$a \in S$ or $a \notin S$

Sets : S ① Well-definedness ② Distinct Elements ③ Unordered

↓

rule

Ordered set : $(S, <)$ Order : ① One and only one of : $x, y \in S$

$x < y$ or $y < x$ or $y = x$

② transitivity : $x, y, z \in S$, $x < y$, $y < z$

$\Rightarrow x < z$

↓

Field

axiom : 公理

Defn (Field, 域) A field F is a set endowed with two operations:

① Addition : "+"

② multiplication "×"

which satisfy the following axioms

Axioms for addition (加法公理)

① For any $x, y \in F \Rightarrow x + y \in F$ (运算封闭)

② Commutative (交换律) $x + y = y + x$ $\forall x, y \in F$

③ Associative (结合律) $(x+y)+z = x + (y+z)$ $\forall x, y, z \in F$

④ Contains 0, that is $x \in F$, $x + 0 = x \in F$, $\forall x \in F$

⑤ For each $x \in F$, there exists an element $-x \in F$, $x + (-x) = 0$

$x - y = x + (-y)$

Axioms for multiplication (乘法公理)

① For any $x, y \in F \Rightarrow x \cdot y \in F$ (运算封闭)

② Commutative (交换律) $x \cdot y = y \cdot x$ $\forall x, y \in F$

③ Associative (结合律) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ $\forall x, y, z \in F$

④ Contains 1, that is $x \cdot 1 = x \in F$, $\forall x \in F$

⑤ If $x \in F$, $x \neq 0$, there exists an element $\frac{1}{x} \in F$, $x \cdot \frac{1}{x} = 1$

$x / y = x \cdot \frac{1}{y}$

Consequence 1: Distribution law (分配律)

$x \cdot (y+z) = x \cdot y + x \cdot z$ $\forall x, y, z \in F$

Notational convention: $x+(-y) =: x-y$ $\quad x \cdot x =: x^2 \cdots$

$x \cdot \frac{1}{y} =: x/y$

Thm (Consequences of axioms of addition) for any $x, y, z \in \mathbb{F}$

① if $x+y = x+z \Rightarrow y = z$

② if $x+y = x \Rightarrow y = 0$

③ if $x+y = 0 \Rightarrow y = -x$

④ $-(-x) = x$

Proof: ① $x+y = x+z \Rightarrow -x+(x+y) = -x+(x+z)$

③ associative law $\Rightarrow (-x+x)+y = (-x+x)+z$

⑤ $x+(-x)=0 \Rightarrow 0+y = 0+z$

④ Contains 0 $\Rightarrow y = z$

② $x+y = x \Rightarrow -x+(x+y) = -x+x$

$\Rightarrow (-x+x)+y = -x+x$

$\Rightarrow 0+y = 0$

$\Rightarrow y = 0$

③ $x+y=0 \Rightarrow -x+(x+y) = -x+0$

$\Rightarrow (-x+x)+y = -x+0$

$\Rightarrow 0+y = -x+0$

$\Rightarrow y = -x$

④ omitted

Thm (Consequence of axioms of multiplication) for any $x, y, z \in \mathbb{F}$

① if $x \neq 0$, $x \cdot y = x \cdot z \Rightarrow y = z$

② if $x \neq 0$, $x \cdot y = x \Rightarrow y = 1$

③ if $x \neq 0$, $x \cdot y = 1 \Rightarrow y = \frac{1}{x}$

④ if $x \neq 0$, then $\frac{1}{1/x} = x$

Proof: omitted.

$$
\begin{tikzcd}
 & \text{set} \arrow[d] \arrow[dl, bend right] & \\
\text{Field} \arrow[dr] & \text{ordered set} \arrow[d] & \\
 & \text{ordered Field} &
\end{tikzcd}
$$

Thm (Consequence of axioms of field) For any $x, y, z \in \mathbb{F}$

① $0 \cdot x = 0$

② if $x \neq 0$, $y \neq 0$, then $x \cdot y \neq 0$.

