Sequence → convergence properties.

① Monotone convergence theorem : 单调 + 有界 ⟺ 数列收敛
② Bolzano - Wierstrass theorem : 有界 ⇒ ∃ 收敛子列.
③ Bolzano - Cauchy sequence : $\forall \varepsilon > 0\ \exists N \in \mathbb{N},\ |x_n - x_m| < \varepsilon\ \forall n, m \geq N$
$\Rightarrow \{x_n\}$ converges

Series 级数

The ordered sum of sequence.

Sequence 数列 x_n
Series 级数 $S_m = \sum_{n=1}^{m} a_n$

Defn (Convergence of series) Let $\{x_n\}$ be a sequence, then if

$$\sum_{n=1}^{\infty} x_n = A$$

we say the series $S_m = \sum_{n=1}^{m} x_n$ converges.

Note: if the sequence $\{S_m\} := \sum_{n=1}^{m} x_n$ converges, then the series converges.

Thm (Uniqueness) If a series $\sum_{n=1}^{\infty} x_n$ converges, then the sum is unique.

Thm (Arithmetic operations) Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be convergent series. then

① $\sum_{n=1}^{\infty} (x_n \pm y_n) = \sum_{n=1}^{\infty} x_n \pm \sum_{n=1}^{\infty} y_n$

② $\sum_{n=1}^{\infty} c x_n = c \sum_{n=1}^{\infty} x_n$

Thm. Let $M \geq 1$ be an integer, then $\sum_{n=1}^{\infty} x_n$ converges if and only if

$$\sum_{n=1}^{\infty} x_{M+n} = \sum_{n=M+1}^{\infty} x_n$$

converges

Note: The tail's behaviour determines the convergence property of a series.

Telescoping 望远

Telescoping series (缩项级数)

$$\sum_{n=1}^{\infty} (x_n - x_{n+1}) = (x_1 - \lim_{n\to\infty} x_n)$$

E.g.: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \lim_{n\to\infty} \frac{1}{n+1} = 1$

Geometric series (等比级数)

$$\sum_{n=1}^{\infty} a^{n-1} = \lim_{n\to\infty} \frac{1-a^n}{1-a}$$

E.g.: $\sum_{n=1}^{k} a^{n-1} = \frac{1-a^k}{1-a} \quad (a \neq 1)$

$\lim_{k\to\infty} \frac{1-a^k}{1-a} = \frac{1}{1-a}$ if $|a| < 1$

and it diverges if $|a| \geq 1$

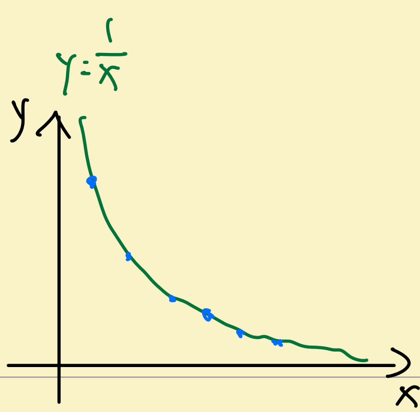
Harmonic series (调和级数)

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} \cdots$$

$\lim_{n\to\infty} \frac{1}{n} = 0$

Fact: $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge!

Proof: $\sum_{n=1}^{\infty} \frac{1}{n} \Longleftrightarrow \int_1^{+\infty} \frac{1}{n}\, dn = \lim_{n\to\infty} \ln n - \overset{0}{\cancel{\ln 1}} = +\infty$



$$\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\cdots$$
$$\geq 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)\cdots$$
$$= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\cdots$$
$$= 1+\sum_{k=1}^{\infty}\frac{1}{2} = +\infty$$

p-Harmonic series (p-调和级数)

$$\sum_{n=1}^{\infty}\frac{1}{n^p}\ (p>0)$$ Converges if $p>1$; Diverges if $0<p\leq 1$

Proof: If $0<p<1$, we have $\frac{1}{n^p}>\frac{1}{n}$. since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, we have $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges.

If $p>1$,
$$\sum_{n=1}^{\infty}\frac{1}{n^p} = 1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}+\cdots$$
$$\leq 1+\left(\frac{1}{2^p}+\frac{1}{2^p}\right)+\left(\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}\right)+\cdots$$
$$= 1+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+\cdots$$
$$= \frac{1}{1-2^{1-p}} < +\infty$$

Hence $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges.

Alternating Harmonic series 交错调和级数

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$$ it converges.

