

MATH 173 PROBLEM SET 7

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Problem 1.

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Solution. This problem is straightforward. Since $\overline{\Omega}$ is closed, and $c > 0$, there exists constants $C_1, C_2 > 0$ such that $C_1 < c(x) < C_2$. Assume $C_1 < 1$ and $C_2 > 1$. If not, we can always choose smaller C_1 and larger C_2 . So, for any $u \in C^1(\overline{\Omega})$,

$$\begin{aligned} \|u\|_{H_c^1(\Omega)}^2 &= \int_{\Omega} |u(x)|^2 dx + \int_{\Omega} c(x) |\nabla u(x)|^2 dx \\ &\geq C_1^2 \int_{\Omega} |u(x)|^2 dx + \int_{\Omega} C_1^2 |\nabla u(x)|^2 dx \\ &= C_1^2 \|u\|_{H^1(\Omega)}^2. \end{aligned}$$

Similarly,

$$\begin{aligned} \|u\|_{H_c^1(\Omega)}^2 &= \int_{\Omega} |u(x)|^2 dx + \int_{\Omega} c(x) |\nabla u(x)|^2 dx \\ &\leq C_2^2 \int_{\Omega} |u(x)|^2 dx + \int_{\Omega} C_2^2 |\nabla u(x)|^2 dx \\ &= C_2^2 \|u\|_{H^1(\Omega)}^2. \end{aligned}$$

So, the statement follows for all $u \in C^1(\overline{\Omega})$. By continuity and density, it follows that the statement holds for $u \in H^1(\Omega)$. \square

Problem 2.

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Solution.

- (a) Suppose $v = u + w = u' + w'$ where $u, u' \in M$, $w, w' \in M^\perp$. Then, we see

$$u' - u = u' + w - v = w - w'.$$

But $u' - u \in M$ and $w - w' \in M^\perp$ and $M \cap M^\perp = \{0\}$. So, $u' - u = w - w' = 0$. Thus, the decomposition is unique.

We see that $u = u + 0$. By uniqueness, $P(v) = u = P(u) = P(P(v))$, so $P = P^2$. □

- (b) Let $v = u + w$ and $v' = u' + w'$ with $u, u' \in M$, $w, w' \in M^\perp$. We see

$$\langle Pv, v' \rangle = \langle u, u' + w' \rangle = \langle u, u' \rangle + \langle u, w' \rangle = \langle u, u' \rangle = \langle u, u' \rangle + \langle w, u' \rangle = \langle u + w, u' \rangle = \langle v, Pv' \rangle,$$

So $P = P^*$ by definition. □

- (c) Since T is linear, $T(H)$ is a subspace. For any sequence v_j in $T(H)$ where $v_j \rightarrow v$ in H , we know $T(v_j) = v_j$ because $T^2 = T$. So, by continuity of T , we have $T(v) = v$, so $v \in T(H)$ and we can conclude $T(H)$ is closed.

Let $u = T(v)$ and let $w = v - u$. Note $u \in T(H)$. For any $y = T(x) \in T(H)$, we have

$$\langle y, w \rangle = \langle T(x), v - u \rangle = \langle x, T(v - u) \rangle = \langle x, T(v) - T(u) \rangle = \langle x, u - u \rangle = 0$$

because $T = T^2 = T^*$. So, $w \in T(H)^\perp$ and thus T is the projection. □

Problem 3.

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Solution. In this problem, we will use H^1 to denote $H^1((0, 1))$ to reduce clutter.

