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Bolzano-Weierstrass Theorem: Every bounded sequence contains a convergent subsequence.

Cauchy Convergence Criterion

$\forall \varepsilon>0, \exists N \in \mathbb{N}_+$, if $n>N$ $|x_n - A| < \varepsilon$.

$\forall \varepsilon>0, \exists N \in \mathbb{N}_+$, if $n, m>N$ $|x_n - x_m| \le \varepsilon$. ← Cauchy sequence

Completeness 完备性 $S = \{x \in \mathbb{Q}, x^2 < 2\}$ $x = \sqrt{2}$

E.g. $a_1 = 1$ $a_{n+1} = \frac{1}{2}(a_n + \frac{2}{a_n})$ $\lim_{n\to\infty} a_n = \sqrt{2}$

$a_2 = \frac{3}{2}$

$a_3 = \frac{17}{12}$

① Prove boundedness

$a_{n+1} = \frac{1}{2}(a_n + \frac{2}{a_n}) \ge \sqrt{a_n \cdot \frac{2}{a_n}} = \sqrt{2}$ $\frac{x+y}{2} \ge \sqrt{xy}$

② Prove monotonically decrease

$a_{n+1} - a_n = \frac{1}{2}(a_n + \frac{2}{a_n}) - a_n = \frac{1}{2}(\frac{2}{a_n} - a_n) = \frac{2-a_n^2}{2a_n} < 0$

$2 - a_n^2 < 0$

$a_{n+1} = \frac{1}{2}(a_n + \frac{2}{a_n})$ $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n + \frac{2}{a_n})$

$\lim_{n\to\infty} a_{n+1} = L$ $L \in \mathbb{Q}$ $L = \frac{1}{2}(L + \frac{2}{L}) \Rightarrow L^2 = 2$

$L = \sqrt{2}$

级數

Series

The ordered sum of sequence

sequene 數列 a_n

Series 级數 $\sum_{k=1}^{\infty} a_n$

E.g. $\sum_{k=1}^{\infty} (-1)^k$ $e = \sum_{n=1}^{\infty} \frac{1}{n!}$

Defn (Convergence of Series) Let $\{a_n\}$ be a sequene, then we write

$$\sum_{k=1}^{\infty} a_k = c$$

any say the series converges.

Consider $s_n = \sum_{k=1}^{n} a_k$. If the sequence $\{s_n\}$ converges to c, then the series converges.

Thm (Uniqueness) If a series $\sum_{n=1}^{\infty} a_n$ converges, then the sum is unique

$s_m = \sum_{n=1}^{m} a_n$ $\lim_{m\to\infty} s_m = A$

Thm: If series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then the series

$$\sum_{n=1}^{\infty} (a_n \underline{+} b_n) = \sum_{n=1}^{\infty} a_n \underline{+} \sum_{n=1}^{\infty} b_n$$

Thm: If a series $\sum_{n\geq 1}^{\infty} a_n$ converges, then

$$\sum_{n=1}^{\infty} c \cdot a_n = c \sum_{n=1}^{\infty} a_n$$

is also converges.

Thm Let $M \geq 1$ be an integer. Then series $\sum_{n\geq 1}^{\infty} a_n$ converges if and only if

$$\sum_{k=1}^{\infty} a_{M+k} = \sum_{k=M+1}^{\infty} a_k$$ converges.

The tail's behavior determines the convergence/divergence of the series.

缩项级数

Telescoping series

$$\sum_{k=1}^{\infty} (a_k - a_{k+1})$$

E.g. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$

几何级数，等比级数

Geometric series

$$\sum_{k=1}^{\infty} r^{k-1} = \lim_{n\to\infty} \frac{1-r^n}{1-r} = \frac{1}{1-r} \qquad |r|<1$$

$\sum_{k=1}^{n} r^{k-1} = \frac{1-r^n}{1-r} \quad (r \neq 1)$ and it diverges when $|r| \geq 1$

