① Continuity: (1) $f(x_0)$ exists; (2) $\lim\limits_{x \to x_0}$ exists; (3) $\lim\limits_{x \to x_0} f(x) = f(x_0)$

$\iff \lim\limits_{x \to x_0} f(x) = f(\lim\limits_{x \to x_0} x)$

② Continuity: $\delta$-$\varepsilon$-definition. $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $|x - x_0| < \delta$

$\Rightarrow |f(x) - f(x_0)| < \varepsilon$.

③ Uniform Continuity (一致连续) $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $\forall x_1, x_2 \in S$

and $|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$.

E.g.: $f(x) = \frac{1}{x}$ is continuous on $(0,1)$

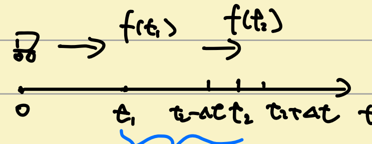
but is not uniformly continuous on $(0,1)$

E.g.: $f(x) = x^2$ is continuous on $\mathbb{R}$

but is not uniformly continuous on $\mathbb{R}$.

Differentiation (可微性, 可导性)

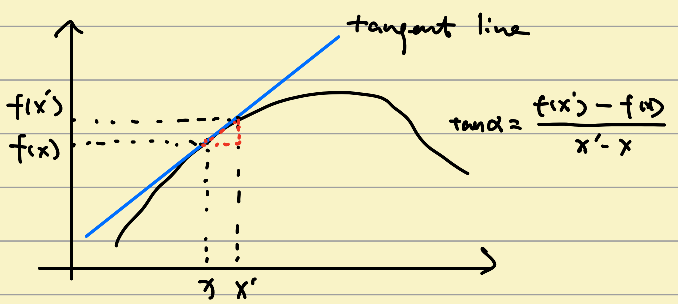
速度 $= \dfrac{\text{距离}}{\text{时间}} = \dfrac{f(t_2) - f(t_1)}{t_2 - t_1}$




当 $t_2$ 的速度: $\lim\limits_{\Delta t \to 0} \dfrac{f(t_2 + \Delta t) - f(t_2)}{\Delta t} = \lim\limits_{\Delta t \to 0} \dfrac{f(t_2) - f(t_2 - \Delta t)}{\Delta t}$

Geometric perspective

$\lim\limits_{x' \to x} \dfrac{f(x') - f(x)}{x' - x}$



**Defn**: A function $f:(a,b)\to\mathbb{R}$ is differentiable at a point $x_0\in(a,b)$ if the limit

$$f'(x_0):=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

exists. Then the limit is called the derivative of $f$, denoted as $f'$.

$$f'(x_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0)-f(x_0-\Delta x)}{\Delta x}$$

$$=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$ By letting $\Delta x = x-x_0$

E.g.: $f(x)=x^2$

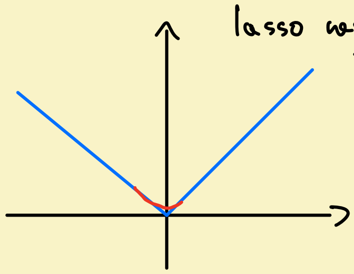
$$f'(x_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{(x_0+\Delta x)^2-x_0^2}{\Delta x}=\lim_{\Delta x\to 0}\frac{2x_0\cdot\cancel{\Delta x}+\Delta x^{\cancel{2}}}{\cancel{\Delta x}}$$

$$=\lim_{\Delta x\to 0} 2x_0+\Delta x = 2x_0$$

regularizer

lasso regression



E.g.: $f(x)=|x|$ $\quad x_0=0$

$$\lim_{\Delta x\to 0^+}\frac{|0+\Delta x|-|0|}{\Delta x}=\lim_{\Delta x\to 0^+}\frac{|\Delta x|}{\Delta x}=1$$

$$\lim_{\Delta x\to 0^-}\frac{|0+\Delta x|-|0|}{\Delta x}=\lim_{\Delta x\to 0^-}\frac{|\Delta x|}{\Delta x}=-1$$

subderivative

By the definition of derivative, the limit $f'(0)$ does not exist.

