MATH 173 Problem Set 3

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Problem 1. TODO ◁

Solution. As the hint suggests, $u' = 0$ means by definition $u(\phi) = 0$ for any $\phi \in C_c^\infty(\mathbb{R})$. For any $\psi \in C_c^\infty(\mathbb{R})$, let $\phi_0 \in C_c^\infty(\mathbb{R})$ be a bump function such that $\int_\mathbb{R} \phi_0(x)dx = 1$. Let $\hat{\psi} = \psi - \phi_0 \int_\mathbb{R} \psi(x)dx$. We see that

$$\int_\mathbb{R} \hat{\psi}(x)dx = \int_\mathbb{R} \psi(x)dx - \int_R \psi(x)dx \cdot \int_\mathbb{R} \phi_0(x)dx = \int_\mathbb{R} \psi(x)dx - \int_\mathbb{R} \psi(x)dx = 0.$$

So, we can let

$$\phi(x) = \int_{-\infty}^x \hat{\psi}(x)dx.$$

We see $\hat{\psi}$ has compact support and is in $C_c^\infty(\mathbb{R})$ (since it is the sum of two compact support functions). Since $\int_\mathbb{R} \hat{\psi}(x)dx = 0$, we know ϕ must have compact support as well and be in $C_c^\infty(\mathbb{R})$. Now, let $c = u(\phi_0)$. We see that by linearity of u,

$$u(\psi) = u(\hat{\psi}) + u(\phi_0) \cdot \int_\mathbb{R} \psi(x)dx = u(\phi') + c\int_\mathbb{R} \psi(x)dx = 0 + c\int_\mathbb{R} \psi(x)dx = c\int_\mathbb{R} \psi(x)dx,$$

which is exactly what we wanted to prove. □

Problem 2.TODO ◁

Solution. First, let us define $f(x-ct)$ in a way that aligns with the case that f is a nice function. We see that if f were nice, then

$$\begin{aligned} f(x-ct)(\phi) &= \int_{\mathbb{R}^2} f(x-ct)\phi(t,x)dxdt \\ &= \int_{\mathbb{R}^2} f(z)\phi(s,z+cs)dzds \\ &= \int_{\mathbb{R}^2} f(z)\phi(s,z+cs)dsdz \\ &= \int_{z\in\mathbb{R}} f(z)\int_{s\in\mathbb{R}} \phi(s,z+cs)dsdz \\ &= f\left(\int_{t\in\mathbb{R}} \phi(t,z+ct)dt\right). \end{aligned}$$

So, we see that $f(x-ct)(\phi) = f(\Phi)$ where $\Phi = \int_{\mathbb{R}} \phi(s,z+cs)ds$. Note that $\Phi \in C_c^\infty(\mathbb{R})$ because the integral of a smooth function is smooth and ϕ is compactly supported. So, we can define $u = f(x-ct)$.

Now, we must show that u satisfies the PDE. This is done by simply writing out our definition and using the linearity of f. More precisely,

$$\begin{aligned} (u_t + cu_x)(\phi) &= -f\left(\int_s \phi_t(s,z+cs)ds\right) - cf\left(\int_s \phi_x(s,z+cs)ds\right) \\ &= -f\left(\int_s \left[\phi_t(s,z+cs) + s\phi_x(s,z+cs)\right] ds\right) \\ &= -f\left(\int_s \partial_s\phi(s,z+cs)ds)\right) \\ &= -f(0) \\ &= 0. \end{aligned}$$

Note that we used the fundamental theorem of calculus, that ϕ has compact support, and that f is linear in the above computation.

Thus, $u = f(x-ct)$ by definition and u solves the PDE in the sense of distribution, as we wanted to show. □

Problem 3. TODO ◁

Solution.

