Sequences and Series

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Contents

Preface

The calculus is one of man-kinds most significant scientific achievements, transforming previously intractable physical problems into often routine calculations. Although its roots trace back into antiquity, it was developed in the late 17th century by Newton, when developing his laws of motion and gravitation, and Leibniz, who developed the notation we still use today. Analysis is the branch of mathematics that underpins the theory behind the calculus, placing it on a firm logical foundation through the introduction of the notion of a limit.

Mathematics is a practical subject. It is best learned by doing it, rather than watching or reading about someone else doing it.

Chapter 0

Foundations

0.1 Sets

We start with the “definition” of a set.

Definition 0.1.1

A **set** is a collection of objects. These objects are called **elements** or **members** of the set. A set with no objects is called the **empty set** and is denoted by $\emptyset$.

The above definition is not rigorous (not all classes/collections can be sets, and certain axioms should be made about a set), but it is sufficient for the purpose here.

Often when we are dealing with *finite* sets (sets that have finitely many elements) we simply describe them by listing their elements inside “curly brackets”. For example, the set whose elements are the numbers 1, 6 and 2 is written

$$\{1, 6, 2\}.$$

It is not important how we order this list, indeed

$$\{1, 6, 2\} = \{6, 1, 2\} = \cdots \quad \text{etc.}.$$

Definition 0.1.2

For a set A we write $x \in A$ if x **is an element** of A. By $x \notin A$ we mean that x **is not an element** of A.

For more complicated sets we need some more sophisticated notation. We typically denote a set whose members have a certain property P by

$$\{x : x \text{ has property } P\}.$$

This notation is sometimes called **set-builder notation**.

Here we recall the basic operations of sets

Definition 0.1.3: Operations

1. A **union** of two sets A and B is defined as
$$A \cup B := \{x : x \in A \text{ or } x \in B\}.$$
2. An **intersection** of two sets A and B is defined as
$$A \cap B := \{x : x \in A \text{ and } x \in B\}.$$
3. A *complement of B relative to A* (or **set-theoretic difference** of A and B) is defined as
$$A \setminus B := \{x : x \in A \text{ and } x \notin B\}.$$
4. We say **complement** of B and write B^c instead of $A \setminus B$ if the set A is the entire universe, and is understood from context.
5. We say sets A and B are **disjoint** if $A \cap B = \emptyset$.

0.1.1 Number systems

Example 0.1.1

Here we write

- $\{1, 2, 3, \dots\}$ is called the set of **Natural Numbers** and is denoted $\mathbb{N}$.
- $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ is called the set of **Integers** and is denoted $\mathbb{Z}$.
- $\{m/n : m \in \mathbb{Z}, n \in \mathbb{N}\}$ is called the set of **Rational Numbers** (or **Rationals**) and is denoted $\mathbb{Q}$.

Definition 0.1.4: Subset

If A and B are sets with the property that every element of A belongs to B; i.e.
$$x \in A \implies x \in B,$$
then we say that A is a *subset* of B (or A is *contained in* B). If A is a subset of B we often write $A \subseteq B$. The empty set $\emptyset$ is a subset of any nonempty set.
If $A \subseteq B$ and $A \neq B$, then A is said to be a proper subset of B, denoted as $A \subset B$.

By $B \supseteq A$ we mean $A \subseteq B$; by $B \supset A$ we mean $A \subset B$.
Notice that
$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}.$$

Remark 0.1.1. *We assume that all the familiar arithmetics involving rational numbers are* *known, and will use (and in fact have used) them freely: e.g.,* $\frac{m}{n} + \frac{p}{q} = \frac{mq+np}{nq}$ *for integers* m, n, p, q *with* $n, q \neq 0$.

0.2 Ordered set

Definition 0.2.1: Order

Let S be a set. An order on S is a relation, denoted by $\prec$, with the following properties:

- For $x, y \in S$, one and only one of the following holds:

$$x \prec y \text{ (“}x\text{ precedes }y\text{”) }, x = y,\ y \prec x \text{ (“}y\text{ precedes }x\text{”)}.$$

- For $x, y, z \in S$, if $x \prec y$ and $y \prec z$, then $x \prec z$.

Here the order $\prec$ is just a relation, does not necessarily mean “strictly smaller than”.

Examples of $\prec$:

Example 0.2.1

Consider the set of rational numbers $\mathbb{Q}$, and $>$ means “(strictly) bigger than”. Then $>$ is an order on $\mathbb{Q}$. Indeed, for any $x, y \in \mathbb{Q}$, either $x > y$, or $x = y$ or $y > x$ holds, and only one of them holds; moreover, if $x > y$ and $y > z$ then $x > z$.

Example 0.2.2

Consider the set of rational numbers $\mathbb{Q}$, and $<$ means “smaller than”. Then $<$ is an order on $\mathbb{Q}$. Indeed, for any $x, y \in \mathbb{Q}$, either $x < y$, or $x = y$ or $y < x$ holds, and only one of them holds; moreover, if $x < y$ and $y < z$ then $x < z$.

Examples of relations that are not orders:

Example 0.2.3

Consider the set of rational numbers $\mathbb{Q}$. Then $\leq$ is not an order on $\mathbb{Q}$. Indeed, for any $x \in \mathbb{Q}$, tale $y = x$, all of then three assertions hold: $x \leq y$, $x = y$, $y \leq x$ holds, but the definition requires exactly one of the three holds.

Example 0.2.4

Consider the set of rational numbers $\mathbb{Q}$. Then $\neq$ is not an order on $\mathbb{Q}$. Indeed, if $x \neq y$ and $y \neq z$, it does not necessarily hold that $x \neq z$. For example, take $x = z = 1$ and $y = 0$.

Example 0.2.5

Let $S = 2^{\mathbb{Q}}$, which means the set of all subsets of $\mathbb{Q}$. Then $\subset$ is not an order on S. Indeed, consider $A = \{1, 2\}$ and $B = \{2, 3\}$. Then $A \subset B$ does not hold; $B \subset A$ does not hold; and $A = B$ does not hold. So the first requirement in the definition of an order is not satisfied by $\subset$, despite the second requirement in the definition of an order holds for $\subset$.)

Definition 0.2.2: Ordered set

A set S endowed with an order $\prec$ is called an ordered set, denoted as $(S, \prec)$. When the order is clear, simply call S an ordered set.

Suppose $(S, \prec)$ or for brevity S is an order set in this section.

- Convenient to introduce notations:
  - $\preceq$: $x \preceq y$ if and only if $x \prec y$ or $x = y$.
  - $\succ$; $y \succ x$ (y succeeds (is after) x) if and only if $x \prec y$.
  - $\succeq$: $y \succeq x$ if and only if $y \succ x$ or $y = x$.

Definition 0.2.3: Upper bound

Consider an ordered set S and $\emptyset \neq E \subset S$. If there is $y \in S$ such that for all $x \in E$, it holds that $x \preceq y$, then E is said to be bounded above, and y is called an upper bound of E.

Example 0.2.6

Consider the ordered set $(\mathbb{Q}, <)$. Then $\{0, 1, 2\}$ is bounded from above, with an upper bound e.g., 3.

Definition 0.2.4: Least upper bound

Suppose S is an ordered set, and $\emptyset \neq E \subset S$ is bounded above. Suppose there exists some $\alpha \in S$ such that

- α is an upper bound of E.
- For any $\gamma \prec \alpha$, γ is not an upper bounded of E.

Then α is called the least upper bound of E (in S), or the supremum of E, denoted as

$$\alpha = \sup E.$$

In the above definition, one may ask if $\sup E$ is unique, if it exists. The answer is affirmative. Indeed, if β is also a least upper bound of E, then the definition of an order implies one and only one of the following holds:

- $\alpha \prec \beta$, in which case, α is not an upper bound (by definition of least upper bound), which cannot be true;
- $\beta \prec \alpha$, in which case, β is not an upper bound (by definition of least upper bound), which cannot be true;
- $\beta = \alpha$.

Consider the ordered set $(\mathbb{Q}, <)$. Then the least upper bound of $E \subset \mathbb{Q}$, if exists, is just the smallest element of the set of all upper bounds of E in $\mathbb{Q}$.

Example 0.2.7

Consider the ordered set $(\mathbb{Q}, <)$. The set $\{0, 1, 2\} \subset \mathbb{Q}$ has the least upper bound 2. Indeed, the set of upper bounds of $\{0, 1, 2\}$ in $\mathbb{Q}$ is $\{x \in \mathbb{Q} : \ x \geq 2\}$, and the smallest element of this set is given by 2.

The next example shows that not all bounded above subsets of $\mathbb{Q}$ have a least upper bound in $\mathbb{Q}$.

We need a preliminary observation:

Lemma 0.2.1

There is no rational number $x \in \mathbb{Q}$ satisfying $x^2 = 2$.

Proof. Suppose $x \in \mathbb{Q}$ such that $x^2 = 2$. Write $x = \frac{m}{n}$ with m and n being coprime[1]. So $(\frac{m}{n})^2 = 2$ or $m^2 = 2n^2$. Hence m^2 is divisible by 2 (i.e. m^2 is an even number) and so m is divisible by 2 (for if m is odd, then m^2 is odd too). Write $m = 2k$ and so $(2k)^2 = 2n^2$. Divide by 2 and note that $2k^2 = n^2$, and hence n is divisible by 2. But that is a contradiction against m, n being coprime. $\square$

Example 0.2.8

Consider the ordered set $(\mathbb{Q}, <)$, and let $E = \{p \in \mathbb{Q} : \ p > 0, p^2 < 2\}$. Then E is bounded above by e.g., 4. The collection of the upper bounds of E in $\mathbb{Q}$ is given by $F = \{p \in \mathbb{Q} : \ p > 0, p^2 > 2\}$, because no positive rational number p satisfies $p^2 = 2$. We show that F does not have a smallest element. That means, for any $p \in F$, we can find some $q \in F$ such that $q < p$. This would in turn imply that E does not have a least upper bound in $\mathbb{Q}$.

Proof. Let $p \in F$ be fixed. Then we construct the required $q \in F$ satisfying $q < p$. That means, we need a positive rational number $q < p$ satisfying $q^2 > 2$.

Since $p^2 > 2$, we have $p^2 - 2 > 0$. We would like to find some positive $x \in \mathbb{Q}$ such that $q = p - x(p^2 - 2)$, that is,

- $p - x(p^2 - 2) > 0$;

[1] In number theory, two integers a and b are said to be coprime (also written co-prime) if the only positive integer (factor) that divides both of them is 1.

- $(p - x(p^2 - 2))^2 > 2$.

Take $x = \frac{1}{p+2}$. Then

$$p - x(p^2 - 2) = p - \frac{p^2 - 2}{p + 2} = \frac{p^2 + 2p - p^2 + 2}{p + 2} = \frac{2p + 2}{p + 2} > 0;$$

and

$$(p - x(p^2 - 2))^2 = (p - \frac{p^2 - 2}{p + 2})^2 = \left(\frac{p^2 + 2p - p^2 + 2}{p + 2}\right)^2 = \left(\frac{2(p + 1)}{p + 2}\right)^2$$

and

$$\begin{aligned} \left(\frac{2(p + 1)}{p + 2}\right)^2 &> 2 \Leftrightarrow 4(p^2 + 1 + 2p) > 2(p^2 + 4 + 4p) \\ \Leftrightarrow \quad 2p^2 &> 4 \Leftrightarrow p^2 > 2, \end{aligned}$$

which is true. □

Definition 0.2.5: Least upper bound property

An ordered set S is said to have the least upper bound property if any nonempty bounded above $E \subset S$ has a least upper bound in S.

The previous example demonstrates that $(\mathbb{Q}, <)$ does not have the least upper bound property.

We can similarly define lower bound, and greatest lower bound.

Definition 0.2.6: Lower bound

Consider an ordered set S and $\emptyset \neq E \subset S$. If there is $y \in S$ such that for all $x \in E$, it holds that $x \succeq y$, then E is said to be bounded below, and y is called a lower bound of E.

Definition 0.2.7: Greatest lower bound

Suppose S is an ordered set, and $\emptyset \neq E \subset S$ is bounded below. Suppose there exists some $\alpha \in S$ such that

- α is a lower bound of E.
- For any $\gamma \succ \alpha$, γ is not a lower bounded of E.

Then α is called the greatest lower bound of E, or the infimum of E, denoted as

$$\alpha = \inf E.$$

If the ordered set has the least upper bound property, then it also has the greastest lower bound in the sense of the following theorem.

Theorem 0.2.1: Greatest lower bound property

Suppose S is an ordered set and has the least upper bound property. Then for any nonempty $B \subset S$, which is bounded below, it has the greatest lower bound in S. Moreover, if L is the set of lower bounds of B in S, then

$$\sup L = \inf B,$$

both exist.

0.3 Field

Definition 0.3.1: Field

A field is a set F endowed with two operations: one is called addition $+$ and one is called multiplication $\cdot$ (also denoted as $\times$), which satisfy the following axioms:

- Axioms for addition:
  - For any $x, y \in F$, $x + y \in F$.
  - Commutative: $x + y = y + x$.
  - Associative: $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.
  - F contains an element, called 0, such that $0 + x = x$ for all $x \in F$.
  - For each $x \in F$, there is an element $-x \in F$ such that $x + (-x) = 0$.
- Axioms for multiplication:
  - For any $x, y \in F$, $x \cdot y \in F$. (When there is no confusion, we write $x \cdot y$ as xy).
  - Commutative:$x \cdot y = y \cdot x$ for all $x, y \in F$.
  - Associative: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in F$.
  - F contains an element, called $1 \neq 0$, such that $1 \cdot x = x$ for all $x \in F$.
  - If $x \in F$ and $x \neq 0$, then there is an element $\frac{1}{x} \in F$ such that $x \cdot \frac{1}{x} = 1$.
- Distributive law: $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in F$.

It is conventional to use the following notations:

- $x + (-y) =: x - y$.
- $x \cdot \frac{1}{y} =: \frac{x}{y} =: x/y$.
- $(x + y) + z =: x + y + z$.
- $(x \cdot y) \cdot z =: x \cdot y \cdot z$.
- $x \cdot x =: x^2$, $x \cdot x \cdot x =: x^3, \ldots$.

- $x + x =: 2x$, $x + x + x =: 3x, \ldots$.

Example 0.3.1

$\mathbb{Q}$ with the usual addition and multiplication is a field, 0, 1, $\frac{1}{x}$ are understood in the usual sense. The set of natural numbers $\mathbb{N}$ with the usual addition and multiplication is, clearly, not a field.

