Introduction to interest rate models

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fixed-income security

- fixed-income security: promises to make a series of interest payments in fixed amounts and to repay the principal amount at maturity
- Bonds are rated based on their relative probability of default (failure to make promised payments).

fixed-income security

- The features of a fixed-income security include specification of:
- The issuer of the bond.
- The maturity date of the bond.
- The par value (principal value to be repaid).
- Coupon rate and frequency.
- Currency in which payments will be made

Issuers of Bonds

- Corporations: corporate bonds
- Sovereign national governments: U.S. Treasury bonds
- Non-sovereign governments: state bonds
- Quasi-government entities: Federal National Mortgage Association (Fannie Mae)
- Supranational entities: World Bank, the European Investment Bank, and the International Monetary Fund (IMF)
- Special purpose entities: asset-backed securities

Bond Maturity

- The **maturity date** of a bond is **the date on which the principal is to be repaid**. Once a bond has been issued, the time remaining until maturity is referred to as the **term to maturity** or **tenor** of a bond.
- Bonds that have no maturity date are called **perpetual bonds**
- Bonds with original maturities of one year or less are referred to as **money market securities**.
- Bonds with original maturities of more than one year are referred to as **capital market securities**

Par Value

- The **par value** of a bond is the **principal amount** that will be repaid at maturity.
- **Premium**: a bond that is selling for more than its par value
- **Discount**: a bond that is selling at less than its par value
- **At par**: a bond that is selling for exactly its par value
- **zero-coupon bonds** or **pure discount bonds**: a bond pays no interest prior to maturity

Collateral and credit enhancements

- **Unsecured bonds**: a claim to the overall assets and cash flows of the issuer.
- **Secured bonds**: backed by a claim to specific assets of a corporation
- **Mortgage-backed security (MBS)**: The underlying assets are a pool of mortgages. The most common type of securitized bond

Bullet structure

- A typical bond has a **bullet structure**
- Consider a $1,000 face value 5-year bond with an annual coupon rate of 5%. With **a bullet structure**, the bond's promised payments at the end of each year would be as follows.

Year	1	2	3	4	5
PMT	$50	$50	$50	$50	$1,050
Principal remaining	$1,000	$1,000	$1,000	$1,000	$0

Fully amortizing structure

- Fully amortizing structure: the principal is fully paid off when the last periodic payment is made.
- Consider a $1,000 face value 5-year bond with an annual coupon rate of 5%. With a Fully amortizing, the bond's promised payments at the end of each year would be as follows.

Year	1	2	3	4	5
PMT	$230.97	$230.97	$230.97	$230.97	$230.98
Principal remaining	$819.03	$629.01	$429.49	$219.99	$0

Floating-rate bonds and LIBOR

- Some bonds pay periodic interest that depends on a current market rate of interest. These bonds are called floating-rate notes (FRN) or floaters.
- The most widely used reference rate for floating-rate bonds was the **London Interbank Offered Rate (LIBOR)**
- As an example, consider a floating-rate note that pays the London Interbank Offered Rate (LIBOR) plus a margin of 0.75% (75 basis points) annually. If 1-year LIBOR is 2.3% at the beginning of the year, the bond will pay 2.3% + 0.75% = 3.05% of its par value at the end of the year. The new 1-year rate at that time will determine the rate of interest paid at the end of the next year. Most floaters pay quarterly and are based on a quarterly (90-day) reference rate.

Reset date

- The values of floating rate notes (FRNs) are more stable than those of fixed-rate debt of similar maturity
- because the coupon interest rates are reset periodically based on a reference rate
- Recall that the coupon rate on a floating-rate note is the reference rate plus or minus a margin
- The coupon rate for the next period is set using the market reference rate (MRR) for the reset period

EXAMPLE: Valuation of a floating-rate note

- A $100,000 floating rate note is based on a 180-day MRR with a quoted margin of 120 basis points. On a reset date with 5 years remaining to maturity, the 180-day MRR is quoted as 3.0% (annualized) and the required rate of return (based on the issuer's current credit rating) is 4.5% (annualized). What is the market value of the floating rate note?

EXAMPLE: Valuation of a floating-rate note

- The current annualized coupon rate on the note is 3.0% + 1.2% = 4.2%, so the next semiannual coupon payment will be 4.2% / 2 = 2.1% of face value. The required return in the market (discount margin) as an effective 180-day discount rate is 4.5% / 2 = 2.25%.
- Using a face value of 100%, 10 coupon payments of 2.1%, and a discount rate per period of 2.25%, we can calculate the present value of the floating rate note as:
- N = 10; I/Y = 2.25%; FV = 100; PMT = 2.1; CPT PV = 98.67

Yield-to-maturity (YTM)

- The market discount rate appropriate for discounting a bond's cash flows is called the bond's **yield-to-maturity (YTM)**
- If we know a bond's yield-to-maturity, we can calculate its value, and if we know its value (market price), we can calculate its yield-to-maturity.

