Sequence and Series

GitHub.com/styluck/sas　　　80% exam　　20% homeworks

基础：集合，排序，实数的完备性。

☆ 序列：ε-N 极限定义 → 有界性，极限定律，
单调序列、Weierstrass定理，柯西收敛

级数：部分和序列收敛。

幂级数：收敛半径，→ 指数函数，Taylor/Maclaurin展开。

Definition → 　**Defn:** A set is a collection of objects. These objects are called elements or members.

element：元素。

A set with no objects is called an empty set, denoted ϕ

Defn: For a set A, if x is an element of A, we write

$\in$: in

$$x \in A$$

if x is not an element of A, we write

$$x \notin A$$

The properties of a set

① Well-definedness (确定性，良定义性)

either $x \in A$ or $x \notin A$

② Distinct elements (互异性)

{ } : curely brackets

e.g. $A = \{1, 6, 3\}$ 　 $B = \{1, 3, 6, 6\} = \{1, 3, 6\}$

③ Unordered (无序性)

$A = \{1, 6, 3\}$ 　 $B = \{1, 3, 6\}$ 　 $A = B$

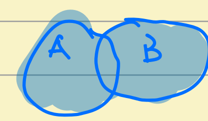
Set-builder notation: $\{x : x \text{ has property } P\}$

$\{x \mid x \text{ has property } P\}$

e.g: $\{x : x < 2,\ x \in \mathbb{Z}\}$ $\{x \in \mathbb{Z} : x < 2\}$.

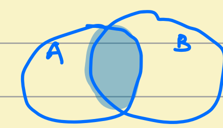
Defn: Set operation:

① Union (并集)



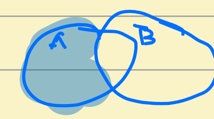
$$A \cup B := \{x : x \in A \text{ or } x \in B\}$$

② Intersection (交集)



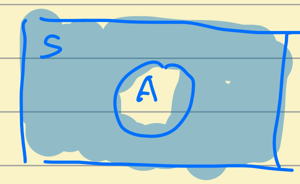
$$A \cap B := \{x : x \in A \text{ and } x \in B\}$$

③ Set difference (集合减法)



$$A \setminus B := \{x : x \in A \text{ and } x \notin B\}$$

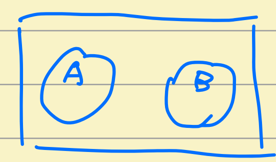
④ Complement (补集)



$$A^c := \{x \in S : x \notin A\}$$

$A^c = S \setminus A$

Disjoint: $A \cap B = \phi$



Number Systems

① Natural Numbers $\mathbb{N}$ $\{0, 1, 2, 3, \cdots\}$

② Integers $\mathbb{Z}$ $\{\cdots -3, -2, -1, 0, 1, 2, 3 \cdots\}$

Ratio 比例数

③ Rational Numbers $\mathbb{Q}$ $\{\frac{m}{n} : m, n \in \mathbb{Z},\ n \neq 0\}$

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$$

$\subset$ subset (真子集)

$\subseteq$ subset equal (子集)

Defn: If A is the subset of B, then

$$\forall x \in A \Rightarrow x \in B$$

Fact: $\phi \subseteq A$

$(\mathbb{Q}, >)$

Ordered set (有序集)

$\prec$: precede

Defn: Let A be a set. An order on A is a relation, denote by $\prec$, with the following properties:

Ordered ① (有序性) For any $x, y \in A$, one and only one of the following holds:

$$x \prec y, \quad x = y, \quad y \prec x$$

Transitivity ② (传递性) For any $x, y, z \in A$, if $x \prec y$, and $y \prec z$, then $x \prec z$.

E.g. $(\mathbb{Q}, <)$ ordered ① for any $x, y \in \mathbb{Q}$ $x<y$, or $x=y$, $y<x$

transitivity ② for any $x, y, z \in \mathbb{Q}$, if $x<y$, $y<z$, $\Rightarrow x<z$

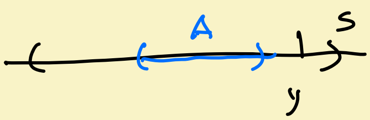
E.g. $(\mathbb{Q}, >)$ ordered ① for any $x, y \in \mathbb{Q}$ $x>y$, or $x=y$, $y>x$

transitivity ② for any $x, y, z \in \mathbb{Q}$, if $x>y$, $y>z$, $\Rightarrow x>z$

E.g. $(\mathbb{Q}, \geq)$ ① $x, y \in \mathbb{Q}$ $x \geq y$ and $x=y$ and $y \geq x$

↳ not an ordered set.

