MATH 173 Problem Set 4

Stepan (Styopa) Zharkov

April 27, 2022

Problem 1. TODO ◁

Solution. For this problem, we'll follow the first itinerary suggested by the hint. We know

$$f(x) = \int_0^x f'(t)dt.$$

Let $\chi_{[0,x]}$ be the characteristic function of $[0,x]$. We now see that by the Cauchy-Schwartz inequality,

$$\begin{aligned}
\int_0^1 f(x)^2 dx &= \int_0^1 \left(\int_0^x f'(t)dt \right) dx \\
&= \int_0^1 \left(\int_0^1 \chi_{[0,x]} f'(t)dt \right)^2 dx \\
&\leq \int_0^1 \left(\int_0^1 \chi_{[0,x]}^2 dt \right) \left(\int_0^1 f'(t)^2 dt \right) dx \\
&= \left(\int_0^1 f'(t)^2 dt \right) \int_0^1 \left(\int_0^1 \chi_{[0,x]}^2 dt \right) dx \\
&= \left(\int_0^1 f'(t)^2 dt \right) \int_0^1 x dx \\
&= \frac{1}{2} \int_0^1 f'(t)^2 dt.
\end{aligned}$$

So, an absolute constant of $C = 1/2$ works. □

Problem 2. TODO ◁

Solution.

(a) First, we see that for any $\varepsilon > 0$, there exists an R such that $|u(0, R)| \leq \varepsilon$. So,

$$\sup_{x\in\mathbb{R}^n, t\in[0,t]} u(t, x) \geq -\varepsilon$$

and thus

$$\sup_{x\in\mathbb{R}^n, t\in[0,t]} u(t, x) \geq 0.$$

Also,

$$\sup_{x\in\mathbb{R}^n, t\in[0,t]} u(t, x) \geq \sup_{x\in\mathbb{R}^n} u(0, x)$$

so

$$\sup_{x\in\mathbb{R}^n, t\in[0,t]} u(t, x) \geq \max\{0, \sup_{x\in\mathbb{R}^n} u(0, x)\}.$$

The other direction is more involved. Let

$$C = \sup_{x\in\mathbb{R}^n, t\in[0,t]} u(t, x).$$

If $C = 0$, then we are done so let $C > 0$. For any $\varepsilon > 0$ such that $\varepsilon < C$, let x_0, t_0 be such that $u(x_0, t_0) > C - \varepsilon$. We know there exists an R such that

$$\sup_{|x|>R, t\in[0,t]} u(t, x) < C - \varepsilon.$$

Let $R_0 > \max t_0, R$. We then see by the maximum principle on the hypercylinder with the R_0-bal as the base that u achieves its supremum in the cylinder either on the base $\{(0, x) : |x| \leq R_0\}$ or on the wall $\{(x, t) : t \in [0, T], |x| = R_0\}$. However we saw that the supremum on the wall is less than $C - \varepsilon < u(t_0, x_0)$, which is inside the cylinder. So, the supremum is attained on the base and there exits an x such that $u(0, x) \geq u(t_0, x_0) \geq C - \varepsilon$. Since ε was arbitrarily small, we have shown that RHS is at least C, so we can conclude that

$$\sup_{x\in\mathbb{R}^n, t\in[0,t]} u(t, x) = \max\{0, \sup_{x\in\mathbb{R}^n} u(0, x)\},$$

as we wanted. □

Problem 3. TODO ◁

Solution. TODO

Problem 4. TODO $\triangleleft$

Solution. TODO

Problem 5. TODO $\triangleleft$

Solution. TODO

Problem 6. TODO ◁

Solution. TODO

Problem 7. TODO ◁

Solution. TODO