Yield-to-maturity (YTM)

- Consider a newly issued 10-year, $1,000 par value, 10% coupon, annual-pay bond. The coupon payments will be $100 at the end of each year the $1,000 par value will be paid at the end of year 10. First, let's value this bond assuming the appropriate discount rate is 10%. The present value of the bond's cash flows discounted at 10% is:

$$\frac{100}{1.1}+\frac{100}{1.1^2}+\frac{100}{1.1^3}+\ldots\ldots+\frac{100}{1.1^9}+\frac{1,100}{1.1^{10}}=1,000$$

Yield-to-maturity (YTM)

- Now let's value that same bond with a discount rate of 8%:

$$\frac{100}{1.08}+\frac{100}{1.08^2}+\frac{100}{1.08^3}+\ldots\ldots+\frac{100}{1.08^9}+\frac{1,100}{1.08^{10}}=1,134.20$$

- If the market discount rate for this bond were 8%, it would sell at a premium of $134.20 above its par value.
- **When bond yields decrease, the present value of a bond's payments, its market value, increases.**

Yield-to-maturity (YTM)

- If we discount the bond's cash flows at 12%, the present value of the bond is:

$$\frac{100}{1.12} + \frac{100}{1.12^2} + \frac{100}{1.12^3} + \ldots\ldots + \frac{100}{1.12^9} + \frac{1,100}{1.12^{10}} = 887.00$$

- If the market discount rate for this bond were 12%, it would sell at a discount of $113 to its par value.
- **When bond yields increase, the present value of a bond's payments, its market value, decreases.**

Yield-to-maturity (YTM)

- Calculating the value of a bond with semiannual coupon payments.
- Rather than $100 per year, the security will pay $50 every six months.
- With an annual YTM of 8%, we need to discount the coupon payments at 4% per period

$$\frac{50}{1.04}+\frac{50}{1.04^2}+\frac{50}{1.04^3}+\ldots\ldots+\frac{50}{1.04^{19}}+\frac{1{,}050}{1.04^{20}} = 1{,}135.90$$

The relationships between price and yield

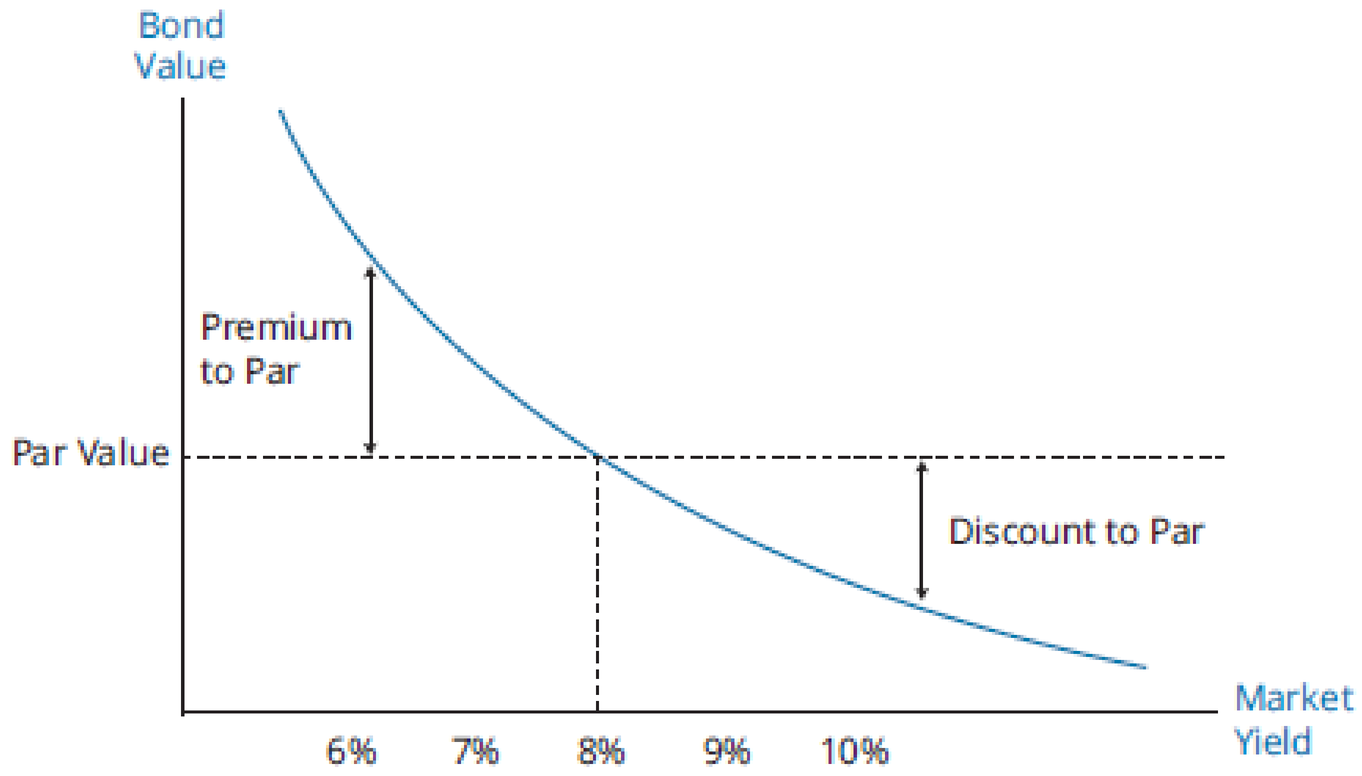
- 1. At a point in time, a decrease (increase) in a bond's YTM will increase (decrease) its price.
- 2. If a bond's coupon rate is greater than its YTM, its price will be at a premium to par value. If a bond's coupon rate is less than its YTM, its price will be at a discount to par value.
- 3. The percentage decrease in value when the YTM increases by a given amount is smaller than the increase in value when the YTM decreases by the same amount (the price-yield relationship is convex).

