If $\lim_{n\to\infty} x_n = A$, $\lim_{n\to\infty} x_n = B$ $\Rightarrow$ $A = B$ Uniqueness

If $\lim_{n\to\infty} x_n = A$ then there exists $M \in \mathbb{R}$ such for all $x_n$,

$$|x_n| < M$$

$$\lim_{n\to\infty}(x_n \pm y_n) = \lim_{n\to\infty} x_n \pm \lim_{n\to\infty} y_n$$

$$\lim_{n\to\infty}(x_n y_n) = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n$$

$$\lim_{n\to\infty} c\cdot x_n = c \cdot \lim_{n\to\infty} x_n$$

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} x_n \Big/ \lim_{n\to\infty} y_n \quad \lim_{n\to\infty} y_n \neq 0, \ y_n \neq 0.$$

$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$, and $x_n \le z_n \le y_n$ $\Rightarrow$ $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = \lim_{n\to\infty} y_n$

E.g. $\lim_{n\to\infty} \frac{2^n}{n!}$ since $0 \le \frac{2^n}{n!} \le \frac{4}{n}$

$$\lim_{n\to\infty} \frac{4}{n} = 0 \Rightarrow \lim_{n\to\infty} \frac{2^n}{n!} = 0$$

$\lim_{n\to\infty} \frac{n!}{n^n}$ since $0 \le \frac{n!}{n^n} \le \frac{1}{n}$

$$\lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \lim_{n\to\infty} \frac{n!}{n^n} = 0$$

零序列

Sequence that converges to zero is often referred to as null sequence

If $s > 0$ $\frac{1}{n^s} \to 0$ as $n \to \infty$

If $|\lambda| < 1$, $\lambda^n \to 0$ as $n \to \infty$

If $s \in \mathbb{R}$ and $|\lambda| > 1$, $\frac{n^s}{\lambda^n} \to 0$ as $n \to \infty$

If $s \in \mathbb{R}$, $\frac{n^s}{n!} \to 0$ as $n \to \infty$

If $\lambda \in \mathbb{R}$, $\frac{\lambda^n}{n!} \to 0$ as $n \to \infty$

$$\frac{n!}{n^n} \to 0 \quad \text{as } n \to \infty$$

$n! > 100^n$ $n \to \infty$ $a_n = \frac{5^n}{n!}$

Monotone Sequence : sequence gets steadily larger or smaller.

monotonic sequence

Defn : A sequence $\{x_n\}$ is strictly increasing if $x_n < x_{n+1}$ holds for all $n \in \mathbb{Z}_+$.

: A sequence $\{x_n\}$ is strictly decreasing if $x_n > x_{n+1}$ holds for all $n \in \mathbb{Z}_+$.

: A sequence $\{x_n\}$ is nondecreasing if $x_n \leq x_{n+1}$ holds for all $n \in \mathbb{Z}_+$.

: A sequence $\{x_n\}$ is nonincreasing if $x_n \geq x_{n+1}$ holds for all $n \in \mathbb{Z}_+$.

Thm (Monotone Convergence Theorem)

Suppose that $\{x_n\}$ is a monotonic sequence. Then $\{x_n\}$ is convergent if and only if $\{x_n\}$ is bounded.

Proof: Suppose that $\{x_n\}$ is monotonic, if $\{x_n\}$ is convergent, by

$\Rightarrow$ the properties of convergent sequence, $\{x_n\}$ is bounded.

$\Leftarrow$ Since $\{x_n\}$ is bounded, assume that $\{x_n\}$ is non-decreasing. Then its supremum exists.

$$L := \sup\{x_n ; \mathbb{Z}^+\}$$


Then $L-\varepsilon$ is not an upper bound for $\{x_n\}$.

It follows that there exists $N \in \mathbb{N}$, such that $x_N > L-\varepsilon$.

Since $\{x_n\}$ is non-decreasing, we have for any $n \geq N$,

$$L-\varepsilon < x_N \leq x_n \leq L < L+\varepsilon$$

$$\Rightarrow \quad |x_n - L| < \varepsilon$$

Hence, $\{x_n\}$ is convergent.

E.g. Show that $x_n = \left(1+\frac{1}{n}\right)^n$ is convergent.

Proof: Bernoulli inequality : $(1+x)^n \geq 1+nx \quad \forall n \in \mathbb{Z}_+ \quad x \geq -1$

$$\frac{x_{n+1}}{x_n} = \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n} = \frac{\left(\frac{n+2}{n+1}\right)^{n+1}}{\left(\frac{n+1}{n}\right)^n} = \frac{(n+2)^{n+1}\cdot n^n}{(n+1)^{2n+1}}$$

$$= \left[\frac{(n+2)n}{(n+1)^2}\right]^n \frac{n+2}{n+1} = \left[1-\frac{1}{n^2+2n+1}\right]^n \frac{n+2}{n+1}$$

$$\geq \left(1-\frac{n}{n^2+2n+1}\right)\frac{n+2}{n+1} = \frac{n^3+3n^2+3n+2}{n^3+3n^2+3n+1} > 1$$

$$y_n = \left(1+\frac{1}{n}\right)^{n+1}$$

$$\frac{y_{n-1}}{y_n} = \frac{\left(1+\frac{1}{n-1}\right)^n}{\left(1+\frac{1}{n}\right)^{n+1}} = \frac{n^{2n+1}}{(n-1)^n(n+1)^{n+1}} = \left(\frac{n^2}{n^2-1}\right)^n \frac{n}{n+1}$$

$$= \left(1+\frac{1}{n^2-1}\right)^n \frac{n}{n+1}$$

$$\geq \left(1+\frac{n}{n^2-1}\right)\frac{n}{n+1} = \frac{n^3+n^2-n}{n^3+n^2-n-1} > 1$$

