Let $\{a_k\}$ be a sequence. A series is $s = \sum_{k=1}^{\infty} a_k$

Ratio test: $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$, series converges

Root test: $\lim_{k\to\infty} \sqrt[k]{a_k} < 1$, series converges.

Integral test: $f: [1, +\infty) \to \mathbb{R}$ $\lim_{x\to\infty} \int_1^x f(x)\,dx \Rightarrow$ series converges

Alternating Series test $\{a_k\} \to 0$ as $n \to 0$, $\sum_{k=1}^{\infty} (-1)^{k-1} a_k$ converges,

Power series

A power series is series of the form:

$$\sum_{n=0}^{\infty} a_n x^n \qquad \sum_{n=1}^{\infty} a_n$$

Set of convergence : $S := \{ x \in \mathbb{R} \mid \sum_{n=0}^{\infty} a_n x^n \text{ converges} \}$

$S \neq 0$, because $0 \in S$

Consider a function $f: S \to \mathbb{R}$

$$f(x) := \sum_{n=0}^{\infty} a_n x^n \qquad x \in S$$

E.g. $f(x) = \sum_{n=0}^{\infty} x^n \Rightarrow f(x) = \frac{1}{1-x}$ $x \in (-1, 1)$

$\Rightarrow S = (-1, 1)$

Thm: Given a power series $\sum_{n=0}^{\infty} a_n x^n$, it either converges absolutely for any $x \in \mathbb{R}$, or there exists $R \in [0, +\infty)$, such that

(1) when $|x| < R$, it converges absolutely.

(2) when $|x| > R$, it diverges.

$|x| < R \qquad (-R, R) \subseteq S \subseteq [-R, R]$

$S = (-R, R)$ or $S = [-R, R)$ or $S = (-R, R]$ or $S = [-R, R]$

E.g. $\sum_{n=1}^{\infty} \frac{x^n}{n}$ when $|x| > 1$ $\lim_{n \to \infty} \frac{x^n}{n} = +\infty$, so it diverges for $|x| > 1$

when $|x| < 1$ $\lim_{n \to \infty} \frac{x^n}{n} = 0$, so it converges for $|x| < 1$

when $x = 1$, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

when $x = -1$ $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges

The set of convergence $S = [-1, 1)$

Convergence test

Ratio test: $\lim_{k \to \infty} \frac{a_{k+1}}{a_k} < 1$, series converges

Ratio test: Consider the power series $\sum_{n=0}^{\infty} a_n x^n$ Suppose that

$$\frac{|a_{n+1}|}{|a_n|} \to l \quad \text{as } n \to \infty$$

Then the radius of convergence

$$R = \begin{cases} 0 & \text{if } l = +\infty \\ \frac{1}{l} & \text{if } l \in \mathbb{R} \setminus \{0\} \\ \infty & \text{if } l = 0 \end{cases}$$

Proof: Suppose that the power series $\sum_{n=0}^{\infty} a_n x^n$ satisfies that

$$\frac{|a_{n+1}|}{|a_n|} \to l \quad \text{as} \quad n \to \infty$$

$$\lim_{n \to \infty} \frac{|a_{n+1} \cdot x^{n+1}|}{|a_n \cdot x^n|} = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} \cdot |x| = |x| \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|}$$

$$= |x| \cdot l$$

$$\left.\begin{array}{ll} \text{It converges} & |x|\cdot \ell < 1 \\ \text{It diverges} & |x|\cdot \ell > 1 \end{array}\right\} \text{according to the ratio test of series.}$$

① if $\ell \neq 0$, $R = \frac{1}{\ell}$ ② if $\ell = 0$ then $R = \infty$

③ $\ell = \infty$, we have $\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \infty$, then $R = 0$

E.x. $\sum\limits_{n=0}^{\infty} (-1)^n n^2 x^n$ $\sum\limits_{n=0}^{\infty} a_n x^n$ $a_n = (-1)^n n^2$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = 1$$

$\therefore$ Radius of convergence $R = \frac{1}{1} = 1$

$(-1,1) \subseteq S \subseteq [-1,1]$

$$\left.\begin{array}{lll} x=1 & \sum\limits_{n=0}^{\infty} (-1)^n n^2 & \lim\limits_{n\to\infty} (-1)^n n^2 = \pm\infty \\ x=-1 & \sum\limits_{n=0}^{\infty} n^2 & \lim\limits_{n\to\infty} n^2 = +\infty \end{array}\right\} \text{diverges}$$

Then the set of convergence is $S = (-1,1)$

E.g. $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$ $\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0$

The radius of convergence is ∞. so $S = \mathbb{R}$

Root test: $\lim_{k\to\infty} \sqrt[k]{a_k} < 1$, series converges.

