RAC 2 Lecture Notes

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Contents

1 Basic materials

We begin by recalling some notation. We denote by $\mathbb{R}$ the set of real numbers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{Z}$ the set of integers, and $\mathbb{N} = \{1, 2, \ldots\}$ the set of natural numbers.
- The set of rational numbers $\mathbb{Q}$: can be denoted as $\frac{p}{q}$, where p and q ($q \neq 0$) are integers. 1
- The set of real numbers $\mathbb{R}$: includes both the rational numbers and irrational numbers (a real number that cannot be expressed as a ratio (or fraction) of two integers).

1.1 Some set theory

- If an element a belongs to a set A, we write $a \in A$; and if not we write $a \notin A$.
- If A is a subset of B (perhaps equal to B), we write

$$A \subseteq B \quad (\text{or} \quad B \supseteq A).$$

- Let A and B two subsets of a set X. Then,

$$A = B \qquad \text{iff}^{1} \qquad A \subseteq B \qquad \text{and} \qquad B \subseteq A.$$

- Let A and B be two sets. Then,

(1) The *union* of A and B, $A \cup B$, is the set defined by

$$A \cup B = \{x \in X \ : \ x \in A \quad \text{or} \quad x \in B\}.$$

(2) The *intersection* of A and B, $A \cap B$, is the set defined by

$$A \cap B = \{x \in X \ : \ x \in A \quad \text{and} \quad x \in B\}.$$

If $A \cap B = \emptyset$, then A and B have no points in common and A and B are said to be *disjoint*.

- Let A be a subset of a set X. Then, the *complement of A in X*, A^c (also $X - A$ or $X \backslash A$) is the set

$$A^c = \{x \in X \quad \text{such that} \quad x \notin A\}.$$

- Let I be any set (finite or not). For each element $i \in I$, we are given a subset A_i of X, then we denote the union and intersection of the collection of sets $\{A_i\}_{i \in I}$ by

$$\bigcup_{i \in I} A_i := \{x \in X : x \in A_i \text{ for some } \ i\} \qquad \text{and} \qquad \bigcap_{i \in I} A_i := \{x \in X : x \in A_i \text{ for all } i\},$$

respectively.

The following identities are known as the *De Morgan's Laws*,

(1)

$$\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} (A_i^c)$$

(1)

$$\left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} (A_i^c).$$

[1] Iff means "if and only if".

1.2 Intervals

An *interval* is a subset of $\mathbb{R}$ taking one of the following forms, where a and b are arbitrary real numbers and $a < b$.

$$\begin{aligned}
(a,b) &:= \{x \in \mathbb{R} : a < x < b\} && \text{(this is called an } \textit{open} \text{ interval)} \\
(a,b] &:= \{x \in \mathbb{R} : a < x \leq b\} \\
[a,b) &:= \{x \in \mathbb{R} : a \leq x < b\} \\
[a,b] &:= \{x \in \mathbb{R} : a \leq x \leq b\} && \text{(this is called a } \textit{closed} \text{ interval)} \\
(a,\infty) &:= \{x \in \mathbb{R} : x > a\} && \text{(not an } \textit{open} \text{ interval or a } \textit{closed} \text{ interval)} \\
[a,\infty) &:= \{x \in \mathbb{R} : x \geq a\} \\
(-\infty,b) &:= \{x \in \mathbb{R} : x < b\} \\
(-\infty,b] &:= \{x \in \mathbb{R} : x \leq b\} \\
(-\infty,\infty) &:= \mathbb{R}
\end{aligned}$$

The interval $[0,\infty)$ will also be denoted by $\mathbb{R}_+$, and $(0,\infty)$ will be denoted by $\mathbb{R}_{++}$.

1.3 Functions

Suppose that $X \subseteq \mathbb{R}$. A *real function* $f : X \to Y \subset \mathbb{R}$ is a rule which assigns to every real number $x \in X$ a unique real number $y \in Y$. If the number $y \in Y$ corresponds to the number $x \in X$, then we write $y = f(x)$.

- The *domain* of f, $Dom(f)$, is the set X.

- The *image* or the *range* of f is the set

$$f(X) = \{f(x) : x \in X\}$$

(sometimes we will also use the notation $Im(f)$ for $f(X)$.

- The *graph* of f is the set of points in $\mathbb{R}^2$ defined by

$$\{(x, f(x)) \in \mathbb{R}^2 : x \in X\}.$$

Example 1.1. The first two items are standard examples of real functions. The last item shows that sequences are essentially the same as functions.

1. $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = x^2$.
2. $f : [0,\infty) \to \mathbb{R}$, given by $f(x) = \sqrt{x}$.
3. If $(a_n)_{n\in\mathbb{N}}$ is a real sequence, then we may define a function $f : \mathbb{N} \to \mathbb{R}$ by $f(n) = a_n$ for every $n \in \mathbb{N}$.

Definition 1.2. Let $f : X \to \mathbb{R}$ be a real function.

1. The function f is said to be injective (or one-to-one) if for all a and b in X, $f(a) = f(b)$ implies $a = b$. Equivalently, if $a \neq b$, then $f(a) \neq f(b)$. In this case f is called an injection.

2. The function f is said to be surjective if $f(X) = \mathbb{R}$. In this case f is called a surjection.

3. The function f is said to be bijective if it is both injective and surjective. In this case f is called a bijection.

1.4 Euclidean distance on the real line

Definition 1.3. On $\mathbb{R}$, we define the *Euclidean distance on the real line* to be the function $d : \mathbb{R} \times \mathbb{R} \longrightarrow [0, \infty)$ defined by:

$$d(x, y) = |x - y|, \qquad \forall\, x, y \in \mathbb{R}.$$

- For any given real numbers x and $y \in \mathbb{R}$, we call $d(x, y)$ or $|x - y|$ the "distance between x and y".
- We call $\mathbb{R}$ with this distance the "Euclidean space on the real line".
- We will often write $|\cdot|$ meaning the Euclidean distance on the real line.

Proposition 1.4. *The Euclidean distance on $\mathbb{R}$ satisfies the following properties: for any $x, y, z \in \mathbb{R}$*

1. $|x - y| \geq 0$ *and* $|x - y| = 0$ *if and only if* $x = y$.
2. $|x - y| = |y - x|$ *(Symmetry)*
3. $|x - y| \leq |x - z| + |z - y|$ *(Triangle inequality for real numbers)*

Remark 1.5. Many of the concepts introduced in the "Real Analysis and the Calculus"-module, such as the notion of convergence of sequences of real numbers and that of the continuity of a real function, **are based on the idea of "distance" between real numbers and the associated idea of "closeness".** For example, recall the following definition: Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers and $l \in \mathbb{R}$. We say that (a_n) converges to l as $n \to \infty$, and write $a_n \to l$ as $n \to \infty$, if: Given any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - l| < \varepsilon \qquad \text{whenever} \qquad n \geq N.$$

In words, the sequence (a_n) converges to l as $n \to \infty$ if one can make the distance between the numbers of the sequence and l as small as we please by taking elements in the sequence associated to an index sufficiently large. What is it that makes the notion/definiton of convergence work? *Answer:* the notion of "closeness", that is the notion of "distance".

The above remark suggests that if we can find an appropriate notion of "distance" on a non-empty set X (not necessarily a subset of real numbers), we may give the definition of what it means for a sequence of elements of the set X to be a convergent sequence in X.

1.5 Open, closed and bounded sets

When studying the notions of continuity or differentiability for real functions, and also when studying the properties of real functions, we will often talk about open/closed intervals or closed bounded intervals. One may ask:

Q1. What do these notions (open/closed/bounded sets) mean?

Q2. What is the difference between the interval $(0, 1)$, $(0, 1]$, and $[0, 1]$?

In order to answer these two questions we need to introduce a preliminary definition.

Definition 1.6. Given $x_0 \in \mathbb{R}$, an *open interval centred at* x_0 is a set of real numbers of the form

$$(x_0 - r, x_0 + r) \qquad \text{for some} \qquad r > 0$$

Notice that

$$(x_0 - r, x_0 + r) = \{x \in \mathbb{R} \ : \ |x - x_0| < r\} = \{x \in \mathbb{R} \ : \ d(x, x_0) < r\}.$$

Thus, one can describe an open interval centred at a point in terms of the euclidean distance on the real line.

Definition 1.7. Let $\mathbb{R}$ be the set of real numbers.

(i) A subset $U \subseteq \mathbb{R}$ is an open set if:
For every $x \in U$, there exists $\varepsilon > 0$ (which may depend on x) such that

$$(x - \varepsilon, x + \varepsilon) \subseteq U.$$

In words, a set U of real numbers is an open set if it has the property that for every point x in U there is an open interval centred at the point that is contained in U. The open interval may vary with x, and it may be very small.

(ii) A subset $F \subseteq \mathbb{R}$ is a closed set if the complement of F in $\mathbb{R}$, F^c , is an open set.

(iii) A subset $Y \subseteq \mathbb{R}$ is a bounded set if there exists $K \in \mathbb{R}$ such that

$$|x| \leq K \qquad \text{for all } x \in Y.$$

We will continue to see some examples.

Examples.

1. Let $a < b$ be real numbers. The open interval (a, b) is open in $\mathbb{R}$

Indeed, given any $x \in (a, b)$, we can take $\varepsilon = \min\{|x - a|, |x - b|\}$, so that

$$(x - \varepsilon, x + \varepsilon) \subseteq (a, b).$$

Thus, (a, b) is an open set.

2. Let $a < b$ be real numbers. The closed interval $[a, b]$ is a closed set.

Indeed, notice that $[a, b]^c = (-\infty, a) \cup (b, \infty)$. We want to show that $(-\infty, a) \cup (b, \infty)$ is an open set. Now,

If $x \in (-\infty, a)$, take $\varepsilon = |x - a|/2 > 0$ (for example), then

$$(x - \varepsilon, x + \varepsilon) \subseteq (-\infty, a) \subseteq (-\infty, a) \cup (b, \infty).$$

If $x \in (b, \infty)$, take $\varepsilon = |x - b|/2 > 0$ (for example), then

$$(x - \varepsilon, x + \varepsilon) \subseteq (b, \infty) \subseteq (-\infty, a) \cup (b, \infty).$$

Thus, $(-\infty, a) \cup (b, \infty)$ is an open set, and therefore $[a, b]$ is a closed set.

3. Let $a, b \in \mathbb{R}$. Then, arguing in a similar way as above, we have that: $(-\infty, a)$ and (b, ∞) are open sets. Also, $(-\infty, a]$ and $[b, \infty)$ are closed sets.

4. Let $a < b$ be real numbers. The interval $[a, b)$ is not open and is not closed.

Indeed, to see that $[a, b)$ is not an open set, it suffices to find a point in the interval $[a, b)$ such that any open interval centred at the point contains points outside the set $[a, b)$. Consider the point a in $[a, b)$, then for any $\varepsilon > 0$

$$(a - \varepsilon, a + \varepsilon)$$

contains points which are not in the set $[a, b)$ (notice that all the points in $(a - \varepsilon, a)$ are not in $[a, b)$).

To see that $[a, b)$ is not a closed set we need to show that $[a, b)^c = (-\infty, a) \cup [b, \infty)$ is not an open set. Indeed, observe that there exists $b \in [a, b)^c$ such that any open interval centred at b has points outside the set $[a, b)^c$.

5. All of the intervals considered in **1,2 and 4** are bounded sets. For example: notice that for any $x \in (a, b)$, we have that $|x| \leq (|a| + |b|)$.

Exercise. Is the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ an open set? is it a closed set? is it a bounded set? (*Why?*)

The following theorems establish how open/closed sets behave with respect to taking union and intersections of open/closed sets.

Theorem 1.8.

(i) The union of an arbitrary family/collection of open sets is an open set.

(ii) The intersection of a finite number of open sets is an open set.

Proof. (*Non-examinable*)
Proof of (i). Let $\{U_i\}_{i \in I}$ be an arbitrary collection of open sets U_i. We want to show that $\bigcup_{i \in I} U_i$ is an open set. Indeed:

Given $x \in \bigcup_{i\in I}$, then there exists an index $i_o \in I$ such that $x \in U_{i_0}$ and since by hypothesis we know that U_{i_0} is an open set, then there exists $\varepsilon > 0$ such that

$$(x-r, x+r) \subseteq U_{i_0},$$

and since $U_{i_0} \subseteq \bigcup_{i\in I}$, we obtain that

$$(x-r, x+r) \subseteq U_{i_0} \subseteq \bigcup_{i\in I} U_i.$$

Thus, $\bigcup_{i\in I} U_i$ is an open set.

Proof of (ii). Let $\{U_1, \dots U_N\}$ be a finite collection of N-open sets U_i, $i \in \{1, \dots N\}$. We want to show that $\bigcap_{i=1}^N U_i$ is an open set. Indeed:

Given $x \in \bigcap_{i=1}^N U_i$, then for all $i \in \{1, \dots N\}$ we have that $x \in U_i$, and since each U_i is an open set (by hypothesis) we have that

$$\begin{aligned}
&\text{there exists } \varepsilon_1 > 0 \text{ such that } (x-\varepsilon_1, x+\varepsilon_1) \subseteq U_1,\\
&\vdots\\
&\text{there exists } \varepsilon_N > 0 \text{ such that } (x-\varepsilon_N, x+\varepsilon_N) \subseteq U_N.
\end{aligned}$$

Take $\varepsilon = \min(\varepsilon_1, \cdots, \varepsilon_N)/2 > 0$, then

$$(x-\varepsilon, x+\varepsilon) \subseteq U_i, \qquad \text{for all } i \in \{1, 2, \dots N\},$$

Thus

$$(x-\varepsilon, x+\varepsilon) \subseteq \bigcap_{i=1}^N U_i.$$

The above argument shows that $\bigcap_{i=1}^N U_i$ is an open set. □

As a straightforward consequence of Theorem 1.8 and the De Morgan's Laws, we obtain the following:

Theorem 1.9.

(i) The intersection of an arbitrary family/collection of closed sets is a closed set.

(ii) The union of a finite number of closed sets is a closed set.

Both Theorems 1.8 and 1.9 are useful tools to generate open/closed sets by taking (appropriate) unions or intersection of open/closed sets.

Example. Since for all $n \in \mathbb{N}$ the interval $(-n, n)$ is an open sets, using the above theorem we conclude that $\bigcup_{n=1}^\infty (-n, n) = \mathbb{R}$ is an open set.

2 Limits of functions

In this section we will define what we mean by expressions like

$$\lim_{x \to x_0} f(x) \qquad \text{or} \qquad \lim_{x \to \infty} f(x)$$

and their properties.

2.1 Limit at a point

We start with the definition of a limit at $x_0 \in \mathbb{R}$.

Definition 2.1. Let f be a real-valued function of a real variable. Suppose that $b < x_0 < c$ and that f is defined on the open interval (b, c) except possibly at x_0. We say that the limit of f, as x tends to x_0, is A if and only if: for any $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$|f(x) - A| < \varepsilon,$$

whenever $0 < |x - x_0| < \delta$. In this case, we write

$$\lim_{x \to x_0} f(x) = A\,.$$

Remark 2.2.

1. $\lim_{x \to x_0} f(x) = A$ means that as x goes to x_0, eventually $f(x)$ will be close enough to A. Or put it in another way, for any $\varepsilon > 0$, whenever x and x_0 are close within a distance of δ, $f(x)$ and A are close within a distance of ε. The choice of δ usually depends on ε and x_0.

2. The precise value of b and c are not important, as long as $x_0 \in (b, c)$. When looking for $\lim_{x \to x_0} f(x)$, we only care about the behavior of $f(x)$ when x is close to x_0.

3. The condition on x can be written as $x \in (b, c)$ and

$$x_0 - \delta < x < x_0 + \delta, \qquad x \neq x_0.$$

   When $\delta > 0$ is small enough, we may have $(x_0 - \delta, x_0 + \delta) \subset (b, c)$.

4. The exclusion of $x = x_0$ should be seen as an advantage. An inequality is required to be true for all x satisfying some condition but there is not need to check $x = x_0$. It may happen to be true that $|f(x) - A| < \varepsilon$ when $x = x_0$ but it is irrelevant to the definition. For instance, in the definition of a derivative

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

   must require that the value for $x = x_0$ be excluded since the expression is not defined there.

Here are some examples illustrating how to verify a limit directly from the definition.

Example 2.3. Show that

$$\lim_{x \to 8} x = 8.$$

Proof. For any $\varepsilon > 0$. We want to show that there is a $\delta > 0$ such that, whenever $|x-8| < \delta$, we have $|x-8| < \varepsilon$. We can take $\delta = \varepsilon$ (actually any positive $\delta < \varepsilon$ also works). Then by definition, $\lim\limits_{x \to 8} x = 8$. $\square$

Example 2.4. Show that
$$\lim_{x \to 8}(10x - 2) = 78.$$

Proof. For any $\varepsilon > 0$. We want to show that there is a $\delta > 0$ such that whenever $|x-8| < \delta$, we have
$$|(10x-2) - 78| < \varepsilon \iff |x-8| < \frac{\varepsilon}{10}$$
Thus, we can take $\delta = \frac{\varepsilon}{10}$ (actually any positive $\delta < \frac{\varepsilon}{10}$ also works). Then, by definition, $\lim\limits_{x \to 8}(10x-2) = 78$. $\square$

The definition gives no hints as how to compute the limit. It can be used only to verify the correctness of a limit statement.

