Derivatives and Asset valuation

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https://github.com/styluck/matfin

DERIVATIVES

- A **derivative** is a security that derives its value from the value of another security or a variable (such as an interest rate or stock index value) at some specific future date.
- The security or variable that determines the value of a derivative security is referred to as the **underlying** for the derivative.
- The value of a derivative at a point in time is derived from the value of the underlying (asset or variable) on which the derivative contract is based.

Forward contract

- A relatively simple example of a derivative is a **forward contract** that specifies the price at which one party agrees to buy or sell an underlying security at a specified **future date**.

Forward contract

- Consider a **forward contract** to buy 100 shares of ABC at \$30 per share three months from now:
- ABC shares are the **underlying asset** for the forward contract.
- \$30 is the **forward price** in the contract.
- The date of the future transaction, when the shares will be exchanged for cash, is referred to as the **settlement date (maturity date)** of the forward contract.
- 100 shares is the **contract size** of the forward contract.
- The forward price is set so that the forward contract has **zero value** to both parties at **contract initiation**; neither party pays at the initiation of the contract.

Example: ABC's forward contract

- Case 1. The **spot price**, or the market price of the underlying, is $30, equal to the forward price of $30.
- Neither party has profits or losses on the forward contract. Ignoring transactions costs, the party selling ABC shares could buy them back at $30 per share and the party buying the shares could sell them at $30 per share.

Example: ABC's forward contract

- Case 2. The spot price of ABC shares is $40, greater than the forward price of $30.
- The party buying 100 ABC shares for $3,000 at settlement can sell those shares at the spot price for $4,000, realizing a profit of $1,000 on the forward contract. The party that must deliver the ABC shares delivers shares with a market value of $4,000 and receives $3,000, realizing a $1,000 loss on the forward contract.

Example: ABC’s forward contract

- Case 3. The spot price of ABC shares is $25 at settlement, less than the forward price of $30.
- The party buying 100 ABC shares for $3,000 at settlement can sell those shares at the market price of $25 to get $2,500 and realize a loss of $500 on the forward contract. The party that must deliver the ABC shares delivers shares with a market value of $2,500 and receives $3,000, realizing a $500 gain on the forward contract.

Forward contract

- To summarize, the buyer of the shares in a forward contract will have gains when the market price of the shares at settlement is greater than the forward price, and losses when the market price of the shares at settlement is less than the forward price.
- The party that must deliver the shares in a forward contract will have gains when the market price of the shares at settlement is less than the forward price, and losses when the market price of the shares at settlement is greater than the forward price.
- <u>The gains of one party equal the losses of the other party at settlement.</u>

Exposures

- In a forward contract between two parties, one party (the buyer) commits to buy and the other party (the seller) commits to sell a physical or financial asset at a **specific price** on a **specific date** (the settlement date) in the future.
- The buyer has **long exposure** to the underlying asset in that he will make a profit on the forward if the price of the underlying at the settlement date exceeds the forward price, and have a loss if the price of the underlying at the settlement date is less than the forward price. The results are opposite for the seller of the forward, who has **short exposure** to the underlying asset.

Futures contract

- A **futures contract** is quite similar to a forward contract but is standardized and exchange traded.
- The primary ways in which forwards and futures differ are that futures **trade in a liquid secondary market**, are subject to greater regulation, and trade in markets with more disclosure (transparency).
- Futures are **backed by a central clearinghouse** and require **daily cash settlement** of gains and losses, so that counterparty credit risk is minimized.

Margin and mark-to-market

- On a futures exchange, **margin** is cash or other acceptable collaterals that both the buyer and seller must deposit.
- At the end of each trading day, the margin balance in a futures account is adjusted for any gains and losses in the value of the futures position based on the new settlement price, a process called the **mark-to-market** or marking-to-market.

SWAPS

- Swaps are agreements to exchange a series of payments on multiple settlement dates over a specified time period (e.g., quarterly payments for two years). At each settlement date, the two payments are netted so that only one net payment is made. The party with the greater liability at each settlement date pays the net difference to the other party.
- Swaps trade in a **dealer market** and the parties are exposed to counterparty credit risk, unless the market has a **central counterparty structure** to reduce counterparty risk. In this case, margin deposits and mark-to-market payments may also be required to further reduce counterparty risk.

SWAPS

- We can illustrate the basics of a swap with a simple **fixed-for-floating interest rate swap** for two years with quarterly interest payments based on a notional principal amount of $10 million. In such a swap, one party makes quarterly payments at a fixed rate of interest (the swap rate) and the other makes quarterly payments based on a floating market reference rate.
- The swap rate is set so that the swap has zero value to each party at its inception. As expectations of future values of the market reference rate change over time, the value of the swap can become positive for one party and negative for the other party.

Example: a swap

- Consider an interest rate swap with a notional principal amount of \$10 million, a fixed rate of 2%, and a floating rate of the 90-day secured overnight financing rate (SOFR). At each settlement date, the fixed-rate payment will be
  - \$10 million × 0.02/4 = \$50,000.
- The floating-rate payment at the end of the first quarter will be based on 90-day SOFR at the initiation of the swap, so that both payments are known at the inception of the swap.

Example: a swap

- If, at the end of the first quarter, 90-day SOFR is 1.6%, the floating-rate payment at the second quarterly settlement date will be
  - $10 million × 0.016 / 4 = $40,000.
- The fixed-rate payment is again $50,000, so at the end of the second quarter the fixed-rate payer will pay the net amount of $10,000 to the other party.

Options

- The two types of options of interest to us here are **put options** and **call options** on an underlying asset. We introduce them using option contracts for 100 shares of a stock as the underlying asset.
- A put option gives the buyer the right (but not the obligation) to sell 100 shares at a specified price (the **exercise price**, also referred to as the **strike price**) for specified period of time, the **time to expiration**. The put seller (also called the writer of the option) takes on the obligation to purchase the 100 shares at the price specified in the option, if the put buyer exercises the option.

Example: an option

- If the exercise price of the puts is $25 at the expiration of the option, and the shares are trading at or above $25, the put holder will not exercise the option. There is no reason to exercise the put and sell shares at $25 when they can be sold for more than $25 in the market. This is the outcome for any stock price greater than or equal to $25. Regardless of whether the stock price at option expiration is $25 or $1,000, the put buyer lets the option expire, and the put seller keeps the proceeds from the sale.
- If the stock price is below $25, the put buyer will exercise the option and the put seller must purchase 100 shares for $25 from the put buyer. On net, the put buyer essentially receives the difference between the stock price at expiration and $25 (times 100 shares).

Options

- Unlike forwards, futures, and swaps, options are sold at a price (they do not have zero value at initiation). The price of an option is also referred to as the **option premium**.
- At expiration the payoff (value) of a call option to the owner is Max (0, S – X), where S is the price of the underlying at expiration and X is the exercise price of the call option. The Max ( ) function tells us that if S < X at expiration, the option value is zero, that is, it expires worthless and will not be exercised.
- At expiration the payoff (value) of a put option to the owner is Max (0, X – S), where S is the price of the underlying at expiration and X is the exercise price of the put option. A put has a zero value at expiration unless X – S is positive.

