Github.com/styluck/sas

80% exam   20% homeworks

Defn: **A set** is a collection of objects. These objects are called **elements** or **members**.

A set with no objects is called the **empty set**, denoted **ϕ**

Defn: For a set A, if **x is an element** of A, we write **$x \in A$**

if x is **not an element** of A, we write **$x \notin A$**

① Well-definedness 确定性   either $x \in A$ or $x \notin A$

② Distinct elements 互异性 $A = \{1, 6, 3\}$ $B = \{1, 6, 6, 3\}$

$A = B$ $= \{1, 6, 3\}$

③ Unordered 无序性 $\{1, 6, 3\} = \{3, 1, 6\} = \{1, 3, 6\}$

curely brackets.

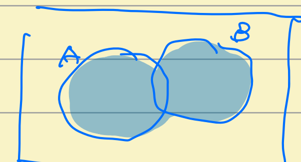
Set-builder notation: $\{x : x \text{ has property } P\}$

$\{x \mid x \text{ has property } P\}$

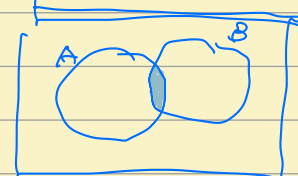
$\{x : x < 2\}$ $\{x : x < 2, x \in \mathbb{Q}\}$ $\{x \in \mathbb{Q} : x < 2\}$

Defn: Set Operation:

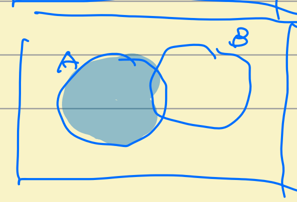
Union 并集: $A \cup B := \{x : x \in A \text{ or } x \in B\}$



intersection 交集: $A \cap B := \{x : x \in A \text{ and } x \in B\}$

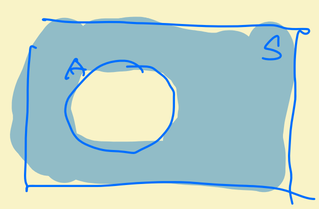


Set difference 集合减法: $A \setminus B := \{x : x \in A \text{ and } x \notin B\}$



Complement 补集: $A^c := \{x \in S : x \notin A\}$

$A^c = S \setminus A$



disjoint : $A \cap B = \emptyset$

Number systems. 自然数

Natural Numbers $\mathbb{N}$ $\{0, 1, 2, 3, \dots\}$

Integers 整数 $\mathbb{Z}$ $\{\dots, -3, -2, -1, 0, 1, 2, 3 \dots\}$

Rational Numbers $\mathbb{Q}$ $\{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$

Ratio $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$ $\subset$ subset 真子集

$\subseteq$ subset equal 子集

Defn: If A is the subset of B, then
$\forall x \in A \Rightarrow x \in B$

$\emptyset \subseteq A$

Ordered set

Defn: Let S be a set. An order on S is a relation, denote by $\prec$, with the following properties.

有序性 ordered

① For any $x, y \in S$, one and only one of the following holds
$x \prec y$ $x = y$ $y \prec x$

传递性 transitivity

② For any $x, y, z \in S$, if $x \prec y$, and $y \prec z$, $\Rightarrow$ $x \prec z$.

Example 1 : $(\mathbb{Q}, >)$ ① Ordered: for any $x, y \in \mathbb{Q}$ $x > y$ or $x = y$ or $y > x$
② Trans : for any $x, y, z \in \mathbb{Q}$, if $x > y$, $y > z$ $\Rightarrow$ $x > z$.

Example 2 : $(\mathbb{Q}, \geq)$ ① Not ordered: there exists $x, y \in \mathbb{Q}$
$x \geq y$, and, $x = y$. and $y \geq x$

Defn: An ordered set is a set that is endow with an order "$\prec$"
Denoted by $(S, \prec)$

Defn: (Upper bound) Consider an ordered set S and $E \subset S$, $E \neq \phi$.
There exists $y \in S$ such that for any $x \in E$, we have $x \preceq y$.
then y is called the upper bound of E.

Example: $S = (\mathbb{Z}, <)$ $\quad E = \{0, 1, 2\} \subset S$. $\quad 3$ is an upper bound

Defn: (Least upper bound 最小上界) Suppose S is an ordered set, and $\phi \neq E \subset S$ is bounded above. Suppose there exists $\alpha \in S$ such that
① α is an upper bound of E
② For any $\beta \prec \alpha$, $\beta \in S$, β is not an upper bound of E.
Then α is called the least upper bound of E is S. or the supremum 上确界

$$\alpha = \sup E$$

If supremum exists, then it is unique. That is
if $\alpha = \sup E$, and $\beta = \sup E$, then we have $\alpha = \beta$

$S = (\mathbb{Q}, <)$, $E = \{0, 1, 2\} \subset \mathbb{Q}$ the upper bound is $\{x \in \mathbb{Q} : x \geq 2\}$
the supremum is 2.

