Series → convergence property.

Defn: ($\varepsilon$-N definition) $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, such that $n \geq N \Rightarrow |x_n - A| < \varepsilon$

Properties of limits.

① Uniqueness $\lim_{n\to\infty} x_n = A$, $\lim_{n\to\infty} x_n = B \Rightarrow A = B$

② Boundedness $\lim_{n\to\infty} x_n = A$, $\exists M > 0$ such that $|x_n| < M$ $\forall n \in \mathbb{N}$

③ (保号性) $\lim_{n\to\infty} x_n = A > 0$ $\exists N \in \mathbb{N}$ such $\forall n \geq N \Rightarrow x_n > 0$

④ Arithmetic operations (四则运算) $+, -, \times, \div$

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Null sequence (零序列)

A sequence $\{x_n\}$ that converges to zero is often referred as null sequence.

E.g.: ① $s > 0$, $\lim_{n\to\infty} \frac{1}{n^s} = 0$

② $|\lambda| < 1$, $\lim_{n\to\infty} \lambda^n = 0$

③ $s \in \mathbb{R}$, $|\lambda| > 1$, $\lim_{n\to\infty} \frac{n^s}{\lambda^n} = 0$

④ $s \in \mathbb{R}$, $\lim_{n\to\infty} \frac{n^s}{n!} = 0$

⑤ $\lambda \in \mathbb{R}$, $\lim_{n\to\infty} \frac{\lambda^n}{n!} = 0$

⑥ $\lim_{n\to\infty} \frac{n!}{n^n} = 0$

## Monotone sequence (单调序列)

A sequence get steadily larger or smaller is called a monotone sequence.

**Defn:** A sequence $\{x_n\}$ is monotonically increasing if $x_n \le x_{n+1}$ holds for all $n \in \mathbb{N}$.

A sequence $\{x_n\}$ is monotonically decreasing if $x_n \ge x_{n+1}$ holds for all $n \in \mathbb{N}$.

E.g.: $\{1^n\}$ is both monotonically increasing and monotonically decreasing.

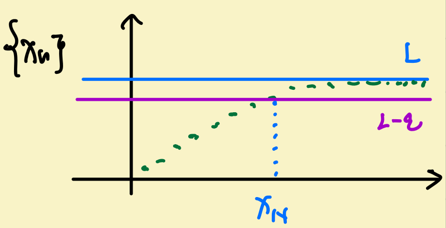
A sequence $\{x_n\}$ is strictly increasing if $x_n < x_{n+1}$ holds for all $n \in \mathbb{N}$.

A sequence $\{x_n\}$ is strictly decreasing if $x_n > x_{n+1}$ holds for all $n \in \mathbb{N}$.

Bolzano–Wierestrass theorem

**Thm** (Monotone convergence theorem)

Suppose that $\{x_n\}$ is a monotonic sequence. Then $\{x_n\}$ is convergent if and only if $\{x_n\}$ is bounded.




Proof: Suppose that $\{x_n\}$ is monotonic

($\Rightarrow$) if $\{x_n\}$ is convergent, by the properties of convergent sequence, $\{x_n\}$ is bounded.

($\Leftarrow$) if $\{x_n\}$ is bounded, since $\{x_n\}$ is monotonic, without lose of generality, assume that $\{x_n\}$ is monotonically increasing. Then its supremum exists. Let

$$L := \sup\{x_n\}.$$

Then $L-\varepsilon$ is not an upper bound of $\{x_n\}$. There exists $N \in \mathbb{N}$, such that $x_N > L-\varepsilon$. Since $\{x_n\}$ is non-decreasing, we have $\forall n \ge N$

$$L-\varepsilon < x_N < x_n \le L < L+\varepsilon$$

$$\Rightarrow \quad |x_n - L| < \varepsilon$$

Hence $\{x_n\}$ is convergent.

