L'Hospital's rule $\leftarrow$ Cauchy's MVT

$\begin{cases} ① & \frac{0}{0}, \frac{\infty}{\infty} \\ ② & \lim\limits_{x\to x_0} \frac{f'(x)}{g'(x)} = A \text{ exists} \end{cases}$ ③ f, g are differentiable $A \in \mathbb{R} \cup \{-\infty, +\infty\}$

$$\Rightarrow \lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$$

Ex. ① $\lim\limits_{x\to 0^+} x\cdot \ln x$ $\quad 0\cdot\infty \Rightarrow \frac{\infty}{\infty}$

$$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -x$$

② $\lim\limits_{x\to 0^+} (\sin x)^{\frac{a}{1+\ln x}}$ $\quad 0^0 \Rightarrow e^{\frac{\infty}{\infty}}$

$$= \exp\left(\lim_{x\to 0^+} \frac{a}{1+\ln x} \sin x\right)$$

③ $\lim\limits_{x\to +\infty} \left(x+\sqrt{1+x^2}\right)^{\frac{1}{\ln x}}$ $\quad \infty^0 \Rightarrow e^{\frac{\infty}{\infty}}$

$$= \exp\left(\lim_{x\to +\infty} \left(\frac{1}{\ln x}\right) \ln\left(x+\sqrt{1+x^2}\right)\right)$$

④ $\lim\limits_{x\to +\infty} \frac{x+\sin x}{x}$ $\quad \lim\limits_{x\to +\infty} \frac{1+\cos x}{1}$

Taylor's theorem

$$f(x) \approx f(a) + f'(a)(x-a)$$ first order approximation

Ex. $f(x) = \sqrt{x+3}$, let $a=1$, then $f(a) = 2$

$f'(x) = \frac{1}{2}\frac{1}{\sqrt{x+3}}$ $\quad f'(a) = \frac{1}{4}$

Find $\sqrt{3.98} \approx 1.99499$

$f(a) + f'(a)(0.98-1)$

$= 2 + \frac{1}{4}(-0.02) = 1.995$

Find $\sqrt{4.05} \approx 2.01246$

$= 2 + \frac{1}{4}(0.05) = 2.0125$

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

$$\vdots$$

Taylor polynomial 多项式

Defn: Suppose that f is differentiable with its derivative f', and f' is itself differentiable, with its derivative f'', ...

$$f,\ f',\ f'',\ f''',\ f^{(4)} \dots f^{(n)}.$$

If $f^{(n)}$ exists for all positive integers n, then we say f is infinitely differentiable.

$f(x) = e^x \quad f'(x) = e^x \quad \dots$

$f(x) = x \quad f'(x) = 1 \quad f''(x) = 0 \quad \dots$

$f(x) = \sin x \quad f'(x) = \cos x \quad \dots$

Taylor polynomial: Suppose that f is n-order differentiable, then its Taylor polynomial at $x = x_0$ is

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Thm. If f is n-order differentiable at $x=x_0$, then

$$f(x) = T_n(x) + o((x-x_0)^n) \to \text{Peano residual}$$

$$\Rightarrow f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + o((x-x_0)^n)$$

Proof: Let $R_n(x) = f(x) - T_n(x)$ $\quad Q_n(x) := (x-x_0)^n$

to prove $\lim_{x\to x_0} \frac{R_n(x)}{Q_n(x)} = 0$

$$R_n(x_0) = f(x_0) - \left(f(x_0) + f'(x_0)(x_0-x_0) + \frac{f''(x_0)}{2}(x_0-x_0)^2 \cdots \right) = 0$$

$$R_n'(x_0) = f'(x_0) - \left(f'(x_0) + f''(x_0)(x_0-x_0) + \cdots\right) = 0$$

$\vdots$

$$R_n^{(n)}(x_0) = 0$$

$$Q_n(x_0) = (x_0-x_0)^n = 0$$

$$Q_n'(x_0) = n(x_0-x_0)^{n-1} = 0$$

$\vdots$

$$Q_n^{(n-1)}(x_0) = n!\,(x_0-x_0) = 0$$

$$Q_n^{(n)}(x_0) = n!$$

$$\Rightarrow \lim_{x\to x_0} \frac{R_n(x)}{Q_n(x)} = \lim_{x\to x_0} \frac{R_n'(x)}{Q_n'(x)} = \cdots = \lim_{x\to x_0} \frac{R_n^{(n)}(x)}{Q_n^{(n)}(x)} = \frac{0}{n!} = 0.$$

