

MATH 173 PROBLEM SET 6

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Problem 1. TODO

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Solution.

(a) This problem is straightforward.

$$\begin{aligned}\overline{\mathcal{F}(\phi)(y)} &= \overline{\int_{\mathbb{R}^n} e^{-ix \cdot y} \phi(x) dx} \\ &= \int_{\mathbb{R}^n} \overline{e^{-ix \cdot y} \phi(x)} dx \\ &= \int_{\mathbb{R}^n} e^{ix \cdot y} \overline{\phi(x)} dx \\ &= (2\pi)^n (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot y} \overline{\phi(x)} dx \\ &= (2\pi)^n \mathcal{F}^{-1} \overline{\phi}(y)\end{aligned}$$

This is what we wanted to show.

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(b) We know the Fourier inversion formula holds for Schwartz functions. By part (a) and the equation about interchanging fourier transform under the integral that we saw in class, we know

$$\begin{aligned}(2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\phi} \bar{\psi} &= \int_{\mathbb{R}^n} \hat{\phi} \check{\psi} \\ (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\phi} \bar{\psi} &= \int_{\mathbb{R}^n} \check{\phi} \hat{\psi} \\ (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\phi} \bar{\psi} &= \int_{\mathbb{R}^n} \phi \bar{\psi}.\end{aligned}$$

Setting $\psi = \phi$, we see that

$$\int_{\mathbb{R}^n} |\hat{\phi}|^2 = (2\pi)^2 \int_{\mathbb{R}^n} |\phi|^2,$$

as we wanted.

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Problem 2. TODO

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Solution.

- (a) Let $\tilde{u}(t, x)$ be defined as in the problem. On $(0, +\infty) \times (0, +\infty)$, \tilde{u} is the same as u , so it satisfies the equation $\tilde{u}_t = \tilde{u}_{xx}$. On $(0, +\infty) \times (-\infty, 0)$, we see that

$$\tilde{u}_t(t, x) = -u_t(t, -x) = -u_{xx}(t, -x) = \tilde{u}_{xx}(t, x).$$

So, $\tilde{u}_t = \tilde{u}_{xx}$ on all of $(0, +\infty) \times (0, +\infty) \cup (0, +\infty) \times (-\infty, 0)$ and is C^2 there. Now, consider the points along $x = 0$. We can define $\tilde{u}(t, 0) = 0$. We see that

$$\lim_{x \rightarrow 0} u(t, x) = u(t, 0) = 0,$$

and thus $\lim_{x \rightarrow 0} \tilde{u}(t, x) = 0$ from both sides, and is equal to $\tilde{u}(t, 0)$. So, \tilde{u} is continuous in $[0, +\infty) \times \mathbb{R}$.

Now, let's consider differentiability. Let $t > 0$. We see that

$$\begin{aligned} \lim_{h \rightarrow +0} \frac{\tilde{u}(t, h) - \tilde{u}(t, 0)}{h} &= \lim_{h \rightarrow +0} \frac{\tilde{u}(t, h)}{h} \\ &= \lim_{h \rightarrow +0} \frac{u(t, h)}{h} \\ &= \lim_{h \rightarrow -0} \frac{u(t, -h)}{-h} \\ &= \lim_{h \rightarrow -0} \frac{\tilde{u}(t, h)}{h}. \end{aligned}$$

Thus, the derivative from both sides matches up. We see that $\tilde{u}_t(t, 0) = 0$. Since u is continuously differentiable on the border, both components of the derivative are continuous, so u is differentiable on $(0, +\infty) \times \mathbb{R}$.

