

Sequence: $\{a_n\}$ ← it is a set with infinitely many elements

Series: the ordered sum of sequence

$$\sum_{n=1}^{\infty} a_n$$

$S_m = \sum_{n=1}^m a_n$ $\{S_m\}$ is a sequence.

Convergence tests

① Comparison test: $0 \leq a_n \leq b_n$ $\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges

$\sum_{n=1}^{\infty} a_n$ diverges $\Rightarrow \sum_{n=1}^{\infty} b_n$ diverges.

② Ratio test: $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L < 1$

E.g.: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{2}{n} + \frac{1}{n^2}} = 1$

If $L < 1$: $\sum_{n=1}^{\infty} a_n$ converges

If $L > 1$: $\sum_{n=1}^{\infty} a_n$ does not converge.

If $L = 1$: $\sum_{n=1}^{\infty} a_n$ we do not know.

E.g.: $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{(2n+2)(2n+1)} \left[\frac{(n+1)!}{(n!)^2} \right]^2$$

$$= \lim_{n \rightarrow \infty} \frac{1}{(2n+2)(2n+1)} [n+1]^2 = \frac{1}{4} < 1$$

Root Tests: If $a_n > 0$ for all $n \in \mathbb{N}$, and

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = L < 1$$

the the series $\sum_{n=1}^{\infty} a_n$ converges.

Proof: Suppose that $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} < 1$, then there exists $\alpha \in \mathbb{R}$ such that, for any $n \geq N$ and $N \in \mathbb{N}$ that is large enough, we have.

$$\sqrt[n]{a_n} < \alpha < 1$$

$$\Rightarrow a_n < \alpha^n$$

Since $\alpha < 1$, the series $\sum_{n=1}^{\infty} \alpha^n$ converges. By comparison test, we have $\sum_{n=1}^{\infty} a_n$ converges.

E.g. $\sum_{n=1}^{\infty} \frac{1}{n!}$

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n!}} = 1$$

$$\lim_{n \rightarrow \infty} \ln \sqrt[n]{\frac{1}{n!}} = \frac{1}{n} \ln \frac{1}{n!} = 0$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n^2}} = 1$$

If $L < 1$, $\sum_{n=1}^{\infty} a_n$ converges.

If $L > 1$, $\sum_{n=1}^{\infty} a_n$ diverges.

If $L = 1$, we do not know.

Power series

E.g.: $\sum_{n=1}^{\infty} n r^n$

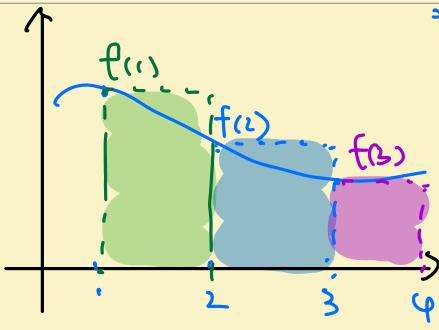
$$r > 0$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{n r^n} = \lim_{n \rightarrow \infty} \sqrt[n]{n} r = 1 \cdot r$$

① If $0 < r < 1$, then $\sum_{n=1}^{\infty} n r^n$ converges.

② If $r > 1$, then $\sum_{n=1}^{\infty} n r^n$ diverges.

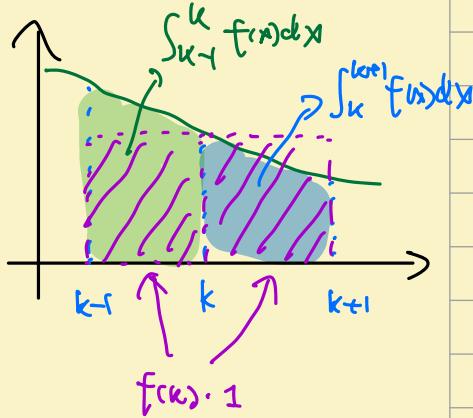
③ If $r = 1$, we do not know.



* Integral test Let f be a nonnegative decreasing function on $[1, +\infty)$
 $\sum_{n=1}^{\infty} f(n)$ converges if and only if

$$\lim_{t \rightarrow \infty} \int_1^t f(x) dx = \int_1^{\infty} f(x) dx \text{ converges.}$$

Proof: Since f is decreasing.



$$\int_k^{\infty} f(x) dx < f(k) \cdot 1 < \int_{k+1}^k f(x) dx$$

$$\int_{k+1}^k f(x) dx < f(k+1) \cdot 1 < \int_{k+2}^{k+1} f(x) dx$$

$$\int_1^{n+1} f(x) dx < \sum_{k=1}^n f(k) < f(1) + \int_1^n f(x) dx$$

Then if $\int_1^n f(x) dx$ converges as $n \rightarrow \infty$, then $\sum_{k=1}^{\infty} f(k)$ also converges.

