Real Analysis: Answer to Exercise 2

Question 1. **Solution**: (a) Let

$$A_n = \left(-1 - \frac{1}{n}, 1 + \frac{1}{n}\right), \quad \bigcap_{n=1}^{\infty} A_n = [-1, 1].$$

$$B_n = \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right], \quad \bigcup_{n=1}^{\infty} B_n = (-1, 1).$$

Question 2. **Solution:** $|x - a|$ is the distance between x and a. If this distance is less than any positive number ϵ (or less than any $1/n$, where $n > 0$) then the distance must be 0. But if the distance from x to a is 0, then $x = a$.

Question 3. **Solution**: (a) Let $f(x) = (3x^3 - 5x^2 + 7x - 13)/(2x^3 - \pi)$. Dividing the top and bottom by x^3 we have

$$f(x) = \frac{3x^3 - 5x^2 + 7x - 13}{2x^3 - \pi} = \frac{3 - 5/x + 7/x^2 - 13/x^3}{2 - \pi/x^3}.$$

Since $5/x$, $7/x^2$, $13/x^3$ and π/x^3 all tend to 0 and $x \to \infty$, we see that

$$\frac{3x^3 - 5x^2 + 7x - 13}{2x^3 - \pi} \to \frac{3 - 0 + 0 - 0}{2 - 0} = \frac{3}{2}.$$

To prove that this is indeed the limit, let $\epsilon > 0$. Then

$$\begin{aligned}\left|f(x) - \frac{3}{2}\right| &= \left|\frac{-10x^2 + 14x - 26 + 3\pi}{4x^3 - 2\pi}\right| = \frac{|-10x^2 + 14x - 26 + 3\pi|}{|4x^3 - 2\pi|} \\ &\leq \frac{|-10x^2 + 14x - 26 + 3\pi|}{|2x^3|},\end{aligned}$$

since $2x^3 \leq 2x^3 + (2x^3 - 2\pi) = 4x^3 - 2\pi$, as long as $\pi \leq x^3$, so we are dividing by less on the right hand side. Hence,

$$\begin{aligned}\left|f(x) - \frac{3}{2}\right| \leq \frac{|-10x^2 + 14x - 26 + 3\pi|}{|2x^3|} &= \left|\frac{-5}{x} + \frac{7}{x^2} + \frac{-13}{x^3} + \frac{3\pi}{2x^3}\right| \\ = \left|\frac{-5}{x}\right| + \left|\frac{7}{x^2}\right| + \left|\frac{-13}{x^3}\right| + \left|\frac{3\pi}{2x^3}\right| &= \frac{5}{|x|} + \frac{7}{|x^2|} + \frac{13}{|x^3|} + \frac{3\pi}{|2x^3|}\end{aligned}$$

by the triangle inequality. Since we are assuming $\pi \le x^3$, certainly we have $1 \le x$, so that

$$\left|f(x)-\frac{3}{2}\right| \le \frac{5}{|x|}+\frac{7}{|x^2|}+\frac{13}{|x^3|}+\frac{3\pi}{|2x^3|} \le \frac{5}{x}+\frac{7}{x}+\frac{13}{x}+\frac{3\pi}{2x} \le \frac{50}{x}.$$

Let $X_\epsilon > \max\{50/\epsilon, \pi^{1/3}\}$. Then if $x > X_\epsilon$, then $x^3 > \pi$ and $x > 50/\epsilon$, so that $|f(x)-3/2| \le 50/x < \epsilon$, as required.

(b) Let $f(x) = 2x^2-3x+5$. As $x \to 1$, $f(x) = 2x^2-3x+5 \to 2.1^1 - 3.1 + 5 = 4$. To prove this, let $\epsilon > 0$.

$$\begin{aligned}|f(x)-4| &= |2x^2-3x+5-4| = |2x^2-3x+1| = |(2x-1)(x-1)| \\ &= |2x-1||x-1|.\end{aligned}$$

Since we want the limit as $x \to 1$, we can choose $|x-1| < 1/4$, or $3/4 < x < 5/4$, so that $1/2 < 2x-1 < 3/2$. But then

$$|f(x)-4| = |2x-1||x-1| < 3|x-1|/2.$$

Now choose $\delta < 2\epsilon/3$ and $\delta < 1/4$. Then if $|x-1| < \delta$, $|x-1| < 1/4$, so that $|f(x)-4| = |2x-1||x-1| < 3|x-1|/2 < 3\delta/2 < \epsilon$, as required.

(c) Let $f(x) = 1/(1-x)$. As $x \to 2$, $f(x) = 1/(1-x) \to 1/(1-2) = 1/-1 = -1$. To prove this, let $\epsilon > 0$.

$$|f(x)-(-1)| = \left|\frac{1}{1-x}+1\right| = \left|\frac{2-x}{1-x}\right| = \frac{|2-x|}{|1-x|}.$$

For $|2-x| < 1/4$, $-1/4 < x-2 < 1/4$, so that $3/4 < x-1 < 5/4$, hence $3/4 < |x-1| = |1-x|$, so

$$|f(x)-(-1)| = \frac{|2-x|}{|1-x|} < \frac{|2-x|}{3/4} = \frac{4}{3}|x-2|.$$

Let $\delta < 3\epsilon/4$ and $\delta < 1/4$, then if $|x-2| < \delta$, $|f(x)-(-1)| < \frac{4}{3}|x-2| < \epsilon$, as required.

