Introduction to mathematical finance

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https://github.com/styluck/mat_fin

Syllabus

- **Theory of Interest Rates:** this involves mathematical representations of financial concepts, such as present value, future value, annuities, and perpetuities.
- **Introduction to Interest Rate Models:** it includes mathematical models those are used to represent and understand the movements of interest rates over time.
- **Data and financial modelling:** it explains how to use mathematical models to model financial assets in real world.

Syllabus

- **Measures of Investment Risk:** This section deals with different metrics and methods used to quantify and assess the risk associated with investments.
- **Modern Portfolio Theory:** The Modern Portfolio Theory (MPT) was developed by Harry Markowitz and focuses on the optimal allocation of assets in a portfolio.
- **Asset Valuation:** Asset valuation involves determining the intrinsic value of financial instruments such as stocks, bonds, and other securities.

Resources for the module

- **Lectures** (PowerPoint Slides + hand-written lecture notes)
- **Problem sheets and example class**
  - 1 per two week
- **Assessments**
  - 2 written assessments, a total of 20%
  - Final exam, 3 hours, 80%
- **Contacts**
  - Email: xiaolh@jnu.edu.cn
  - https://github.com/styluck/mat_fin

Reference

- Shreve, Steven. Stochastic calculus for finance I: the binomial asset pricing model. Springer Science & Business Media, 2005.
- Kellison, Stephen G. The theory of interest, 2006.
- Fabozzi, Frank J., and Francesco A. Fabozzi. Bond markets, analysis, and strategies. MIT Press, 2021.
- Tuckman, Bruce, and Angel Serrat. Fixed income securities: tools for today's markets. John Wiley & Sons, 2022.
- CFA Institute 2023 CFA® Program Curriculum: Level I

1. EAY AND COMPOUNDING FREQUENCY

The time value of money

- The time value of money (TVM): a fundamental concept in finance that recognizes the idea that the value of money changes over time due to factors such as interest rates, inflation, and opportunity costs.

The time value of money

- Which would you choose?
- Option 1: $100, 2 years from now.
- Option 2: $100, 3 years from now.
- Option 3: $100, 1 year from now.

The time value of money

- Which would you choose?
- Option 1: $100, 2 years from now.
- Option 2: $100, 3 years from now.
- Option 3: $100, 1 year from now.
- 3 is better.

The time value of money

- Which would you choose?
- Option 1: \$100, now.
- Option 2: \$104, 1 years from now.
- Option 3: \$114, 3 year from now.
- There is a time value of money.

Evaluation: PV and FV

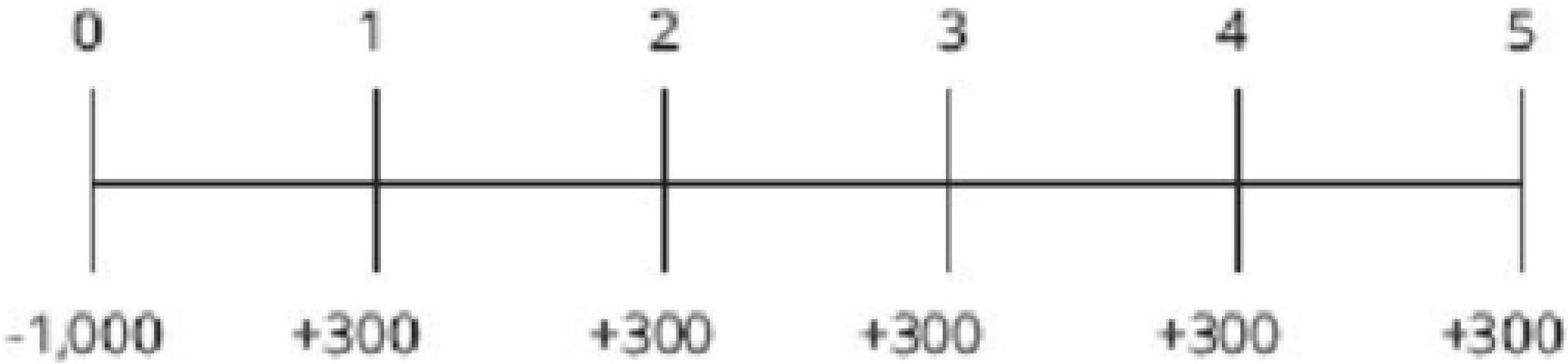
- **The future value (FV)**
  - an investment’s cashflows as a result of the effects of compound interest.
  - Computing FV involves projecting the cash flows forward, on the basis of an appropriate compound interest rate, to the end of the investment’s life.
- **The present value (PV)**
  - it brings the cash flows from an investment back to the beginning of the investment’s life based on an appropriate compound rate of return
- **PV and/or FV of investment’s cash flows**
  - comparing investment alternatives because the value of the investment’s cash flows must be measured at some common point in time,
  - at the end of the investment horizon (FV) or at the beginning of the investment horizon (PV)

