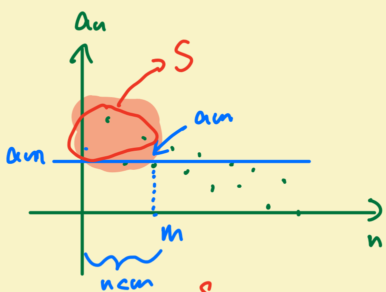
① Monotonicity: 4 types

② Monotone convergence theorem

Monotone sequence + bounded $\Leftrightarrow$ convergent sequence

③ Subsequence: (i) A sequence $\{x_{n_i}\}$, (ii) $\{n_i\}$ is strictly increasing



Thm: Every sequence $\{x_n\}$ contains a monotonic subsequence.

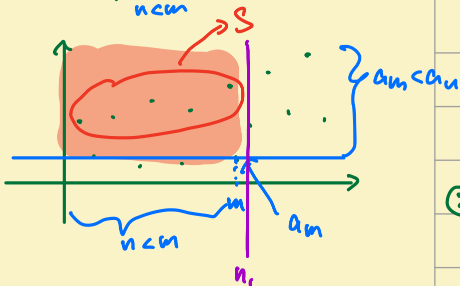
构造一个子终存在单调递减

子列的集合.

$\{(-1)^n\}$ $\{(-1)^n \cdot \frac{1}{n}\}$

Proof: consider the set $S = \{ n \in \mathbb{Z}^+ : \forall n < m,\ a_n \geq a_m \}$

① Suppose S is finite. Let $n_1 \in \mathbb{N}$ such that $n_1 > \max S$ then $n_1 \notin S$. By the definition of S, $n_2 > n_1 \Rightarrow a_{n_2} > a_{n_1}$. Since $n_2 \notin S$, $n_3 > n_2 \Rightarrow a_{n_3} \geq a_{n_2}$. Continue in this way, we obtain an increasing subsequence $\{a_{n_k}\}$.



② Suppose S is infinite. By the definition of S, we have $n_{k+1} > n_k \Rightarrow a_{n_{k+1}} < a_{n_k} \in S$. then $\{a_{n_k}\}$ is a decreasing subsequence.

如果说 $\{x_n\}$ 收敛, 则

$\lim_{n\to\infty} x_n = A \quad |A| \neq +\infty$

存在.

$\lim_{n\to\infty} x_n = \pm\infty$ 不是收敛!

Thm: If $\{x_n\}$ converges to A, then every subsequence of $\{x_n\}$ converges to A.

Proof: Since $\{x_n\}$ converges to A, then there exists $N \in \mathbb{N}$ such that, for any $n \geq N$ and $\varepsilon > 0$, we have

$$|x_n - A| < \varepsilon .$$

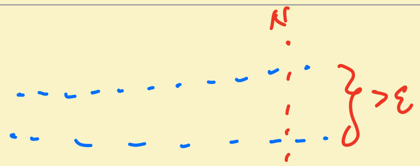
Let $\{x_{n_k}\}$ be an arbitrary subsequence of $\{x_n\}$. If $k \geq N$ then $n_k \geq N$

$$|x_{n_k} - A| < \varepsilon . \quad \forall k \geq N$$

By the definition of a convergent sequence, the subsequence converges.

$n \uparrow \quad n_k \uparrow$

n	1	2	3	4	5	6	7	8	9
n_k	1	2		4		6		8	
	n_1	n_2		n_3		n_4		n_5	



Corollary: If a sequence $\{x_n\}$ possesses two subsequences which converges to distinct limits, then $\{x_n\}$ does not converge.

Thm (Bolzano - Weierstrass theorem) Every bounded sequence contains a convergent subsequence.

$\{(-1)^n\}$

Proof: Since every sequence contains a monotonic subsequence, and this subsequence is bounded. By the monotone convergence theorem, the subsequence converges.

Bolzano-Cauchy convergence criterion

Comment: By B-C convergence criterion, we can show that a sequence converges without knowing the value of the limit in advance.