Theorem 0.3.1: Consequences of axioms of addition

Axioms of addition imply: for $x, y, z \in F$ where F is a field,

(a) If $x + y = x + z$, then $y = z$.

(b) If $x + y = x$, then $y = 0$. (So 0 is unique in F.)

(c) If $x + y = 0$, then $y = -x$. (So $-x$ is unique in F corresponding to x.)

(d) $-(-x) = x$.

Proof. (a) Add $(-x)$, which exists due to the axioms of addition, to te both sides of $x+y = x+z$ and use the associative law: the left hand side becomes $(-x)+(x+y) = ((-x)+x)+y = 0 + y = y$; and the right hand side becomes $(-x) + (x + z) = ((-x) + x) + z = 0 + z = z$. Thus, $y = z$.

(b) Follows from (a) by putting $z = 0$.

(c) Follows from (a) by taking $z = -x$.

(d) We know $x + (-x) = 0$. (c) implies $x = -(-x)$. $\square$

Theorem 0.3.2: Consequences of axioms of multiplication

Axioms of multiplication imply: for $x, y, z \in F$ where F is a field,

(a) If $x \neq 0$, $x \cdot y = x \cdot z$, then $y = z$.

(b) If $x \neq 0$, $x \cdot y = x$, then $y = 1$. (So 1 is unique in F.)

(c) If $x \neq 0$, $x \cdot y = 1$, then $y = \frac{1}{x}$. (So $\frac{1}{x}$ is unique in F corresponding to $x \neq 0$.)

(d) If $x \neq 0$, then $\frac{1}{1/x} = x$.

The proof of this statement is left as an exercise.

Remark 0.3.1. *When there is no danger of confusion, we write $x \cdot y$ as xy in what follows.*

Theorem 0.3.3: Consequences of axioms of field

Axioms of field imply: for $x, y, z \in F$ where F is a field,

(a) $0x = 0$.

(b) If $x, y \neq 0$,then $xy \neq 0$.

(c) (-x)y =-(xy)=x(-y).

(d) $(-x)(-y) = xy$.

Proof. (a) We know if $x+y = x$ then $y = 0$ (this is part (b) of consequences of axioms of addition). We use this to prove (a) as follows. Since by distributive law, 0x+0x=(0+0)x=0x, we see $0x = 0$.

(b) Suppose for contradiction that $xy = 0$. Then $(\frac{1}{x})xy\frac{1}{y} = \frac{1}{x}0\frac{1}{y}$. But the left hand side is 1 and the right hand side is 0 (by part (a)). This is a desired contradiciton (as $1 \neq 0$), meaning $xy \neq 0$.

(c) Prove (-x)y=-(xy) as follows. We need to show $(-x)y + (xy) = 0$. The left hand side is, by distributive law $(-x + x)y = 0y = 0$, the last equality valid by (a). This implies $-(xy) = (-x)y$.

Prove $x(-y) = -(xy)$ similarly: $x(-y) + (xy) = x(-y + y) = x0 = 0x = 0$.

(d) By part (c) $(-x)(-y) = -(x(-y)) = -(-(xy)) = (xy)$, where the last equalty holds by part (d) of consequences of axioms of addition. ☐

is an order, so order set satisfied

Definition 0.3.2: Ordered field

An ordered field F is a field, on which an order $\prec$ is endowed, and satisfies

- If $y \prec z$, then $x + y \prec x + z$ for all $x, y, z \in F$.
- If $x, y \in F$, $0 \prec x$ and $0 \prec y$, then $0 \prec x \cdot y$.

In an ordered field, if $0 \prec x$, then we call $x \in F$ positive; and if $x \prec 0$, then we call $x \in F$ negative.

The following statement is a consequence of the defintion of ordered field.

Theorem 0.3.4: Properties of ordered field

Let F be an ordered filed with order $\prec$ and $x, y \in F$. Then the following hold.

(a) If $x \succ 0$, then $-x \prec 0$ and vice versa.

(b) If $x \succ 0$ and $y \prec z$, then $xy \prec xz$.

(c) If $x \prec 0$ and $y \prec z$, then $xy \succ xz$.

(d) If $x \neq 0$, then $x^2 \succ 0$. (In particular, $1 \succ 0$.)

(e) If $0 \prec x \prec y$, then $0 \prec \frac{1}{y} \prec \frac{1}{x}$.

Proof. (a) By the first additional requirement in the definition of ordered field, $0 \prec x$ implies $0 + (-x) \prec x + (-x)$, i.e., $-x \prec 0$.

If $x \prec 0$, then similarly, $(-x) + x \prec (-x) + 0$ and so $0 \prec -x$.

(b) Since $y \prec z$, $(z-y) \succ 0$, and so by the second additional requirement in the definition of ordered field, $x(z-y) \succ 0$. That is, $xz - xy \succ 0$. By the first extra requirement in the definition of ordered field, this implies $xz - (xy) + (xy) \succ 0 + xy = xy$, and thus $xz \succ xy$.

(c) $x \prec 0$ implies $-x \succ 0$ (by part (a)). Part (b) implies $(-x)y \prec (-x)z$. Thus, $-(xy) \prec -(xz)$. Now the first additional requirement in definition of ordered field implies $xy - (xy) \prec (xy) - (xz)$, i.e., $0 \prec (xy) - (xz)$. Again using the first additional requirement in definition of ordered filed: $0 + (xz) \prec (xy) - (xz) + (xz)$, i.e., $xz \prec xy$ as required.

(d) If $x \succ 0$, then by part (b), $x^2 = x \cdot x \succ x0 = 0$.

If $x \prec 0$, then $-x \succ 0$. Part (d) of consequences of axioms of field implies $(-x)(-x) = x \cdot x = x^2$. Now the previous reasoning applied to $(-x) \succ 0$ yields $(-x)(-x) \succ 0$, and so $x^2 \succ 0$.

(e) Firstly show $\frac{1}{y} \succ 0$. Since $0 \prec y$, part (c) asserts that if $\frac{1}{y} \prec 0$, then $1 = y \cdot \frac{1}{y} \prec 0$, which is a contradiction because $1 \succ 0$. Therefore, $\frac{1}{y} \succ 0$.

Next show $\frac{1}{y} \prec \frac{1}{x}$ as follows. Since $0 \prec x$, we see $0 \prec \frac{1}{x}$. Now by part (b), from $x \prec y$, we get

$$\frac{1}{x}\frac{1}{y}x \prec \frac{1}{x}\frac{1}{y}y.$$

Using the commutative and associative law of multiplication, we get $\frac{1}{y} \prec \frac{1}{x}$ as required. $\square$

0.4 Real field and real numbers

Now we can define the set of real numbers, $\mathbb{R}$, as an ordered field with the least upper bound property.

Definition 0.4.1: Real field

An ordered (with the order $<$) field with the least upper bound property and contains $\mathbb{Q}$ (endowed with the usual addition and multiplication) as a subfield is called the real field, denoted as $\mathbb{R}$ or $(-\infty, \infty)$. Here by subfield we mean that applying the addition and multiplication to the elements of $\mathbb{Q}$ is the same as applying the usual addition and multiplication to elements of $\mathbb{Q}$.

Any element of $\mathbb{R}$ is called a real number. Of course, $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. The last inclusion is strict because, as was demonstrated earlier, $\mathbb{Q}$ does not have the least upper bound property.

While we do not explore any further details, we make the following remark.

- The real field really exists and is unique in a certain sense. One way of constructing the real field starting from $\mathbb{Q}$ is the method of Dedekind cut.

A useful property of $\mathbb{R}$ is the following one.

Theorem 0.4.1: Archimedian property

Suppose $x, y \in \mathbb{R}$ and $0 < x$. Then there is some positive integer n such that

$$nx > y.$$

The next statement further reinforces the relationship between $\mathbb{R}$ in relation to $\mathbb{Q}$.

Theorem 0.4.2: Denseness of $\mathbb{Q}$ in $\mathbb{R}$

Suppose $x, y \in \mathbb{R}$, and $x < y$. Then there is some rational number $p \in \mathbb{Q}$ such that

$$x < p < y.$$

We have seen that there is no positive natural number $x \in \mathbb{Q}$ satisfying $x^2 = 2$. The next statement asserts that this equation has a unique positive real solution.

Theorem 0.4.3: nth root of a real number

For each positive real number $x > 0$, and positive integer n, there is one and only one positive real number $y \in \mathbb{R}$ satisfying $y^n = x$.
In this case, we denote $y := x^{\frac{1}{n}}$ or $y := \sqrt[n]{x}$. When $n = 2$, $\sqrt[2]{x}$ is written as $\sqrt{x}$.

In particular, $\sqrt{2}$ is not a rational number, but a real number. A real number that is not rational is called irrational.

Theorem 0.4.4

If a and b are positive real numbers and n is a positive integer, then $(a \cdot b)^{1/n} = a^{1/n} \cdot b^{1/n}$. When there is no confusion, we omit the $\cdot$, and the above is written as $(ab)^{1/n} = a^{1/n}b^{1/n}$.

We may use the above two theorems to show the following result.

Example 0.4.1

Let $b > 0$ be a real number, m, n be two positive integers. Show $(b^{\frac{1}{n}})^m = (b^m)^{\frac{1}{n}}$. We introduce notation

$$b^{\frac{m}{n}} := \left(b^{\frac{1}{n}}\right)^m = (b^m)^{\frac{1}{n}}.$$

In an exercise, we show that if $m/n = p/q$ for positive integers m, n, p, q, then $b^{\frac{m}{n}} = b^{\frac{p}{q}}$. This makes the above notation made legitimately.

Proof. We only need to show $((b^{\frac{1}{n}})^m)^n = b^m$ as follows. By definition,

$$((b^{\frac{1}{n}})^m)^n = \underbrace{((b^{\frac{1}{n}})^m)((b^{\frac{1}{n}})^m)\ldots((b^{\frac{1}{n}})^m)}_{n \text{ times}}$$

and

$$(b^{\frac{1}{n}})^m = \underbrace{(b^{\frac{1}{n}})(b^{\frac{1}{n}})\ldots(b^{\frac{1}{n}})}_{m \text{ times}}.$$

Therefore,

$$\begin{aligned}((b^{\frac{1}{n}})^m)^n &= \underbrace{\underbrace{(b^{\frac{1}{n}})(b^{\frac{1}{n}})\ldots(b^{\frac{1}{n}})}_{m \text{ times}}\underbrace{(b^{\frac{1}{n}})(b^{\frac{1}{n}})\ldots(b^{\frac{1}{n}})}_{m \text{ times}}\ldots\underbrace{(b^{\frac{1}{n}})(b^{\frac{1}{n}})\ldots(b^{\frac{1}{n}})}_{m \text{ times}}}_{n \text{ times}} \\ &= \underbrace{\underbrace{(b^{\frac{1}{n}})(b^{\frac{1}{n}})\ldots(b^{\frac{1}{n}})}_{n \text{ times}}\underbrace{(b^{\frac{1}{n}})(b^{\frac{1}{n}})\ldots(b^{\frac{1}{n}})}_{n \text{ times}}\ldots\underbrace{(b^{\frac{1}{n}})(b^{\frac{1}{n}})\ldots(b^{\frac{1}{n}})}_{n \text{ times}}}_{m \text{ times}} \\ &= b^m,\end{aligned}$$

where we used associative and commutative laws of multiplication. □

Example 0.4.2

If $a, b > 0$ are positive real numbers, n is a positive integer, such that $a^n > b$, then $a > b^{\frac{1}{n}}$.

Proof. It is impossible that $a = b^{\frac{1}{n}}$ as the latter term is the unique positive real solution to equation $x^n = b$. Assume for contradiction that $a < b^{\frac{1}{n}}$. Then

$$\begin{aligned} & a^n = (a)(a^{n-1}) < b^{\frac{1}{n}}a^{n-1} = b^{\frac{1}{n}}(a)(a^{n-2}) \\ < \ & b^{\frac{1}{n}}b^{\frac{1}{n}}a^{n-2} = b^{\frac{2}{n}}a^{n-2} = b^{\frac{2}{n}}(a)(a^{n-3}) \\ & \vdots \\ < \ & b^{\frac{n-1}{n}}(a) \\ < \ & b^{\frac{n-1}{n}}b^{\frac{1}{n}} = (b^{1/n})^{n-1}(b^{1/n}) = \left(b^{\frac{1}{n}}\right)^n = b, \end{aligned}$$

which is a contradiction because $a^n > b$. □

Note that the above proof also shows that

Lemma 0.4.1

For positive real numbers b, c, and positive integer n, $c < b$ if and only if $c^{1/n} < b^{1/n}$.

For the if part, this can be seen by putting $c^{1/n} = a$ in the blue part of the previous proof. The only if part follows from the previous example.

Intervals

There are certain special sets of real numbers that appear particularly often. These are called **intervals**, and take one of the following forms where a and b denote fixed real numbers:

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$$

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$$

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

$$(a, \infty) = \{x \in \mathbb{R} : a < x\}$$

$$(-\infty, b) = \{x \in \mathbb{R} : x < b\}$$

$$[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$$

$$(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$$

The sets with only round brackets are called **open intervals**, and the sets with only square brackets are called **closed intervals**.

In relevance to this module, we make the following remark.

Remark 0.4.1. *Below, unless specified otherwise, when we talk about least upper bound or greatest lower bound of a subset of* $\mathbb{R}$*, we always mean them in* $\mathbb{R}$*.*

Finally,we reintroduce the least upper bound and greatest lower bound of a subset of $\mathbb{R}$ with the more familiar notation of $<$. The notation $\prec$ should not appear again in the rest of this block.

Definition 0.4.2: Supremum and Infimum

Let $E \subset \mathbb{R}$.

1. If there exists $b \in \mathbb{R}$ such that $x \leq b$ for all $x \in E$, then we say E is **bounded above** and b is an **upper bound** of E.

2. If there exists $b \in \mathbb{R}$ such that $x \geq b$ for all $x \in E$, then we say E is **bounded below** and b is a **lower bound** of E.

3. If there exists an upper bound b_0 of E such that whenever b is any upper bound for E we have $b_0 \leq b$, then b_0 is called the **least upper bound** or the **supremum** of E. We write
$$\sup E := b_0.$$

4. Similarly, if there exists a lower bound b_0 of E such that whenever b is any lower bound for E we have $b_0 \geq b$, then b_0 is called the **greatest lower bound** or the **infimum** of E. We write
$$\inf E := b_0.$$

5. When a set E is both bounded above and bounded below, we say simply that E is **bounded**.

Example 0.4.3: A supremum or infimum for E (even if they exist) need not be in E

- For example, the set $E := \{x \in \mathbb{Q} : x < 1\}$ has a least upper bound of 1 (Why?). Indeed, 1 is clearly an upper bound of E. Moreover, if $y \in \mathbb{Q}$ is such that $y < 1$, then by the denseness of $\mathbb{Q}$ in $\mathbb{R}$, there would be some $z \in \mathbb{Q}$ such that $y < z < 1$. This rational number z is in E, meaning that y cannot be an upper bound of E. Therefore, 1 is the least upper bound of E (in $\mathbb{R}$).

