Power series $\sum_{n=0}^{\infty} a_n x^n$ $\qquad \sum_{n=1}^{\infty} a_n$

Ratio test $\dfrac{|a_{n+1}|}{|a_n|} \to l$ as $n \to \infty$

The radius of convergence $R = \begin{cases} 0 & \text{if } l = +\infty \\ \frac{1}{l} & \text{if } l \in \mathbb{R} \setminus \{0\} \\ \infty & \text{if } l = 0 \end{cases}$

Root test $|a_n|^{\frac{1}{n}} \to l$ as $n \to \infty$

The radius of convergence $R = \begin{cases} 0 & \text{if } l = +\infty \\ \frac{1}{l} & \text{if } l \in \mathbb{R} \setminus \{0\} \\ \infty & \text{if } l = 0 \end{cases}$

$e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$ $\qquad e = \sum_{k=0}^{\infty} \frac{1}{k!}$

binomial theorem:

$(a+b)^n = \sum_{k=0}^{n} C_k^n a^n \cdot b^{n-k}$

$C_k^n = \dfrac{n!}{k!(n-k)!}$

$$(1+\tfrac{1}{n})^n = \sum_{k=0}^{n} C_k^n (\tfrac{1}{n})^k \cdot 1^{n-k} = \sum_{k=0}^{n} C_k^n (\tfrac{1}{n})^k$$

$$= 1 + n \cdot \tfrac{1}{n} + \frac{n(n-1)}{2!} \times \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \times \frac{1}{n^3} \cdots$$

$$= 1 + 1 + \tfrac{1}{2!}(1-\tfrac{1}{n}) + \tfrac{1}{3!}(1-\tfrac{1}{n})(1-\tfrac{2}{n}) \cdots$$

$$= 1 + 1 + \tfrac{1}{2!} + \tfrac{1}{3!} + \cdots + \tfrac{1}{n!} \quad \text{as } n \to \infty$$

$$\Rightarrow \quad \lim_{n \to \infty} (1+\tfrac{1}{n})^n = \sum_{k=0}^{\infty} \frac{1}{k!}$$

Maclaurin and Taylor Series.

$\sum_{n=0}^{\infty} a_n (x-c)^n \to$ Power series at point c.

$\sum_{n=0}^{\infty} a_n (x-c)^n$ $\qquad \left|\dfrac{a_{n+1}}{a_n}\right| \to l \in (0, +\infty)$ as $n \to \infty \Rightarrow R = \frac{1}{l}$

$\sum_{n=0}^{\infty} a_n x^n$ $\quad R \to x \in S \quad (-R, R) \subseteq S \subseteq [-R, R]$

$|x-c| < R \Rightarrow c-R < x < R+c$

Terminology 术语: $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ if f converges for $x \in (c-R, c+R)$

Then the power series is said to be a power series representation of f on $(c-R, c+R)$.

E.g. $f(x) = \frac{1}{1-x}$ $\quad \sum_{n=0}^{\infty} x^n$ for $x \in (-1,1)$

Defn: (Taylor series) If the function f has a power series representation on the interval $(c-R, c+R)$ then the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$0! = 1$
$$= \frac{f(c)}{0!} + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3 \cdots$$

is called the Taylor series of the function f about c.

Thm (Taylor series 泰勒) Suppose that the power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

converges to a function $f(x)$ for all $x \in (c-R, c+R)$, $R \in [0, +\infty]$, Then

$$a_n = \frac{f^{(n)}(c)}{n!}$$

Proof: If $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ then the mth derivative of f is

$$f^{(m)}(x) = \sum_{n=m}^{\infty} n(n-1)\cdots(n-m+1)\, a_n (x-c)^{n-m}$$

$$f^{(m)}(x) = m!\, a_m + \sum_{n=m+1}^{\infty} n(n-1)\cdots(n-m+1)\, a_n (x-c)^{n-m}$$

The above equation holds for $|x-c| < R$, Let $x = c$, we get

$$f^{(m)}(x) = m!\, a_m$$

$$\Rightarrow a_m = \frac{f^{(m)}(x)}{m!}$$

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Maclaurin series of f : The Talyor series when c=0.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + f'(0)\cdot x + \frac{1}{2!} f''(0)\cdot x^2 + \frac{1}{3!} f'''(0)\cdot x^3 \cdots$$

E.g.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \frac{x^4}{n!} \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$f(x) = e^x$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} \leftarrow \begin{cases} f'(x) = e^x & f'(0) = 1 \\ f''(x) = e^x & f''(0) = 1 \\ \vdots & \end{cases}$$

E.g.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n=2k \\ (-1)^k & \text{if } n=2k+1 \end{cases}$$

$f'(x) = \cos x$ $\quad f'(0) = 1$

$f''(x) = -\sin x$ $\quad f''(0) = 0$

$f^{(3)}(x) = -\cos x$ $\quad f^{(3)}(0) = -1$

$f^{(4)}(x) = \sin x$ $\quad f^{(4)}(0) = 0$

Ratio test + comparison test

$$\left| f^{(n)}(0) \frac{x^n}{n!} \right| \le \frac{|x^n|}{|n!|} \qquad \frac{\frac{|x^{n+1}|}{(n+1)!}}{\frac{|x^n|}{n!}} = \frac{x}{n+1}$$

$\frac{x}{n+1} \to 0$ as $n \to \infty$

E.g.

