① 时间固定，终端自由
② 时间自由，终端固定

$$\begin{cases} \dot{x} = f(x(t), \alpha(t)) & t>0 \\ x(0) = x_0 \end{cases} \qquad \alpha(t): [0,+\infty) \to \mathbb{R}^m$$

收益函数： $P[\alpha(\cdot)] = \int_0^T \underbrace{r(x(t),\alpha(t))\,dt}_{\text{运行收益}} + \underbrace{g(x(T))}_{\text{终端收益}}$

PMP

(ODE) $\dot{x}^*(t) = \nabla_p H(x^*(t), p^*(t), \alpha^*(t))$

(ADJ) $\dot{p}^*(t) = -\nabla_x H(x^*(t), p^*(t), \alpha^*(t))$

(PMP) $H(x^*(t), p^*(t), \alpha^*(t)) = \max_{\alpha\in A} H(x^*(t), p^*(t), \alpha)$

① 时间固定，终端自由

(i) $H(\cdot x^*(t), p^*(t), \alpha^*(t))$ 是关于 t 的常数

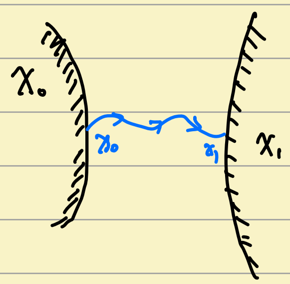
$T>0$ fixed (ii) 终端条件： $p^*(T) = \nabla g(x^*(T))$

② 时间自由，终端固定

$H(x^*(t), p^*(t), \alpha^*(t)) \equiv 0 \quad \forall t$

4.5 截面条件 (时间自由，终端自由)

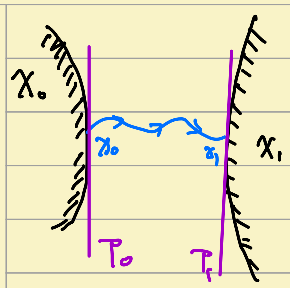
$$\begin{cases} \dot{x} = f(x(t), \alpha(t)) & t>0 \\ x_0 \in X_0 \end{cases}$$



$X_0 \subset \mathbb{R}^n$：初始点集， $X_1 \subset \mathbb{R}^n$：终点集

目标： 最优化收益函数 $P[\alpha(\cdot)] = \int_0^\tau r(x(t),\alpha(t))\,dt$

$\tau = \tau[\alpha(\cdot)]$：首次到达 $x_1 \in X_1$ 的时间



T_0: 过 x_0 处 关于 X_0 的切平面

T_1: 过 x_1 处 关于 X_1 的切平面

Thm 4.5 (PMP + 截面条件) 令 α^* 为最优控制, x^* 为对应的最优状态, 且

$$x_0 = x^*(0), \quad x_1 = x^*(\tau^*)$$

则存在 p^* 使得

(ODE) $$\dot{x}^*(t) = \nabla_p H(x^*(t), p^*(t), \alpha^*(t))$$

(ADJ) $$\dot{p}^*(t) = -\nabla_x H(x^*(t), p^*(t), \alpha^*(t))$$

(PMP) $$H(x^*(t), p^*(t), \alpha^*(t)) = \max_{a\in A} H(x^*(t), p^*(t), a)$$

此外, p^* 满足截面条件 (transversality condition)

$$\begin{cases} p^*(\tau^*) \perp T_1 \\ p^*(0) \perp T_0 \end{cases}$$

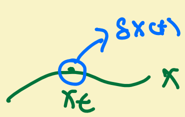
Proof:

$x=2$

$x+\delta x = 2.1$

Hamiltonian: $H(x,p,a) = f(x,a)p + r(x,a)$

$$\dot{x}(t) = f(x,a)$$



构造变分 (相当于在当前状态加一个扰动)

$$\delta x(t) \qquad \delta\alpha(t) \qquad \delta\tau$$

则有: $\delta\dot{x}(t) = \nabla_x f(x^*,\alpha^*)\delta x(t) + \nabla_\alpha f(x^*,\alpha^*)\delta\alpha(t) \quad \forall 0<t<\tau^*$

$P[\alpha(\cdot)] = \int_0^\tau r(x(t),\alpha(t))dt$

收益函数的一阶变分:

(A) $$\delta P = \int_0^{\tau^*}\left[\nabla_x r(x,\alpha)\cdot\delta x(t) + \nabla_\alpha r(x,\alpha)\,\delta\alpha(t)\right]dt + r(x^*(\tau^*),\alpha^*(\tau^*))\delta\tau$$

