Sequences & Series: Answer to Exercise 3

1. Judge the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. If it converges, find its sum.

   Answer:

   We can decompose the general term as $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

   The sum of the series is:

$$
\begin{aligned}
S_N &= \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) \\
&= 1 - \frac{1}{N+1}
\end{aligned}
$$

   As $N \to \infty$, $\lim_{N\to\infty} S_N = 1$. So the series converges and its sum is 1.

2. Judge the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (Hint: you can make use of the convergence conclusion of p-harmonic series).

   Answer:

   Series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$. By the convergence of p-series, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

3. Judge the convergence of the series$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ (Hint: you can make use of the convergence conclusion of p-harmonic series).

   Answer:

   Series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2} < 1$. By the divergence of p-series, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

4. Judge the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, including the situations of absolute convergence and conditional convergence.

   Answer:

   Series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ This is an alternating series. Let $b_n = \frac{1}{n}$. Then $b_{n+1} = \frac{1}{n+1} < \frac{1}{n} = b_n$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$

   By the Alternating Series Test, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

   For absolute convergence, consider the series $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$, which is the harmonic series and diverges. So the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent.

5. Judge the convergence of the series using the ratio test: $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

   Answer:

   We use the ratio test. Let $a_n = \frac{n}{2^n}$:

$$
\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}}\right| \\
&= \lim_{n\to\infty} \frac{n+1}{2n} \\
&= \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2n}\right) \\
&= \frac{1}{2} < 1
\end{aligned}
$$

   By the ratio test, the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges.