Measures of investment risk

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https://github.com/styluck/matfin

sources of returns

- There are three sources of returns from investing in a fixed-rate bond:
- 1. Coupon and principal payments.
- 2. Interest earned on coupon payments that are reinvested over the investor’s holding period for the bond.
- 3. Any capital gain or loss if the bond is sold prior to maturity.

sources of risks

- There are three sources of risks from investing in a fixed-rate bond:
- 1. credit risk
- 2. reinvestment risk
- 3. market price risk

sources of returns

- There are five results may happen:
- 1. An investor who holds a fixed-rate bond to maturity will earn an annualized rate of return equal to the YTM of the bond when purchased.
- 2. An investor who sells a bond prior to maturity will earn a rate of return equal to the YTM at purchase if the YTM at sale has not changed since purchase.
- 3. If the market YTM for the bond, our assumed reinvestment rate, increases (decreases) after the bond is purchased but before the first coupon date, a buy-and-hold investor's realized return will be higher (lower) than the YTM of the bond when purchased

sources of returns

- 4. If the market YTM for the bond, our assumed reinvestment rate, increases after the bond is purchased but before the first coupon date, a bond investor will earn a rate of return that is lower than the YTM at bond purchase if the bond is held for a short period.
- 5. If the market YTM for the bond, our assumed reinvestment rate, decreases after the bond is purchased but before the first coupon date, a bond investor will earn a rate of return that is lower than the YTM at bond purchase if the bond is held for a long period.

Case1: held to maturity

- Consider a 6% annual-pay three-year bond purchased at a YTM of 7% and held to maturity.
- With an annual YTM of 7%, the bond’s purchase price is $973.76.
- At maturity, the investor will have received coupon income and reinvestment income equal to the future value of an annuity of three $60 coupon payments calculated with an interest rate equal to the bond’s YTM. This amount is:
- $60(1.07)^2 + 60(1.07) + 60 = \$192.89$

Case1: held to maturity

- We can easily calculate the amount earned from reinvestment of the coupons as:
- 192.89 − 3(60) = $12.89
- Adding the maturity value of $1,000 to $192.89, we can calculate the investor's rate of return over the three-year holding period as

$$\left(\frac{1{,}192.89}{973.76}\right)^{\frac{1}{3}} - 1 = 7\%$$

Case1: held to maturity

- We can calculate an investor's rate of return on the same bond purchased at a YTM of 5%.
- Price at purchase:
- N = 3; I/Y = 5; FV = 1,000; PMT = 60; CPT → PV = –1,027.23
- Coupons and reinvestment income:
- 60(1.05)2 + 60(1.05) + 60 = $189.15 or
- N = 3; I/Y = 5; PV = 0; PMT = 60; CPT → FV = –189.15

Case1: held to maturity

- Holding period return:

$$\left(\frac{1,189.15}{1,027.23}\right)^{\frac{1}{3}} - 1 = 5\%$$

Case1: held to maturity

- Conclusion: for a fixed-rate bond that does not default and has a reinvestment rate equal to the YTM, an investor who holds the bond until maturity will earn a rate of return equal to the YTM at purchase, regardless of whether the bond is purchased at a discount or a premium.

Case 2: sold prior to maturity

- 2. An investor who **sells a bond prior to maturity** will earn a rate of return equal to the YTM at purchase if the YTM at sale has not changed since purchase.
- For such an investor, we call the time the bond will be held the investor's **investment horizon**.
- The value of a bond that is sold at a discount or premium to par will move to the par value of the bond by the maturity date.
- At dates between the purchase and the sale, the value of a bond at the same YTM as when it was purchased is its **carrying value.**

EXAMPLE: Capital gain or loss on a bond

- An investor purchases a 20-year bond with a 5% semiannual coupon and a yield to maturity of 6%. Five years later the investor sells the bond for a price of 91.40. Determine whether the investor realizes a capital gain or loss, and calculate its amount.