Time Lines

- A time line is simply a **diagram** of the cash flows associated with a TVM problem.
- A cash flow that occurs in the present (today) is put **at time zero**.
- Cash outflows (payments) are given **a negative sign**, and cash inflows (receipts) are given **a positive sign**.
- **Discounting**: cashflows are moved **to the beginning** of the investment period to calculate the PV
- **Compounding**: **to the end** of the period

Time Lines

- A time line for an investment that costs $1,000 today (outflow) and will return a stream of cash payments (inflows) of $300 per year at the end of each of the next five years.



Time-value of money: interest rates

- An **interest rate** is the fee for using money and can be expressed as the **amount of interest due per period**, as a proportion of the amount borrowed, normally in annual percentage terms.
- The interest rate is determined by the **supply** (lenders or savers) and **demand** (borrowers or investors) for money, although its level is also determined by **macroeconomic policy**.

Time-value of money: interest rates

- Why does I.R. (interest rates) exist?
  - Opportunity cost.
  - Preference for consuming now.
  - Inflation
  - Risk
  - Everybody does it.

Time-value of money: interest rates

- **The required rate of return**
  - for a particular investment
  - the return that investors and savers require to get them to willingly lend their funds
  - the market rate of return
- **The discount rates**
  - Interest rates are also referred to as discount rates
  - the terms **discount rates** and **interest rates** are often used interchangeably

Time-value of money: interest rates

- **The opportunity cost**
  - we can also view interest rates as the opportunity cost of current consumption.
  - If the market rate of interest on 1-year securities is 5%, earning an additional 5% is the opportunity forgone when current consumption is chosen rather than saving (postponing consumption).

Time-value of money: interest rates

- **The real risk-free rate**
  - The real risk-free rate of interest is a theoretical rate on a single-period loan that has **no expectation of inflation** in it.
  - Since expected inflation in future periods is not zero, the rates we observe on U.S. Treasury bills (T-bills), for example, are **risk-free rates but not real rates of return**. T-bill rates are nominal risk-free rates because they contain an inflation premium.
  - The approximate relation here is:

    nominal risk-free rate = real risk-free rate + expected inflation rate

The types of risk

- These types of risk are:
  - **Default risk.** The risk that a borrower will not make the promised payments in a timely manner.
  - **Liquidity risk.** The risk of receiving less than fair value for an investment if it must be sold for cash quickly.
  - **Maturity risk.** As we will cover in detail in the section on debt securities, the prices of longer term bonds are more volatile than those of shorter-term bonds. Longer maturity bonds have more maturity risk than shorter-term bonds and require a maturity risk premium.