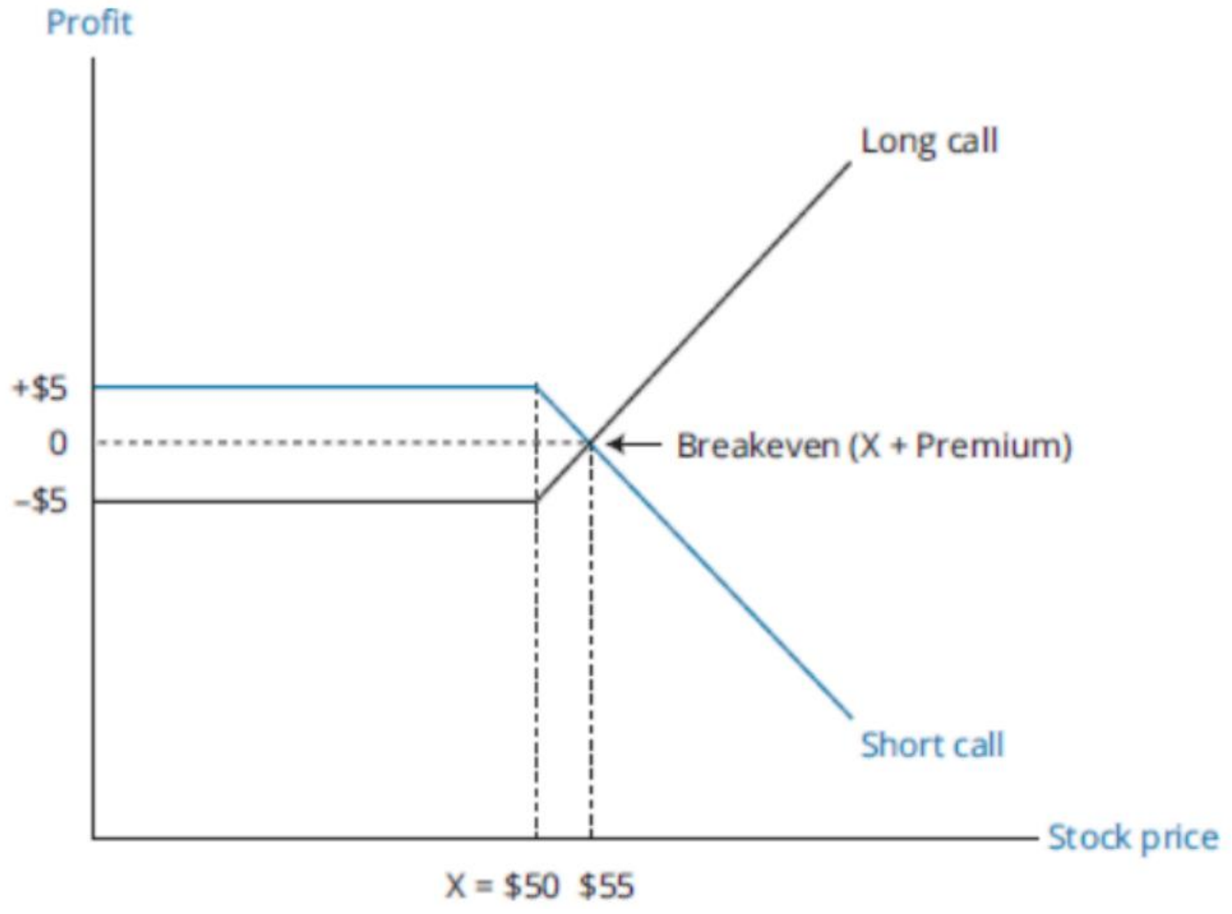
Options

- Because the seller (writer) of an option receives the option premium, the profit to the option seller at expiration is the amount of the premium received minus the option payoff at expiration.
- The writer loses the payoff at expiration and will have a loss on the option if the payoff is greater than the premium received. Note the risk exposures of call and put buyers and writers. The buyer of a put or call has **no further obligation**, so the maximum loss to the buyer is simply the amount they paid for the option. The writer of a call option has exposure to an **unlimited loss** because the maximum price of the underlying, S, is (theoretically) unlimited, so that the payoff S – X is unlimited. The payoff on a put option is X – S, so if the lower limit on S is zero, the maximum payoff on a put option is the exercise price, X.

Example: Call Option Profits and Losses

- Consider a call option with a premium of $5 and an exercise price of $50. This means the buyer pays $5 to the writer. At expiration, if the price of the stock is less than or equal to the $50 exercise price, the option has zero value, the buyer of the option is out $5, and the writer of the option is ahead $5. When the stock's price exceeds $50, the option starts to gain (breakeven will come at $55, when the value of the stock equals the exercise price plus the option premium.
- Conversely, as the price of the stock moves upward, the seller of the option starts to lose (negative figures will start at $55, when the value of the stock equals the exercise price plus the option premium)

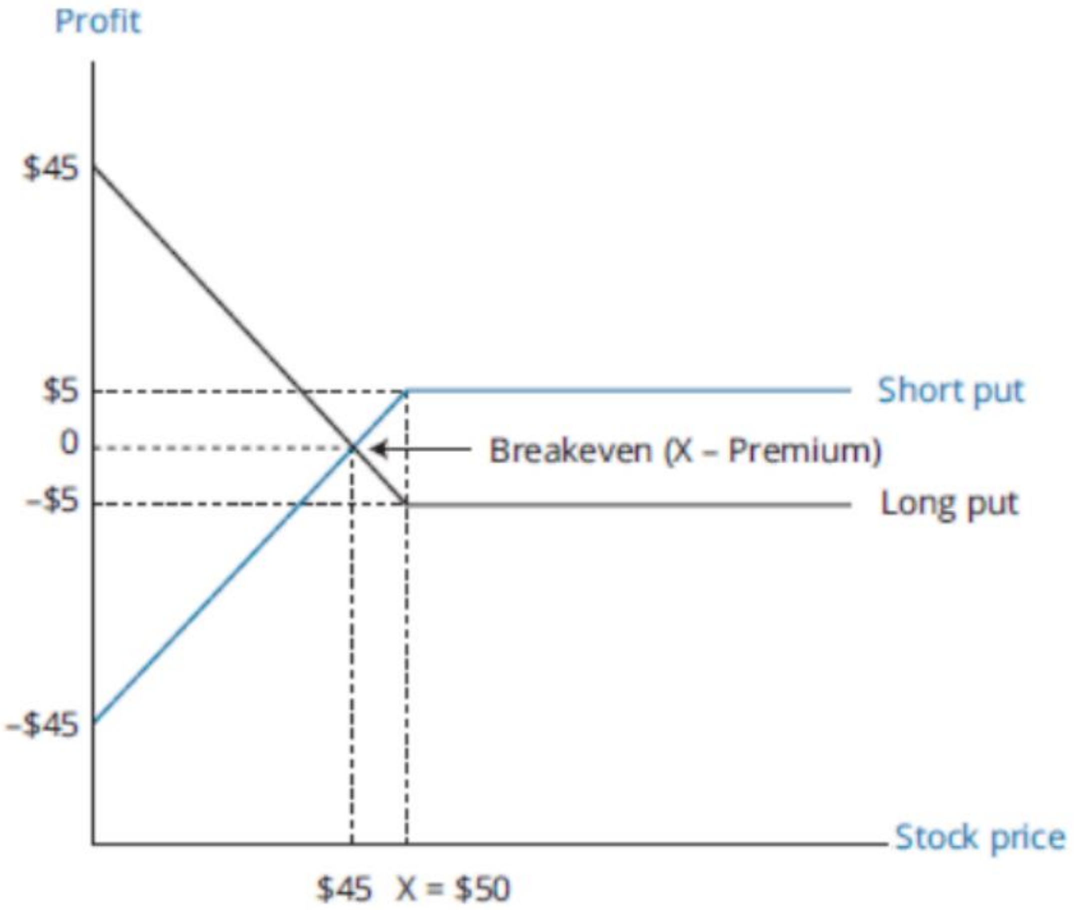
Call Option Profits and Losses



Example: Put Option Profits and Losses

- To examine the profits and losses associated with trading put options, consider a put option with a $5 premium and a $50 exercise price. The buyer pays $5 to the writer. When the price of the stock at expiration is greater than or equal to the $50 exercise price, the put has zero value.
- The buyer of the option has a loss of $5 and the writer of the option has a gain of $5. When the stock price is less than $50, the put option has a positive payoff. Breakeven will come at $45, when the value of the stock equals the exercise price less the option premium. At a stock price below $45, the put seller will have a loss.

Put Option Profits and Losses



EXAMPLE: Option profit calculations

- Suppose that both a call option and a put option have been written on a stock with an exercise price of $40. The current stock price is $42, and the call and put premiums are $3 and $0.75, respectively.
- Calculate the profit to the long and short positions for both the put and the call with an expiration day stock price of $35 and with a price at expiration of $43.

EXAMPLE: Option profit calculations

- Profit will be computed as ending option value – initial option cost.
- Stock at $35:
  - Long call: $0 – $3 = –$3. The option has no value, so the buyer loses the premium paid.
  - Short call: $3 – $0 = $3. Because the option has no value, the call writer's gain equals the premium received.
  - Long put: $5 – $0.75 = $4.25. The buyer paid $0.75 for an option that is now worth $5.
  - Short put: $0.75 – $5 = –$4.25. The seller received $0.75 for writing the option, but the option will be exercised so the seller will lose $5 at expiration.

EXAMPLE: Option profit calculations

- Stock at $43:
  - Long call: –$3 + $3 = $0. The buyer paid $3 for the option, and it is now in the money by $3. Hence, the net profit is zero.
  - Short call: $3 – $3 = $0. The seller received $3 for writing the option and now faces a –$3 valuation because the buyer will exercise the option, for a net profit of zero.
  - Long put: –$0.75 – $0 = –$0.75. The buyer paid $0.75 for the put option and the option now has no value.
  - Short put: $0.75 – $0 = $0.75. The seller received $0.75 for writing the option and it has zero value at expiration.

Exposures

- A buyer of puts or a seller of calls has **short exposure** to the underlying (will profit when the price of the underlying asset decreases).
- A buyer of calls or a seller of puts has **long exposure** to the underlying (will profit when the price of the underlying asset increases).

