Sequence → a set on the real field
→ a list without stopping
→ a function : $\mathbb{N}_+ \to \mathbb{R}$

Concept : finite, infinite

countable : if a set is countable if its element can be put into a one-to-one correspondence with the set of natural numbers $\mathbb{N}$

E.g. 1, 3, 5, 7, 9

$a_n = f(n) = 2n-1$ → Formula of n

$a_1 = 1$, $a_n = a_{n-1} + 2$ → Recursion formula

E.g. Arithmetic progression 等差数列.

c, $c+d$, $c+2d$ ...

Formula of n : $a_n = c + (n-1)d$

Recursion formula : $a_n = a_{n-1} + d$ $a_1 = c$

E.g: Geometric progression 等比数列.

c, $c \cdot r$, $c \cdot r^2$, ...

Formula of n : $a_n = c \cdot r^{n-1}$

Recursion formula: $a_n = r \cdot a_{n-1}$ $a_1 = c$

Study objective : convergence, bounded.

ε-δ definition

$\lim_{n\to\infty} a_n$

Defn: (ε-N definition) We define $\{a_n\}$ converges to A if for any $\varepsilon > 0$, there exists a natural number $N > 0$, such that whenever $n \geq N$. $|a_n - A| < \varepsilon$

Notation: $\lim_{n\to\infty} a_n = A$

E.g.: Prove $\lim\limits_{n\to\infty} \frac{1}{n^\alpha} = 0$, $\alpha > 0$

$\left|\frac{1}{n^\alpha} - 0\right| < \varepsilon \quad \forall \varepsilon > 0$

$\Rightarrow \frac{1}{n^\alpha} < \varepsilon \Rightarrow n > \frac{1}{\varepsilon^{\frac{1}{\alpha}}}$

$\Updownarrow$

$n > N$

Proof: Since $\left|\frac{1}{n^\alpha} - 0\right| = \frac{1}{n^\alpha}$. for any $\varepsilon > 0$, take

$$N = \left[\frac{1}{\varepsilon^{\frac{1}{\alpha}}}\right] + 1$$

for any $n > N$, we have $\frac{1}{n^\alpha} < \frac{1}{N^\alpha} < \varepsilon$

E.g.: Prove $\lim\limits_{n\to\infty} \frac{3n^2}{n^2 - 3} = 3$

$\left|\frac{3n^2}{n^2-3} - 3\right| = \left|\frac{9}{n^2-3}\right| < \varepsilon$

$\left|\frac{9}{n^2-3}\right| \leq \frac{9}{n} < \varepsilon$

Proof: Since $\left|\frac{3n^2}{n^2-3} - 3\right| = \left|\frac{9}{n^2-3}\right|$ suppose $n \geq 3$ then

$\frac{9}{n^2-3} \leq \frac{9}{n}$. for any $\varepsilon > 0$, take $N = \left[\frac{9}{\varepsilon}\right] + 1$

for any $n > N$, we have $\left|\frac{3n^2}{n^2-3} - 3\right| < \varepsilon$.

E.g.: Prove $\lim\limits_{n\to\infty} \frac{a^n}{n!} = 0$

Proof: If $a = 0$. obviously the equation holds.

If $a \neq 0$. since $\left|\frac{a^n}{n!} - 0\right| = \frac{|a|^n}{n!} = \frac{|a|}{1} \cdot \frac{|a|}{2} \cdot \frac{|a|}{3} \cdots \frac{|a|}{n} \leq K \frac{|a|}{n}$

set $k = [|a|] + 1$, then $K = \frac{|a|}{1} \cdot \frac{|a|}{2} \cdot \frac{|a|}{3} \cdots \frac{|a|}{k} \geq 1$

$$\frac{|a|}{1} \cdot \frac{|a|}{2} \cdot \frac{|a|}{3} \cdots \frac{|a|}{n} = \underbrace{\frac{|a|}{1} \cdot \frac{|a|}{2} \cdots \frac{|a|}{k}}_{>1} \cdot \underbrace{\frac{|a|}{k+1} \cdots \frac{|a|}{n-1}}_{<1} \cdot \frac{|a|}{n} \leq K \frac{|a|}{n}$$

$K \frac{|a|}{n} < \varepsilon$

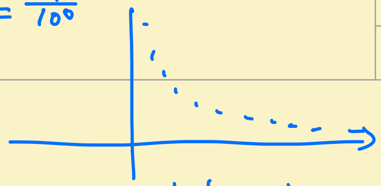
$\Rightarrow n > \frac{K|a|}{\varepsilon}$

for any $\varepsilon > 0$, take $N = \max\left\{k, \frac{K|a|}{\varepsilon}\right\}$

then for any $n > N$, $\left|\frac{a^n}{n!} - 0\right| < \varepsilon$.

$a_n = \frac{1}{n} \to 0$

$\varepsilon = \frac{1}{100}$



$\left|\frac{1}{n} - 0\right| < \varepsilon$

Remark: ① ε is arbitrary.

② N corresponds to ε.

③ $\{a_n\} \subset (A-\varepsilon, A) \cup (A, A+\varepsilon)$, $\forall n > N$

Defn: We say $\{a_n\}$ converges to A, if there are at most N elements of $\{a_n\}$ are not in $(A-\varepsilon, A)\cup(A, A+\varepsilon)$

E.g: Prove $\lim_{n\to\infty} n^2$ is divergent

Proof: take $\varepsilon=1$ Suppose $\lim_{n\to\infty} n^2 = A$, but there exist infinite number of elements of $\{n^2\}$ such that $|n^2-A|>\varepsilon=1$. Hence $\{n^2\}$ is not convergent.