The relationships between price and yield

- 4. Other things equal, the price of a bond with a lower coupon rate is more sensitive to a change in yield than is the price of a bond with a higher coupon rate.
- 5. Other things equal, the price of a bond with a longer maturity is more sensitive to a change in yield than is the price of a bond with a shorter maturity

The relationships between price and yield

- Market Yield vs. Bond Value for an 8% Coupon Bond



Relationship Between Price and Maturity

- Prior to maturity, a bond can be selling at a significant discount or premium to par value.
- Regardless of its required yield, the price will converge to par value as maturity approaches.

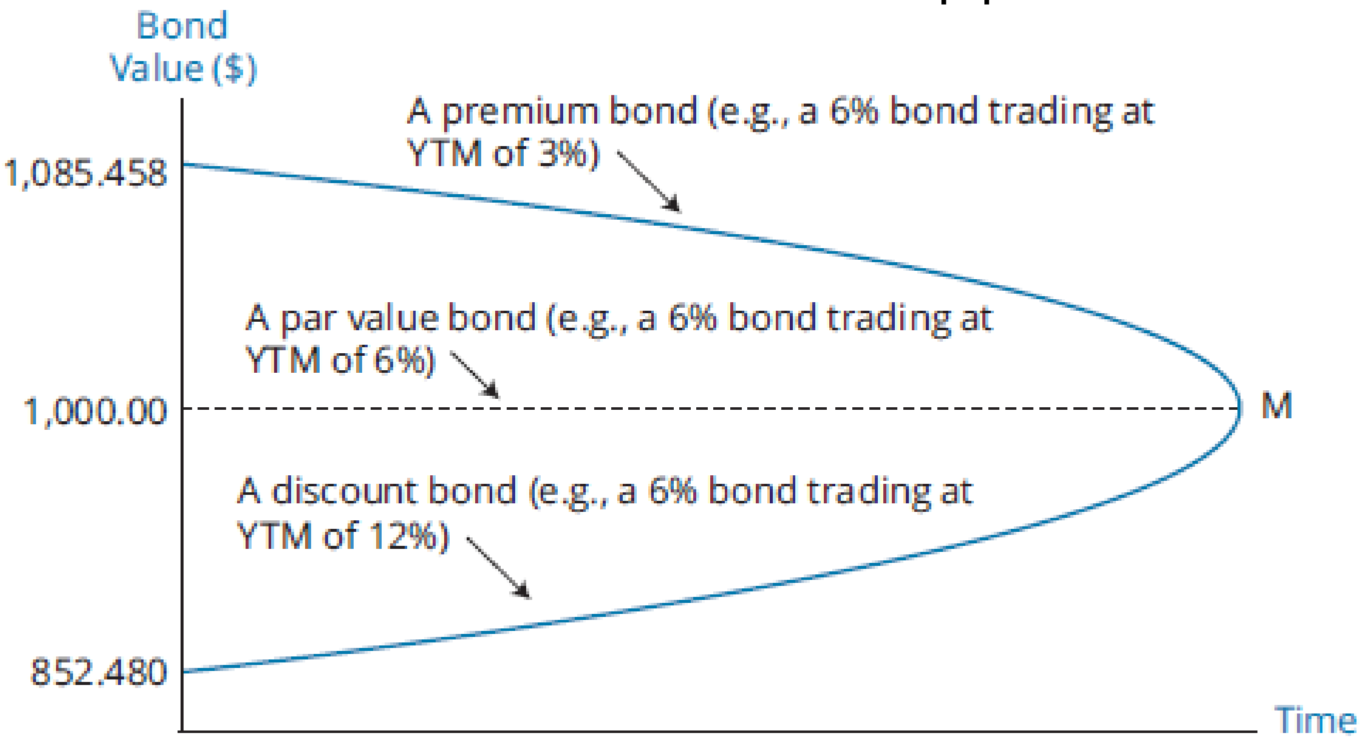
Relationship Between Price and Maturity

- Consider a bond with $1,000 par value and a 3-year life paying 6% semiannual coupons. The bond values corresponding to required yields of 3%, 6%, and 12% as the bond approaches maturity are

Time to Maturity (in years)	YTM = 3%	YTM = 6%	YTM = 12%
3.0	$1,085.46	$1,000.00	$852.48
2.5	1,071.74	1,000.00	873.63
2.0	1,057.82	1,000.00	896.05
1.5	1,043.68	1,000.00	919.81
1.0	1,029.34	1,000.00	945.00
0.5	1,014.78	1,000.00	971.69
0.0	1,000.00	1,000.00	1,000.00

Relationship Between Price and Maturity

- Consider a bond with $1,000 par value and a 3-year life paying 6% semiannual coupons. The bond values corresponding to required yields of 3%, 6%, and 12% as the bond approaches maturity are



The full price

- For most bond trades, the settlement date, which is when cash is exchanged for the bond, will fall between coupon payment dates.
- The value of a bond between coupon dates can be calculated, using its current YTM, as the value of the bond on its last coupon date (PV) times (1 + YTM / # of coupon periods per year)t/T,
  - where
  - t is the number of days since the last coupon payment, and
  - T is the number of days in the coupon period.
- this value is referred to as the **full price** of the bond.

EXAMPLE: Calculating the full price of a bond

- A 5% bond makes coupon payments on June 15 and December 15 and is trading with a YTM of 4%. The bond is purchased and will settle on August 21 when there will be four coupons remaining until maturity. Calculate the full price of the bond using actual days.

EXAMPLE: Calculating the full price of a bond