  Note that, 1 is not in the set E.

- On the other hand, if we take $G := \{x \in \mathbb{Q} : x \leq 1\}$, then the least upper bound of G is clearly also 1, and in this case $1 \in G$.

- On the other hand, the set $P := \{x \in \mathbb{Q} : x \geq 0\}$ has no upper bound (why?). Indeed, for any positive real number $y > 0$, the Archimedian property of $\mathbb{R}$ implies that for some positive integer n, $y < n \cdot 1 = n$. But $n \in P$. This means, y cannot be an upper bound of P. Therefore, P can have no a least upper bound. On the other hand 0 is the greatest lower bound of P.

Recall that $\mathbb{R}$ has the least upper bound property, which is also called the completeness of $\mathbb{R}$:

Theorem 0.4.5: Least upper bound property of $\mathbb{R}$

A nonempty set of real numbers that is bounded above (or below) has a least upper bound (respectively, greatest upper bound).

It says that if E is nonempty and bounded above (or below), then $\sup E$ (respctively, $\inf E$) exists and is a real number.

Example 0.4.4

Let $A = \{x \in \mathbb{R} : \; x > 0, x^2 > 5\}$. Prove that A is non-empty, bounded below, and $\inf A = \sqrt{5}$.

Proof. Clearly, $\sqrt{7} \in A$ therefore A is non-empty. All elements of A are positive, therefore A is bounded below. Consequently, the completeness of $\mathbb{R}$ implies $\inf A \in \mathbb{R}$ exists.

It remains to show that the greatest lower bound of A is $\sqrt{5}$. For all $x \in A$ we have $x > \sqrt{5}$, therefore $\sqrt{5}$ is a lower bound of A.

We argue by contradiction. Suppose $\sqrt{5} < \inf A$. Pick a real number a such that $\sqrt{5} < a < \inf A$ (which can be done by the denseness of $\mathbb{Q}$ in $\mathbb{R}$). Then a is an element of A (since a is a positive real whose square is > 5), which is smaller then the infimum of A; contradiction (because $\inf A$ is a lower bound). $\square$

Chapter 1

Sequence

Greek mathematics used various methods involving infinite processes. There was no significant change until the time of Sir Isaac Newton. Newton further developed technics involving infinite processes, which then regarded as a legitimate mathematical tool. At that time, it was recognised that careful rules needed to be developed governing the use of such infinite processes and the calculus to ensure the validity of the final results, which is when *real analysis* came into play. Nowadays it is known that real analysis is the very foundation of Calculus and every subject that follows. This rigorous subject stems from the work of eighteenth and nineteenth-century mathematicians.

1.1 Sequences

A sequence (of real numbers, of sets, of functions, of anything other things) is a list. In this module, a sequence must **continue without stopping**, a finite list is **not** called a sequence. Here we give the rigorous definition.

Definition 1.1.1: Sequence ★★★★★

A sequence of real numbers is a function[a]

$$f : \mathbb{Z}^+ \to \mathbb{R}.$$

[a] A function f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E. In the above definition, $D = \mathbb{Z}^+$ and $E = \mathbb{R}$.

Thus the sequence is the function defined on the set of positive integers.

Remark 1.1.1. In terms of writing out the sequence f, we have several ways,

- As a list

  $$f(1), f(2), \cdots, f(n), \cdots$$

  with the three dots indicating that the list is to continue indefinitely.

- As a list (with the subscript notation)

  $$f_1, f_2, \cdots, f_n, \cdots$$

- The expression
$$\{f(n)\} \quad \text{or} \quad (f(n))$$
with the understanding that the index n ranges over all the positive integers.

- The expression
$$\{f_n\} \quad \text{or} \quad (f_n)$$
with the same understanding of the index n.

1.1.1 Sequence examples

In order to specify some sequences we need to point out what every term in the sequence is. An explicit formula of the general terms is the best option, such like,

$$(a_n) \text{ where } a_n = 2n + 1.$$

In some cases, a formula relating the nth term to some preceding terms is also preferable. Such formulas are referred as *recursion formulas*. For instance

$$x_n = 2 + x_{n-1}.$$

A sequence is just a list of numbers and there is no general ways to get a formula, for example,

$$-3, \ln 2, 4, -900, \ln 17, 200000, \cdots .$$

However, there are some special sequences which have simple formula representation and are of particular importance in our course.

Arithmetic Progressions

The sequence is the one in which each term is obtained from the preceding term by adding a fixed amount. Such sequence is called arithmetic progressions,

$$c, c + d, c + 2d, c + 3d, \cdots , c + (n - 1)d, \cdots$$

is the general form of arithmetic progression. The number d is called the common difference.

The above arithmetic progression could be written as

$$x_n = c + (n - 1)d$$

or as a recursion formula

$$x_1 = c \qquad x_n = x_{n-1} + d.$$

Geometric Progressions

A sequence is such that each term is obtained from the preceding term by multiplying a fixed amount. The sequence

$$c, cr, cr^2, cr^3, \cdots , cr^{n-1}, \cdots$$

is the general geometric progression. The number r is called the common ratio.

Every geometric progression could be written by a formula

$$x_n = cr^{n-1}$$

or a recursion formula

$$x_1 = c \qquad x_n = rx_{n-1}.$$

Sequence of partial sums

Given a sequence

$$x_1, x_2, x_3, \cdots$$

A new sequence can be constructed upon it

$$s_n = \sum_{k=1}^{n} x_n.$$

The new sequence is called the sequence of partial sums of the sequence $\{x_n\}$.

1.2 Convergence

The sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n}, \cdots$$

is getting closer and closer to the number 0. How should we define this properly? This was first answered by Augustin Cauchy in the 1820s. He defined the notion of limit by mathematical statement about inequalities.

1.2.1 ε-N definition

The following is the single most important definition you will ever see.

Definition 1.2.1: Limit of a Sequence ★★★★★

We define that $\{a_n\}$ *converges* to a number ℓ if for every $\varepsilon > 0$, there exist a natural number N such that

$$|a_n - \ell| < \varepsilon$$

whenever $n \geq N$.

Remark 1.2.1. We would like to add some remarks concerning the definition.

1. If $\{a_n\}$ converges to ℓ, we write

$$\lim_{n\to\infty} a_n = \ell.$$

   A sequence that converges is said to be *convergent*. A sequence that fails to converge is said to *diverge*.

2. In the definition, the number N depends on ε. If there is a need to emphasise this dependence, one would write $N(\varepsilon)$ rather than N. This N is not necessarily to be a natural number, any real number will do the work. If N is found, then any real numbers that are larger than N would also work. The definition only requires one to find some N but *not necessarily* the smallest N that would work.

3. The main point of the definition is that the N can be determined, no matter how small a number ε is chosen.

Example 1.2.1: ★★★★

Assume that $x_n = C$ for all $n \in \mathbb{N}$, where C is a constant. Show that

$$\lim_{n\to\infty} x_n = C.$$

Proof. Given $\varepsilon > 0$, we have

$$|x_n - C| = 0 < \varepsilon, \quad \forall\, n \geq 1.$$

It follows from the definition that $\lim\limits_{n\to\infty} x_n = C$. □

Example 1.2.2: ★★★★★

Prove that

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

by the definition.

Rough work. Given $\varepsilon > 0$. We need to find a number N (or $N(\varepsilon)$ if you prefer) so that every term in the sequence on and after the Nth term is closer to 0 than ε, that is, so that

$$\left|\frac{1}{n} - 0\right| < \varepsilon$$

for $n = N, N+1, N+2, \cdots$. It is convenient to *work backward* and determine how large n should be for this. This will follow if

$$\frac{1}{n} < \varepsilon$$

or

$$n > \frac{1}{\varepsilon}.$$

The smallest n for that this statement holds could be our N[1]. Thus we could use

$$N = \left[\frac{1}{\varepsilon}\right] + 1^{2}.$$

Thus the proof is completed. □

Proof. For any $\varepsilon > 0$, set $N = \left[\frac{1}{\varepsilon}\right] + 1$. Then for any $n \geq N$, we have

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} = \varepsilon.$$

Thus we finish the proof by the definition. □

---

[1] As a matter of fact, any integer N with $N > \frac{1}{\varepsilon}$ will do the work.

[2] Here $[\cdot]$ means that

$$[x] = \max\{n; n \leq x\}$$

for all $x > 0$. It then follows that $[x] + 1 > x$ for $x > 0$. Therefore, it holds $\frac{1}{N} < \varepsilon$, since

$$\frac{1}{N} = \frac{1}{\left[\frac{1}{\varepsilon}\right] + 1} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon.$$

Remark 1.2.2. The key to the proof is to determine the existence of the integer N, such that for all $n \geq N$ a inequality holds. Thus the proof can be reduced to solve an inequality. However, it is easier than just solve the inequality under consideration since we don't care the optimal range of the n such that this inequality would hold.

Example 1.2.3

Show that
$$\lim_{n\to\infty} \frac{1}{\ln n} = 0.$$

Rough work. We want to show $\frac{1}{\ln n} < \varepsilon$ for any given ε when n large enough, i.e. $n \geq N$. It suffice to make sure that $n \geq e^{\frac{1}{\varepsilon}}$. Therefore, the number N would be any number that larger than $e^{\frac{1}{\varepsilon}}$. ☐

Proof. Given $\varepsilon > 0$, we take
$$N = \left[e^{\frac{1}{\varepsilon}}\right] + 1.$$
When $n \geq N$, we have the chain of inequalities $n \geq N > e^{\frac{1}{\varepsilon}}$, and thus
$$\frac{1}{\ln n} \leq \frac{1}{\ln N} < \frac{1}{\ln(e^{\frac{1}{\varepsilon}})} = \varepsilon.$$
The desired assertion follows from the definition. ☐

Remark 1.2.3. Since we have the formula
$$\log_a n = \frac{\ln n}{\ln a},$$
the above example shows that, for $a > 1$, we would have
$$\lim_{n\to\infty} \frac{1}{\log_a n} = 0.$$

Example 1.2.4: ★★★★★

Show that the sequence $\{(-1)^n\}$ diverges.

Proof. The proof is obtained by contradiction. If the sequence $\{(-1)^n\}$ converges and
$$\lim_{n\to\infty} (-1)^n = A.$$

From the definition there exists $N \in \mathbb{N}$, such that when $n > N$ we have

$$|(-1)^n - A| < 1.$$

By checking the above inequality against odd n and even n, one would obtain

$$|A + 1| < 1 \text{ and } |A - 1| < 1.$$

It then follows from the triangle inequality that

$$2 = |(A + 1) - (A - 1)| \leq |A + 1| + |A - 1| < 1 + 1 = 2.$$

But this cannot be so. From this contradiction the statement follows. □

The definition gives us a way to verify the convergence of a sequence. However, the definition doesn't provide a way of finding limits. In the previous examples, it is easy to guess that the limit is 0. Unfortunately, in general, it usually is not obvious what the limit should be. For instance

$$\lim_{n\to\infty} \frac{n!}{n^n}$$

or

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

As a matter of fact, the definition does not only just offer a rigorous way to verify a limit if given, but also enable us to develop a theory of how to find limits. The related topics will be discussed in the following sections.

1.2.2 Uniqueness of sequence limits

To develop a theory of limits, the first step is to make sure that the definition has defined limit unambiguously.

Theorem 1.2.1: Uniqueness ★★★

Suppose that

$$\lim_{n\to\infty} x_n = A \text{ and } \lim_{n\to\infty} x_n = B$$

are both true. Then $A = B$.

Proof. Proof by the definition. Let ε be any positive number. Then, by definition, we can find a number N_1 so that

$$|x_n - A| < \varepsilon$$

for $n \geq N_1$. We also be able to find a number N_2 so that

$$|x_n - B| < \varepsilon$$

for $n \geq N_2$. Let $N_3 = \max\{N_1, N_2\}$. Then both inequalities

$$|x_{N_3} - A| < \varepsilon \text{ and } |x_{N_3} - B| < \varepsilon$$

are true.

It follows that

$$|A - B| \leq |x_{N_3} - A| + |x_{N_3} - B| < 2\varepsilon$$

so that

$$|A - B| < 2\varepsilon.$$

But ε can be any positive number. This could only be true if $A = B$, which conclude the proof. □

1.2.3 Divergence to ∞

A sequence that fails to converge is said to diverge. There is one special way of divergence, i.e. divergence to infinity, that is of particular importance. It is worth to describe it in more details.

The sequence $x_n = n$ diverges since the terms get larger and larger. It is usually written as

$$\lim_{n\to\infty} x_n = \infty.$$

However, we don't say that this sequence "converges to ∞" but rather it "diverges to ∞".

Definition 1.2.2: Divergence to ∞ ★★★★

Let $\{x_n\}$ be a sequence of real numbers. We say that $\{x_n\}$ diverges to ∞ and write

$$\lim_{n\to\infty} x_n = \infty.$$

or

$$x_n \to \infty \text{ as } n \to \infty$$

provided that for every number M there is an integer N so that

$$x_n > M$$

for all $n \geq N$.

The key point of the definition is that the choice of N is guaranteed no matter how large M is chosen.

Example 1.2.5: ★★★★

Prove

$$\frac{n^2+1}{n+1} \to \infty \text{ as } n \to \infty$$

using the definition.

Proof. Let M be any positive number. Take the integer $N = [M] + 2$, then, for all $n \geq N$, we have

$$\frac{n^2+1}{n+1} \geq \frac{n^2-1}{n+1} = n - 1 \geq N - 1 = [M] + 1 > M.$$

Then the proof is completed. □

Here we choose $N = [M] + 2$ and it happens to work. As a matter of fact, such N can be obtained by solve the inequality

$$\frac{n^2+1}{n+1} \geq M$$

Write the above as

$$n + \frac{1}{n+1} - \frac{n}{n+1} \geq M.$$

Since $\frac{n}{n+1} < 1$ and thus the above would be true as long as $n \geq [M] + 2$.

1.3 Properties of Limits

1.3.1 Boundedness properties of Limits

A sequence is bounded if its range is a bounded set. It follows that

Definition 1.3.1: Bounded sequence

A sequence $\{x_n\}$ is bounded if there is a real number $M > 0$ so that every term in the sequence satisfies

$$|x_n| \leq M,$$

or $x_n \in [-M, M]$.