Example 2.5. Show that
$$\lim_{x \to 8} x^2 = 64.$$

Proof. For any $\varepsilon > 0$. We want to show that there is a $\delta > 0$, such that whenever $|x-8| < \delta$, we have $|x^2 - 64| < \varepsilon$. Note that
$$|x^2 - 64| = |x-8||x+8|,$$
if we require $\delta \leq 1$, we obtain that $|x - 8| < 1$ which implies that $|x| < 9$ and hence $|x+8| \leq 17$. Thus, if $17|x-8| < \varepsilon$, then, $|x^2 - 64| \leq 17|x-8| < \varepsilon$.

So, we require $\delta \leq \frac{\varepsilon}{17}$ and $\delta \leq 1$. The choice $\delta = \min\left(\frac{\varepsilon}{17}, 1\right)$ will work.

Thus, by definition, $\lim\limits_{x \to 8} x^2 = 64$. $\square$

Here we introduce the squeeze theorem, as the squeeze theorem for the limits of sequences, it will be frequently used.

Theorem 2.6 (Squeeze Theorem (or Sandwich Theorem)). *Suppose that, for some $\alpha > 0$,*
$$f(x) \leq h(x) \leq g(x)$$
for all $x \in (x_0 - \alpha, x_0) \cup (x_0, x_0 + \alpha)$, and that
$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = A$$
exist. Then $\lim\limits_{x \to x_0} h(x) = A$.

Proof. For any $\varepsilon > 0$, by the definition of a limit and $\lim\limits_{x \to x_0} f(x) = A$, there exists $\delta_1 > 0$ with $\delta_1 < \alpha$, such that whenever $0 < |x - x_0| < \delta_1$,
$$|f(x) - A| < \varepsilon \iff A - \varepsilon < f(x) < A + \varepsilon$$

$\lim_{x\to x_0} g(x) = A$ means that there exists $\delta_2 > 0$, with $\delta_2 < \alpha$ such that whenever $0 < |x - x_0| < \delta_2$

$$|g(x) - A| < \varepsilon \iff A - \varepsilon < g(x) < A + \varepsilon\,.$$

So, take $\delta = \min(\delta_1, \delta_2)$, we see that whenever $0 < |x - x_0| < \delta$,

$$\begin{aligned} A - \varepsilon < f(x) < A + \varepsilon \\ A - \varepsilon < g(x) < A + \varepsilon\,, \end{aligned}$$

It follows that

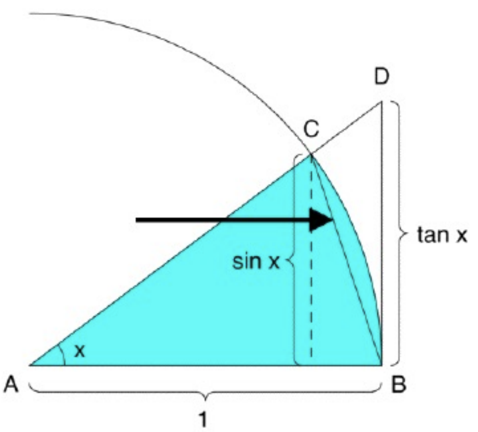
$$A - \varepsilon < f(x) \le h(x) \le g(x) < A + \varepsilon,$$

whenever $0 < |x - x_0| < \delta$. Therefore by definition we conclude $\lim\limits_{x\to x_0} h(x) = A$. □

Theorem 2.7.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1\,.$$

Proof. In the following picture



Let $|AB| = |AC| = 1$ and $\angle CAB = x \in (0, \frac{\pi}{2})$, then one height of the triangle ABC is $\sin x$ and $DB = \tan x$. Therefore,

(i) The area of the triangle ABC is $\frac{\sin x}{2}$.

(ii) The area of the segment ABC is $\frac{x}{2}$.

(iii) The area of the triangle ABD is $\frac{\tan x}{2}$.

Clearly, we have

$$\frac{\sin x}{2} < \frac{x}{2} < \frac{\tan x}{2},$$

for all $x \in (0, \frac{\pi}{2})$. It follows that

$$\cos x < \frac{\sin x}{x} < 1.$$

Since $\cos x$, $\frac{\sin x}{x}$ and 1 are all even function thus the above inequality holds for $x \in (-\frac{\pi}{2}, 0)$ as well. Therefore, for all $0 < |x| < \frac{\pi}{2}$, we conclude that

$$\left|\frac{\sin x}{x} - 1\right| < 1 - \cos x = 2\sin^2\frac{x}{2} \le 2\left(\frac{x}{2}\right)^2 \le \frac{x^2}{2}.$$

The conclusion follows from the squeeze theorem. □

Remark 2.8. The limit $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$ is very important. From the proof demonstrated above, we have the following three useful consequences:

1. $|\sin x| < |x|$, for all $x \neq 0$;
2. $\lim_{x\to 0} \sin x = 0$;
3. $\lim_{x\to 0} \cos x = 1$.
4. $\lim_{x\to 0} \dfrac{\cos x - 1}{x} = 0$.

The following example is very important in showing the continuity of exponential functions.

Example 2.9. Let $a > 1$, show that $\lim_{x\to 0} a^x = 1$.

Proof. For any $\varepsilon \in (0, 1)$, set $\delta = \min\{-\log_a(1 - \varepsilon), \log_a(1 + \varepsilon)\}$. Then, we have

$$x < \log_a(1 + \varepsilon), \quad -x < -\log_a(1 - \varepsilon),$$

for all $0 < |x| < \delta$. It follows that

$$\log_a(1 - \varepsilon) < x < \log_a(1 + \varepsilon).$$

Therefore,

$$1 - \varepsilon < a^x < 1 + \varepsilon,$$

i.e.

$$|a^x - 1| < \varepsilon.$$

By the definition it follows that $\lim_{x\to 0} a^x = 1$. □

2.2 One-sided limits and limits as x tends to ∞

Definition 2.10. Suppose that the domain of f contains $(a - \alpha, a)$ for some $\alpha > 0$ and $A \in \mathbb{R}$. If for all $\varepsilon > 0$, there exists $\delta > 0$ with $\delta < \alpha$, such that

$$|f(x) - A| < \varepsilon$$

holds for all $a - \delta < x < a$, then we say the limit of f, as x tends to a from below or the left, is A. In this case we write $f(x) \to A$ as $x \to a^-$, or

$$\lim_{x\to a^-} f(x) = A.$$

Similarly, we can define $\lim_{x\to a^+} f(x)$.

From the above definition, we have the following fact:

Proposition 2.11. *$\lim_{x\to a} f(x) = A$ if and only if both $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and*

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = A\,.$$

Example 2.12. Prove that $\lim_{x\to 0^-} e^{1/x} = 0$.

Proof. For any $\varepsilon \in (0,1)$, set $\delta = -\frac{1}{\ln\varepsilon} > 0$. For all x such that $-\delta < x < 0$, we have

$$\left|e^{1/x} - 0\right| = e^{1/x} < e^{-1/\delta} = \varepsilon.$$

By the definition we obtain $\lim_{x\to 0^-} e^{1/x} = 0$. $\square$

Definition 2.13. Suppose that the domain of f contains (α, ∞) for some $\alpha \in \mathbb{R}$ and $A \in \mathbb{R}$. If for all $\varepsilon > 0$, there exists $K > \alpha$, such that

$$|f(x) - A| < \varepsilon$$

holds for all $x > K$, then we say the limit of f, as x tends to ∞, is A. In this case we write $f(x) \to A$ as $x \to \infty$, or

$$\lim_{x\to\infty} f(x) = A.$$

Similarly, we can define $\lim_{x\to-\infty} f(x)$ and $\lim_{|x|\to\infty} f(x)$.

For the limits as x tends to ∞, the squeeze theorem holds as well.

Example 2.14. Prove $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$.

Proof. For $x \geq 1$, we have

$$\left(1+\frac{1}{[x]+1}\right)^{[x]} \leq \left(1+\frac{1}{x}\right)^x \leq \left(1+\frac{1}{[x]}\right)^{[x]+1},$$

i.e.

$$\left(1+\frac{1}{[x]+1}\right)^{[x]+1}\left(1+\frac{1}{[x]+1}\right)^{-1} \leq \left(1+\frac{1}{x}\right)^x \leq \left(1+\frac{1}{[x]}\right)^{[x]}\left(1+\frac{1}{[x]}\right).$$

Note that $[x] \in \mathbb{Z}^+$. By using the following sequence limit

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

and the squeeze theorem we obtain the desired result. $\square$

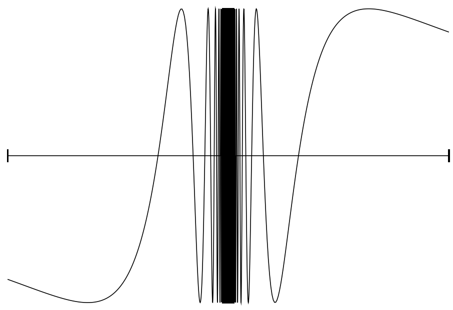
2.3 Non-existence of a limit

Similar to the sequence case, a limit of a function, as x tends to some points or ∞, does not necessarily exist.

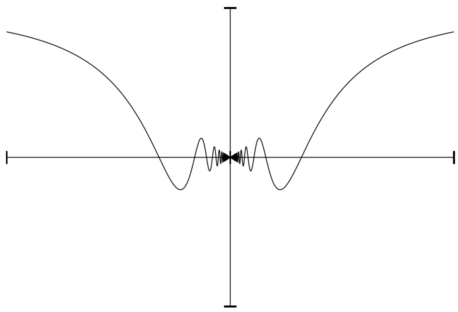
Due to the definition of limits, if $\lim_{x\to x_0} f(x) \neq A$, either the limit $\lim_{x\to x_0} f(x)$ does not exist or exists but not A, then there exists $\varepsilon_0 > 0$, such that for any $\delta > 0$ there exists x_δ with $0 < |x_\delta - x_0| < \delta$, we have

$$|f(x_\delta) - A| > \varepsilon_0.$$

Example 2.15. Show that $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ does not exist, but $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$ exists.



$\sin(\frac{1}{x})$



$x\sin(\frac{1}{x})$

Proof. Let us only work with the first one, since the second limit follows from the squeeze theorem.

The proof is obtained by contradiction. Assume that the first limit exists, say

$$\lim_{x\to 0}\sin\left(\frac{1}{x}\right) = A \in \mathbb{R}.$$

Then, there exists $\delta > 0$, such that

$$\left|\sin\left(\frac{1}{x}\right) - A\right| < \frac{1}{2}, \qquad \forall\ 0 < |x| < \delta.$$

By Archimedean properties we have $n \in \mathbb{Z}^+$ large enough such that $\frac{1}{2n\pi+\frac{\pi}{2}} < \frac{1}{2n\pi} < \delta$. Thus it follows that

$$|\sin(2n\pi) - A| < \frac{1}{2}$$

and

$$\left|\sin\left(2n\pi + \frac{\pi}{2}\right) - A\right| < \frac{1}{2},$$

since $\sin(2n\pi) = 0$ and $\sin\left(2n\pi + \frac{\pi}{2}\right) = 1$, we have $|A| < \frac{1}{2}$ and $|1-A| < \frac{1}{2}$, a contradiction. $\square$

A special case of non-existence of a limit is divergence to ∞, or it is usually referred as 'the limit is ∞'. '∞' is not a number, so if we say that 'the limit is ∞', we mean that the limit does not exist.

Definition 2.16. Assume there exists $\alpha > 0$ such that $(x_0 - \alpha, x_0) \cup (x_0, x_0 + \alpha) \subset \mathbf{dom}(f)$. We say that the limit of f, as x tends to x_0, is ∞ if and only if: for any $M > 0$, there exists a $\delta > 0$, such that

$$f(x) > M,$$

whenever $0 < |x - x_0| < \delta$. And we write

$$\lim_{x \to x_0} f(x) = \infty$$

Similarly, one can define $\lim_{x \to x_0} f(x) = -\infty$. Furthermore, one may define the limit of f is ∞, as $x \to x_0^-$, $x \to x_0^+$, $x \to \infty$, and $x \to -\infty$.

Example 2.17. Prove $\lim_{x \to 0^-} \frac{1}{x} = -\infty$.

Proof. For any $M > 0$, set $\delta = \frac{1}{M}$. We have

$$\frac{1}{x} < \frac{1}{-\delta} = -M$$

for all $-\delta < x < 0$. Thus by definition we conclude $\lim_{x \to 0^-} \frac{1}{x} = -\infty$. □

2.4 Properties of limits

The properties of limits of functions are similar to those of sequences. Here we only list them without proofs, which could be obtained by mimicking what we have done for limits of sequences.

Theorem 2.18. *In the following, the abbreviation* lim *denotes the limit with one of the following situations: as $x \to x_0$, $x \to x_0^-$, $x \to x_0^+$, $x \to \infty$, and $x \to -\infty$.*

1. *Uniqueness. If $\lim f(x) = a$ and $\lim f(x) = b$, then $a = b$.*
2. *Squeeze theorem. If $f(x) \leq g(x) \leq h(x)$ and both $\lim f(x) = \lim h(x) = l$ exist, then $\lim g(x) = l$.*
3. *Boundedness. If $\lim_{x \to x_0}$ exists, then there exists $\delta > 0$, such that f is bounded on $(x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)$.*
4. *Algebra of limits. Suppose $\lim f(x)$ and $\lim g(x)$ exist and c is a constant. Then,*
   - *(a)* $\lim cf(x) = c \lim f(x)$
   - *(b)* $\lim(f(x) \pm g(x)) = \lim f(x) \pm \lim g(x)$.
   - *(c)* $\lim[f(x)g(x)] = \lim f(x) \lim g(x)$.
   - *(d)* $\lim \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)}$, *provided $g(x) \neq 0$ and $\lim g(x) \neq 0$.*
5. *If $f(x) \geq 0$ or $f(x) > 0$, then $\lim f(x) \geq 0$ if the limit exists.*

Remark 2.19. The proof of the last property is by contradiction. Suppose $\lim_{x\to x_0} f(x) = A < 0$, then we take $\varepsilon = -\frac{A}{2}$, so there is a $\delta > 0$ such that whenever $|x - x_0| < \delta$, $|f(x) - A| < -\frac{A}{2}$, that is, $\frac{3A}{2} < f(x) < \frac{A}{2} < 0$, which is a contradiction. For $f(x) > 0$, one can NOT conclude that $\lim f(x) > 0$. For example, $f(x) = \frac{1}{x} > 0$ for all $x > 0$, but $\lim_{x\to\infty} f(x) = 0$.

We also have some simple facts about the limits.

Proposition 2.20. *Let c be a constant, assume that $\lim f(x)$ exist, n is a positive integer, a is any real number, then:*

$$\begin{aligned}
&\lim[f(x)^n] = [\lim f(x)]^n\,,\\
&\lim c = c\\
&\lim_{x\to a} x = a\\
&\lim_{x\to a} x^n = a^n\\
&\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}\\
&\lim \sqrt[n]{f(x)} = \sqrt[n]{\lim f(x)}
\end{aligned}$$

Example 2.21. Find

$$\lim_{x\to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

Solution:

$$\begin{aligned}
\lim_{x\to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} &= \frac{\lim_{x\to -2}(x^3 + 2x^2 - 1)}{\lim_{x\to -2}(5 - 3x)}\\
&= \frac{\lim_{x\to -2} x^3 + 2\lim_{x\to -2} x^2 - \lim_{x\to -2} 1}{\lim_{x\to -2} 5 - 3\lim_{x\to -2} x}\\
&= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)}\\
&= -\frac{1}{11}
\end{aligned}$$

Example 2.22. Find

$$\lim_{x\to 1} \frac{x^2 - 1}{x - 1}\,.$$

Solution: We can not apply the quotient law, since $\lim_{x\to 1}(x - 1) = 0$, but we can write

$$\frac{x^2 - 1}{x - 1} = x + 1\,,$$

the above equation holds whenever $x \neq 1$, and when we are taking the limit as $x \to 1$, we dont consider what happens when x is actually equal to 1, thus,

$$\lim_{x\to 1} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1}(x + 1) = 2\,.$$

Example 2.23. Find

$$\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}$$

Solution:

$$\begin{aligned}
\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2} &= \lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}\cdot\frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}\\
&= \lim_{t\to 0}\frac{(t^2+9)-9}{t^2(\sqrt{t^2+9}+3)}\\
&= \lim_{t\to 0}\frac{t^2}{t^2(\sqrt{t^2+9}+3)}\\
&= \lim_{t\to 0}\frac{1}{\sqrt{t^2+9}+3}\\
&= \frac{1}{3+3} = \frac{1}{6}
\end{aligned}$$

Example 2.24. Prove that $\lim_{x\to\infty}\dfrac{3x^2+2x-1}{x^2-2} = 3.$

Proof. For $x > 2$, we have[2]

$$\begin{aligned}
\lim_{x\to\infty}\frac{3x^2+2x-1}{x^2-2} &= \lim_{x\to\infty}\frac{3+\frac{2}{x}-\frac{1}{x^2}}{1-\frac{2}{x^2}}\\
&= \frac{\lim_{x\to\infty}\left(3+\dfrac{2}{x}-\dfrac{1}{x^2}\right)}{\lim_{x\to\infty}\left(1-\dfrac{2}{x^2}\right)} = 3.
\end{aligned}$$

□

One of the basic tools in constructing complicated functions out of simple ones is composition.

