Power series : convergence test

Ratio test : $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \ell$ $\quad R = \begin{cases} 0 & \text{if } \ell = +\infty \\ \frac{1}{\ell} & \ell \in \mathbb{R}\setminus\{0\} \\ \infty & \text{if } \ell = 0 \end{cases}$

$$(-R, R) \subseteq S \subseteq [-R, R]$$

Root test: Consider the power series $\sum_{n=0}^{\infty} a_n x^n$, Suppose that $|a_n|^{\frac{1}{n}} \to \ell$ as $n \to +\infty$

Then

$$R = \begin{cases} 0 & \text{if } \ell = +\infty \\ \frac{1}{\ell} & \text{if } \ell = \mathbb{R}\setminus\{0\} \\ +\infty & \text{if } \ell = 0 \end{cases}$$

Proof: Suppose $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \ell$.

$$\lim_{n\to\infty} \sqrt[n]{a_n x^n} = |a_n|^{\frac{1}{n}} \sqrt[n]{x^n} = |a_n|^{\frac{1}{n}} \cdot |x_n| = \ell \cdot |x_n|$$

If $\ell \cdot |x| < 1$, the series $\sum_{n\geq 0}^{\infty} \sqrt[n]{a_n x^n}$ converges

⋮

Three important power series

① the exponential function
② Taylor series
③ Maclaurin series

$\lim_{n\to\infty} (1+\frac{1}{n})^n = e$

Defn: The exponential function is

$$\exp(x) = \sum_{n\geq 0}^{\infty} \frac{x^n}{n!}$$ $\quad a_n = \frac{1}{n!}$

when $n=0$, we have $\frac{x^0}{0!} := 1$

$e^{x+y} = e^x \cdot e^y$

Thm: (Properties of exponential function)

① $\exp(0) = 1$

Proof: $\exp(0) = \sum_{n=0}^{\infty} \frac{0^n}{n!} = \frac{0^0}{0!} + \frac{0^1}{1!} + \frac{0^2}{2!} + \cdots = 1$

② For any $x, y \in \mathbb{R}$, $\exp(x+y) = \exp(x) \cdot \exp(y)$

Proof: $\exp(x) \cdot \exp(y) = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{y^n}{n!}\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!}$

$$\exp(x+y) = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!}$$

$(a+b)^n = \sum_{k=0}^{n} C_k^n a^k \cdot b^{n-k}$

$$= \sum_{n=0}^{\infty} \frac{C_k^n x^k \cdot y^{n-k}}{n!}$$

$C_k^n = \frac{n!}{k!(n-k)!}$

$$= \sum_{n=0}^{\infty} \frac{\cancel{n!}}{k!(n-k)!} \frac{x^k \cdot y^{n-k}}{\cancel{n!}}$$

③ $\forall x \in \mathbb{R}$. $\exp(x) > 0$

Proof: $\exp(0) = 1$, $\exp(x+y) = \exp(x) \cdot \exp(y)$

Let $y = -x$, $x > 0$, then $\exp(x) > 0$ by definition.

$$\therefore \exp(y) = \frac{\exp(x+y)}{\exp(x)} = \frac{\exp(0)}{\exp(x)} = \frac{1}{\exp(x)} > 0$$

$(x^n)' = nx^{n-1}$

④ $f'(x) = f(x) = \exp(x)$

Proof: $\frac{d}{dx}\exp(x) = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)' = \sum_{n \geq 1}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{\cancel{n} x^{n-1}}{\cancel{n} \cdot (n-1)!}$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} = \exp(x)$$

First term is a constant!

⑤ $\forall x \in \mathbb{R}$, $\exp(-x) = \frac{1}{\exp(x)}$

Proof: See ③

$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

$C_k^n = \frac{n!}{k!\,(n-k)!}$

$C_2^n = \frac{n(n-1)\cdot\cancel{(n-2)!}}{2!\ \cancel{(n-2)!}}$

⑥ $\exp(1) = e$

Proof: Since $\exp(1) = \sum_{n=0}^{\infty}\frac{1}{n!}$

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} C_k^n\, 1^k \cdot \left(\frac{1}{n}\right)^{n-k} = \sum_{k=0}^{n} C_k^n \left(\frac{1}{n}\right)^k \cdot 1^{n-k} = \sum_{k=0}^{n} C_k^n \left(\frac{1}{n}\right)^k$$

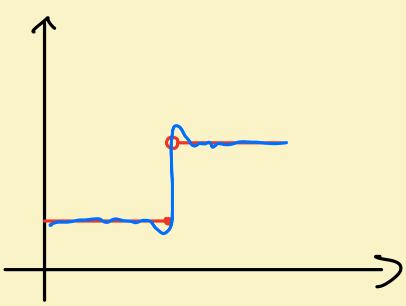
$$= 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\times\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\cdot\frac{1}{n^3}\ \cdots\ \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots\frac{1}{n!}+0 = \sum_{n=0}^{\infty}\frac{1}{n!}$$

$\exp(x) := \sum_{n=0}^{\infty}\frac{x^n}{n!}$

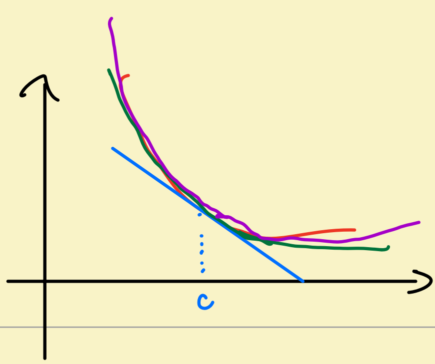
Taylor series



Defn: (Taylor series) If a function f has a power series representation on the interval $(c-R, c+R)$ then the power series is

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n \qquad a_n = \frac{f^{(n)}(c)}{n!}$$

which is called the Taylor series of the function f at point c.