One important feature of convergent sequences is the boundedness.

Theorem 1.3.1: Boundedness ★★★★★

Every convergent sequence is bounded.

Proof. Suppose that $x_n \to A$ as $n \to \infty$. Then there exists an integer N so that

$$|x_n - A| \leq 1$$

whenever $n \geq N$. By triangle inequality we have

$$|x_n| = |x_n - A + A| \leq |x_n - A| + |A| \leq |A| + 1$$

for all $n \geq N$.

Now we write

$$M = \max\{|x_1|, |x_2|, \cdots, |x_{N-1}|, |A| + 1\}$$

we must have

$$|x_n| \leq M$$

for every term in the sequence. Thus we finish the proof. $\square$

Here we implicitly set $\varepsilon = 1$ and the definition of convergence guarantees the existence of the integer N.

As a consequence of this theorem we have

Corollary 1.3.1: ★★★★★

An unbounded sequence must diverge.

This turns out to be a very effective test to show the divergence of a sequence sometime.

Example 1.3.1: ★★★★★

Show the sequence
$$s_n = \sum_{k=1}^{n} \frac{1}{k}$$
diverges.

Proof. By Theorem 1.3.1, it is divergent if it is unbounded.

To show that it is unbounded, we only look at the terms with index $1, 2, 4, 8, \cdots$,

$$s_1 = 1$$
$$s_2 = 1 + \frac{1}{2}$$
$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) \geq 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right)$$
$$s_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \geq 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right)$$

and in general we have that

$$s_{2^n} = 1 + \sum_{j=1}^{n} \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{k} \,{}^{3} \geq 1 + \frac{n}{2}$$

for all $n \in \mathbb{N}$. More precisely, we have

$$\sum_{k=2^{n-1}+1}^{2^n} \frac{1}{k} \geq \sum_{k=2^{n-1}+1}^{2^n} \frac{1}{2^n} = \frac{2^n - 2^{n-1}}{2^n} = \frac{1}{2}.$$

Thus the sequence is unbounded and thus divergent. □

1.3.2 Algebra of Limits

Most sequences we are likely to encounter are obtained through combining simpler sequences by the usual arithmetic operator (addition, subtraction, multiplication, and division). For instance

$$\frac{3n^2 + 2n + 9}{6n^2 + 4n + 6}. \tag{1.3.1}$$

To find the limit of such sequence we need to investigate how the limit operation is influenced by algebraic operator.

[3] This equality can be verified by induction easily. Please check it for yourself.

Theorem 1.3.2: Algebra of Limits ★★★★★

Suppose that $\{x_n\}$ and $\{y_n\}$ are convergent sequences and C is a real number. Then

- Sums and Difference of limits

$$\lim_{n\to\infty}(x_n \pm y_n) = \lim_{n\to\infty} x_n \pm \lim_{n\to\infty} y_n.$$

- Products of limits

$$\lim_{n\to\infty}(x_n y_n) = \left(\lim_{n\to\infty} x_n\right) \times \left(\lim_{n\to\infty} y_n\right).$$

  In particularly, we have, for any constant C,

$$\lim_{n\to\infty}(Cx_n) = C\left(\lim_{n\to\infty} x_n\right).$$

- Quotients of limits: suppose further that $y_n \neq 0$ for all n and that

$$\lim_{n\to\infty} y_n \neq 0.$$

  Then

$$\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = \frac{\lim\limits_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}.$$

Proof. We only prove the second statement (products of limits) since the proofs for others are similar.

Suppose $x_n \to A$, $y_n \to B$ as $n \to \infty$, and $M > 0$ is an upper bound for $\{|y_n|\}$. By definition, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that both

$$|x_n - A| < \frac{\varepsilon}{2M}$$

and

$$|y_n - B| < \frac{\varepsilon}{2|A|+1}$$

hold, provided $n \geq N$. Then we have

$$\begin{aligned}
|x_n y_n - AB| &= |x_n y_n - Ay_n + Ay_n - AB| \\
&\leq |x_n - A||y_n| + |A||y_n - B| \\
&\leq |x_n - A|M + |A||y_n - B| \\
&\leq \frac{\varepsilon}{2M} \times M + \frac{\varepsilon}{2|A|+1} \times |A| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus the second part of the theorem follows by the definition of limit. □

Let us apply the rules of algebra of limits to find out the limit of the sequence given at the beginning of this section

Example 1.3.2: ★★★★★

Find the limit of the sequence (1.3.1).

Solution (Wrong attempt). Many students intend to apply Theorem 1.3.2 directly to the expression (1.3.1) like this

$$\begin{aligned}
\lim_{n\to\infty} \frac{3n^2+2n+9}{6n^2+4n+6} &= \frac{\lim_{n\to\infty}(3n^2+2n+9)}{\lim_{n\to\infty}(6n^2+4n+6)} \\
&= \frac{\lim_{n\to\infty}(3n^2)+\lim_{n\to\infty}(2n)+\lim_{n\to\infty} 9}{\lim_{n\to\infty}(6n^2)+\lim_{n\to\infty}(4n)+\lim_{n\to\infty}(6)} \\
&= \frac{\infty}{\infty} =?.
\end{aligned}$$

□

What can we conclude from this? Nothing! The first step of the above calculation is wrong, even it seems like that we just apply the Theorem 1.3.2 with $x_n = 3n^2+2n+9$ and $y_n = 6n^2+4n+6$, since the precondition of applying the Theorem 1.3.2 is the convergence of both $\{x_n\}$ and $\{y_n\}$.

Solution. Here is the right way to do

$$\begin{aligned}
\lim_{n\to\infty} \frac{3n^2+2n+9}{6n^2+4n+6} &= \lim_{n\to\infty} \frac{3+\frac{2}{n}+\frac{9}{n^2}}{6+\frac{4}{n}+\frac{6}{n^2}} \\
&= \frac{\lim_{n\to\infty}\left(3+\dfrac{2}{n}+\dfrac{9}{n^2}\right)}{\lim_{n\to\infty}\left(6+\dfrac{4}{n}+\dfrac{6}{n^2}\right)} \\
&= \frac{\lim_{n\to\infty} 3 + 2\lim_{n\to\infty}\dfrac{1}{n} + 9\lim_{n\to\infty}\dfrac{1}{n^2}}{\lim_{n\to\infty} 6 + 4\lim_{n\to\infty}\dfrac{1}{n} + 6\lim_{n\to\infty}\dfrac{1}{n^2}} \\
&= \frac{3+2\cdot 0+9\cdot 0}{6+4\cdot 0+6\cdot 0} = \frac{1}{2}.
\end{aligned}$$

Thus the limit should be $\frac{1}{2}$. □

1.3.3 Order Properties of Limits

We have discussed the algebraic properties of limits in the previous section, it turns out that the limit operator preserves the algebraic structure. There is another aspect of structure of the real number system that is equally important and that is the order structure. Does the limit operation preserve the order structure as well? The answer is yes. One marvellous consequence of this fact is the *squeeze theorem*, which turns out to be one of the most useful tools in finding limits.

Theorem 1.3.3: Order

Suppose that $\{x_n\}$ is convergent sequence and that

$$x_n \geq 0$$

for all n. Then

$$\lim_{n\to\infty} x_n \geq 0.$$

Proof. Set $A = \lim_{n\to\infty} x_n$. It follows that, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$|x_n - A| < \varepsilon.$$

From the above we obtain that for all $n \geq N$,

$$A > x_n - \varepsilon \geq -\varepsilon.$$

Since ε is any positive number, this only valids if $A \geq 0$. Thus we complete the proof. $\square$

Corollary 1.3.2

Suppose that $\{x_n\}$ and $\{y_n\}$ are convergent sequences and that

$$x_n \leq y_n$$

for all n. Then

$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n.$$

Proof. Write $z_n = y_n - x_n$, then apply the above theorem. $\square$

Theorem 1.3.4: Squeeze Theorem ★★★★★

Suppose that $\{x_n\}$ and $\{y_n\}$ are convergent sequences, that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$$

and that

$$x_n \le z_n \le y_n$$

for all n. Then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = \lim_{n\to\infty} y_n.$$

Proof. Let $A := \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$. It follows that for any $\varepsilon > 0$ we have $N_1, N_2 \in \mathbb{Z}^+$, such that

$$-\varepsilon < x_n - A < \varepsilon, \quad n \ge N_1;$$

and

$$-\varepsilon < y_n - A < \varepsilon, \quad n \ge N_2.$$

Set $N = \max\{N_1, N_2\}$, then both of the above ineqaulities hold for all $n \ge N$. Since $x_n \le z_n \le y_n$ we conclude that

$$-\varepsilon < x_n - A \le z_n - A \le y_n - A < \varepsilon, \quad n \ge N.$$

It follows that

$$-\varepsilon < z_n - A < \varepsilon, \quad n \ge N.$$

Thus the theorem follows by the definition of limit. □

Example 1.3.3

Let θ be some real number and find the limit of the sequence $\{x_n\}$, where

$$x_n = \frac{\sin n\theta}{n}.$$

Solution. Even the values of $\sin n\theta$ are quite unpredictable, we know those values lie inside the interval $[-1, 1]$. Thus

$$-\frac{1}{n} \le \frac{\sin n\theta}{n} \le \frac{1}{n}.$$

Both the outer sequences converge to 0 and hence the inside sequence must converge to 0 by the squeezed theorem. □

Example 1.3.4

Find the limit $\lim\limits_{n\to\infty} \dfrac{2^n}{n!}$.

Solution. Let $x_n = \dfrac{2^n}{n!}$. We have

$$0 \le x_n \le \frac{4}{n}.$$

Since $\lim\limits_{n\to\infty} \dfrac{4}{n} = 0$ we have $\lim\limits_{n\to\infty} \dfrac{2^n}{n!} = 0$ by the squeeze theorem. □

Example 1.3.5

Let

$$x_n = \frac{n!}{n^n}.$$

Show that

$$\lim_{n\to\infty} x_n = 0.$$

Proof. It is easy to see that

$$0 < x_n < \frac{1}{n}.$$

Then the claim follows from the squeez theorem. □

Example 1.3.6: Roots ★★★★

Let

$$x_n = \sqrt[n]{n}.$$

Find the limit $\lim\limits_{n\to\infty} x_n$.

Proof. Let $x_n = 1 + h_n$, where $h_n > 0$ for $n > 1$. We have, for $n > 1$

$$n = (1 + h_n)^n \ge \frac{n(n-1)}{2} h_n^2.$$

It follows that

$$0 < h_n < \sqrt{\frac{2}{n-1}}, \qquad \text{for } n > 1.$$

Thus we have

$$1 \le x_n = 1 + h_n \le 1 + \sqrt{\frac{2}{n-1}}.$$

Since $\lim\limits_{n\to\infty} \left(1 + \sqrt{\dfrac{2}{n-1}}\right) = 1$ we have $\lim\limits_{n\to\infty} x_n = 1$ by the squeeze theorem. □

1.3.4 Some frequently occurring null sequences

Sequences which converge to zero (such as the one in the last example) are often referred to as *null sequences*. Before we continue with the general theory of sequences, we take a brief look at some important and frequently occurring examples of such sequences.

Theorem 1.3.5: Common null sequences ★★★★★

(i) If $s > 0$ then $\frac{1}{n^s} \to 0$ as $n \to \infty$.

(ii) If $|\lambda| < 1$ then $\lambda^n \to 0$ as $n \to \infty$.

(iii) If $s \in \mathbb{R}$ and $|\lambda| > 1$ then $\frac{n^s}{\lambda^n} \to 0$ as $n \to \infty$.

(iv) If $s \in \mathbb{R}$ then $\frac{n^s}{n!} \to 0$ as $n \to \infty$.

(v) If $\lambda \in \mathbb{R}$ then $\frac{\lambda^n}{n!} \to 0$ as $n \to \infty$.

(vi) We have $\frac{n!}{n^n} \to 0$ as $n \to \infty$.

Due to time constraints we leave the proof of this as exercises. Only (iii) and (iv) need proofs, all others have been convered by different examples.

Informal remark. The above theorem tells us that there is a definite hierarchy on rate of growth of certain functions of n. This can be expressed as follows from the most rapid to the slowest growing:

(i)	Factorial growth,	$n!$,	
(ii)	Exponential growth,	λ^n, $(\lambda > 1)$	e.g. 2^n
(iii)	Polynomial growth,	n^s, $(s > 0)$	e.g. n^3
(iv)	Constants,	c	

Factorial terms will always "eventually beat" a power or exponent. As a specific example, eventually $n! > 100^n$. Similarly, powers of n, e.g. (n^3), never grow as rapidly as exponents of n, e.g. (2^n). [4]

However, although the sequences in the above theorem converge to zero, there is nothing to say that they are *decreasing*. For example

$$a_n := \frac{5^n}{n!}.$$

1.3.5 Monotone Sequences

One important special group of sequences is that of monotone ones. The terms of such sequences get steadily larger or smaller. The analysis of such sequences is much easier than for general sequences, but much deeper idea (completeness axiom of real numbers) is involved.

[4] These observations are useful as a *guide to your intuition* about what a given sequence might behave like for large n.

Definition 1.3.2

A sequence $\{x_n\}$ is (strictly) increasing if

$$x_n < x_{n+1}$$

holds for all $n \in \mathbb{Z}_+$. A sequence $\{y_n\}$ is (strictly) decreasing if

$$y_n > y_{n+1}$$

holds for all $n \in \mathbb{Z}_+$.

Sometimes we encounter sequences that increase or decrease in a less strictly manner. The following definition is to describe such situations.

Definition 1.3.3

A sequence $\{x_n\}$ is nondecreasing if

$$x_n \leq x_{n+1}$$

holds for all $n \in \mathbb{Z}_+$. A sequence $\{y_n\}$ is nonincreasing if

$$y_n \geq y_{n+1}$$

holds for all $n \in \mathbb{Z}_+$.

Note that every strictly increasing sequence is also nondecreasing but not conversely. Monotonic sequences are those fall into any one of these four categories (increasing, decreasing, nonincreasing, and nondecreasing). We have the following results related to the convergence issues for a monotonic sequence

Theorem 1.3.6: Monotone Convergence Theorem ★★★★★

Suppose that $\{x_n\}$ is a monotonic sequence. Then, $\{x_n\}$ is convergent if and only if $\{x_n\}$ is bounded.