Thm (Taylor's theorem) Suppose that f is n-order differentiable at $x_0 \in (a,b)$. Then there exists $\xi \in (a,b)$ such that

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}}_{\text{Lagrangian residual.}}$$

Proof: Set $F(x) = f(x) - [f(x_0) + f'(x_0)(x-x_0) \cdots \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n]$

$G(x) = (x-x_0)^{n+1}$

To prove $F(x_0) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$ $\quad \Rightarrow \frac{F(x_0)}{G(x_0)} = \frac{f^{(n+1)}(\xi)}{(n+1)!}$

$$= \frac{f^{(n+1)}(\xi)}{(n+1)!} G(x_0)$$

$$\frac{f''(x_0)}{2}(x-x_0)^2$$

$G(x_0) = (x-x_0)^{n+1}$

By Cauchy's MVT, we have

$$\frac{F(x_0)}{G(x_0)} = \frac{F(x_0) - \cancel{F(x)}^{\to 0}}{G(x_0) - \cancel{G(x)}^{\to 0}} = \frac{F'(\xi)}{G'(\xi)} \qquad \xi \in (a,b)$$

$$F'(x_0) = -\Big[\cancel{f'(x_0)} + \cancel{f''(x_0)(x-x_0)} - \cancel{f'(x_0)} + \frac{f'''(x_0)}{2}(x-x_0)^2 + \cancel{f''(x_0)(x-x_0)} + \cdots + \frac{f^{(n+1)}(x_0)}{n!}(x-x_0)^n\Big]$$

$$= -\frac{f^{(n+1)}(x_0)}{n!}(x-x_0)^n$$

$$G'(x_0) = (n+1)(x-x_0)^n$$

Then we have

$$\frac{F'(\xi)}{G'(\xi)} = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

Maclaurin series

Defn: When $x_0 = 0$, the Taylor series is known as Maclaurin series.

E.g.: $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots (-1)^{2n-1}\frac{x^{2n-1}}{(2n-1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + (-1)^{2n}\frac{x^{2n}}{2n!}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots (-1)^{n-1}\frac{x^n}{n} \qquad (x > -1)$$

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!}x^2 + \cdots \frac{a(a-1)\cdots(a-n+1)}{n!}x^n$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n \qquad (|x| < 1$$

$(n+1)! > 10^6 \cdot e^{\xi} > 10^7$

E.g.: ① Compute e approximately to 10^{-6}.

$f(x)=e^x$

$x \in (a,b)$

$$e = f(1) \approx 1+1+\frac{1}{2}+\frac{1}{3!}+\cdots+\frac{1}{n!}+\frac{e^{\xi}}{(n+1)!}$$

$\xi \in (0.95, 1.05)$

$1 \in (0.95, 1.05)$

Let $\frac{e^{\xi}}{(n+1)!} < 10^{-6} \Rightarrow n=9$

$\exists \xi \in (a,b)$ such that

Cauchy MVT

$$e \approx 1+1+\frac{1}{2}+\frac{1}{6}+\cdots+\frac{1}{9!} \approx 2.718285$$

② $$\lim_{x\to 0}\frac{\sin x - x + \frac{x^3}{6}}{x^5} = \lim_{x\to 0}\frac{\cos x - 1 + \frac{x^2}{2}}{5x^4}$$

$$= \lim_{x\to 0}\frac{-\sin x + x}{20x^3} = \lim_{x\to 0}\frac{-\cos x + 1}{60x^2} = \lim_{x\to 0}\frac{\sin x}{120x}$$

$$= \lim_{x\to 0}\frac{1}{120}\cdot\cos x = \frac{1}{120}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^{2n-1}\frac{x^{2n-1}}{(2n-1)!} + o(x^{2n-1})$$

$$\lim_{x\to 0}\frac{\sin x - x + \frac{x^3}{6}}{x^5} = \lim_{x\to 0}\frac{\cancel{x} - \cancel{\frac{x^3}{3!}} + \frac{x^5}{5!} + o(x^5) - \cancel{x} + \cancel{\frac{x^3}{6}}}{x^5}$$