Taylor series at point c:

$$f(x) = \cancel{\frac{f(c)}{0!}}\ f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots$$

$$f_1(x) = f(c) + f'(c)(x-c)$$

$$f_2(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2$$

Thm (Uniqueness of Taylor series) Suppose that

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

converges to a function $f(x)$. for all $x \in (c-R, c+R)$, with some $R \in (0, +\infty]$. Then

$$a_n = \frac{f^{(n)}(c)}{n!}$$

Proof: If $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$, then the derivative of f at c

$$f'(x) = \cancel{(a_0)'}^{\,0} + \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$$

$$f''(x) = \cancel{(a_1)''}^{\,0} + \cancel{(a_2(x-c))''}^{\,0} + \sum_{n=2}^{\infty} n(n-1) a_n (x-c)^{n-2}$$

$$\vdots$$

$$f^{(m)}(x) = \sum_{n=m}^{\infty} n(n-1)(n-2)\cdots(n-m)\, a_n (x-c)^{n-m}$$

$$= m!\, a_m + \sum_{n=m+1}^{\infty} n(n-1)(n-2)\cdots(n-m-1)\, a_n (x-c)^{n-m-1}$$

The above equation holds for $x \in (c-R, c+R)$. Let $x = c$,

$$f^{(m)}(c) = m!\, a_m \quad \Rightarrow \quad a_m = \frac{f^{(m)}(c)}{m!}$$

Maclaurin series: The Taylor series at $c = 0$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \qquad a_n = \frac{f^{(n)}(0)}{n!}$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

E.g. ① $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \frac{x^n}{n!} + \cdots$

$$\exp(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$\exp'(0) = e^0 = 1$

$\exp''(0) = e^0 = 1$

$\vdots$

holds for $(0-R, 0+R)$ with $R = +\infty$

② $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots = \sum_{n\geq 0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

Proof: $\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n=2k \\ (-1)^k & \text{if } n=2k+1 \end{cases} \Longleftarrow \begin{cases} f'(x) = \cos x & f'(0) = 1 \\ f''(x) = -\sin x & f''(0) = 0 \\ f^{(3)}(x) = -\cos x & f^{(3)}(0) = -1 \\ f^{(4)}(x) = \sin x & f^{(4)}(0) = 0 \end{cases}$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

③ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

Proof: $\cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n=2k-1 \\ (-1)^k & \text{if } n=2k \end{cases} \Longleftarrow \begin{cases} f'(x) = -\sin x & f'(0) = 0 \\ f''(x) = -\cos x & f''(0) = -1 \\ f'''(x) = \sin x & f'''(0) = 0 \\ f^{(4)}(x) = \cos x & f^{(4)}(0) = 1 \end{cases}$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

④ Convergence radius of $\sin x$ and $\cos x$

Ratio test + comparison test

Since $|f^{(n)}(0)| \leq 1$, we have $\left| f^{(n)}(0) \frac{x^n}{n!} \right| \leq \left| \frac{x^n}{n!} \right|$

$$\frac{\left| \frac{x^{n+1}}{(n+1)!} \right|}{\left| \frac{x^n}{n!} \right|} = \left| \frac{x}{n+1} \right| \to 0 \quad \text{as } n \to \infty$$

$R = +\infty$

⑤ $f(x) = e^x$ Taylor series at $c = 1$

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$ $\quad$ $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$e^x = e^{(x-1)+1} = e^{x-1} \cdot e = e\left(1 + (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3!} + \cdots\right)$

$= e \cdot 1$

E.x. Calculate Maclaurin series of $f(x) = e^{-3x} \cdot \cos(2x)$ as far as terms involving x^4

$\cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

$e^{-3x} = 1 + (-3x) + \frac{(-3x)^2}{2} + \frac{(-3x)^3}{3!} + \frac{(-3x)^4}{4!} \cdots$

$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \cdots$

$f(x) = \left(1 + (-3x) + \frac{(-3x)^2}{2} + \frac{(-3x)^3}{3!} + \frac{(-3x)^4}{4!}\right)\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!}\right)$

$= \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!}\right) + \left(-3x + 6x^2\right) + \left(\frac{9}{2}x^2 - 9x^4\right)$

$+ \frac{(-3x)^3}{3!} + \frac{(-3x)^4}{4!}$

E.x. $\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^4}$

$= \lim_{x \to 0} \frac{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + \cdots\right)}{x^4}$

$= \lim_{x \to 0} \frac{-\frac{1}{12}x^4}{x^4} = -\frac{1}{12}$