Proof. Assume $\{x_n\}$ is non-decreasing. (If it is non-increasing, set $y_n = -x_n$ and work with y_n instead.) The 'only if' part follows form Theorem 1.3.1. Now we turn to the 'if' part. Since $\{x_n\}$ is bounded, thus its supremum exists[5]. Set

$$L := \sup\{x_n : n \in \mathbb{Z}^+\}.$$

For any $\varepsilon > 0$, since L is the least upper bound thus $L - \varepsilon$ is not an upper bound for the set $\{x_n : n \in \mathbb{Z}^+\}$. It follows that there exists an $N \in \mathbb{N}$, such that $x_N > L - \varepsilon$. Due to

[5] The existence of such L is guaranteed by the completeness axiom of real numbers, or, if you prefer, the definition of the real number system.

the fact that the sequence $\{x_n\}$ is non-decreasing, we have

$$L - \varepsilon < x_N \leq x_n \leq L < L + \varepsilon,$$

for all $n \geq N$. Put it differently,

$$L - \varepsilon < x_n < L + \varepsilon,$$

for all $n \geq N$. Thus the theorem is proved. □

The convergence issue for a monotonic sequence is straightforward. However, to prove such a result we need to exploit a much deeper and fundamental fact - the completeness axiom of real numbers. Later we will see that the monotone convergence theorem will become our primary theoretical tool in investigating convergence of sequence[6].

The Monotone Convergence Theorem tells us that if we wish to prove that a given sequence converges, then it is enough to prove that it is both increasing and bounded above (or decreasing and bounded below). This often turns out to be a fruitful strategy, as we illustrate now.

Example 1.3.7: Natural constant e ★★★★★

Let

$$x_n = \left(1 + \frac{1}{n}\right)^n.$$

Show that $\{x_n\}$ is convergent.

To prove this, we need the following Bernoulli inequality,

$$(1+x)^n \geq 1 + nx, \quad \forall n \in \mathbb{Z}^+ \text{ and } x \geq -1.$$

This inequality can be proved by induction, thus we leave it as an exercise.

Proof. For any $n \in \mathbb{Z}^+$, by Bernoulli inequality,

$$\begin{aligned}
\frac{x_{n+1}}{x_n} = \frac{n^n(n+2)^{n+1}}{(n+1)^{2n+1}} &= \left(\frac{n^2+2n}{(n+1)^2}\right)^n \frac{n+2}{n+1} \\
&= \left(1 - \frac{1}{n^2+2n+1}\right)^n \frac{n+2}{n+1} \\
&\geq \left(1 - \frac{n}{n^2+2n+1}\right) \frac{n+2}{n+1} \\
&= \frac{n^3+3n^2+3n+2}{n^3+3n^2+3n+1} > 1.
\end{aligned}$$

[6]As a matter of fact, the Monotone Convergence Theorem is equivalent to the Completeness axiom of real numbers. Both of them are most fundamental properties of real numbers.

Set $y_n = \left(1 + \dfrac{1}{n}\right)^{n+1}$ for all $n \in \mathbb{Z}^+$. Similarly as the above, for any positive integer $n \geq 2$, by Bernoulli inequality again,

$$\begin{aligned}\frac{y_{n-1}}{y_n} &= \frac{n^{2n+1}}{(n-1)^n(n+1)^{n+1}} = \left(\frac{n^2}{n^2-1}\right)^n \frac{n}{n+1} \\ &= \left(1 + \frac{1}{n^2-1}\right)^n \frac{n}{n+1} \\ &\geq \left(1 + \frac{n}{n^2-1}\right) \frac{n}{n+1} \\ &= \frac{n^3+n^2-n}{n^3+n^2-n-1} > 1.\end{aligned}$$

It follows that $\{x_n\}$ and $\{y_n\}$ are increasing and decreasing respectively. Furthermore, for any $n \in \mathbb{Z}^+$, we have

$$2 = x_1 \leq x_n \leq y_n \leq y_1 = 4.$$

Thus we conclude that both $\{x_n\}$ and $\{y_n\}$ are bounded. By Monotone Convergence Theorem we prove that both $\{x_n\}$ and $\{y_n\}$ are convergent.

Since $y_n = x_n(1 + \frac{1}{n})$, by taking $n \to \infty$ on both sides, it follows that both sequences converge to a same limit. $\square$

Denote the limit by

$$e := \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n,$$

which is called the exponential constant.

1.3.6 Compound interest

Jacob Bernoulli discovered this constant e in 1683 by studying a question about compound interest:[7]

An account starts with \$1.00 *and pays* 100% *interest per year. If the interest is credited once, at the end of the year, the value of the account at year-end will be* \$2.00. *What happens if the interest is computed and credited more frequently during the year?*

If the interest is credited twice in the year, the interest rate for each 6 months will be 50%, so the initial 1 is multiplied by 1.5 twice, yielding $\$1.00 \times 1.5^2 = 2.25$ at the end of the year. Compounding quarterly yields $\$1.00 \times 1.25^4 = 2.4414...$, and compounding monthly yields $\$1.00 \times (1+1/12)^{12} = 2.613035...$ If there are n compounding intervals, the interest for each interval will be $100\%/n$ and the value at the end of the year will be $\$1.00 \times (1 + 1/n)^n$.

[7] https://en.wikipedia.org/wiki/E_(mathematical_constant)#Compound_interest

1.4 Bolzano-Weierstrass Theorem

1.4.1 Subsequences

Definition 1.4.1: ★★★★

Let $\{x_n\}$ be a sequence. Then a subsequence of this sequence is a sequence

$$\{x_{n_i}\}_{i=1}^{\infty}$$

with $\{n_i\}$ being an increasing sequence of positive integers.

Example 1.4.1

We consider the sequence

$$1, 1, 1, 1, \cdots$$

to be a subsequence of the sequence

$$\{(-1)^n\}.$$

Because it contains just the even number index terms of the original sequence. Here $n_1 = 2$, $n_2 = 4$, $n_3 = 5$, $\cdots$.

Example 1.4.2

We consider the unbounded sequence $\{x_n\}$ with

$$x_n = \begin{cases} 0, & \text{for odd } n; \\ n, & \text{for even } n. \end{cases}$$

It has a subsequence

$$0, 0, 0, \cdots,$$

or

$$\{x_{2k-1}\}_{k=1}^{\infty}.$$

In the above example the letter n and k are so-called dummy variables. In particular, the expressions $\{x_{2k-1}\}_{k=1}^{\infty}$ and $\{x_{2j-1}\}_{j=1}^{\infty}$ describe the same sequence. We could also have written $\{x_{2n-1}\}_{n=1}^{\infty}$ for this particular subsequence of $\{x_n\}_{n=1}^{\infty}$.

Theorem 1.4.1: ★★★★

If $\{x_n\}$ is a sequence of real numbers converging to ℓ, then every subsequence of x_n converges to ℓ.

Proof. Since $\{x_n\}$ converges to ℓ there exists $N \in \mathbb{Z}^+$ such that

$$|x_n - \ell| < \varepsilon \quad \text{for all } n \geq N.$$

Let $\{x_{n_k}\}$ being a subsequence of $\{x_n\}$. Hence, if $k \geq N$ then $n_k \geq N$. With this observation we obtain

$$|x_{n_k} - \ell| < \varepsilon \quad \text{for all } k \geq N.$$

Thus we finished the proof. □

Corollary 1.4.1: ★★★★

If a seqence $\{x_n\}$ possesses two subsequences which converge to distinct limits, then $\{x_n\}$ does not converge.

Proof. Suppose for a contradiction that $\{x_n\}$ is a convergent sequence which has two subsequences converging to distinct limits, which contradicts with the last theorem. □

Example 1.4.3

Prove that the sequence $\{a_n\}$ given by $a_n = (-1)^n$ does not converge.

Proof. Consider the subsequences $\{a_{2n}\}$ and $\{a_{2n-1}\}$. Since $a_{2n} = 1$ for all $n \in \mathbb{Z}^+$, we have that $a_{2n} \to 1$ as $n \to \infty$. Since $a_{2n-1} = -1$ for all $n \in \mathbb{Z}^+$, we have that $a_{2n} \to -1$ as $n \to \infty$. Since $1 \neq -1$, we conslude that $\{a_n\}$ does not converge. □

1.4.2 Bolzano-Weierstrass

Given a sequence can we always select a subsequence that is monotonic? The answer is yes.

Theorem 1.4.2: ★★★★★

Every sequence contains a monotonic subsequence.

Proof. Consider the set

$$S := \{n \in \mathbb{Z}^+ : a_m \leq a_n \quad \text{for all } m > n\}.$$

Since we don't know whether it is a finite set or an infinite set. Let us consider these two possibilities in turn.

First assume that S is finite. Let n_1 be any natural number that is larger than all of the elements of S (since S is finite). Then we have $n_1 \notin S$. Hence, by the definition of S, there exists $n_2 > n_1$ such that $a_{n_2} > a_{n_1}$. Similarly, since $n_2 \notin S$ there exists $n_3 > n_2$ such that $a_{n_3} > a_{n_2}$. Continuing in this way we obtain an increasing subsequence $\{a_{n_k}\}$, as desired.

Now we turn to the case of infinite S. We may write $S := \{n_k : k \in \mathbb{Z}^+\}$ where $\{n_k\}$ is a strictly increasing sequence of natural numbers. We claim that the sequence $\{a_{n_k}\}$ is non-increasing. To prove this, let $k \in \mathbb{Z}^+$ be arbitrary and observe that since we have that $a_m \leq a_{n_k}$ for all $m > n_k$. Since $n_{k+1} > n_k$, in particular we have that $a_{n_{k+1}} \leq a_{n_k}$. Thus $\{a_{n_k}\}$ is non-increasing. $\square$

Given a sequence can we always select a subsequence that is convergent? With the previous theorem, we have more confidence to say yes. But not quite yes without a further condition - boundedness.

Theorem 1.4.3: Bolzano-Weierstrass ★★★★★

Every bounded sequence contains a convergent subsequence.

1.5 Cauchy Convergence Criterion

What are the necessary and sufficient conditions for a sequence to converge? It is always too ambitious to ask such kind of questions. The answer is more likely either too close to the definition or too special, such like

- A sequence $\{x_n\}$ is convergent if and only if there exists $A \in \mathbb{R}$, so that for all $\varepsilon > 0$ there exists N with the property that

  $$|x_n - A| < \varepsilon$$

  for all $n \geq N$.

- A monotonic sequence $\{x_n\}$ is convergent if and only if it is bounded.

The first characterisation is nothing else but the definition of convergence, which is applicable to all sequences. However, it is not quite easy to be used since before applying this characterisation one needs to determine the limit A, which, in general, is not easy to get. While the second one is too special, it only applies to monotonic sequences.

A more useful characterisation is due to Cauchy.

Theorem 1.5.1: Cauchy Convergence Criterion ★★★

A sequence $\{x_n\}$ is convergent iff for each $\varepsilon > 0$ there exists an integer N with the property that

$$|x_n - x_m| \leq \varepsilon$$

for all $n \geq N$ and $m \geq N$.

It says that a sequence converges iff the terms of the sequence are eventually arbitrarily close to each other. The main advantage of the Cauchy Criterion, comparing the definition characterisation, is that it describes a convergent sequence without knowing the actual value of the limit. The proof of this theorem appeal to Bolzano-Weierstrass theorem.

Proof. See Lebl, *Basic analysis: introduction to real analysis*, Theorem 2.4.5. □

1.5.1 Completeness of the real number system*

Now we have seen the following sequence of implications

$$\begin{aligned} & \text{Completeness Axiom of reals} \\ \Longrightarrow\ & \text{Monotonic Convergence Theorem} \\ \Longrightarrow\ & \text{Bolzano-Weierstrass Theorem} \\ \Longrightarrow\ & \text{Cauchy Convergence Criterion} \end{aligned}$$

It turns out that all these statements are equivalent to each other. Each of them is a characterisation of the completeness of the real number system. As a matter of fact, these statements are defining properties and building blocks of reals. It is these properties that single out the real number system from other dense ordered fields, such like $\mathbb{Q}$.

Example 1.5.1

Give examples to show that Completeness Axiom of reals, Monotonic Convergence Theorem, Bolzano-Weierstrass Theorem, and Cauchy Convergence Criterion don't hold if we restrict ourselves in $\mathbb{Q}$.

Chapter 2

Series

2.1 Series

The ordered sum of a sequence is called a series and is usually denoted by

$$\sum_{k=1}^{\infty} a_k.$$

Now that we have developed a theory of limits of sequences, we will use this theory to develop a theory of series. However, before giving a precise meaning of such infinite summation, it does NOT mean much. For example

$$\sum_{k=1}^{\infty} (-1)^k,$$

which is meaningless since it is NOT convergent, or it is divergent. It turns out that such infinite summation only make sense (or converges) for some particular sequences, not all of them. To further clarify this point, we have

Definition 2.1.1: ★★★★★

Let $\{a_k\}$ be a sequence of real numbers. Then we write

$$\sum_{k=1}^{\infty} a_k = c$$

and say that the series converges if the sequence of partial sums of the series

$$s_n = \sum_{k=1}^{n} a_k$$

converges to c. Otherwise, it is said to be divergent.

It reduces the study of convergence/divergence of series to the study of convergence/divergence of **sequences**. We can apply what we have for sequences to develop a theory of series.

2.1.1 Properties

The first theorem we obtain directly from the sequence theory is the following

Theorem 2.1.1: ★★★

If a series $\sum_{k=1}^{\infty} a_k$ converges, then the sum is unique.

Since the convergence or divergence of a series depends on the convergence or divergence of the sequence of partial sum

$$s_n = \sum_{k=1}^{n} a_k$$

and the value of the series is the limit of the above sequence. Keep this in mind, we can similarly obtain

Theorem 2.1.2: ★★★

If series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge, then so does the series

$$\sum_{k=1}^{\infty} (a_k + b_k)$$

and

$$\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$$

Theorem 2.1.3: ★★★

If series $\sum_{k=1}^{\infty} a_k$ converges, then so does the series $\sum_{k=1}^{\infty} ca_k$ and

$$\sum_{k=1}^{\infty} ca_k = c \sum_{k=1}^{\infty} a_k$$

Theorem 2.1.4: ★★★

Let $M \geq 1$ be any integer. Then the series $\sum_{k=1}^{\infty} a_k$ converges if and only if the series $\sum_{k=1}^{\infty} a_{M+k} \left(= \sum_{k=M+1}^{\infty} a_k \right)$ converges.

Remark 2.1.1. This last theorem asserts that it is the behaviour of the tail that determines the convergence/divergence of the series. Thus in question of convergence, similar as in the sequence case, one can ignore the first finite many terms of the series (though of course those elements do influence which value the series converges to).

