

Sequences & Series: Answer to Exercise 2

1. Given the sequence $a_n = \frac{n}{n+1}$, determine whether the sequence converges. If it does, find its limit.

Answer:

$$\begin{aligned}\lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \frac{n}{n+1} \\ &= \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} \\ &= 1\end{aligned}$$

2. Given the sequence $a_n = \left(1 + \frac{1}{n}\right)^n$, determine whether the sequence converges. If it does, find its limit.

Answer:

The limit of this sequence is e . See the note for detailed proof.

3. Given the sequence $a_n = \frac{\sin n}{n}$, determine whether the sequence converges. If it does, find its limit.

Answer:

Sequence $a_n = \frac{\sin n}{n}$ We know that $-1 \leq \sin n \leq 1$. So, $\left|\frac{\sin n}{n}\right| \leq \frac{1}{n}$. Since $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$, by the Squeeze Theorem, $\lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0$. So the sequence converges to 0.

4. Given the sequence $a_n = \frac{(-1)^n n}{n+1}$, determine whether the sequence converges. If it does, whether it is absolutely convergent or conditionally convergent.

Answer:

We consider the subsequences of even and odd terms:

For even terms ($n = 2k, k \in \mathbb{N}$), $a_{2k} = \frac{2k}{2k+1} \rightarrow 1$ as $k \rightarrow \infty$

For odd terms ($n = 2k+1, k \in \mathbb{N}$), $a_{2k+1} = \frac{-(2k+1)}{2k+2} \rightarrow -1$ as $k \rightarrow \infty$

Since the subsequences converge to different limits, the sequence $\{a_n\}$ does not converge.

5. Given the sequence $a_n = \sqrt{n+1} - \sqrt{n}$, determine whether the sequence converges. If it does, find its limit.

Answer:

We rationalize the expression:

$$\begin{aligned} a_n &= \sqrt{n+1} - \sqrt{n} \\ &= \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{1}{\sqrt{n+1} + \sqrt{n}} \end{aligned}$$

As $n \rightarrow \infty$, $\lim_{n \rightarrow \infty} a_n = 0$. So the sequence converges to 0.