单纯形法的矩阵描述

$$\begin{aligned} \max\ & c^T x \\ s.t\ & Ax \le b \\ & x \ge 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max\ & c^T x + c_s^T x_s \\ s.t\ & Ax + I x_s = b \\ & x, x_s \ge 0 \end{aligned}$$

系数矩阵 $(A, I) \to (B, N)$　　B：基变量的系数矩阵

N：非基变量的系数矩阵

$$\Rightarrow \begin{aligned} \max\ & c_B^T x_B + c_N^T x_N \\ s.t\ & B x_B + N x_N = b \\ & x_B, x_N \ge 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max\ & c_B^T x_B + c_N^T x_N \\ s.t\ & I x_B + B^{-1} N x_N = B^{-1} b \\ & x_B, x_N \ge 0 \end{aligned}$$

$$\Rightarrow \begin{aligned} \max\ & c_B^T x_B + c_N^T x_N \\ s.t\ & x_B = B^{-1} b - B^{-1} N x_N \\ & x_B, x_N \ge 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max\ & c_B^T (B^{-1} b - B^{-1} N x_N) + c_N^T x_N \\ s.t\ & x_B = B^{-1} b - B^{-1} N x_N \\ & x_B, x_N \ge 0 \end{aligned}$$

$$\Rightarrow \begin{aligned} \max\ & c_B B^{-1} b + (c_N - c_B B^{-1} N) x_N \\ s.t\ & x_B = B^{-1} b - B^{-1} N x_N \\ & x_B, x_N \ge 0 \end{aligned}$$

可行解：$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} B^{-1} b \\ 0 \end{pmatrix}$

目标值：$z = c_B B^{-1} b + (c_N - c_B B^{-1} N) x_N$

↳ 非基变量的系数决定目标值是否能进一步增大

对偶(dual)：同一个事物，从不同角度或立场观察，有两种对立的表述

E.g. 1: 周长一定，面积最大的矩形是正方形。
面积一定，周长最小的矩形是正方形。

最优解

E.g. 2: 某工厂甲要生产两种产品。

	x_1	x_2	资源限制
设备工时	1	2	8
材料A	4	0	16
材料B	0	4	12
利润	2	3	

$\Rightarrow$

$$\begin{aligned} \max\ & 2x_1 + 3x_2 \\ \text{s.t.}\ & x_1 + 2x_2 \le 8 \\ & 4x_1 \le 16 \\ & 4x_2 \le 12 \\ & x_{1-2} \ge 0 \end{aligned}$$

工厂甲将现有设备和资源出租给工厂乙，乙如何最小化租赁成本？

	x_1	x_2	资源限制	
设备工时	1	2	8	y_1
材料A	4	0	16	y_2
材料B	0	4	12	y_3

$$\begin{aligned} \min\ & 8y_1 + 16y_2 + 12y_3 \\ \text{s.t.}\ & y_1 + 4y_2 \ge 2 \\ & 2y_1 + 4y_3 \ge 3 \\ & y_{1-3} \ge 0 \end{aligned}$$

原问题 (primal problem)

$$\begin{aligned} \max\ & 2x_1 + 3x_2 \\ \text{s.t.}\ & x_1 + 2x_2 \le 8 \\ & 4x_1 \le 16 \\ & 4x_2 \le 12 \\ & x_{1-2} \ge 0 \end{aligned}$$

$c = (2, 3)$

$A = \begin{pmatrix} 1 & 2 \\ 4 & 0 \\ 0 & 4 \end{pmatrix}$

$b = (8, 16, 12)$

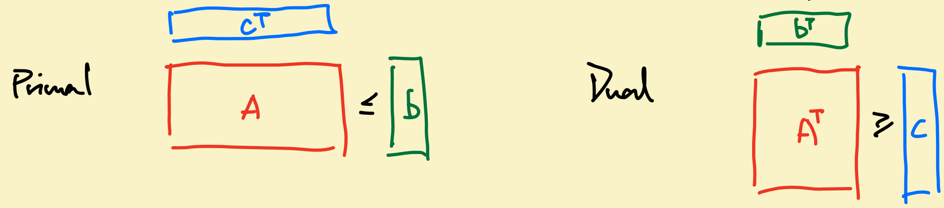
对偶问题 (Dual problem)

$$\begin{aligned} \min\ & 8y_1 + 16y_2 + 12y_3 \\ \text{s.t.}\ & y_1 + 4y_2 \ge 2 \\ & 2y_1 + 4y_3 \ge 3 \\ & y_{1-3} \ge 0 \end{aligned}$$

$A^T = \begin{pmatrix} 1 & 4 & 0 \\ 2 & 0 & 4 \end{pmatrix}$

$$\begin{aligned} \max\ & c^T x \\ \text{s.t.}\ & Ax \le b \\ & x \ge 0 \end{aligned}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t.}\ & A^T y \ge c \\ & y \ge 0 \end{aligned}$$