- A 5% bond makes coupon payments on June 15 and December 15 and is trading with a YTM of 4%. The bond is purchased and will settle on August 21 when there will be four coupons remaining until maturity. Calculate the full price of the bond using actual days.
- Step 1: Calculate the value of the bond on the last coupon date (coupons are semiannual, so we use 4 / 2 = 2% for the periodic discount rate):
- N = 4; PMT = 25; FV = 1,000; I/Y = 2; CPT → PV = –1,019.04

EXAMPLE: Calculating the full price of a bond

- A 5% bond makes coupon payments on June 15 and December 15 and is trading with a YTM of 4%. The bond is purchased and will settle on August 21 when there will be four coupons remaining until maturity. Calculate the full price of the bond using actual days.
- Step 2: Adjust for the number of days since the last coupon payment:
- Days between June 15 and December 15 = 183 days.
- Days between June 15 and settlement on August 21 = 67 days.
- Full price = $1{,}019.04 \times (1.02)^{67/183} = 1{,}026.46$.

The flat price

- The accrued interest since the last payment date can be calculated as the coupon payment times the portion of the coupon period that has passed between the last coupon payment date and the settlement date of the transaction.
- For the bond in the previous example, the accrued interest on the settlement date of August 21 is:
  - \$25 (67 / 183) = \$9.15
- The **full price** minus the accrued interest is referred to as the **flat price** of the bond
  - flat price = full price − accrued interest

The flat price

- So for the bond in our example,
    - the flat price = 1,026.46 − 9.15 = 1,017.31.

- So far, in calculating accrued interest, we used the actual number of days between coupon payments and the actual number of days between the last coupon date and the settlement date. This actual/actual method is **used most often with government bonds**.
- The **30/360 method** is **most often used for corporate bonds**. This method assumes that there are 30 days in each month and 360 days in a year.

EXAMPLE: Accrued interest

- An investor buys a $1,000 par value, 4% annual-pay bond that pays its coupons on May 15. The investor’s buy order settles on August 10. Calculate the accrued interest that is owed to the bond seller, using the 30/360 method and the actual/actual method.