We can now assume that $\tilde{u} = K_t * \tilde{g}$. Writing this out, we have

$$\begin{aligned} \tilde{u}(t, x) &= (K_t * \tilde{g})(x) \\ &= \int (4\pi t)^{-1/2} e^{-\frac{|x-y|^2}{4t}} \tilde{g}(y) dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{|x-y|^2}{4t}} \tilde{g}(y) dy + \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^0 e^{-\frac{|x-y|^2}{4t}} \tilde{g}(y) dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{(x-y)^2}{4t}} g(y) dy - \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{(x+y)^2}{4t}} g(y) dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^\infty g(y) \left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right) dy. \end{aligned}$$

Restricting to to half, we see that

$$u(t, x) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty g(y) \left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right) dy.$$

We can verify that the boundary conditions match. □

(b) This is very similar to part (a), but this time let us define

$$\tilde{u}(t, x) = \begin{cases} u(t, x) & \text{if } x \geq 0 \\ u(t, -x) & \text{if } x < 0 \end{cases}$$

to be the even extension.

We know \tilde{u} is C^2 and satisfies the equation on $(0, +\infty) \times (0, +\infty) \cup (0, +\infty) \times (-\infty, 0)$ for the same reason as in part (a).

We see that

$$\lim_{x \rightarrow 0} u(t, x) = u(t, 0),$$

and thus $\lim_{x \rightarrow 0} \tilde{u}(t, x) = u(t, 0)$ from both sides, and is equal to $\tilde{u}(t, 0)$. So, \tilde{u} is continuous in $[0, +\infty) \times \mathbb{R}$.

Now, we notice that

$$\begin{aligned} \lim_{h \rightarrow +0} \frac{\tilde{u}(t, h) - \tilde{u}(t, 0)}{h} &= \lim_{h \rightarrow +0} \frac{u(t, h) - u(t, 0)}{h} \\ &= 0 \\ &= \lim_{h \rightarrow +0} -\frac{u(t, h) - u(t, 0)}{h} \\ &= \lim_{h \rightarrow -0} -\frac{u(t, -h) - u(t, 0)}{h} \\ &= \lim_{h \rightarrow -0} \frac{\tilde{u}(t, h) - \tilde{u}(t, 0)}{h}. \end{aligned}$$

Thus, the derivative from both sides matches up. We see that $\tilde{u}_t(t, 0) = 0$. Since u is continuously differentiable on the border, both components of the derivative are continuous, so u is differentiable on $(0, +\infty) \times \mathbb{R}$.

Making a similar assumption, we see that this time

$$\begin{aligned} \tilde{u}(t, x) &= (K_t * \tilde{g})(x) \\ &= \int (4\pi t)^{-1/2} e^{-\frac{|x-y|^2}{4t}} \tilde{g}(y) dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{|x-y|^2}{4t}} \tilde{g}(y) dy + \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^0 e^{-\frac{|x-y|^2}{4t}} \tilde{g}(y) dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{(x-y)^2}{4t}} g(y) dy + \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{(x+y)^2}{4t}} g(y) dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^\infty g(y) \left(e^{-\frac{(x-y)^2}{4t}} + e^{-\frac{(x+y)^2}{4t}} \right) dy. \end{aligned}$$

Restricting to to half, we see that

$$u(t, x) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty g(y) \left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right) dy.$$

We can verify that the boundary condition matches. □

Problem 3. TODO

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Solution. This problem is full of tricks and surprises. First, consider $v(t, x) = u(t, x) - a(t)$. This means that

$$v_t = u_t - a'(t) = u_{xx} - a'(t) = v_{xx} - a'(t).$$

So,

$$v_t - v_{xx} = -a'(t)$$

with $v(t, 0) = 0$ and $v(0, x) = 0$. Also, $a(0) = 0$. So, we have an inhomogeneous heat equation.

Let $S_t(\phi)$ be the operator in problem 2. More precisely, let

$$S_t(\phi)(x) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty \phi(y) \left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right) dy.$$

