$\varepsilon-\delta$ definition: $\forall \varepsilon>0, \exists \delta>0$, such that $|x-x_0|<\delta \Rightarrow |f(x)-A|<\varepsilon$

$\lim_{x\to x_0} f(x)=A$ $\lim_{x\to+\infty} f(x)=A$ $\lim_{x\to-\infty} f(x)=A$ $\lim_{|x|\to+\infty} f(x)=A$

$\lim_{x\to x_0^-} f(x)=A$ $\lim_{x\to x_0^+} f(x)=A$

Properties of limits

① Uniqueness $\lim_{x\to x_0} f(x)=a$ $\lim_{x\to x_0} f(x)=b \Leftrightarrow a=b$

② Boundedness $\lim_{x\to x_0} f(x)$ exists $\Rightarrow |f(x)|<M \ \forall x\in(x_0-\delta,x_0)\cup(x_0,x_0+\delta)$

③ 保号性 $\lim_{x\to x_0} f(x)=A>0 \Rightarrow \forall x\in(x_0-\delta,x_0)\cup(x_0,x_0+\delta)$ $f(x)>0$

④ 保不等式性 $\lim_{x\to x_0} f(x)$ $\lim_{x\to x_0} g(x)$ exists,
and $f(x)\le g(x)$ $\forall x\in(x_0-\delta,x_0)\cup(x_0,x_0+\delta) \Rightarrow \lim_{x\to x_0} f(x)\le\lim_{x\to x_0} g(x)$

⑤ Sandwich Theorem: $f(x)\le h(x)\le g(x)$ $\forall x\in(x_0-\delta,x_0)\cup(x_0,x_0+\delta)$ and
$\lim_{x\to x_0} f(x)=\lim_{x\to x_0} g(x)=A \Rightarrow \lim_{x\to x_0} h(x)=A$

⑥ Arithmetic operations $+, -, \times, \div$

⑦ $\lim_{y\to y_0} f(y)=A$ $\lim_{x\to x_0} g(x)=y_0 \Rightarrow \lim_{x\to x_0} f(g(x))=A$

Thm (Heine's theorem) $\lim_{x\to x_0} f(x)=A \Longleftrightarrow \lim_{n\to\infty} f(x_n)=A$ $\forall \{x_n\}\to x_0$

Infinitesimal quantity : $\lim_{x\to x_0} f(x) = 0$

① higher order : $\lim_{x\to x_0} f(x) = 0 \quad \lim_{x\to x_0} g(x) = 0$, f(x) higher order than g(x)

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0$$

② Same order : $\lim_{x\to x_0} f(x) = 0 \quad \lim_{x\to x_0} g(x) = 0, \Rightarrow \lim_{x\to x_0} \frac{f(x)}{g(x)} = C \neq 0$

③ Equivalent : $\lim_{x\to x_0} f(x) = 0 \quad \lim_{x\to x_0} g(x) = 0, \Rightarrow \lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$

Infinite quantity (无穷大量)

Defn : If f is well-defined on $(x_0-\varepsilon, x_0) \cup (x_0, x_0+\varepsilon)$, and

$$\lim_{x\to x_0} |f(x)| = +\infty$$

then we say f is an infinite quantity as $x \to x_0$.

E.g.: $\lim_{x\to 0} \frac{1}{x^2} = +\infty \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty \quad \lim_{x\to 0^+} \frac{1}{x} = +\infty \quad \lim_{x\to 0} \left|\frac{1}{x}\right| = \infty$

Properties : If $\lim_{x\to x_0} f(x) = +\infty \quad \lim_{x\to x_0} g(x) = +\infty \quad \lim_{x\to x_0} h(x) = A \neq 0 \; |A| < +\infty$

then we have

① $\lim_{x\to x_0} \left(f(x) + g(x)\right) = +\infty \qquad \lim_{x\to x_0} \left(f(x) + h(x)\right) = +\infty$

② $\lim_{x\to x_0} f(x)\cdot g(x) = +\infty \qquad \lim_{x\to x_0} f(x)\cdot h(x) = +\infty$.

Commonly used equivalent infinitesimal quantity.

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1 \rightarrow f(x) \sim g(x) \text{ at } x_0$$

① At $x_0 = 0$

$\sin x \sim x \qquad \arcsin x \sim x$

$\tan x \sim x \qquad \arctan x \sim x$

$\ln(x+1) \sim x \qquad e^x - 1 \sim x$

$a^x - 1 \sim x \ln a$

$1 - \cos x \sim \frac{1}{2}x^2$

$\sqrt[n]{1+x} - 1 \sim \frac{1}{n}x$

② At $x_0 = 1$ $\quad \ln x \sim x - 1$

E.g. ① $\lim_{x \to 0} \frac{\arctan x}{\sin 4x} = \lim_{x \to 0} \frac{x}{4x} = \frac{1}{4}$ $\qquad \arctan x \sim x \qquad \sin x \sim x$

