Sequences & Series: Exercise 1

Due date: 2025-11-25

Definition 1. A field is a set F endowed with two operations: one is called addition $+$ and one is called multiplication $\cdot$ (also denoted as $\times$), which satisfy the following axioms:

- Axioms for addition:
  - For any $x, y \in F$, $x + y \in F$.
  - Commutative: $x + y = y + x$.
  - Associative: $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.
  - F contains an element, called 0, such that $0 + x = x$ for all $x \in F$.
  - For each $x \in F$, there is an element $-x \in F$ such that $x + (-x) = 0$.
- Axioms for multiplication:
  - For any $x, y \in F$, $x \cdot y \in F$. (When there is no confusion, we write $x \cdot y$ as xy).
  - Commutative: $x \cdot y = y \cdot x$ for all $x, y \in F$.
  - Associative: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in F$.
  - F contains an element, called $1 \neq 0$, such that $1 \cdot x = x$ for all $x \in F$.
  - If $x \in F$ and $x \neq 0$, then there is an element $\frac{1}{x} \in F$ such that $x \cdot \frac{1}{x} = 1$.
- Distributive law: $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in F$.

Exercise 1. Use **Definition 1** to prove the following theorem:

Theorem 1. The axioms of addition imply the following for $x, y, z \in F$ where F is a field:

(a) If $x + y = x + z$, then $y = z$.

(b) If $x + y = x$, then $y = 0$. (So 0 is unique in F.)

(c) If $x + y = 0$, then $y = -x$. (So $-x$ is unique in F corresponding to x.)

(d) $-(-x) = x$.

Exercise 2. Use **Definition 1** to prove the following theorem:

Theorem 2. The axioms of multiplication imply the following for $x, y, z \in F$ where F is a field:

(a) If $x \neq 0$ and $x \cdot y = x \cdot z$, then $y = z$.

(b) If $x \neq 0$ and $x \cdot y = x$, then $y = 1$. (So 1 is unique in F.)

(c) If $x \neq 0$ and $x \cdot y = 1$, then $y = 1/x$. (So $1/x$ is unique in F corresponding to $x \neq 0$.)

(d) If $x \neq 0$, then $\frac{1}{1/x} = x$.

Exercise 3. Use **Definition 1** to prove the following theorem:

Theorem 3. The axioms of a field imply the following for $x, y, z \in F$ where F is a field:

(a) $0x = 0$.

(b) If $x \neq 0,\ y \neq 0$, then $xy \neq 0$.

(c) $(-x)y = -(xy) = x(-y)$.

(d) $(-x)(-y) = xy$.