No-arbitrage pricing

- In contrast to valuing risky assets as the (risk-adjusted) present value of expected future cash flows, the valuation of derivative securities is based on a **no-arbitrage condition**. **Arbitrage** refers to a transaction in which an investor purchases one asset or portfolio of assets at one price and simultaneously sells an asset or portfolio of assets that has the same future payoffs, regardless of future events, at a higher price, realizing a risk-free gain on the transaction.
- While arbitrage opportunities may be rare, the reasoning is that when they do exist, they will be exploited rapidly. Therefore, we can use a no-arbitrage condition to determine the current value of a derivative, based on the known value of a portfolio of assets that has the same future payoffs as the derivative, regardless of future events. Because there are transaction costs of exploiting an arbitrage opportunity, small differences in price may persist when the arbitrage gain is less than the transaction costs of exploiting it.

No-arbitrage pricing

- We can illustrate no-arbitrage pricing with a 1-year forward contract, with a forward price of $F_0(1)$, on an ABC share that pays no dividends and is trading at a current price, S_0, of $30.
- Consider two strategies to own an ABC share at t =1:

No-arbitrage pricing

- Portfolio 1: Buy a pure discount bond with a yield of 5% that pays $F_0(1)$ at $t = 1$. The current cost of the bond is $F_0(1)/1.05$. Additionally, enter a forward contract on one ABC share at $F_0(1)$ as the buyer. The forward has a zero cost, so the cost of Portfolio 1 is $F_0(1)/1.05$.
- At $t = 1$ the bond pays $F_0(1)$, which will buy an ABC share at the forward price, so that the payoff on Portfolio 1 is the value of one share at $t = 1$, S_1.

No-arbitrage pricing

- Portfolio 2: Buy a share of ABC at S0 = 30 and hold it for one year. Cost at t = 0 is \$30.
- At t = 1 the value of the ABC share is S_1 and this is the payoff for Portfolio 2.
- The no-arbitrage condition (law of one price) requires that two portfolios with the same payoff in the future for any future value of ABC have the same cost today. Because our two portfolios have a payoff of S_1, they must have the same cost at t = 0 to prevent arbitrage. That is, $F_0(1)/1.05 = \$30$, so we can solve for the no-arbitrage forward price as $F_0(1) = 30(1.05) = 31.50$.

No-arbitrage pricing

- To better understand the no-arbitrage condition, we will consider two situations in which the forward price is not at its no-arbitrage value:
  - $F_0(1) > 31.50$ and $F_0(1) < 31.50$.
- If the forward contract price is 32 ($F_0(1) > 31.50$), the profitable arbitrage is to sell the forward (because the forward price is “too high”) and buy a share of stock. At t = 1, deliver the share under the forward contract and receive 32, for a return of 32/30 – 1 = 6.67%, which is higher than the risk-free rate.
- We can also view this transaction as borrowing 30 at the risk-free rate (5%) to buy the ABC share at t = 0, and at t = 1 paying 31.50 to settle the loan. The share delivered under the forward has a contract price of 32, so the arbitrageur has an arbitrage profit of 32 – 31.50 = 0.50 with no risk and no initial cost.

No-arbitrage pricing

- If the forward contract price is 31 ($F_0(1) < 31.50$), the profitable arbitrage is to buy the forward and sell short an ABC share at t = 0. The proceeds of the short sale, 30, can be invested at the risk-free rate to produce 30(1.05) = 31.50 at t = 1.
- The forward contract requires the purchase of a share of ABC for 31, which the investor can return to close out the short position. The profit to an arbitrageur is 31.50 – 31 = 0.50. With no cash investment at t = 0, the investor receives an arbitrage profit of 0.50 at t = 1.

Replication

- When the forward price is “too high,” the arbitrage is to sell the forward and buy the underlying asset. When the forward price is “too low,” the arbitrage is to buy the forward and sell (short) the underlying asset. In either case, the actions of arbitrageurs will move the forward price toward its no-arbitrage level until arbitrage profits are no longer possible.
- **Replication** refers to creating a portfolio with cash market transactions that has the same payoffs as a derivative for all possible future values of the underlying.

Replication

- A long forward on an ABC share can be replicated by borrowing 30 at 5% to purchase an ABC share, and repaying the loan on the settlement date of the forward. At settlement ($t = 1$), the payoff on the replication is $S_1 – 30(1.05) = S_1 – 31.50$ (value of one share minus the repayment of the loan), the same as the payoff on a long forward at 31.50, for any value of ABC shares at settlement.
- A short forward on an ABC share can be replicated by shorting an ABC share and investing the proceeds of 30 at 5%. At settlement the investor receives 31.50 from the investment of short sale proceeds, and must buy a share of ABC for S_1. The payoff on the replicating portfolio is $31.50 – S_1$, the same as the payoff on a short forward at 31.50, for any value of ABC shares at settlement.

Replication

- These replications allow us to calculate the no-arbitrage forward price of an asset, just as we did in our example using ABC shares. Because our replicating portfolio for a long forward has the same payoff as a long forward at time = T, the payoff at settlement on a portfolio that is long the replicating portfolio and short the forward must be zero to prevent arbitrage. For this strategy, when the forward is priced at its no-arbitrage value the payoff at time = T is:
  - $S_T – S_0(1 + R_f)^T – [S_T – F_0(T)] = 0$
  - so that $S_0(1 + R_f)^T + F_0(T) = 0$ and $F_0(T) = S_0(1 + R_f)^T$.

Replication

- For a portfolio that is short the replicating portfolio and long the forward, the payoff at time T is:

  - $S_0(1 + R_f)^T - S_T + [S_T - F_0(T)] = 0$
  - so that $S_0(1 + R_f)^T - F_0(T) = 0$ and $F_0(T) = S_0(1 + R_f)^T$.

- The forward price that will prevent arbitrage is $S_0(1 + R_f)^T$, just as we found in our example of a forward contract on an ABC share.

Spot price and expected future price

- When we derived the no-arbitrage forward price for an asset as $F_0(T) = S_0(1 + R_f)^T$, we assumed there were no benefits of holding the asset and no costs of holding the asset, other than the opportunity cost of the funds to purchase the asset (the risk-free rate of interest).
- Any additional costs or benefits of holding the underlying asset must be accounted for in calculating the no-arbitrage forward price. There may be additional costs of owning an asset, especially with commodities, such as storage and insurance costs. For financial assets, these costs are very low and not significant.

Spot price and expected future price

- There may also be monetary benefits to holding an asset, such as dividend payments for equities and interest payments for debt instruments. Holding commodities may have non-monetary benefits, referred to as **convenience yield**. If an asset is difficult to sell short in the market, owning it may convey benefits in circumstances where selling the asset is advantageous. For example, a shortage of the asset may drive prices up temporarily, making sale of the asset in the short term profitable.
- We denote the present value of any costs of holding the asset from time 0 to settlement at time T (e.g., storage, insurance, spoilage) as PV_0(cost), and the present value of any cash flows from the asset or convenience yield over the holding period as PV_0(benefit).

Spot price and expected future price

- Consider first a case where there are storage costs of holding the asset, but no benefits. For an asset with no costs or benefits of holding the asset, we established the no-arbitrage forward price as $S_0(1 + R_f)^T$, the cost of buying and holding the underlying asset until time T.
- When there are storage costs to hold the asset until time T, an arbitrageur must both buy the asset and pay the present value of storage costs at t = 0. This increases the no-arbitrage price of a 1-year forward to $[S_0 + PV_0(\text{cost})](1 + R_f)^T$. Here we see that costs of holding an asset increase its no-arbitrage forward price.

Spot price and expected future price

- Next consider a case where holding the asset has benefits, but no costs. Returning to our example of a 1-year forward on a share of ABC stock trading at 30, now consider the costs of buying and holding an ABC share that pays a dividend of $1 during the life of the forward contract. In this case, an arbitrageur can now borrow the present value of the dividend (discounted at Rf), and repay that loan when the dividend is received. The cost to buy and hold ABC stock with an annual dividend of $1 is $[30 – PV_0(1)](1.05) = 30(1.05) – 1$. This illustrates that benefits of holding an asset decrease its no-arbitrage forward price.
- The no-arbitrage price of a forward on an asset that has both costs and benefits of holding the asset is simply $[S_0 + PV_0(\text{costs}) – PV_0(\text{benefit})](1 + R_f)^T$.

Spot price and expected future price

- We can also describe these relationships when costs and benefits are expressed as continuously compounded rates of return.
- Recall that given a stated annual rate of r with continuous compounding, the effective annual return is er – 1, and the relationships between present and future values of S for a 1-year period are $FV = Se^{r}$ and $PV = Se^{-r}$.
- For a period of T years, $FV = Se^{rT}$ and $PV = Se^{-rT}$. With continuous compounding the following relationships hold:

Spot price and expected future price

- With no costs or benefits of holding the underlying asset, the no-arbitrage price of a forward that settles at time T is S_0e^{rT}, where r is the stated annual risk-free rate with continuous compounding.
- With storage costs at a continuously compounded annual rate of c, the no-arbitrage forward price until time T is $S_0e^{(r+c)T}$.
- With benefits, such as a dividend yield, expressed at a continuously compounded annual rate of b, the no-arbitrage forward price is until time T is $S_0e^{(r+c-b)T}$.