- (a) Consider the operator $T : H^1 \rightarrow H^1$ where $Tf = f - \int_0^1 f$. It's easy to see that T is linear since integration is linear. We see that for any f ,

$$\begin{aligned} \|Tf\|_{H^1} &= \int_0^1 \left| f - \int_0^1 f \right|^2 + \int_0^1 |\nabla f|^2 \\ &\leq \int_0^1 \left| f - \int_0^1 f \right|^2 + \int_0^1 |\nabla f|^2 \\ &\leq \int_0^1 \left(2|f|^2 + 2 \left| \int_0^1 f \right|^2 \right) + \int_0^1 |\nabla f|^2 \\ &= 4 \int_0^1 |f|^2 + \int_0^1 |\nabla f|^2 \\ &\leq 4 \int_0^1 |f|^2 + 4 \int_0^1 |\nabla f|^2 \\ &= 4 \|f\|_{H^1}^2. \end{aligned}$$

So, T is bounded. It's easy to see that $T(f) \in M$ by definition, and if $f \in M$, then $T(f) = f$. So, $T = T^2$ and $T(H^1) = M$. Also,

$$\begin{aligned} \langle Tf, g \rangle &= \langle f - \int_0^1 f, g \rangle \\ &= \int_0^1 (f\bar{g} + \nabla f \overline{\nabla g}) - \int_0^1 \left(\int_0^1 f \right) \bar{g} \\ &= \int_0^1 (f\bar{g} + \nabla f \overline{\nabla g}) - \left(\int_0^1 \bar{g} \right) \left(\int_0^1 f \right) \\ &= \int_0^1 (f\bar{g} + \nabla f \overline{\nabla g}) - \int_0^1 \left(\int_0^1 \bar{g} \right) f \\ &= \langle f, g - \int_0^1 g \rangle \\ &= \langle f, Tg \rangle. \end{aligned}$$

So, $T = T^*$. Now, by problem 2c, $M = T(H)$ is closed.

Since T is the projection onto M , we know $M^\perp = \ker T$. That is f such that $f - \int_0^1 f = 0$. Thus, f must be constant. We see that any constant function is in the kernel, so M^\perp is the constant functions. \square

- (b) First, let us derive a useful fact. Let g be a function, h, w be distributions. We know from homework that $w' = 0$ as distributions implies $w = c$ as functions. Now, we know $\left(\int_0^x \right)' = g$ by FTC, and the same is true as distributions. So, if $h' = g$ as distributions, then $\left(h - \int_0^x g \right)' = 0$ as distributions. Plugging in for w , we see $h = \int_0^x g + C$. We will use this fact.

Problem 4.

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Solution. Let

$$J^*g = \int_0^x \int_0^t g(s) ds dt - x \int_0^1 \int_0^t g(s) ds dt.$$

It's easy to see that J^* is linear and thi image is indeed in H_0^1 because it is differentiable and is 0 on the endpoints. Now, we show it is bounded. Note that

$$(J^*g)' = \int_0^x g(s) ds - \int_0^1 \int_0^t g(s) ds dt.$$

We see by Cauchy-Schwartz

$$\begin{aligned} \int_0^1 \left| \int_0^x g(s) ds \right|^2 dx &\leq \int_0^1 \int_0^x |g(s)|^2 ds dx \\ &\leq \int_0^1 \int_0^1 |g(s)|^2 ds dx \\ &= \int_0^1 |g(s)|^2 ds \\ &= \|g\|_{L^2}^2. \end{aligned}$$

Similarly,

$$\begin{aligned} \int_0^1 \left| \int_0^1 \int_0^t g(s) ds \right|^2 dx &\leq \int_0^1 \int_0^1 \int_0^t |g(s)|^2 ds dx \\ &\leq \int_0^1 \int_0^1 \int_0^1 |g(s)|^2 ds dx \\ &= \int_0^1 |g(s)|^2 ds \\ &= \|g\|_{L^2}^2. \end{aligned}$$

So, by the triangle inequality on the L^2 norm,

$$\|J^*g\|_{H_0^1} = \|(J^*g)'\|_{L^2} \leq 2\|g\|_{L^2}.$$

Thus, J^* is bounded. The last thing to check is that J^* is actually the adjoint. For that, we importantly note that $(J^*g)'' = -g$. Integrating by parts,

$$\begin{aligned} \langle Jf, g \rangle_{L^2} &= \langle f, g \rangle_{L^2} \\ &= \int f \bar{g} \\ &= - \int f \overline{(J^*g)''} \\ &= \int f' \overline{(J^*g)'} \\ &= \langle f, J^*g \rangle_{H_0^1}. \end{aligned}$$

So, J^* is the adjoint. □

Problem 5.