② $\lim_{x \to 0} \frac{\tan x - \sin x}{\sin(x^3)} = \lim_{x \to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{\sin(x^3)} = \lim_{x \to 0} \frac{\frac{\sin x}{\cos x}(1 - \cos x)}{\sin x^3}$

$= \lim_{x \to 0} \frac{1}{\cos x} \frac{x \cdot \frac{1}{2}x^2}{x^3} = \frac{1}{2}$

E.g.: ① $\lim_{x \to \frac{\pi}{2}} 2(\sin x - \cos x - x^2) = 2 - \frac{\pi^2}{2}$

② $\lim_{x \to 0} \frac{x^2 - 1}{2x^2 - x - 1} = 1$

③ $\lim_{x \to 1} \frac{x^2 - 1}{2x^2 - x - 1} = \lim_{x \to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}(2x+1)} = \lim_{x \to 1} \frac{x+1}{2x+1} = \frac{2}{3}$

④ $\lim_{x \to 0} \frac{(x-1)^3 + (1-3x)}{x^2 + 2x^3} = \lim_{x \to 0} \frac{x^3 - 3x^2 + 3x - 1 + 1 - 3x}{x^2 + 2x^3} = \lim_{x \to 0} \frac{x^2(x-3)}{x^2(1+2x)} = -3$

(5) $\lim_{x\to 4} \frac{\sqrt{1+2x}-3}{\sqrt{x}-2} = \lim_{x\to 4} \frac{(\sqrt{1+2x}-3)(\sqrt{1+2x}+3)}{(\sqrt{x}-2)(\sqrt{1+2x}+3)} = \lim_{x\to 4} \frac{(2x+1-9)}{(\sqrt{x}-2)(\sqrt{1+2x}+3)}$

$= \lim_{x\to 4} \frac{2(x-4)(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{x}+2)(\sqrt{1+2x}+3)} = \lim_{x\to 4} \frac{2\cancel{(x-4)}}{\cancel{(x-4)}} \frac{\sqrt{x}+2}{\sqrt{1+2x}+3} = \frac{4}{3}$

(6) $\lim_{x\to 0} \frac{\sqrt{a^2+x}-a}{x}$ $(a>0)$ $= \lim_{x\to 0} \frac{(\sqrt{a^2+x}-a)(\sqrt{a^2+x}+a)}{x(\sqrt{a^2+x}+a)} = \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{a^2+x}+a)} = \frac{1}{2a}$

(7) $\lim_{x\to 1} \frac{x^n-1}{x^m-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x^{n-1}+x^{n-2}+\cdots+x+1)}{\cancel{(x-1)}(x^{m-1}+x^{m-2}+\cdots+x+1)} = \frac{n}{m}$

2. Find the limits by using squeeze theorem.

(1) $\lim_{x\to -\infty} \frac{x-\cos x}{x} = \lim_{x\to -\infty} 1-\frac{\cos x}{x} = 1$ $\quad 1-\frac{1}{x} \le 1-\frac{\cos x}{x} \le 1+\frac{1}{x}$

(2) $\lim_{x\to +\infty} \frac{x\cdot \sin x}{x^2-4} = 0$ $\quad \frac{-x}{x^2-4} \le \frac{x\sin x}{x^2-4} \le \frac{x}{x^2-4}$

(3) $\lim_{x\to +\infty} \frac{[x]}{x} = 1$ $\quad x-1 \le [x] \le x+1$

3. Based on $\lim_{x\to 0} \frac{\sin x}{x} = 1$, find the following limits.

(1) $\lim_{x\to 0} \frac{\sin 2x}{x} = \lim_{x\to 0} 2\frac{\sin 2x}{2x} = 2$

$\sin x^3 = \sin(x^3)$

$\sin^2 x = (\sin x)^2$

(2) $\lim_{x\to 0} \frac{\sin x^3}{\sin^2 x} = \lim_{x\to 0} \frac{x^2}{\sin^2 x} \frac{\sin x^3}{x^3}\cdot x = 0$

(3) $\lim_{x\to 0} \frac{\tan x}{x} = \frac{\sin x}{x}\cdot\frac{1}{\cos x} = 1$

(4) $\lim_{x\to \frac{\pi}{2}} \frac{\cos x}{x-\frac{\pi}{2}}$ Let $y = x-\frac{\pi}{2}$ $\quad y\to 0$ as $x\to\frac{\pi}{2}$ $\quad x = y+\frac{\pi}{2}$

$= \lim_{y\to 0} \frac{\cos y+\frac{\pi}{2}}{y} = \lim_{y\to 0}\frac{-\sin y}{y} = -1$

(5) $\lim_{x\to 0} \frac{\tan x-\sin x}{x^3} = \lim_{x\to 0} \frac{\sin x-\sin x\cdot\cos x}{x^3\cos x} = \lim_{x\to 0} \frac{\sin x}{x} \frac{1-\cos x}{x^2} \frac{1}{\cos x} = \frac{1}{2}$

⑥ $\lim_{x\to 0} \frac{\arctan x}{x}$ Let $y=\arctan x$ then $x=\tan y$ $y\to 0$ as $x\to 0$