Alternating series tests: if a sequence $\{a_n\}$ monotonically decreases to zero as $n \rightarrow \infty$, then the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

converges.

Proof: Let $S_m = \sum_{n=1}^m (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 \dots - a_m$

$$\uparrow S_{2m} = \sum_{n=1}^m (a_{2n-1} - a_{2n}) = (a_1 - a_2) + (a_3 - a_4) \dots + (a_{2m-1} - a_{2m})$$

$$\downarrow S_{2m+1} = a_1 - \sum_{n=1}^m (a_{2n-1} - a_{2n}) = a_1 - (a_2 - a_3) - (a_4 - a_5) \dots - (a_{2m+1} - a_{2m})$$

Since $\{a_n\}$ monotonically decrease to 0, we have S_m monotonically increases, and S_{m+1} monotonically decreases. We have

$$a_1 - a_2 \leq S_m \leq S_{m+1} \leq a_1$$

Hence, both S_{2m} and S_{2m+1} converge.

Since $S_{2m} - S_{2m-1} = a_{2m} \rightarrow 0$ as $m \rightarrow \infty$, we conclude

$$\lim_{n \rightarrow \infty} S_n = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

converges.

E.g.: $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$, hence the series converges.

$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n}}$ $\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$, hence the series converges.

Power series

A power series is series of the form:

$$\sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=1}^{\infty} a_n$$

Set of convergence: $S := \{x \in \mathbb{R} \mid \sum_{n=0}^{\infty} a_n x^n \text{ converges}\}$

Note: $S \neq \emptyset$, because we always have $0 \in S$.

Consider a function $f: S \rightarrow \mathbb{R}$

$$f(x) := \sum_{n=0}^{\infty} a_n x^n \quad x \in S$$

E.g.: $f(x) = \sum_{n=0}^{\infty} x^n$. If $x \in (-1, 1)$ $\Rightarrow f(x) = \frac{1}{1-x}$ converges.

Then we have the set of convergence: $S = (-1, 1)$.

The Radius of convergence.

Then: Given a power series $\sum_{n=0}^{\infty} a_n x^n$, it either

- ① converges absolutely for $x \in \mathbb{R}$, or
- ② there exists $R \in [0, +\infty)$ such that
 - (a) when $|x| < R$, it converges absolutely.
 - (b) when $|x| > R$, it diverges.

We call R the radius of convergence.

The set of convergence: $S = \{x \in \mathbb{R} \mid \sum_{n=0}^{\infty} a_n x^n \text{ converges}\}$.

$$f(x) = \frac{1}{1-x}$$

$$|x| < R \Rightarrow (-R, R) \subseteq S \subseteq [-R, R]$$

That is, either $S = (-R, R)$ or $S = [-R, R]$ or $S = [-R, R)$.

E.g.: $\sum_{n=1}^{\infty} \frac{1}{n} x^n$ when $|x| > 1 \quad \lim_{n \rightarrow \infty} \frac{x^n}{n} = +\infty$
when $|x| < 1 \quad \lim_{n \rightarrow \infty} \frac{1}{n} x^n = 0$

Hence, the radius of convergence $R = 1$

when $x = 1$, $\sum_{n=1}^{\infty} \frac{1}{n} x^n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

when $x = -1$, $\sum_{n=1}^{\infty} \frac{1}{n} x^n = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$, converges.

Therefore, the set of converges is $S = [-1, 1]$

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} < 1$$

Convergence test

Ratio test: Consider the power series $\sum_{n=0}^{\infty} a_n x^n$. If

$$\frac{|a_{n+1}|}{|a_n|} \rightarrow l \quad \text{as } n \rightarrow \infty$$

Then the radius of convergence.

$$R = \begin{cases} 0 & \text{if } l = +\infty \\ \frac{1}{l} & \text{if } l \in \mathbb{R} \setminus \{0\} \\ \infty & \text{if } l = 0 \end{cases}$$

$$R = \begin{cases} 0 & \text{if } l = +\infty \\ \frac{1}{l} & \text{if } l \in \mathbb{R} \setminus \{0\} \\ \infty & \text{if } l = 0 \end{cases}$$

Proof: Suppose $\frac{|a_{n+1}|}{|a_n|} \rightarrow l$ as $n \rightarrow \infty$. Then we have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1} \cdot x^{n+1}}{a_n \cdot x^n} \right| = \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} \cdot |x| = l \cdot |x|$$

By the ratio test of series.

① If $|x| \cdot l < 1$, it converges.

② If $|x| \cdot l > 1$, it diverges.

Therefore

① $l = 0, |x| \leq +\infty \Rightarrow R = +\infty$

② $l \neq 0$ and $l < +\infty, |x| < \frac{1}{l}$, then $R = \frac{1}{l}$

③ $l = +\infty, x = 0$, then $R = 0$.