EXAMPLE: Capital gain or loss on a bond

- An investor purchases a 20-year bond with a 5% semiannual coupon and a yield to maturity of 6%. Five years later the investor sells the bond for a price of 91.40. Determine whether the investor realizes a capital gain or loss, and calculate its amount.
- Any capital gain or loss is based on the bond's carrying value at the time of sale, when it has 15 years (30semiannual periods) to maturity. The carrying value is calculated using the bond's YTM at the time the investor purchased it.
- Because the selling price of 91.40 is greater than the carrying value of 90.20, the investor realizes a capital gain of 91.40 – 90.20 = 1.20 per 100 of face value.

Case 2: sold prior to maturity

- **Capital gains or losses** at the time a bond is sold are measured relative to this **carrying value**.
- Bonds held to maturity have **NO** capital gain or loss. Bonds sold prior to maturity at the same YTM as at purchase will also have **NO** capital gain or loss.

Case 2: sold prior to maturity

- Using the 6% three-year bond fromour earlier examples
- When the bond is purchased at a YTM of 7% (for $973.76), we have:
- Price at sale (at end of year 2, YTM = 7%):
- 1,060 / 1.07 = 990.65 or
- N = 1; I/Y = 7; FV = 1,000; PMT = 60; CPT → PV = –990.65
- which is the carrying value of the bond.
- Coupon interest and reinvestment income for two years:
- 60(1.07) + 60 = $124.20

Case 2: sold prior to maturity

- Investor's annual compound rate of return over the two-year holding period is:

$$\left(\frac{124.20 + 990.65}{973.76}\right)^{\frac{1}{2}} - 1 = 7\%$$

- This result can be demonstrated for the case where the bond is purchased at a YTM of 5%

Case 3: YTM changed after purchasing

- 3. If the market YTM for the bond, our assumed reinvestment rate, increases (decreases) after the bond is purchased but before the first coupon date, a **buy-and-hold** investor’s realized return will be higher (lower) than the YTM of the bond when purchased

Case 3: YTM changed after purchasing

- For a three-year 6% bond purchased at par (YTM of 6%), first assume that the YTM and reinvestment rate increases to 7% after purchase but before the first coupon payment date. The bond's annualized holding period return is calculated as:
- Coupons and reinvestment interest:
- 60(1.07)2 + 60(1.07) + 60 = $192.89
- Investor's annual compound holding period return:

$$\left(\frac{1{,}192.89}{1{,}000}\right)^{\frac{1}{3}} - 1 = 6.06\%$$

Case 3: YTM changed after purchasing

- If the YTM decreases to 5% after purchase but before the first coupon date, we have the following.
- Coupons and reinvestment interest:
- 60(1.05)2 + 60(1.05) + 60 = $189.15
- Investor's annual compound holding period return:

$$\left(\frac{1,189.15}{1,000}\right)^{\frac{1}{3}} - 1 = 5.94\%$$

Case 4&5: Duration effects

- 4. If the market YTM for the bond, our assumed reinvestment rate, **increases** after the bond is purchased but before the first coupon date, a bond investor will earn a rate of return that is lower than the YTM at bond purchase if the bond is **held for a short period**.
- 5. If the market YTM for the bond, our assumed reinvestment rate, **decreases** after the bond is purchased but before the first coupon date, a bond investor will earn a rate of return that is lower than the YTM at bond purchase if the bond is **held for a long period.**

Case 4&5: Duration effects

- Consider a three-year 6% bond purchased at par by an investor with a one-year investment horizon. If the YTM increases from 6% to 7% after purchase and the bond is sold after one year, the rate of return can be calculated as follows.
- Bond price just after first coupon has been paid with YTM = 7%:
- There is no reinvestment income and only one coupon of $60 received so the holding period rate of return is simply:

$$\left(\frac{981.92 + 60}{1{,}000}\right) - 1 = 4.19\%$$

Case 4&5: Duration effects

- If the YTM decreases to 5% after purchase and the bond is sold at the end of one year, the investor's rate of return can be calculated as follows.
- Bond price just after first coupon has been paid with YTM = 5%:
- And the holding period rate of return is simply:

$$\left(\frac{1{,}018.59 + 60}{1{,}000}\right) - 1 = 7.86\%$$

Case 4&5: Duration effects

- To summarize:
- short investment horizon: market price risk > reinvestment risk
- long investment horizon: reinvestment risk > market price risk