1. This part follows directly from what we have done in class. We can factor

$$u_{xx} - u_{xt} - 6u_{tt} = (\partial_x - 3\partial_t)(\partial_t + 2\partial_t)u,$$

so the general C^2 solution is

$$u(x,t) = g(2x-t) + f(3x+t)$$

for some C^2 functions f, g. □

2. This problem is similar to problem 2, but here we should apply a trick instead of mindlessly taking derivatives. We will split up $(\partial_x - 3\partial_t)(\partial_t + 2\partial_t)$ and apply them to the parts of u in different orders. More precisely, we see that

$$\begin{aligned}(\partial_x - 3\partial_t)(\partial_t + 2\partial_t)u &= (\partial_x - 3\partial_t)(\partial_t + 2\partial_t)(g(2x-t) + f(3x+t)) \\ &= (\partial_t + 2\partial_t)(\partial_x - 3\partial_t)f(3x+t) + (\partial_x - 3\partial_t)(\partial_t + 2\partial_t)g(2x-t).\end{aligned}$$

By a similar derivation to that in problem 2, we can define

$$f(3x+t)(\phi) = f\left(\frac{1}{3}\int_s \phi(s,(z-s)/3)ds\right)$$

and

$$g(2x-t)(\phi) = g\left(\frac{1}{2}\int_s \phi(s,(z+s)/2)ds\right)$$

Now, we see that

$$\begin{aligned}(\partial_t + 2\partial_t)(\partial_x - 3\partial_t)f(3x+t)(\phi) &= (\partial_t + 2\partial_t)(\partial_x - 3\partial_t)f\left(\frac{1}{3}\int_s \phi(s,(z-s)/3)ds\right) \\ &= -(\partial_t + 2\partial_t)f\left(\int_s \left[\frac{1}{3}\phi_x(s,(z-s)/3) - \phi_t(s,(z-s)/3)\right]ds\right) \\ &= -(\partial_t + 2\partial_t)f\left(\int_s \partial_s\phi(s,(z-s)/3)ds\right) \\ &= -(\partial_t + 2\partial_t)f(0) \\ &= -(\partial_t + 2\partial_t)0 \\ &= 0,\end{aligned}$$

where we used the fundamental theorem of calculus along with the fact that ϕ must have compact support.

Similarly,

$$\begin{aligned}(\partial_x - 3\partial_t)(\partial_t + 2\partial_t)g(2x-t)(\phi) &= (\partial_x - 3\partial_t)(\partial_t + 2\partial_t)g\left(\frac{1}{2}\int_s \phi(s,(z+s)/2)ds\right) \\ &= -(\partial_t - 3\partial_t)g\left(\int_s \left[\frac{1}{2}\phi_x(s,(z+s)/2) + \phi_t(s,(z-s)/2)\right]ds\right) \\ &= -(\partial_t - 3\partial_t)g\left(\int_s \partial_s\phi(s,(z+s)/2)ds\right) \\ &= -(\partial_t + 2\partial_t)f(0) \\ &= -(\partial_t + 2\partial_t)0 \\ &= 0,\end{aligned}$$

where, again, we used the fundamental theorem of calculus along with the fact that ϕ must have compact support.

From this, we can conclude that

$$u_{xx} - u_{xt} - 6u_{tt} = (\partial_t + 2\partial_t)(\partial_x - 3\partial_t)f(3x+t) + (\partial_x - 3\partial_t)(\partial_t + 2\partial_t)g(2x-t) = 0 + 0 = 0.$$

So $v + w$ does indeed solve the PDE in part (a). $\square$

Problem 4. TODO ◁

Solution. This problem can be solved using the standard method for second order hyperbolic PDEs of two variables with constant coefficients that we saw in class.

