Set, ordered set, field → Axiom

Archimedian property: $\forall x, y \in \mathbb{R}$, $x > 0$, $\exists n \in \mathbb{Z}_+$ such that $nx > y$

Denseness $x, y \in \mathbb{R}$ $x < y$ $\Rightarrow \exists p \in \mathbb{Q}$ such that $x < p < y$

Sequence : continue without stopping.

Finite, infinite

Countable 可数的 : if a set is countable if its elements can be put into a one-to-one correspondence (bijection) with the set of natural numbers $\mathbb{N}$

Explicit formula : $a_n = 2n+1$

recursion formula : $x_n = 2 + x_{n-1}$

Defn: (ε-N definition) for any $\varepsilon > 0$, there exists a natural number N, such that, whenever $n \geq N$, $\Rightarrow |a_n - \ell| < \varepsilon$.

Notation: $\lim_{n\to\infty} a_n = \ell$

E.g. 1 Prove $\lim_{n\to\infty} \frac{1}{n^\alpha} = 0$, $\alpha > 0$

Proof: since $\left|\frac{1}{n^\alpha} - 0\right| = \frac{1}{n^\alpha}$, for any $\varepsilon > 0$, take $N = \left[\frac{1}{\varepsilon^{\frac{1}{\alpha}}}\right] + 1$

for any $n > N$, we have $\frac{1}{n^\alpha} < \frac{1}{N^\alpha} < \varepsilon$.

E.g. 2 Prove $\lim_{n\to\infty} \frac{3n^2}{n^2-3} = 3$

since $\left|\frac{3n^2}{n^2-3} - 3\right| = \frac{9}{n^2-3}$ suppose $n \geq 3$ $\frac{9}{n^2-3} \leq \frac{9}{n}$

for any $\varepsilon > 0$, take $N = \left[\frac{9}{\varepsilon}\right] + 1$

for any $n > N$, we have $\left|\frac{3n^2}{n^2-3} - 3\right| < \varepsilon$

E.g. 3 Prove $\lim_{n\to\infty} \frac{a^n}{n!} = 0$

Prove $a=0$. Obviously the equation holds.

$a \neq 0$. set $k = [|a|]+1$ $\neq n$

$$\left| \frac{a^n}{n!} - 0 \right| = \frac{|a^n|}{n!} = \frac{|a| \cdot |a| \cdot \cdots |a|}{1 \cdot 2 \cdot \cdots n} \leq K \frac{|a|}{n}$$

$K = \frac{|a| \cdot |a| \cdot \cdots |a|}{1 \cdot 2 \cdot \cdots k} > 1$, then for any $\varepsilon > 0$,

$N = \max\left\{k, \frac{K|a|}{\varepsilon}\right\}$, then for any $n > N$, $\left|\frac{a^n}{n!} - 0\right| \leq K\frac{|a|}{n} < \varepsilon$

① ε is arbitrary.

② N corresponds to ε. $n > N$, $n \geq N$

③ $\{a_n\} \in (\ell-\varepsilon, \ell) \cup (\ell, \ell+\varepsilon)$ for all $n > N$

Defn: We say that $\{a_n\}$ converges to ℓ, if there are at most N elements of $\{a_n\}$ are not in $(\ell-\varepsilon, \ell) \cup (\ell, \ell+\varepsilon)$.

E.g. 1 $\{n^2\}$ is divergent.

Take $\varepsilon = 1$. Suppose $\lim_{n\to\infty} n^2 = a$, but there are infinite number of elements of $\{n^2\}$ such that $|n^2 - a| > 1$. Hence, $\{n^2\}$ is divergent.

E.g. 2. $\{(-1)^n\}$ is divergent.

Take $\varepsilon = 1$. suppose $\lim_{n\to\infty} (-1)^n = 1$, for all $n = 2k+1$, $k \in \mathbb{N}_+$ we have $|(-1)^n - 1| = 2 > 1$, Hence, $\{(-1)^n\}$ is divergent.

E.g. 3. Let $\lim_{n\to\infty} x_n = a$, $\lim_{n\to\infty} y_n = b$, Construct $\{z_n\}$, as follows

$$x_1, y_1, x_2, y_2, x_3, y_3, \dots, x_n, y_n, \dots$$

Prove that $\{z_n\}$ is convergent if and only if $a = b$.