由于 $0 = \int_0^{\tau^*}[\dot{p}\cdot\delta x - p\,\delta\dot{x}]dt$ (B), 且

$$-\int_0^{\tau^*} p\cdot\delta\dot{x}\,dt = -\int_0^{\tau^*} p[\nabla_x f\,\delta x(t) + \nabla_\alpha f\,\delta\alpha(t)]\,dt \qquad (C)$$

把(C)代入(B)，与(A)相加，可得

$$(D)\quad \delta P = \int_0^{\tau^*}[\nabla_x r\,\delta x(t) + \nabla_\alpha r\,\delta\alpha(t)]\,dt + r(x^*,\alpha^*)\,\delta\tau + \underbrace{\int_0^{\tau^*} p\,\delta\dot{x}\,dt}_{\text{分部积分}}$$
$$-\int_0^{\tau^*} p[\nabla_x f\,\delta x(t) + \nabla_\alpha f\,\delta\alpha(t)]\,dt$$

$$\int_0^{\tau^*} p\cdot\delta\dot{x}\,dt = p\cdot\delta x\Big|_0^{\tau^*} - \int_0^{\tau^*}\dot{p}\cdot\delta x\,dt \qquad (E)$$

把(E)代回(D)，可得

$$\delta P = \int_0^{\tau^*}[(\nabla_x r + \nabla_x f\cdot p - \dot{p})\,\delta x + (\nabla_\alpha r + \nabla_\alpha\cdot p)\,\delta\alpha]\,dt$$
$$+ p\cdot\delta x\Big|_0^{\tau^*} + r(x^*,\alpha^*)\,\delta\tau$$

由于 $\dot{p}(t) = -\nabla_x H = -\nabla_x f(x,\alpha)\cdot p + \nabla_x r(x,\alpha)$

代入并化简，可得

$$\delta P = \int_0^{\tau^*}\nabla_\alpha H(x^*,p^*,\alpha^*)\,\delta\alpha\,dt + p\cdot\delta x\Big|_0^{\tau^*} + r(x^*,\alpha^*)\,\delta\tau$$

如果 x^*, α^*, p^* 为最优，$\delta P \le 0$

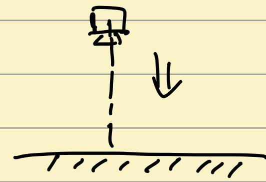
由于 $x_0 \in X_0$，$x_1 \in X_1$，则 $\delta x(0) \in T_0$ $\quad \delta x(\tau^*) \in T_1$

$$p(0)\cdot\delta x(0) = 0, \qquad \forall\,\delta x(0) \in T_0$$
$$p(\tau^*)\cdot\delta x(\tau^*) \ge 0 \qquad \forall\,\delta x(1) \in T_1$$

因此 $\quad p(0) \perp T_0 \qquad p(\tau^*) \perp T_1$

Example 4.4.4.

月球着陆器



初始高度: h_0　初始速度 v_0, 初始质量 x_0

t 时刻: 高度 $h(t)$　速度 $v(t)$　质量 $m(t)$

控制: $\alpha(t) \in [0,1]$　推力大小

目标: 使飞船平稳降落, 且耗费燃料最少.

$\tau = \tau[\alpha(\cdot)]$　使得 $h(\tau) = v(\tau) = 0$

状态变化: $\begin{cases} \dot h(t) = v(t) \\ \dot v(t) = -g + \dfrac{\alpha(t)}{m(t)} \\ \dot m(t) = -k\alpha(t) \end{cases}$ $\Rightarrow \quad \alpha = \dfrac{\dot m}{k}$　　g: 重力加速度

初始值: $\begin{cases} h(0) = h_0 > 0 \\ v(0) = v_0 \\ m(0) = m_0 > 0 \end{cases}$　　$= \int_0^\tau r(x,\alpha)\,dt$

收益函数: $P[\alpha(\cdot)] = \dfrac{m(\tau) - m_0}{k} = -\int_0^\tau \alpha(t)\,dt$

写成标准式:
$$x(t) = \begin{pmatrix} h(t) \\ v(t) \\ m(t) \end{pmatrix} \qquad f = \begin{pmatrix} v(t) \\ -g + \dfrac{\alpha(t)}{m(t)} \\ -k\alpha(t) \end{pmatrix}$$