(d) Let $f(x) = (1/x)\sin x$. We claim $f(x) \to 0$ as $x \to \infty$. Let $\epsilon > 0$.

$$|f(x)-0| = \frac{|\sin x|}{x} \le \frac{1}{x}.$$

Let $X_\epsilon = 1/\epsilon$, then for $x > X_\epsilon$, $|f(x)-0| \le 1/x < 1/X_\epsilon = \epsilon$. Hence $f(x) \to 0$ as $x \to \infty$.

(e) Let $f(x) = x\sin(1/x)$. We claim $f(x) \to 0$ as $x \to 0$. Let $\epsilon > 0$.

$$|f(x)-0| = |x||\sin(1/x)| \le |x|.$$

Let $\delta = \epsilon$, then for $0 < |x| < \delta$, $|f(x)-0| \le |x| < \delta = \epsilon$, as required.

Question 4. **Solution**: None of these are true.

(a) Let $f(x) = 1/(1+x^2)$ and $g(x) = 2/(1+x^2)$, then $f(x) < g(x)$ for all $x \in \mathbb{R}$, but $f(x) \to 0$ and $g(x) \to 0$ as $x \to \infty$.

(b) Let $f(x) = x^3$, for all $x \in \mathbb{R}$, and $g(x) = 1/x^2$ for all $x \neq 0$. Then $\lim_{x \to 0} f(x) = 0$, $\lim_{x \to 0} g(x) = \infty$, but $f(x)g(x) = x$ for all $x \neq 0$, so that $\lim_{x \to 0} f(x)g(x) = 0$.

(c) Let

$$f(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0. \end{cases}$$

Then the left and right limits as $x \to 0$ are both 0 but $f(x) = 1$.

(d) Let $g(x) = 0$ for all $x \in \mathbb{R}$ and let

$$f(x) = \begin{cases} 0, & \text{if } x \neq 0. \\ 1, & \text{if } x = 0. \end{cases}$$

Then $f\big(g(x)\big) = 1$ for all $x \in \mathbb{R}$, so that $\lim_{x \to 0} f\big(g(x)\big) = 1$, so we are done putting $a = 0$, $b = 0$ and $c - 1$.

Question 5. **Solution**:

1. This is just says exactly that $f(x) \to \ell$ as $x \to a$.
2. This tells us that f is bounded on any interval $[-N, N]$, that is to say that for any $N > 0$, there is a $K > 0$ such that $-K \leq f(x) \leq K$ whenever $-N \leq x \leq N$.

   To see this, note first that $0 < |x - a| < \delta$ if and only if x is in the interval $(a - \delta, a + \delta)$ and $x \neq a$.

   Now fix $N > 0$ and then choose $\delta_N > 0$ so that the interval $(a - \delta_N, a + \delta_N)$ contains the interval $[-N, N]$. The condition in the question holds for all $\delta > 0$, so for δ_N it tells us that there is an $\epsilon > 0$ such that for all $x \in (a - \delta_N, a + \delta_N)$, where $x \neq a$, we have $f(x) \in (\ell - \epsilon, \ell + \epsilon)$, i.e. $\ell - \epsilon < f(x) < \ell + \epsilon$. Now choose K large enough that the interval $[-K, K]$ contains the value $f(a)$ and the interval $(\ell - \epsilon, \ell + \epsilon)$.

   We have shown that for any $N > 0$, there is $K > 0$ such that $-N \leq x \leq N$, $-K \leq f(x) \leq K$.
3. This tells us that f is bounded on all of $\mathbb{R}$, that is to say that there is a $K > 0$ such that $-K \leq f(x) \leq K$ for all x.

   To see this take $\epsilon > 0$ as given by the condition and choose K such that the interval $[-K, K]$ contains both $f(a)$ and the interval $(\ell - \epsilon, \ell + \epsilon)$. If $x = a$, then $-K < f(x) < K$. Now take any $x \neq a$. There is a $\delta > 0$ such that $a - \delta < x < a + \delta)$, i.e. $0 < |x - a| < \delta$. So by the condition in the question, $|f(x) - \ell| < \epsilon$. This means that $\ell - \epsilon < f(x) < \ell + \epsilon$, so that, again, $-K < f(x) < K$.
4. Let $\delta > 0$ be as given by the condition. Then whenever x is not equal to a but is in the interval $(a - \delta, a + \delta)$, the condition says that for all $\epsilon > 0$, $|f(x) - \ell| < \epsilon$. From Question 3 on Problem Sheet 1, we know that this means that $f(x) = \ell$. So we know that $f(x) = \ell$ for all x in $(a - \delta, a + \delta)$ except possibly when $x = a$.