EXAMPLE: No-arbitrage price with continuous compounding

- Consider a stock index trading at 1,550 with a dividend yield of 1.3% (continuously compounded rate) when the risk-free rate is 3% (continuously compounded rate). Calculate the no-arbitrage 6-month forward price of the stock index.

EXAMPLE: No-arbitrage price with continuous compounding

- Consider a stock index trading at 1,550 with a dividend yield of 1.3% (continuously compounded rate) when the risk-free rate is 3% (continuously compounded rate). Calculate the no-arbitrage 6-month forward price of the stock index.
- The no-arbitrage price of a long 6-month forward is
  - $1{,}550 \times e^{(0.03 - 0.013)(0.5)} = 1{,}563.23$.

Net cost of carry

- The **net cost of carry** (or simply cost of carry or carry) is the benefits of holding the asset minus the costs of holding the asset (including the opportunity cost of funds, R_f).
- When the benefits (cash flow yield or convenience yield) exceed the costs (including the opportunity cost of funds) of holding the asset, the forward price will be less than the spot price.

FORWARD CONTRACT VALUATION

- Consider a forward contract that is initially priced at its no-arbitrage value of $F_0(T) = S_0(1 + R_f)^T$. At initiation, the value of such a forward is:
  - $V_0(T) = S_0 - F_0(T)\,(1 + R_f)^{-T} = 0.$
- At any time during its life, the value of the forward contract to the buyer will be:
  - $V_t(T) = S_t - F_0(T)\,(1 + R_f)^{-(T-t)}$
- This is simply the current spot price of the asset minus the present value of the forward contract price.

FORWARD CONTRACT VALUATION

- This value can be realized by selling the asset short at S_t and investing $F_0(T)\,(1 + R_f)^{-(T-t)}$ in a pure discount bond at R_f. These transactions end any exposure to the forward; at settlement, the proceeds of the bond will cover the cost of the asset at the forward price, and the asset can be delivered to cover the short position.
- At expiration, time T, the value of a forward to the buyer is
    - $S_T - F_0(T)(1 + R_f)^{-(T-T)} = S_T - F_0(T)$.
- The long buys an asset valued at S_T for the forward contract price of $F_0(T)$, gaining if $S_T > F_0(T)$, losing if $S_T < F_0(T)$. If the forward buyer has a gain, the forward seller has an equal loss, and vice versa.

FORWARD CONTRACT VALUATION

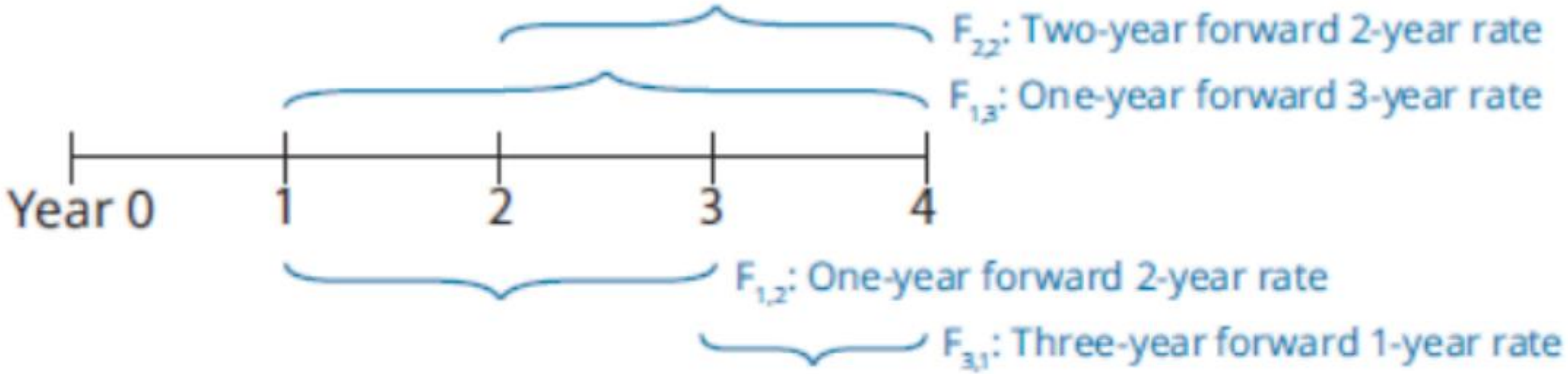
- In the more general case, when there are costs and benefits of holding the underlying asset, the value of a forward to the buyer at time = $t < T$ is:
  - $V_t(T) = [S_t + PV_t\ (\text{costs}) - PV_t(\text{benefit})] - F_0(T)\ (1 + R_f)^{-(T-t)}$

The forward rates

- Forward rates are yields for future periods. The rate of interest on a 1-year loan to be made two years from today is a forward rate.
- The notation for forward rates must identify both the length of the loan period and how far in the future the money will be loaned (or borrowed). 1y1y or $F_{1,1}$ is the rate for a 1-year loan one year from now; 2y1y or $_{F2,1}$ is the rate for a 1-year loan to be made two years from now; the 2- year forward rate three years from today is 3y2y or $F_{3,2}$; and so on.

The forward rates

- For money market rates the notation is similar, with 3m6m denoting a 6-month rate three months in the future.
- Recall that spot rates are zero-coupon rates. We will denote the YTM (with annual compounding) on a zero-coupon bond maturing in n years as Z_n.



The implied forward rates

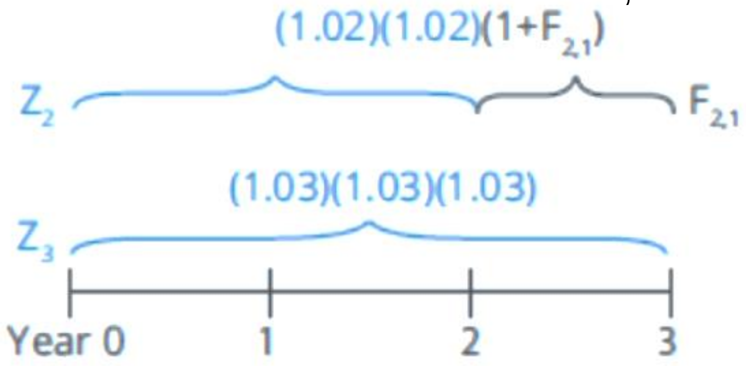
- An **implied forward rate** is the forward rate for which the following two strategies have the same yield over the total period:
- Investing from $t = 0$ to the forward date, and rolling over the proceeds for the period of the forward.
- Investing from $t = 0$ until the end of the forward period.
- As an example, lending for two years at Z_2 would produce the same ending value as lending for one year at Z_1 and, at $t = 1$, lending the proceeds of that loan for one year at $F_{1,1}$. That is, $(1 + Z_2)^2 = (1 + Z_1)(1 + F_{1,1})$. When this condition holds, $F_{1,1}$ is the implied (no-arbitrage) forward rate.

EXAMPLE: Implied forward rate

- Consider two zero-coupon bonds, one that matures in two years and one that matures in three years, when $Z_2 = 2\%$ and $Z_3 = 3\%$. Calculate the implied 1-year forward rate two years from now, $F_{2,1}$.

EXAMPLE: Implied forward rate

- Lending for three years at Z_3 should be equivalent to lending for two years at Z_2 and then for the third year at $F_{2,1}$.



- Lending \$100 for two years at Z_2 (2%) results in a payment of \$100$(1.02)^2$ = \$104.04 at t = 2, while lending \$100 for three years at Z_3 (3%) results in a payment of \$100$(1.03)^3$ = \$109.27. The forward interest rate $F_{2,1}$ must be 109.27/104.04 – 1 = 5.03%, the implied forward rate from t = 2 to t = 3.

FUTURES VALUATION

- While the price of a forward contract is constant over its life when no mark-to-market gains or losses are paid, its value will fluctuate with changes in the value of the underlying. The payment at settlement of the forward reflects the difference between the (unchanged) forward price and the spot price of the underlying.
- The price and value of a futures contract both change when daily mark-to-market gains and losses are settled. Consider a futures contract on 100 ounces of gold at $1,870 purchased on Day 0. The following illustrates the changes in contract price and value with daily mark-to-market payments.