Theorem 2.25. *Suppose $\lim_{y\to y_0} f(y) = A$ and $\lim_{x\to x_0} g(x) = y_0$. Furthermore, there exists $\delta > 0$ such that $g(x) \neq y_0$ for all $x \in (x_0-\delta, x_0)\cup(x_0, x_0+\delta)$. Then, $\lim_{x\to x_0} f(g(x)) = A$.*

Proof. For any $\varepsilon > 0$, since $\lim_{y\to y_0} f(y) = A$ there exists $\eta > 0$ such that

$$|f(y) - A| < \varepsilon$$

holds for all $0 < |y - y_0| < \eta$. Since $\lim_{x\to x_0} g(x) = y_0$ there exists $\delta > 0$ such that

$$|g(x) - y_0| < \eta$$

holds for all $0 < |x - x_0| < \delta$. Therefore, for all $0 < |x - x_0| < \delta$, we have

$$|f(g(x)) - A| < \varepsilon.$$

By definition we have $\lim_{x\to x_0} f(g(x)) = A$. □

[2]In the proof, we make use of the limit $\lim_{x\to\infty}\frac{1}{x} = 0$, which can be verified by definition easily.

Remark 2.26. The idea of Theorem 2.25 is that $g(x) \to y_0$ as $x \to x_0$, so in $\lim_{x\to x_0} f(g(x))$, we can replace $g(x)$ by y and obtain $\lim_{x\to x_0} f(g(x)) = \lim_{y\to y_0} f(y)$.

Remark 2.27. In the above theorem, with minor changes, one may replace x_0 and y_0 by $x_0^{\pm}$, $y_0^{\pm}$, and $\pm\infty$.

Example 2.28. Prove $\lim\limits_{x\to-\infty} \left(1+\frac{1}{x}\right)^x = e.$

Proof. Set $y = -(x+1)$. Then, as $x \to -\infty$, we have $y \to +\infty$. Therefore,

$$\begin{aligned}
\lim_{x\to-\infty} \left(1+\frac{1}{x}\right)^x = \lim_{y\to\infty} \left(1-\frac{1}{y+1}\right)^{-(y+1)} &= \lim_{y\to\infty} \left(\frac{y}{y+1}\right)^{-(y+1)} \\
&= \lim_{y\to\infty} \left(1+\frac{1}{y}\right)^{y+1} = e.
\end{aligned}$$

□

Example 2.29. Prove $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = e.$

Proof. Set $y = \frac{1}{x}$. Then, as $x \to 0$, we have $y \to \pm\infty$ (or $|y| \to \infty$). Therefore,

$$\lim_{x\to 0} (1+x)^{\frac{1}{x}} = \lim_{|y|\to\infty} \left(1+\frac{1}{y}\right)^y = e.$$

□

Theorem 2.30. $\lim\limits_{x\to x_0} f(x) = l$ *if and only if, for any sequence* $\{x_n\} \subset \mathbf{dom}(f)$ *satisfying* $x_n \neq x_0$ *and* $x_n \to x_0$ *as* $n \to \infty$*, we have* $\lim\limits_{n\to\infty} f(x_n) = l$.

Proof. $\Longrightarrow$. If $\lim\limits_{x\to x_0} f(x) = l$, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $0 < |x - x_0| < \delta$, we have

$$|f(x) - A| < \varepsilon.$$

Since $x_n \to \infty$ as $n \to \infty$ there exists $N \in \mathbb{Z}^+$ such that $|x_n - x_0| < \delta$ whenever $n > N$. It follows that when $n > N$, we have

$$0 < |x_n - x_0| < \delta,$$

then

$$|f(x_n) - A| < \varepsilon.$$

Therefore, we conclude that $\lim\limits_{n\to\infty} f(x_n) = l$.

$\Longleftarrow$. Prove by contradiction. If $\lim\limits_{x\to x_0} f(x) \neq l$, either the limit $\lim\limits_{x\to x_0} f(x)$ does not exist or exists but not l, then there exists $\varepsilon_0 > 0$, such that for any $\delta > 0$ there exists x_δ with $0 < |x_\delta - x_0| < \delta$, we have

$$|f(x_\delta) - l| > \varepsilon_0.$$

By setting $\delta_n = \frac{1}{n}$, we can choose a sequence $\{x_n\}$, such that

$$0 < |x_n - x_0| < \frac{1}{n}, \quad |f(x_n) - l| \geq \varepsilon_0.$$

This sequence $\{x_n\}$ satisfies $x_n \neq x_0$ and $x_n \to x_0$ as $n \to \infty$, but $\lim\limits_{n\to\infty} f(x_n) \neq l$. This contradicts with the precondition. Therefore $\lim\limits_{x\to x_0} f(x) = l$. $\square$

Example 2.31. Show $\lim\limits_{x\to 0} \sin\left(\dfrac{1}{x}\right)$ does not exist.

Proof. Let us define the sequence $x_n := \frac{1}{\pi n + \frac{\pi}{2}}$. It is not hard to see that $\lim x_n = 0$. Furthermore,

$$\sin\left(\frac{1}{x_n}\right) = \sin\left(\pi n + \frac{\pi}{2}\right) = (-1)^n.$$

Therefore, $\{\sin(1/x_n)\}$ does not converge. Thus, by by the Theorem 1.1.5 $\lim\limits_{x\to 0} \sin\left(\dfrac{1}{x}\right)$ does not exist. $\square$

3 Continuous functions

In your high-school the concept of a continuous function is that if you draw its graph without lifting the pen from the paper. This intuition may be useful in simple situations, but we require rigorous treatment.

3.1 Definition

Definition 3.1. Suppose there exists $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta)$ is a subset of $\mathbf{dom}(f)$. We say f is continuous at x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

If f is continuous at all $x \in \mathbf{dom}(f)$, then we say f is a continuous function. If $\lim_{x \to x_0} f(x) \neq f(x_0)$, we say that f is discontinuous at x_0.

Remark 3.2. Notice that the definition implicitly requires three things if f is continuous at x_0:

(1) $f(x_0)$ is defined (that is, x_0 is in the domain of f)

(2) $\lim_{x \to x_0} f(x)$ exists.

(3) $\lim_{x \to x_0} f(x) = f(x_0)$.

Remark 3.3.

1. If f is a continuous at x_0 and the sequence $\{x_n\} \subset (x_0 - \delta, x_0 + \delta)$ has limit x_0 as $n \to \infty$, then we have

   $$\lim_{n \to \infty} f(x_n) = f\left(\lim_{n \to \infty} x_n\right).$$

   By Theorem 2.25, if f is continuous at x_0 and $\lim_{z \to z_0} g(z) = x_0$, then

   $$\lim_{z \to z_0} f(g(z)) = f\left(\lim_{z \to z_0} g(z)\right) = f(x_0)\,.$$

   In particular, if f is continuous everywhere in the domain, then

   $$\lim_{z \to z_0} f(g(z)) = f\left(\lim_{z \to z_0} g(z)\right).$$

2. Similarly, one can define left continuous and right continuous by using one-sided limits. For instance, when we say f is continuous on a closed interval $[a, b]$, if f is continuous on (a, b) and left/right continuous at two end points $\{a, b\}$.

3. The $\varepsilon - \delta$ definition of continuous at x_0 is: for any $\varepsilon > 0$, there exists $\delta > 0$ such that

   $$|f(x) - f(x_0)| < \varepsilon,$$

   for all $|x - x_0| < \delta$, here we assume that all $x \in \mathbf{dom}(f)$.

Example 3.4 (Continuity of constant functions)**.** Prove that the constant function $f(x) = C$ is continuous.

Proof. For any $x_0 \in \mathbb{R}$, we have

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} C = C = f(x_0).$$

The conclusion follows by the definition. $\square$

By using the facts $\lim_{x \to 0} \sin x = 0$ and $\lim_{x \to 0} \cos x = 1$, we show both $\sin x$ and $\cos x$ are continuous.

Example 3.5 (Continuity of trigonometric functions)**.** Prove that $\sin x$ and $\cos x$ are continuous.

Proof. For any $x_0 \in \mathbb{R}$, by Remark 1.1.1 we have

$$\begin{aligned}
\lim_{x \to x_0} \sin x &= \lim_{x \to x_0} \sin(x_0 + (x - x_0)) \\
&= \lim_{x \to x_0} \left[\sin x_0 \cos(x - x_0) + \cos x_0 \sin(x - x_0)) \right] \\
&= \sin x_0 \lim_{x \to x_0} \cos(x - x_0) + \cos x_0 \lim_{x \to x_0} \sin(x - x_0)) \\
&= \sin x_0.
\end{aligned}$$

Thus $\sin x$ is continuous. Similarly, we can show $\cos x$ is also continuous. $\square$

Example 3.6 (Continuity of exponential functions)**.** Prove that a^x with $a > 0$ is continuous.

Proof. For $a > 1$, we have

$$\lim_{x \to x_0} a^x = a^{x_0} \lim_{x \to x_0} a^{x - x_0} = a^{x_0} \lim_{x \to 0} a^x = a^{x_0}.$$

For $0 < a < 1$, we have $\frac{1}{a} > 1$ and

$$\lim_{x \to x_0} a^x = a^{x_0} \lim_{x \to x_0} a^{x - x_0} = a^{x_0} \lim_{x \to 0} a^x = a^{x_0} \frac{1}{\lim_{x \to 0} \left(\frac{1}{a} \right)^x} = a^{x_0}.$$

Thus we finish the proof. $\square$

Example 3.7. $f(x) = |x|$ is continuous on $\mathbb{R}$.

3.2 Continuity of Elementary functions

First recall the convention of elementary functions

Definition 3.8 (Elementary Function)**.** An elementary function is a function of one variable which is the composition of a finite number of arithmetic operations $(+, -, \times, \div)$, exponentials, trigonometries, inverse trigonometries, logarithms, constants, and power functions.

In this section, we would like to prove a beautiful result, which says that all elementary functions are continuous (on their domain). To this purpose, we need the following theorem

Theorem 3.9. *1. If both $f(x)$ and $g(x)$ are continuous at x_0, then $f(x) \pm g(x)$, $f(x)g(x)$, and $\frac{f(x)}{g(x)}$ with $g(x_0) \neq 0$ are all continuous at x_0.*

2. If $f(y)$ and $g(x)$ are continuous at $g(x_0)$ and x_0 respectively, then

$$\lim_{x \to x_0} f(g(x)) = f(g(x_0)),$$

i.e. $f(g(x))$ is continuous at x_0.

3. If f is continuous on $[a, b]$ and f^{-1} is its inverse, then f^{-1} is also continuous.

Proof. Due to Theorem 2.18 and 2.25, only 3 needs a proof.

We prove by contradiction. Let x_0 is in the domain of f. Choose $\delta > 0$ small enough such that $N_\delta(x_0) := (x_0 - \delta, x_0 + \delta) \subset \mathbf{dom}(f)$ with $f(N_\delta(x_0))$ bounded (Theorem 1.1.3 - 3).

Assume that f^{-1}, which maps $f(N_\delta(x_0))$ to $N_\delta(x_0)$, is not continuous at $y_0 \in f(N_\delta(x_0))$. Then, there exists a sqeuence $\{y_n\} \subset f(N_\delta(x_0))$ converging to y_0 such that $\{f^{-1}(y_n)\} \subset N_\delta(x_0)$ does not converge to x_0. Then by Bolzano-Weierstrass there exist a subsequence $\{f^{-1}(y_{n_k})\}$ that converges to, say $x_1 \neq x_0$. By the continuity of f, we have

$$f(x_1) = \lim_{k \to \infty} f(f^{-1}(y_{n_k})) = \lim_{k \to \infty} y_{n_k} = y_0.$$

This contradicts with that f has inverse. □

Example 3.10. Prove that $\ln x$ is continuous.

Proof. Since e^x is continuous, thus the desired result follows from Theorem 3.9. □

Example 3.11 (Continuity of power functions)**.** Prove that x^α with $\alpha \in \mathbb{R}$ is continuous on $[0, \infty)]$.[3] [4] [5]

[3]With a common convention that $0^0 = 1$.

[4]We exclude $x < 0$ since for some power α the domain of x^α may be only $[0, \infty)$. For instance, when $\alpha = \frac{1}{2}$.

[5]When the domain of x^α is $\mathbb{R}$, the continuity can be easily extended to $(-\infty, 0)$.

Proof. Assume $x_0 > 0$. We have

$$\lim_{x\to x_0} x^\alpha = \lim_{x\to x_0} e^{\ln(x^\alpha)} = \lim_{x\to x_0} e^{\alpha \ln x} = e^{\alpha \lim\limits_{x\to x_0} \ln x} = e^{\alpha \ln x_0} = x_0^\alpha.$$

Thus we finish the proof.

It is easy to check the power function is right continuous at $x_0 = 0$. □

Theorem 3.12. *Elementary functions are continuous (on their domain).*

Proof. The proof is obtained by collecting some proofs that we have done. For the six types of basic elementary functions, we have

1. Constant functions: Example 3.4.
2. Trigonometries: Example 3.5 and Theorem 3.9 - 1.
3. Inverse trigonometries: Theorem 3.9 - 3.
4. Exponentials: Example 3.6.
5. Logarithms: Theorem 3.9 - 3.
6. Power functions: Example 3.11.

Since an elementary function is obtained from basic elementary functions with finite many times of compositions, taking inverse, and arithmetic operations. Thus the proof is completed by using the theorem 3.9. □

Next we state an important limit which plays a crucial rule in differential calculus.

Theorem 3.13.

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1\,.$$

Proof. Set $y = e^x$. Then, as $x \to 0$, we have $y \to 1$ by Example 3.6. Therefore,

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{y\to 1} \frac{y-1}{\ln y}.$$

Then, set $z = y - 1$. Thus as $y \to 1$ we have $z \to 0$.

$$\lim_{y\to 1} \frac{y-1}{\ln y} = \lim_{z\to 0} \frac{z}{\ln(1+z)} = \frac{1}{\lim\limits_{z\to 0} \ln(1+z)^{\frac{1}{z}}} = \frac{1}{\ln \lim\limits_{z\to 0}(1+z)^{\frac{1}{z}}} = 1.$$

Where we used Example 2.29 in the last step. □

3.3 Discontinuous functions

First let us introduce a corollary of Theorem 2.30.

Corollary 3.14. *Let f be a function. Suppose that for some $x_0 \in \mathbf{dom}(f)$, there exists a sequence $\{x_n\}$, $x_n \in \mathbf{dom}(f)$, and $\lim x_n = x_0$ such that $\{f(x_n)\}$ does not converge to $f(x_0)$ (or does not converge at all), then f is not continuous at x_0.*

Example 3.15. The function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0, \end{cases}$$

is not continuous at 0.

Proof. Take the sequence $\{-\frac{1}{n}\}$. Then $f(-\frac{1}{n}) = -1$ and so $\lim f(-\frac{1}{n}) = -1$, but $f(0) = 1$. □

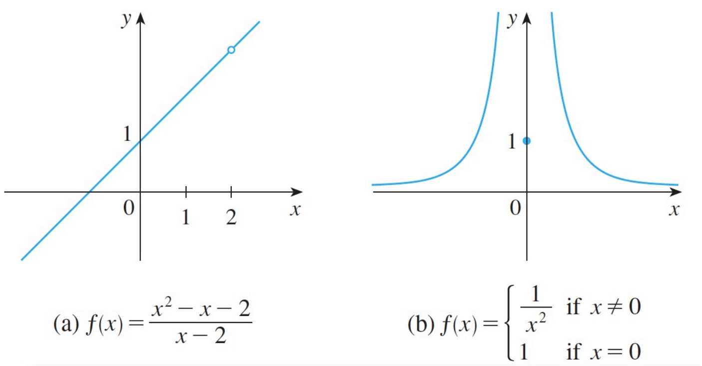
Example 3.16. The *Dirichlet function*,

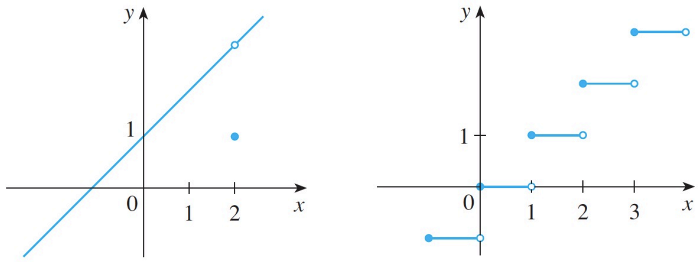
$$f(x) := \begin{cases} 1 & \text{if } x \text{ is rational}, \\ 0 & \text{if } x \text{ is irrational}, \end{cases}$$

is discontinuous at all $x_0 \in \mathbb{R}$.