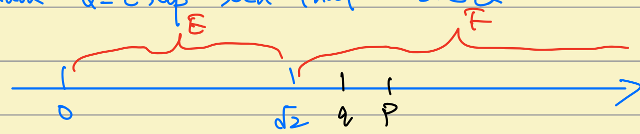
Lemma: There is no rational number $x \in \mathbb{Q}$, satisfying $x^2 = 2$ 至偏的
Proof: Suppose $x \in \mathbb{Q}$ such that $x^2 = 2$. $x = \frac{m}{n}$ with m and n coprime

$x = \frac{m}{n}$ $\quad x^2 = 2 \Rightarrow \left(\frac{m}{n}\right)^2 = 2 \Rightarrow m^2 = 2n^2$
m^2 is divisible by $2 \Rightarrow m$ is divisible by 2. Write $m = 2k$
$(2k)^2 = 2n^2 \Rightarrow 2k^2 = n^2 \Rightarrow m, n$ not coprime.
$\Rightarrow$ contradiction.

Example: $S=(\mathbb{Q},<)$ $\quad E=\{x\in\mathbb{Q}: x>0,\ p^2<2\}$

There is no supremum $\alpha = E\sup$ such that $\alpha\in\mathbb{Q}$

$F=\{p\in\mathbb{Q}: p>0,\ p^2>2\}$



Fixed any $p\in F$ then p is an upper bound of E.

Construct $q=p-x(p^2-2)$ $\quad x\in\mathbb{Q}$, then $q\in\mathbb{Q}$

We need to prove $q\in F$, that is $q>0$, $q^2>2$

Take $x=\frac{1}{p+2}$ $\qquad p-q=\frac{1}{p+2}(p^2-2)>0$

$$q=p-x(p^2-2)=p-\frac{p^2-2}{p+2}=\frac{p^2+2p-p^2+2}{p+2}=\frac{2p+2}{p+2}>0$$

$$q^2=(p-x(p^2-2))^2=\left(\frac{2(p+1)}{p+2}\right)^2>2 \iff 4(p+1)^2>2(p+2)^2$$

$$\iff 4(p^2+2p+1)>2(p^2+4p+4)$$

$$4p^2+8p+4>2p^2+8p+8$$

$$\Rightarrow 2p^2>4 \Rightarrow p^2>2$$

Then we proved $q>0$ and $q^2>2$ $\Rightarrow q\in F$

Defn (Least upper bound property) An ordered set is said to have the Least upper bound Property, if any nonemty subset E bounded above has a least upper bound in S.

Example: $\mathbb{Q}$ has no least upper bound property.

Defn (Lower bound) Consider an ordered set S and $\phi\neq E\subset S$. If there exists $y\in S$ such that for any $x\in E$, we have $x\geq y$. Then E is said to be bounded below. and y is a lower bound of E

Defn (Greatest lower bound). Suppose S is an ordered set, and $\phi \neq E \subset S$ is bounded below. Suppose there exists $\alpha \in S$ such that

① α is a lower bound of E

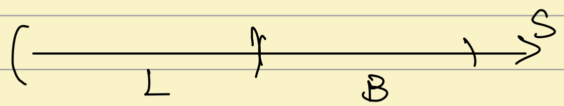
② For any $\beta > \alpha$, then β is not a lower bound of E.

α is called infimum 下确界

$$\alpha = \inf E$$

The infimum is unique.

Thm (Greatest lower bound property).



$$\alpha = \inf B$$

$$L := \{x \in S : x \text{ is a lower bound of } B\}$$

then $\sup L = \inf B$ both exist

Field : A field is a set endowed with two operations:

① addition $+$

② multiplication $\times$., which satisfy the following axioms 公理

Axioms for addition: 加法公理

① For any $x, y \in F$, $\Rightarrow x+y \in F$

② Commutative 交换律 $x+y = y+x$

③ Associative 结合律 $(x+y)+z = x+(y+z)$

④ Contains 0, that is $x \in F$, $x+0 = x$

⑤ For each $x \in F$, there exist an element $-x \in F$, $x+(-x) = 0$

$x - y = x + (-y)$

$\forall$: for all

Axioms for multiplication:

① For any $x, y \in F$, $x \cdot y \in F$

② Commutative $x \cdot y = y \cdot x$

③ Contains 1. such that $x \cdot 1 = x$ for all $x \in F$

④ If $x \in F$, $x \neq 0$, there exists an element $\frac{1}{x} \in F$, such that $x \cdot \frac{1}{x} = 1$

Distribution law 分配律

$$x \cdot (y+z) = x \cdot y + x \cdot z \quad \forall x, y, z \in F$$

$x \cdot \frac{1}{y} = \frac{x}{y}$ $\quad x \cdot x = x^2$