Macaulay Duration

- **Duration** is used as a measure of a bond's interest rate risk or sensitivity of a bond's full price to a change in its yield. The measure was first introduced by Frederick Macaulay and his formulation is referred to as **Macaulay duration**.
- A bond's (annual) Macaulay duration is calculated as the weighted average of the number of years until each of the bond's promised cash flows is to be paid, where the weights are the present values of each cash flow as a percentage of the bond's full value.

Macaulay Duration

- Consider a newly issued three-year 4% annual-pay bond with a yield to maturity of 5%. The present values of each of the bond's promised payments, discounted at 5%, and their weights in the calculation of Macaulay duration, are shown in the following table.

$C_1 = 40$	$PV_1 = 40 / 1.05$	$= 38.10$	$W_1 = 38.10 / 972.77$	$= 0.0392$
$C_2 = 40$	$PV_2 = 40 / 1.05^2$	$= 36.28$	$W_2 = 36.28 / 972.77$	$= 0.0373$
$C_3 = 1{,}040$	$PV_3 = 1{,}040 / 1.05^3$	$= \underline{898.39}$	$W_3 = 898.39 / 972.77$	$= \underline{0.9235}$
		972.77		1.0000

Macaulay Duration

- Now that we have the weights, and because we know the time until each promised payment is to be made, we can calculate the Macaulay duration for this bond:
- 0.0392(1) + 0.0373(2) + 0.9235(3) = 2.884 years

Modified Duration

- Modified duration (ModDur) is calculated as Macaulay duration (MacDur) divided by one plus the bond’s yield to maturity. For the bond in our earlier example, we have:
- ModDur = 2.884 / 1.05 = 2.747
- Modified duration provides an approximate percentage change in a bond’s price for a 1% change in yield to maturity. The price change for a given change in yield to maturity can be calculated as:
- approximate percentage change in bond price = –ModDur × ΔYTM