Proof. Suppose x_0 is rational. Take a sequence $\{x_n\}$ of irrational numbers such that $\lim x_n = x_0$. Then $f(x_n) = 0$ and so $\lim f(x_n) = 0$, but $f(x_0) = 1$. If x_0 is irrational, take a sequence of rational numbers $\{x_n\}$ that converges to x_0. Then $\lim f(x_n) = 1$, but $f(x_0) = 0$. □

The picture below shows some functions with discontinuities. The kind of discontinuity illustrated in parts (a) and (c) is called removable because we could remove the discontinuity by redefining f at just the single number 2. [The function $g(x) = x + 1$ is continuous.] The discontinuity in part (b) is called an infinite discontinuity. The discontinuities in part (d) are called jump discontinuities because the function jumps from one value to another.





(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

(d) $f(x) = [\![x]\!]$

3.4 Properties of continuous functions on $[a, b]$

Recall that f is continuous on a closed interval $[a, b]$, if f is

1. continuous at every point on (a, b),
2. left continuous at b, i.e. $\lim\limits_{x \to b^-} f(x) = f(b)$,
3. right continuous at a, i.e. $\lim\limits_{x \to a^+} f(x) = f(a)$.

For such continuous functions defined on closed and bounded intervals have some nice and useful properties.

Theorem 3.17 (Boundedness). *Let f be continuous on $[a, b]$. Then f is bounded.*

Proof. Let us prove this theorem by contradiction. Suppose f is not bounded, then for each $n \in \mathbb{N}$, there is an $x_n \in [a, b]$, such that

$$|f(x_n)| \geq n.$$

Now $\{x_n\}$ is a bounded seuqence as $a \leq x_n \leq b$. By the Bolzano-Weierstrass theorem, there is a convergent subsequence $\{x_{n_k}\}$. Let $x_0 := \lim\limits_{k \to \infty} x_{n_k}$. Since $a \leq x_{n_k} \leq b$ for all k, then $a \leq x_0 \leq b$. The limit $\lim f(x_{n_k})$ does not exist because the sequence is not bounded ($|f(x_{n_k})| \geq n_k \geq k$). On the other hand $f(x_0)$ is a finite number and

$$f(x_0) = f\left(\lim_{k \to \infty} x_{n_k}\right).$$

Thus f is not continuous at x_0. Therefore, from this contradiction, we conclude that f is bounded. $\square$

Theorem 3.18 (Minimum-maximum theorem). *Let $f\colon [a, b] \to \mathbb{R}$ be a continuous function. Then f achieves both minimum and maximum on $[a, b]$.*

Proof. We have shown that f is bounded. Therefore, the set $f([a,b]) = \{f(x) : x \in [a,b]\}$ has a supremum. From what we know about suprema, there are sequences $\{f(x_n)\}$, where x_n is in $[a,b]$, such that

$$\lim_{n\to\infty} f(x_n) = \sup f([a,b]).$$

We know $\{x_n\}$ is bounded (their elements belong to a bounded interval $[a,b]$). We apply the Bolzano-Weierstrass theorem. Hence there exist convergent subsequences $\{x_{n_i}\}$. Let

$$x_0 := \lim_{i\to\infty} x_{n_i}.$$

Then as $a \le x_{n_i} \le b$, we have that $a \le x_0 \le b$. We apply that a limit of a subsequence is the same as the limit of the sequence, and we apply the continuity of f to obtain

$$\sup f([a,b]) = \lim_{n\to\infty} f(x_n) = \lim_{i\to\infty} f(x_{n_i}) = f\left(\lim_{i\to\infty} x_{n_i}\right) = f(x_0).$$

Therefore, f achieves a maximum at x_0.

Similarly, we can prove that f achieves its minimum on $[a,b]$. □

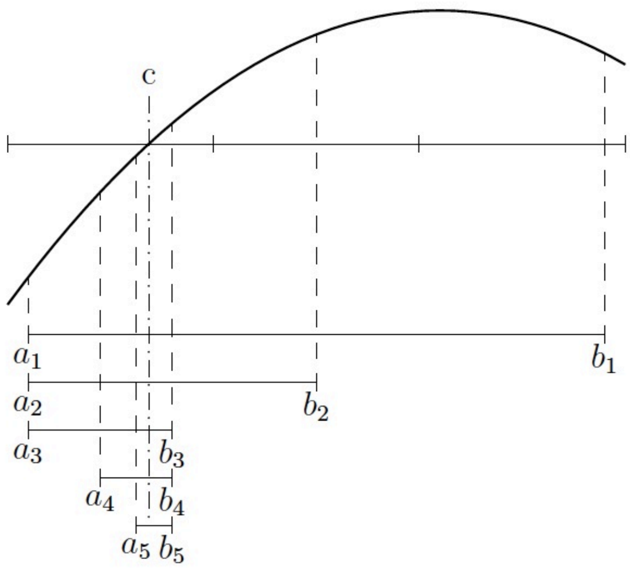
Here the minimum (maximum) is also called absolute minimum (absolute maximum). This is to distinguish with “local minimum” (“local maximum”), which means that there is a $\delta > 0$ such that $f(a) \le f(x)$ $(f(a) \ge f(x))$ for all $x \in (a-\delta, a+\delta)$.

The following intermediate value theorem is one of the **cornerstones** of analysis.

Theorem 3.19 (Intermediate Value Theorem)**.** *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Suppose $f(a) < 0$ and $f(b) > 0$. Then there exists a number $c \in (a,b)$ such that $f(c) = 0$.*

Proof. We define two sequences $\{a_n\}$ and $\{b_n\}$ inductively:

1. Let $a_1 := a$ and $b_1 := b$.
2. If $f\left(\frac{a_n+b_n}{2}\right) \ge 0$, let $a_{n+1} := a_n$ and $b_{n+1} := \frac{a_n+b_n}{2}$.
3. If $f\left(\frac{a_n+b_n}{2}\right) < 0$, let $a_{n+1} := \frac{a_n+b_n}{2}$ and $b_{n+1} := b_n$.



bisection method

See above for an example defining the first five steps. From the definition of the two sequences it is obvious that if $a_n < b_n$, then $a_{n+1} < b_{n+1}$. Thus by induction we have $a_n < b_n$ for all n. Furthermore, $a_n \leq a_{n+1}$ and $b_n \geq b_{n+1}$ for all n, that is the sequences are monotone. As $a_n < b_n \leq b_1 = b$ and $b_n > a_n \geq a_1 = a$ for all n, the sequences are also bounded. Therefore, the sequences converge. Let $c := \lim\, a_n$ and $d := \lim\, b_n$. We now want to show that $c = d$. We notice

$$b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}.$$

By induction we see that

$$b_n - a_n = \frac{b_1 - a_1}{2^{n-1}} = 2^{1-n}(b-a).$$

As $2^{1-n}(b-a)$ converges to zero, we take the limit as n goes to infinity to get

$$d - c = \lim_{n\to\infty} (b_n - a_n) = \lim_{n\to\infty} 2^{1-n}(b-a) = 0.$$

In other words $d = c$.

By construction, for all n we have

$$f(a_n) < 0 \qquad \text{and} \qquad f(b_n) \geq 0.$$

We use the fact that $\lim\, a_n = \lim\, b_n = c$ and the continuity of f to take limits in those inequalities to get

$$f(c) = \lim\, f(a_n) \leq 0 \qquad \text{and} \qquad f(c) = \lim\, f(b_n) \geq 0.$$

As $f(c) \geq 0$ and $f(c) \leq 0$, we conclude $f(c) = 0$. Obviously, $a < c < b$. □

Example 3.20. Let $f(x)$ be a polynomial of odd degree. Then f has a real root.

Proof. Assume the coefficient of the leading term is positive, then we have

$$\lim_{x\to\infty} f(x) = \infty$$

and

$$\lim_{x\to-\infty} f(x) = -\infty.$$

Therefore, the conclusion follows from the intermediate value theorem. □

3.5 Uniform continuity

Definition 3.21 (Uniform continuity)**.** Let $S \subset \mathbb{R}$, and let $f\colon S \to \mathbb{R}$ be a function. Suppose for any $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $x, c \in S$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$. Then we say f is *uniformly continuous.*

Remark 3.22.

1. The negation of the definition of uniformly continuous is: *There exists an $\varepsilon_0 > 0$ such that for every $\delta > 0$, there exist points x, y in S with $|x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon_0$.*

2. f is not uniformly continuous on S if and only if there exist sequences $\{a_n\}$ and $\{b_n\}$ in S with $\lim\limits_{n\to\infty}(a_n - b_n) = 0$, $\varepsilon_0 > 0$, such that

$$|f(a_n) - f(b_n)| \geq \varepsilon_0.$$

It is not hard to see that a uniformly continuous function must be continuous. The only difference in the definitions is that for a given $\varepsilon > 0$ we pick a $\delta > 0$ that works for all $c \in S$. That is, δ can no longer depend on c, it only depends on ε.

Example 3.23. The function $f\colon (0,1) \to \mathbb{R}$, defined by $f(x) := \frac{1}{x}$ is not uniformly continuous, but it is continuous.

Proof. Given $\varepsilon > 0$, then for $\varepsilon > |\frac{1}{x} - \frac{1}{y}|$ to hold we must have

$$\varepsilon > |\frac{1}{x} - \frac{1}{y}| = \frac{|y-x|}{|xy|} = \frac{|y-x|}{xy},$$

or

$$|x - y| < xy\varepsilon.$$

Therefore, to satisfy the definition of uniform continuity we would have to have $\delta \leq xy\varepsilon$ for all x, y in $(0,1)$, but that would mean that $\delta \leq 0$. Therefore there is no single $\delta > 0$.

Another way to show this is to set $a_n = \frac{1}{n+1}$ and $b_n = \frac{1}{n}$. We have

$$\lim_{n\to\infty}(a_n - b_n) = 0,$$

but

$$|f(a_n) - f(b_n)| = 1, \quad \forall\ n.$$

Therefore $f(x)$ is not uniformly continuous on $(0,1)$. $\square$

Example 3.24. $f\colon [0,1] \to \mathbb{R}$, defined by $f(x) := x^2$ is uniformly continuous.

On the other hand, $f\colon \mathbb{R} \to \mathbb{R}$, defined by $f(x) := x^2$ is not uniformly continuous.

Proof. Note that $0 \leq x, c \leq 1$. Then

$$|x^2 - c^2| = |x+c||x-c| \leq (|x| + |c|)\,|x - c| \leq (1+1)\,|x-c|\,.$$

Therefore given $\varepsilon > 0$, let $\delta := \frac{\varepsilon}{2}$. If $|x - c| < \delta$, then $|x^2 - c^2| < \varepsilon$.

Set $a_n = n + \frac{1}{n}$ and $b_n = n$. We have

$$\lim_{n\to\infty}(a_n - b_n) = 0,$$

but

$$|f(a_n) - f(b_n)| = (n + 1/n)^2 - n^2 \geq 2, \quad \forall\ n.$$

Therefore $f(x)$ is not uniformly continuous on $\mathbb{R}$. $\square$

We have seen that if f is defined on an interval that is either not closed or not bounded, then f can be continuous, but not uniformly continuous. For a closed and bounded interval $[a, b]$, we can, however, make the following statement.

Theorem 3.25. *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Then f is uniformly continuous.*

Proof. We prove the statement by contradiction. Suppose f is not uniformly continuous. We will prove that there is some $c \in [a,b]$ where f is not continuous. Let us negate the definition of uniformly continuous:

There exists an $\varepsilon_0 > 0$ such that for every $\delta > 0$, there exist points x, y in S with $|x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon_0$.

So for the $\varepsilon_0 > 0$ above, we find sequences $\{x_n\}$ and $\{y_n\}$ such that $|x_n - y_n| < \frac{1}{n}$ and such that $|f(x_n) - f(y_n)| \geq \varepsilon_0$. By Bolzano-Weierstrass, there exists a convergent subsequence $\{x_{n_k}\}$. Let $c := \lim x_{n_k}$. As $a \leq x_{n_k} \leq b$, then $a \leq c \leq b$. Write

$$|y_{n_k} - c| = |y_{n_k} - x_{n_k} + x_{n_k} - c| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - c| < \frac{1}{n_k} + |x_{n_k} - c| \,.$$

As $\frac{1}{n_k}$ and $|x_{n_k} - c|$ both go to zero when k goes to infinity, $\{y_{n_k}\}$ converges and the limit is c. We now show that f is not continuous at c. We estimate

$$\begin{aligned} |f(x_{n_k}) - f(c)| &= |f(x_{n_k}) - f(y_{n_k}) + f(y_{n_k}) - f(c)| \\ &\geq |f(x_{n_k}) - f(y_{n_k})| - |f(y_{n_k}) - f(c)| \\ &\geq \varepsilon_0 - |f(y_{n_k}) - f(c)| \,. \end{aligned}$$

Or in other words

$$|f(x_{n_k}) - f(c)| + |f(y_{n_k}) - f(c)| \geq \varepsilon_0.$$

At least one of the sequences $\{f(x_{n_k})\}$ or $\{f(y_{n_k})\}$ cannot converge to $f(c)$. Thus f cannot be continuous at c. $\square$

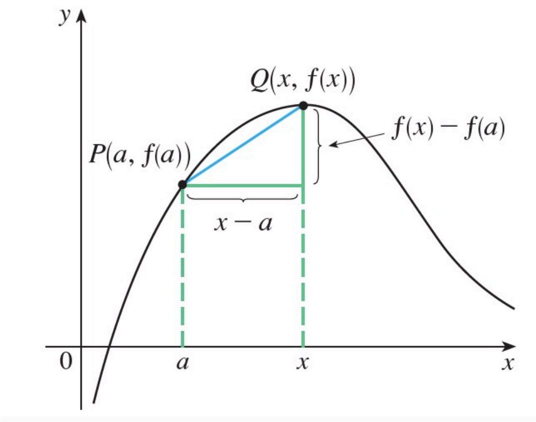
4 Differentiation

We begin with some basic ideas of the derivative.

Tangents

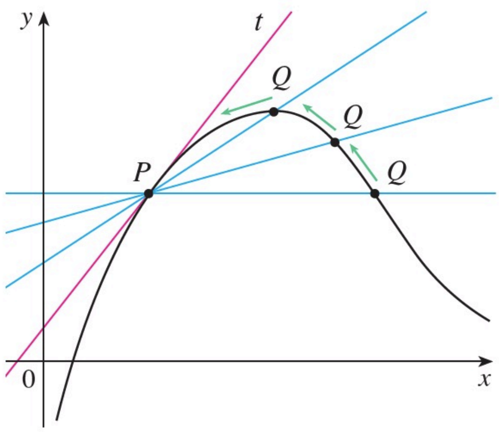
If a curve C has equation $y = f(x)$ and we want to find the tangent line to C at the point $P = (a, f(a))$, then we consider a nearby point $Q = (x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}.$$



Secant line PQ

Then we let Q approach P along the curve C by letting x approach a. If m_{PQ} approaches a number m, then we define the tangent t to be the line through P with slope m. (This amounts to saying that the tangent line is the limiting position of the secant line PQ as Q approaches P.



Q approaches P.

Definition 4.1. The tangent line to the curve $y = f(x)$ at the point $P = (a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Another way to write m is

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Velocities

Suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t. The function f that describes the motion is called the position function of the object. In the time interval from $t = a$ to $t = a + h$, the change in position is $f(a+h) - f(a)$. The average velocity over this time interval is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}.$$

Now suppose we compute the average velocities over shorter and shorter time intervals $[a, a+h]$. In other words, we let h approach 0. we define the velocity (or instantaneous velocity) $v(a)$ at time $t = a$ to be the limit of these average velocities

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Limits of this form arise whenever we calculate a rate of change in any of the sciences or engineering, such as a rate of reaction in chemistry or a marginal cost in economics. Since this type of limit occurs so widely, it is given a special name and notation.

4.1 Definition and basic properties

Definition 4.2. A real function $f : (a, b) \to \mathbb{R}$ is differentiable at a point $x_0 \in (a, b)$ if the limit

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. If this limit exists, it is called the derivative of f at x_0 , and denoted $f'(x_0)$

The limit in this definition can be also written as

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Often we write $y = f(x)$, and then an alternative notation for $f'(x_0)$ is

$$\left(\frac{dy}{dx}\right)_{x=x_0} \text{ or } \left.\frac{dy}{dx}\right|_{x=x_0}.$$

Example 4.3. Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = |x|$. Then f is differentiable at all points in $\mathbb{R} \backslash \{0\}$ and f is not differentiable at 0.

Answer. When $x_0 > 0$ and h is small, that is, when $|h| < x_0$, we see that $x_0 + h > 0$. So

$$\frac{f(x_0+h)-f(x_0)}{h} = \frac{x_0+h-x_0}{h} = \frac{h}{h} = 1$$

Therefore,

$$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = 1$$

When $x_0 < 0$ and h is small, that is, when $|h| < |x_0|$, we see that $x_0 + h < 0$. So

$$\frac{f(x_0+h)-f(x_0)}{h} = \frac{-(x_0+h)-(-x_0)}{h} = \frac{-h}{h} = -1$$

Therefore

$$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = -1$$

We have now shown that f is differentiable at all points in $\mathbb{R}\backslash\{0\}$.