FUTURES VALUATION

Day 0	Price = Settlement Price of 1,870	MTM Value = 0
Day 1	Settlement Price = 1,875	MTM value = \$500
	\$500 addition to margin account	
	New futures price = 1,875	MTM value = 0
Day 2	Settlement price = 1,855	MTM value = –\$2,000
	\$2,000 deduction from margin account	
	New futures price = 1,855	MTM value = 0

FUTURES VALUATION

- Interest rate futures contracts are available on many market reference rates. We may view these as exchange-traded equivalents to forward rate agreements. One key difference is that interest rate futures are quoted on a price basis. For a market reference rate (MRR) from time A to time B, an interest rate futures price is stated as follows:
    - futures price = $100 – (100 \times MRR_{A,\ B–A})$

FUTURES VALUATION

- For example, if the futures price for a 6-month rate six months from now is 97, then $MRR_{6m,\ 6m} = 3\%$.
- Like other futures contracts, interest rate futures are subject to daily mark-to-market. The **basis point value (BPV)** of an interest rate futures contract is defined as:
    - BPV = notional principal × period × 0.01%
- If the contract in our example is based on notional principal of €1,000,000, its BPV is €1,000,000 × (0.0001 / 2) = €50. This means a one basis point change in the MRR will change the futures contract value by €50.

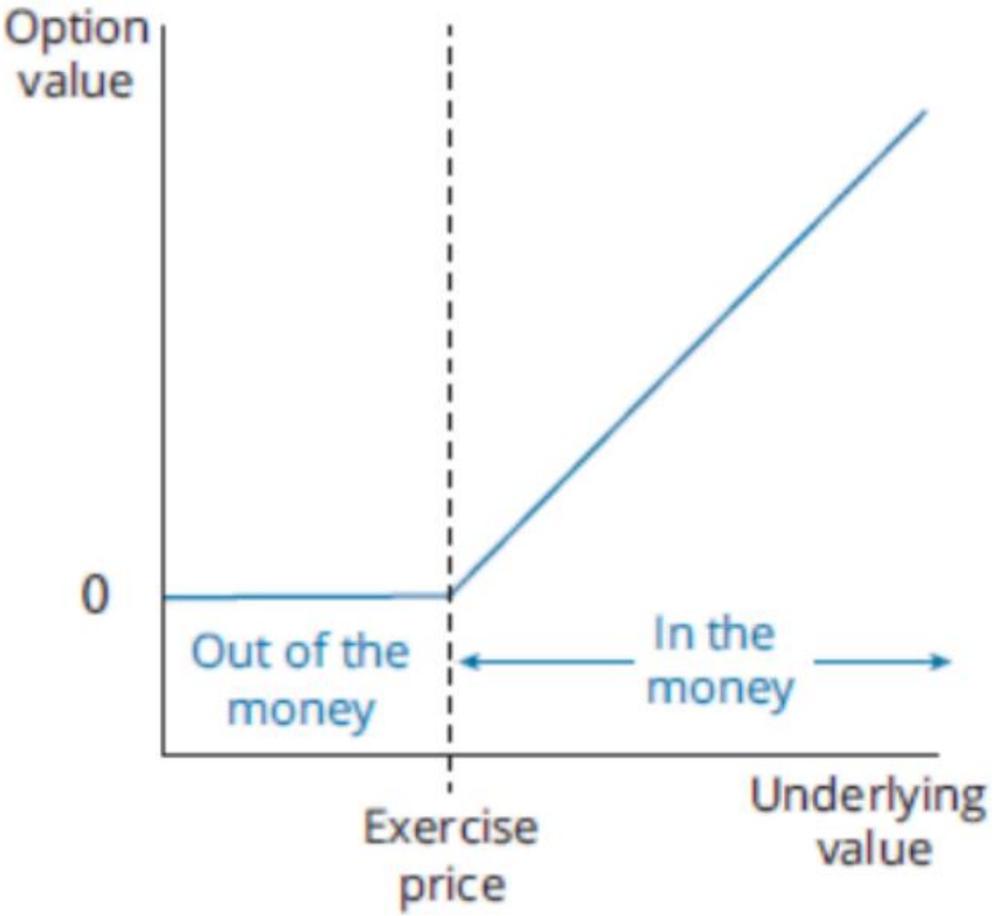
Moneyness

- **Moneyness** refers to whether an option is in the money or out of the money.
- If immediate exercise of the option would generate a positive payoff, it is **in the money**.
- If immediate exercise would result in a loss (negative payoff), it is **out of the money**.
- When the current asset price equals the exercise price, exercise will generate neither a gain nor loss, and the option is **at the money**.

Moneyness: call option

- The following describes the conditions for a call option to be in, out of, or at the money. S is the price of the underlying asset and X is the exercise price of the option.
- In-the-money call options. If $S - X > 0$, a call option is in the money. $S - X$ is the amount of the payoff a call holder would receive from immediate exercise, buying a share for X and selling it in the market for a greater price S.
- Out-of-the-money call options. If $S - X < 0$, a call option is out of the money.
- At-the-money call options. If $S = X$, a call option is said to be at the money.

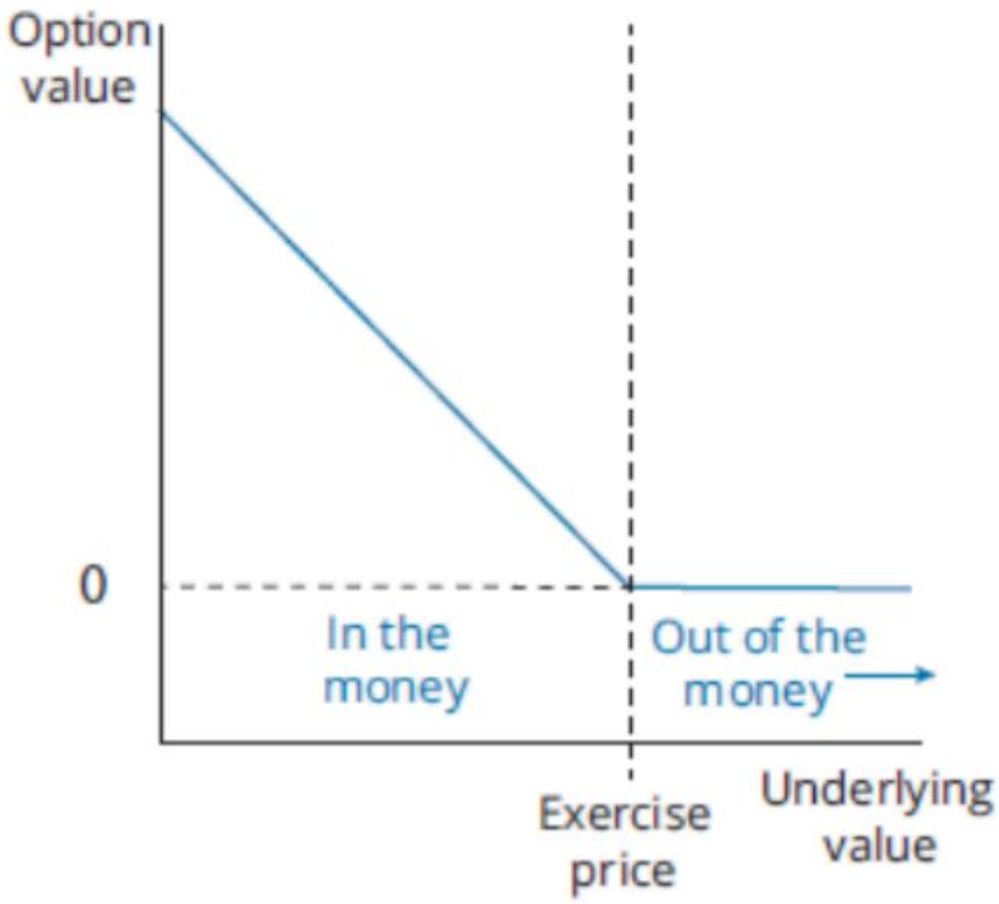
Moneyness: call option



Moneyness: put option

- The following describes the conditions for a put option to be in, out of, or at the money.
- In-the-money put options. If $X - S > 0$, a put option is in the money. $X - S$ is the amount of the payoff from immediate exercise, buying a share for S and exercising the put to receive X for the share.
- Out-of-the-money put options. When the stock's price is greater than the exercise price, a put option is said to be out of the money. If $X - S < 0$, a put option is out of the money.
- At-the-money put options. If $S = X$, a put option is said to be at the money.

Moneyness: put option



EXAMPLE: Moneyness

- Consider a July 40 call and a July 40 put, both on a stock that is currently selling for $37/share. Calculate how much these options are in or out of the money.

EXAMPLE: Moneyness

- Consider a July 40 call and a July 40 put, both on a stock that is currently selling for $37/share. Calculate how much these options are in or out of the money.
- The call is $3 out of the money because $S - X = -\$3.00$. The put is $3 in the money because $X - S = \$3.00$.