**Thm**. If a function is differentiable at a point $x_0$, then it is continuous at $x_0$.

Proof: Suppose that $f$ is differentiable at $x_0$, then

$$f'(x_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

exists. To prove $\lim_{x\to x_0} f(x) = f(\lim_{x\to x_0} x)$

First, we know that $f(x_0)$ exists, then we have

$$\lim_{x\to x_0} f(x) = f(x_0)$$

Let $x = x_0+\Delta x$, then we $\lim_{\Delta x\to 0} f(x_0+\Delta x) = f(x_0)$ holds.

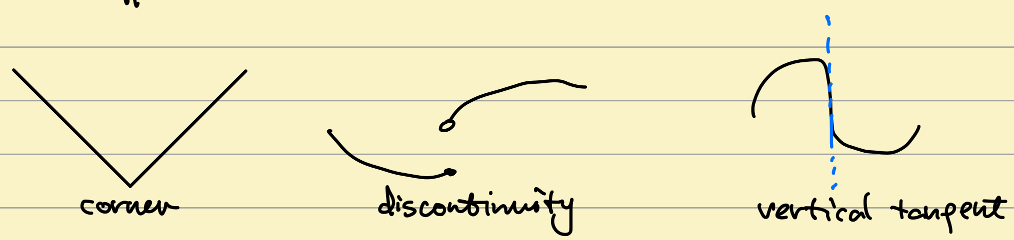
$$\lim_{\Delta x\to 0}\left(f(x_0+\Delta x)-f(x_0)\right) = \lim_{\Delta x\to 0}\left(\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\cdot\Delta x\right)$$

$$= \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\cdot\lim_{\Delta x\to 0}\Delta x$$

$$= f'(x_0)\cdot 0 = 0$$

Then we proved that $\lim_{x\to x_0} f(x) = f(\lim_{x\to x_0} x)$ holds.

Corollary: If a function $f$ is not continuous at $x_0$, then it is not differentiable at $x_0$.




E.g. ① $(x^n)' = n x^{n-1}$

$$\frac{(x_0+\Delta x)^n - x_0^n}{\Delta x} = \frac{(x_1)^n - x_0^n}{x_1 - x_0}$$

$$\frac{x_1^n - x_0^n}{x_1 - x_0} = \frac{(x_1 - x_0)}{x_1 - x_0}\left(x_1^{n-1} + x_1^{n-2}x_0 + x_1^{n-3}x_0^2 + \cdots x_1\cdot x_0^{n-2} + x_0^{n-1}\right)$$

$$= x_1^{n-1} + x_1^{n-2}x_0 + x_1^{n-3}x_0^2 + \cdots x_1\cdot x_0^{n-2} + x_0^{n-1}$$

$$\lim_{x_1\to x_0} x_1^{n-1} + x_1^{n-2}x_0 + x_1^{n-3}x_0^2 + \cdots x_1\cdot x_0^{n-2} + x_0^{n-1}$$

$$= n\cdot x_0^{n-1}$$

$\sin(a+b)$
$= \sin a \cos b + \sin b \cos a$

② $(\sin x)' = \cos x$

$$\frac{\sin(x+\Delta x) - \sin x}{\Delta x} = \frac{\sin x \cdot \cos\Delta x + \sin\Delta x \cdot \cos x - \sin x}{\Delta x}$$

$$= \frac{\sin\Delta x}{\Delta x}\cos x + \sin x \frac{\cos\Delta x - 1}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\sin\Delta x}{\Delta x}\cos x = \cos x$$

$$\lim_{\Delta x \to 0} \frac{\cos\Delta x - 1}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos^2\Delta x - 1}{\Delta x^2} \cdot \frac{\Delta x}{\cos\Delta x + 1} = \frac{1}{2} \cdot 0 = 0$$

$$\therefore \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x} = \cos x$$

$\cos(a+b)$
$= \cos a \cdot \cos b - \sin a \cdot \sin b$

③ $(\cos x)' = -\sin x$

$$\frac{\cos(x+\Delta x) - \cos x}{\Delta x} = \frac{\cos x \cos\Delta x - \sin x \sin\Delta x - \cos x}{\Delta x}$$

$$= \frac{\cos x(\cos\Delta x - 1)}{\Delta x} - \sin x \frac{\sin\Delta x}{\Delta x}$$

$$\lim_{\Delta x \to 0} \sin x \frac{\sin\Delta x}{\Delta x} = \sin x$$

$$\lim_{\Delta x \to 0} \cos x \frac{\cos\Delta x - 1}{\Delta x} = 0$$

$$\lim_{\Delta x \to 0} \frac{\cos(x+\Delta x) - \cos x}{\Delta x} = -\sin x$$

④ $(\log_a x)' = \frac{1}{x}\log_a e$

$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

$$\frac{\log_a(x+\Delta x) - \log_a x}{\Delta x} = \frac{1}{\Delta x}\log_a \frac{x+\Delta x}{x}$$

