

# Sequences & Series: Exercise 1

Due date: 2024-11-26

**Definition 1.** A field is a set  $F$  endowed with two operations: one is called addition  $+$  and one is called multiplication  $\cdot$  (also denoted as  $\times$ ), which satisfy the following axioms:

- Axioms for addition:
  - For any  $x, y \in F$ ,  $x + y \in F$ .
  - Commutative:  $x + y = y + x$ .
  - Associative:  $(x + y) + z = x + (y + z)$  for all  $x, y, z \in F$ .
  - $F$  contains an element, called 0, such that  $0 + x = x$  for all  $x \in F$ .
  - For each  $x \in F$ , there is an element  $-x \in F$  such that  $x + (-x) = 0$ .
- Axioms for multiplication:
  - For any  $x, y \in F$ ,  $x \cdot y \in F$ . (When there is no confusion, we write  $x \cdot y$  as  $xy$ ).
  - Commutative:  $x \cdot y = y \cdot x$  for all  $x, y \in F$ .
  - Associative:  $(x \cdot y) \cdot z = x \cdot (y \cdot z)$  for all  $x, y, z \in F$ .
  - $F$  contains an element, called  $1 \neq 0$ , such that  $1 \cdot x = x$  for all  $x \in F$ .
  - If  $x \in F$  and  $x \neq 0$ , then there is an element  $\frac{1}{x} \in F$  such that  $x \cdot \frac{1}{x} = 1$ .
- Distributive law:  $x \cdot (y + z) = x \cdot y + x \cdot z$  for all  $x, y, z \in F$ .

**Exercise 1.** Use **Definition 1** to prove the following theorem:

**Theorem 1.** The axioms of addition imply the following for  $x, y, z \in F$  where  $F$  is a field:

- (a) If  $x + y = x + z$ , then  $y = z$ .
- (b) If  $x + y = x$ , then  $y = 0$ . (So 0 is unique in  $F$ .)
- (c) If  $x + y = 0$ , then  $y = -x$ . (So  $-x$  is unique in  $F$  corresponding to  $x$ .)
- (d)  $-(-x) = x$ .

**Exercise 2.** Use **Definition 1** to prove the following theorem:

**Theorem 2.** The axioms of multiplication imply the following for  $x, y, z \in F$  where  $F$  is a field:

- (a) If  $x \neq 0$  and  $x \cdot y = x \cdot z$ , then  $y = z$ .
- (b) If  $x \neq 0$  and  $x \cdot y = x$ , then  $y = 1$ . (So 1 is unique in  $F$ .)
- (c) If  $x \neq 0$  and  $x \cdot y = 1$ , then  $y = 1/x$ . (So  $1/x$  is unique in  $F$  corresponding to  $x \neq 0$ .)
- (d) If  $x \neq 0$ , then  $\frac{1}{1/x} = x$ .

**Exercise 3.** Use **Definition 1** to prove the following theorem:

**Theorem 3.** The axioms of a field imply the following for  $x, y, z \in F$  where  $F$  is a field:

- (a)  $0x = 0$ .
- (b) If  $x \neq 0$ ,  $y \neq 0$ , then  $xy \neq 0$ .
- (c)  $(-x)y = -(xy) = x(-y)$ .
- (d)  $(-x)(-y) = xy$ .