Root test: Consider the power series $\sum_{n=0}^{\infty} a_n x^n$ suppose that

$$|a_n|^{\frac{1}{n}} \to l \quad \text{as} \quad n \to \infty$$

Then

$$R = \begin{cases} 0 & \text{if} \quad l = +\infty \\ \frac{1}{l} & \text{if} \quad l \in \mathbb{R} \setminus \{0\} \\ \infty & \text{if} \quad l = 0 \end{cases}$$

Proof: Suppose $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = l$

$$\lim_{n\to\infty} \sqrt[n]{a_n x^n} = \sqrt[n]{a_n} \sqrt[n]{|x^n|} = |a_n|^{\frac{1}{n}} \cdot |x| = l \cdot |x|$$

if $l \cdot |x| < 1$, it converges.

$l \cdot |x| > 1$, it diverges

Hence if $l \neq 0$ then $R = \frac{1}{l}$, If $l = 0$ then $R = +\infty$,

If $l = \infty$ then $R = 0$

Differentiability of power series

Thm. Suppose $\sum_{n=0}^{\infty} a_n x^n$ is a power series with radius of convergence R

then the function : $(-R, R) \to \mathbb{R}$ is given by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

is differentiable, and its derivative:

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

The exponential function

The exponential function : $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\quad a_n = \frac{1}{n!}$

when $n=0$ $\frac{x^0}{0!} := 1$

Properties :

① $\exp(0) = 1$

Proof: $\exp(0) = \sum_{n=0}^{\infty} \frac{0^n}{n!} = \frac{0^0}{0!} + \underbrace{\frac{0^1}{1!} + \frac{0^2}{2!} + \cdots}_{=0} = 1$

② For any $x, y \in \mathbb{R}$, $\exp(x+y) = \exp(x) \cdot \exp(y)$

$e^{x+y} = e^x \cdot e^y$

Proof $e^x \cdot e^y = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{y^n}{n!}\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!}$

$$e^{(x+y)} = \sum_{n=0}^{\infty} (x+y)^n / n!$$

$(a+b)^n = \sum_{k=0}^{n} C_k^n a^k b^{n-k}$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} C_k^n x^k y^{n-k} / n!$$

$C_k^n = \frac{n!}{k!(n-k)!}$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\cancel{n!}}{k!(n-k)!} x^k y^{n-k} / \cancel{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!}$$

Therefore $e^x \cdot e^y = e^{(x+y)}$

③ $\forall x \in \mathbb{R}$, $\exp(x) > 0$

Proof: $\exp(0) = 1$ $\quad \exp(x+y) = \exp(x) \cdot \exp(y)$

$x > 0$ $\quad \exp(x) > 0$

Let $y = -x$ $\quad \exp(y) = \frac{\exp(x+y)}{\exp(x)} = \frac{1}{\exp(x)} > 0$

$(x^n)' = nx^{n-1}$

④ $f'(x) = f(x) = \exp(x)$

Proof: $\frac{d}{dx}\exp(x) = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)' = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{\cancel{n}x^{n-1}}{\cancel{n}\cdot(n-1)!}$

$= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \exp(x)$

⑤ For any $x \in \mathbb{R}$, $\exp(-x) = \frac{1}{\exp(x)}$

Proof $\exp(x)\cdot\exp(-x) = \exp(x-x) = \exp(0) = 1$

⑥ $\exp(1) = e$

$\exp(1) = \sum_{k=0}^{\infty} \frac{1}{k!}$ $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$

Proof: We need to prove $\sum_{k=0}^{\infty} \frac{1}{k!} = \lim_{n\to\infty}(1+\frac{1}{n})^n$

$(1+\frac{1}{n})^n = \sum_{k=0}^{n} C_k^n (\frac{1}{n})^k \cdot 1^{n-k} = \sum_{k=0}^{n} C_k^n (\frac{1}{n})^k$

$= 1 + n\times\frac{1}{n} + \frac{n(n-1)}{2!}\times\frac{1}{n^2}$

$+ \frac{n(n-1)(n-2)}{3!}\times\frac{1}{n^3} \cdots + \frac{1}{n^n}$

binomial theorem

$(a+b)^n = \sum_{k=0}^{n} C_k^n a^k \cdot b^{n-k}$

$C_k^n = \frac{n!}{k!(n-k)!}$

$= 1 + 1 + \frac{1}{2!}(1-\frac{1}{n}) + \frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n})$

$\cdots \frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{k-1}{n}) + \cdots \frac{1}{n^n}$

Hence $\lim_{n\to\infty}(1+\frac{1}{n})^n = 1+1+\frac{1}{2!}+\frac{1}{3!}\cdots = \sum_{n=0}^{\infty}\frac{1}{n!}$