2.1.2 Examples

There is one type of series, called telescoping series,

$$\sum_{k=1}^{\infty}(a_k - a_{k+1}),$$

缩项级数，它可以写成如下形式

which are easy to sum.

Example 2.1.1: Telescoping Series ★★★★

Consider the series

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)} = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1.$$

Similarly, one can consider the following more general form

$$\sum_{k=1}^{\infty}\frac{a}{(k)(k+n_0)}$$

with $n_0 \in \mathbb{Z}^+$ and $a \in \mathbb{R}$.

Another type of series, which is easy to sum as well, is the geometric series.

Example 2.1.2: Geometric Series ★★★★

Another frequently used but easily computed class of series is the geometric series. From the formula

几何级数（Geometric Series）也称为等比级数

$$\sum_{k=1}^{n} r^{k-1} = \frac{1-r^n}{1-r} \quad (r \neq 1),$$

we see that

$$\sum_{k=1}^{\infty} r^{k-1} = \lim_{n\to\infty}\frac{1-r^n}{1-r} = \frac{1}{1-r}$$

provided $|r| < 1$. And the series diverges for $|r| \geq 1$.

Example 2.1.3: Harmonic Series

Consider the sequence $\{\frac{1}{k}\}$, which converges to 0. However, the corresponding series

调和级数

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots ,$$

which is called the harmonic series, diverges.

Example 2.1.4: p-Harmonic Series ★★★★★

Consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots ,$$

for any parameter $0 < p < \infty$ is called the p-harmonic series. We have considered the case $p = 1$ which reduces to the harmonic series. What about the other value of p?

Proof. First we observe that if $0 < p < 1$, then

$$\frac{1}{k} < \frac{1}{k^p}.$$

Since the former series has unbounded partial sums, so does this p-harmonic series.

For $p > 1$, it turns out that the series converges. To prove this we group the terms in the same manner as we did for the harmonic series

$$\begin{aligned} 1 + \left[\frac{1}{2^p} + \frac{1}{3^p}\right] + \left[\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right] + \left[\frac{1}{8^p} + \cdots + \frac{1}{15^p}\right] + \cdots \\ \le 1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \cdots \\ \le \frac{1}{1 - 2^{1-p}} \end{aligned}$$

since the second line is a convergent geometric series with ratio 2^{1-p}. We have obtained an upper bound for the partial sums

$$\sum_{k=1}^{n} \frac{1}{k^p}$$

for all $p > 1$. The series converges since the partial sums are both increasing and bounded above. $\square$

Example 2.1.5: Alternating Harmonic Series ★★★★★

Consider the sequence $\{(-1)^{k-1}\frac{1}{k}\}$, which converges to 0 as well. The corresponding series 交错调和级数

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots,$$

which is called the alternating harmonic series, converges.

Proof. By denoting the partial sums by

$$s_n = \sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k},$$

we see that the subsequence $\{s_{2n}\}$ is increasing and $\{s_{2n-1}\}$ is decreasing. Furthermore, we have

$$\frac{1}{2} \leq s_{2n} \leq s_{2n-1} \leq 1.$$

By the Monotonic Convergence Theorem, we have that both limits

$$\lim_{n\to\infty} s_{2n} \quad \text{and} \quad \lim_{n\to\infty} s_{2n-1}$$

exist. Finally, since

$$s_{2n} - s_{2n-1} = -\frac{1}{2n} \to 0$$

we can conclude that $\lim\limits_{n\to\infty} s_n$ exists. □

2.1.3 Necessary conditions of convergence

The following result is a necessary condition of convergence. It is usually called null test or zero test for convergence. If a series fails the null test, which means that the sequence of the terms doesn't converge to zero, then the series must diverge. See next section for more discussion.

Theorem 2.1.5: ★★★★★

If $\sum\limits_{k=1}^{\infty} a_k$ converges, then $a_k \to 0$ as $k \to \infty$.

Proof. Denote the sequence of partial sums of a convergent series $\sum\limits_{k=1}^{\infty} a_k$ by $\{s_n\}$. It follows from the definition that there exists $C \in \mathbb{R}$, such that

$$\lim_{n\to\infty} s_n = C.$$

Then,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = C - C = 0.$$

□

The converse is not true, such as harmonic series. To show convergence of a series it is not enough to know that the terms tend to zero. In the following sections, we shall see that the rate at which the terms tend to zero and the manner in which the terms tend to zero also matter a lot.

2.1.4 Absolute Convergence

We have seen that for the convergence/divergence of a nonnegative series it can be reduced to the boundedness of the sequence of its partial sums. Such reduction does not hold for general series with both positive and negative terms. For a general series

$$\sum_{k=1}^{\infty} a_k,$$

one may consider

$$\sum_{k=1}^{\infty} |a_k|,$$

whose terms are all nonnegative. If we could show the later series is convergent, so is the former one.

Theorem 2.1.6: ★★★★★

If the series $\displaystyle\sum_{k=1}^{\infty} |a_k|$ converges, then so does the series $\displaystyle\sum_{k=1}^{\infty} a_k$

This is only a sufficient condition for convergence of a series.

Definition 2.1.2: ★★★★★

A series $\displaystyle\sum_{k=1}^{\infty} a_k$ is said to be absolutely convergent if $\displaystyle\sum_{k=1}^{\infty} |a_k|$ converges.

In order to distinguish convergence from absolute convergence, we refer to the former as non-absolutely convergence, or conditional convergence.

Definition 2.1.3

A series $\displaystyle\sum_{k=1}^{\infty} a_k$ is said to be non-absolutely (or conditionally) convergent if it converges but the series $\displaystyle\sum_{k=1}^{\infty} |a_k|$ diverges.

Every absolutely convergent series is convergent, but not conditional convergent.

Example 2.1.6

The alternating harmonic series is conditionally convergent.

2.1.5 Good properties of absolutely convergent series*

The first two theorems in this section essentially say that

- You may rearrange the terms in an absolutely convergent series without affecting the limit.
- You may rearrange the terms in a conditionally convergent series and build a convergent series that converges to any limit you choose.

Furthermore

- You can make sense of the "product" of two absolutely convergent series.

A rearrangement of a series is defined as follows. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be any permutation of the natural numbers. That is σ is a bijection of $\mathbb{N}$ to itself. If $b_n := a_{\sigma(n)}$ then $\{b_n\}$ is said to be a *rearrangement* of $\{a_n\}$.

These results are more formally stated as follows:

Theorem 2.1.7: Dirichlet's Theorem: rearrangements of series

Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series of real numbers. If $\{b_n\}$ is any rearrangement of $\{a_n\}$ then

1. $\displaystyle\sum_{n=1}^{\infty} b_n$ is an absolutely convergent series.
2. $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$.

Proof. [See Spivak, Calculus, pg 402]
[See Hart, Theorem 3.4.4]

Theorem 2.1.8: Conditional convergence

If the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges but does *not* converge absolutely (i.e. $\displaystyle\sum_{n=1}^{\infty} a_n$ is conditionally convergent) and $\gamma \in \mathbb{R}$ is *any real number*, then there exists a rearrangement $\{b_n\}$ of the sequence $\{a_n\}$ so that

$$\sum_{n=1}^{\infty} b_n = \gamma.$$

Proof. [See Spivak, Calculus, pg 401]

2.1.6 Products of series*

Looking back at the section on the algebra of limits for series we see that *so far we have no rule to calculate the product of two series*, such as

$$\left(\sum_{n=1}^{\infty} a_n\right) \times \left(\sum_{n=1}^{\infty} b_n\right)$$

How would we do this?

It is natural to expect that the normal rules of algebra apply to products of infinite sums

$$(a_1 + a_2 + a_3 + \cdots) \times (b_1 + b_2 + b_3 + \cdots),$$

allowing us to express this as a sum of terms of the form

$$a_n b_m$$

where n and m take all possible values in $\mathbb{N}$. Using the "$\cdots$" notation, these terms are

$$\begin{array}{ccccc} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 & \cdots \\ a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 & \cdots \\ a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 & \cdots \\ a_4b_1 & a_4b_2 & a_4b_3 & a_4b_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

When we form the sum (or "double series")

$$\sum_{n,m=1}^{\infty} a_n b_m$$

from these terms, *in what order should we take them?*

We need to ensure that the order is unimportant. This is precisely where the concept of absolute convergence comes to our rescue.

One way to arrange them is to ensure that $n + m = k$ for each $k \geq 2$,

e.g. $k = 2$

$$a_1 b_1$$

e.g. $k = 3$

$$a_1 b_2 + a_2 b_1$$

e.g. $k = 4$

$$a_1 b_3 + a_2 b_2 + a_3 b_1$$

In general

$$\sum_{r=1}^{k-1} a_r b_{k-r}, \quad k \geq 2,$$

which corresponds to taking diagonals in the above array. This is known as the *Cauchy product*

$$\sum_{k=2}^{\infty} \sum_{r=1}^{k-1} a_r b_{k-r},$$

and equals

$$a_1b_1 + (a_1b_2 + a_2b_1) + (a_1b_3 + a_2b_2 + a_3b_1) + \cdots .$$

Theorem 2.1.9

Assume $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ are absolutely convergent series of real numbers. Then

$$\sum_{n,m=1}^{\infty} a_n b_m$$

is absolutely convergent[a] and

$$\sum_{n,m=1}^{\infty} a_n b_m = \left(\sum_{n=1}^{\infty} a_n\right) \times \left(\sum_{n=1}^{\infty} b_n\right).$$

[a]As a consequence of this the double series

$$\sum_{n,m=1}^{\infty} a_n b_m$$

is rearrangement-invariant, meaning that it doesn't matter in which order we do the summation. A popular choice would be the Cauchy product, described above.

2.2 Convergence Tests

2.2.1 Null test

If a series $\sum_{k=1}^{\infty} a_k$ converges then $a_k \to 0$. If the terms don't tend to zero then the series doesn't converges. This should be the first step when consider any series.

Theorem 2.2.1: Null test ★★★★

If the terms of the series $\sum_{k=1}^{\infty} a_k$ do not converge to zero, then the series diverges.

For example,

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{k}}$$

diverges. To see this, it only needs to observe that the terms tend to 1 as $k \to \infty$. If a sequence $\{a_n\}$ does converge to zero, then the series $\sum_{k=1}^{\infty} a_k$ may or may not be convergent; it depends on the series. For example, we have seen that the harmonic series is divergent despite the fact that $\{\frac{1}{n}\}$ converges to zero.

2.2.2 Comparison Tests

Theorem 2.2.2: Comparison Tests ★★★★

Given two series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ such that $0 \leq a_k \leq b_k$ for all k.

- If the larger series converges, then so does the smaller series.
- If the smaller series diverges, then so does the larger series.

Example 2.2.1

Show the convergence of the series

$$\sum_{k=1}^{\infty} \frac{k+3}{k^3 + 2k^2 + 3k + 1}.$$

Solution. Note that

$$\frac{k+3}{k^3 + 2k^2 + 3k + 1} = \frac{1 + 3/k}{k^2(1 + 2/k + 3/k^2 + 1/k^3)} \leq \frac{4}{k^2}.$$

Since the series $\sum_{k=1}^{\infty} \frac{4}{k^2}$ converges, and so does our series by comparison test. □

Example 2.2.2

Show the divergence of the series

$$\sum_{k=1}^{\infty} \sqrt{\frac{k+4}{k^2+3k+1}}.$$

Solution. Note that

$$\sqrt{\frac{k+4}{k^2+3k+1}} = \sqrt{\frac{1+4/k}{k(1+3/k+1/k^2)}} \geq \sqrt{\frac{1}{5k}}.$$

Since the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges, and so does our series by comparison test. □

2.2.3 Ratio Tests

Theorem 2.2.3: Ratio Tests ★★★★

If terms of the series $\sum_{k=1}^{\infty} a_k$ are all positive and the ratios

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$$

then the series is convergent.

Proof. If

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$$

then there is a number $\alpha < 1$ so that there exists N large enough s.t.

$$\frac{a_{k+1}}{a_k} < \alpha$$

for $k \geq N$. It follows that

$$a_{N+k} < a_{N+k-1}\alpha < \cdots < a_N\alpha^k.$$

By comparison test we conclude the series $\sum_{k=N}^{\infty} a_k$ converges, so does $\sum_{k=1}^{\infty} a_k$ by the definition.

□

If terms of the series $\sum_{k=1}^{\infty} a_k$ are all positive and the ratios

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = L.$$

- If $L < 1$, then the series $\sum_{k=1}^{\infty} a_k$ is convergent.
- If $L > 1$, then the series $\sum_{k=1}^{\infty} a_k$ is divergent. Moreover, the terms $a_k \to \infty$.
- If $L = 1$, then the series $\sum_{k=1}^{\infty} a_k$ may diverge or converge. Could you give some of such examples? (Check $\sum \frac{1}{k^2}$ and $\sum \frac{1}{k}$.)

Example 2.2.3

Show the convergence of the series $\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$

Solution. Note that

$$\frac{a_{k+1}}{a_k} = \frac{((k+1)!)^2}{(2k+2)!}\frac{(2k)!}{(k!)^2} = \frac{(k+1)^2}{(2k+2)(2k+1)} \to \frac{1}{4}.$$

Consequently, this is a convergent series. □

2.2.4 Root Tests

Theorem 2.2.4: Root Tests ★★★★

If terms of the series $\sum_{k=1}^{\infty} a_k$ are all nonnegative and the roots

$$\lim_{k\to\infty} \sqrt[k]{a_k} < 1$$

then the series is convergent.

Proof. If

$$\lim_{k\to\infty} \sqrt[k]{a_k} < 1$$

then there is a number $\alpha < 1$ so that there exists N large enough s.t.

$$\sqrt[k]{a_k} < \alpha$$

for $k \geq N$. It follows that

$$a_k < \alpha^k$$

for $k \geq N$. By comparison test we conclude the series $\sum_{k=N}^{\infty} a_k$ converges, so does $\sum_{k=1}^{\infty} a_k$ by the definition. □

If terms of the series $\sum_{k=1}^{\infty} a_k$ are all positive and the roots

$$\lim_{k\to\infty} \sqrt[k]{a_k} = L.$$

- If $L < 1$, then the series $\sum_{k=1}^{\infty} a_k$ is convergent.
- If $L > 1$, then the series $\sum_{k=1}^{\infty} a_k$ is divergent. Moreover, the terms $a_k \to \infty$.
- If $L = 1$, then the series $\sum_{k=1}^{\infty} a_k$ may diverge or converge. Could you give some of such examples? (Check $\sum \frac{1}{k^2}$ and $\sum \frac{1}{k}$.)