Finally, when $x_0 = 0$, we see that

$$\frac{f(h)-f(0)}{h} = \frac{|h|-0}{h} = \frac{|h|}{h}$$

and so

$$\lim_{h\to 0^+}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} 1 = 1$$

and

$$\lim_{h\to 0^-}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^-} -1 = -1\,.$$

Thus, $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$ does not exist at $x_0 = 0$(see Proposition 2.11). Thus f is not differentiable at 0, as claimed.

Example 4.4. Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^{1/3}$. Is f differentiable at 0? If so, find $f'(0)$, and if not, explain why not.

Answer: Observe that

$$\frac{f(0+h)-f(0)}{h} = \frac{f(h)}{h} = \frac{h^{1/3}}{h} = \frac{1}{h^{2/3}}$$

and $1/h^{2/3}$ increases without bound as $h \to 0$. Thus $f'(0)$ does not exist.

The following is an important result which says that differentiability at a point implies continuity at this point.

Theorem 4.5. *If a function is differentiable at a point $x_0 \in \mathbb{R}$, then it is continuous at x_0.*

Proof. Since f is differentiable at x_0,

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

exists. By the algebra of limits,

$$\lim_{h\to 0}(f(x_0+h)-f(x_0)) = \lim_{h\to 0}\left(\frac{f(x_0+h)-f(x_0)}{h}\cdot h\right)$$
$$= \lim_{h\to 0}\left(\frac{f(x_0+h)-f(x_0)}{h}\right)\cdot\left(\lim_{h\to 0} h\right) = f'(x_0)\cdot 0 = 0\,.$$

This means

$$\lim_{h\to 0} f(x_0+h) = f(x_0)$$

and therefore

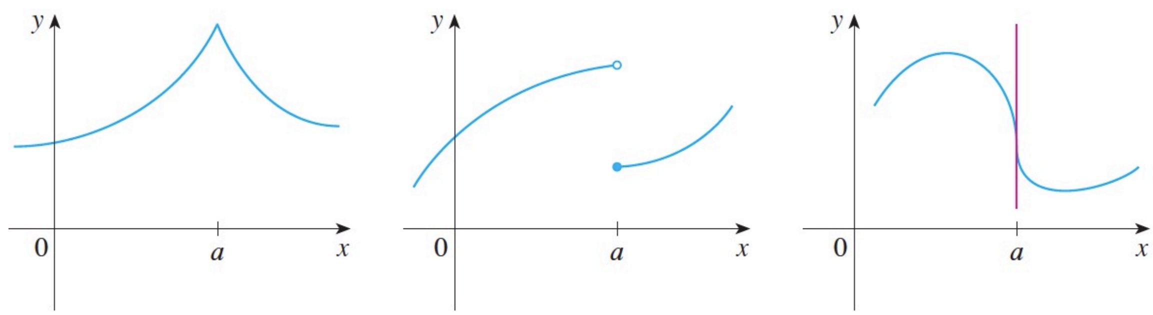
$$\lim_{x\to x_0} f(x) = f(x_0)\,.$$

It follows that f is continuous at x_0. □

Corollary 4.6. *If f is not continuous at a, then it is not differentiable at a.*

Remark 4.7. The converse of Theorem 4.5 is *not true*. Indeed, consider the absolute value function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$. It is true that f is continuous on $\mathbb{R}$. However, by Example 4.3 we know that f is not differentiable at 0.

The following picture gives three ways that the function can fail to be differentiable at a.



(a) A corner (b) A discontinuity (c) A vertical tangent

4.2 Differentiation rules

So far we have considered the derivative of a function $y = f(x)$ at a fixed number a. Here we change our point of view and let the number a vary. If we replace a by a variable x, we obtain

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\,,$$

given any number x for which this limit exists. So we can regard f' as a new function, called the derivative of f.

We can also write

$$f'\,,\quad \frac{df}{dx}\,,\quad \frac{dy}{dx}\,,\quad \frac{d}{dx}f(x)\,,\quad Df(x)\,,\quad Df\,.$$

Definition 4.8. A function f is differentiable on an open interval (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every point in the interval.

Differentiation rules:

1. Derivative of a Constant Function

$$\frac{d}{dx}(C) = 0\,.$$

Proof.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{C - C}{h} = \lim_{h\to 0} 0 = 0$$

□

2. The Power Rule: If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}\,.$$

Proof. Now we only prove the case when n is a positive integer.

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{x^n - a^n}{x - a} \\ &= \lim_{x\to a} \left(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}\right) \\ &= a^{n-1} + a^{n-2}a + \cdots + aa^{n-2} + a^{n-1} \\ &= na^{n-1} \end{aligned}$$

□

3. The Constant Multiple Rule: Let $c \in \mathbb{R}$ be a constant, and f be differentiable at x, then, cf is also differentiable at x and

$$(cf)'(x) = cf'(x)\,.$$

Proof.

$$\lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h} = c \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = cf'(x)\,.$$

Thus, cf is differentiable at x_0 and $(cf)'(x_0) = cf'(x_0)$. □

4. The Sum Rule: If f and g are both differentiable at x, then $f + g$ is also differentiable at x and

$$(f + g)'(x) = f'(x) + g'(x)\,.$$

Thus, writing $f - g$ as $f + (-1) \times g$ and apply (3) by taking $c = -1$, we obtain that

$$(f - g)'(x) = f'(x) - g'(x)\,.$$

Proof. For $h \neq 0$,

$$\frac{(f+g)(x+h)-(f+g)(x)}{h}=\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}$$

so, by the properties of limits, it follows that

$$\begin{aligned}\lim_{h\to 0}\frac{(f+g)(x+h)-(f+g)(x)}{h}&=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\&=f'(x)+g'(x)\end{aligned}.$$

Hence, $f+g$ is differentiable at x and $(f+g)'(x)=f'(x)+g'(x)$. □

5. Product Rule: If f and g are both differentiable at x, then, fg is also differentiable at x and
$$(fg)'(x)=f'(x)g(x)+f(x)g'(x)\,.$$

Proof.

$$\begin{aligned}(fg)'(x)&=\lim_{h\to 0}\frac{(fg)(x+h)-(fg)(x)}{h}\\&=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\&=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}\\&=\lim_{h\to 0}\left[\frac{f(x+h)g(x+h)-f(x+h)g(x)}{h}+\frac{f(x+h)g(x)-f(x)g(x)}{h}\right]\\&=\lim_{h\to 0}\left[f(x+h)\frac{g(x+h)-g(x)}{h}\right]+\lim_{h\to 0}\left[g(x)\frac{f(x+h)-f(x)}{h}\right]\\&=\lim_{h\to 0}f(x+h)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}+g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\&=f(x)g'(x)+g(x)f'(x)\ ,\end{aligned}$$

since f is continuous at x. □

6. Quotient Rule: If f and g are differentiable at x, then, $\frac{f}{g}$ is also differentiable provided $g(x)$ is not 0, and,
$$\left(\frac{f}{g}\right)'(x)=\frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}\,.$$

Proof.

$$\begin{aligned}(f/g)'(x)&=\lim_{h\to 0}\frac{(f/g)(x+h)-(f/g)(x)}{h}\\&=\lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h}\\&=\lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x+h)}{g(x)g(x+h)h}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{1}{g(x)g(x+h)} \\
&\quad \times \lim_{h\to 0}\left[\frac{f(x+h)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(x+h)}{h}\right] \\
&= \frac{1}{(g(x))^2}\lim_{h\to 0}\left[\frac{f(x+h)g(x)-f(x)g(x)}{h}-\frac{f(x)g(x+h)-f(x)g(x)}{h}\right] \\
&= \frac{1}{(g(x))^2}\left[g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}-f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right] \\
&= \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2},
\end{aligned}
$$

since g is continuous at x. ☐

7. Differentiation of Trigonometric Functions: the functions $\sin : \mathbb{R} \to \mathbb{R}$ and $\cos : \mathbb{R} \to \mathbb{R}$ are differentiable on $\mathbb{R}$ and,

$$(\sin x)' = \cos x \quad (\cos x)' = -\sin x\,.$$

By writing

$$\tan x = \frac{\sin x}{\cos x}\,, \quad \cot x = \frac{\cos x}{\sin x}\,, \quad \sec x = \frac{1}{\cos x}\,, \quad \csc x = \frac{1}{\sin x}$$

and applying the quotient rule, we obtain that

$$(\tan x)' = \sec^2 x\,, \quad (\cot x)' = -\csc^2 x\,, \quad (\sec x)' = \sec x \tan x\,, \quad (\csc x)' = -\csc x \cot x\,.$$

Proof. From the trigonometric identity $\sin(x+h) = \cos h \sin x + \sin h \cos x$,

$$
\begin{aligned}
\frac{\sin(x+h)-\sin(x)}{h} &= \frac{\cos h \sin x + \sin h \cos x - \sin x}{h} \\
&= \frac{\sin h}{h}\cos x - \frac{1-\cos h}{h}\sin x \\
&= \frac{\sin h}{h}\cos x - \frac{1-\cos^2 h}{(1+\cos h)h}\sin x \\
&= \frac{\sin h}{h}\cos x - \frac{\sin h}{1+\cos h}\,\frac{\sin h}{h}\sin x\,.
\end{aligned}
$$

The sine and cosine functions are continuous, and $\lim_{h\to 0}\sin h = 0$ and $\lim_{h\to 0}\cos h = 1$. We also know that $\lim_{h\to 0}\frac{(\sin h)}{h} = 1$, so by the algebra of limits we conclude that

$$\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} = \cos x,$$

$(\cos x)' = -\sin x$ can be proved similarly. ☐

8. Differentiation of e^x: The function e^x is differentiable on $\mathbb{R}$, and

$$(e^x)' = e^x\,.$$

Proof.

$$\lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = e^x \lim_{h\to 0}\frac{e^h-1}{h} = e^x$$

by 3.13. □

9. Differentiation of $\ln x$: $\ln x$ is differentiable on $(0,\infty)$ and

$$(\ln x)' = \frac{1}{x}.$$

Proof. Since

$$\frac{\ln(x+h)-\ln x}{h} = \frac{\ln(1+\frac{h}{x})}{h},$$

let $\ln(1+\frac{h}{x}) = u$, then $u = x(e^u - 1)$, and $u \to 0$ as $h \to 0$, by Theorem 2.25 we have

$$\lim_{h\to 0}\frac{\ln(1+\frac{h}{x})}{h} = \lim_{u\to 0}\frac{u}{x(e^u-1)} = \frac{1}{x}.$$

□

Note that by writing

$$\log_a x = \frac{\ln x}{\ln a},$$

we obtain that

$$(\log_a x)' = \frac{1}{x\ln a}.$$

Now we come to the differentiation for composite functions that are differentiable at a point. Recall that the composed function $g\circ f$ is defined by $g\circ f(x) = g(f(x))$ for all x (for which this makes sense). It is helpful to have a preliminary lemma.

Lemma 4.9 (Alternative criterion for differentiability at a point)**.** *A function $f:(a,b)\to\mathbb{R}$ is differentiable at $x_0\in(a,b)$ if and only if there exists a function $F:(a,b)\to\mathbb{R}$ that satisfies*

$$f(x) = f(x_0) + (x-x_0)F(x) \quad \textit{for all } x\in(a,b), \tag{4.1}$$

and is continuous at x_0. Furthermore, if f is differentiable at x_0 then the function F is given by

$$F(x) = \begin{cases} \dfrac{f(x)-f(x_0)}{x-x_0} & \textit{if } x\neq x_0 \\ f'(x_0) & \textit{if } x = x_0. \end{cases}$$

Proof. A function F satisfies (4.1) if and only if

$$F(x) = \frac{f(x)-f(x_0)}{x-x_0}$$

for all $x\in(a,b)\setminus\{x_0\}$. If this holds, then F is continuous at x_0 if and only if $\lim_{x\to x_0}F(x)$ exists and is equal to $F(x_0)$. However, this is true if and only if f is differentiable at x_0 and $f'(x_0) = F(x_0)$. □

Theorem 4.10 (Chain Rule)**.** *Suppose that $f : (a, b) \to (c, d)$ is differentiable at $x_0 \in (a, b)$ and that $g : (c, d) \to \mathbb{R}$ is differentiable at $f(x_0)$. Then the composed function $g \circ f$ is differentiable at x_0, and*

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0).$$

Proof. By the alternative criterion for differentiability, applied to the function g and the point $f(x_0)$, there is a function $G : (c, d) \to \mathbb{R}$, continuous at $f(x_0)$, such that

$$g(y) = g(f(x_0)) + (y - f(x_0))G(y) \quad \text{for all } y \in (c, d)$$

and $G(f(x_0)) = g'(f(x_0))$. By substituting $f(x_0 + h)$ for y, we deduce that

$$g(f(x_0 + h)) - g(f(x_0)) = (f(x_0 + h) - f(x_0))\, G(f(x_0 + h)).$$

Hence, for all $h \neq 0$,

$$\begin{aligned}\frac{g \circ f(x_0 + h) - g \circ f(x_0)}{h} &= \frac{g(f(x_0 + h)) - g(f(x_0))}{h} \\ &= G(f(x_0 + h))\, \frac{f(x_0 + h) - f(x_0)}{h}.\end{aligned}$$

Since f is differentiable at x_0,

$$\frac{f(x_0 + h) - f(x_0)}{h} \to f'(x_0)$$

as $h \to 0$, and by Theorem 3.9 (composition of continuous functions),

$$G(f(x_0 + h)) \to G(f(x_0)) = g'(f(x_0))$$

as $h \to 0$. Here we have also used the fact that f is continuous at x_0, which follows from Theorem 4.5.

Hence by the algebra of limits,

$$\lim_{h \to 0} \frac{g \circ f(x_0 + h) - g \circ f(x_0)}{h} = g'(f(x_0))f'(x_0).$$

□

Thus when x is a variable the chain rule can be written as

$$(g(f(x)))' = g'(f(x))f'(x)$$

If $y = g(u)$ and $u = f(x)$, so $y = g(f(x))$, another way to write the chain rule is

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example 4.11. Find the derivatives of the following functions.

$$1 : f(x) = 4x^3 - 5x^2 + 2x + 1$$

$$2 : y = \frac{x^2 + x - 2}{x^3 + 6}$$

$$3 : y = \frac{1}{x}$$
$$4 : y = x^2 \sin x$$

Solution: For 1,

$$f'(x) = 12x^2 - 10x + 2$$

For 2,

$$\frac{dy}{dx} = \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3+6)^2}$$

For 3: we can write $y = x^{-1}$, then,

$$\frac{dy}{dx} = -1x^{-2} = \frac{-1}{x^2}\,.$$

For 4:

$$y' = 2x \sin x + x^2 \cos x\,.$$

Example 4.12. Differentiate

$$1 : y = \sin(x^2), \quad 2 : y = \sin^2 x\,.$$

Solution:

$$1 : y' = 2x\cos(x^2)\,, \quad 2 : y' = 2\sin x \cos x\,.$$

Example 4.13. Find $f'(x)$ if

$$f(x) = \frac{1}{\sqrt[3]{x^2 + x + 1}}$$

Solution: Write

$$f(x) = (x^2 + x + 1)^{-\frac{1}{3}}\,,$$

then

$$f'(x) = -\frac{1}{3}(x^2 + x + 1)^{-\frac{4}{3}}(2x + 1)\,.$$

Example 4.14. Differentiate

$$f(x) = (2x+1)^5(x^3 - 2x + 3)^3$$

Solution: Apply product rule, and then the chain rule:

$$f'(x) = 10(2x+1)^4(x^3 - 2x + 3)^3 + 3(2x+1)^5(x^3 - 2x + 3)^2(3x^2 - 2)$$

Example 4.15. Differentiate

$$y = \sin(\cos(\tan x))\,.$$

Solution:

$$y' = -\cos(\cos(\tan x))\sin(\tan x)\sec^2 x\,.$$

Example 4.16 (Differentiation of a^x)**.**

$$(a^x)' = a^x \ln a\,.$$

Proof. Write

$$a^x = e^{x \ln a}\,,$$

so

$$(a^x)' = \left(e^{x \ln a}\right)' = e^{x \ln a}(x \ln a)' = e^{x \ln a} \ln a = a^x \ln a\,.$$

□

Proof of the Power Rule when n is not necessarily a positive integer. Write

$$x^n = e^{n \ln x}\,,$$

thus,

$$\frac{d}{dx} x^n = \frac{d}{dx} e^{n \ln x} = e^{n \ln x}\,(n \ln x)' = e^{n \ln x}\frac{n}{x} = n x^{n-1}\,.$$

□

To study the differentiation of inverse functions, we state the Inverse Function Theorem.

Theorem 4.17. *Suppose that $f : [a,b] \to [c,d]$ is continuous and strictly increasing, that $f(a) = c$ and that $f(b) = d$. Then the inverse function $f^{-1} : [c,d] \to [a,b]$ exists, is continuous, strictly increasing and surjective. The same holds if both occurrences of increasing are replaced by decreasing.*

The proof for existence of f^{-1} is easy, the continuity of f^{-1} is from Theorem 3.9 (3).