E.g.: Prove $\lim_{n\to\infty}(-1)^n$ is divergent.

Proof: take $\varepsilon=\frac{1}{2}$. suppose $\lim_{n\to\infty}(-1)^n=1$, but for all $n=2k+1$, $k\in\mathbb{N}$, we have $|(-1)^n-1|=2>\varepsilon=\frac{1}{2}$. Hence, $\{(-1)^n\}$ diverges.

Defn: (Divergence to ∞) We say $\{a_n\}$ diverges to ∞ $-\infty$ if for any positive number M negative, there exists a natural number N, such that for all $n\geq N$, we have $a_n>M$ $a_n<M$

E.g.: Prove $\frac{n^2+1}{n+1}\to\infty$ as $n\to\infty$.

Proof: Let M be any positive number. Take $N=[M]+2$. then for any $n\geq N$, we have

$$\frac{n^2+1}{n+1}\geq\frac{n^2-1}{n+1}=n-1\geq N-1=[M]+1>M.$$

① 唯一性

② 有界性

③ 保号性.

Properties of convergent sequences

Thm (Uniqueness) If

$$\lim_{n\to\infty}a_n=A \quad \text{and} \quad \lim_{n\to\infty}a_n=B$$

then $A=B$.

Proof: Fix any $\varepsilon>0$. Then by definition. we can find $N_1\in\mathbb{N}$ such that for any $n>N_1$, we have

$$|a_n-A|<\frac{\varepsilon}{2}$$

By definition, we can find $N_2 \in \mathbb{N}$ such that for any $n > N_2$, we have
$$|a_n - B| < \frac{\varepsilon}{2}$$
Take $N = \max\{N_1, N_2\}$, then
$$|A-B| = |(a_n - A) - (a_n - B)| \le |a_n - A| + |a_n - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
Hence $A = B$.

Defn (Boundedness) A sequence $\{a_n\}$ is bounded, if there is a real number $M > 0$ such that
$$|a_n| < M \quad \forall n \in \mathbb{N}$$

Thm (Boundedness) Every convergent sequence is bounded.

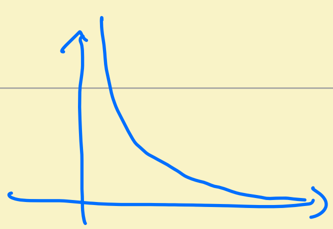
Proof: Suppose that $a_n \to A$ as $n \to \infty$. Then there exists $N \in \mathbb{N}$ such that, for any $\varepsilon > 0$,
$$|a_n - A| < \varepsilon \qquad \text{whenever } n > N$$
Since
$$|a_n| = |a_n - A + A| \le |a_n - A| + |A| \le |A| + \varepsilon$$
for any $n \ge N$. Now we define
$$M := \max\{|a_1|, |a_2|, \dots, |a_{N-1}|, |A| + \varepsilon\}$$
Then $|a_n| \le M$ for any $n \in \mathbb{N}$.

Corollary：推论

Corollary: An unbounded sequence is divergent.

E.g. $\{(-1)^n\}$ is bounded but not convergent.

harmonic series
调和级数



E.g. Show that $a_n = \sum_{k=1}^{n} \frac{1}{k}$ diverges.

Proof:
$$a_n = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \dots \frac{1}{8}\right) + \dots + \left(\frac{1}{2^{m-1}+1} + \dots \frac{1}{2^m}\right) + \dots$$
$$\ge 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \dots + \frac{1}{8}\right) + \dots \left(\frac{1}{2^m} + \dots \frac{1}{2^m}\right) + \dots$$

$$= 1 + \frac{1}{2} + 2\cdot\frac{1}{4} + 4\cdot\frac{1}{8} + \cdots + 2^{m-1}\cdot\frac{1}{2^m} + \cdots$$

$$= 1 + \underbrace{\frac{1}{2} + \frac{1}{2} \cdots \frac{1}{2}}_{m} + \cdots$$

$$= 1 + \frac{m-1}{2}$$

As $n \to \infty$, we have $m \to \infty$, then

$$a_n \to \infty$$

The sequence $\{a_n\}$ is unbounded and therefore divergent.

Thm: If $\lim\limits_{n\to\infty} a_n = A > 0$, then there exists $N \in \mathbb{N}$, such that for any $n > N$, we have $a_n > 0$.

Proof: Suppose $a_n > A' > 0$, take $\varepsilon = A - A'$, since a_n is convergent, there exists $N \in \mathbb{N}$, such that $n > N$, we have

$$|a_n - A| < \varepsilon = A - A' \quad \Rightarrow a_n > 0$$

Thm (Arithmetic operations) Suppose $\{a_n\}$ and $\{b_n\}$ are convergent, and $c \in \mathbb{R}$, then

① $\lim\limits_{n\to\infty} (ca_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$

② $\lim\limits_{n\to\infty} (a_n \cdot b_n) = \left(\lim\limits_{n\to\infty} a_n\right) \cdot \left(\lim\limits_{n\to\infty} b_n\right)$ in particular $\lim\limits_{n\to\infty} ca_n = c \cdot \lim\limits_{n\to\infty} a_n$

③ $\lim\limits_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$ if $b_n \neq 0 \ \forall n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} b_n \neq 0$.