EXAMPLE: Accrued interest

- An investor buys a $1,000 par value, 4% annual-pay bond that pays its coupons on May 15. The investor's buy order settles on August 10. Calculate the accrued interest that is owed to the bond seller, using the 30/360 method and the actual/actual method.
- The annual coupon payment is 4% × $1,000 = $40.
- Using the 30/360 method, interest is accrued for 30 – 15 = 15 days in May; 30 days each in June and July; and 10 days in August, or 15 + 30 + 30 + 10 = 85 days.

$$\text{accrued interest (30/360 method)} = \frac{85}{360} \times \$40 = \$9.44$$

EXAMPLE: Accrued interest

- An investor buys a $1,000 par value, 4% annual-pay bond that pays its coupons on May 15. The investor's buy order settles on August 10. Calculate the accrued interest that is owed to the bond seller, using the 30/360 method and the actual/actual method.
- Using the actual/actual method, interest is accrued for 31 – 15 = 16 days in May; 30 days in June; 31 days in July; and 10 days in August, or 16 + 30 + 31 + 10 = 87 days.

$$\text{accrued interest (actual/actual method)} = \frac{87}{365} \times \$40 = \$9.53$$

Matrix pricing

- Matrix pricing is a method of estimating the required yield-to-maturity (or price) of bonds that are currently not traded or infrequently traded.
- The procedure is to use the YTMs of traded bonds that have credit quality very close to that of a nontraded or infrequently traded bond and are similar in maturity and coupon, to estimate the required YTM.

EXAMPLE: Pricing an illiquid bond

- You are estimating the value of a nontraded 4% annual-pay, A+ rated bond that has three years remaining until maturity. You have obtained the following yields-to-maturity on similar corporate bonds:
  - A+ rated, 2-year annual-pay, YTM = 4.3%
  - A+ rated, 5-year annual-pay, YTM = 5.1%
  - A+ rated, 5-year annual-pay, YTM = 5.3%
- Estimate the value of the nontraded bond.

EXAMPLE: Pricing an illiquid bond

simple *linear interpolation*.

- Answer:
- Step 1: Take the average YTM of the 5-year bonds: (5.1 + 5.3) / 2 = 5.2%.
- Step 2: Interpolate the 3-year YTM based on the 2-year and average 5-year YTMs:

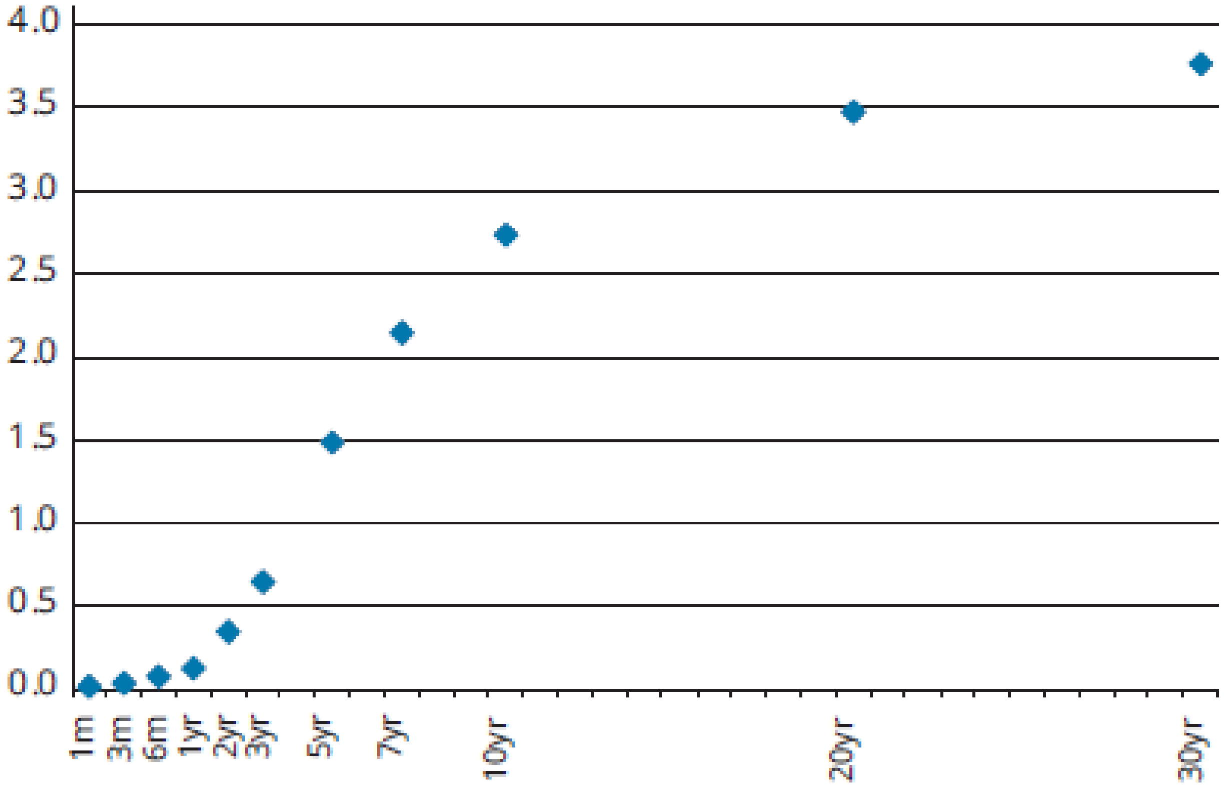
4.3% + (5.2% – 4.3%) × [(3 years – 2 years) / (5 years – 2 years)]
= 4.6%