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

E.g. $\frac{1}{1-x} = 1+x+x^2+x^3+\cdots+x^n+\cdots = \sum_{n=0}^{\infty} x^n \qquad |x|<1$

at $x=0$

$f(x) = \frac{1}{1+x} = (1+x)^{-1} \qquad f'(x) = -1\cdot(1+x)^{-2} \qquad = -1$

$f''(x) = (-1)(-2)\times(1+x)^{-3} \qquad = 2$

$f'''(x) = (-1)\cdot(-2)\cdot(-3)\cdot(1+x)^{-4} = -6$

$f^{(n)}(0) = (-1)^n \cdot n! \qquad \vdots \qquad = 4!$

$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n \cdot n! \frac{x^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \cdot x^n = 1-x+x^2-x^3\cdots$

E.g $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots (-1)^{n+1}\frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n}$

E.g. $f(x) = e^x$ Taylor series at $x=1$

$e^t = 1+t+\frac{t^2}{2}+\cdots\frac{t^n}{n!}+\cdots \qquad t = x-1$

$e^x = e^{(x-1)+1} = e\cdot e^{x-1} = e\left(1+(x-1)+\frac{(x-1)^2}{2}+\cdots\frac{(x-1)^n}{n!}+\cdots\right)$

$= e\cdot e(x-1) + e\frac{(x-1)^2}{2} + \cdots + e\frac{(x-1)^n}{n!}$

E.g. $f(x) = e^{-3x}$

$e^t = 1+t+\frac{t^2}{2}+\cdots\frac{t^n}{n!}+\cdots$

Let $t=-3x \qquad e^{-3x} = 1+(-3x)+\frac{(-3x)^2}{2}+\cdots\frac{(-3)^n}{n!}x^n+\cdots$

E.g $f(x) = \cos 2x$

$\cos t = 1-\frac{t^2}{2}+\frac{t^4}{4!}+\cdots+(-1)^n\frac{t^{2n}}{(2n)!}+\cdots = \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n)!}$

let $t=2x \qquad \cos 2x = 1-\frac{(2x)^2}{2}+\frac{(2x)^4}{4!}+\cdots(-1)^n\frac{(2x)^{2n}}{(2n)!}\cdots$

E.g. $f(x) = \frac{1}{3x+2}$ at $x=-2$ $\quad \sum_{n=0}^{\infty} \frac{f^{(n)}(-2)}{n!}(x+2)^n$

$$\frac{1}{1-t} = 1+t+t^2+t^3+\cdots+t^n+\cdots = \sum_{n=0}^{\infty} t^n \qquad |t|<1$$

$$3x+2 = 3(x+2)-4 = -4\left(1-\frac{3}{4}(x+2)\right)$$

$$t = \frac{3(x+2)}{4}$$

$$f(x) = \frac{1}{3x+2} = -\frac{1}{4}\,\frac{1}{1-\frac{3}{4}(x+2)} = -\frac{1}{4}\left[1+\frac{3}{4}(x+2)+\frac{9}{16}(x+2)^2\cdots\left(\frac{3}{4}\right)^n(x+2)^n\cdots\right]$$

E.g $f(x) = e^{-\frac{x^2}{2}}$ find $f^{(98)}(0)$ and $f^{(99)}(0)$

$$e^t = 1+t+\frac{t^2}{2}+\cdots\frac{t^n}{n!}+\cdots \qquad \text{let } t=-\frac{x^2}{2}$$

$$e^{-\frac{x^2}{2}} = 1-\frac{x^2}{2}+\frac{x^4}{2^2\cdot 2!}+\cdots+(-1)^n\frac{x^{2n}}{2^n n!}+\cdots$$

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$$

$$f^{(98)}(0)\cdot\frac{x^{98}}{98!} = (-1)^{49}\frac{x^{98}}{2^{49}\,49!}$$

$$\Rightarrow f^{(98)}(0) = -\frac{98!}{2^{49}\cdot 49!}$$

$$f^{(99)}(0)\frac{x^{99}}{99!} = 0\cdot x^{99}$$

$$\Rightarrow f^{(99)}(0) = 0$$

Ex. Calculate Maclaurin series of $f(x) = e^{-3x} - \cos(2x)$ as far as terms involving x^4.

$$e^t = 1+t+\frac{t^2}{2}+\cdots\frac{t^n}{n!}+\cdots$$

$$\cos t = 1-\frac{t^2}{2}+\frac{t^4}{4!}+\cdots+(-1)^n\frac{t^{2n}}{(2n)!}+\cdots = \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n)!}$$

$$e^{-3x} = 1-3x+\frac{9}{2}x^2-\frac{9}{2}x^3+\frac{(-3)^4}{4!}x^4+\cdots$$

$$\cos 2x = 1-\frac{4x^2}{2}+\frac{16x^4}{4!}+\cdots$$

$$f(x) = -3x+\frac{13}{2}x^2-\frac{9}{2}x^3+\frac{65}{24}x^4\cdots$$

Ex. Calculate Maclaurin series of $f(x) = e^{-3x} \cdot \cos(2x)$ as far as terms involving x^4.

$$e^{-3x} = 1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \frac{(-3)^4}{4!}x^4 + \cdots$$

$$\cos 2x = 1 - \frac{4x^2}{2} + \frac{16x^4}{4!} + \cdots$$

$$\left(1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \frac{(-3)^4}{4!}x^4\right)\left(1 - 2x^2 + \frac{2}{3}x^4\right)$$

$$= 1 - 3x + \left(\frac{9}{2} - 2\right)x^2 + \left(6 - \frac{9}{2}\right)x^3 + \left(-9 + \frac{2}{3} + \frac{(-3)^4}{4!}\right)x^4$$

$$\lim_{x\to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^4}$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots$$

$$e^{-\frac{x^2}{2}} = 1 - \frac{x^2}{2} + \frac{x^4}{8} + \cdots$$

$$= \lim_{x\to 0} \frac{1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots - \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + \cdots\right)}{x^4}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{12}x^4}{x^4} = -\frac{1}{12}$$