Theorem 4.18. *Suppose that f satisfies the conditions of the Inverse Function Theorem, and that $x_0 \in (a,b)$ and $y_0 = f(x_0)$. If in addition f is differentiable at x_0 and $f'(x_0) \neq 0$, then f^{-1} is differentiable at y_0, and*

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Proof. Take x close to, but different from x_0, and write $y = f(x)$ as well as $y_0 = f(x_0)$; equivalently, $x = f^{-1}(y)$ and $x_0 = f^{-1}(y_0)$.

Since f is differentiable at x_0,

$$\frac{f(x) - f(x_0)}{x - x_0} \to f'(x_0) \quad \text{as } x \to x_0.$$

Hence

$$\frac{y - y_0}{f^{-1}(y) - f^{-1}(y_0)} \to f'(x_0) \quad \text{as } x \to x_0.$$

Because $f'(x_0) \neq 0$, when x is close enough to x_0, the quotient on the left hand side is nonzero, and by the algebra of limits,

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} \to \frac{1}{f'(x_0)} \quad \text{as } x \to x_0.$$

Since f is continuous, $y \to y_0$ when $x \to x_0$; since f^{-1} is also continuous, $x \to x_0$ when $y \to y_0$. Thus $y \to y_0$ exactly when $x \to x_0$, and it follows that

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} \to \frac{1}{f'(x_0)} \quad \text{as } y \to y_0.$$

This means that f^{-1} is indeed differentiable at y_0 and that its derivative is $\frac{1}{f'(x_0)}$, which is what we were required to show. $\square$

Note that we may also write

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}.$$

An easy way to remember this is to write $y = f(x)$, and then $x = f^{-1}(y)$, so

$$(f^{-1})'(y) = \frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1} = \frac{1}{f'(x)}\,.$$

Example 4.19 (Differentiation of inverse trigonometric functions)**.** Recall that the function $\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ is differentiable at every point in $(-\frac{\pi}{2}, \frac{\pi}{2})$. By Theorem 4.17, the inverse function arcsin is a well-defined continuous function on $[-1, 1]$. Write $f(x) = \sin(x)$ and $g(y) = \arcsin(y)$, and take $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $y = f(x)$. Then $y \in (-1, 1)$ if and only if $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

For such y, by Theorem 4.18, g is differentiable at y, and

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{\cos(x)} = \frac{1}{\sqrt{1 - \sin^2(x)}} = \frac{1}{\sqrt{1 - y^2}}.$$

(we take the positive square root because cos is positive when $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.) So

$$\frac{d}{dy}(\arcsin y) = \frac{1}{\sqrt{1 - y^2}}.$$

The function $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to (-\infty, \infty)$ is differentiable at every point in $(-\frac{\pi}{2}, \frac{\pi}{2})$. By Theorem 4.17 applied to any closed subinterval of $(-\frac{\pi}{2}, \frac{\pi}{2})$, the inverse function arctan is a well-defined continuous function on $(-\infty, \infty)$.

Write $y = f(x) = \tan(x)$ and $x = g(y) = \arctan(y)$, and take $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{\sec^2 x} = \frac{1}{1 + \tan^2 x} = \frac{1}{1 + y^2}\,.$$

So,

$$\frac{d}{dy}\,(\arctan y) = \frac{1}{1 + y^2}\,.$$

By writing

$$\arccos x = \frac{\pi}{2} - \arcsin x \quad \text{and} \quad \mathrm{arccot} x = \frac{\pi}{2} - \arctan x$$

we obtain that

$$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1 - x^2}} \quad \text{and} \quad \frac{d}{dx}\mathrm{arccot} x = \frac{-1}{1 + x^2}\,.$$

We summarize the differentiation formulas as follows:

$$
\begin{array}{ll}
(cf)' = cf' \quad c \text{ is a constant} & \\
(f+g)' = f' + g' & (f-g)' = f' - g' \\
(fg)' = f'g + fg' & \left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2} \\
[f(g(x))]' = f'(g(x))g'(x)\,. &
\end{array}
$$

$$
\begin{aligned}
&\frac{d}{dx}(C) = 0\,, \quad C \text{ is a constant} \\
&\frac{d}{dx}(x^\alpha) = \alpha x^{\alpha-1}\,, \quad \alpha \text{ is any constant}
\end{aligned}
$$

$$
\begin{array}{ll}
\dfrac{d}{dx}(e^x) = e^x\,, & \dfrac{d}{dx}(a^x) = a^x \ln a\,, \\
\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}\,, & \dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \ln a}\,, \\
\dfrac{d}{dx}(\sin x) = \cos x\,, & \dfrac{d}{dx}(\cos x) = -\sin x\,, \\
\dfrac{d}{dx}(\tan x) = \sec^2 x\,, & \dfrac{d}{dx}(\cot x) = -\csc^2 x\,, \\
\dfrac{d}{dx}(\sec x) = \sec x \tan x\,, & \dfrac{d}{dx}(\csc x) = -\csc x \cot x\,, \\
\dfrac{d}{dx}(\arcsin x) = \dfrac{1}{\sqrt{1-x^2}}\,, & \dfrac{d}{dx}(\arccos x) = \dfrac{-1}{\sqrt{1-x^2}}\,, \\
\dfrac{d}{dx}(\arctan x) = \dfrac{1}{1+x^2}\,, & \dfrac{d}{dx}(\text{arccot} x) = \dfrac{-1}{1+x^2}\,,
\end{array}
$$

4.3 Implicit differentiation and logarithmic differentiation

The functions that we have met so far can be described by expressing one variable explicitly in terms of another variablefor example,

$$y = \sqrt{x^3+1}\,, \qquad \text{or} \qquad y = x \sin x$$

or, in general, $y = f(x)$. Some functions, however, are defined implicitly by a relation between x and y such as

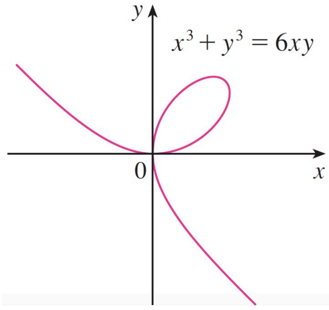
$$1: \quad x^2 + y^2 = 25\,.$$

or

$$2: \quad x^3 + y^3 = 6xy\,.$$

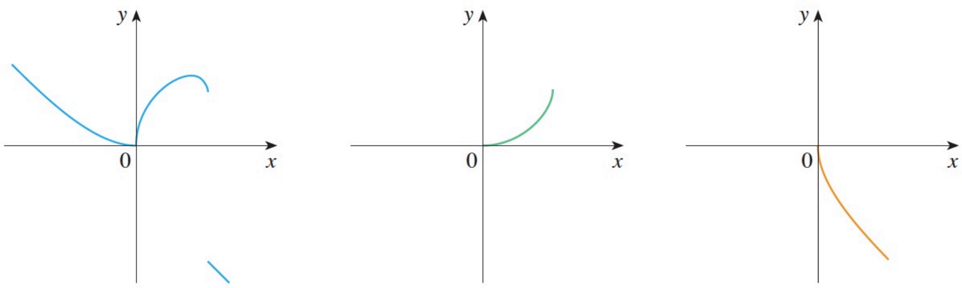
In some cases it is possible to solve such an equation for y as an explicit function (or several functions) of x. For instance, if we solve Equation 1 for y, we get $y = \pm\sqrt{25-x^2}$, so two of the functions determined by the implicit Equation l are $f_1(x) = \sqrt{25-x^2}$ and $f_2(x) = -\sqrt{25-x^2}$, The graphs of f_1 and f_2 are the upper and lower semicircles of the circle $x^2 + y^2 = 25$.

Its not easy to solve Equation 2 for y explicitly as a function of x by hand. (A computer algebra system has no trouble, but the expressions it obtains are very complicated.) Nonetheless, 2 is the equation of a curve called the folium of Descartes shown below.



folium of Descartes

Equation 2 implicitly defines y as several functions of x. The graphs of three such functions are shown below:



Graphs of three functions defined by the folium of Descartes

When we say that f is a function defined implicitly by Equation 2, we mean that the equation

$$x^3 + (f(x))^3 = 6xf(x)$$

is true for all values of x in the domain of f.

Fortunately, we dont need to solve an equation for y in terms of x in order to find the derivative of y. Instead we can use the method of implicit differentiation. This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for y'. In the examples and exercises of this section it is always assumed that the given equation determines y implicitly as a differentiable function of x so that the method of implicit differentiation can be applied.

Example 4.20. 1. Find y' if $x^3 + y^3 = 6xy$.

2. Find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point $(3, 3)$.

Solution: For 1: Differentiating both sides of $x^3 + y^3 = 6xy$ with respect to x, regarding y as a function of x, and using the Chain Rule on the term y^3 and the Product Rule on the term $6xy$, we get

$$3x^2 + 3y^2y' = 6xy' + 6y$$

or

$$x^2 + y^2 y' = 2xy' + 2y\,.$$

We now solve for y',

$$\begin{aligned} y^2 y' - 2xy' &= 2y - x^2 \\ \left(y^2 - 2x\right) y' &= 2y - x^2 \\ y' &= \frac{2y - x^2}{y^2 - 2x} \end{aligned}$$

For 2: when $x = y = 3$, $y' = -1$, so an equation of the tangent to the folium at $(3,3)$ is

$$y - 3 = -1 \times (x - 3) \quad \text{or} \quad x + y = 6\,.$$

Example 4.21. Find y' if $\sin(x + y) = y^2 \cos x$.

Solution: Differentiating implicitly with respect to x and remembering that y is a function of x, we get

$$\cos(x + y) \cdot (1 + y') = y^2(-\sin x) + (\cos x)\,(2yy')\ ,$$

solve y' to get

$$y' = \frac{y^2 \sin x + \cos(x + y)}{2y \cos x - \cos(x + y)}\,.$$

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called logarithmic differentiation.

Example 4.22. Differentiate

$$y = \frac{x^{3/4}\sqrt{x^2 + 1}}{(3x + 2)^5}\,.$$

Solution: We take logarithms of both sides of the equation and use the properties of logarithms to simplify:

$$\ln y = \frac{3}{4} \ln x + \frac{1}{2} \ln(x^2 + 1) - 5 \ln(3x + 2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{2x}{x^2 + 1} - 5 \cdot \frac{3}{3x + 2}\,,$$

Solving $\frac{dy}{dx}$ to get

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2}\right)\,,$$

from the expression of y in terms of x, we obtain

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2 + 1}}{(3x + 2)^5}\left(\frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2}\right)\,.$$

Example 4.23. Differentiate

$$y = x^{\sqrt{x}}\,.$$

Solution: Since both the base and the exponent are variable, we use logarithmic differentiation:

$$\begin{aligned}
\ln y &= \ln x^{\sqrt{x}} = \sqrt{x}\ln x \\
\frac{y'}{y} &= \sqrt{x}\cdot\frac{1}{x} + (\ln x)\frac{1}{2\sqrt{x}} \\
y' &= y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2+\ln x}{2\sqrt{x}}\right)\,.
\end{aligned}$$

Note that another way to solve the problem is to write $y = e^{\sqrt{x}\ln x}$ and apply the chain rule.

4.4 Rolle's theorem and mean value theorems

One of the most important theorems in the theory of functions of a real variable is Rolle's Theorem. It underpins a lot of what we know about functions.

Theorem 4.24 (Rolle's Theorem)**.** *Suppose that the function f is continuous on $[a,b]$ and differentiable on (a,b), and that $f(a) = f(b)$. Then there is a point $c \in (a,b)$ such that $f'(c) = 0$.*

Proof. By the Minimum-Maximum Theorem (Theorem 3.18), since f is continuous on $[a,b]$, the supremum $\sup(f([a,b]))$ and the infimum $\inf(f([a,b]))$ exist, and they are both attained. We write them as M and m.

Since $m \le f(a) = f(b) \le M$ by definition, then it must be true that

(1) $f(a) = f(b) < M$, or that

(2) $m < f(a) = f(b)$, or that

(3) $m = f(a) = f(b) = M$.

Consider first the possibility that $M > f(a)$. In this case, by the Boundedness Theorem, there is a point $c \in [a,b]$ such that $f(c) = M$; further, $c \in (a,b)$, since $M > f(a) = f(b)$. Now $f'(c)$ exists since f is differentiable on (a,b). We claim that $f'(c) = 0$. Indeed, since $f(x) \le f(c)$ for all $[a,b]$, we have

$$f'(c) = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}\,,$$

when $h \to 0^+$, $\frac{f(c+h)-f(c)}{h} \le 0$, thus,

$$f'(c) = \lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h} \le 0\,,$$

when $h \to 0^-$, $\frac{f(c+h)-f(c)}{h} \ge 0$, thus

$$f'(c) = \lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h} \ge 0\,,$$

which means that $f'(c) \geq 0$ and $f'(c) \leq 0$, that is, $f'(c) = 0$.

The possibility that $m < f(a)$ is treated similarly.

The last possibility is that $m = M$. In this case, the function f is constant, and the derivative of a constant function is 0, so $f'(c) = 0$ for all $c \in (a, b)$.

Thus in each case, there is a point c with the required property. □

Corollary 4.25. *If $f : (a, b) \to \mathbb{R}$ is differentiable, then between any two different zeros of f there is at least a zero of f'.*

Proof. If x_1 and x_2 are two distinct zeros of f, then $f(x_1) = f(x_2)$, so, by Rolle's Theorem, there exists c in the interval (x_1, x_2) (or in (x_2, x_1) if $x_2 < x_1$) such that $f'(c) = 0$, that is, c is a root of f' lying between x_1 and x_2. □

Corollary 4.26. *If $f : (a, b) \to \mathbb{R}$ is differentiable and has n distinct zeros, then f' has* at least *$n - 1$ distinct zeros.*

Equivalently, if f' has m distinct zeros, then f has at most *$m + 1$ distinct zeros.*

Theorem 4.27 (Mean Value Theorem)**.** *Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Proof. Define $\phi : [a, b] \to \mathbb{R}$ by

$$\phi(x) = f(x) - \Big(\frac{f(b) - f(a)}{b - a}\Big)x.$$

Then ϕ is continuous on $[a, b]$ and differentiable on (a, b), by the algebra of continuous and differentiable functions. Furthermore,

$$\begin{aligned}
\phi(a) - \phi(b) &= f(a) - \Big(\frac{f(b) - f(a)}{b - a}\Big)a - \Bigg(f(b) - \Big(\frac{f(b) - f(a)}{b - a}\Big)b\Bigg) \\
&= f(a) - f(b) - \Big(\frac{f(b) - f(a)}{b - a}\Big)(a - b) \\
&= 0
\end{aligned}$$

and hence $\phi(a) = \phi(b)$. By Rolle's Theorem, there exists $c \in (a, b)$ such that $\phi'(c) = 0$. Hence

$$f'(c) - \frac{f(b) - f(a)}{b - a} = 0,$$

which is equivalent to the desired conclusion. □

Corollary 4.28. *Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).*

1. *If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.*

2. *If $f'(x) > 0$ for all $x \in (a,b)$, then f is strictly increasing on $[a,b]$.*

3. *If $f'(x) < 0$ for all $x \in (a,b)$, then f is strictly decreasing on $[a,b]$.*

Proof. For (1), take an arbitrary point $x \in (a,b]$, and consider f as a function on $[a,x]$; it is automatic that f is continuous on $[a,x]$ and differentiable on (a,x).

By the Mean Value Theorem applied to f on $[a,x]$, there exists $c \in (a,x)$ such that

$$\frac{f(x)-f(a)}{x-a} = f'(c) = 0.$$

Therefore $f(x) = f(a)$. Since x was chosen arbitrarily, f is constant on $[a,b]$.

For (2), (3), take $x_1, x_2 \in [a,b]$ such that $x_1 < x_2$, and consider f as a function on $[x_1,x_2]$; it is automatic that f is continuous on $[x_1,x_2]$ and differentiable on (x_1,x_2). By the Mean Value Theorem applied to f on $[x_1,x_2]$, there exists $c \in (x_1,x_2)$ such that

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c).$$

In case (2), $f'(x) > 0$ for all $x \in (a,b)$ by hypothesis, so in particular $f'(c) > 0$, and hence $f(x_2) > f(x_1)$; that is, f is strictly increasing. In case (3), we argue similarly to conclude that f is strictly decreasing. □

Remark 4.29. The converse of Parts (2) and (3) of Corollary 4.28 are not true. For instance, suppose that $f(x) = x^3$; then f is strictly increasing, but $f'(0) = 0$. However, it can be shown that if f is increasing and differentiable, then $f'(x) \geq 0$ for all x.

Example 4.30. Show that, if $-\pi/4 \leq a < b \leq \pi/4$, then

$$|\tan b - \tan a| < 2|b-a|.$$

Answer: The function $x \mapsto \tan x$ is continuous and differentiable on $(-\pi/2, \pi/2)$, and hence continuous on $[a,b]$ and differentiable on (a,b), because $[a,b] \subset (-\pi/2,\pi/2)$ (and $(a,b) \subset (-\pi/2,\pi/2)$). Hence by the Mean Value Theorem, there exists $c \in (a,b)$ such that

$$\frac{\tan b - \tan a}{b-a} = \sec^2 c,$$

and this implies that

$$|\tan b - \tan a| = (\sec^2 c)|b-a|.$$

Since $\cos^2 x > 1/2$ for all $x \in (-\pi/4, \pi/4)$,

$$|\tan b - \tan a| < 2|b-a|,$$

as claimed.