Modified Duration

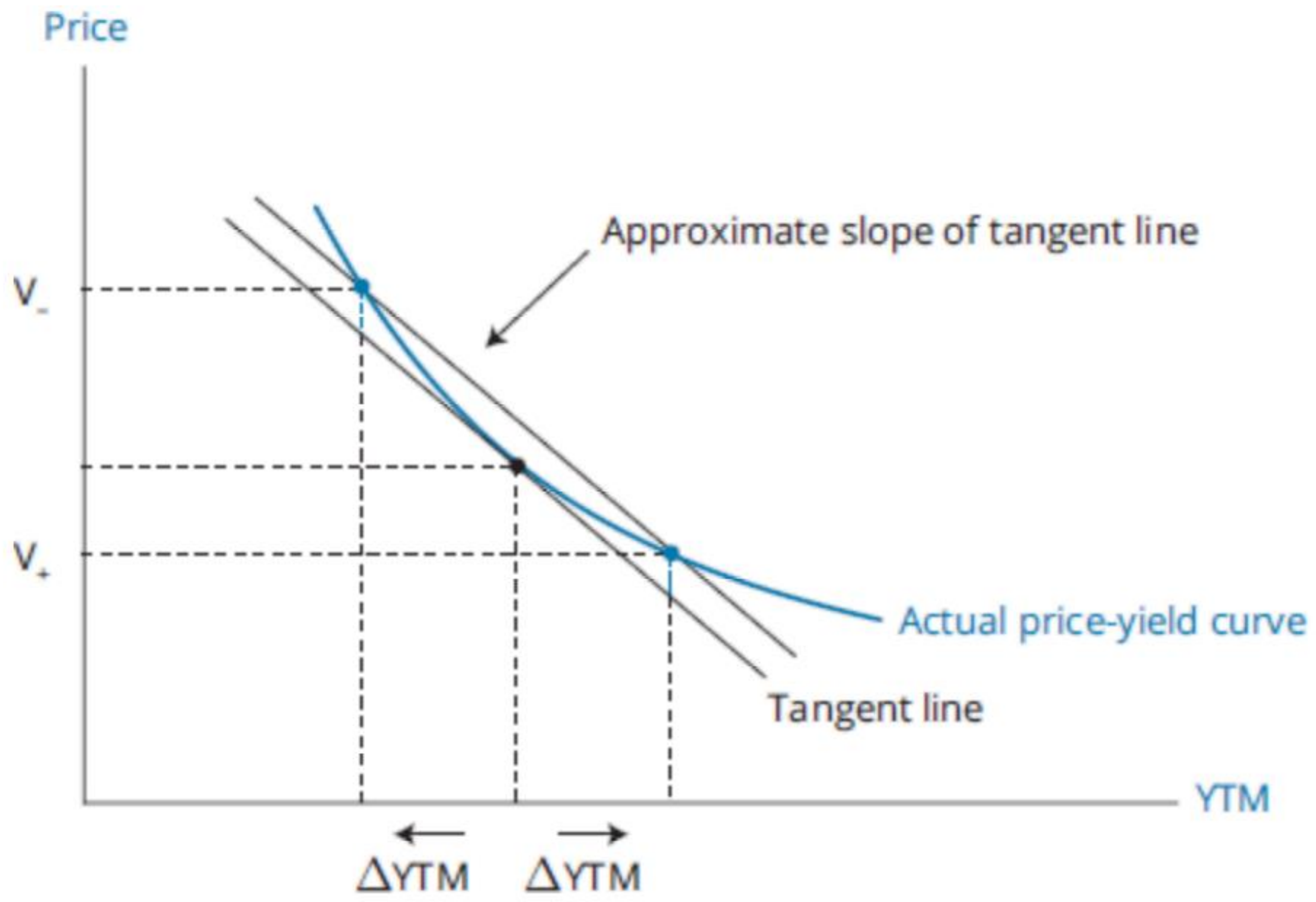
- Based on a ModDur of 2.747, the price of the bond should fall by approximately 2.747 × 0.1% = 0.2747% in response to a 0.1% increase in YTM. The resulting price estimate of $970.098 is very close to the value of the bond calculated directly using a YTM of 5.1%, which is $970.100.
- For an annual-pay bond, the general form of modified duration is:
  - $ModDur = MacDur / (1 + YTM)$
- For a semiannual-pay bond with a YTM quoted on a semiannual bond basis:
  - $ModDur_{SEMI} = MacDur_{SEMI} / (1 + YTM / 2)$

Approximate Modified Duration

- We can approximate modiϐied duration directly using bond values for an increase in YTM and for a decrease in YTM of the same size.
- The calculation of approximate modiϐied duration is based on a given change in YTM. V_ is the price of the bond if YTM is decreased by ΔYTM and V_+ is the price of the bond if the YTM is increased by ΔYTM.

$$\text{approximate modified duration} = \frac{V_- - V_+}{2 \times V_0 \times \Delta \text{YTM}}$$

Approximate Modified Duration



EXAMPLE: Calculating approximate modiβied duration

- A bond is trading at a full price of 980. If its yield to maturity increases by 50 basis points, its price will decrease to 960. If its yield to maturity decreases by 50 basis points, its price will increase to 1,002. Calculate the approximate modiβied duration.