E.x. $\max \; 10x_1 + 8x_2 + 6x_3$ $\qquad c = (10, 8, 6)$

$$
\begin{aligned}
\text{s.t} \quad & 8x_1 + 4x_2 + 5x_3 \le 40 \\
& 6x_1 + 5x_2 + 3x_3 \le 30 \\
& 2x_1 + 3x_2 + 4x_3 \le 20 \\
& x_1 - \tfrac{1}{2}x_3 \le 0 \\
& x_{1-3} \ge 0
\end{aligned}
$$

$$
A = \begin{pmatrix} 8 & 4 & 5 \\ 6 & 5 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & -\frac{1}{2} \end{pmatrix} \le \underbrace{\begin{pmatrix} 40 \\ 30 \\ 20 \\ 0 \end{pmatrix}}_{b}
$$

$$
\begin{aligned}
\min \quad & 40y_1 + 30y_2 + 20y_3 \\
& 8y_1 + 6y_2 + 2y_3 + y_4 \ge 10 \\
& 4y_1 + 5y_2 + 3y_3 \ge 8 \\
& 5y_1 + 3y_2 + 4y_3 - \tfrac{1}{2}y_4 \ge 6 \\
& y_{1-4} \ge 0
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & c^T x \\
\text{s.t} \quad & Ax = b \\
& x \ge 0
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
\max \quad & c^T x \\
\text{s.t} \quad & Ax \le b \quad y' \\
& -Ax \le -b \quad y'' \\
& x \ge 0
\end{aligned}
$$

$$
\Rightarrow \quad
\begin{aligned}
\min \quad & b^T y' - b^T y'' \\
\text{s.t.} \quad & A^T y' - A^T y'' \ge c \\
& y', y'' \ge 0
\end{aligned}
\quad \xRightarrow{y = y' - y''} \quad
\begin{aligned}
\min \quad & b^T y \\
\text{s.t} \quad & A^T y \ge c \\
& y \gtrless 0
\end{aligned}
$$

结论：原问题的等式约束对应的对偶变量无符号限制.

$$
\begin{aligned}
\max \quad & c^T x \\
\text{s.t} \quad & Ax \le b \\
& x \ge 0
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
\max \quad & c^T x + 0^T x_s \\
\text{s.t} \quad & Ax + I x_s = b \\
& x, x_s \ge 0
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
& b^T y \\
\text{s.t} \quad & A^T y \ge c \\
& I y \ge 0
\end{aligned}
$$

更通用情形:

$$
\begin{aligned}
\max\quad & c^T x \\
s.t.\quad & A_1 x \le b_1 \\
& A_1 x = b_1 \\
& A_1 x \ge b_1 \\
& x \ge 0
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
\max\quad & c^T x + 0^T x_s + 0^T x_t \\
s.t.\quad & A_1 x + I x_s = b_1 \\
& A_2 x = b_2 \\
& A_3 x - I x_t = b_3 \\
& x, x_s, x_t \ge 0
\end{aligned}
$$

$$
\begin{aligned}
\min\quad & b_1^T y_1 + b_2^T y_2 + b_3^T y_3 \\
s.t.\quad & A_1^T y_1 + A_2^T y_2 + A_3^T y_3 \ge c \\
& I y_1 \ge 0 \\
& -I y_3 \ge 0 \\
& y_1, y_2, y_3 \gtrless 0
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
\min\quad & b_1^T y_1 + b_2^T y_2 + b_3^T y_3 \\
s.t.\quad & A_1^T y_1 + A_2^T y_2 + A_3^T y_3 \ge c \\
& y_1 \ge 0 \\
& y_3 \le 0 \\
& y_2 \gtrless 0
\end{aligned}
$$

原问题		对偶问题	
目标	max	目标	min
变量数	n	约束数	n
变量符号	≥ 0	约束符号	$\ge$
	≤ 0		$\le$
	$\gtrless$		$=$
约束数	m	变量数	m
约束符号	$\le$	变量符号	≥ 0
	$\ge$		≤ 0
	$=$		$\gtrless$

E.x.

$$
\begin{aligned}
\min\quad & 2x_1 + 3x_2 - 5x_3 + x_4 \\
s.t.\quad & x_1 + x_2 - 3x_3 + x_4 \ge 5 \\
& 2x_1 + 2x_3 - x_4 \le 4 \\
& x_2 + x_3 + x_4 = 6 \\
& x_1 \le 0 \quad x_{2-3} \ge 0 \quad x_4 \gtrless 0
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
\max\quad & 5y_1 + 4y_2 + 6y_3 \\
s.t.\quad & y_1 + 2y_2 \ge 2 \\
& y_1 + y_3 \le 3 \\
& -3y_1 + 2y_2 + y_3 \le -5 \\
& y_1 - y_2 + y_3 = 1 \\
& y_1 \ge 0 \quad y_2 \le 0 \quad y_3 \gtrless 0
\end{aligned}
$$

$$c = [2 \quad 3 \quad -5 \quad 1]$$

$$A = \begin{bmatrix} 1 & 1 & -3 & 1 \\ 2 & 0 & 2 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 5 \\ 4 \\ 6 \end{bmatrix}$$

Prop: LP对偶问题的对偶是原问题.

