

## Sequences & Series: Answer to Exercise 2

- Given the sequence  $a_n = \frac{n}{n+1}$ , determine whether the sequence converges. If it does, find its limit.

**Answer:**

$$\begin{aligned}\lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \frac{n}{n+1} \\ &= \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} \\ &= 1\end{aligned}$$

- Given the sequence  $a_n = \left(1 + \frac{1}{n}\right)^n$ , determine whether the sequence converges. If it does, find its limit.

**Answer:**

The limit of this sequence is  $e$ . See the note for detailed proof.

- Given the sequence  $a_n = \frac{\sin n}{n}$ , determine whether the sequence converges. If it does, find its limit.

**Answer:**

Sequence  $a_n = \frac{\sin n}{n}$ . We know that  $-1 \leq \sin n \leq 1$ . So,  $\left|\frac{\sin n}{n}\right| \leq \frac{1}{n}$ . Since  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ , by the Squeeze Theorem,  $\lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0$ . So the sequence converges to 0.

- Given the sequence  $a_n = \frac{(-1)^n n}{n+1}$ , determine whether the sequence converges. If it does, whether it is absolutely convergent or conditionally convergent.

**Answer:**

We consider the subsequences of even and odd terms:

For even terms ( $n = 2k, k \in \mathbb{N}$ ),  $a_{2k} = \frac{2k}{2k+1} \rightarrow 1$  as  $k \rightarrow \infty$

For odd terms ( $n = 2k+1, k \in \mathbb{N}$ ),  $a_{2k+1} = \frac{-(2k+1)}{2k+2} \rightarrow -1$  as  $k \rightarrow \infty$

Since the subsequences converge to different limits, the sequence  $\{a_n\}$  does not converge.

5. Given the sequence  $a_n = \sqrt{n+1} - \sqrt{n}$ , determine whether the sequence converges. If it does, find its limit.

**Answer:**

We rationalize the expression:

$$\begin{aligned}a_n &= \sqrt{n+1} - \sqrt{n} \\&= \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} \\&= \frac{1}{\sqrt{n+1} + \sqrt{n}}\end{aligned}$$

As  $n \rightarrow \infty$ ,  $\lim_{n \rightarrow \infty} a_n = 0$ . So the sequence converges to 0.