- Step 3: Price the nontraded bond with a YTM of 4.6%:
- N = 3; PMT = 40; FV = 1,000; I/Y = 4.6; CPT → PV = –983.54
- The estimated value is $983.54 per $1,000 par value.

Yield curve

- A **yield curve** shows yields by maturity. Yield curves are constructed for yields of various types
- it’s very important to understand exactly which yield is being shown.
- **The term structure** of interest rates refers to the yields at different maturities (terms) for securities or interest rates.

Yield curve



1 month	0.02	5 year	1.49
3 month	0.04	7 year	2.15
6 month	0.08	10 year	2.74
1 year	0.13	20 year	3.48
2 year	0.35	30 year	3.77
3 year	0.65		

Source: www.treasury.gov/resource-center

Spot rates

- **Spot rates** are the market discount rates for **a single payment** to be received in the future.
- The discount rates for zero-coupon bonds are spot rates and we sometimes refer to spot rates as **zero-coupon rates** or simply **zero rates**.

$$\frac{CPN_1}{1 + S_1} + \frac{CPN_2}{(1 + S_2)^2} + \ldots\ldots\ldots + \frac{CPN_N + FV_N}{(1 + S_N)^N} = PV$$

EXAMPLE: Valuing a bond using spot rates

- Given the following spot rates, calculate the value of a 3-year, 5% annual-coupon bond.
  - Spot rates
  - 1-year: 3%
  - 2-year: 4%
  - 3-year: 5%

EXAMPLE: Valuing a bond using spot rates

- Given the following spot rates, calculate the value of a 3-year, 5% annual-coupon bond.
  - Spot rates
  - 1-year: 3%
  - 2-year: 4%
  - 3-year: 5%

$$\frac{50}{1.03}+\frac{50}{(1.04)^2}+\frac{1{,}050}{(1.05)^3} = 48.54 + 46.23 + 907.03 = \$1{,}001.80$$

EXAMPLE: Valuing a bond using spot rates

- This price, calculated using spot rates, is sometimes called the no-arbitrage price of a bond because if a bond is priced differently there will be a profit opportunity from arbitrage among bonds.
- Because the bond value is slightly greater than its par value, we know its YTM is slightly less than its coupon rate of 5%. Using the price of 1,001.80, we can calculate the YTM for this bond as:

- N = 3; PMT = 50; FV = 1,000; PV = –1,001.80; CPT → I/Y = 4.93%

Forward rates

- **Forward rates** are yields for future periods. The rate of interest on a 1-year loan that would be made two years from now is a forward rate.
- A forward yield curve shows the **future** rates for bonds or money market securities for the same maturities for annual periods in the future.
- Typically, the forward curve would show the yields of 1-year securities for each future year, quoted on a semiannual bond basis.

EXAMPLE: Computing spot rates from forward rates

- If the current 1-year spot rate is 2%, the 1-year forward rate one year from today (1y1y) is 3%, and the 1-year forward rate two years from today (2y1y) is 4%, what is the 3-year spot rate?

