Sequences & Series: Answer to Exercise 1

Definition 1. A field is a set F endowed with two operations: one is called addition $+$ and one is called multiplication $\cdot$ (also denoted as $\times$), which satisfy the following axioms:

- Axioms for addition:
  - For any $x, y \in F$, $x + y \in F$.
  - Commutative: $x + y = y + x$.
  - Associative: $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.
  - F contains an element, called 0, such that $0 + x = x$ for all $x \in F$.
  - For each $x \in F$, there is an element $-x \in F$ such that $x + (-x) = 0$.
- Axioms for multiplication:
  - For any $x, y \in F$, $x \cdot y \in F$. (When there is no confusion, we write $x \cdot y$ as xy).
  - Commutative: $x \cdot y = y \cdot x$ for all $x, y \in F$.
  - Associative: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in F$.
  - F contains an element, called $1 \neq 0$, such that $1 \cdot x = x$ for all $x \in F$.
  - If $x \in F$ and $x \neq 0$, then there is an element $\frac{1}{x} \in F$ such that $x \cdot \frac{1}{x} = 1$.
- Distributive law: $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in F$.

Exercise 1. Use **Definition 1** to prove the following theorem:

Theorem 1. The axioms of addition imply the following for $x, y, z \in F$ where F is a field:

(a) If $x + y = x + z$, then $y = z$.

(b) If $x + y = x$, then $y = 0$. (So 0 is unique in F.)

(c) If $x + y = 0$, then $y = -x$. (So $-x$ is unique in F corresponding to x.)

(d) $-(-x) = x$.

Proof. (a) Given $x + y = x + z$, we can add $-x$ to both sides of the equation.

$$\begin{aligned}
(-x) + (x + y) &= (-x) + (x + z) \\
((-x) + x) + y &= ((-x) + x) + z \\
0 + y &= 0 + z \\
y &= z
\end{aligned}$$

(b) If $x + y = x$, then add $-x$ to both sides:

$$\begin{aligned}
(-x) + (x + y) &= (-x) + x \\
((-x) + x) + y &= 0 \\
0 + y &= 0 \\
y &= 0
\end{aligned}$$

This shows that 0 is unique.
(c) If $x + y = 0$, then add $-x$ to both sides:

$$\begin{aligned}
(-x) + (x + y) &= (-x) + 0 \\
((-x) + x) + y &= -x \\
0 + y &= -x \\
y &= -x
\end{aligned}$$

This shows that $-x$ is unique.
(d) Since $x + (-x) = 0$, we know that $-x$ is the additive inverse of x. Also, $(-x) + (-(-x)) = 0$, which implies that $-(-x)$ is the additive inverse of $-x$. Since the additive inverse of an element is unique, we have $-(-x) = x$

□

Exercise 2. Use **Definition 1** to prove the following theorem:

Theorem 2. The axioms of multiplication imply the following for $x, y, z \in F$ where F is a field:

(a) If $x \neq 0$ and $x \cdot y = x \cdot z$, then $y = z$.

(b) If $x \neq 0$ and $x \cdot y = x$, then $y = 1$. (So 1 is unique in F.)

(c) If $x \neq 0$ and $x \cdot y = 1$, then $y = 1/x$. (So $1/x$ is unique in F corresponding to $x \neq 0$.)

(d) If $x \neq 0$, then $\frac{1}{1/x} = x$.

Proof. (a) Given $x \neq 0$ and $x \cdot y = x \cdot z$, we can multiply both sides by $\frac{1}{x}$

$$
\begin{aligned}
\frac{1}{x}(x \cdot y) &= \frac{1}{x}(x \cdot z) \\
\left(\frac{1}{x} \cdot x\right) y &= \left(\frac{1}{x} \cdot x\right) z \\
1 \cdot y &= 1 \cdot z \\
y &= z
\end{aligned}
$$

(b) If $x \neq 0$ and $x \cdot y = x$, then multiply both sides by $\frac{1}{x}$

$$
\begin{aligned}
\frac{1}{x}(x \cdot y) &= \frac{1}{x} \cdot x \\
\left(\frac{1}{x} \cdot x\right) y &= 1 \\
1 \cdot y &= 1 \\
y &= 1
\end{aligned}
$$

This shows that 1 is unique.

(c) If $x \neq 0$ and $x \cdot y = 1$, then $y = \frac{1}{x}$ by the definition of the multiplicative inverse. To show uniqueness, assume there is another element z such that $x \cdot z = 1$. Then $x \cdot y = x \cdot z$, and by part (a), $y = z$

(d) If $x \neq 0$, we know that $x \cdot \frac{1}{x} = 1$. Let $y = \frac{1}{x}$, then $y \neq 0$ and $y \cdot \frac{1}{y} = 1$. Since $x \cdot y = 1$, we have $\frac{1}{y} = x$, that is $\frac{1}{1/x} = x$

□

Exercise 3. Use **Definition 1** to prove the following theorem:

Theorem 3. The axioms of a field imply the following for $x, y, z \in F$ where F is a field:

(a) $0x = 0$.

(b) If $x \neq 0,\ y \neq 0$, then $xy \neq 0$.

(c) $(-x)y = -(xy) = x(-y)$.

(d) $(-x)(-y) = xy$.

Proof. (a) $0x = (0 + 0)x = 0x + 0x$. Adding $-0x$ to both sides gives $0x - 0x = (0x + 0x) - 0x$, which implies $0 = 0x$

(b) Suppose $x \neq 0$ and $y \neq 0$. If $xy = 0$, then multiply both sides by $\frac{1}{x}$ (since $x \neq 0$): $\frac{1}{x}(xy) = \frac{1}{x} \cdot 0$, which implies $\left(\frac{1}{x} \cdot x\right) y = 0$, or $y = 0$, which is a contradiction. So $xy \neq 0$

(c) $(-x)y + xy = (-x + x)y = 0y = 0$, so $(-x)y = -(xy)$

Also, $x(-y) + xy = x(-y + y) = x \cdot 0 = 0$, so $x(-y) = -(xy)$

(d) $(-x)(-y) = -(x(-y)) = -(-(xy)) = xy$

□