Example 2.2.4

Show the convergence/divergence of the series $\sum_{k=1}^{\infty} kr^k$ for $r > 0$.

Solution. Note that

$$\lim_{k\to\infty} \left(kr^k\right)^{1/k} = \lim_{k\to\infty} \sqrt[k]{k}r = r.$$

So the series converges for all $0 < r < 1$ and diverges for all $r > 1$. The case $r = 1$ is inconclusive for the root test. However, when $r = 1$, we have $\sum_{k=1}^{\infty} kr^k = \sum_{k=1}^{\infty} k$, which is obviously divergent. □

Note that this exercise can also be solved by ratio test. Could you determine the convergence of $\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$ by root test?

2.2.5 Integral Tests

Theorem 2.2.5: Integral Tests ★★★★

Let f be a nonnegative decreasing function on $[1, \infty)$. Then

$$\lim_{X\to\infty}\int_1^X f(x)dx$$

converges if and only if the series $\sum_{k=1}^{\infty} f(k)$ converges.

Proof. Since f is decreasing we have

$$\int_k^{k+1} f(x)dx \le f(k) \le \int_{k-1}^{k} f(x)dx.$$

Thus

$$\int_1^{n+1} f(x)dx \le \sum_{k=1}^{n} f(k) \le f(1) + \int_1^{n} f(x)dx.$$

The series converges if and only if the partial sums are bounded. □

Example 2.2.5

Determine for what value of p the following p-harmonic series, i.e. $\sum_{k=1}^{\infty}\frac{1}{k^p}$, is convergent.

Solution. First we compute[1]

$$\lim_{X\to\infty}\int_1^X \frac{dx}{x^p} = \lim_{X\to\infty}\frac{1}{p-1}\left(1-\frac{1}{X^{p-1}}\right),$$

which converges to $\frac{1}{p-1}$ for $p > 1$ and diverges for $0 < p < 1$.

For $p = 1$, we have

$$\lim_{X\to\infty}\int_1^X \frac{dx}{x} = \lim_{X\to\infty} \ln X.$$

Thus, the series diverges for $p = 1$. □

[1] The only thing you need to know is the formular

$$\int_1^X x^{-p}dx = \frac{1}{p-1}\left(1-\frac{1}{X^{p-1}}\right),$$

for $X > 1$. I suppose that you have seen this formular. If not, you will learn it very soon.

Example 2.2.6

Determine for what value of p the following series, i.e. $\sum_{k=1}^{\infty} \frac{1}{k(\ln k)^p}$, is convergent.

Solution. First we have

$$\lim_{X\to\infty} \int_e^X \frac{dx}{x(\ln x)^p} = \lim_{X\to\infty} \frac{1}{p-1}\left(1 - \frac{1}{(\ln X)^{p-1}}\right),$$

which converges to $\frac{1}{p-1}$ for $p > 1$ and diverges for $0 < p < 1$. For $p = 1$, we have

$$\lim_{X\to\infty} \int_e^X \frac{dx}{x\ln x} = \lim_{X\to\infty} \ln\ln X.$$

Thus, the series diverges for $p = 1$. □

2.2.6 Alternating Series Test

In this section, we consider the series with terms of changing signs. If such series is not absolutely convergent or if it is not easy to check, what should we do? Usually, there is no general way to determine its convergence. However, there is a special kind of series with alternating positive and negative terms can be handled easily provided that the absolute value of its terms decreases. Recall the alternating harmonic series

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k},$$

which is not absolute convergent. However, we have showed that it is convergent. Similar idea used their can be used to show

Theorem 2.2.6: Alternating Series Test ★★★★★

The series

$$\sum_{k=1}^{\infty}(-1)^{k-1}a_k,$$

where the terms alternate in sign, converges if the sequence $\{a_k\}$ decreases monotonically to zero.

Proof. By denoting the partial sums by

$$s_n = \sum_{k=1}^{n}(-1)^{k-1}a_k,$$

we see that the subsequence $\{s_{2n}\}$ is increasing and $\{s_{2n-1}\}$ is decreasing by writing

$$s_{2n} = \sum_{k=1}^{n}(a_{2k-1} - a_{2k}),$$

and

$$s_{2n-1} = a_1 - \sum_{k=2}^{n} (a_{2k-2} - a_{2k-1}).$$

Furthermore, we have

$$a_1 - a_2 \leq s_{2n} \leq s_{2n-1} \leq a_1.$$

By the Monotonic Convergence Theorem, we have that both limits

$$\lim_{n\to\infty} s_{2n} \quad \text{and} \quad \lim_{n\to\infty} s_{2n-1}$$

exist. Finally, since

$$s_{2n} - s_{2n-1} = a_{2n} \to 0,$$

as $n \to \infty$. We can conclude that $\lim_{n\to\infty} s_n$ exists.

□

Example 2.2.7: Alternating Harmonic Series

Consider the sequence $\{(-1)^{k-1}\frac{1}{k}\}$, which converges to 0 as well. The corresponding series

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots ,$$

which is called the alternating harmonic series, converges. Since

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

diverges, we conclude that $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k}$ converges conditionally.

Example 2.2.8: Alternating p-Harmonic Series

Consider the series

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots ,$$

for any parameter $0 < p < \infty$ is called the alternating p-harmonic series. We have considered the case $p = 1$ which reduces to the alternating harmonic series. What about the other value of p? It can be proven, similar as the alternating harmonic series, that the alternating p-harmonic series is converges for all $0 < p < \infty$. Furthermore, it is conditionally convergent for $0 < p \leq 1$, absolutely convergent for $p > 1$.(Exercise!)

Example 2.2.9: Alternating Series

Consider the series

$$\sum_{k=2}^{\infty}(-1)^{k-1}\frac{1}{k(\ln k)^p}$$

for any parameter $-\infty < p < \infty$. We have considered the case $p \geq 0$ which reduces to the alternating series. What about the value of $p < 0$?

Chapter 3

Power Series

For us a power series is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n$$

where $\{a_n\}$ is a sequence of real numbers and $x \in \mathbb{R}$. We consider $\{a_n\}$ as fixed for this series, and x as a variable. Clearly the series might converge for some values of x and not for others. Given a power series, we would of course like to be able to establish which values of x lead to convergence, and which lead to divergence.

Define the subset of real numbers

$$S := \left\{ x \in \mathbb{R} : \sum_{n=0}^{\infty} a_n x^n \text{ converges} \right\}.$$

Note that $S \neq \emptyset$, because $0 \in S$. The set S is also referred as *set of convergence* of the series. We may consider the power series $\sum_{n=0}^{\infty} a_n x^n$ as a *function*

$$f : S \longrightarrow \mathbb{R},$$

defined by

$$f(x) := \sum_{n=0}^{\infty} a_n x^n, \quad x \in S.$$

Please note that the domain of this function is S, where the left hand side makes sense.

Example 3.0.1

Consider the geometric series

$$f(x) = \sum_{n=0}^{\infty} x^n.$$

We already know that for the geometric series, $S = (-1, 1)$. Moreover, in this particular case we have a "closed form" expression for f, namely

$$f(x) = \frac{1}{1-x}, \quad \forall x \in S = (-1, 1).$$

3.1 Radius of Convergence

Question. Can the set of convergence S be *any* subset of $\mathbb{R}$, or does it have to take some special form? For example, can we find a power series such that the corresponding S is the union of two disjoint intervals?

The answer to this question is beautifully simple.

Theorem 3.1.1: ★★★★★

Given a power series $\sum_{n=0}^{\infty} a_n x^n$, either it converges absolutely for all $x \in \mathbb{R}$, or there exists $R \in [0, \infty)$ such that

(1) it converges absolutely when $|x| < R$

(2) it diverges when $|x| > R$.

Proof. Interested readers can find a proof in Section 10.2 of [4], for instance Theorem 10.8 and Theorem 10.9. □

We refer to R as the *Radius of Convergence* of the power series. If the power series is absolutely convergent for all $x \in \mathbb{R}$ then we say that it has infinite radius of convergence, and write $R = \infty$.

Remark 3.1.1

We can restate the theorem as

$$(-R, R) \subseteq S \subseteq [-R, R]$$

and the power series converges absolutely in $(-R, R)$. In particular we see that S is always an *interval*.

Sometimes $(-R, R) \subsetneq S \subsetneq [-R, R]$; that is, it might be the case that $S = [-R, R)$ or $S = (-R, R]$.

Example 3.1.1: ★★★★★

Indeed, consider for example the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

1. When $|x| > 1$ the general term $x^n/n \not\to 0$, and so the series diverges for $|x| > 1$ by the null sequence test.
2. When $|x| < 1$ the series converges (absolutely) by the comparison test because $|x^n/n| \leq |x|^n$ for all $n \in \mathbb{N}$.

It follows that $R = 1$.

1. For $x = 1$ we have the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges.
2. For $x = -1$ we have the alternating series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ which converges by the alternating series test.

It follows that $S = [-1, 1)$ for this particular power series.

3.2 Convergence Test

Fortunately there are some very easy-to-use theorems that allow us to find the radius of convergence of a given power series with ease (most of the time). These simply follow from applying the ratio and root tests to a general power series.

3.2.1 Ratio Test

Theorem 3.2.1: Ratio test for power series ★★★★

Consider the power series
$$\sum_{n=0}^{\infty} a_n x^n.$$
Suppose that
$$\frac{|a_{n+1}|}{|a_n|} \to \ell, \quad \text{as} \quad n \to \infty.$$
Then
$$R = \begin{cases} 0 & \text{if} \ \ \ell = \infty, \\ \frac{1}{\ell} & \text{if} \ \ \ell \in \mathbb{R}\backslash\{0\}, \\ \infty & \text{if} \ \ \ell = 0. \end{cases}$$

Proof. Suppose that $\ell \in \mathbb{R}$. By the ratio test of series Theorem 2.2.3, the series
$$\sum_{n=0}^{\infty} a_n x^n$$
converges absolutely if
$$\lim_{n\to\infty} \frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = |x| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x|\ell < 1,$$
and $\sum_{n=0}^{\infty} |a_n x^n|$ diverges if
$$\lim_{n\to\infty} \frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = |x| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x|\ell > 1.$$
Hence if $\ell \neq 0$, then $R = \frac{1}{\ell}$. If $\ell = 0$ then $|x|\ell < 1$ for all $x \in \mathbb{R}$, and so $R = \infty$.

If $\ell = \infty$, i.e if
$$\frac{|a_{n+1}|}{|a_n|} \to \infty$$
as $n \to \infty$, then the power series diverges for all $x \neq 0$ by the null test. Hence $R = 0$. □

Example 3.2.1

Find the radius of convergence of the power series
$$\sum_{n=0}^{\infty} (-1)^n n^2 x^n.$$
What is the convergence set S for this power series?

Solution. We shall use the ratio test Theorem 3.2.1. Let $a_n = (-1)^n n^2$ and observe that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^2}{n^2} = \left(1 + \frac{1}{n}\right)^2 \to 1$$

as $n \to \infty$, by the algebra of limits. Hence the radius of convergence

$$R = \frac{1}{1} = 1.$$

Thus $(-1, 1) \subseteq S \subseteq [-1, 1]$, and so it remains to determine whether the power series converges for $x = \pm 1$. If $x = 1$, then the power series becomes the series

$$\sum_{n=1}^{\infty} (-1)^n n^2,$$

which diverges by the null sequence test as $(-1)^n n^2 \not\to 0$. Thus $1 \notin S$. Similarly $-1 \notin S$, and so

$$S = (-1, 1).$$

□

Example 3.2.2

Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

What is the convergence set S for this power series?

Solution. We shall use the ratio test[1]. Let $a_n = 1/n!$ and observe that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0$$

as $n \to \infty$. Hence $R = \infty$ by the ratio test for power series, and so the power series converges (absolutely) for all $x \in \mathbb{R}$; i.e. $S = \mathbb{R}$. □

[1] Please note there are two ratio tests that we have learned. One is the ratio tests for series Theorem 2.2.3, another one is the ratio tests for power series Theorem 3.2.1. Reasonable readers should understand them in their context.

3.2.2 Root Test

Theorem 3.2.2: Root test for power series ★★★★

Consider the power series

$$\sum_{n=0}^{\infty} a_n x^n.$$

Suppose that

$$|a_n|^{\frac{1}{n}} \to \ell, \quad \text{as} \quad n \to \infty.$$

Then

$$R = \begin{cases} 0 & \text{if } \ \ell = \infty, \\ \frac{1}{\ell} & \text{if } \ \ell \in \mathbb{R}\backslash\{0\}, \\ \infty & \text{if } \ \ell = 0. \end{cases}$$

The proof is very similar to the proof of the ratio test for power series, and so we leave it as an exercise.

3.3 Differentiability of power series

In this brief section we discuss the calculus of power series. Due to time constraints we omit the proofs of the statements we make.

Suppose that

$$\sum_{n=0}^{\infty} a_n x^n$$

is a power series with convergence set S (we know already that $(-R, R) \subseteq S \subseteq [-R, R]$, where $R > 0$ is the radius of convergence). Define the function $f : S \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Question. Is f differentiable? If so, is it true that

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad ?$$

This would certainly be true if f were a *finite* sum of powers of x, as f would simply be a polynomial, and we know that polynomials are differentiable. The function f is an *infinite* sum of powers of x, and it is not immediately apparent what the answer to this question should be.

Theorem 3.3.1: Differentiability of power series ★★★★

uppose

$$\sum_{n=0}^{\infty} a_n x^n$$

is a power series with radius of convergence R. Then the function $f : (-R, R) \to \mathbb{R}$ given by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

is differentiable, and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Remark. Implicit in the statement of the above theorem is that if a power series

$$\sum_{n=0}^{\infty} a_n x^n$$

has radius of convergence R then the power series

$$\sum_{n=1}^{\infty} n a_n x^{n-1}$$

also has radius of convergence R.

Many of the special functions that we work with in mathematics, such as the exponential and trigonometric functions, are often most conveniently defined as power series.

3.4 The exponential function

The exponential function $\exp : \mathbb{R} \to \mathbb{R}$ is defined by

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

By the ratio test, the radius of convergence of the power series is ∞, and so exp is indeed defined on the whole of $\mathbb{R}$. Moreover, by the previous theorem, exp is differentiable on $\mathbb{R}$ and

$$\frac{\mathrm{d}}{\mathrm{d}x} \exp(x) = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \exp(x) \quad \forall x \in \mathbb{R}. \tag{†}$$

Note also that by definition,

$$\exp(0) = 1. \tag{††}$$

Remark. It can be proved that there exists only one differentiable function on $\mathbb{R}$ satisfying both the differential equation (†) and the condition (††).