调和级数

Harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \qquad \lim_{k\to\infty} \frac{1}{k} = 0$$

$$\sum_{k=1}^{\infty} \frac{1}{k} \iff \int_1^{+\infty} \frac{1}{k}\,dk = \lim_{b\to\infty} \ln b - \ln 1 = \infty$$

$$\sum_{k=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$

$$\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

p-Harmonic Series

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \qquad 0 < p < 1 \qquad p > 1$$

$0<p<1$: $\frac{1}{k^p} > \frac{1}{k}$ → diverges

$p>1$:

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} + \cdots$$

$$\leq 1 + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right) + \cdots$$

$$\leq 1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \cdots$$

$$\leq \frac{1}{1 - 2^{1-p}}$$

Alternating Harmonic Series 交替调和级数

$$\sum_{k=1}^{\infty} (-1)^{k-1}\frac{1}{k} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}\cdots$$

$$S_n = \sum_{k=1}^{n} (-1)^{k-1}\frac{1}{k} \qquad S_{2n-1} = \sum_{k=1}^{2n-1} (-1)^{k-1}\frac{1}{k} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}$$

$$S_{2n} = \sum_{k=1}^{2n} (-1)^{k-1}\frac{1}{k} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}$$

S_{2n-1} $\quad 1,\ 1\left(-\frac{1}{2}+\frac{1}{3}\right),\ 1\left(-\frac{1}{2}+\frac{1}{3}\right)\left(-\frac{1}{4}+\frac{1}{5}\right)$ $\downarrow$

S_{2n} $\quad 1-\frac{1}{2},\ 1-\frac{1}{2}+\left(\frac{1}{3}-\frac{1}{4}\right),\ 1-\frac{1}{2}+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)$ $\uparrow$

$$\frac{1}{2} \le S_{2n} \le S_{2n-1} \le 1$$

Necessary conditions of convergence.

Thm: If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$, an $n\to\infty$

Proof: $S_n = \sum_{k=1}^{n} a_k \qquad \lim_{n\to\infty} S_n = c \Rightarrow \lim_{n\to\infty} a_n = 0$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k\right) = \lim_{n\to\infty}(S_n - S_{n-1}) = c-c=0$$

Absolute convergence 绝对收敛

Defn: If the series $\sum_{k=1}^{\infty}|a_k|$ converges. then we say $\sum_{k=1}^{\infty} a_k$ is Absolutely convergent.

Thm: If the series $\sum_{k=1}^{\infty}|a_k|$ converges, then so does the series $\sum_{k=1}^{\infty} a_k$.

条件收敛

Conditionally convergent

Defn: A series $\sum_{k=1}^{\infty} a_k$ is said to be conditionally convergent if it converges, but the series $\sum_{k=1}^{\infty} |a_k|$ diverges.

E.g. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$

Convergence tests

Thm (Comparison tests). Given two series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, such that $0 \le a_k \le b_k$, for any k

If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges

If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

E.g. $\sum_{k=1}^{\infty} \frac{k+3}{k^3+2k^2+3k+1}$

$$\frac{k+3}{k^3+2k^2+3k+1} = \frac{1+\frac{3}{k}}{k^2\left(1+\frac{2}{k}+\frac{3}{k^2}+\frac{1}{k^3}\right)} \le \frac{4}{k^2 \cdot 1}$$

$\sum_{k=1}^{\infty} \frac{4}{k^2} = 4 \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, then the series is convergent.

E.g. $\sum_{k=1}^{\infty} \sqrt{\frac{k+4}{k^2+3k+1}}$

$$\sqrt{\frac{k+4}{k^2+3k+1}} = \sqrt{\frac{1+\frac{4}{k}}{k\left(1+\frac{3}{k}+\frac{1}{k^2}\right)}} \ge \sqrt{\frac{1}{k \cdot 5}}$$

$\frac{1}{\sqrt{5}} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges, then the series diverges.

Thm: (Ratio Tests) If $a_n > 0$ for any $n > 0$, and

$$\lim_{k \to \infty} \frac{a_{k+1}}{a_k} < 1$$

then the series $\sum_{k=1}^{\infty} a_k$ converges.