Series: The ordered sum of sequence.

Converge: $\sum_{k=1}^{\infty} a_k = c < \infty$

Properties: Uniqueness, $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

$$\sum_{n=1}^{\infty} c \cdot a_n = c \cdot \sum_{n=1}^{\infty} a_n$$

Convergence tests

Comparison test: Given two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ such that

$0 \le a_n \le b_n$, for any n,

If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.

E.g. $\sum_{k=1}^{\infty} \frac{k+3}{k^3+3k^2+3k+1} \le \frac{4}{k^2} \Rightarrow \sum_{k=1}^{\infty} \frac{k+3}{k^3+3k^2+3k+1}$ converges

$\sum_{k=1}^{\infty} \sqrt{\frac{k+4}{k^2+3k+1}} \ge \sqrt{\frac{1}{5k}} \Rightarrow$ diverges.

Ratio Tests

If terms of the series $\sum_{k=1}^{\infty} a_k$ are all positive and the ratios

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$$

then the series is convergent.

Proof: If $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$, there is a number α such that

$$\frac{a_{k+1}}{a_k} < \alpha < 1$$

for an $N \in \mathbb{N}$ that is large enough and $k \geq N$.

$$a_{N+k} < \alpha \cdot a_{N+k-1} < \alpha^2 a_{N+k-2} \cdots < a_N \cdot \alpha^k$$

Geometric series

Since $\sum_{k=1}^{\infty} a_N \cdot \alpha^k$ converges as $\alpha < 1$, by comparison test, we have

$\sum_{k=1}^{\infty} a_{N+k}$ converges, so does $\sum_{k=1}^{\infty} a_k$.

$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = L$ we require $a_k > 0$ for any $k \in \mathbb{N}$.

If $L < 1$, then the series $\sum_{k=1}^{\infty} a_k$ is convergent.

If $L > 1$, then the series $\sum_{k=1}^{\infty} a_k$ is divergent. since $a_k \to \infty$ as $k \to \infty$

If $L = 1$ then the series $\sum_{k=1}^{\infty} a_k$ may diverge or converge

E.g. $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges $\quad \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{k}{k+1} = 1$

$\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges $\quad \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(k+1)^2}{k} = 1$

E.g. $\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$ $\quad \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{((k+1)!)^2}{(k!)^2} \frac{(2k)!}{(2k+2)!}$

$$= \lim_{k\to\infty} \frac{(k+1)^2}{(2k+2)(2k+1)} = \frac{1}{4} < 1$$

Root Tests If terms of the series $\sum_{k=1}^{\infty} a_k$ are all nonnegative and

$$\lim_{k\to\infty} \sqrt[k]{a_k} < 1$$

then the series is convergent.

Proof: If $\lim_{k\to\infty} \sqrt[k]{a_k} < 1$, then there is a number α such that

$$\sqrt[k]{a_k} < \alpha < 1$$

for any $k \geq N$ and $N \in \mathbb{N}$ is large enough,

$$a_k < \alpha^k$$

By comparison test $\sum_{k=1}^{\infty} \alpha^k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

$\lim_{k\to\infty} \sqrt[k]{a_k} = L$

If $L < 1$, the series $\sum_{k=1}^{\infty} a_k$ is convergent.

If $L > 1$, the series $\sum_{k=1}^{\infty} a_k$ is divergent.

If $L = 1$, the series $\sum_{k=1}^{\infty} a_k$ may diverges or converges.

E.g. $\sum_{k=1}^{\infty} \frac{1}{k}$ $\qquad \lim_{k\to\infty} \sqrt[k]{\frac{1}{k}} = 1$

$\sum_{k=1}^{\infty} \frac{1}{k^2}$ $\qquad \lim_{k\to\infty} \sqrt[k]{\frac{1}{k^2}} = 1$

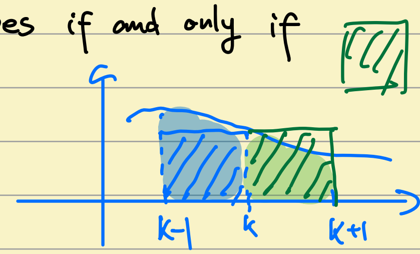
E.g. $\sum_{k=1}^{\infty} k r^k$ $\qquad r > 0$ $\qquad \lim_{k\to\infty} \sqrt[k]{k r^k} = \lim_{k\to\infty} \sqrt[k]{k}\, r = r$

If $0 < r < 1$, then $\sum_{k=1}^{\infty} k r^k$ converges,

If $r \geq 1$, then $\sum_{k=1}^{\infty} k r^k$ diverges,

Integral tests Let f be a nonnegative decreasing function on $[1,\infty)$

$\lim\limits_{x\to\infty}\int_1^x f(x)\,dx = \int_1^\infty f(x)\,dx$ converges if and only if $\sum\limits_{k=1}^{\infty} f(k)$ converges.