Theorem 3.4.1: Properties of $\exp$

The exponential function has the following properties:

1. $\exp(0) = 1$.
2. For any $x, y \in \mathbb{R}$, $\exp(x+y) = \exp(x)\exp(y)$.
3. For any $x \in \mathbb{R}$, $\exp(x) > 0$.
4. For any $x \in \mathbb{R}$, $\exp(-x) = \frac{1}{\exp(x)}$.
5. $\exp(1) = e$, where e denotes the exponential constant.

Remark 3.4.1. In particular, the above theorem tells us that exp is a homomorphism from the group $(\mathbb{R}, +)$ into the group $(\mathbb{R}_+, \cdot)$, where $\mathbb{R}_+ := (0, \infty)$. (Actually, exp is an isomorphism of groups!)

Notation. It can be proved that $exp(x) = e^x$ for all $x \in \mathbb{R}$.

Proof. Point (1) is a consequence of the definition of exp.

We now prove (2). By definition,

$$\exp(x+y) = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!}.$$

Recall the binomial formula

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k y^{n-k}.$$

Thus, we have that

$$\exp(x+y) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^k}{k!} \frac{y^{n-k}}{(n-k)!}.$$

The right hand side of the above formula is the Cauchy product of the two absolutely convergent series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $\sum_{n=0}^{\infty} \frac{y^n}{n!}$. Thus

$$\exp(x+y) = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \times \left(\sum_{n=0}^{\infty} \frac{y^n}{n!} \right) = \exp(x)\exp(y).$$

Points (3) and (4) follow from (1) and (2). Indeed, by definition of exp, we have that

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \geq 1 > 0 \quad \forall x \geq 0.$$

Moreover, by (1) and (2)

$$1 = \exp(0) = \exp(x - x) = \exp(x)\exp(-x) \quad \forall x \in \mathbb{R}.$$

It follows that $\exp(x) > 0$ for all $x \in \mathbb{R}$, and

$$\exp(-x) = \frac{1}{\exp(x)}.$$

In order to prove that $e = \exp(1)$, we are required to show that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{\infty}\frac{1}{k!}$$

By using the binomial theorem,

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} C_n^k\left(\frac{1}{n}\right)^k = \sum_{k=0}^{n}\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) \le \sum_{k=0}^{n}\frac{1}{k!}.$$

Thus

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \le \sum_{k=0}^{\infty}\frac{1}{k!} = \exp(1).$$

It remains to prove that

$$e \ge \exp(1).$$

The first step is to show that for all m and n

$$\left(1+\frac{1}{n}\right)^{n+m} \ge \sum_{k=0}^{m}\frac{1}{k!}.$$

By the Binomial theorem,

$$\left(1+\frac{1}{n}\right)^{n+m} = \sum_{k=0}^{n+m}\frac{1}{k!}\frac{(n+m)!}{(n+m-k)!}\frac{1}{n^k} \ge \sum_{k=0}^{m}\frac{1}{k!}\frac{(n+m)!}{(n+m-k)!}\frac{1}{n^k},$$

where we have thrown away the last n terms of the sum. So

$$\begin{aligned}\left(1+\frac{1}{n}\right)^{n+m} &\ge \sum_{k=0}^{m}\frac{1}{k!}\frac{(n+m)(n+m-1)\cdots(n+m-k+1)}{n^k}\\ &\ge \sum_{k=0}^{m}\frac{1}{k!}.\end{aligned}$$

Now fix m and let $n\to\infty$. Then,

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^m \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{m+n} \ge \sum_{k=0}^{m}\frac{1}{k!}$$

We have proved that

$$e \geq \sum_{k=0}^{m} \frac{1}{k!} \quad \forall m \in \mathbb{N}.$$

Then by taking the limit as $m \to \infty$, we get

$$e \geq \sum_{k=0}^{\infty} \frac{1}{k!} = \exp(1),$$

as required to finish the proof. $\square$

Remark 3.4.2. The above calculation gives another proof of

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e,$$

other than the one given in Example 1.3.5.

3.5 Maclaurin and Taylor Series

As we have seen above, power series provide a deep connection between the theory of series (developed in these notes) and the theory of functions (Block 2). In particular, and as we shall see, the theory of the *calculus* provides us with a way of expressing a variety of "well-behaved" functions as power series. The series constructed in this way are referred to as Maclaurin or Taylor series. Due to time constraints out treatment of Maclaurin/Taylor series will be mainly methodological (i.e. without thorough proofs).

Often it is convenient to express a power series in terms of powers of $(x - c)$ for some real number c, rather than simply powers of x. Such a series, that is, a series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots,$$

is called a *power series about the point c*.

For a power series in this form, Theorem 3.1 becomes:

Theorem 3.5.1: Radius of Convergence

Given a power series

$$\sum_{n=0}^{\infty} a_n(x-c)^n,$$

we have

(1) $\displaystyle\sum_{n=0}^{\infty} a_n(x-c)^n$ converges absolutely when $|x-c|<R$

(2) $\displaystyle\sum_{n=0}^{\infty} a_n(x-c)^n$ diverges when $|x-c|>R$.

Here R is the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n x^n,$$

as defined at the beginning of Section 3.

Remarks. Just to be clear:

- If $R=0$ the power series converges at $x=c$ only.
- If $R\in\mathbb{R}\backslash\{0\}$ then the power series
  - converges (absolutely) for every x such that $|x-c|<R$,
  - diverges for every x such that $|x-c|>R$, and
  - may or may not converge when $|x-c|=R$, i.e. when $x=c-R$ or $x=c+R$.

  In this case the power series defines a function from $(c-R,c+R)$ to $\mathbb{R}$.
- If $R=\infty$ the power series converges (absolutely) for all x, and defines a function on the whole of $\mathbb{R}$.

Terminology. Suppose that f is a function and that the power series

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

converges to $f(x)$ for each $x\in(c-R,c+R)$. Then the power series is said to be a *power series representation* of f on $(c-R,c+R)$.

Example 3.5.1

The function $f(x)=\frac{1}{1-x}$ has power series representation $\sum x^n$ for $x\in(-1,1)$. This representation does not hold for $|x|\geq 1$ as we have already discussed. Notice here that the power series is about 0 - i.e. $c=0$.

Theorem 3.5.2: Taylor Series

Suppose that the power series
$$\sum_{n=0}^{\infty} a_n(x-c)^n$$
converges to a function $f(x)$ for all $c-R<x<c+R$ where $0<R\leq\infty$. Then
$$a_n = \frac{f^{(n)}(c)}{n!}$$
for each $n \in \mathbb{N}$.

Proof. [2] If
$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n,$$
then differentiating m times (as we may by our theorem on the differentiability of power series) we get
$$f^{(m)}(x) = \sum_{n=m}^{\infty} n(n-1)\cdots(n-m+1)a_n(x-c)^{n-m},$$
so that
$$f^{(m)}(x) = m!a_m + \sum_{n=m+1}^{\infty} n(n-1)\cdots(n-m+1)a_n(x-c)^{n-m}$$
whenever $|x-c|<R$. Setting $x=c$ in this last expression we see that the terms in the sum for $n\geq m+1$ vanish, leaving
$$f^{(m)}(c) = m!a_m;$$
i.e. $a_m = \frac{f^{(m)}(c)}{m!}$, as claimed. $\square$

[2] Not examinable

Definition 3.5.1: ★★★★★

If the function f has a power series representation on the interval $(c-R, c+R)$, then the power series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \\ &= \frac{f(c)}{0!} + \frac{f'(c)(x-c)}{1!} + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \cdots \end{aligned}$$

is called the *Taylor Series of the function f about c.*
In the particular case that $c = 0$, then Taylor series of f is usually called the *Maclaurin series* of f:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \\ &= \frac{f(0)}{0!} + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots \end{aligned}$$

Tips for finding Taylor and Maclaurin series.
In applications it can be quite painstaking to calculate the series directly by differentiating the function repeatedly. It is often quicker to manipulate known series. Because power series representations of functions are unique, *any valid method of calculating it will suffice.*

3.6 Taylor and Maclaurin Series of Common Functions

Some common expansions: ★★★★★

1. For any $x \in \mathbb{R}$

$$e^x = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} x^n/n!$$

2. For any $x \in \mathbb{R}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n+1)!$$

3. For any $x \in \mathbb{R}$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}/(2n)!$$

4. The Binomial Theorem: for any $\alpha \in \mathbb{R}$ and x such that $|x| < 1$

$$\begin{aligned} (1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \cdots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n + \cdots \\ &= \sum_{n\geq 0} \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n \end{aligned}$$

5. From 4. we have, for any x such that $|x| < 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$$
$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots = \sum_{n=0}^{\infty} (-x)^n$$

6. For any x such that $|x| < 1$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots = \sum_{n\geq 1} (-1)^{n+1}\frac{x^n}{n}$$

3.7 Manipulation of Power Series

Example 3.7.1

(a) Find the Taylor series for $f(x) = e^x$ about $x = 1$.
(b) Calculate the Maclaurin series of $f(x) = e^{-3x}$.
(c) Calculate the Maclaurin series of $f(x) = \cos(2x)$.
(d) Calculate the Taylor series of $f(x) = \dfrac{1}{3x+2}$ about $x = -2$.

(a) We know that

$$e^t = 1 + t + \frac{t^2}{2} + \cdots + \frac{t^n}{n!} + \cdots$$

for any t. We want an expansion in terms of $(x-1)$. Putting $t = (x-1)$ in the above, we get

$$e^x = e^{(x-1)+1} = ee^{(x-1)} = e\left(1 + (x-1) + \frac{(x-1)^2}{2} + \cdots + \frac{(x-1)^n}{n!} + \cdots\right)$$
$$= e + e(x-1) + \cdots + \frac{e}{n!}(x-1)^n + \cdots$$

(b) We know that

$$e^t = 1 + t + \frac{t^2}{2} + \cdots + \frac{t^n}{n!} + \cdots$$

Putting $t = -3x$, we get

$$e^{-3x} = 1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \cdots + \frac{(-3)^n}{n!}x^n + \cdots$$

The expansion is valid for all x.

(c) We know that

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

Replacing x with $2x$, we get

$$\begin{aligned}\cos 2x &= 1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{4!} + \cdots + (-1)^n \frac{(2x)^{2n}}{(2n)!} + \cdots \\ &= 1 - \frac{4x^2}{2} + \frac{16x^4}{4!} + \cdots + (-1)^n \frac{2^{2n}x^{2n}}{(2n)!} + \cdots\end{aligned}$$

The expansion is valid for all x.

(d) We know that for any x such that $|x| < 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$$

Now we want an expansion involving powers of $(x - -2) = (x+2)$. Since

$$3x + 2 = 3(x+2) - 6 + 2 = 3(x+2) - 4 = -4\big(1 - 3(x+2)/4\big)$$

we have, replacing x by $3(x+2)/4$ in the expansion of $1/(1-x)$

$$\begin{aligned}\frac{1}{3x+2} &= -\frac{1}{4}\frac{1}{1-3(x+2)/4} \\ &= -\frac{1}{4}\left(1 + \frac{3(x+2)}{4} + \frac{9(x+2)^2}{16} + \cdots + \frac{3^n(x+2)^n}{4^n} + \cdots\right) \\ &= -\frac{1}{4} - \frac{3}{16}(x+2) - \cdots - \frac{3^n}{4^{n+1}}(x+2)^n + \cdots\end{aligned}$$

The expansion is valid for $|3(x+2)/4| < 1$, i.e. for $|x+2| < 4/3$.

3.7.1 Term by term addition of power series:

Let $\sum_{n=0}^{\infty} a_n(x-c)^n$ have radius of convergence $R_1 > 0$ and $\sum_{n=0}^{\infty} b_n(x-c)^n$ have radius of convergence $R_2 > 0$. By the linearity of summation (or the "Algebra of Limits for Sums") the sum of the two series is

$$\sum_{n=0}^{\infty}(a_n + b_n)(x-c)^n,$$

and has radius of convergence $R \geq \min\{R_1, R_2\}$.

Example 3.7.2

Calculate the Maclaurin series of $f(x) = e^{-3x} - \cos(2x)$ as far as terms involving x^4.

From above we have

$$\begin{aligned}e^{-3x} &= 1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \frac{(-3)^4}{4!}x^4 + \cdots \\ \cos 2x &= 1 - \frac{4x^2}{2} + \frac{16x^4}{4!} + \cdots\end{aligned}$$

so that

$$\begin{aligned}
f(x) &= e^{-3x} - \cos(2x) \\
&= \left(1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \frac{(-3)^4}{4!}x^4 + \cdots\right) \\
&\quad - \left(1 - \frac{4x^2}{2} + \frac{16x^4}{4!} + \cdots\right) \\
&= -3x + \frac{13}{2}x^2 - \frac{9}{2}x^3 + \frac{65}{24}x^4 + \cdots
\end{aligned}$$

3.7.2 Term by term multiplication of power series:

Let $\sum_{n=0}^{\infty} a_n(x-c)^n$ have radius of convergence $R_1 > 0$ and $\sum_{n=0}^{\infty} b_n(x-c)^n$ have radius of convergence $R_2 > 0$. The product of the two series is

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$$

where

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0,$$

and has radius of convergence $R \geq \min\{R_1, R_2\}$.

Example 3.7.3

Calculate the Maclaurin series of $f(x) = e^{-3x}\cos(2x)$ as far as terms involving x^4.

It is often easier to work directly, rather than use the above (Cauchy product) formula. We know

$$\begin{aligned}
e^{-3x} &= 1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \frac{(-3)^4}{24}x^4 + \cdots \\
\cos 2x &= 1 - \frac{4x^2}{2} + \frac{16x^4}{24} + \cdots
\end{aligned}$$

so that

$$\begin{aligned}
f(x) &= e^{-3x}\cos(2x) \\
&= \left(1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \frac{(-3)^4}{24}x^4 + \cdots\right)\left(1 - \frac{4x^2}{2} + \frac{16x^4}{24} + \cdots\right) \\
&= 1 - 3x + \left(\frac{9}{2} - \frac{4}{2}\right)x^2 + \left(\frac{12}{2} - \frac{9}{2}\right)x^3 + \left(-\frac{36}{4} + \frac{16}{24} + \frac{(-3)^4}{24}\right)x^4 + \cdots \\
&= 1 - 3x + \frac{5}{2}x^2 + \frac{3}{2}x^3 - \frac{119}{24}x^4 + \cdots
\end{aligned}$$

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