Hamiltonian: $H(x,p,\alpha) = f(x,\alpha)\cdot p + r(x,\alpha)$
$$= \begin{pmatrix} v \\ -g + \frac{\alpha}{m} \\ -k\alpha \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} - \alpha$$
$$= v\cdot p_1 + (-g + \tfrac{\alpha}{m})p_2 - k\alpha p_3 - \alpha.$$

(ADJ)　$$\nabla_x H = \begin{pmatrix} H_h \\ H_v \\ H_m \end{pmatrix} = \begin{pmatrix} 0 \\ p_1 \\ -\dfrac{\alpha p_2}{m^2} \end{pmatrix} = \begin{pmatrix} \dot p_1 \\ \dot p_2 \\ \dot p_3 \end{pmatrix}$$

(PMP) $\alpha^* = \underset{\alpha\in[0,1]}{\arg\max}\, H(x(t), p(t), \alpha)$

$$= \underset{\alpha\in[0,1]}{\arg\max}\left\{ v\cdot p_1 + (-g + \frac{\alpha}{m})p_2 - k\alpha p_3 - \alpha \right\}$$

$$= v\cdot p_1 - g p_2 + \underset{\alpha\in[0,1]}{\arg\max}\left\{ \alpha\left(-1 + \frac{p_2}{m} - k p_3\right)\right\}$$

令 $S(t) = -1 + \frac{p_2}{m} - k p_3$

则最优控制：

$$\alpha(t) = \begin{cases} 1 & \text{if } S(t) > 0 \\ 0 & \text{if } S(t) \le 0 \end{cases}$$

假设存在转换时间 t^*

$$\alpha(t) = \begin{cases} 0 & \text{if } 0 \le t \le t^* \\ 1 & \text{if } t^* \le t \le \tau \end{cases}$$

考虑 $0 < t \le t^*$，此时飞船自由落体

由于(ODE)有

$$\begin{cases} \dot h(t) = v(t) \\ \dot v(t) = -g \\ \dot m(t) = 0 \end{cases} \Rightarrow \begin{cases} h(t) = -\frac{1}{2}gt^2 + v_0 t + h_0 \\ v(t) = -gt + v_0 \\ m(t) = m_0 \end{cases}$$

考虑 $t^* \le t \le \tau^*$，此时推进器全开减速

由于(ODE)，有

$$\begin{cases} \dot h(t) = v(t) \\ \dot v(t) = -g + \frac{1}{m} \\ \dot m(t) = -k \end{cases} \qquad m(t) = m_0 - k(t - t^*)$$

$$\dot v(t) = -g + \frac{1}{m} \Rightarrow v(\tau) - v(t) = \int_t^\tau \left(-g + \frac{1}{m(s)}\right) ds$$

代入 $v(\tau)=0$，可得

$$-v(t) = -g(\tau - t) + \int_t^\tau \frac{1}{m_0 - k(s - t^*)} ds$$

令 $u = m_0 + k(t^* - s)$

$$I(t) = \int_t^{\tau} \frac{1}{m(s)} ds = \int_{m(t)}^{m(\tau)} \frac{1}{u}\left(-\frac{1}{k}\right) du = -\frac{1}{k} \ln u \Big|_{m(t)}^{m(\tau)} = -\frac{1}{k}\left(\ln m(\tau) - \ln m(t)\right)$$

则

$$v(t) = g(\tau - t) + \frac{1}{k} \ln \frac{m(\tau)}{m(t)}$$

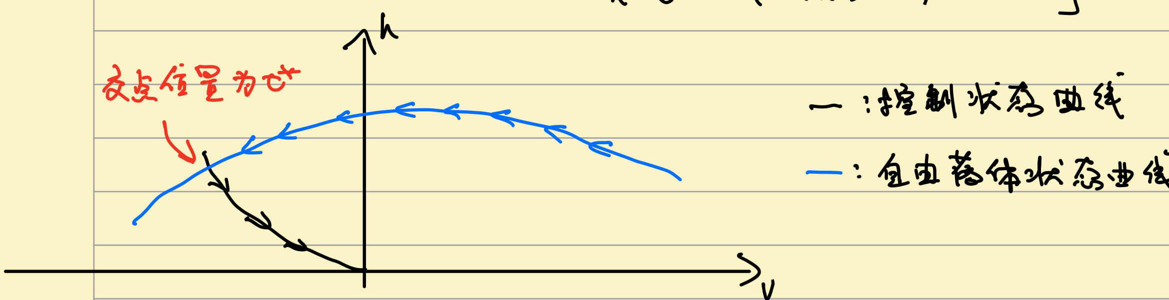
$$= g(\tau - t) + \frac{1}{k} \ln \frac{m_0 - k(\tau - t^*)}{m_0 - k(t - t^*)} \qquad t \in [t^*, \tau]$$