EXAMPLE: Computing spot rates from forward rates

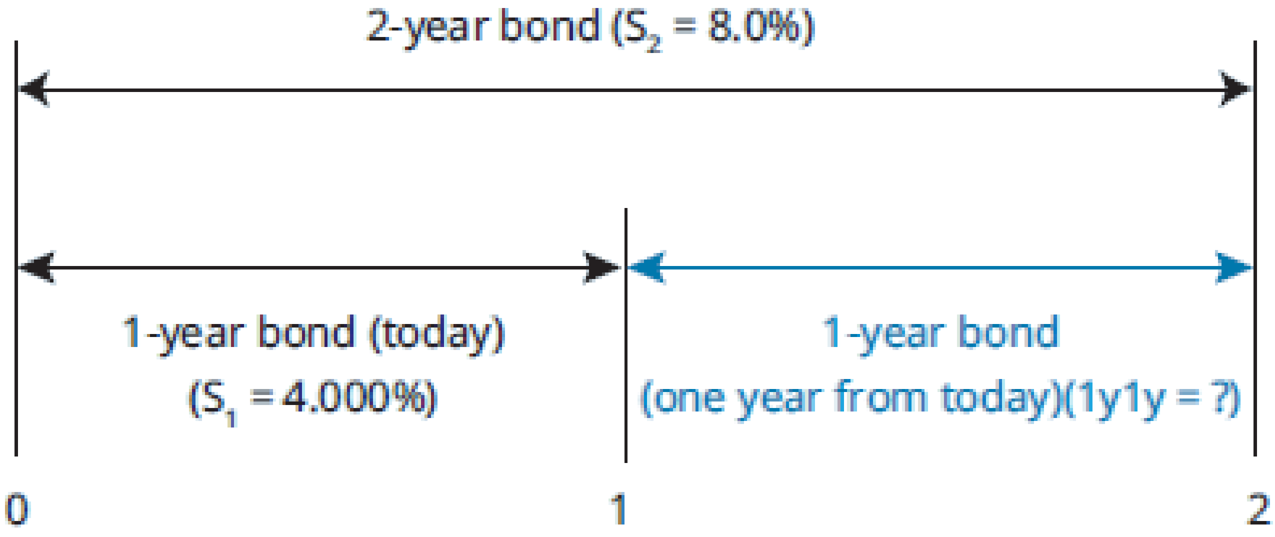
- If the current 1-year spot rate is 2%, the 1-year forward rate one year from today (1y1y) is 3%, and the 1-year forward rate two years from today (2y1y) is 4%, what is the 3-year spot rate?
- Answer:
- This relation is illustrated as (1 + S3)3 = (1 + S1)(1 + 1y1y)(1 + 2y1y). Thus, S3 = [(1 + S1)(1 + 1y1y)(1 + 2y1y)]1/3 − 1, which is the geometric mean return we covered in Quantitative Methods
    - S3 = [(1.02)(1.03)(1.04)]1/3 − 1 = 2.997%

EXAMPLE: Computing a forward rate from spot rates

- The 2-period spot rate, S2, is 8%, and the 1-period spot rate, S1, is 4%. Calculate the forward rate for one period, one period from now, 1y1y.

EXAMPLE: Computing a forward rate from spot rates

- The 2-period spot rate, S2, is 8%, and the 1-period spot rate, S1, is 4%. Calculate the forward rate for one period, one period from now, 1y1y.
- Answer:



- (1 + S2)2 = (1 + S1)(1 + 1y1y), we can get

EXAMPLE: Computing a forward rate from spot rates

- The 2-period spot rate, S2, is 8%, and the 1-period spot rate, S1, is 4%. Calculate the forward rate for one period, one period from now, 1y1y.
- Answer: $$\frac{(1+S_2)^2}{(1+S_1)} = (1 + 1y1y)$$
- Or, because we know that both choices have the same payoff in two years: $$(1.08)^2 = (1.04)(1 + 1y1y)$$ $$(1 + 1y1y) = \frac{(1.08)^2}{(1.04)}$$ $$1y1y = \frac{(1.08)^2}{(1.04)} - 1 = \frac{1.1664}{1.04} - 1 = 12.154\%$$

Yield spread

- A yield spread is the difference between the yields of two different bonds. Yield spreads are typically quoted in basis points.
- A yield spread relative to a benchmark bond is known as a benchmark spread. For example, if a 5-year corporate bond has a yield of 6.25% and its benchmark, the 5-year Treasury note, has a yield of 3.50%, the corporate bond has a benchmark spread of 625 − 350 = 275 basis points.

EXAMPLE: Estimating the spread for a new 6-year, A rated bond issue

- Consider the following market yields:
  - 5-year, U.S. Treasury bond, YTM 1.48%
  - 5-year, A rated corporate bond, YTM 2.64%
  - 7-year, U.S. Treasury bond, YTM 2.15%
  - 7-year, A rated corporate bond, YTM 3.55%
  - 6-year U.S. Treasury bond, YTM 1.74%
- Estimate the required yield on a newly issued 6-year, A rated corporate bond.

EXAMPLE: Estimating the spread for a new 6-year, A rated bond issue

- Answer:
- 1. Calculate the spreads to the benchmark (Treasury) yields.
  - Spread on the 5-year corporate bond is 2.64 – 1.48 = 1.16%.
  - Spread on the 7-year corporate bond is 3.55 – 2.15 = 1.40%.
- 2. Calculate the average spread because the 6-year bond is the midpoint of five and seven years:
  - average spread = (1.16 + 1.40) / 2 = 1.28%
- 3. Add the average spread to the YTM of the 6-year Treasury (benchmark) bond.
  - 1.74 + 1.28 = 3.02%, which is our estimate of the YTM on the newly issued 6-year,

REVISION

- a. describe basic features of a fixed-income security
- **Basic features of a fixed income security include the issuer, maturity date, par value, coupon rate, coupon frequency, and currency.**