Then, by Duhamel's principle,

$$v(t, x) = \int_0^t S_{t-s}(f(s, \cdot)) ds$$

where $f(t, x) = -a'(t)$. Expanding, and changing variables, we have

$$\begin{aligned} v(t, x) &= \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int_0^\infty -a'(s) \left(e^{-\frac{(x-y)^2}{4(t-s)}} - e^{-\frac{(x+y)^2}{4(t-s)}} \right) dy ds \\ &= \int_0^t \frac{-a'(s)}{\sqrt{4\pi(t-s)}} \left[\int_0^x e^{-\frac{(y)^2}{4(t-s)}} dy + \int_0^\infty e^{-\frac{(y)^2}{4(t-s)}} dy - \int_x^\infty e^{-\frac{(y)^2}{4(t-s)}} dy \right] ds \\ &= \int_0^t \frac{-a'(s)}{\sqrt{4\pi(t-s)}} \left[\int_0^x e^{-\frac{(y)^2}{4(t-s)}} dy + \int_0^x e^{-\frac{(y)^2}{4(t-s)}} dy \right] ds \\ &= \int_0^t \frac{-2a'(s)}{\sqrt{4\pi(t-s)}} \int_0^x e^{-\frac{(y)^2}{4(t-s)}} dy ds \end{aligned}$$

Now, using the hint, we can change variables and apply integration by parts to see that

$$\begin{aligned} v(t, x) &= \int_0^t \frac{-2a'(s)}{\sqrt{4\pi(t-s)}} \int_0^{x(4(t-s))^{-1/2}} e^{-z^2} dz ds \\ &= \int_0^t \frac{-2a(s)}{\sqrt{\pi}} e^{\frac{-x^2}{4(t-s)}} \left(-(t-s)^{-3/2} \cdot \frac{1}{4} \right) ds \\ &= \frac{x^2}{\sqrt{4\pi}} \int_0^t (t-s)^{-3/2} e^{\frac{-x^2}{4(t-s)}} a(s) ds \\ &= \frac{x^2}{\sqrt{4\pi}} \int_0^t (s)^{-3/2} e^{\frac{-x^2}{4(s)}} a(t-s) ds. \end{aligned}$$

Thus, we can conclude that

$$u(t, x) = a(t) + \frac{x^2}{\sqrt{4\pi}} \int_0^t (s)^{-3/2} e^{\frac{-x^2}{4(s)}} a(t-s) ds,$$

which is not exactly what we wanted, but I suspect there is a typo in the problem. □

Problem 4. TODO

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Solution.

(a) By d'Alembert's formula, we know

$$\begin{aligned} u(x, t) &= \frac{1}{2}(\phi(x + ct) + \phi(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(\sigma) d\sigma \\ &= \frac{1}{2}((x + ct)^2 + (x - ct)^2) + \frac{1}{2c} \int_{x-ct}^{x+ct} 1 d\sigma \\ &= x^2 + (ct)^2 + t^2. \end{aligned}$$

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(b) For this problem, we can repeatedly use the fundamental theorem of calculus. We assume that u_x vanishes at infinity. We see

$$\begin{aligned} \int u(t, x) dx &= \int \left(u(0, x) + \int_0^t u_t(\tau, x) d\tau \right) dx \\ &= \int \left(u(0, x) + \int_0^t \left(u_t(0, x) + \int_0^\tau u_{tt}(s, x) ds \right) d\tau \right) dx \\ &= \int \left(u(0, x) + \int_0^t \left(u_t(0, x) + \int_0^\tau c^2 u_{xx}(s, x) ds \right) d\tau \right) dx \\ &= \int u(0, x) dx + \int \int_0^t u_t(0, x) d\tau dx + \int \int_0^t \int_0^\tau c^2 u_{xx}(s, x) ds d\tau dx \\ &= \int \phi(x) dx + \int \int_0^t \psi(x) d\tau dx + \int \int_0^t \int_0^\tau c^2 u_{xx}(s, x) ds d\tau dx \\ &= \int \phi(x) dx + t \int \psi(x) d\tau dx + \int_0^t \int_0^\tau \int c^2 u_{xx}(s, x) dx ds d\tau \\ &= \int \phi(x) dx + t \int \psi(x) d\tau dx + \int_0^t \int_0^\tau 0 ds d\tau \\ &= \int \phi(x) dx + t \int \psi(x) d\tau dx \end{aligned}$$

Problem 5. TODO

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Solution. TODO

Problem 6. TODO

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Solution. TODO

Problem 7. TODO

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Solution. TODO