The types of risk

- Each of these risk factors is associated with **a risk premium** that we add to the nominal risk-free rate to adjust for greater default risk, less liquidity, and longer maturity relative to a very liquid, short-term, default risk-free rate such as that on T-bills. We can write:

nominal rate of interest = nominal risk-free rate

+ default risk premium

+ liquidity premium

+ maturity risk premium

Calculating PV and FV

- **Future value**
  - the amount to which a current deposit will grow over time when it is placed in an account paying compound interest. The FV, also called the compound value, is simply an example of compound interest at work.
  - The formula for the FV of a single cash flow is:

$$FV = PV(1 + I/Y)^N$$

where:

PV = amount of money invested today (the present value)

I/Y = rate of return per compounding period

N = total number of compounding periods

Calculating PV and FV

- **Present Value**
  - It is the amount of money that must be invested today, at a given rate of
  - return over a given period of time, in order to end up with a specified FV.
  - The interest rate used in the discounting process is commonly referred to as the discount rate but may also be referred to as the opportunity cost, required rate of return, and the cost of capital.
  - It represents the annual compound rate of return that can be earned on an investment.

$$PV = FV \times \left[\frac{1}{(1 + I/Y)^N}\right] = \frac{FV}{(1 + I/Y)^N}$$

Example: FV of a single sum

- Calculate the FV of a $200 investment at the end of two years if it earns an annually compounded rate of return of 10%.

Example: PV of a single sum

- Given a discount rate of 10%, calculate the PV of a $200 cash flow that will be received in two years.

Annuities

- An annuity is a stream of equal cash flows that occurs at equal intervals over a given period.
- There are two types of annuities: **ordinary annuities** and **annuities due**.
- ordinary annuity: It is characterized by cash flows that occur **at the end** of each compounding period.
- annuity due: payments or receipts occur **at the beginning** of each period

Example: FV of an ordinary annuity

- What is the future value of an ordinary annuity that pays $200 per year at the end of each of the next three years, given the investment is expected to earn a 10% rate of return?
- use the following formula:
  - $\mathrm{FV} = \mathrm{PMT} \times \left(\frac{(1 + \mathrm{I/Y})^{\mathrm{n}} - 1}{\mathrm{I/Y}} \right)$
    - Where:
    - - ($Pmt = \$200$) is the annual payment (ordinary annuity),
    - - ($r = 10\%$) is the interest rate per period, and
    - -($n = 3$) is the number of periods.

Example: PV of an ordinary annuity

- What is the PV of an annuity that pays $200 per year at the end of each of the next three years, given a 10% discount rate?
- use the formula:

$$PV = \frac{\text{PMT} \times (1 - (1 + r)^{-n})}{r}$$

  - Where:
  - - ($Pmt = \$200$) is the annual payment (ordinary annuity)
  - - ($r = 10\backslash\%$) is the discount rate per period, and,
  - - ($n = 3$) is the number of periods.

Example: PV of an ordinary annuity beginning later than t = 1

- What is the present value of four \$100 end-of-year payments if the first payment is to be received three years from today and the appropriate rate of return is 9%?

Example: PV of a bond’s cash flows

- A bond will make coupon interest payments of 70 euros (7% of its face value) at the end of each year and will also pay its face value of 1,000 euros at maturity in six years. If the appropriate discount rate is 8%, what is the present value of the bond's promised cash flows?

Example: FV of an annuity due

- What is the future value of an annuity that pays $200 per year at the beginning of each of the next three years, commencing today, if the cash flows can be invested at an annual rate of 10%?

Example: PV of an annuity due

- Given a discount rate of 10%, what is the present value of an annuity that makes $200 payments at the beginning of each of the next three years, starting today?

Present Value of a Perpetuity

- A **perpetuity** is a financial instrument that pays a fixed amount of money at set intervals over an **infinite** period of time.
- A perpetuity is a **perpetual annuity**.
- Preferred stocks are examples of perpetuities since they promise fixed interest or dividend payments **forever**.

$$PV_{perpetuity} = \frac{PMT}{I/Y}$$

Example: PV of a perpetuity

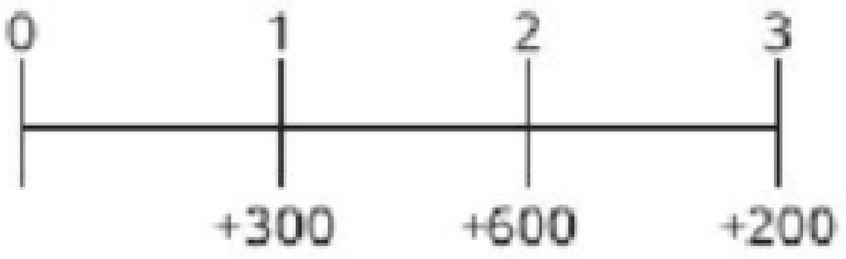
- Kodon Corporation issues preferred stock that will pay $4.50 per year in annual dividends beginning next year and plans to follow this dividend policy forever. Given an 8% rate of return, what is the value of Kodon’s preferred stock today?