Exercise value

- We define the **exercise value** (or **intrinsic value**) of an option as the maximum of zero and the amount that the option is in the money. That is, the exercise value is the amount an option is in the money, if it is in the money, or zero if the option is at or out of the money. The exercise value is the value of the option if exercised immediately.
- Prior to expiration, an option has **time value** in addition to its exercise value. The time value of an option is the amount by which the **option premium** (price) exceeds the exercise value and is sometimes called the speculative value of the option. This relationship can be written as:
  - option premium = exercise value + time value

Option Valuation: no-arbitrage pricing

- The following is some terminology that we will use to determine the minimum and maximum values for European options:

S_t = the price of the underlying stock at time t
X = the exercise price of the option
T–t = the time to expiration
c_t = the price of a European call at any time t prior to expiration at time = T
p_t = the price of a European put at any time t prior to expiration at time = T
Rf = the risk-free rate

Upper Bound for Call Options

- The maximum value of a European call option at any time t is the time-t share price of the **underlying stock**.
- This makes sense because no one would pay more for the right to buy an asset than the asset's market value. It would be cheaper simply to buy the underlying asset.
- At time $t = 0$, the upper boundary condition for European call options is $c_0 \leqslant S_0$, and at any time t during a European call option's life, the upper boundary condition is $c_t \leqslant S_t$.

Upper Bound for Put Options

- Logically the value of a put option cannot be more than its **exercise price**. This would be its exercise value if the underlying stock price goes to zero.
- However, because European puts cannot be exercised prior to expiration, their maximum value is the present value of the exercise price discounted at the risk-free rate.
- Even if the stock price goes to zero and is expected to stay at zero, the put buyer will not receive the intrinsic value, X, until the expiration date.

Upper Bound for Put Options

- At time t = 0, the upper boundary condition can be expressed for European put options as:
  - $p_0 \leqslant X\,(1 + R_f)^{-T}$

- At any time t during a European put option's life, the upper boundary condition is:
  - $p_t \leqslant X\,(1 + R_f)^{-(T-t)}$

Lower Bounds for Options

- Theoretically, no option will sell for less than its intrinsic value and no option can take on a negative value. For European options, however, the lower bound is not so obvious because these options are not exercisable immediately.
- To determine the lower bounds for European options, we can examine the value of a portfolio in which the option is combined with a long or short position in the stock and a pure discount bond.

Lower Bounds for Options

- For a European call option, construct the following portfolio:
  - A long at-the-money European call option with exercise price X, expiring at time T.
  - A long discount bond priced to yield the risk-free rate that pays X at option expiration.
  - A short position in one share of the underlying stock priced at
    - $S_0 = X$.
- The current value of this portfolio is
  - $c_0 – S_0 + X(1 + R_f)^{–T}$.

Lower Bounds for Options

- At expiration time T, this portfolio will pay $c_T - S_T + X$. That is, we will collect $c_T = \max[0, S_T - X]$ on the call option, pay ST to cover our short stock position, and collect X from the maturing bond.
- If $S_T \geqslant X$, the call is in-the-money, and the portfolio will have a zero payoff because the call pays $S_T - X$, the bond pays +X, and we pay $-S_T$ to cover our short position. That is, the time $t = T$ payoff is: $S_T - X + X - S_T = 0$.
- If $S_T < X$, the call is out-of-the-money, and the portfolio has a positive payoff equal to $X - S_T$ because the call value, c_T , is zero, we collect X on the bond, and pay $-S_T$ to cover the short position. So, the time $t = T$ payoff is: $0 + X - S_T = X - S_T$.

Lower Bounds for Options

- To prevent arbitrage, any portfolio that has no possibility of a negative payoff cannot have a negative value. Thus, we can state the value of the portfolio at time $t = 0$ as:
  - $c_0 - S_0 + X(1 + R_f)^{-T} \geqslant 0$
- which allows us to conclude that:
  - $c_0 \geqslant S_0 - X(1 + R_f)^{-T}$
- Combining this result with the earlier minimum on the call value of zero, we can write:
  - $c_0 \geqslant \text{Max}[0, S_0 - X(1 + R_f)^{-T}]$
- Note that $X(1 + R_f)^{-T}$ is the present value of a pure discount bond with a face value of X.

Lower Bounds for Options

- For a European put option we can derive the minimum value by forming the following portfolio at time t = 0:
- A long at-the-money put option with exercise price X, expiring at T.
- A short position on a risk-free bond priced at $X(1 + R_f)^{-T}$, equivalent to borrowing $X(1 + R_f)^{-T}$.
- A long position in a share of the underlying stock priced at S0.
- At expiration time T, this portfolio will pay $p_T + S_T - X$. That is, we will collect $p_T = \text{Max}[0, X - S_T]$ on the put option, receive ST from the stock, and pay X on the bond (loan).
- If ST > X, the payoff will equal: $p_T + S_T - X = S_T - X$.
- If ST ⩽ X, the payoff will be zero.

Lower Bounds for Options

- Again, a no-arbitrage argument can be made that the portfolio value must be zero or greater, because there are no negative payoffs to the portfolio.
- At time t = 0, this condition can be written as:
    - $p_0 + S_0 - X(1 + R_f)^{-T} \geqslant 0$
- and rearranged to state the minimum value for a European put option at time t = 0 as:
    - $p_0 \geqslant X(1 + R_f)^{-T} - S_0$
- We have now established the minimum bound on the price of a European put option as:
    - $p_0 \geqslant \text{Max}[0, X(1 + R_f)^{-T} - S_0]$

Lower Bounds for Options

Option	Minimum Value	Maximum Value
European call	$c_t \geq \text{Max}[0, S_t - X(1 + Rf)^{-(T-t)}]$	S_t
European put	$p_t \geq \text{Max}[0, X / X(1 + Rf)^{-(T-t)} - S_t]^{(T-t)}$	$X(1 + Rf)^{-(T-t)}$

Factors that determine the value of an option

- 1. **Price of the underlying asset**. For call options, the higher the price of the underlying, the greater its exercise value and the higher the value of the option. Conversely, the lower the price of the underlying, the less its exercise value and the lower the value of the call option. In general, call option values increase when the value of the underlying asset increases.
- For put options this relationship is reversed. An increase in the price of the underlying reduces the value of a put option.

Factors that determine the value of an option

- 2. **The exercise price.** A higher exercise price decreases the values of call options and a lower exercise price increases the values of call options.
- A higher exercise price increases the values of put options and a lower exercise price decreases the values of put options.

Factors that determine the value of an option

- 3. **The risk-free rate of interest**. An increase in the risk-free rate will increase call option values, and a decrease in the risk-free rate will decrease call option values.
- An increase in the risk-free rate will decrease put option values, and a decrease in the risk-free rate will increase put option values.

Factors that determine the value of an option

- 4. **Volatility of the underlying**. Volatility is what makes options valuable. If there were no volatility in the price of the underlying asset (its price remained constant), options would always be equal to their exercise values and time or speculative value would be zero.
- An increase in the volatility of the price of the underlying asset increases the values of both put and call options and a decrease in volatility of the price of the underlying decreases both put values and call values.

Factors that determine the value of an option

- 5. **Time to expiration**. Because volatility is expressed per unit of time, longer time to expiration effectively increases expected volatility and increases the value of a call option. Less time to expiration decreases the time value of a call option so that at expiration its value is simply its exercise value.
- For most put options, longer time to expiration will increase option values for the same reasons. For some European put options, however, extending the time to expiration can decrease the value of the put. In general, the deeper a put option is in the money, the higher the risk-free rate, and the longer the current time to expiration, the more likely that extending the option's time to expiration will decrease its value.

Factors that determine the value of an option

- 6. **Costs and benefits of holding the asset**. If there are benefits of holding the underlying asset (dividend or interest payments on securities or a convenience yield on commodities), call values are decreased and put values are increased. The reason for this is most easily understood by considering cash benefits. When a stock pays a dividend, or a bond pays interest, this reduces the value of the asset. Decreases in the value of the underlying asset decrease call values and increase put values.
- Positive storage costs make it more costly to hold an asset. We can think of this as making a call option more valuable because call holders can have long exposure to the asset without paying the costs of actually owning the asset. Puts, on the other hand, are less valuable when storage costs are higher.