Proof: Let $S_k=\sum_{n=1}^{k}(-1)^{n-1}\frac{1}{n}$, then

$$S_{2k-1}=\sum_{n=1}^{2k-1}(-1)^{n-1}\frac{1}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}$$

$k=1$: $S_{2k-1}=1$

$k=2$ $S_{2k-1}=1-\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$

$k=3$ $S_{2k-1}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}$

monotonically decreasing

$S_{2k} = \sum_{n=1}^{2k} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}$

$k=1 \qquad S_{2k} = 1 - \frac{1}{2}$

$k=2 \qquad S_{2k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$

$k=3 \qquad S_{2k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}$

} monotonically increasing

$\Rightarrow \quad \frac{1}{2} \le S_{2k} \le S_{2k-1} \le 1$ By monotone convergence theorem, we conclude S_k converges.

Thm (Necessary condition of convergent series)

If $\sum_{n=1}^{\infty} x_n$ converges, then $x_n \to 0$ as $n \to +\infty$.

Proof: Let $S_k = \sum_{n=1}^{k} x_n$. then $x_k = S_k - S_{k-1}$

Hence $\lim_{k\to\infty} x_k = \lim_{k\to\infty} (S_k - S_{k-1}) = (A - A) = 0$

as $\sum_{n=1}^{\infty} x_n = A$ is convergent series.

Absolute convergence (绝对收敛)

Defn: If the series $\sum_{n=1}^{\infty} |x_n|$ converges, then we say $\sum_{n=1}^{\infty} x_n$ is absolutely convergent.

E.g.: Alternating Harmonic series converges, but not absolutely converges

Thm: If the series $\sum_{n=1}^{\infty} |x_n|$ converges, then $\sum_{n=1}^{\infty} x_n$ converges.

Conditional convergence (条件收敛)

Defn: A series $\sum_{n=1}^{\infty} x_n$ is said to be conditionally convergent if it converges but $\sum_{n=1}^{\infty} |x_n|$ does not.

Convergence tests

① Telescoping $\sum_{n=1}^{\infty}(x_n - x_{n+1})$

② Geometric $\sum_{n=1}^{\infty} a^n$

③ Harmonic $\sum_{n=1}^{\infty} \frac{1}{n}$

④ p-Harmonic $\sum_{n=1}^{\infty} \frac{1}{n^p}$

⑤ Alternating Harmonic $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$

Thm (Comparison test) Given two series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ such that $0 \le x_n \le y_n$, $\forall n \in \mathbb{N}$

① If $\sum_{n=1}^{\infty} y_n$ converges, then $\sum_{n=1}^{\infty} x_n$ converges.

② If $\sum_{n=1}^{\infty} x_n$ diverges, then $\sum_{n=1}^{\infty} y_n$ diverges.

E.g.: Prove $\sum_{k=1}^{\infty} \frac{k+3}{k^3+2k^2+3k+1}$ converges.

Proof: $\frac{k+3}{k^3+2k^2+3k+1} = \frac{1+\frac{3}{k}}{k^2+2k+3+\frac{1}{k}} = \frac{1+\frac{3}{k}}{k^2(1+\frac{2}{k}+\frac{3}{k^2}+\frac{1}{k^3})} \le \frac{1+3}{k^2 \cdot 1}$

$\sum_{n=1}^{\infty} \frac{4}{n^2} = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, then the series converges.

E.g.: Prove $\sum_{n=1}^{\infty} \sqrt{\frac{n+4}{n^2+3n+1}}$ diverges

Proof: $\sqrt{\frac{n+4}{n^2+3n+1}} = \sqrt{\frac{1+\frac{4}{n}}{n+3+\frac{1}{n}}} = \sqrt{\frac{1+\frac{4}{n}}{n(1+\frac{3}{n}+\frac{1}{n^2})}} \ge \sqrt{\frac{1}{n(1+3+1)}}$

Since $\frac{1}{\sqrt{5}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges, then the series diverges.

Thm (Ratio test) If $x_n \geq 0$ for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} < 1$$

then the series $\sum_{n=1}^{\infty} x_n$ converges.

Proof: Suppose that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} < 1$, there exists $\alpha > 0$ such that

$$\frac{x_{n+1}}{x_n} < \alpha < 1$$

There exists $N \in \mathbb{N}$ and for any $n \geq N$

$$x_{N+n} < x_{N+n-1} \cdot \alpha < x_{N+n-2} \cdot \alpha^2 \cdots < x_N \cdot \alpha^n$$

Since $\sum_{k=1}^{\infty} x_N \cdot \alpha^k$ (is a geometric series with $\alpha < 1$) converges, by comparison test, we have $\sum_{k=1}^{\infty} x_{N+k}$ converges. So does $\sum_{n=1}^{\infty} x_n$.

Note: ① $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L$, we require $x_n \geq 0$ for any $n \in \mathbb{N}$.

② If $L < 1$, then $\sum_{n=1}^{\infty} x_n$ converges.

If $L > 1$, then $\sum_{n=1}^{\infty} x_n$ diverges.

If $L = 1$, we do not know whether $\sum_{n=1}^{\infty} x_n$ converges or not.