③ $(-x) \cdot y = -(x \cdot y) = x \cdot (-y)$

④ $(-x) \cdot (-y) = x \cdot y$

Proof: ① Let $y = 0$ $\quad (y+y) \cdot x \overset{\text{distribution law}}{=} y \cdot x + y \cdot x$

$\Rightarrow (0+0) \cdot x = \cancel{0 \cdot x} + 0 \cdot x = \cancel{0 \cdot x}$

$\Rightarrow 0 \cdot x = 0$

② Suppose by contradiction: $x \neq 0$, $y \neq 0$, but $x \cdot y = 0$.

$\frac{1}{x} \cdot (x \cdot y) \cdot \frac{1}{y} = \frac{1}{x} \cdot 0 \cdot \frac{1}{y} = 0$

but $\left(\frac{1}{x} \cdot x\right) \cdot \left(y \cdot \frac{1}{y}\right) = 1 \cdot 1 \neq 0$

Contradiction $\Rightarrow x \cdot y \neq 0$.

③ $(-x) \cdot y = -(x \cdot y) \Rightarrow (-x) \cdot y + x \cdot y = -(x \cdot y) + x \cdot y$

$\Rightarrow (-x) \cdot y + x \cdot y = 0$

$\Rightarrow (-x + x) \cdot y = 0$

$\Rightarrow 0 \cdot y = 0$

then we proved $(-x) \cdot y = -(x \cdot y)$ holds.

$x \cdot (-y) = -(x \cdot y) \Rightarrow x \cdot (-y) + x \cdot y = -(x \cdot y) + x \cdot y$

$x \cdot (-y) + x \cdot y = 0$

$x \cdot (-y + y) = 0$

$x \cdot 0 = 0$

then we proved $x \cdot (-y) = -(x \cdot y)$ holds.

④ $(-x) \cdot (-y) = -(x \cdot (-y)) = -(-(x \cdot y)) = xy$

Defn: (Ordered field) An ordered field $\mathbb{F}$ is a field, with order "<" endowed, and satisfies: set + axioms

① If $y < z$, then $x + y < x + z$, $\forall x, y, z \in \mathbb{F}$

② $\forall x, y \in \mathbb{F}$, $0 < x$ and $0 < y$, then $0 < x \cdot y$

Notation:

If $0<x \Rightarrow x\in F$ is positive

If $x<0 \Rightarrow x\in F$ is negative

Thm (Properties of ordered field) $(F,<)$ is ordered field, $\forall x,y\in F$ then

① If $0<x \Rightarrow -x<0$; if $x<0 \Rightarrow 0<-x$

② If $0<x$ and $y<z \Rightarrow x\cdot y < x\cdot z$

③ If $x<0$ and $y<z \Rightarrow x\cdot z < x\cdot y$

④ If $x\neq 0 \Rightarrow 0<x^2 := x\cdot x$

⑤ If $0<x<y \Rightarrow 0<\frac{1}{y}<\frac{1}{x}$

Proof: ①–④ omitted.

⑤ Prove $0<\frac{1}{y}$. Since $0<1=y\cdot\frac{1}{y}$ by ②, we have $0<\frac{1}{y}$

Prove $\frac{1}{y}<\frac{1}{x}$. Since $0<x \Rightarrow 0<\frac{1}{x}$, from $x<y$, by ③

$\Rightarrow (\frac{1}{x}\cdot\frac{1}{y})\, x < (\frac{1}{x}\cdot\frac{1}{y})\, y \Rightarrow \frac{1}{y}\cdot(\frac{1}{x}\cdot x) < \frac{1}{x}(\frac{1}{y}\cdot y)$

$\Rightarrow \frac{1}{y}<\frac{1}{x}$

Real field and real numbers

Defn: (Real field, 实数域) Real field is an ordered field ①, with the least upper bound property ②, and contains $\mathbb{Q}$ as a subfield ③

Notation: $\mathbb{R}$ or $(-\infty, +\infty)$

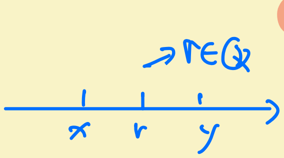
Thm (Archimedean property, 阿基米德性质) suppose $x,y\in\mathbb{R}$, $x>0$, then there exists a positive integer number n such that

$$nx>y.$$

Thm (Denseness of $\mathbb{Q}$ in $\mathbb{R}$, 有理数在实数域的稠密性)