Factors that determine the value of an option

Increase in:	Effect on Call Option Values	Effect on Put Option Values
Price of underlying asset	Increase	Decrease
Exercise price	Decrease	Increase
Risk-free rate	Increase	Decrease
Volatility of underlying asset	Increase	Increase
Time to expiration	Increase	Increase, except some European puts
Costs of holding underlying asset	Increase	Decrease
Benefits of holding underlying asset	Decrease	Increase

PUT–CALL PARITY

- Our derivation of **put-call parity** for European options is based on the payoffs of two portfolio combinations: a fiduciary call and a protective put.
- A **fiduciary** call is a combination of a call with exercise price X and a pure-discount, riskless bond that pays X at maturity (option expiration). The payoff for a fiduciary call at expiration is X when the call is out of the money, and X + (S − X) = S when the call is in the money.
- A **protective** put is a share of stock together with a put option on the stock. The expiration date payoff for a protective put is (X – S) + S = X when the put is in the money, and S when the put is out of the money

PUT–CALL PARITY

- If at expiration S is greater than or equal to X:
  - The protective put pays S on the stock while the put expires worthless, so the payoff is S.
  - The fiduciary call pays X on the bond portion while the call pays (S − X), so the payoff is X + (S − X) = S.
- If at expiration X is greater than S:
  - The protective put pays S on the stock while the put pays (X − S), so the payoff is S + (X − S) = X.
  - The fiduciary call pays X on the bond portion while the call expires worthless, so the payoff is X.

PUT–CALL PARITY

- In either case, the payoff on a protective put is the same as the payoff on a fiduciary call. Our no-arbitrage condition holds that portfolios with identical payoffs regardless of future conditions must sell for the same price to prevent arbitrage.
- We can express the put–call parity relationship as:

$$c + X(1 + Rf)^{-T} = S + p$$

PUT–CALL PARITY

- Equivalencies for each of the individual securities in the put–call parity relationship can be expressed as:

$$S = c - p + X(1 + Rf)^{-T}$$

$$p = c - S + X(1 + Rf)^{-T}$$

$$c = S + p - X(1 + Rf)^{-T}$$

$$X(1 + Rf)^{-T} = S + p - c$$

- Note that the options must be European style and the puts and calls must have the same exercise price and time to expiration for these relations to hold.

PUT–CALL PARITY

- The single securities on the left-hand side of the equations all have exactly the same payoffs as the portfolios on the right-hand side. The portfolios on the right-hand side are the synthetic equivalents of the securities on the left. For example, to synthetically produce the payoff for a long position in a share of stock, use the following relationship:

$$S = c - p + X(1 + Rf)^{-T}$$

- This means that the payoff on a long stock can be synthetically created with a long call, a short put, and a long position in a risk-free discount bond.

EXAMPLE: Call option valuation using put–call parity

- Suppose that the current stock price is \$52 and the risk-free rate is 5%. You have found a quote for a 3-month put option with an exercise price of \$50. The put price is \$1.50, but due to light trading in the call options, there was not a listed quote for the 3-month, \$50 call. Estimate the price of the 3-month call option.

EXAMPLE: Call option valuation using put–call parity

- Rearranging put–call parity, we find that the call price is:

$$\text{call} = \text{put} + \text{stock} - \text{present value (X)}$$

$$\text{call} = \$1.50 + \$52 - \frac{\$50}{1.05^{0.25}} = \$4.11$$

- This means that if a 3-month, $50 call is available, it should be priced at (within transaction costs of) $4.11 per share.

Explain put–call forward parity

- **Put–call-forward parity** is derived with a forward contract rather than the underlying asset itself.
- Consider a forward contract on an asset at time T with a contract price of $F_0(T)$. At contract initiation the forward contract has zero value.
- At time T, when the forward contract settles, the long must purchase the asset for $F_0(T)$. The purchase (at time = 0) of a pure discount bond that will pay $F_0(T)$ at maturity (time = T) will cost $F_0(T)(1 + R_f)^{-T}$.

Explain put–call forward parity

- By purchasing such a pure discount bond and simultaneously taking a long position in the forward contract, an investor has created a synthetic asset.
- At time = T the proceeds of the bond are just sufficient to purchase the asset as required by the long forward position.
- Because there is no cost to enter into the forward contract, the total cost of the synthetic asset is the present value of the forward price, $F_0(T)(1 + R_f)^{-T}$.

Explain put–call forward parity

- The put–call-forward parity relationship is derived by substituting the synthetic asset for the underlying asset in the put–call parity relationship. Substituting $F_0(T)(1 + R_f)^{-T}$ for the asset price S_0 in $S + p = c + X(1 + R_f)^{-T}$ gives us:

$$F_0(T)(1 + Rf)^{-T} + p_0 = c_0 + X(1 + Rf)^{-T}$$

- which is put–call-forward parity at time 0, the initiation of the forward contract, based on the principle of no arbitrage. By rearranging the terms, put–call forward parity can also be expressed as:

$$p_0 - c_0 = [X - F_0(T)](1 + Rf)^{-T}$$

Application of Options Theory to Corporate Finance

- We can view the claims of a firm's equity holders and debt holders as a call option and a put option, respectively.
- Consider a firm that has a value of V_t at time = t and has issued debt in the form of a zero-coupon bond that will pay D at time = T.
- At time = T, if $V_T > D$ the equity holders receive $V_T - D$ and if $V_T < D$, the firm is insolvent and equity holders receive nothing. The payoff to the equity holders at time = T can be written as $Max(0, V_T - D)$ which is equivalent to a call option with the firm value as the underlying and an exercise price of D.

Application of Options Theory to Corporate Finance

- At time = T, if $V_T > D$ the debt holders receive D and if $V_T < D$, the firm is insolvent and debt holders receive V_T. The payoff to the debt holders at time = T can be written as Max(V_T, D).
- This is equivalent to a portfolio that is long a risk-free bond that pays D at t = T, and short (has sold) a put option on the value of the firm, V_T, with an exercise price of D.
- If $V_T > D$ the portfolio pays D and the put expires worthless, and if $V_T < D$ the portfolio pays $D – (D – V_T) = V_T$ and the debtholders effectively pay $D – V_T$ on the short put position.

BINOMIAL MODEL FOR OPTION VALUES

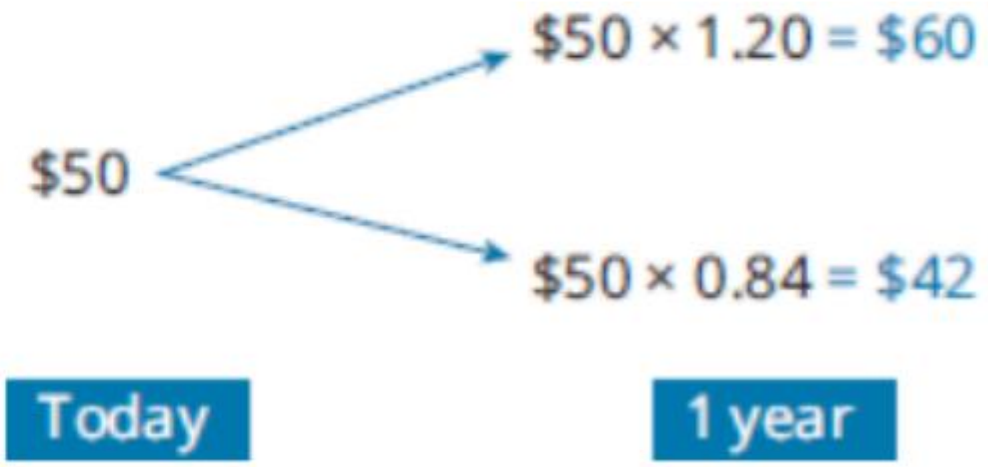
- Recall from Quantitative Methods that a binomial model is based on the idea that, over the next period, some value will change to one of two possible values (binomial). To construct a one-period binomial model for pricing an option, we need:
  - A value for the underlying at the beginning of the period.
  - An exercise price for the option. The exercise price can be different from the value of the underlying. We assume the option expires one period from now.
  - Returns that will result from an up-move and a down-move in the value of the underlying over one period.
  - The risk-free rate over the period.

BINOMIAL MODEL FOR OPTION VALUES

- For now we do not need to consider the probabilities of an up-move or a down-move. Later in this reading we will examine one-period binomial models with risk-neutral probabilities.
- As an example, we can model a call option with an exercise price of \$55 on a stock that is currently valued (S_0) at \$50. Let us assume that in one period the stock's value will either increase (S_1^u) to \$60 or decrease (S_1^d) to \$42.
- We state the return from an up-move (R^u) as \$60/\$50 = 1.20, and the return from a down-move (R^d) as \$42/\$50 = 0.84.

BINOMIAL MODEL FOR OPTION VALUES

- The call option will be in the money after an up-move or out of the money after a down-move.
- Its value at expiration after an up-move, c_1^u, is Max(0, \$60 – \$55) = \$5. Its value after a down-move, c_1^d, is Max(0, \$42 – \$55) = 0.



BINOMIAL MODEL FOR OPTION VALUES

- Now we can use no-arbitrage pricing to determine the initial value of the call option (c_0).
- We do this by creating a portfolio of the option and the underlying stock, such that the portfolio will have the same value following either an up-move (V_1^u) or a down-move (V_1^d) in the stock.
- For our example, we would write the call option and buy a number of shares of the stock that we will denote as h. We must solve for the h that results in $V_1^u = V_1^d$.