The Mean Value Theorem is very useful for proving inequalities of this type.

We now come to a useful generalization of the Mean Value Theorem, known as the Generalized Mean Value Theorem, or Cauchy's Mean Value Theorem.

Theorem 4.31 (Generalized Mean Value Theorem)**.** *Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b), and that $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. We define a function $\phi : [a, b] \to \mathbb{R}$ by

$$\phi(x) = f(x) - \left(\frac{f(b) - f(a)}{g(b) - g(a)} \right) g(x) \quad \forall x \in [a, b].$$

First, observe that $g(b) \neq g(a)$, and thus ϕ is well-defined (indeed, if $g(b) = g(a)$, using Rolle's theorem we have that there exists $c \in (a, b)$ such that $g'(c) = 0$, but this contradicts the fact that we are assuming that $g'(x) \neq 0$ for all $x \in (a, b)$. Thus $g(b) \neq g(a)$).

Notice also that ϕ is continuous on $[a, b]$ and differentiable on (a, b), by the algebra of continuous and differentiable functions. Furthermore, $\phi(a) = \phi(b)$. Hence by Rolle's Theorem, there exists $c \in (a, b)$ such that $\phi'(c) = 0$; that is,

$$f'(c) - \left(\frac{f(b) - f(a)}{g(b) - g(a)} \right) g'(c) = 0,$$

which is equivalent to the desired conclusion, since $g'(c) \neq 0$. □

The following argument is *erroneous*. By the Mean Value Theorem there exists a point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

and

$$\frac{g(b) - g(a)}{b - a} = g'(c);$$

dividing the first identity by the second gives the desired conclusion.

This argument is erroneous because, in general, the point c that works for f is different to the point c that works for g.

4.5 L'Hôpital's rule

Our main application of the Generalized Mean Value Theorem is L'Hôpital's Rule, which is mainly used to calculate the limit of the form $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ such that $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$, which is called the "$\frac{0}{0}$" form.

Corollary 4.32 (L'Hôpital's Rule)**.** *Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains x_0 (except possibly at x_0). Suppose that*

$$\lim_{x \to x_0} f(x) = 0 \quad \textit{and} \quad \lim_{x \to x_0} g(x) = 0$$

or that

$$\lim_{x \to x_0} f(x) = \pm\infty \quad \textit{and} \quad \lim_{x \to x_0} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$".) Then

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is $-\infty$ or ∞.)

Remark 4.33. 1. The "$\frac{0}{0}$" and "$\frac{\infty}{\infty}$" are called indeterminate forms. L'Hôpital's Rule only works for indeterminate forms. For example, consider

$$\lim_{x\to 1}\frac{2x-1}{x}=\frac{\lim_{x\to 1}(2x-1)}{\lim_{x\to 1}x}=1\,,$$

however, if we take the derivative of the top and bottom, we get

$$\lim_{x\to 1}\frac{(2x-1)'}{(x)'}=\lim_{x\to 1}\frac{2}{1}=2\,,$$

which is wrong.

2. LHospitals Rule is also valid for one-sided limits and for limits at infinity or negative infinity; that is, "$x\to x_0$" can be replaced by any of the symbols "$x\to x_0^+$", "$x\to x_0^-$", "$x\to\infty$", or "$x\to-\infty$".

Proof. We will only prove the "$\frac{0}{0}$" form as $x\to x_0$. We want to show that

$$\frac{f(x)}{g(x)}\to A\quad\text{as } x\to x_0.$$

To this end, we take arbitrary $\epsilon\in\mathbb{R}^+$. Since

$$\frac{f'(x)}{g'(x)}\to A\quad\text{as } x\to x_0,$$

there exists $\delta\in\mathbb{R}^+$ such that

$$\left|\frac{f'(x)}{g'(x)}-A\right|<\epsilon\quad\text{whenever } 0<|x-x_0|<\delta. \tag{4.2}$$

Now suppose that $a<x<b$ and $x_0<x<x_0+\delta$. Since $\lim_{x\to x_0}f(x)=\lim_{x\to x_0}g(x)=0$, we can define $f(x_0)=g(x_0)=0$, so the functions f and g are continuous on $[x_0,x]$ and differentiable on (x_0,x), by hypothesis, and so by the Generalized Mean Value Theorem, there exists $c\in(x_0,x)$ such that

$$\frac{f(x)}{g(x)}=\frac{f(x)-f(x_0)}{g(x)-g(x_0)}=\frac{f'(c)}{g'(c)}\,.$$

Now $x_0<x<x_0+\delta$ and $x_0<c<x$, so $x_0<c<x_0+\delta$, and hence by (4.2),

$$\left|\frac{f(x)}{g(x)}-A\right|=\left|\frac{f'(c)}{g'(c)}-A\right|<\epsilon.$$

Therefore

$$\frac{f(x)}{g(x)} \to A \quad \text{as } x \to x_0+.$$

A similar argument, where we consider x in $(x_0 - \delta, x_0)$, gives

$$\frac{f(x)}{g(x)} \to A \text{ as } x \to x_0-,$$

and the desired conclusion follows. □

Example 4.34. Find

$$\lim_{x\to 0} \frac{1-\cos x}{x^2}\,.$$

Answer It is obvious that both $1-\cos x$ and x^2 are differentiable on $\mathbb{R}$, and this is a "$\frac{0}{0}$" form and $g'(x) \neq 0$ whenever $x \neq 0$. Hence, by L'Hôpital's Rule, we get

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{(1-\cos x)'}{(x^2)'} = \lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2}.$$

Example 4.35. Find

$$\lim_{x\to\infty} \frac{e^x}{x^2}$$

Solution: This is an "$\frac{\infty}{\infty}$" form.

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

so the limit is ∞, that is, the limit does not exist.

Generally, for any polynomial $P(x)$ with degree n, using L'Hôpital's rule n times, we see that

$$\lim_{x\to\infty} \frac{e^x}{P(x)} = \infty\,,$$

this shows that when x becomes large, the exponential function e^x (or any a^x with $a > 1$) grows much faster than any polynomial.

Example 4.36. Prove

$$\lim_{x\to 0} \frac{x^2 - 2(1-\cos(x))}{x^2(1-\cos(x))} = \frac{1}{6}.$$

Answer: Both the top and bottom are differentiable and this is a "$\frac{0}{0}$" form. If we apply L'Hôpital's Rule directly, the derivative of the bottom is complicated. Instead, let us start by writing

$$\begin{aligned}
\lim_{x\to 0} \frac{x^2 - 2(1-\cos x)}{x^2(1-\cos x)} &= \lim_{x\to 0} \frac{x^2 - 2(1-\cos x)}{x^4} \frac{x^2}{1-\cos x} \\
&= \lim_{x\to 0} \frac{x^2 - 2(1-\cos x)}{x^4} \lim_{x\to 0} \frac{x^2}{1-\cos x}
\end{aligned}$$

$$
\begin{aligned}
&= 2\lim_{x\to 0}\frac{[x^2-2(1-\cos x)]'}{(x^4)'}\\
&= 2\lim_{x\to 0}\frac{2x-2\sin x}{4x^3}\\
&= \lim_{x\to 0}\frac{x-\sin x}{x^3} \quad \text{another "}\frac{0}{0}\text{" form}\\
&= \lim_{x\to 0}\frac{[x-\sin x]'}{(x^3)'}\\
&= \lim_{x\to 0}\frac{1-\cos x}{3x^2}\\
&= \frac{1}{6}\,.
\end{aligned}
$$

Example 4.37. Evaluate

$$\lim_{x\to 0^+} x\ln x\,.$$

This is called indeterminate form of "$0\cdot\infty$" type.

Solution:

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0\,.$$

Example 4.38. Find

$$\lim_{x\to \frac{\pi}{2}^-}(\sec x-\tan x)$$

This is called indeterminate form of "$\infty-\infty$" type.

Solution: We write

$$\lim_{x\to \frac{\pi}{2}^-}(\sec x-\tan x) = \lim_{x\to \frac{\pi}{2}^-}\frac{1-\sin x}{\cos x} = \lim_{x\to \frac{\pi}{2}^-}\frac{-\cos x}{-\sin x} = 0\,.$$

Example 4.39. Calculate

$$\lim_{x\to 0^+}(1+\sin(4x))^{\cot x}\,.$$

This is called indeterminate form of "1^∞" type.

Solution:

$$\lim_{x\to 0^+}(1+\sin(4x))^{\cot x} = \lim_{x\to 0^+} e^{\cot x(1+\sin(4x))} = e^{\lim_{x\to 0^+}\cot x\ln(1+\sin(4x))}\,,$$

so the limit in the exponent is

$$\lim_{x\to 0^+}\cot x\ln(1+\sin(4x)) = \lim_{x\to 0^+}\frac{\ln(1+\sin(4x))}{\sin x} = \lim_{x\to 0^+}\frac{\frac{4\cos x}{1+\sin(4x)}}{\cos x} = 4\,,$$

thus,

$$\lim_{x\to 0^+}(1+\sin(4x))^{\cot x} = e^4\,.$$

Example 4.40. Find

$$\lim_{x\to\infty} x^{\frac{1}{x}} \quad \text{and} \quad \lim_{x\to 0^+} x^x .$$

The first one is called indeterminate form of "∞^0" type and the second is called indeterminate form of "0^0" type.

Solution:

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{\ln x}{x}} = e^{\lim_{x\to\infty} \frac{\ln x}{x}} = e^{\lim_{x\to\infty} \frac{\frac{1}{x}}{1}} = 1 .$$

Similarly,

$$\lim_{x\to 0^+} x^x = 1 .$$

4.6 Taylor's theorem

4.6.1 Linear approximation and differentials

We have seen that a curve lies very close to its tangent line near the point of tangency. In fact, by zooming in toward a point on the graph of a differentiable function, we noticed that the graph looks more and more like its tangent line. This observation is the basis for a method of finding approximate values of functions.

The idea is that it might be easy to calculate a value $f(a)$ of a function, but difficult (or even impossible) to compute nearby values of f . So we settle for the easily computed values of the linear function L whose graph is the tangent line of f at $(a, f(a))$. In other words, we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a. An equation of this tangent line is

$$y = f(a) + f'(a)(x - a) .$$

and the approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the linear approximation or tangent line approximation of f at a. The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x - a)$$

is called the linearization of f at a.

Example 4.41. Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$.

Solution: The derivative of $f(x) = \sqrt{x+3}$ is

$$f'(x) = \frac{1}{2\sqrt{x+3}} ,$$

and so we have $f(1) = 2$ and $f'(1) = \frac{1}{4}$. We see that the linearization is

$$L(x) = f(1) + f'(1)(x - 1) = \frac{7}{4} + \frac{x}{4} .$$

That is, the corresponding linear approximation for $\sqrt{x+3}$ when x it close to 1 is

$$\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4} .$$

In particular, we have

$$\sqrt{3.98} \approx \frac{7}{4} + \frac{0.98}{4} = 1.995 \qquad \sqrt{4.05} \approx \frac{7}{4} + \frac{1.05}{4} = 2.0125\,.$$

The ideas behind linear approximations are sometimes formulated in the terminology and notation of differentials. If $y = f(x)$, where f is a differentiable function, then the differential dx is an independent variable; that is, dx can be given the value of any real number. The differential dy is then defined in terms of dx by the equation

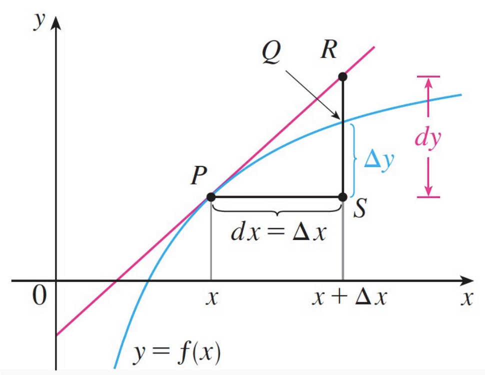
$$dy = f'(x)dx$$

So dy is a dependent variable; it depends on the values of x and dx. If dx is given a specific value and x is taken to be some specific number in the domain of f , then the numerical value of dy is determined.

The geometric meaning of differentials is shown below. Let $P(x, f(x))$ and $Q(x + \Delta x, f(x + \Delta x))$ be points on the graph of f and let $dx = \Delta x$. The corresponding change in y is

$$\Delta y = f(x + \Delta x) - f(x)$$

The slope of the tangent line PR is the derivative $f'(x)$. Thus the directed distance from S to R is $f'(x)dx = dy$. Therefore dy represents the amount that the tangent line rises or falls (the change in the linearization) when x changes by an amount dx, whereas Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by an amount $dx = \Delta x$.



In a nutshell, $dx = \Delta x$ is the (actual) change of x, and dy is used to approximate the actual change of Δy, i.e. $dy \approx \Delta y$.

4.6.2 Taylor polynomials

Differential is used to approximate the function by linear functions. We can also use polynomial to approximate the function to get even better approximation. This kind of polynomial is the so-called Taylor polynomials.

For reasons that will soon become clear, we would like to have a definition of differentiability for a function defined on *any* interval, not just an *open* interval.

Definition 4.42. A function $f : [a, b] \to \mathbb{R}$ is *differentiable* if it is differentiable on the open interval (a, b), *and* both of the limits

$$\lim_{h \to 0+} \frac{f(a + h) - f(a)}{h} \quad \text{and} \quad \lim_{h \to 0-} \frac{f(b + h) - f(b)}{h}$$

exist. We denote these one sided limits by $f'(a)$ and $f'(b)$ respectively.

Similarly a function $f : [a, b) \to \mathbb{R}$ is differentiable if it is differentiable on (a, b) and the right hand limit

$$\lim_{h \to 0+} \frac{f(a+h) - f(a)}{h}$$

exists, and a function $f : (a, b] \to \mathbb{R}$ is differentiable if it is differentiable on (a, b) and the left hand limit

$$\lim_{h \to 0-} \frac{f(b+h) - f(b)}{h}$$

exists.

The notion of *higher derivatives* will also be important for us.

Definition 4.43. Suppose that f is differentiable with derivative f' on an interval, and that f' is itself differentiable. Then we denote the derivative of f' by f'', and call it the *second derivative* of f. Continuing in this way (differentiability permitting), we obtain functions

$$f, f', f'', f''', \ldots, f^{(n)}$$

each of which is the derivative of the one before. We call $f^{(n)}$ the *nth derivative of f*. If $f^{(n)}$ exists for *all* positive integers n, then we say that f is *infinitely differentiable.*

If $y = f(x)$, some other notation for the n-th derivatives of $f(x)$ are

$$\frac{d^n y}{dx^n}, \quad \frac{d^n f}{dx^n}, \quad \frac{d^n f(x)}{dx^n}, \quad \frac{d^n f}{dx^n}(x).$$

Note that you can not write $\frac{dy^n}{dx^n}$. The reason is that, we treat differentiation $\frac{d}{dx}$ as an operator, and apply this operator to $f(x)$ n times.

Theorem 4.44 (Taylor's Theorem)**.** *Suppose that $f, f', f'', \ldots, f^{(n)}$ all exist and are continuous on $[a, b]$, and that $f^{(n+1)}$ exists on (a, b). Then there exists $c \in (a, b)$ such that*

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Equivalently, if $f, f', \ldots, f^{(n)}$ exist and are continuous on $[a, a+t]$ and $f^{(n+1)}$ exists on $(a, a+t)$, then there exists $\theta \in (0, 1)$ such that

$$f(a+t) = f(a) + f'(a)t + \frac{f''(a)}{2}t^2 + \cdots + \frac{f^{(n)}(a)}{n!}t^n + \frac{f^{(n+1)}(a+\theta t)}{(n+1)!}t^{n+1}.$$

Proof. Define the function $\phi : [a, b] \to \mathbb{R}$ by

$$\phi(x) = f(x) + (b-x)f'(x) + \cdots + \frac{(b-x)^n}{n!}f^{(n)}(x) + K\left(\frac{b-x}{b-a}\right)^{n+1},$$

where

$$K = f(b) - f(a) - f'(a)(b-a) - \cdots - \frac{f^{(n)}(a)}{n!}(b-a)^n.$$

Since $f, f', \ldots, f^{(n)}$ exist and are continuous on $[a, b]$, it follows that ϕ is continuous on $[a, b]$. Since $f^{(n+1)}$ exists on (a, b), it follows that ϕ is differentiable on (a, b). Moreover, $\phi(a) = \phi(b)$, and hence by Rolle's Theorem, there exists $c \in (a, b)$ such that $\phi'(c) = 0$.