EXAMPLE: Calculating approximate modiϐied duration

- A bond is trading at a full price of 980. If its yield to maturity increases by 50 basis points, its price will decrease to 960. If its yield to maturity decreases by 50 basis points, its price will increase to 1,002. Calculate the approximate modiϐied duration.
- The approximate modiϐied duration is

$$\frac{1{,}002 - 960}{2 \times 980 \times 0.005} = 4.29$$

- and the approximate change in price for a 1% change in YTM is 4.29%.

Effective Duration

- he calculation of effective duration is the same as the calculation of approximate modiβied duration with the change in YTM, Δy, replaced by Δcurve, the change in the benchmark yield curve used with a bond pricing model to generate V_{-} and V_{+}. The formula for calculating effective duration is:

$$\text{effective duration} = \frac{V_{-} - V_{+}}{2 \times V_0 \times \Delta\text{curve}}$$

Affects on duration

- Other things equal, an increase in a bond’s maturity will (usually) increase its interest rate risk.
- Other things equal, an increase in the coupon rate of a bond will decrease its interest rate risk.
- Other things equal, an increase (decrease) in a bond’s YTM will decrease (increase) its interest rate risk.

Money duration

- The **money duration** of a bond position (also called dollar duration) is expressed in currency units.
- money duration = annual modiβied duration × full price of bond position
- The **price value of a basis point** (PVBP) is the money change in the full price of a bond when its YTM changes by one basis point, or 0.01%.
- We can calculate the PVBP directly for a bond by calculating the average of the decrease in the full value of a bond when its YTM increases by one basis point and the increase in the full value of the bond when its YTM decreases by one basis point.

EXAMPLE: Money duration

- 1. Calculate the money duration on a coupon date of a $2 million par value bond that has a modiβied duration of 7.42 and a full price of 101.32, expressed for the whole bond and per $100 of face value.
- 2. What will be the impact on the value of the bond of a 25 basis points increase in its YTM?

EXAMPLE: Money duration

- 1. The money duration for the bond is modiβied duration times the full value of the bond:
- 7.42 × $2,000,000 × 101.32% = $15,035,888
- The money duration per $100 of par value is:
- 7.42 × 101.32 = $751.79
- 2. $15,035,888 × 0.0025 = $37,589.72
- The bond value decreases by $37,589.72.

EXAMPLE: Calculating the price value of a basis point

- A newly issued, 20-year, 6% annual-pay straight bond is priced at 101.39. Calculate the price value of a basis point for this bond assuming it has a par value of $1 million.