Thm (Bolzano - Cauchy) A sequence $\{x_n\} \subset \mathbb{R}$ converges if and only if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon \qquad \forall\ n \geq N \text{ and } m \geq N.$$

Note: A sequence $\{x_n\}$ satisfying the condition above is called a Cauchy sequence

Proof: ($\Rightarrow$) Given any $\varepsilon > 0$, if $\{x_n\}$ converges to A, then there exists $N \in \mathbb{N}$, such that

$$|x_n - A| < \frac{\varepsilon}{2} \qquad \forall n > N$$

B-W thm:

subsequence + bounded

$\Updownarrow$

subsequence convergence

Now for each $n, m \geq N$,

$$|x_n - x_m| = |(x_n - A) - (x_m - A)| \leq |x_n - A| + |x_m - A| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

then $\{x_n\}$ is a Cauchy sequence.

($\Leftarrow$) If $\{x_n\}$ is a Cauchy sequence, Given any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|x_n - x_N| < \varepsilon \qquad \forall\, n > N$$

then we have $x_N - \varepsilon < x_n < x_N + \varepsilon$, $\forall n > N$. As a result, $\forall n \geq 1$

$$\min\{x_{N} - \varepsilon, x_1, x_2, \cdots x_{N-1}\} \leq x_n \leq \max\{x_{N+\varepsilon}, x_1, x_2, \cdots x_{N-1}\}$$

then $\{x_n\}$ is bounded. By B-W theorem, there exists a subsequence $\{x_{n_k}\}$ which converges to A. By the definition, there exists $K \in \mathbb{N}$ such that for any $\varepsilon > 0$,

$$|x_{n_k} - A| < \frac{\varepsilon}{2} \qquad \forall\, k \geq K \; (n_k > n_K)$$

Take $n_k \geq \max\{n_K, N\}$ we have

$$|x_n - A| = |x_n - x_{n_k} + x_{n_k} - A|$$

$$\leq |x_n - x_{n_k}| + |x_{n_k} - A| < |x_n - x_{n_k}| + \frac{\varepsilon}{2} < \varepsilon + \frac{\varepsilon}{2}$$

$\forall n \geq N$

Then $\{x_n\}$ converges.

Comment: The Cauchy sequence converges only if it is on a complete space. For example, the real space.

Counter example: $S = \{x \in \mathbb{Q}, x^2 < \}$

Given a sequence $a_{n+1} = \frac{1}{2}(a_n + \frac{2}{a_n})$ $\qquad \lim_{n \to \infty} a_n = \sqrt{2}$

① Prove boundedness: $a_{n+1} = \frac{1}{2}(a_n + \frac{2}{a_n}) \geq \sqrt{a_n \cdot \frac{2}{a_n}} = \sqrt{2}$

$\frac{x+y}{2} \geq \sqrt{xy}$

② Prove monotonicity: $a_{n+1} - a_n = \frac{1}{2}(a_n + \frac{2}{a_n}) - a_n$

$$= \frac{1}{2}\left(\frac{2}{a_n} - a_n\right) = \frac{2 - a_n^2}{2a_n} < \frac{2 - \sqrt{2}^2}{2a_n} = 0$$

$\therefore\ a_{n+1} - a_n < 0 \quad \forall n \geq 1$. Hence $\{a_n\}$ converges.

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = A \qquad \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$$

$$\Rightarrow A = \frac{1}{2}\left(A + \frac{2}{A}\right)$$

$$\Rightarrow A = \sqrt{2}$$

Series (级数)

The ordered sum of sequence

sequence a_n

series $\sum_{k\geq1}^{\infty} a_n$

E.g.: $\sum_{k=1}^{\infty} (-1)^k$ not converge $\qquad \sum_{n\geq1}^{\infty} \frac{1}{n!} \approx e$ converges.

Defn: (Convergence of series) Let $\{x_n\}$ be a sequence, then if we can write

$$\sum_{n\geq1}^{\infty} x_n = c \qquad |c| \neq +\infty$$

then we say the series $\sum_{n=1}^{\infty} x_n$ converges.