Real Analysis: Answer to Exercise 3

Question 1. **Solution**: (a) As $x \to 0$, $x^2 + 1 \to 1$ and $3x - 5 \to -5$,
so $\lim_{x\to 1} f(x) = \lim_{x\to 1}(x^2+1)/(3x-5) = 1/(-5) = -1/5$ by the Algebra of Limits.

(b) Note that

$$\frac{7x}{\sin(4x)} = \frac{7}{4}.\frac{4x}{\sin(4x)}$$

and that as $x \to 0$, $4x \to 0$. So, letting $u = 4x$, we see that

$$\lim_{x\to 0} \frac{7x}{\sin(4x)} = \lim_{u\to 0} \frac{7}{4}.\frac{u}{\sin(u)} = \frac{7}{4}.$$

(c) Note first that

$$\begin{aligned}
&\frac{\sqrt{3x-2}-\sqrt{5x-6}}{\sqrt{2x-1}-\sqrt{x+1}} \\
=&\frac{\sqrt{3x-2}-\sqrt{5x-6}}{\sqrt{2x-1}-\sqrt{x+1}}.\frac{\sqrt{3x-2}+\sqrt{5x-6}}{\sqrt{3x-2}+\sqrt{5x-6}}.\frac{\sqrt{2x-1}+\sqrt{x+1}}{\sqrt{2x-1}+\sqrt{x+1}} \\
=&\frac{(3x-2)-(5x-6)}{(2x-1)-(x+1)}.\frac{\sqrt{2x-1}+\sqrt{x+1}}{\sqrt{3x-2}+\sqrt{5x-6}} \\
=&\frac{-2(x-2)}{x-2}.\frac{\sqrt{2x-1}+\sqrt{x+1}}{\sqrt{3x-2}+\sqrt{5x-6}} = -2\frac{\sqrt{2x-1}+\sqrt{x+1}}{\sqrt{3x-2}+\sqrt{5x-6}}
\end{aligned}$$

So

$$\lim_{x\to 2} \frac{\sqrt{3x-2}-\sqrt{5x-6}}{\sqrt{5x-1}-\sqrt{x+7}} = \lim_{x\to 2} -2\frac{\sqrt{2x-1}+\sqrt{x+1}}{\sqrt{3x-2}+\sqrt{5x-6}}$$

But as $x \to 2$, $\sqrt{2x-1}+\sqrt{x+1} \to \sqrt{3}+\sqrt{3} = 2\sqrt{3}$ and $\sqrt{3x-2}+\sqrt{5x-6} \to \sqrt{4}+\sqrt{4} = 4$, so

$$\lim_{x\to 2} \frac{\sqrt{3x-2}-\sqrt{5x-6}}{\sqrt{5x-1}-\sqrt{x+7}} = -2.\frac{2\sqrt{3}}{4} = -\sqrt{3},$$

by the Algebra of Limits.

(d) For $x \neq k\pi$, $\sin x \neq 0$ and $1 - \cos x \neq 0$, so

$$\frac{\sin x}{1-\cos x} = \frac{\sin x}{1-\cos x}\frac{1+\cos x}{1+\cos x} = \frac{\sin x(1+\cos x)}{1-\cos^2 x} = \frac{1+\cos x}{\sin x},$$

as $1 - \cos^2 x = \sin^2 x$. Hence

$$\frac{x\sin x}{1-\cos x} = \frac{x\sin x(1+\cos x)}{\sin^2 x} = \frac{x(1+\cos x)}{\sin x}.$$

So, by the Algebra of Limits,

$$\lim_{x\to 0}\frac{x\sin x}{1-\cos x} = \big(\lim_{x\to 0}\frac{x}{\sin x}\big)\big(\lim_{x\to 0} 1+\cos x\big) = (1)(1+1) = 2.$$

Question 2. **Solution**:

$$|f(x)| \le 2|x| \implies \lim_{x\to 0} f(x) = 0\,.$$

Question 3. **Solution**: f continuous on $[-2,5]$, we see that

$$\begin{aligned} \lim_{x\to -1^+} f(x) = f(-1) &\implies a-b=0 \\ \lim_{x\to 2^-} f(x) = f(2) &\implies 4a+2b=c \\ \lim_{x\to 2^+} f(x) = f(2) &\implies 4=c \end{aligned}$$

Thus, $a=\frac{2}{3}$ and $b=\frac{2}{3}$.

Question 4. **Solution**: Given any $\epsilon>0$, we want to find $\delta>0$ such that for any x such that $|x-1|<\delta$, we have $|f(x)-f(1)|<\epsilon$.

Take $\delta=\min\left(\frac{\epsilon}{3},1\right)$. For $|x-1|<\delta$, when x is rational, $|f(x)-1|=|x-1|<\delta\le\epsilon/3<\epsilon$. For $|x-1|<\delta$ and when x is irrational, $|f(x)-1|=|x^2-1|=|x+1||x-1|\le 3|x-1|<\epsilon$. Thus f is continuous at $x=1$.

$f(2)=2$, there exists a sequence $a_n\to 2$, such that a_n are irrational, so $f(a_n)=a_n^2\to 4$, thus, f is not continuous at 2.

Question 5. **Solution**:

$$f(x)=\begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}$$

and take $g(x)=1-f(x)$.

Question 6. **Solution**: If x is irrational then there is a sequence $\{r_n\}$ such that r_n are rational and $r_n\to x$. So,

$$f(x)=\lim_{n\to\infty} f(r_n)=\lim_{n\to\infty} g(r_n)=g(x)\,.$$

Question 7. **Solution**: It is easy to see that

$$p(x) = \frac{1}{2}\left(f(x) + g(x) + |f(x) - g(x)|\right)$$

and

$$q(x) = \frac{1}{2}\left(f(x) + g(x) - |f(x) - g(x)|\right) ,$$

and we know that $h(x) = |x|$ is continuous.

Question 8. **Solution**: Define

$$F(x) = \begin{cases} f(x) & 0 < x < 1 \\ 0 & x = 0 \text{ or } x = 1 \,. \end{cases}$$

Thus $F(x)$ is continuous on $[0,1]$ and achieves both absolute max and min.

If $\max F(x) = M = f(x_M) > 0$, then f achieves an absolute max.

If $\min F(x) = m = f(x_m) < 0$, then f achieves an absolute min.

If $\max F(x) = \min F(x) = 0$, then f is constant, and achieves both max and min.

Question 9. **Solution**: Since f is continuous on $[a,b]$, f achieves both min and max on $[a,b]$. Let $m = \min_{a \le x \le b} f(x)$ and $M = \max_{a \le x \le b} f(x)$. We claim that

$$f([a,b]) = [m, M].$$

For any $y \in (m, M)$, intermediate value theorem guarantees that there is an $x \in [a,b]$ such that $f(x) = y$. Thus, $f([a,b]) = [m,M]$, which is a closed and bounded interval.

If $m = M$, then $f([a,b])$ is a single number.

If g is surjective and continuous, then the image $(0,1)$ should be closed, which is a contradiction. Thus, g is not continuous.