$= \lim_{y\to 0} \frac{y}{\tan y} = 1$

⑦ $\lim_{x\to +\infty} x\sin\frac{1}{x} = 1$ Let $y=\frac{1}{x}$ then $x=\frac{1}{y}$ $y\to 0$ as $x\to+\infty$

⑧ $\lim_{x\to a} \frac{\sin^2 x - \sin^2 a}{x-a}$

$\sin a+\sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$

$\sin a-\sin b = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$

$\sin 2a = 2\sin a\cdot\cos a$

$= \lim_{x\to a} \frac{(\sin x+\sin a)(\sin x-\sin a)}{x-a}$

$= \lim_{x\to a} \frac{2\sin\left(\frac{x+a}{2}\right)\cos\left(\frac{x-a}{2}\right)\cdot 2\sin\left(\frac{x-a}{2}\right)\cos\left(\frac{x+a}{2}\right)}{x-a}$

$= \lim_{x\to a} \frac{2\sin\left(\frac{x+a}{2}\right)\cos\left(\frac{x+a}{2}\right)\cdot 2\sin\left(\frac{x-a}{2}\right)\cos\left(\frac{x-a}{2}\right)}{x-a}$

$= \lim_{x\to a} \frac{\sin(x+a)\sin(x-a)}{x-a} = \lim_{x\to a}\sin(x+a) = \sin 2a$

$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$

4. Based on $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e$, find the following limits.

① $\lim_{x\to 0}(1+2x)^{\frac{1}{x}} = \lim_{x\to 0}(1+2x)^{\frac{1}{2x}\cdot 2} = \lim_{x\to 0}\left[(1+2x)^{\frac{1}{2x}}\right]^2 = e^2$

② $\lim_{x\to 0}(1+\alpha x)^{\frac{1}{x}}$ $(\alpha>0)$ $= e^{\alpha}$

③ $\lim_{x\to 0}(1+\tan x)^{\cot x}$ Let $y=\cot x$, then $\tan x = \frac{1}{y}$.

$|y|\to+\infty$ as $x\to 0$

$= \lim_{|y|\to+\infty}\left(1+\frac{1}{y}\right)^y = e$

④ $\lim_{x\to 0}\left(\frac{1+x}{1-x}\right)^{\frac{1}{x}} = \lim_{x\to 0}\left(\frac{1-x+2x}{1-x}\right)^{\frac{1}{x}} = \lim_{x\to 0}\left(1+\frac{2x}{1-x}\right)^{\frac{1}{x}}$

Let $y=\frac{1-x}{2x}$ $x=\frac{1}{2y+1}$ $|y|\to+\infty$ as $x\to 0$

$= \lim_{|y|\to+\infty}\left(1+\frac{1}{y}\right)^{2y+1} = \lim_{|y|\to+\infty}\left[\left(1+\frac{1}{y}\right)^y\right]^2\left(1+\frac{1}{y}\right) = e^2\cdot 1$

$\sin x \sim x \quad \arcsin x \sim x$

$\tan x \sim x \quad \arctan x \sim x$

$1-\cos x \sim \frac{1}{2}x^2$

$\sqrt[n]{1+x} - 1 \sim \frac{1}{n}x$

$\ln(x+1) \sim x \quad e^x - 1 \sim x$

$a^x - 1 \sim x\ln a$

⑤ $\lim_{x\to+\infty}\left(\frac{3x+2}{3x-1}\right)^{2x-1} = \lim_{x\to+\infty}\left(1+\frac{1}{x-\frac{1}{3}}\right)^{2x-1}$ Let $y = x-\frac{1}{3}$

$= \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{2y-\frac{1}{3}} = e^2 \cdot 1$

⑥ $\lim_{x\to\infty}\left(1+\frac{\alpha}{x}\right)^{\beta x}$ Let $y=\frac{1}{\alpha}x$ then $x=\alpha y$

$= \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{\beta\alpha y} = e^{\alpha\beta}$

5. Using the equivalent infinitesimal quantity to find the limits.

① $\lim_{x\to\infty}\frac{x\arctan\frac{1}{x}}{x-\cos x} = \lim_{x\to\infty}\frac{1}{x-\cos x} = 0$

② $\lim_{x\to 0}\frac{\sqrt{1+x^2}-1}{1-\cos x} = \lim_{x\to 0}\frac{\sqrt{1+x^2}-1}{\frac{1}{2}x^2}\cdot\frac{\frac{1}{2}x^2}{1-\cos x} = \lim_{x\to 0}\frac{(\sqrt{1+x^2}-1)(\sqrt{1+x^2}+1)}{\frac{1}{2}x^2(\sqrt{1+x^2}+1)}$

$= \lim_{x\to 0}\frac{1+x^2-1}{\frac{1}{2}x^2(\sqrt{1+x^2}+1)} = \lim_{x\to 0}\frac{1}{\frac{1}{2}(\sqrt{1+x^2}+1)} = 1$