$$= \lim_{x\to 0}\frac{\frac{x^5}{5!}}{x^5} + \lim_{x\to 0}\cancel{\frac{o(x^5)}{x^5}} \to 0$$

$$= \lim_{x\to 0}\frac{1}{120}$$

③ $$\lim_{x\to 0}\frac{\sqrt{1+x^2} - \cos x - 1 + \frac{x^2}{2}}{x^4}$$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!}x^2 \cdots$

$$\lim_{x\to x_0}\frac{o((x-x_0)^n)}{(x-x_0)^n} = 0$$

$$= 1 + \frac{x^2}{2} - \frac{x^4}{8} + o(x^4)$$
$$-1 + \frac{x^2}{2} - \frac{x^4}{4!} + o(x^4)$$
$$-1 + \frac{x^2}{2}$$

$$\lim_{x\to 0}\frac{o(x^4)}{x^4} = 0$$

$$= \frac{-1 + \frac{3}{2}x^2 - \frac{x^4}{6}}{x^4} = -\frac{1}{6}$$

$\sqrt{1+x^2} = (1+t)^{\frac{1}{2}}$

$t = x^2$

$o(x^4) + o(x^4) = o(x^4)$

$o(x^4) + o(x^3) = o(x^3)$

⑤ $\lim\limits_{x\to 0} \dfrac{e^x \sin x - x(1+x)}{x^3} = \lim\limits_{x\to 0} \dfrac{e^x \sin x + e^x \cos x - (1+2x)}{3x^2}$

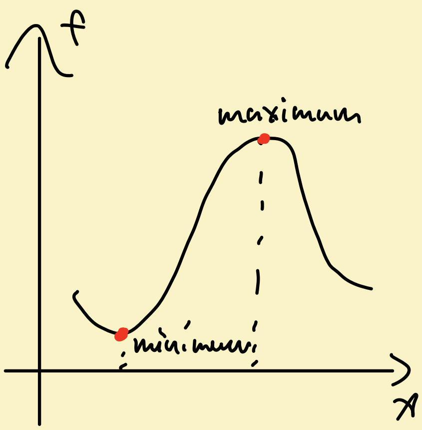
$= \lim\limits_{x\to 0} \dfrac{\cancel{e^x \sin x} + e^x \cos x - \cancel{e^x \sin x} + e^x \cos x - 2}{6x} = \lim\limits_{x\to 0} \dfrac{1}{3} \dfrac{e^x \cos x - 1}{x}$

$= \lim\limits_{x\to 0} \dfrac{1}{3} (\underbrace{e^x \cos x}_{\to 1} - \underbrace{e^x \sin x}_{\to 0}) = \dfrac{1}{3}$

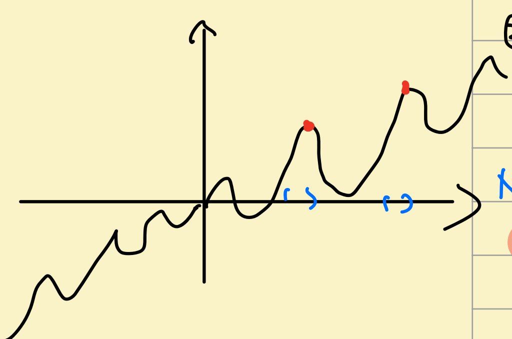
$\lim\limits_{x\to 0} \dfrac{\left(1 + x + \frac{x^2}{2} + \cancel{\frac{x^3}{6}} + \cdots\right)\left(x - \frac{x^3}{6} + \cdots\right) - x - x^2}{x^3}$

$= \dfrac{1}{2} - \dfrac{1}{6} = \dfrac{1}{3}$

Maximum and minimum value



$f(x) = x \cdot \sin x$



Defn: Suppose a function f with its domain D, and $x_0 \in D$. Then $f(x_0)$ is a

① Maximum value of f on D, if for any $x \in D$, $f(x_0) \geq f(x)$

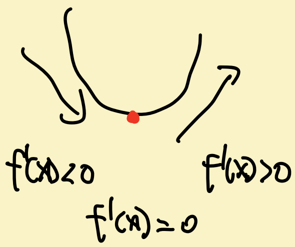
② Minimum value of f on D, if for any $x \in D$, $f(x_0) \leq f(x)$