E.g.: Show that $x_n = (1+\frac{1}{n})^n$ is convergent.

$$\frac{x_{n+1}}{x_n} = \frac{(1+\frac{1}{n+1})^{n+1}}{(1+\frac{1}{n})^n} = \frac{(\frac{n+2}{n+1})^{n+1}}{(\frac{n+1}{n})^n} = \frac{(n+2)^{n+1}\cdot n^n}{(n+1)^{2n+1}} = \frac{(n+2)^n \cdot n^n}{[(n+1)^2]^n}\,\frac{n+2}{n+1}$$

$$= \left[\frac{(n+2)n}{(n+1)^2}\right]^n \frac{n+2}{n+1} = \left[\frac{n^2+2n}{n^2+2n+1}\right]^n \frac{n+2}{n+1} = \left[1-\frac{1}{n^2+2n+1}\right]^n \frac{n+2}{n+1}$$

Bernoulli inequality : $(1+x)^n \geq 1+nx \quad \forall n \in \mathbb{N} \quad x \geq -1$

$$\left[1-\frac{1}{n^2+2n+1}\right]^n \frac{n+2}{n+1} \geq \left(1-\frac{n}{n^2+2n+1}\right)\frac{n+2}{n+1} = \frac{n^3+3n^2+3n+2}{n^3+3n^2+3n+1} > 1$$

Hence $\{x_n\}$ is monotonically increasing and bounded below: $2 = x_1 \leq x_n$

Claim: $y_n = (1+\frac{1}{n})^{n+1}$ is monotonically decreasing.

$$\frac{y_{n-1}}{y_n} = \frac{(1+\frac{1}{n-1})^n}{(1+\frac{1}{n})^{n+1}} = \frac{n^{2n+1}}{(n-1)^n(n+1)^{n+1}} = \left(\frac{n^2}{n^2-1}\right)^n \frac{n}{n+1}$$

$$= \left(1+\frac{1}{n^2-1}\right)^n \frac{n}{n+1} \geq \left(1+\frac{n}{n^2-1}\right)\frac{n}{n+1} = \frac{n^3+n^2-n}{n^3+n^2-n-1} > 1$$

Hence $\{y_n\}$ is monotonically decreasing and bounded above $y_n \leq y_1 = 4$

$$\Rightarrow \quad 2 = x_1 \leq x_n \leq y_n \leq y_1 \leq 4 \quad \Rightarrow \{x_n\} \text{ is bounded.}$$

By monotone convergence theorem, both $\{x_n\}$ and $\{y_n\}$ are convergent

In fact $\lim\limits_{n\to\infty}(1+\frac{1}{n})^n = e$.

E.g.: compounded interest 复利 2025 $1 million

2026 $2 million

$\$1 \times (1+100\%) = \$2$

$\$1 \times (1+50\%)^2 = \$2.25$

$\$1 \times (1+25\%)^4 = \$2.4414$

$\$1 \times (1+100\%/12)^{12} = \$2.613035$

$\vdots$

$\$1 \times (1+\frac{100\%}{n})^n \Longleftrightarrow \$1 \times (1+\frac{r\%}{n})^n \to \$1 \cdot e^{r\%}$ as $n\to\infty$

$\$1 \times (1+r\%)^2$

$\vdots$

$\$1 \cdot (e^{r\%})^2 \quad t=2$

$\Downarrow$

$\$1 \cdot e^{r\% T} \quad t=T$

**Subsequence** 子列

**Defn**: (subsequence) Let $\{x_n\}$ be a sequence, Then a subsequence of $\{x_n\}$ is a sequence.

$$\{x_{n_i}\}_{i=1}^{\infty}$$

**Fact**: ① a subsequence is also a sequence. (infinite elements)

② $\{n_i\}$ is an strictly increasing sequence with positive integers.

E.g.: $\{(-1)^n\}$ Let $n_i = 2k$ $k \in \mathbb{N}$

to get subsequence: $\{1^n\}$ : $1, 1, \cdots 1, \cdots$

E.g.: $x_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}$ $0, 2, 0, 4, 0, 6, \cdots$

$\{x_{2k-1}\}$ : $0, 0, \cdots, 0, \cdots$

**Thm**: Every sequence $\{x_n\}$ contains a monotonic subsequence.