First, we can factor

$$u_{xx} + 3u_{xy} - 4u_{yy} = (\partial_x - \partial_y)(\partial_x + 4\partial_y)u.$$

So, as we know, the general solution has the form

$$u(t, x) = f(4x - y) + g(x + y).$$

Now, we just need to use the initial conditions to find the specific f and g. We see that

$$f(3x) + g(2x) = \sin x$$

and that

$$4f'(3x) + g'(2x) = 0.$$

Taking the derivative of the former gives us that

$$3f'(3x) + 2g'(2x) = \cos x.$$

Now, we have a system of linear equations, so we can solve to see that

$$3f'(3x) = -\frac{3}{5}\cos x \text{ and } g'(2x) = \frac{8}{5}\cos x.$$

Since $f(3x) + g(2x) = \sin x$, the constant during integration must be the same, so we can assume it to vanish and

$$f(x) = -\frac{3}{5}\sin(x/3) \text{ and } g(x) = \frac{8}{5}\sin(x/2).$$

Plugging this back in, we see that

$$u(x, t) = -\frac{3}{5}\sin\left(\frac{4x - y}{3}\right) + \frac{8}{5}\sin\left(\frac{x + y}{2}\right),$$

which is the solution we were after. □

Problem 5. TODO ◁

Solution. We assume $t \geq 0$, as always. First, lets' consider the case $c = 0$. Then, we have $u_{tt} = 0$, so $u_t(x,t)$ is constant along the line $\{(t, x_0)\}$ for any x_0. So, $u(t,x)$ has is a line of constant slope along any $\{(t, x_0)\}$. The starting point and the slope are defined by the initial conditions, so we see that

$$u(t,x) = \phi(x) + t\psi(x).$$

This vanishes when $\phi(0) + t\psi = 0$. Since ϕ, ψ are positive, ϕ must be 0, so $|x| \geq 1$. Any point where $|x| \geq 1$ works.

We can also check that u is linear (and thus C^∞) everywhere except for the places where of ϕ and ψ are not C^∞. In other words u is C^∞ when $x \neq -1, 0, 1$.

Now, let's look at the nonzero case. We know from the notes that the solution is

$$u(t,x) = \frac{1}{2}(\phi(x+ct) + \phi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\sigma) d\sigma.$$

Notice that we can assume $c > 0$ because $c < 0$ would also flip the integral, leading to the same solution.

We see that ϕ, ψ are non-negative, so this vanishes only when $\phi(x-ct) = 0, \phi(x+c) = 0$, and $\psi = 0$ on the interval $(x - ct, x + ct)$. For the analysis of when u vanishes, we will only consider $c = 1$. Instead of writing out the cases, consider the following picture of where this is true:
TODO: image
We see that u is C^∞ everywhere but the discontinuities of ϕ, ψ. In other words, u is C^∞ when $x + ct \neq -1, 0$, or 1. □

Problem 6. TODO ◁

Solution. As the hint suggests, consider $v(x_0, x_n, t) = u(x_0, x_n, t) - u(x_0, -x_n, t)$. Notice that

$$v_{tt} - c^2 \Delta_x v = u_{tt}(x_0, x_n, t) - u_{tt}(x_0, -x_n, t) - c^2 \Delta_x u(x_0, x_n, t) + c^2 \Delta_x u(x_0, -x_n, t) = f(x_0, x_n, t) - f(x_0, -x_n, t) = 0.$$

We also see that

$$v(x, x_n, 0) = u(x_0, x_n, 0) - u(x_0, -x_n, 0) = \phi(x_0, x_n) - \phi(x_0, -x_n)$$

and

$$v_t(x, x_n, 0) = u_t(x_0, x_n, 0) - u_t(x_0, -x_n, 0) = \psi(x_0, x_n) - \psi(x_0, -x_n).$$

So, v solves the equation $v_{tt} - c^2 \Delta_x v = 0$ with 0 initial conditions. We know that the only solution to this is $v = 0$. Thus, we see that $u(x_0, x_n, t) - u(x_0, -x_n, t) = v(x_0, x_n, t) = 0$, so

$$u(x_0, x_n, t) = u(x_0, -x_n, t),$$

and u is even with respect to x_n, exactly as we wanted. □

Problem 7. TODO $\triangleleft$

Solution. TODO $\square$