Set : S ① Well-definedness ② Distinct Elements ③ Unordered

↓

Ordered Set : $(S, <)$ ① One and only one of the following :

$x, y \in S$ $x<y$ or $y<x$ or $y=x$

② transitivity : $x, y, z \in S$, $x<y$, $y<z$

$\Rightarrow x<z$

Field ① Axioms for addition

② Axioms for multiplication

Thm (Consequences of axioms of addition) for any $x, y, z \in F$

① if $x+y=x+z \Rightarrow y=z$

② if $x+y=x \Rightarrow y=0$

③ if $x+y=0 \Rightarrow y=-x$

④ $-(-x)=x$

Proof: ① $x+y=x+z \Rightarrow (-x)+(x+y)=(-x)+(x+z)$ associative law

$\Rightarrow (-x+x)+y=(-x+x)+z$

$\Rightarrow y=z$

② $x+y=x \Rightarrow (-x)+(x+y)=(-x)+x$

$\Rightarrow y=0$

③, ④

Thm (Consequences of axioms of multiplication) for any $x, y, z \in F$

① if $x\neq 0$, $x\cdot y=x\cdot z \Rightarrow y=z$

② if $x\neq 0$, $x\cdot y=x \Rightarrow y=1$

③ if $x\neq 0$, $x\cdot y=1 \Rightarrow y=\frac{1}{x}$

④ if $x\neq 0$, then $\frac{1}{1/x}=x$

Proof is omitted.

Thm (Consequences of axiom of field) For any $x, y, z \in F$.

① $0 \cdot x = 0$

② if $x \neq 0$, $y \neq 0$, then $x \cdot y \neq 0$.

③ $(-x) \cdot y = -(x \cdot y) = x \cdot (-y)$

④ $(-x) \cdot (-y) = x \cdot y$.

Proof: ① Let $y = 0$ $\quad (y+y) \cdot x = y \cdot x + y \cdot x$

$\Rightarrow (0+0) \cdot x = 0 \cdot \cancel{x} + 0 \cdot x = 0 \cancel{x} \Rightarrow 0 \cdot x = 0$

② Suppose by contradiction $x \cdot y = 0$, $\frac{1}{x} \cdot (xy) \cdot \frac{1}{y} = \frac{1}{x} \cdot 0 \cdot \frac{1}{y}$

$\Rightarrow (\frac{1}{x} \cdot x) \cdot (y \cdot \frac{1}{y}) = 1 \cdot 1 = 0$ contradiction, then $x \cdot y \neq 0$

③ $(-x) \cdot y = -(x \cdot y) \Rightarrow (-x) \cdot y + x \cdot y = -(x \cdot y) + x \cdot y$

$\Rightarrow -x \cdot y + x \cdot y = 0$

$\Rightarrow (-x+x) \cdot y = 0 \Rightarrow 0 \cdot y = 0$

then we proved $(-x) \cdot y = -(x \cdot y)$ holds.

$x(-y) = -(xy) \Rightarrow x(-y) + (xy) = -(xy) + (xy)$

$\Rightarrow x(-y) + (xy) = x(-y+y) = x \cdot 0 = 0$

then we proved $x(-y) = -(xy)$ holds.

④ $(-x) \cdot (-y) = -(x \cdot (-y)) = -(-(xy)) = xy$

Defn: (Ordered field) An ordered field F is field (set + axioms), with order "$<$" endowed, and satisfies:

① If $y < z$, then $x + y < x + z$, for any $x, y, z \in F$

② $x, y \in F$, $0 < x$ and $0 < y$, then $0 < x \cdot y$

If $0 < x \Rightarrow x \in F$ is positive

If $x < 0 \Rightarrow x \in F$ is negative

Thm (Properties of ordered field). $(F, <)$ is an ordered field, $x, y \in F$, then

① If $0 < x \Rightarrow -x < 0$; if $x < 0$, $\Rightarrow 0 < -x$

② If $0 < x$ and $y < z \Rightarrow xy < xz$

③ If $x < 0$ and $y < z \Rightarrow xz < xy$

④ If $x \neq 0 \Rightarrow 0 < x^2 = x \cdot x$

⑤ If $0 < x < y \Rightarrow 0 < \frac{1}{y} < \frac{1}{x}$

Proof: ① – ④ are omitted.

⑤ Prove $0 < \frac{1}{y}$. since $0 < 1 = y \cdot \frac{1}{y}$ by ③, we have $0 < \frac{1}{y}$

Prove $\frac{1}{y} < \frac{1}{x}$. since $0 < x$, and $0 < \frac{1}{x}$. from $x < y$

$$\left(\frac{1}{x}\frac{1}{y}\right)\cdot x < \left(\frac{1}{x}\frac{1}{y}\right)\cdot y \Rightarrow \frac{1}{y}\left(\frac{1}{x}\cdot x\right) < \frac{1}{x}\left(\frac{1}{y}\cdot y\right)$$

$$\Rightarrow \frac{1}{y} < \frac{1}{x}$$

Real field and real numbers

Defn. (Real field) Real field is an ordered field①, with the least upper bound property②, and contains $\mathbb{Q}$ as a subfield③.