Example: PV of a deferred perpetuity

- Assume the Kodon preferred stock in the preceding examples is scheduled to pay its first dividend **in four years**, and is non-cumulative (i.e., does not pay any dividends for the first three years). Given an 8% required rate of return, what is the value of Kodon's preferred stock today?

UNEVEN CASH FLOWS

- Using a rate of return of 10%, compute the future value/present value of the 3-year uneven cash flow stream described above at the end of the third year.



Computing an annuity payment needed to achieve a given FV

- At an expected rate of return of 7%, how much must be deposited at the end of each year for the next 15 years to accumulate $3,000?

Computing a loan payment

- Suppose you are considering applying for a $2,000 loan that will be repaid with equal end-ofyear payments over the next 13 years. If the annual interest rate for the loan is 6%, how much will your payments be?

Computing the number of periods in an annuity

- How many $100 end-of-year payments are required to accumulate $920 if the discount rate is 9%?

Computing the number of years in an ordinary annuity

- Suppose you have a $1,000 ordinary annuity earning an 8% return. How many annual end-of-year $150 withdrawals can be made?

Computing the rate of return for an annuity

- Suppose you have the opportunity to invest \$100 at the end of each of the next five years in exchange for \$600 at the end of the fifth year. What is the annual rate of return on this investment?

Computing the discount rate for an annuity

- What rate of return will you earn on an ordinary annuity that requires a $700 deposit today and promises to pay $100 per year at the end of each of the next 10 years?

The frequencies of compounding

- More frequent compounding **DOES** have an impact on FV and PV computations

The frequencies of compounding

- Compute the FV one year from now of $1,000 today and the PV of $1,000 to be received one year from now using a stated annual interest rate of 6% with a range of compounding periods.

Compounding Frequency	Interest Rate per Period	Effective Annual Rare	Future Value	Present Value
Annual (m = 1)	6.000%	6.00%	$1,060.00	$943,396
Semiannual(m = 2)	3.000	6.090	1,060.90	942.596
Quarterly (m = 4)	1.500	6.136	1,061.36	942.184
Monthly (m =12)	0.500	6.168	1,061.68	941.905
Daily (m = 365)	0.016438	6.183	1,061.83	941.769

Example: Present value with monthly compounding

- Alice would like to have $5,000 saved in an account at the end of three years. If the return on the account is 9% per year with monthly compounding, how much must Alice deposit today in order to reach her savings goal in three years?

Effective annual rate (EAR)

- It is the rate of interest that investors actually realize as a result of compounding.
- EAR may be determined as follows:

$$EAR = (1 + periodic\ rate)^{m} - 1$$

  - where:
  - periodic rate = stated annual rate/m
  - m = the number of compounding periods per year
- EAR for a stated rate of 8% compounded annually is **not the same** as the EAR for 8% compounded semiannually, or quarterly

Example: Computing EARs for a range of compounding frequencies

- Compute the FV one year from now of $1,000 today and the PV of $1,000 to be received one year from now using a stated annual interest rate of 6% with a range of compounding periods.

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RIVISION

- **a. interpret interest rates as required rates of return, discount rates, or opportunity costs.**
- An interest rate can be interpreted as the rate of return required in equilibrium for a particular investment, the discount rate for calculating the present value of future cash flows, or as the opportunity cost of consuming now, rather than saving and investing.

RIVISION

- **b. explain an interest rate as the sum of a real risk-free rate and premiums that compensate investors for bearing distinct types of risk.**
- The real risk-free rate is a theoretical rate on a single-period loan when there is no expectation of inflation. Nominal risk-free rate = real risk-free rate + expected inflation rate.
- Securities may have several risks, and each increases the required rate of return. These include default risk, liquidity risk, and maturity risk.
- The required rate of return on a security = real risk-free rate + expected inflation + default risk premium + liquidity premium + maturity risk premium.

