Sequences & Series: Exercise 2

1. Given the sequence $a_n = \frac{n}{n+1}$, determine whether the sequence converges. If it does, find its limit.

   Answer:

   $$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n}{n+1} \\ &= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} \\ &= 1\end{aligned}$$

2. Given the sequence $a_n = \left(1+\frac{1}{n}\right)^n$, determine whether the sequence converges. If it does, find its limit.

   Answer:

   The limit of this sequence is e. See the note for detailed proof.

3. Given the sequence $a_n = \frac{\sin n}{n}$, determine whether the sequence converges. If it does, find its limit.

   Answer:

   Sequence $a_n = \frac{\sin n}{n}$ We know that $-1 \leqslant \sin n \leqslant 1$. So, $\left|\frac{\sin n}{n}\right| \leqslant \frac{1}{n}$. Since $\lim_{n\to\infty} \frac{1}{n} = 0$, by the Squeeze Theorem, $\lim_{n\to\infty} \frac{\sin n}{n} = 0$. So the sequence converges to 0.

4. Given the sequence $a_n = \frac{(-1)^n n}{n+1}$, determine whether the sequence converges. If it does, whether it is absolutely convergent or conditionally convergent.

   Answer:

   We consider the subsequences of even and odd terms:

   For even terms ($n = 2k, k \in \mathbb{N}$), $a_{2k} = \frac{2k}{2k+1} \to 1$ as $k \to \infty$

   For odd terms ($n = 2k+1, k \in \mathbb{N}$), $a_{2k+1} = \frac{-(2k+1)}{2k+2} \to -1$ as $k \to \infty$

   Since the subsequences converge to different limits, the sequence $\{a_n\}$ does not converge.

5. Given the sequence $a_n = \sqrt{n+1} - \sqrt{n}$, determine whether the sequence converges. If it does, find its limit.

   Answer:

   We rationalize the expression:

$$\begin{aligned} a_n &= \sqrt{n+1} - \sqrt{n} \\ &= \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{1}{\sqrt{n+1} + \sqrt{n}} \end{aligned}$$

   As $n \to \infty$, $\lim_{n\to\infty} a_n = 0$. So the sequence converges to 0.

6. Judge the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. If it converges, find its sum.

   Answer:

   We can decompose the general term as $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

   The sum of the series is:

$$\begin{aligned} S_N &= \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) \\ &= 1 - \frac{1}{N+1} \end{aligned}$$

   As $N \to \infty$, $\lim_{N\to\infty} S_N = 1$. So the series converges and its sum is 1.

7. Judge the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (Hint: you can make use of the convergence conclusion of p-harmonic series).

   Answer:

   Series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$. By the convergence of p-series, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

8. Judge the convergence of the series$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ (Hint: you can make use of the convergence conclusion of p-harmonic series).

   Answer:

   Series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2} < 1$. By the divergence of p-series, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

9. Judge the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, including the situations of absolute convergence and conditional convergence.

   Answer:

   Series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ This is an alternating series. Let $b_n = \frac{1}{n}$. Then $b_{n+1} = \frac{1}{n+1} < \frac{1}{n} = b_n$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$

   By the Alternating Series Test, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

   For absolute convergence, consider the series $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$, which is the harmonic series and diverges. So the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent.

10. Judge the convergence of the series using the ratio test: $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

    Answer:

    We use the ratio test. Let $a_n = \frac{n}{2^n}$:

$$
\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}}\right| \\
&= \lim_{n\to\infty} \frac{n+1}{2n} \\
&= \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2n}\right) \\
&= \frac{1}{2} < 1
\end{aligned}
$$

    By the ratio test, the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges.