MATH 173 Problem Set 4

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Problem 1. TODO ◁

Solution. For this problem, we'll follow the first itinerary suggested by the hint. We know

$$f(x) = \int_0^x f'(t)dt.$$

Let $\chi_{[0,x]}$ be the characteristic function of $[0, x]$. We now see that by the Cauchy-Schwartz inequality,

$$\begin{aligned}
\int_0^1 f(x)^2 dx &= \int_0^1 \left(\int_0^x f'(t)dt \right) dx \\
&= \int_0^1 \left(\int_0^1 \chi_{[0,x]} f'(t)dt \right)^2 dx \\
&\leq \int_0^1 \left(\int_0^1 \chi_{[0,x]}^2 dt \right) \left(\int_0^1 f'(t)^2 dt \right) dx \\
&= \left(\int_0^1 f'(t)^2 dt \right) \int_0^1 \left(\int_0^1 \chi_{[0,x]}^2 dt \right) dx \\
&= \left(\int_0^1 f'(t)^2 dt \right) \int_0^1 x dx \\
&= \frac{1}{2} \int_0^1 f'(t)^2 dt.
\end{aligned}$$

So, an absolute constant of $C = 1/2$ works. □

Problem 2. TODO $\triangleleft$

Solution.

(a) First, we see that for any $\varepsilon > 0$, there exists an R such that $|u(0, R)| \leq \varepsilon$. So,

$$\sup_{x \in \mathbb{R}^n, t \in [0,t]} u(t, x) \geq -\varepsilon$$

and thus

$$\sup_{x \in \mathbb{R}^n, t \in [0,t]} u(t, x) \geq 0.$$

Also,

$$\sup_{x \in \mathbb{R}^n, t \in [0,t]} u(t, x) \geq \sup_{x \in \mathbb{R}^n} u(0, x)$$

so

$$\sup_{x \in \mathbb{R}^n, t \in [0,t]} u(t, x) \geq \max\{0, \sup_{x \in \mathbb{R}^n} u(0, x)\}.$$

The other direction is more involved. Let

$$C = \sup_{x \in \mathbb{R}^n, t \in [0,t]} u(t, x).$$

If $C = 0$, then we are done so let $C > 0$. For any $\varepsilon > 0$ such that $\varepsilon < C$, let x_0, t_0 be such that $u(x_0, t_0) > C - \varepsilon$. We know there exists an R such that

$$\sup_{|x|>R, t \in [0,t]} u(t, x) < C - \varepsilon.$$

Let $R_0 > \max t_0, R$. We then see by the maximum principle on the hypercylinder with the R_0-bal as the base that u achieves its supremum in the cylinder either on the base $\{(0, x) : |x| \leq R_0\}$ or on the wall $\{(x, t) : t \in [0, T], |x| = R_0\}$. However we saw that the supremum on the wall is less than $C - \varepsilon < u(t_0, x_0)$, which is inside the cylinder. So, the supremum is attained on the base and there exits an x such that $u(0, x) \geq u(t_0, x_0) \geq C - \varepsilon$. Since ε was arbitrarily small, we have shown that RHS is at least C, so we can conclude that

$$\sup_{x \in \mathbb{R}^n, t \in [0,t]} u(t, x) = \max\{0, \sup_{x \in \mathbb{R}^n} u(0, x)\},$$

as we wanted. $\square$

(b) If u and u' are solutions that go to 0 at infinity uniformly, then consider $v = u - u'$. We see that $v(0, x) = 0$ and that $v \to 0$ at infinity uniformly as well, so by part (a), we know $v \leq 0$. By a similar reasoning $-v = u' - u \leq 0$ as well. So, $v = 0$ and thus $u = u'$. Thus, the solution in the given class of functions must be unique. $\square$

Problem 3. TODO $\triangleleft$

Solution.

(a) This problem is just computation. By the product of sines formula,

$$\sin(n\pi x)\sin(n\pi x) = \frac{1}{2}(\cos((n-m)\pi x) - \cos((n+m)\pi x)).$$

So, if $m \neq n$, then

$$\begin{aligned}\int_0^1 \sin n\pi x \sin m\pi x dx &= \frac{1}{2}\left[\frac{1}{(n-m)\pi}\sin((n-m)\pi x)\right]_0^1 + \frac{1}{2}\left[\frac{1}{(n+m)\pi}\sin((n+m)\pi x)\right]_0^1 \\ &= \frac{1}{2}(0+0) \\ &= 0.\end{aligned}$$

For the other case, if $m = n$, then

$$\begin{aligned}\int_0^1 \sin n\pi x \sin m\pi x dx &= \frac{1}{2}\int_0^1 (\cos((0)\pi x) - \cos((2n)\pi x))dx \\ &= \frac{1}{2}\left[\cos(0) - \frac{1}{2n\pi}\cos(2n\pi x)\right]_0^1 \\ &= \frac{1}{2}(1-0) \\ &= \frac{1}{2}.\end{aligned}$$

So, we have shown the equality we wanted to show. $\square$

(b) This problem can be solved by heavy computation and using part (a). Instead, we'll use what we have seen in class. Fix s and y. We know

$$u(t,x) = \int_0^1 K(t,x,r)K(s,r,y)dr$$

gives a solution for $u_t = u_{xx}$ with initial conditions $u(0,x) = K(s,x,y)$. We also know that the heat kernel satisfies the heat equation for $t > 0$. Note that this means

$$u'(t,x) = K(t+s,x,y)$$

satisfies $u'_t = u'_{yy}$. Also note that we can check that $u(0,x) = K(s,x,y)$, so the initial conditions are the same as above. Since the solution with the same initial conditions is unique, our two solutions must be the same, so

$$\int_0^1 K(t,x,r)K(s,r,y)dr = K(t+s,x,y),$$

as we wanted. $\square$

Problem 4. TODO $\triangleleft$

Solution. TODO

Problem 5. TODO $\triangleleft$

Solution. TODO

Problem 6. TODO $\triangleleft$

Solution. TODO

Problem 7. TODO $\triangleleft$

Solution. TODO