Proof: ($\Rightarrow$) Since $a = b$, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$, for any $\varepsilon > 0$, there are only finite number of elements are not in the neighbor of $(a-\varepsilon, a) \cup (a, a+\varepsilon)$

($\Leftarrow$) $\lim_{n\to\infty} z_n = A$ for any $\varepsilon > 0$, there are only finite number of elements that are not in the neighbor of $(A-\varepsilon, A) \cup (A, A+\varepsilon)$

$$a = \lim_{n\to\infty} x_n = A = \lim_{n\to\infty} y_n = b \quad \Rightarrow a = b.$$

无穷小数列

Defn: If $\lim_{n\to\infty} a_n = 0$, then we say a_n is an infinite decimal sequence.

Thm: The sequence a_n converges to A, if and only if $\{a_n - A\}$ is an infinite decimal sequence.

唯一性

Thm (Uniqueness) If $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} a_n = b$, then $a = b$.

Proof:

Take $\varepsilon > 0$, there exists N_1 such that $n > N_1 \Rightarrow |x_n - a| < \varepsilon$,

Take $\varepsilon > 0$, there exists N_2 such that $n > N_2 \Rightarrow |x_n - b| < \varepsilon$,

Take $N_3 = \max\{N_1, N_2\} \Rightarrow |x_n - a| < \varepsilon$, $|x_n - b| < \varepsilon$.

$$\Rightarrow |A - B| = |(x_n - A) - (x_n - B)|$$

$$\leq |x_n - A| + |x_n - B| < 2\varepsilon$$

Then we have $A = B$.

Thm (Bounded) Every convergent sequence is bounded. That is There exists $M > 0$, such that for any $n \in \mathbb{N}_+$, we have

$$|a_n| \leq M$$

Proof: Set $\lim_{n\to\infty} a_n = A$. Take $\varepsilon = 1$, there exists N such that $n>N$, $\Rightarrow$

$|a_n - A| < 1$, $\Rightarrow$ $-1 < a_n - A < 1$ $\Rightarrow$ $A-1 < a_n < A+1$

Take $M = \max\{|a_1|, |a_2| \dots |a_N|, |A-1|, |A+1|\}$, then

for any $n \in \mathbb{N}_+$, we have $\{a_n\} \le M$.

E.g. $\{(-1)^n\}$ is bounded but not convergent.

Thm. If $\lim_{n\to\infty} a_n = A > 0$, then there exists N, such that for any $n>N$, we have $a_n > 0$.

Proof: Suppose $a_n > A'$. take $\varepsilon = A - A'$, there exists N, such that $n>N$, we $|a_n - A| < \varepsilon = A - A'$ $\Rightarrow$ $a_n > A'$

Thm (Algebra 代数 of limits)

a) $\lim_{n\to\infty}(cx_n) = c\left(\lim_{n\to\infty} x_n\right)$

b) $\lim_{n\to\infty}(x_n \pm y_n) = \lim_{n\to\infty} x_n \pm \lim_{n\to\infty} y_n$

c) $\lim_{n\to\infty}(x_n y_n) = \left(\lim_{n\to\infty} x_n\right)\cdot\left(\lim_{n\to\infty} y_n\right)$

d) Suppose $y_n \neq 0$ $\lim_{n\to\infty} y_n \neq 0$, then $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$

e) Suppose $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = a$, and $x_n \le y_n \le z_n$ for any $n>N$

then we have $\lim_{n\to\infty} y_n = a$.

E.g. 1 $\lim_{n\to\infty} \sqrt[n]{n} = 1$

Set $\sqrt[n]{n} = 1 + x_n$ $\Rightarrow$ $n = (1+x_n)^n > \frac{n(n-1)}{2} x_n^2$

$\Rightarrow$ $0 < x_n < \sqrt{\frac{2}{n-1}}$

$1 < x_n + 1 < \sqrt{\frac{2}{n-1}} + 1$

$z_n = 1 \le \sqrt[n]{n} \le y_n = \sqrt{\frac{2}{n-1}} + 1$ $\lim_{n\to\infty} y_n = 1$

then we have $\lim_{n\to\infty} \sqrt[n]{n} = 1$

E.g. $\lim_{n\to\infty} \frac{x^n}{x^n+1}$ $\quad x \neq -1$

$|x|<1$ $\quad \lim_{n\to\infty} x^n = 0 \quad \lim_{n\to\infty} \frac{x^n}{x^n+1} = \frac{0}{0+1} = 0$

$|x|>1$ $\quad \lim_{n\to\infty} \frac{x^n}{x^n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{x^n}} = 1$

E.g. $\lim_{n\to\infty} \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right) = \lim_{n\to\infty} \frac{\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{\sqrt{n+1}+\sqrt{n}}$

$= \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{\frac{n+1}{n}}+1} = \frac{1}{2}$