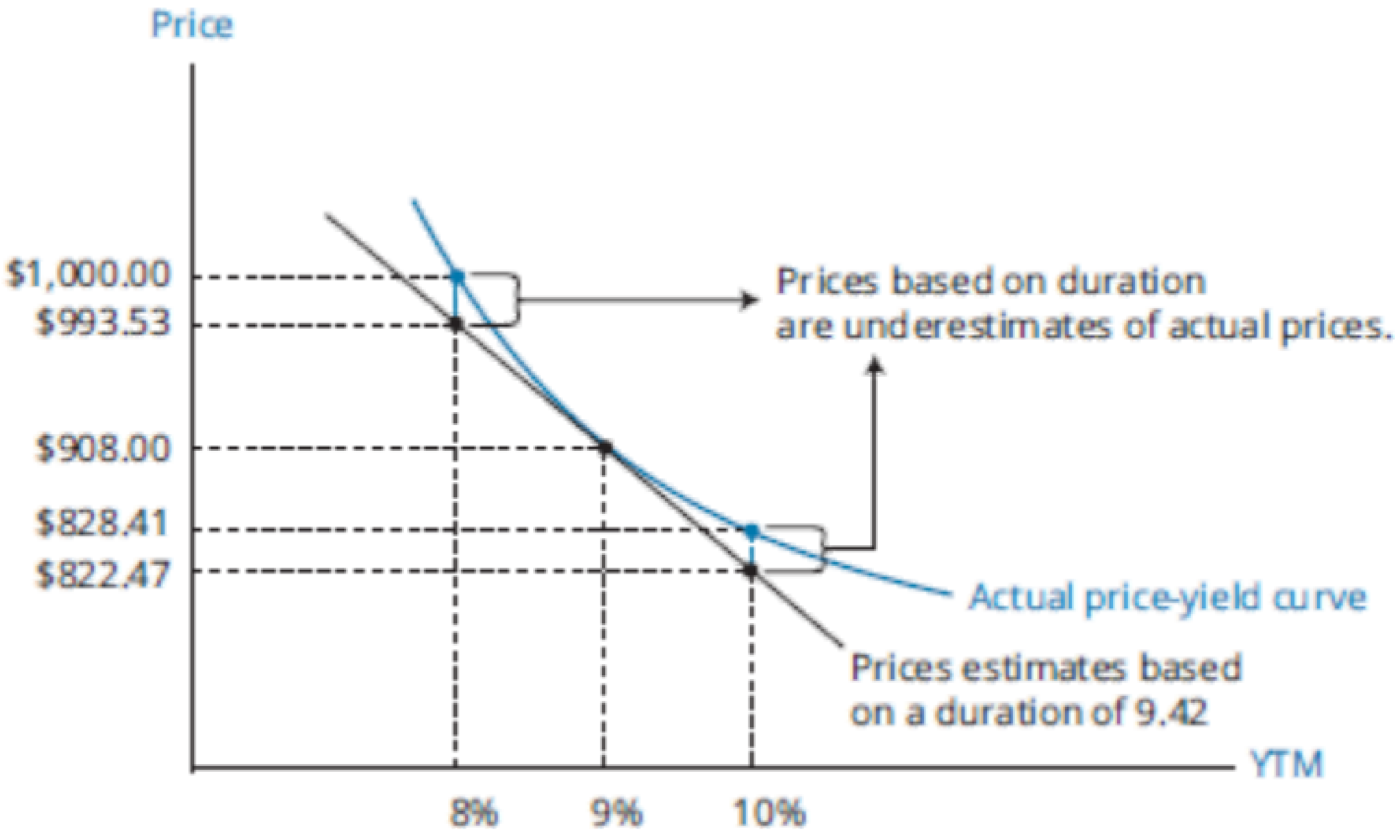
EXAMPLE: Calculating the price value of a basis point

- First we need to ϐind the YTM of the bond:
- N = 20; PV = –101.39; PMT = 6; FV = 100; CPT→I/Y = 5.88
- Now we need the values for the bond with YTMs of 5.89 and 5.87.
- I/Y = 5.89; CPT → PV = –101.273 (V_+)
- I/Y = 5.87; CPT → PV = –101.507 (V_-)
- PVBP (per $100 of par value) = (101.507 – 101.273) / 2 = 0.117
- For the $1 million par value bond, each 1 basis point change in the yield to maturity will change the bond's
- price by 0.117 × $1 million × 0.01 = $1,170.

Convexity

- The modiβied duration is a **linear approximation** of the relationship between yield and price and that, because of the **convexity** of the true price-yield relation, duration-based estimates of a bond’s full price for a given change in YTM will **be increasingly different** from actual prices.

Convexity



Price
$1,000.00
$993.53
$908.00
$828.41
$822.47
Prices based on duration
are underestimates of actual prices.
Actual price-yield curve
Prices estimates based
on a duration of 9.42
YTM
8%
9%
10%

Convexity

- **Convexity** is a measure of the curvature of the price-yield relation. The more curved it is, the greater the convexity adjustment to a duration-based estimate of the change in price for a given change in YTM.
- A bond's convexity can be estimated as:

$$\text{approximate convexity} = \frac{V_- + V_+ - 2V_0}{(\Delta \text{YTM})^2 V_0}$$

Effective convexity

- Effective convexity, like effective duration, must be used for bonds with embedded options.
- The calculation of effective convexity is the same as the calculation of approximate convexity, except that the change in the yield curve, rather than a change in the bond's YTM, is used.

$$\text{approximate effective convexity} = \frac{V_{-} + V_{+} - 2V_0}{(\Delta \text{curve})^2 V_0}$$

Effective convexity

- A bond's convexity is increased or decreased by the same bond characteristics that affect duration.
- A longer maturity, a lower coupon rate, or a lower yield to maturity will all increase convexity, and vice versa.
- For two bonds with equal duration, the one with cash flows that are more dispersed over time will have the greater convexity.