$$= \frac{1}{\Delta x}\log_a\left(1+\frac{\Delta x}{x}\right)$$

$$= \frac{x}{\Delta x}\cdot\frac{1}{x}\log_a\left(1+\frac{\Delta x}{x}\right)$$

$$= \frac{1}{x}\log_a\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}$$

$a = e$

$(\ln x)' = \frac{1}{x}$

$$\lim_{\Delta x\to 0}\log_a\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}} = \log_a e$$

$$\therefore \frac{\log_a(x+\Delta x) - \log_a x}{\Delta x} = \frac{1}{x}\log_a e$$

⑤ $(e^x)' = e^x$

$e^x - 1 \sim x$

$$\lim_{\Delta x\to 0}\frac{e^{x+\Delta x} - e^x}{\Delta x} = e^x \lim_{\Delta x\to}\frac{e^{\Delta x}-1}{\Delta x} = e^x\cdot 1 = e^x$$

## The algebra of differentiation.

① $f(x) = c$, then $f'(x) = 0 \quad \forall x \in dom(f)$

Proof: $f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{c-c}{\Delta x} = 0$

② $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$

Proof: $\lim_{\Delta x\to 0}\frac{[f(x+\Delta x) \pm g(x+\Delta x)] - [f(x) \pm g(x)]}{\Delta x}$

$$= \lim_{\Delta x\to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x} \pm \lim_{\Delta x\to 0}\frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= f'(x) \pm g'(x)$$

③ $[c \cdot f(x)]' = c \cdot f'(x)$

Proof: $\lim_{\Delta x \to 0} \frac{c f(x+\Delta x) - c f(x)}{\Delta x} = c \cdot \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

④ $[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

Proof: $\lim_{\Delta x \to 0} \frac{f(x+\Delta x) \cdot g(x+\Delta x) - f(x) \cdot g(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} [\underline{f(x+\Delta x) \cdot g(x+\Delta x) - f(x+\Delta x) \cdot g(x)} + \underline{f(x+\Delta x) \cdot g(x) - f(x) \cdot g(x)}]$

$\lim_{\Delta x \to 0} \frac{1}{\Delta x} f(x+\Delta x) \left(g(x+\Delta x) - g(x)\right) = f(x) g'(x)$

$\lim_{\Delta x \to 0} \frac{1}{\Delta x} [f(x+\Delta x) - f(x)] g(x) = f'(x) \cdot g'(x)$

$= f(x) \cdot g'(x) + f'(x) \cdot g(x)$

⑤ $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$ $\quad g(x) \neq 0$

Proof: $\lim_{\Delta x \to 0} \frac{\frac{f(x+\Delta x)}{g(x+\Delta x)} - \frac{f(x)}{g(x)}}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) g(x) - f(x) g(x+\Delta x)}{\Delta x\, g(x+\Delta x) \cdot g(x)}$

$\lim_{\Delta x \to 0} \frac{1}{g(x+\Delta x) g(x)} = \frac{1}{[g(x)]^2}$

$\lim_{\Delta x \to 0} \frac{1}{\Delta x} [\underline{f(x+\Delta x) g(x) - f(x) \cdot g(x)} + \underline{f(x) \cdot g(x) - f(x) g(x+\Delta x)}]$

$\lim_{\Delta x \to 0} \frac{1}{\Delta x} [f(x+\Delta x) - f(x)] \cdot g(x) = f'(x) \cdot g(x)$

$\lim_{\Delta x \to 0} \frac{1}{\Delta x} - f(x) [g(x+\Delta x) - g(x)] = - f(x) \cdot g'(x)$