Defn: An ordered set is a set that is endow with an order "$\prec$", denoted by $(S, \prec)$



Defn: (Upper bound, 上界) Consider an ordered set S and $A \subset S$, $A \neq \emptyset$. There exists an element $y \in S$, such that for any $x \in A$, we have $x \preceq y$, then y is called the upper bound of A.

E.g. $S = (\mathbb{Z}, <)$ $A = \{0, 1, 2\} \subset S$, 3 is an upper bound,

4 2

Defn: (least upper bound, 最小上界) Consider an ordered set S and $A \subset S$, $A \neq \emptyset$, and A is bounded above. Suppose there exists $\alpha \in S$, such that:

① α is an upper bound of A.

② For any $\beta \in S$, if $\beta < \alpha$, then β is not an upper bound of A.

上确界.

Then α is called the least upper bound of A, or the supremum

$$\alpha = \sup A$$

Fact: If supremum of an ordered set A exists, then it is unique

That is $\alpha = \sup A$, $\beta = \sup A$, then $\alpha = \beta$.

E.g. $S = (\mathbb{Q}, <)$, $A = \{0, 1, 2\}$, Then its upper bound $\{x \in \mathbb{Q} : x \geq 2\}$

its supremum is 2

E.g. $S = (\mathbb{Q}, <)$, $A = \{x \in \mathbb{Q},\ x > 0,\ x^2 < 2\}$.

Claim: there is no supremum $\alpha = \sup A$ such that $\alpha \in \mathbb{Q}$.

Lemma: There is no $x \in \mathbb{Q}$, satisfying $x^2 = 2$.

Proof: Suppose $x \in \mathbb{Q}$ such that $x^2 = 2$. $x = \frac{m}{n}$ with $m, n \in \mathbb{Z}$, and m and n coprime (互质的)

$x = \frac{m}{n}$, $x^2 = 2 \Rightarrow \left(\frac{m}{n}\right)^2 = 2 \Rightarrow m^2 = 2n^2$

m^2 is divisible by $2 \Rightarrow m$ is divisible by 2

$\Rightarrow$ Write $m = 2k$, $k \in \mathbb{Z}$

$\Rightarrow (2k)^2 = 2n^2 \Rightarrow 2k^2 = n^2 \Rightarrow m, n$ are not coprime

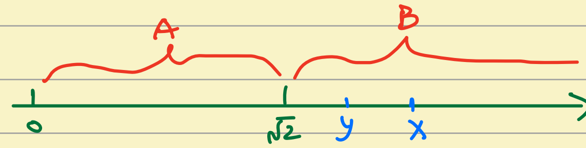
$\Rightarrow$ contradiction !

B：A在Q中全部上界的集合（因为$\sqrt{2}\notin\mathbb{Q}$）

对于任意$x\in B$，都能找到$y<x$且$y\in B$，所以B无最小元，

$\Rightarrow$ A无上确界。

$B=\{x\in\mathbb{Q},\ x>0,\ x^2>2\}$



Fix any $x\in B$, then x is an upper bound of A

Construct $y=x-\alpha(x^2-2)$, $\alpha\in\mathbb{Q}$, then $y\in\mathbb{Q}$

To prove $y\in B$, that is $y>0$, $y^2>2$

Take $\alpha=\frac{1}{x+2}$ $\qquad x-y=\frac{1}{x+2}(x^2-2)>0$

$$y=x-\alpha(x^2-2)=x-\frac{x^2-2}{x+2}=\frac{x^2+2x-x^2+2}{x+2}=\frac{2x+2}{x+2}>0$$

$$y^2=(x-\alpha(x^2-2))^2=\frac{[2(x+1)]^2}{(x+2)^2}>2 \iff 4(x+1)^2>2(x+2)^2$$

$$\iff 4(x^2+2x+1)>2(x^2+4x+4)$$

$$x^2>2$$

$y>0,\ y^2>2 \Rightarrow y\in B$

Defn: (least upper bound property). An ordered set S is said to have the least upper bound property, if any nonempty subset A bounded above has a least upper bound in S.

E.g. $\mathbb{Q}$ has no least upper bound property.

Defn: (Lower bound) Consider an ordered set S and $A\subset S$, $A\neq\emptyset$. There exists an element $y\in S$, such that for any $x\in A$, we have $y\le x$, then y is called the lower bound of A.

Defn: (Greatest lower bound) Consider an ordered set S and $A \subset S$, $A \neq \emptyset$, and A is bounded below. Suppose there exists $\alpha \in S$, such that:

① α is a lower bound of A,

② For any $\beta \in S$, if $\alpha < \beta$, then β is not a lower bound of

下确界

Then α is called the greatest lower bound of A, or the infimum

$$\beta = \inf A$$

Fact: The infimum is unique.