Thm: (Weak duality theorem) 原问题: max, 对偶问题: min
若 x 是原问题的可行解, 且 y 是对偶问题的可行解, 则有
$$c^Tx \le b^Ty.$$

Proof:

$$\begin{aligned} \max\ & c^Tx \\ s.t\ & Ax \le b \\ & x \ge 0 \end{aligned} \qquad\qquad \begin{aligned} \min\ & y^Tb \\ s.t.\ & A^Ty \ge c \\ & y \ge 0 \end{aligned}$$

$$\begin{aligned} Ax \le b &\Rightarrow y^TAx \le y^Tb \\ A^Ty \ge c &\Rightarrow x^TA^Ty \ge x^Tc \end{aligned} \quad \Rightarrow x^Tc \le y^TAx \le y^Tb \Rightarrow x^Tc \le y^Tb$$

Corollary 1: 若原问题有无界解, 则其对偶问题无可行解.
若对偶问题有无界解, 则其原问题无可行解.

Proof: 设原问题有无界解, 对偶问题有可行解. 则存在可行解 x:
$$c^Tx = M, \quad M \to \infty$$
$\therefore \exists \bar{y}$, $\bar{y}^Tb < +\infty \Rightarrow \bar{y}^Tb < M \Rightarrow c^Tx > \bar{y}^Tb \Rightarrow$ contradiction

上述结论的逆不一定成立.

Thm (Strong duality theorem) 若原问题存在最优解, 则对偶问题也存在最优解, 且目标值相等.

Proof:

$$\begin{aligned} \max\ & c^Tx + 0\cdot x_s \\ s.t\ & Ax + Ix_s = b \\ & x, x_s \ge 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max\ & c_B^Tx_B + c_N^Tx_N \\ s.t\ & Bx_B + Nx_N = b \\ & x_B, x_N \ge 0 \end{aligned}$$

可行解: $x_B = B^{-1}b - B^{-1}Nx_N$ 　　目标值: $z = c_B^TB^{-1}b + (c_N^T - c_B^TB^{-1}N)x_N$

最优解满足: $\begin{cases} c_N^T - c_B^T B^{-1} N \le 0 \\ c_B^T - c_B^T B^{-1} B \le 0 \end{cases}$ $\qquad c_B^T - c_B^T I = c_B^T - c_B^T = 0$

$c_i - c_B^T B^{-1} A_i \le 0$ 对于所有的 x_i

$$\Rightarrow \begin{cases} c^T - c_B^T B^{-1} A \le 0 \\ 0^T - c_B^T B^{-1} I \le 0 \end{cases} \qquad \text{Let } y^T = c_B^T B^{-1}$$

$$c^T - c_B^T B^{-1} A \le 0 \Rightarrow c^T - y^T A \le 0 \Rightarrow A^T y \ge c$$

$$0^T - c_B^T B^{-1} I \le 0 \Rightarrow 0^T - y^T \le 0 \Rightarrow y \ge 0$$

$$y^T b = c_B^T B^{-1} b = c^T x$$

Thm (Complementarity slackness theorem) 若 x, y 为 primal, dual 的可行解，则 x, y 为最优解, 充要条件:

① $y^T x_s = 0$

② $x^T y_s = 0$

Proof:

$$x_s = Ax - b \qquad\qquad y_s = A^T y - c$$

$$\begin{array}{lll} \max & c^T x \\ \text{s.t} & Ax + x_s = b \\ & x,\ x_s \ge 0 \end{array} \qquad \begin{array}{lll} \min & b^T y \\ \text{s.t} & A^T y - y_s = c \\ & y,\ y_s \ge 0 \end{array}$$

$$\Rightarrow \begin{array}{l} c^T x = (A^T y - y_s)^T x = y^T A x - \underbrace{y_s^T x}_{0} \\ b^T y = (Ax + x_s)^T y = x^T A y + \underbrace{x_s^T y}_{0} \end{array} \Rightarrow c^T x = b^T y$$

$$\begin{cases} A^T y - c \ge 0,\ x \ge 0 & (A^T y - c)^T x = 0 \\ b - Ax \ge 0 \quad y \ge 0 & (b - Ax)^T y = 0 \end{cases}$$