$= \frac{1}{[g(x)]^2} \cdot \left(f'(x) \cdot g(x) - f(x) \cdot g'(x)\right)$

E.x. $(\tan x)' = \sec^2 x$ hint: $\tan x = \frac{\sin x}{\cos x}$ $\sec x = \frac{1}{\cos x}$

$$\lim_{\Delta x \to 0} \frac{\tan(x+\Delta x) - \tan x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{\sin(x+\Delta x)}{\cos(x+\Delta x)} - \frac{\sin x}{\cos x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x)\cos x - \sin x \cdot \cos(x+\Delta x)}{\cos x \cdot \cos(x+\Delta x) \cdot \Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x - x)}{\Delta x} \cdot \lim_{\Delta x \to 0} \frac{1}{\cos(x+\Delta x) \cdot \cos x}$$

$$= 1 \cdot \frac{1}{(\cos x)^2} = \sec^2 x$$

$(\cot x)' = -\csc^2 x$ hint: $\cot = \frac{\cos x}{\sin x}$ $\csc x = \frac{1}{\sin x}$

$$\lim_{\Delta x \to 0} \frac{\cot(x+\Delta x) - \cot x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{\cos(x+\Delta x)}{\sin(x+\Delta x)} - \frac{\cos x}{\sin x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin x \cos(x+\Delta x) - \sin(x+\Delta x) \cdot \cos x}{\Delta x \sin(x+\Delta x) \cdot \sin x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin[x-(x+\Delta x)]}{\Delta x} \cdot \lim_{\Delta x \to 0} \frac{1}{\sin(x+\Delta x) \cdot \sin x}$$

$$= -1 \cdot \frac{1}{\sin^2 x} = -\csc^2 x$$

$(\sec x)' = \sec x \tan x$

$$\lim_{\Delta x \to 0} \frac{\sec(x+\Delta x) - \sec x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{1}{\cos(x+\Delta x)} - \frac{1}{\cos x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cos x - \cos(x+\Delta x)}{\Delta x (\cos x \cdot \cos(x+\Delta x))} = \sin x \lim_{\Delta x \to 0} \frac{1}{\cos x \cos(x+\Delta x)}$$

$$= \sin x \cdot \frac{1}{\cos^2 x}$$

$$= \tan x \cdot \sec x$$

$(\csc x)' = -\csc x \cot x$ hint: $\csc x = \frac{1}{\sin x}$

$$\lim_{\Delta x \to 0} \frac{\csc(x+\Delta x) - \csc x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{1}{\sin(x+\Delta x)} - \frac{1}{\sin x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin x - \sin(x+\Delta x)}{\Delta x} \cdot \frac{1}{\sin x \sin(x+\Delta x)}$$

$$= -\cos x \cdot \frac{1}{\sin^2 x} = -\csc x \cdot \cot x$$

① $f(x) = c \quad f'(x) = 0$ ② $[f \pm g]' = f' \pm g'$

③ $(c \cdot f)' = c \cdot f'$ ④ $[f \cdot g]' = f'g + f \cdot g'$

⑤ $\left(\frac{f}{g}\right)' = \frac{f'g - f \cdot g'}{g^2}$

## Chain rule

**Lemma** (Alternative criterion for differentiability)

A function $f: (a,b) \to \mathbb{R}$ is differentiable at $x_0 \in (a,b)$ if and only if there exists a function $F: (a,b) \to \mathbb{R}$

$$f(x) = f(x_0) + (x - x_0) \cdot F(x) \qquad (1)$$

such that ① $F(x)$ is continuous at $x_0$,

② If $F(x)$ is differentiable at $x_0$, then

$$F(x) = \begin{cases} \frac{f(x) - f(x_0)}{x - x_0} & \text{if } x \neq x_0 \\ \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} & \text{if } x = x_0 \end{cases}$$

Proof: If $x \in (a,b) \setminus \{x_0\}$ then $F(x) = \frac{f(x) - f(x_0)}{x - x_0}$ that is

$$f(x_0) + (x - x_0) F(x) = f(x_0) + (f(x) - f(x_0)) = f(x)$$

then (1) holds.

To verify $F(x)$ is continuous at $x_0$, we $F(x) \to f'(x)$ as $x \to x_0$

then (1) holds as $x \to x_0$.

Thm (Chain rule). Suppose that $f : (a,b) \to (c,d)$ is differentiable at $x_0$, and $g : (c,d) \to \mathbb{R}$ is differentiable at $f(x_0)$. Then,

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$$

$g \circ f(x)$

$= g(f(x))$

$[g(f(x))]' = \frac{dg}{dx}$

$\frac{dg}{dx} = \frac{dg}{df} \cdot \frac{df}{dx}$

$g(h(f(x)))' = \frac{dg}{dx}$

$\frac{dg}{dx} = \frac{dg}{dh} \cdot \frac{dh}{df} \cdot \frac{df}{dx}$