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Solution.

- (a) This problem is a computation. Since F_ε is rotationally symmetric, we can use the given fact to see that

$$\Delta F_\varepsilon = h''(|x|) + (n-2)|x|^{-1}h'(x)$$

where

$$h(y) = c_n(y^2 + \varepsilon)^{(2-n)/2}.$$

We compute that

$$h'(y) = yc_n(2-n)(y^2 + \varepsilon^2)^{-n/2}.$$

Also,

$$h''(y) = c_n(2-n)(y^2 + \varepsilon^2)^{-n/2} - y^2 c_n(2-n)n(y^2 + \varepsilon^2)^{-(n+2)/2}.$$

So, plugging this in, we have

$$\begin{aligned} \Delta F_\varepsilon &= c_n(2-n)(|x|^2 + \varepsilon^2)^{-n/2} - |x|^2 c_n(2-n)n(|x|^2 + \varepsilon^2)^{-(n+2)/2} - (n-1)(2-n)c_n(|x|^2 + \varepsilon^2)^{-n/2} \\ &= \varepsilon^2 c_n(2-n)(|x|^2 + \varepsilon^2)^{-(n+2)/2} \\ &= -\varepsilon^{-n} c_n(n-2)(|x/\varepsilon|^2 + 1)^{-(n+2)/2} \\ &= \varepsilon^{-n} g(x/\varepsilon), \end{aligned}$$

as we wanted. □

- (b) Since F_ε is bounded by F , for any test function ϕ , the dominated convergence theorem gives us

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} |(F - F_\varepsilon)(\phi)| &= \lim_{\varepsilon \rightarrow 0} \left| \int c_n \left(|x|^{2-n} - (|x|^2 + \varepsilon^2)^{(2-n)/2} \right) \phi(x) dx \right| \\ &= \left| \int c_n \left(\lim_{\varepsilon \rightarrow 0} \left(|x|^{2-n} - (|x|^2 + \varepsilon^2)^{(2-n)/2} \right) \right) \phi(x) dx \right| \\ &= \left| \int c_n(0) \phi(x) dx \right| \\ &= 0. \end{aligned}$$

So, $F_\varepsilon \rightarrow F$ in the sense of distributions indeed.

Now we see that from part (a) and page 59 of chapter 5,

$$\begin{aligned} \Delta F(\phi) &= F(\Delta \phi) \\ &= \lim_{\varepsilon \rightarrow 0} F_\varepsilon(\Delta \phi) \\ &= \lim_{\varepsilon \rightarrow 0} \int \varepsilon^{-n} g(x/\varepsilon) \phi(x) dx \\ &= \delta(\phi) \\ &= \phi(0). \end{aligned}$$

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Problem 6.

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Solution.

- (a) From problem 5(b), and what we did in class, we see since F is compactly supported,

$$\Delta u = \Delta F * f = \delta * f = f.$$

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- (b) We have seen that the Fourier transform of δ is 1, but let us confirm this. We see that for any test function ϕ ,

$$\begin{aligned}\mathcal{F}(\delta)(\phi) &= \delta(\mathcal{F}(\phi)) \\ &= \delta\left(\int \phi(x)e^{-ix \cdot y} dx\right) \\ &= \int \phi(x) dx.\end{aligned}$$

So, by definition, $\mathcal{F}(\delta) = 1$.

Now, note that using our Fourier transform of derivative rules, we have

$$\mathcal{F}(\Delta F) = \sum_{j=1}^n (-iy_j)^2 \mathcal{F}(F) = -|y|^2 \mathcal{F}(F).$$

Since $\mathcal{F}(\Delta F) = \mathcal{F}(\delta) = 1$, we can say

$$\mathcal{F}(F) = -\frac{1}{|y|^2}.$$

□