① $x_n = \left(1+\frac{1}{n}\right)^n$ is increasing

② $y_n = \left(1+\frac{1}{n}\right)^{n+1}$ is decreasing

$$2 = x_1 \leq x_n \leq y_n \leq y_1 = 4 \Rightarrow \{x_n\} \text{ is bounded}$$

By Monotone Convergence theorem, both $\{x_n\}$ and $\{y_n\}$ are convergent.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

E.g. Compound interest 复利

$1 \times (1+1) = 2$

$1 \times (1+0.5)^2 = 2.25$

| | |
|---|---|
| \$1.00 | \$2.00 |
| 2023 | 2024 |

$1 \times (1+0.25)^4 = 2.4414$

$1 \times (1+1/12)^{12} = 2.613035$

$1 \times (1+\frac{1}{365})^{365}$ $\quad 1 \cdot e^{rt}$

## Bolzano-Weierstrass Thm.

Defn (Subsequence) 子列 Let $\{x_n\}$ be a sequence. Then a subsequence of $\{x_n\}$ is a sequence

$$\{x_{n_i}\}_{i=1}^{\infty}$$

with $\{n_i\}$ being an increasing sequence of positive integers.

E.g. $\{(-1)^n\}$ $\quad n_i = 2k \quad k \in \mathbb{Z}_+$

$\Rightarrow \quad 1, 1, 1, \ldots$

E.g.

$$x_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}$$

$0, 2, 0, 4, 0, 6, 0, 8 \cdots$

$\{x_{2k-1}\}_{k=1}^{\infty} = 0, 0, \ldots$

Thm: If $\{x_n\}$ converges to $\ell$, then every subsequence of $\{x_n\}$ converges to $\ell$.

Proof: Since $\{x_n\}$ converges to $\ell$, then there exists $N \in \mathbb{Z}^+$ such that for any $n \geq N \Rightarrow |x_n - \ell| < \varepsilon$.

Let $\{x_{n_k}\}$ be an arbitrary subsequence of $\{x_n\}$. if $k \geq N$ then $n_k \geq N$

$$|x_{n_k} - \ell| < \varepsilon \quad \text{for any } k \geq N.$$

推论

Corollary. If a sequence $\{x_n\}$ pocesses two subsequenes which converges to distinct limits, then $\{x_n\}$ does not converges.

Thm: Every sequence contains a monotonic subsequence.

Proof:

Consider the set $S: \{n \in \mathbb{Z}^+ : \text{for any } m > n,\ a_m \le a_n\}$

① Suppose $S$ is finite, Let $n_1$ be a natural number that is greater than all elements of $S$. Then $n_1 \notin S$. By the definition of $S$, $n_2 > n_1 \Rightarrow a_{n_2} > a_{n_1}$. Since $n_2 \notin S$, $n_3 > n_2 \Rightarrow a_{n_3} > a_{n_2}$. Continuing in this way, we obtain an increasing subsequene $\{a_{n_k}\}$.

② Suppose $S$ is infinite. $S := \{n_k : k \in \mathbb{Z}^+, \text{for any } n_j > n_k,\ a_{n_j} > a_{n_k}\}$

Since $n_{k+1} > n_k \Rightarrow a_{n_{k+1}} < a_{n_k} \in S$ then $\{a_{n_k}\}$ is non-increasing

Thm (Bolzano-Weierstrass Theorem)

Every bounded sequence contains a convergent subsequence.

Cauchy Convergence Criterion

① $\forall \varepsilon > 0\ \exists N \in \mathbb{N},\ \forall n > N \Rightarrow |x_n - l| < \varepsilon$ we need to know the limit.

apply only to monotonic sequence

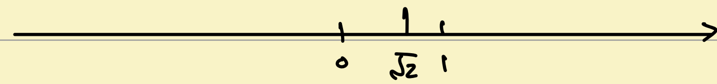
② $\{x_n\}$ monotonic, bounded $\Leftrightarrow$ convergent

Thm (Cauchy Convergence Criterion)

A sequence $\{x_n\}$ is convergent if and only if. for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that

$$\text{for any } n \ge N,\ m \ge N \Rightarrow |x_n - x_m| \le \varepsilon$$

## Completeness of the real number system*




The real space is complete
The set of all rational number is not complete.

Defn*: A space is complete if all the Cauchy sequence converges.