Proof: Since f is decreasing

$$\int_k^{k+1} f(x)\,dx \le 1\cdot f(k) \le \int_{k-1}^{k} f(x)\,dx$$

$$\int_{k-1}^{k} f(x)\,dx \le f(k-1) \le \int_{k-2}^{k-1} f(x)\,dx$$

$$\vdots$$

$$\int_1^{n+1} f(x)\,dx \le \sum_{k=1}^{n} f(k) \le f(1) + \int_1^n f(x)\,dx$$

$n\to\infty$, $\sum\limits_{k=1}^{\infty} f(k)$ converges if and only if it is bounded, that is

$\lim\limits_{n\to\infty}\int_1^n f(k)\,dx$ converges.

E.g. $\sum\limits_{k=1}^{\infty} \frac{1}{k^p}$ $(p>0)$

$$\int_1^x \frac{1}{x^p}\,dx = -\frac{1}{p-1}\frac{1}{x^{p-1}}\Bigg|_1^x = \frac{1}{p-1}\left(1-\frac{1}{x^{p-1}}\right)$$

$$\lim_{x\to\infty}\int_1^x \frac{1}{x^p}\,dx$$

$$= \lim_{x\to\infty}\frac{1}{p-1}\left(1-\frac{1}{x^{p-1}}\right) = \frac{1}{p-1} \quad \text{if } p>1$$

diverges if $0<p<1$

$\lim\limits_{x\to\infty}\ln x$ diverges if $p=1$

$$\int_1^x \frac{1}{x}\,dx = \ln x\Big|_1^x = \ln x$$

E.g. $\sum_{k=1}^{\infty} \frac{1}{k(\ln k)^p}$

$u = \ln x \quad \frac{du}{dx} = \frac{1}{x} \quad du = \frac{1}{x} dx$

$$\int \frac{1}{x(\ln x)^p} dx = \int \frac{1}{(\ln x)^p} \cdot \frac{1}{x} dx = \int \frac{1}{u^p} du$$

$p \neq 1$ $\quad \int \frac{1}{u^p} du = \int u^{-p} du = \frac{u^{-p+1}}{1-p} + C$

$$= \frac{1}{(1-p)} \frac{1}{(\ln x)^{p-1}} + C$$

$p=1$ $\quad \int \frac{1}{u} du = \ln|u| + C = \ln|\ln x| + C$

$$\lim_{X\to\infty} \int_1^X \frac{dx}{x(\ln x)^p} = \int_1^e \frac{dx}{x(\ln x)^p} + \lim_{X\to\infty} \int_e^X \frac{dx}{x(\ln x)^p}$$

if $p > 1$ $\quad = C + \frac{1}{p-1}\left(\frac{1}{(\ln X)^{p-1}} - 1\right) = C + \frac{1}{p-1}$

if $0 < p < 1$ $\quad$ diverges

if $p = 1$ $\quad$ diverges $\quad \lim_{X\to\infty} \ln \ln X = \infty$

Alternating Series Test

$\sum_{k=1}^{\infty} \frac{1}{k}$ diverges $\quad \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k}$ converges

Thm: The series $\sum_{k=1}^{\infty} (-1)^{k-1} a_k$ converges, if the sequence $\{a_n\}$ decreases monotonically to zero.

Proof: $S_n = \sum_{k=1}^{n} (-1)^{k-1} a_n = a_1 - a_2 + a_3 - a_4 \cdots a_n$

$$S_{2n} = \sum_{k=1}^{n} (a_{2k-1} - a_{2k}) = (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) \cdots$$

$$S_{2n-1} = a_1 - \sum_{k=2}^{n} (a_{2k-2} - a_{2k-1}) = a_1 - (a_2 - a_3) - (a_4 - a_5) - (a_6 - a_7) \cdots$$

$$a_1 - a_2 \leq S_{2n} < S_{2n-1} \leq a_1$$

S_{2n} is monotonically increasing, S_{2n-1} is monotonically decreasing

Since S_{2n} and S_{2n-1} is bounded. then S_{2n} and S_{2n-1} converge.

Since $S_{2n} - S_{2n-1} = a_{2n} \to 0$ as $n \to \infty$, we conclude

$$\lim_{n \to \infty} S_n = \sum_{k=1}^{\infty} (-1)^{k-1} a_k$$

is convergent.

E.g. $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k}$ $\quad \lim_{k \to \infty} \frac{1}{k} = 0$, hence this series converges.

$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^3}$ converges

$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{\sqrt{k}}$ $\quad \lim_{k \to \infty} \frac{1}{\sqrt{k}} = 0$, converges. $\quad \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges