Real Analysis: Exercise 3

Due date: 2025-12-02

Question 1.

1. $\lim_{x\to 0} \frac{x^2+1}{3x-5}$.

2. $\lim_{x\to 0} 7x/\sin(4x)$.

3. $\lim_{x\to 2} \frac{\sqrt{3x-2}-\sqrt{5x-6}}{\sqrt{2x-1}-\sqrt{x+1}}$,

4. $\lim_{x\to 0} \frac{x\sin x}{1-\cos x}$.

   Hint: For $x \neq k\pi$, $k \in \mathbb{Z}$: $\frac{\sin x}{1-\cos x} = \frac{1+\cos x}{\sin x}$.

Question 2. Determine whether $\lim_{x\to a} f(x)$ exists and compute the limit if it exists.

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x^2}\right), & \text{if } x \text{ is irrational} \\ -2x, & \text{if } x \text{ is rational} \end{cases}, \quad a = 0\,.$$

Question 3. Let a, b and c be real numbers and $f : [-2, 5] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 0\,, & \text{if } -2 \leq x \leq -1\,, \\ ax^2 + bx\,, & \text{if } -1 < x < 2\,, \\ c\,, & \text{if } x = 2\,, \\ 2x\,, & \text{if } 2 < x \leq 5\,. \end{cases}$$

Find all values of a, b and c such that the function f is continuous on $[-2, 5]$. Justify your answer.

Question 4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x & \text{if } x \text{ is rational}\,, \\ x^2 & \text{if } x \text{ is irrational}\,. \end{cases}$$

Prove that f is continuous at 1 and discontinuous at 2.

Question 5. Give an example of functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that the function h defined by $h(x) := f(x) + g(x)$ is continuous, but f and g are not continuous. Can you find f and g that are nowhere continuous, but h is a continuous function?

Question 6. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Suppose that for all rational numbers r, $f(r) = g(r)$. Show that $f(x) = g(x)$ for all x.

Question 7. Suppose $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be continuous functions. Define $p : X \to \mathbb{R}$ by $p(x) := \max\{f(x), g(x)\}$ and $q : X \to \mathbb{R}$ by $q(x) = \min\{f(x), g(x)\}$. Prove that p and q are continuous.

Question 8. et $f : (0, 1) \to \mathbb{R}$ be a continuous function such that $\lim_{x \to 0} f(x) = \lim_{x \to 1} f(x) = 0$. Show that f achieves either an absolute minimum or an absolute maximum on $(0, 1)$(but perhaps not both)

Question 9. Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function. Prove that the image $f([a, b])$ is a closed and bounded interval or a single number. Thus, prove that if $g : [0, 1] \to (0, 1)$ is a bijection, then g is not continuous.