由于 $h(\tau) = 0$,

$$h(\tau) - h(t) = \int_t^{\tau} v(s) ds$$

$$\Rightarrow h(t) = -\int_t^{\tau} v(s) ds$$

$$= -\frac{g}{2}(\tau - t)^2 - \frac{1}{k^2}\left[m(t)\left(\ln \frac{m(\tau)}{m(t)} + 1\right) - m(\tau)\right]$$



$h(t) = -\frac{1}{2} g t^2 + v_0 t + h_0$

$v(t) = -gt + v_0$ $\Rightarrow t = \frac{1}{g}(v_0 - v_t)$ $\Rightarrow h(t) = h_0 - \frac{1}{2g}(v^2(t) - v_0^2)$

1. $\begin{cases} \dot{x}_1 = x_1 + a \\ \dot{x}_2 = -x_2 + a^2 \end{cases}$

$$P = \int_0^T \left[-(x_1^2 + x_2^2) - \frac{1}{2}a^2\right] dt + \frac{1}{2}(q_1 x_1^2 + q_2 x_2^2)$$

$$H(x, p, a) = f(x, a) \cdot p + r(x, a)$$

$$= (x_1 + a) \cdot p_1 + (-x_2 + a^2) \cdot p_2 - (x_1^2 + x_2^2) - \frac{1}{2}a^2$$

$$= (p_2 - \frac{1}{2})a^2 + p_1 a + (p_1 x_1 - p_2 x_2 - x_1^2 - x_2^2)$$

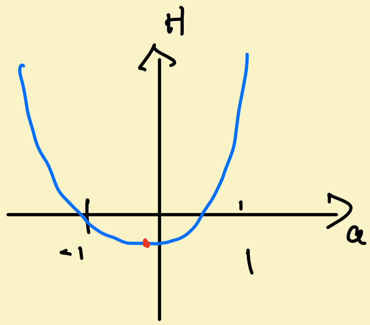
(ODE) $\dot{x} = \nabla_p H = \begin{cases} x_1 + a \\ -x_2 + a^2 \end{cases}$

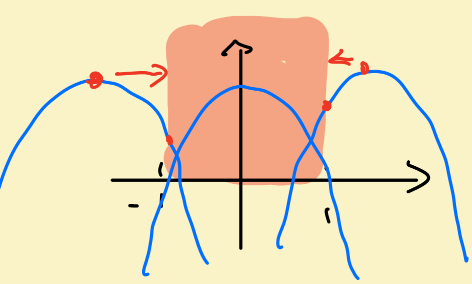
(ADJ) $\dot{p} = -\nabla_x H = \begin{cases} -p_1 + 2x_1 \\ p_2 + 2x_2 \end{cases}$

(PMP) $a^*(t) \in \underset{a \in [-1,1]}{\arg\max}\, H(x(t), p(t), a)$

$$= \underset{a \in [-1,1]}{\arg\max}\, (p_2 - \frac{1}{2})a^2 + p_1 a$$

$$\nabla_a H = 2(p_2 - \frac{1}{2}) + p_1 = 0 \Rightarrow \hat{a} = -\frac{p_1}{2p_2 - 1}$$





Case 1: 若 $p_2 - \frac{1}{2} \geq 0$ $\quad a^*(t) = \text{sign}(p(t))$

Case 2: 令 $\Pi(a) = \max(-1, \min(1, a))$

$$a^*(t) = \Pi\left(-\frac{p_1}{2p_2 - 1}\right)$$

(c) $p(T) = \nabla_x g(x(T)) = \begin{pmatrix} q_1 x_1 \\ q_2 x_2 \end{pmatrix}$

$g(x) = \frac{1}{2}(q_1 x_1^2 + q_2 x_2^2)$

(d) 由于 $H(x^*, p^*, a^*) = \max_a H(x^*, p^*, a)$

$$\frac{d}{dt} H(x^*, p^*, a^*) = \nabla_x H \cdot \frac{dx}{dt} + \nabla_p H \cdot \frac{dp}{dt} + \frac{\partial}{\partial t} H$$

$$= \nabla_x H \cdot \nabla_p H + \nabla_p H \cdot (-\nabla_x H) + \frac{\partial}{\partial t} H$$

$$= 0 + \cancelto{0}{\frac{\partial}{\partial t} H}$$

$$= 0$$

说明 H 关于 t 是常数.