorem, both $\{a_n\}$ and $\{b_n\}$ converge

Suppose $\lim\limits_{n\to\infty} a_n = c$ $\quad \lim\limits_{n\to\infty} b_n = d$

$$b_{n+1} - a_{n+1} = \begin{cases} \text{case ②} & \frac{a_n+b_n}{2} - a_n = \frac{b_n-a_n}{2} \\ \text{case ③} & b_n - \frac{a_n+b_n}{2} = \frac{b_n-a_n}{2} \end{cases} = \frac{b_n-a_n}{2}$$

Then we have

$$b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n) = \left(\frac{1}{2}\right)^2 (b_{n-1} - a_{n-1}) \cdots$$

$$= \frac{1}{2^{n-1}}(b_1 - a_1) = \frac{1}{2^{n-1}}(b-a)$$

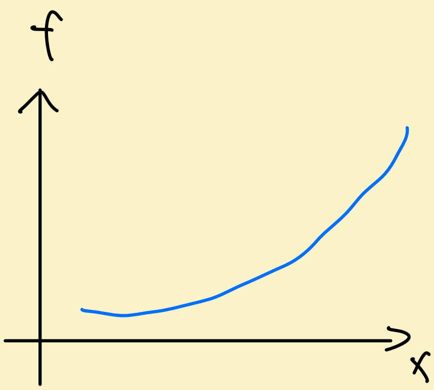
$$\lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^{n-1}}(b-a) = 0$$

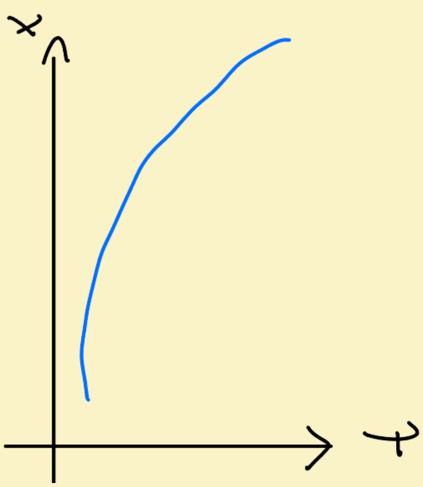
That is
$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n =: x_0 \in (a,b)$$

and
$$\lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} f(a_n) = 0$$

by the fact that $f(b_{n+1}) \geq 0$ and $f(a_{n+1}) \leq 0$. That is

$$f(x_0) = 0$$





Thm (Continuity of the inverse function, 反函數連續性)
Suppose that f is continuous on $[a,b]$. If f is strictly monotonic, then its inverse function f^{-1} is a continuous function on $[f(a), f(b)]$ (or $[f(b), f(a)]$).