B B C C A

Note 5

$\lceil \frac{5}{\varepsilon} \rceil$ 0 conditionally convergent

9. Compute $\left| \frac{5n-2}{2n+1} - \frac{5}{2} \right| = \left| \frac{2(5n-2) - 5(2n+1)}{4n+2} \right| = \left| \frac{9}{4n+2} \right|$

We need $\left| \frac{9}{4n+2} \right| < \varepsilon \iff n > \frac{1}{4} \left(\frac{9}{\varepsilon} - 2 \right)$

Choose $N = \lceil \frac{1}{4} \left(\frac{9}{\varepsilon} - 2 \right) \rceil$

Then, for all $\varepsilon > 0$, for all $n \geq N$, we have $\left| \frac{5n-2}{2n+1} - \frac{5}{2} \right| < \varepsilon$.

10. $\lim_{n\to\infty} \sqrt{n^2 + 9n + 4} - n$

$= \lim_{n\to\infty} \frac{(\sqrt{n^2+9n+4} - n)(\sqrt{n^2+9n+4} + n)}{\sqrt{n^2+9n+4} + n}$

$= \lim_{n\to\infty} \frac{n^2 + 9n + 4 - n^2}{\sqrt{n^2+9n+4} + n} = \lim_{n\to\infty} \frac{9n+4}{\sqrt{n^2+9n+4} + n} = \lim_{n\to\infty} \frac{9 + \frac{4}{n}}{\sqrt{1 + \frac{9}{n} + \frac{4}{n^2}} + 1}$

$= \frac{9}{\sqrt{1} + 1} = \frac{9}{2}$

11. $\frac{1}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3} = \frac{A(n+3) + Bn}{n(n+3)}$

$A(n+3) + Bn = 1 + 0 \cdot n$

$(A+B) \cdot n + A \cdot 3 = 0 \cdot n + 1 \implies A = \frac{1}{3} \quad B = -\frac{1}{3}$

$\frac{1}{n(n+3)} = \frac{1}{3} \left(\frac{1}{n} - \frac{1}{n+3} \right)$

$$S_N = \sum_{n=1}^{N} \frac{1}{n(n+3)} = \frac{1}{3} \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+3} \right)$$

$$= \frac{1}{3} \left[1 - \cancel{\frac{1}{4}} + \frac{1}{2} - \cancel{\frac{1}{5}} + \frac{1}{3} - \cancel{\frac{1}{6}} + \cancel{\frac{1}{4}} - \frac{1}{7} + \cancel{\frac{1}{5}} - \frac{1}{8} \cdots + \frac{1}{N} - \frac{1}{N+3} \right]$$

$$\therefore S_N = \frac{1}{3} \left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{N+1} - \frac{1}{N+2} - \frac{1}{N+3} \right)$$

Let $N \to \infty$ $\quad S_\infty = \frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}\right) = \frac{1}{3} \cdot \frac{11}{6} = \frac{11}{18}$

12. Using ratio test. Let $a_n = \frac{(x-1)^n}{n!}$ then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left(\frac{(x-1)^{n+1}}{(n+1)!} \cdot \frac{n!}{(x-1)^n} \right) = \lim_{n\to\infty} \left(\frac{x-1}{n+1} \right) = 0$$

So the convergence domain is $(-\infty, +\infty)$

13. $$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Note 8

$f(0) = 1$		$f(0) = \cos 0 = 1$
$f'(0) = 0$	Let $f(x) = \cos x$	$f'(0) = -\sin x = 0$
$f''(0) = -1$	then we have	$f''(0) = -\cos x = -1$
$f'''(0) = 0$		$f'''(0) = \sin x = 0$

So the function is $f(x) = \cos x$

Using ratio test

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-1)^{n+1} x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{(-1)^n x^{2n}} \right| = \left| \frac{x^2}{(2n+2)(2n+1)} \right| \to 0$$

as $n \to \infty$

So the radius of convergence is $R = +\infty$.

14. $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$f(x) = (1-x)^{-1}$ $\qquad$ $f(0) = 1$

$f'(x) = (1-x)^{-2}$ $\qquad$ at $x = 0$ $\qquad$ $f'(0) = 1$

$f''(x) = 2(1-x)^{-3}$ $\qquad$ $f''(0) = 2$

$f'''(x) = 3!(1-x)^{-4}$ $\qquad$ $f'''(0) = 3!$

$\vdots$

$f^{(n)}(0) = n!$

$f(x) = \sum_{n=0}^{\infty} x^n$

Consider the radius of convergence $(-1, 1)$

$f: (-1,1) \to \mathbb{R}$ $\quad$ $f(x) = \sum_{n=0}^{\infty} x^n$