RIVISION

- **c. calculate and interpret the future value (FV) and present value (PV) of a single sum of money, an ordinary annuity, an annuity due, a perpetuity (PV only), and a series of unequal cash flows.**
- Future value: FV = $PV(1 + I/Y)^N$ Present value: PV = $FV/(1 + I/Y)^N$
- An annuity is a series of equal cash flows that occurs at evenly spaced intervals over time. Ordinary annuity cash flows occur at the end of each time period. Annuity due cash flows occur at the beginning of each time period.
- Perpetuities are annuities with infinite lives (perpetual annuities):
  - $PV_{perp} = \frac{PMT}{I/Y}$

RIVISION

- **d. demonstrate the use of a time line in modeling and solving time value of money problems.**
- Constructing a time line showing future cash flows will help in solving many types of TVM problems.
- Cash flows occur at the end of the period depicted on the time line. The end of one period is the same as the beginning of the next period.
- For example, a cash flow at the beginning of Year 3 appears at time $t = 2$ on the time line.

RIVISION

- **e. calculate the solution for time value of money problems with different frequencies of compounding.**
- For non-annual time value of money problems, divide the stated annual interest rate by the number of compounding periods per year, m, and multiply the number of years by the number of compounding periods per year.

RIVISION

- f. calculate and interpret the effective annual rate, given the stated annual interest rate and the frequency of compounding.
- The EAR when there are m compounding periods =

$$\left(1+\frac{stated\ annual\ rate}{m}\right)^{m}-1$$

QUIZ 1

- 1. An interest rate is best
  - A. a discount rate or a measure of risk.
  - B. a measure of risk or a required rate of return.
  - C. a required rate of return or the opportunity cost of consumption.
- 2. An interest rate from which the inflation premium has been subtracted is known as:
  - A. a real interest rate.
  - B. a risk-free interest rate.
  - C. a real risk-free interest rate.

QUIZ 2

- 1. The amount an investor will have in 15 years if $1,000 is invested today at an annual interest rate of 9% will be closest
  - A. $1,350.
  - B. $3,518.
  - C. $3,642.
- 2. How much must be invested today, at 8% interest, to accumulate enough to retire a $10,000 debt due seven years from today?
  - A. $5,835.
  - B. $6,123.
  - C. $8,794.
- 3. An investor has just won the lottery and will receive $50,000 per year at the end of each of the next 20 years. At a 10% interest rate, the present value of the winnings is closest
  - A. $425,678.
  - B. $637,241.
  - C. $2,863,750.

QUIZ 2

- 4. An investor is to receive a 15-year, $8,000 annuity, with the first payment to be received today. At an 11% discount rate, this annuity's worth today is closest
  - A. $55,855.
  - B. $57,527.
  - C. $63,855.
- 5. If $1,000 is invested today and $1,000 is invested at the beginning of each of the next three years at 12% interest (compounded annually), the amount an investor will have at the end of the fourth year will be closest
  - A. $4,779.
  - B. $5,353.
  - C. $6,792.
- 6. Terry Corporation preferred stocks are expected to pay a $9 annual dividend forever. If the required rate of return on equivalent investments is 11%, a share of Terry preferred should be worth:
  - A. $81.82.
  - B. $99.00.
  - C. $122.22.

QUIZ 3

- 1. What is the effective annual rate for a credit card that charges 18% compounded monthly?
  - A. 15.38%.
  - B. 18.81%.
  - C. 19.56%.
- 2. Given daily compounding, the growth of $5,000 invested for one year at 12% interest will be closest to:
  - A. $5,600.
  - B. $5,628.
  - C. $5,637.
- 3. An investor is looking at a $150,000 home. If 20% must be put down and the balance is financed at a stated annual rate of 9% over the next 30 years, what is the monthly mortgage payment?
  - A. $799.33.
  - B. $895.21.
  - C. $965.55