Hence

$$0 = \phi'(c) = \frac{(b-c)^n}{n!} f^{(n+1)}(c) - (n+1)K \frac{(b-c)^n}{(b-a)^{n+1}},$$

which can be rewritten as

$$K = \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1},$$

as required. □

Remark 4.45. (1) If f is a polynomial of degree at most n, then $f^{(n+1)}$ is identically zero. Therefore, in this case, Taylor's Theorem says

$$f(a+t) = f(a) + f'(a)\,t + \frac{f''(a)}{2!}\,t^2 + \cdots + \frac{f^{(n)}(a)}{n!}\,t^n.$$

(2) When $n = 0$, Taylor's Theorem is exactly the Mean Value Theorem. It is instructive to compare the proofs of these two theorems, and the Generalized Mean Value Theorem.

(3) Implicitly $b > a$ (and thus $t > 0$). However, the theorem still holds when $b < a$ (and thus $t < 0$) provided that we replace the interval $[a, b]$ by $[b, a]$ (and thus $[a, a+t]$ by $[a+t, a]$), and replace similarly the open intervals that appear.

(4) The value c (and thus θ) depends on f, a, b and n (or f, a, t and n). If any of these are to be varied, we should make this dependence clear by writing, for example, c_n, $c_n(t)$, $\ldots$ (or θ_n, $\theta_n(t)$, $\ldots$).

In conclusion, Taylor's Theorem says that we may approximate certain functions $t \mapsto f(a+t)$ by the polynomial

$$f(a) + f'(a)\,t + \frac{f''(a)}{2}\,t^2 + \cdots + \frac{f^{(n)}(a)}{n!}\,t^n,$$

with an error (or remainder) term

$$R_n(t) = \frac{f^{(n+1)}(a+\theta t)}{(n+1)!} t^{n+1}.$$

Here θ depends on f, n, a and t.

Suppose that f and *all* its derivatives exist on (a, b), and that $x_0 \in (a, b)$. By Taylor's Theorem, for all t such that $x_0 + t \in (a, b)$ and all $n \in \mathbb{N}$,

$$f(x_0 + t) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}\,t^k + R_n(t),$$

where

$$R_n(t) = \frac{f^{(n+1)}(x_0 + \theta t)}{(n+1)!} t^{n+1}$$

for some $\theta \in (0,1)$. *If* $R_n(t) \to 0$ as $n \to \infty$ for all t such that $x_0 + t \in (a,b)$, then the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}\, t^k$$

converges, to $f(x_0+t)$. This series is called the *Taylor Series of f centred at x_0*. Sometimes we set $x = x_0 + t$, and write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

In this case, the restriction $x_0 + t \in (a,b)$ becomes $x \in (a,b)$.

Remark 4.46. Taylor Series centred at 0 (that is, series for which the centre x_0 is 0) are usually known as *Maclaurin Series*, for historical reasons.

Example 4.47. Find the Maclaurin Series of the function $x \mapsto \sin x$.

Answer: Write $f(x) = \sin(x)$. We first observe that f has derivatives of all orders on $\mathbb{R}$, and

$$\begin{aligned} f(x) &= \sin(x) & f^{(4)}(x) &= \sin(x) \\ f'(x) &= \cos(x) & f^{(5)}(x) &= \cos(x) \\ f''(x) &= -\sin(x) & f^{(6)}(x) &= -\sin(x) \\ f'''(x) &= -\cos(x) & &\dots \end{aligned}$$

and so on, repeating periodically. We set $x = 0$. This gives

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n = 2k, \qquad k = 0,1,2\dots \\ (-1)^k & \text{if } n = 2k+1, \qquad k = 0,1,2\dots \end{cases}$$

Now, taking the centre x_0 to be 0 and fixing t,

$$|R_n(t)| = \left|\frac{t^{n+1}}{(n+1)!}\right| \left|f^{(n+1)}(\theta t)\right| \le \frac{|t|^{n+1}}{(n+1)!},$$

since $|\sin(\theta t)| \le 1$ and $|\cos(\theta t)| \le 1$. Therefore, since

$$\frac{|t|^{n+1}}{(n+1)!} \to 0 \quad \text{as } n \to \infty$$

(this is an exercise in its own right!), it follows from the Sandwich Theorem that $R_n(t) \to 0$ as $n \to \infty$, and so

$$\sin(t) = f(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots$$

is the Maclaurin series of the sine function, which is valid *for all* $t \in \mathbb{R}$.

A similar argument yields the Maclaurin series for the cosine function.

Remark 4.48. There are some functions, which have derivatives of all orders, but for which $R_n(t) \not\to 0$ as $n \to \infty$, and these functions cannot be expanded in Taylor Series. A classic example is the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0. \end{cases}$$

This function has derivatives of all orders, and moreover

$$f^{(k)}(0) = 0 \quad \text{for all } k \in \mathbb{N},$$

and so in particular

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\, x^k = 0 \quad \text{for all } x \in \mathbb{R},$$

and this is rather different to $f(x)$.

In discussing Taylor Series above, we dealt with series of the form

$$\sum_{n=0}^{\infty} a_n x^n$$

where $(a_n)_{n \geq 0}$ is a sequence of real numbers. We shall refer to such series as *power series.* We consider the power series $\sum_{n=0}^{\infty} a_n x^n$ as a function

$$S \to \mathbb{R}; \qquad x \mapsto \sum_{n=0}^{\infty} a_n x^n,$$

where S is the *set of convergence* given by

$$S := \left\{ x \in \mathbb{R} : \sum_{n=0}^{\infty} a_n x^n \text{ converges} \right\}.$$

For such a power series, the *radius of convergence* is given by

$$R = \begin{cases} \sup\{|x| : x \in S\} & \text{if } S \text{ is bounded,} \\ \infty & \text{if } S \text{ is unbounded,} \end{cases}$$

and a fundamental result (from 1RAC) is that

$$(-R, R) \subseteq S \subseteq [-R, R]$$

and the power series converges absolutely in $(-R, R)$.

We have the following fact about differentiation term by term for the power series.

Theorem 4.49. *Suppose that*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \,.$$

Then for $x \in (-R, R)$, we have

$$f'(x) = \sum_{n=0}^{\infty} a_n n x^{n-1} \,,$$

that is, the derivative of f is just the sum of the derivative of each term.

4.7 Maximum and Minimum values

Let us start with the definition of maximum and minimum values of a function f.

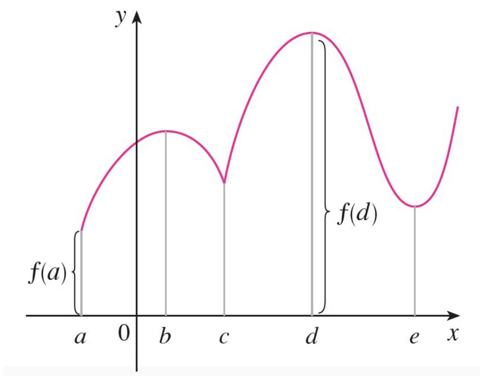
Definition 4.50. Let c be a number in the domain D of a function f. Then $f(c)$ is the

- absolute maximum value of f on D if $f(c) \geq f(x)$ for all x in D.
- absolute minimum value of f on D if $f(c) \leq f(x)$ for all x in D.

An absolute maximum or minimum is sometimes called a global maximum or minimum. The maximum and minimum values of f are called extreme values of f.

Definition 4.51. The number $f(c)$ is a

- local maximum value of f if $f(c) \geq f(x)$ for all x close to c. (i.e., there is a $\delta > 0$ such that for all $x \in (c - \delta, c + \delta)$, $f(c) \geq f(x)$.)
- local maximum value of f if $f(c) \leq f(x)$ for all x close to c. (i.e., there is a $\delta > 0$ such that for all $x \in (c - \delta, c + \delta)$, $f(c) \leq f(x)$.)



In the above graph, $f(a)$ is the absolute minimum, $f(d)$ is the absolute maximum. $f(e), f(c)$ are the local minima, $f(b), f(d)$ are the local maxima.

We next state the Feymat's theorem, the proof is similar with Rolle's theorem and is omitted.

Theorem 4.52 (Fermat's theorem)**.** *If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.*

Definition 4.53. Let f be a real-valued function of a real variable. The point c is called a stationary point of f if $f'(c) = 0$.

The following examples caution us against reading too much into Fermats Theorem: We cant expect to locate extreme values simply by setting $f'(x) = 0$ and solving for x.

Example 4.54. If $f(x) = x^3$, then $f'(x) = 3x^2$, so $f'(0) = 0$. But f has no maximum or minimum at 0.

Example 4.55. The function $f(x) = |x|$ has its (local and absolute) minimum value at 0, but that value cant be found by setting $f'(x) = 0$ because $f'(0)$ does not exist.

Examples 4.54 and 4.55 show that we must be careful when using Fermats Theorem. Example 4.54 demonstrates that even when $f'(c) = 0$ there need not be a maximum or minimum at c. (In other words, the converse of Fermats Theorem is false in general.) Furthermore, there may be an extreme value even when $f'(c)$ does not exist.

Fermats Theorem does suggest that we should at least start looking for extreme values of f at the numbers c where $f'(c) = 0$ or where $f'(c)$ does not exist. Such numbers are given a special name.

Definition 4.56. A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example 4.57. Find the critical numbers of

$$f(x) = x^{\frac{3}{5}}(4 - x)\,.$$

Solution:

$$f'(x) = \frac{12 - 8x}{5x^{\frac{2}{5}}}\,.$$

Therefore $f'(x) = 0$ is $x = \frac{3}{2}$ and $f'(x)$ does not exist when $x = 0$. Thus the critical numbers of $f(x)$ are $\frac{3}{2}$ and 0.

In terms of critical numbers, Fermats Theorem can be rephrased as follows

Proposition 4.58. *If f has a local maximum or minimum at c, then c is a critical number of f.*

To find an absolute maximum or minimum of a continuous function on a closed interval, we note that either it is local or it occurs at an endpoint of the interval. Thus the following three-step procedure always works:

Closed Interval Method:

(1) Find the values of f at the critical numbers of f in (a, b).

(2) Find the values of f at the endpoints of the interval.

(3) The largest of the values from Steps (1) and (2) is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 4.59. Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1 \qquad -\frac{1}{2} \leq x \leq 4\,.$$

Solution: Use the Closed Interval Method:

$$\begin{aligned} f(x) &= x^3 - 3x^2 + 1 \\ f'(x) &= 3x^2 - 6x = 3x(x-2) \end{aligned}$$

Since $f'(x)$ exists for all x, the only critical numbers of f occur when $f'(x) = 0$, that is, $x = 0$ or $x = 2$. Notice that each of these critical numbers lies in the interval $(-\frac{1}{2}, 4)$.

The values of f at these critical numbers are

$$f(0) = 1 \quad f(2) = -3\,,$$

The values of f at the endpoints of the interval are

$$f(-\frac{1}{2}) = \frac{1}{8} \quad f(4) = 17$$

Comparing these four numbers, we see that the absolute maximum value is $f(4) = 17$ and the absolute minimum value is $f(2) = -3$.

The following result characterizes the absolute minimum or maximum in simpler situation.

Proposition 4.60 (First Derivative Test for Absolute Extreme Values)**.** *Suppose that c is a critical number of a continuous function f defined on an interval.*

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

The proof is easy since for example (a) says $f(x)$ is strictly increasing when $x < c$ and strictly decreasing when $x > c$, so c certainly gives the absolute maximum.

We also have a local version of the First Derivative Test.

Proposition 4.61 (First Derivative Test for Local Extreme Values)**.** *Suppose that c is a critical number of a continuous function f.*

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.

f' changes from positive to negative at c means that there is a $\delta > 0$ such that $f'(x) > 0$ for $x \in (c - \delta, c)$ and $f'(x) < 0$ for $x \in (c, c + \delta)$.

The following Second Derivative Test identifies the local maximum and minimum of a function.

Theorem 4.62. *Suppose f'' is continuous near c.*

1. *If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.*
2. *If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.*

4.8 Curve Sketching

4.8.1 Asymptote

Definition 4.63. The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either

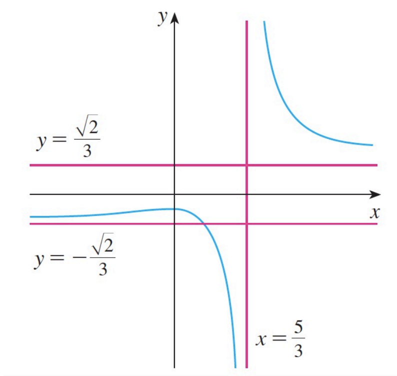
$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L\,.$$

The line $x = a$ is a vertical asymptote if at least one of the following statements is true:

$$\begin{aligned} &\lim_{x\to a^+} f(x) = \infty \qquad && \lim_{x\to a^+} f(x) = -\infty \\ &\lim_{x\to a^-} f(x) = \infty && \lim_{x\to a^-} f(x) = -\infty \end{aligned}$$

The following graph shows the horizontal and vertical asymptotes for the function:

$$y = \frac{\sqrt{2x^2+1}}{3x-5}\,.$$

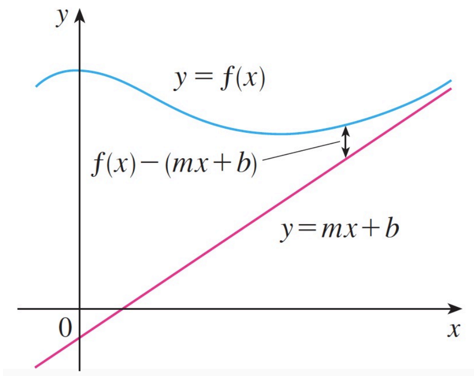


Slant Asymptotes

Some curves have asymptotes that are oblique, that is, neither horizontal nor vertical. If

$$\lim_{x\to\infty\ (\text{or } -\infty)} (f(x) - (kx + b)) = 0\,,$$

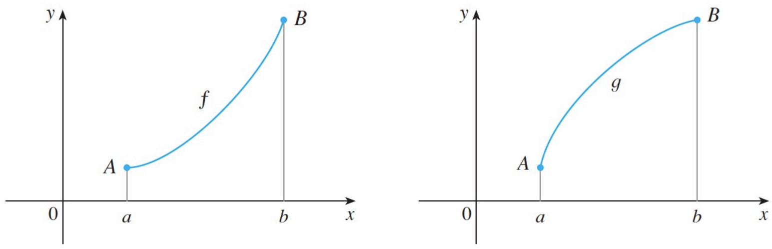
where $k \neq 0$, then the line $y = kx+b$ is called a slant asymptote because the vertical distance between the curve $y = f(x)$ and the line $y = kx + b$ approaches 0, as the following picture shows.



For rational functions, slant asymptotes occur when the degree of the numerator is one more than the degree of the denominator.

4.8.2 Concavity

The two graphs below show two increasing functions on (a, b). Both graphs join point A to point B but they look different because they bend in different directions. How can we distinguish between these two types of behavior?



Definition 4.64. If the graph of f lies above all of its tangents on an interval I, then it is called concave upward on I. If the graph of f lies below all of its tangents on I, it is called concave downward on I.

According to this definition, (a) is concave upward on (a, b) while (b) is concave downward on (a, b).

Concavity test:

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Definition 4.65. A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Guideline for Sketching a Curve:

(A) Domain.

(B) Intercepts: The y-intercept and the x-intercepts, we set $y = 0$ and solve for x. (You can omit this step if the equation is difficult to solve.)

(C) Symmetry: Whether f is even, odd or periodic.

(D) Asymptotes: horizontal asymptotic and vertical asymptotic.

(E) Intervals of Increase or Decrease

(F) Local Maximum and Minimum Values

(G) Concavity and Points of Inflection

(H) Sketch the Curve

Example 4.66. Sketch the graph of

$$f(x) = \frac{x^3}{x^2+1}\,.$$

Solution: The domain is $\mathbb{R}$. The x- and y- intercepts are 0. f is odd, the graph is symmetric about the origin. x^2+1 is never 0, there is no vertical asymptote. $\lim_{x\to\infty} f(x) = \infty$, so there is no horizontal asymptote. We see that

$$f(x) = \frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}\,,$$

This equation suggests that $y = x$ is a candidate for a slant asymptote. In fact,

$$f(x) - x \to 0 \quad \text{as} \quad x \to \pm\infty\,,$$

thus, $y = x$ is the slant asymptote for $f(x)$.

Take the derivative of $f(x)$,

$$f'(x) = \frac{x^2(x^2+3)}{(x^2+1)^2}\,,$$

since $f'(x) > 0$ for all x, $f(x)$ is increasing on $\mathbb{R}$.

Although $f'(0) = 0$, f' does not change sign at 0, so there is no local maximum or minimum.

$$f''(x) = \frac{2x(3-x^2)}{(x^2+1)^3}\,,$$

Since $f''(x) = 0$ when $x = 0$ or $x = \pm\sqrt{3}$, we set up the following chart:

interval	f''	f
$x < -\sqrt{3}$	$+$	CU on $(-\infty, -\sqrt{3})$
$-\sqrt{3} < x < 0$	$-$	CD on $(-\sqrt{3}, 0)$
$0 < x < \sqrt{3}$	$+$	CU on $(0, \sqrt{3})$
$x > \sqrt{3}$	$-$	CD on $(\sqrt{3}, \infty)$

The points of inflection are $(-\sqrt{3}, -\frac{3\sqrt{3}}{4})$, $(0,0)$ and $(\sqrt{3}, \frac{3\sqrt{3}}{4})$.

The graph is sketched below.