BINOMIAL MODEL FOR OPTION VALUES

- The initial value of our portfolio, V_0, is $hS_0 – c_0$ (remember we are short the call option).
- The portfolio value after an up-move, V_1^u, is $hS_1^u-c_1^u$.
- The portfolio value after a down-move, V_1^d, is $hS_1^d-c_1^d$.

BINOMIAL MODEL FOR OPTION VALUES

- In our example, V_1^u=h(\$60)-%5, and V_1^d = h(\$42)-0. Setting V_1^u = V_1^d and solving for h, we get:
  - h(\$60) – \$5 = h(\$42)
  - h(\$60) – h(\$42) = \$5
  - h = \$5 / (\$60 – \$42) = 0.278
- This result, the number of shares of the underlying we would buy for each call option we would write, is known as the **hedge ratio** for this option.

BINOMIAL MODEL FOR OPTION VALUES

- With $V_1^u = V_1^d$, the value of the portfolio after one period is known with certainty. This means we can say that either V_1^u or V_1^d must equal V0 compounded at the risk-free rate for one period. In this example, V_1^d = 0.278($42) = $11.68, or V_1^u = 0.278($60) – $5 = $11.68. Let us assume the risk-free rate over one period is 3%. Then V_0= $11.68 / 1.03 = $11.34.
- Now we can solve for the value of the call option, c_0. Recall that $V_0 = hS_0 - c_0$, so $c_0 = hS_0 - V_0$. Here, c_0 = 0.278($50) – $11.34 = $2.56.

Risk neutrality

- Another approach to constructing a one-period binomial model involves risk-neutral probabilities of an up-move or a down-move. Consider a share of stock currently priced at $30.
- The size of the possible price changes, and the probabilities of these changes occurring, are as follows:

R^u = up-move factor = 1.15

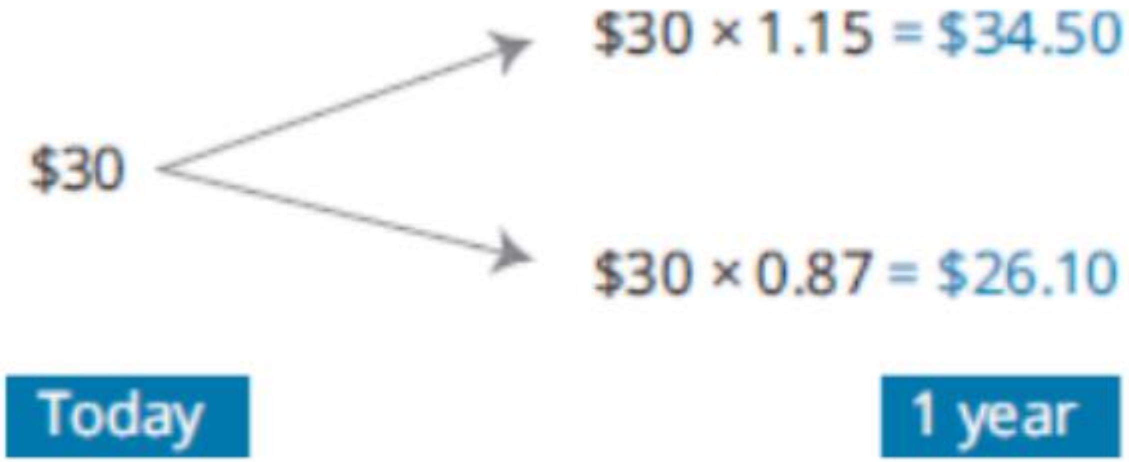
R^d = down-move factor $= \frac{1}{R^u} = \frac{1}{1.15} = 0.87$

π_U = risk-neutral probability of an up-move = 0.715

π_D = risk-neutral probability of a down-move $= 1 - \pi_U = 1 - 0.715 = 0.285$

Risk neutrality

- Note that the down-move factor is the reciprocal of the up-move factor, and the probability of an up-move is one minus the probability of a down-move. The beginning stock value of $30 is to the left, and to the right are the two possible end-of-period stock values, 30 × 1.15 = $34.50 and 30 × 0.87 = $26.10.



Risk neutrality

- The risk-neutral probabilities of an up-move and a down-move are calculated from the sizes of the moves and the risk-free rate:

$$\pi_U = \text{risk-neutral probability of an up-move} = \frac{1 + Rf - R^d}{R^u - R^d}$$

$$\pi_D = \text{risk-neutral probability of a down-move} = 1 - \pi_U$$

where:

Rf = risk-free rate

R^u = size of an up-move

R^d = size of a down-move

Risk neutrality

- We can calculate the value of an option on the stock by:
- Calculating the payoffs of the option at expiration for the up-move and down-move prices.
- Calculating the expected payoff of the option in one year as the (risk-neutral) probability-weighted average of the up-move and down-move payoffs.
- Calculating the PV of the expected payoff by discounting at the risk-free rate.

EXAMPLE: Calculating call option value with risk-neutral probabilities

- Use the binomial tree in Figure 57.2 to calculate the value today of a 1-year call option on a stock with an exercise price of $30. Assume the risk-free rate is 7%, the current value of the stock is $30, and the up-move factor is 1.15.

EXAMPLE: Calculating call option value with risk-neutral probabilities

- First, we need to calculate the down-move factor and risk-neutral the probabilities of the up- and down-moves:

$$R^d = \text{size of down-move} = \frac{1}{R^u} = \frac{1}{1.15} = 0.87$$

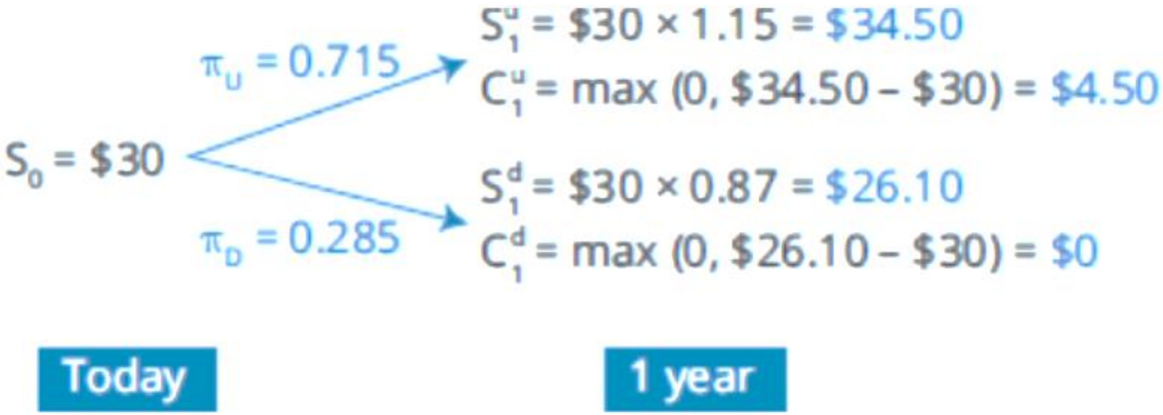
$$\pi_U = \text{risk-neutral probability of an up-move} = \frac{1 + 0.07 - 0.87}{1.15 - 0.87} = 0.715$$

$$\pi_D = \text{risk-neutral probability of a down-move} = 1 - 0.715 = 0.285$$

- Next, determine the payoffs on the option in each state. If the stock moves up to \$34.50, a call option with an exercise price of \$30 will pay \$4.50. If the stock moves down to \$26.10, the call option will expire worthless. Let the stock values for the up-move and down-move be S_1^u and S_1^d and for the call values, c_1^u and c_1^d.

EXAMPLE: Calculating call option value with risk-neutral probabilities

- One-Period Call Option With X = $30
- The expected value of the option in one period is:



  - E(call option value in 1 year) = ($4.50 × 0.715) + ($0 × 0.285) = $3.22
- The value of the option today, discounted at the risk-free rate of 7%, is:

$$C_0 = \frac{\$3.22}{1.07} = \$3.01$$

EXAMPLE: Valuing a one-period put option on a stock

- Use the information in the previous example to calculate the value of a put option on the stock with an exercise price of $30.

EXAMPLE: Valuing a one-period put option on a stock

- If the stock moves up to \$34.50, a put option with an exercise price of \$30 will expire worthless. If the stock moves down to \$26.10, the put option will be worth \$3.90.
- The risk-neutral probabilities are 0.715 and 0.285 for an up- and down-move, respectively. The expected value of the put option in one period is:
  - E(put option value in 1 year) = (\$0 × 0.715) + (\$3.90 × 0.285) = \$1.11
- The value of the option today, discounted at the risk-free rate of 7%, is:

$$p_0 = \frac{\$1.11}{1.07} = \$1.04$$