Notation: $\mathbb{R}$ or $(-\infty, +\infty)$

Thm (Archimedian property) 阿基米德性质

Suppose $x, y \in \mathbb{R}$, $x > 0$. Then there exists a positive integer n such that $nx > y$.

Thm (Denseness of $\mathbb{Q}$ in $\mathbb{R}$) 稠密性

Suppose $x, y \in \mathbb{R}$, and $x < y$. Then there exists a $r \in \mathbb{Q}$ such that

$$x < r < y$$

Proof: $x = a_0 . a_1 a_2 a_3 a_4 \cdots$ 不足近似: $x_n = 11.3471$

$x = 11.347165$ $n = 4$, 过剩近似: $\bar{x}_n = 11.3472$

$$\Rightarrow x_n \le x \le \bar{x}_n$$

Since $x < y$, there exists $n > 0$ such that $\bar{x}_n < y_n$. Let $r = \frac{\bar{x}_n + y_n}{2}$

$$\Rightarrow x \le \bar{x}_n < r < y_n < y$$

We conclude $x < r < y$ and $r \in \mathbb{Q}$.

Thm (n^{th} root of a real number) For any $x \in \mathbb{R}$, $x>0$, and $n \in \mathbb{Z}_+$ there exists one and only one positive real number y satisfies.

$$y^n = x$$

Denote $y = x^{\frac{1}{n}} = \sqrt[n]{x}$

Thm: $(a \cdot b)^{\frac{1}{n}} = a^{\frac{1}{n}} \cdot b^{\frac{1}{n}}$

Intervals: An interval is a subset of $\mathbb{R}$. Given $a, b \in \mathbb{R}$, and $a<b$.

Open interval

$$\left.\begin{aligned} (a, b) &:= \{x \in \mathbb{R} \mid a<x<b\} \\ [a, b] &:= \{x \in \mathbb{R} \mid a \le x \le b\} \\ [a, b) &:= \{x \in \mathbb{R} \mid a \le x < b\} \\ (a, b] &:= \{x \in \mathbb{R} \mid a < x \le b\} \end{aligned}\right\} \text{finite interval}$$

$$\left.\begin{aligned} (a, +\infty) &:= \{x \in \mathbb{R} \mid x > a\} \\ (-\infty, b) &:= \{x \in \mathbb{R} \mid x < b\} \\ [a, +\infty) &:= \{x \in \mathbb{R} \mid x \geq a\} \\ (-\infty, b] &:= \{x \in \mathbb{R} \mid x \le b\} \end{aligned}\right\} \text{infinite interval}$$

Defn (Supremum and Infimum)

Supremum: least upper bound

Infimum: Greatest lower bound

Example: $A = \{x \in \mathbb{R} : x>0,\ x^2>5\}$ Prove that A is non-empty, bounded below, and $\inf A = \sqrt{5}$.

Proof: ① Non-empty. Since $\sqrt{6} \in A$, then A is non-empty.

② bounded below. Since for any $x \in A$, we have $x > -1$, then A is bounded below.

③ $\inf A = \sqrt{5}$. Suppose that $\sqrt{5} < \inf A$, then there exists a number r such that $\sqrt{5} < r < \inf A$. By the definition of A, we have $r \in A$, but $r < \inf A$. contradiction.

Sequence 數列 A sequence is a list without stopping.

Defn (Sequence): A sequence of real number is a function:

$$f: \mathbb{Z}^+ \to \mathbb{R}$$

Example: $f(1), f(2), \cdots f(n), \cdots$

Notation: $f_1, f_2, \cdots f_n, \cdots$

$\{f_n\}$

Formula of n : $f(n) = 2n+1$

Recursion formula : $f_n = 2 + f_{n-1}$

Arithmetic Progression 等差數列

$$c,\ c+d,\ c+2d,\ c+3d,\ \cdots,\ c+(n-1)d.$$

Formula of n: $f(n) = c + (n-1)d$

Recursion formula: $f_n = f_{n-1} + d \qquad f_1 = c$

Geometric Progression 等比數列

$$c,\ cr,\ c\cdot r^2,\ c\cdot r^3,\ \cdots\ c\cdot r^{n-1}$$

Formula of n: $f(n) = c\cdot r^{n-1}$

Recursion formula: $f_n = r\cdot f_{n-1}, \quad f_1 = c$

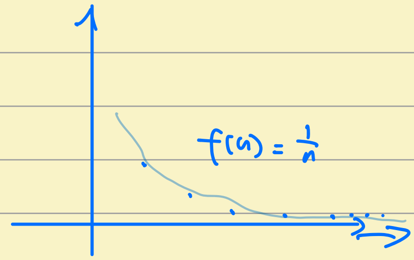
Sequence of partial sums.

Given a sequence : $x_1, x_2, x_3, \cdots$

A new sequence : $f_n = \sum_{k=1}^{n} x_n$

Convergence

$$f(n) = \frac{1}{n}$$



$f(n)$ is getting closer to 0 as $n \to \infty$.

Defn (ε-N definition) We define $\{f_n\}$ converges to A if for any $\varepsilon > 0$, there exists a natural number $N > 0$, such that

$$\text{when } n \geq N \Rightarrow |f_n - A| < \varepsilon .$$