Defn: Suppose a funtion f with its domain D, and $x_0 \in D$, Then $f(x_0)$ is a

① local maximum value of f, if there exists $\delta > 0$, for any $x \in (x_0 - \delta, x_0 + \delta)$, $f(x_0) \geq f(x)$

② local minimum value of f. if there exists $\delta > 0$, for any $x \in (x_0 - \delta, x_0 + \delta)$, $f(x_0) \leq f(x)$

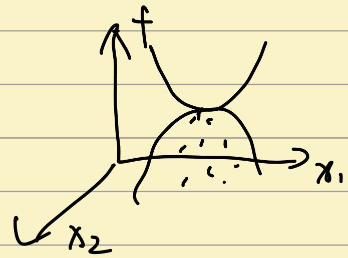
Note: both maximum and minimum values are called extreme value.



Thm (Fermat's theorem) Suppose that f is differentiable on its domain D, and f has a local maximum or minimum at x_0, then $f'(x_0) = 0$.

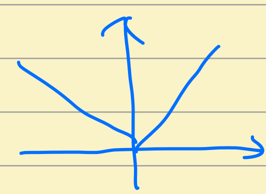
E.g.: $f(x) = x^3$ $\quad x=0 \quad f'(0) = 0$ but $f(0)$ not extreme value

Saddle point



Defn: A critical point of a function f is a number $x_0 \in D$ such that $f'(x_0) = 0$ or $f'(x_0)$ does not exist.

E.g.: $f(x) = |x|$



$x=0$ is a critical point

$\frac{3}{5}x^{-\frac{2}{5}}(4-x) - x^{\frac{3}{5}}$

E.g.: Find the critical point of $f(x) = x^{\frac{3}{5}}(4-x)$

$\frac{12}{5}x^{-\frac{2}{5}} - \frac{3}{5}x^{\frac{3}{5}} - x^{\frac{3}{5}}$

$f'(x) = \frac{12-8x}{5x^{\frac{2}{5}}}$ $\qquad f'(x) = 0 \Rightarrow x = \frac{3}{2}$

$-\frac{8}{5}x^{\frac{3}{5}}$

$\frac{12-8x}{x^{-\frac{2}{5}}}$ $\qquad f'(x)$ does not exist: $x = 0$

$\frac{1}{5}(12x^{-\frac{2}{5}} - 8x^{\frac{3}{5}})$

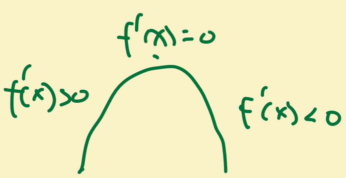
$8x \cdot x^{-\frac{2}{5}}$

Thm: Suppose f is twice-differentiable at x_0.

① If $f'(x_0) = 0$, and $f''(x_0) < 0$, then f has a local maximum at x_0.

$f'(x) \downarrow \Rightarrow f''(x) < 0$

② If $f'(x_0) = 0$, and $f''(x_0) > 0$, then f has a local minimum at x_0.



E.g.: $f(x) = x^2 + x$ find local minimum.

$$f'(x) = 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$$

$$f''(x) = 2$$

E.g. $f(x) = (2x-5)\,x^{\frac{2}{3}} = 2x^{\frac{5}{3}} - 5x^{\frac{2}{3}}$

$$f'(x) = 2\cdot\frac{5}{3}x^{\frac{2}{3}} - 5\cdot\frac{2}{3}x^{-\frac{1}{3}} = \frac{10}{3}\left(x^{\frac{2}{3}} - x^{-\frac{1}{3}}\right)$$

$$= \frac{10}{3}\left(x^1\cdot x^{-\frac{1}{3}} - 1\cdot x^{-\frac{1}{3}}\right) = \frac{10}{3}\,\frac{x-1}{\sqrt[3]{x}}$$

critical point: $x=1$, $x=0$

$$f''(x) = \frac{10}{3}\left(\frac{2}{3}x^{-\frac{1}{3}} + \frac{1}{3}x^{-\frac{4}{3}}\right) \geq 0$$

$x=1$: local minimum

$x=0$: local minimum