Effective convexity

- By taking account of both a bond's duration (ϐirst-order effects) and convexity (second-order effects), we can improve an estimate of the effects of a change in yield on a bond's value, especially for larger changes in yield.

$$\text{change in full bond price} = -\text{annual modified duration}(\Delta \text{YTM}) + \tfrac{1}{2}\,\text{annual convexity}(\Delta \text{YTM})^2$$

EXAMPLE: Estimating price changes with duration and convexity

- Consider an 8% bond with a full price of \$908 and a YTM of 9%. Estimate the percentage change in the full price of the bond for a 30 basis point increase in YTM assuming the bond’s duration is 9.42 and its convexity is 68.33.

EXAMPLE: Estimating price changes with duration and convexity

- Consider an 8% bond with a full price of \$908 and a YTM of 9%. Estimate the percentage change in the full price of the bond for a 30 basis point increase in YTM assuming the bond's duration is 9.42 and its convexity is 68.33.
- The duration effect is –9.42 × 0.003 = 0.02826 = –2.826%.
- The convexity effect is 0.5 × 68.33 × (0.003)2 = 0.000307 = 0.0307%.
- The expected change in bond price is (–0.02826 + 0.000307) = –2.7953%.

Duration and the investment horizon

- Macaulay duration has an interesting application in matching a bond to an investor’s investment horizon.
- When the investment horizon and the bond’s Macaulay duration are matched, a parallel shift in the yield curve prior to the ϐirst coupon payment will not (or will minimally) affect the **investor’s horizon return**.

EXAMPLE: Investment horizon yields

- Consider an eight-year, 8.5% bond priced at 89.52 to yield 10.5% to maturity. The Macaulay duration of the bond is 6. We can calculate the horizon yield for horizons of 3 years, 6 years, and 8 years, assuming the YTM falls to 9.5% prior to the first coupon date.

EXAMPLE: Investment horizon yields

- Sale after 3 years
- Bond price:
- N = 5; PMT = 8.5; FV = 100; I/Y = 9.5; CPT → PV = 96.16
- Coupons and interest on reinvested coupons:
- N = 3; PMT = 8.5; PV = 0; I/Y = 9.5; CPT → FV = 28.00
- Horizon return:
- $[(96.16 + 28.00) / 89.52]^{1/3} - 1 = 11.520\%$

EXAMPLE: Investment horizon yields

- Sale after 6 years
- Bond price:
- N = 2; PMT = 8.5; FV = 100; I/Y = 9.5; CPT → PV = 98.25
- Coupons and interest on reinvested coupons:
- N = 6; PMT = 8.5; PV = 0; I/Y = 9.5; CPT → FV = 64.76
- Horizon return:
- $[(98.25 + 64.76) / 89.52]^{1/6} - 1 = 10.505\%$

EXAMPLE: Investment horizon yields

- Held to maturity, 8 years
- Maturity value = 100
- Coupons and interest on reinvested coupons:
- N = 8; PMT = 8.5; PV = 0; I/Y = 9.5; CPT → FV = 95.46
- Horizon return:
- $[(100 + 95.46) / 89.52]^{1/8} - 1 = 10.253\%$

EXAMPLE: Investment horizon yields

- For an investment horizon equal to the bond's Macaulay duration of 6, the horizon return is equal to the original YTM of 10.5%.
- For a shorter three-year investment horizon, the price increase from a reduction in the YTM to 9.5% dominates the decrease in reinvestment income so the horizon return, 11.520%, is greater than the original YTM.
- For an investor who holds the bond to maturity, there is no price effect and the decrease in reinvestment income reduces the horizon return to 10.253%, less than the original YTM.