Suppose $x,y\in\mathbb{R}$, and $x<y$. Then there exists a $r\in\mathbb{Q}$ such that

$$x<r<y$$



Proof: $x = 11.3471\cdots$

$x = a_0 . a_1 a_2 a_3 a_4 \cdots$

$a_0 = 11$

$a_1 = 3$ $\qquad n = 3$

$a_2 = 4$

$a_3 = 7$

$a_4 = 1$

不足近似：$x_n = 11.347$

过剩近似：$\bar{x}_n = 11.348$

$\Rightarrow \quad x_n \le x \le \bar{x}_n$

Since $x < y$, there exists $n > 0$ such that $\bar{x}_n < y_n$. Let $r = \frac{\bar{x}_n + y_n}{2}$

$\Rightarrow \quad x \le \bar{x}_n < r < y_n \le y.$

We conclude $x < r < y$, and $n \in \mathbb{Q}$

Thm (n^{th} root of a real number) For any $x \in \mathbb{R}$, and $n \in \mathbb{Z}_+$ there exists one and only one positive number y satisfies

$$y^n = x$$

Denote $y = x^{\frac{1}{n}} = \sqrt[n]{x}$

Thm: $(a \cdot b)^{\frac{1}{n}} = a^{\frac{1}{n}} \cdot b^{\frac{1}{n}}$

Defn: (Intervals) An interval is a subset of $\mathbb{R}$. Given $a, b \in \mathbb{R}$, and $a < b$, then

$$\left.\begin{aligned}
(a,b) &:= \{x \in \mathbb{R} \mid a < x < b\} \\
[a,b] &:= \{x \in \mathbb{R} \mid a \le x \le b\} \\
[a,b) &:= \{x \in \mathbb{R} \mid a \le x < b\} \\
(a,b] &:= \{x \in \mathbb{R} \mid a < x \le b\}
\end{aligned}\right\} \text{finite interval}$$

$$\left.\begin{aligned}
(a,+\infty) &:= \{x \in \mathbb{R} \mid x > a\} \\
(-\infty,b) &:= \{x \in \mathbb{R} \mid x < b\} \\
[a,+\infty) &:= \{x \in \mathbb{R} \mid x \ge a\} \\
(-\infty,b] &:= \{x \in \mathbb{R} \mid x \le b\}
\end{aligned}\right\} \text{infinite interval}$$

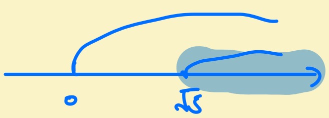
Defn: (Supremum and Infimum , 上确界 和 下确界)

Let $E \subset \mathbb{R}$. If there exists $b \in \mathbb{R}$ such that $x \le b$, $\forall x \in E$, then we say E is bounded above.

If b_0 is an upper bound of E, such that whenever b is an upper bound of E, we have $b_0 \le b$, then b_0 is the supremum of E.

Let $E \subset \mathbb{R}$, If there exists $a \in \mathbb{R}$ such that $x \ge a$, $\forall x \in E$, then we say E is bounded below.

If a_0 is a lower bound of E, such that whenever a is a lower bound of E, we have $a_0 \ge a$, then a_0 is the infimum of E.



E.g.: $A = \{x \in \mathbb{R} \mid x > 0,\ x^2 \ge 5\}$ Prove that A is non-empty, bounded below, and $\inf A = \sqrt{5}$.

Proof: ① Non-empty. since $\sqrt{6} \in A$, then A is non-empty.

② bounded below. since $\forall x \in A$, we have $x > -1$.
then A is bounded below.

③ $\inf A = \sqrt{5}$. Suppose that $\sqrt{5} < \inf A$, then there exist a number r such that $\sqrt{5} < r < \inf A$. By the definition of A. we have $r \in A$. but $r < \inf A$. contradiction. Hence $\inf A = \sqrt{5}$.

Sequence 数列 $\quad f: \mathbb{Z}_{++} \to \mathbb{R}$

速度 ←

面积 ↙

☆ 收敛性