REVISION

- b. describe how cash flows of fixed-income securities are structured
- **A bond with a bullet structure pays coupon interest periodically and repays the entire principal value at maturity.**
- **A bond with an amortizing structure repays part of its principal at each payment date. A fully amortizing structure makes equal payments throughout the bond’s life.**

REVISION

- c. describe the use of interbank offered rates as reference rates in floating-rate debt.
- **Floating-rate notes have coupon rates that adjust based on a reference rate such as LIBOR.**

REVISION

- d. calculate a bond's price given a market discount rate
- **The price of a bond is the present value of its future cash flows, discounted at the bond's yield-to-maturity.**
- **For an annual-coupon bond with N years to maturity:**

$$\text{price} = \frac{\text{coupon}}{(1+\text{YTM})} + \frac{\text{coupon}}{(1+\text{YTM})^2} + \ldots + \frac{\text{coupon} + \text{principal}}{(1+\text{YTM})^N}$$

- **For a semiannual-coupon bond with N years to maturity:**

$$\text{price} = \frac{\text{coupon}}{\left(1+\frac{\text{YTM}}{2}\right)} + \frac{\text{coupon}}{\left(1+\frac{\text{YTM}}{2}\right)^2} + \ldots + \frac{\text{coupon} + \text{principal}}{\left(1+\frac{\text{YTM}}{2}\right)^{N\times 2}}$$

REVISION

- e. identify the relationships among a bond's price, coupon rate, maturity, and market discount rate (yield-to-maturity)
- **A bond's price and YTM are inversely related. An increase in YTM decreases the price and a decrease in YTM increases the price.**
- **A bond will be priced at a discount to par value if its coupon rate is less than its YTM, and at a premium to par value if its coupon rate is greater than its YTM.**
- **Prices are more sensitive to changes in YTM for bonds with lower coupon rates and longer maturities, and less sensitive to changes in YTM for bonds with higher coupon rates and shorter maturities.**
- **A bond's price moves toward par value as time passes and maturity approaches.**

REVISION

- f. define spot rates and calculate the price of a bond using spot rates.
- **Spot rates are market discount rates for single payments to be made in the future.**
- **The no-arbitrage price of a bond is calculated using (no-arbitrage) spot rates as follows:**

$$\text{no-arbitrage price} = \frac{\text{coupon}}{(1+S_1)} + \frac{\text{coupon}}{(1+S_2)^2} + \cdots + \frac{\text{coupon} + \text{principal}}{(1+S_N)^N}$$

REVISION

- d. describe and calculate the flat price, accrued interest, and the full price of a bond.
- **The full price of a bond includes interest accrued between coupon dates. The flat price of a bond is the full price minus accrued interest.**
- **Accrued interest for a bond transaction is calculated as the coupon payment times the portion of the coupon period from the previous payment date to the settlement date.**
- **Methods for determining the period of accrued interest include actual days (typically used for government bonds) or 30-day months and 360-day years (typically used for corporate bonds).**

REVISION

- e. describe matrix pricing.
- **Matrix pricing is a method used to estimate the yield-to-maturity for bonds that are not traded or infrequently traded. The yield is estimated based on the yields of traded bonds with the same credit quality.**
- **If these traded bonds have different maturities than the bond being valued, linear interpolation is used to estimate the subject bond's yield.**

REVISION

- f. calculate annual yield on a bond for varying compounding periods in a year.
- **The effective yield of a bond depends on its periodicity, or annual frequency of coupon payments. For an annual-pay bond the effective yield is equal to the yield-to-maturity.**
- **For bonds with greater periodicity, the effective yield is greater than the yield-to-maturity.**
- **A YTM quoted on a semiannual bond basis is two times the semiannual discount rate.**

REVISION

- h. define the yield curve
- **A yield curve shows the term structure of interest rates by displaying yields across different maturities.**

REVISION

- i. define forward rates and calculate spot rates from forward rates, forward rates from spot rates, and the price of a bond using forward rates.
- **Forward rates are current lending/borrowing rates for short-term loans to be made in future periods.**
- **A spot rate for a maturity of N periods is the geometric mean of forward rates over the N periods. The same relation can be used to solve for a forward rate given spot rates for two different periods.**
- **To value a bond using forward rates, discount the cash flows at times 1 through N by the product of one plus each forward rate for periods 1 to N, and sum them.**

REVISION

- i. define forward rates and calculate spot rates from forward rates, forward rates from spot rates, and the price of a bond using forward rates.
- **For a 3-year annual-pay bond:**

$$\text{price} = \frac{\text{coupon}}{(1+S_1)} + \frac{\text{coupon}}{(1+S_1)(1+1y1y)} + \frac{\text{coupon} + \text{principal}}{(1+S_1)(1+1y1y)(1+2y1y)}$$