Duration gap

- The difference between a bond’s Macaulay duration and the bondholder’s investment horizon is referred to as a duration gap.
- A positive duration gap (Macaulay duration greater than theinvestment horizon) exposes the investor to market price risk from increasing interest rates.
- A negative duration gap (Macaulay duration less than the investment horizon) exposes the investor to reinvestment risk from decreasing interest rates.

REVISION

- **a. calculate and interpret the sources of return from investing in a ϐixed-rate bond.**
- Sources of return from a bond investment include:
- Coupon and principal payments.
- Reinvestment of coupon payments.
- Capital gain or loss if bond is sold before maturity.
- Changes in yield to maturity produce market price risk (uncertainty about a bond’s price) and reinvestment risk (uncertainty about income from reinvesting coupon payments). An increase (a decrease) in YTM decreases (increases) a bond’s price but increases (decreases) its reinvestment income.

REVISION

- **b. deßine, calculate, and interpret Macaulay, modißied, and effective durations.**
- Macaulay duration is the weighted average number of coupon periods until a bond’s scheduled cash ßlows.

$$\text{approximate modified duration} = \frac{V_{-} - V_{+}}{2 \times V_0 \times \Delta YTM}$$

- Effective duration is a linear estimate of the percentage change in a bond’s price that would result from a 1% change in the benchmark yield curve.

$$\text{effective duration} = \frac{V_{-} - V_{+}}{2 \times V_0 \times \Delta\text{curve}}$$

REVISION

- **c. explain how a bond’s maturity, coupon, and yield level affect its interest rate risk.**
- Holding other factors constant:
- Duration increases when maturity increases.
- Duration decreases when the coupon rate increases.
- Duration decreases when YTM increases.

REVISION

- **d. calculate and interpret the money duration of a bond and price value of a basis point (PVBP).**
- Money duration is stated in currency units and is sometimes expressed per 100 of bond value.
- money duration = annual modiβied duration × full price of bond position
- money duration per 100 units of par value = annual modiβied duration × full bond price per 100 of par value
- The price value of a basis point is the change in the value of a bond, expressed in currency units, for a change in YTM of one basis point, or 0.01%.
- PVBP = $[(V_{-} - V_{+}) / 2] \times$ par value × 0.01

REVISION

- **h. calculate and interpret approximate convexity and compare approximate and effective convexity.**
- Convexity refers to the curvature of a bond's price-yield relationship.

$$\text{approximate convexity} = \frac{V_{-} + V_{+} - 2V_0}{(\Delta YTM)^2 V_0}$$

- Effective convexity is appropriate for bonds with embedded options:

$$\text{approximate effective convexity} = \frac{V_{-} + V_{+} - 2V_0}{(\Delta curve)^2 V_0}$$

REVISION

- **i. calculate the percentage price change of a bond for a speciβied change in yield, given the bond's approximate duration and convexity.**
- Given values for approximate annual modiβied duration and approximate annual convexity, the percentage change in the full price of a bond can be estimated as:

$$\%\Delta \text{ full bond price} = -\text{annual modified duration}(\Delta\text{YTM}) + \frac{1}{2}\text{annual convexity}(\Delta\text{YTM})^2$$

REVISION

- **k. describe the relationships among a bond’s holding period return, its duration, and the investment horizon.**
- Over a short investment horizon, a change in YTM affects market price more than it affects reinvestment income.
- Over a long investment horizon, a change in YTM affects reinvestment income more than it affects market price.
- Macaulay duration may be interpreted as the investment horizon for which a bond’s market price risk and reinvestment risk just offset each other.
- duration gap = Macaulay duration − investment horizon