

# MATH 173 PROBLEM SET 7

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## Problem 1.

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**Solution.** This problem is straightforward. Since  $\overline{\Omega}$  is closed, and  $c > 0$ , there exists constants  $C_1, C_2 > 0$  such that  $C_1 < c(x) < C_2$ . Assume  $C_1 < 1$  and  $C_2 > 1$ . If not, we can always choose smaller  $C_1$  and larger  $C_2$ . So, for any  $u \in C^1(\overline{\Omega})$ ,

$$\begin{aligned} \|u\|_{H_c^1(\Omega)}^2 &= \int_{\Omega} |u(x)|^2 dx + \int_{\Omega} c(x) |\nabla u(x)|^2 dx \\ &\geq C_1^2 \int_{\Omega} |u(x)|^2 dx + \int_{\Omega} C_1^2 |\nabla u(x)|^2 dx \\ &= C_1^2 \|u\|_{H^1(\Omega)}^2. \end{aligned}$$

Similarly,

$$\begin{aligned} \|u\|_{H_c^1(\Omega)}^2 &= \int_{\Omega} |u(x)|^2 dx + \int_{\Omega} c(x) |\nabla u(x)|^2 dx \\ &\leq C_2^2 \int_{\Omega} |u(x)|^2 dx + \int_{\Omega} C_2^2 |\nabla u(x)|^2 dx \\ &= C_2^2 \|u\|_{H^1(\Omega)}^2. \end{aligned}$$

So, the statement follows for all  $u \in C^1(\overline{\Omega})$ . By continuity and density, it follows that the statement holds for  $u \in H^1(\Omega)$ .  $\square$

**Problem 2.**

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**Solution.**

- (a) Suppose  $v = u + w = u' + w'$  where  $u, u' \in M$ ,  $w, w' \in M^\perp$ . Then, we see

$$u' - u = u' + w - v = w - w'.$$

But  $u' - u \in M$  and  $w - w' \in M^\perp$  and  $M \cap M^\perp = \{0\}$ . So,  $u' - u = w - w' = 0$ . Thus, the decomposition is unique.

We see that  $u = u + 0$ . By uniqueness,  $P(v) = u = P(u) = P(P(v))$ , so  $P = P^2$ .  $\square$

- (b) Let  $v = u + w$  and  $v' = u' + w'$  with  $u, u' \in M$ ,  $w, w' \in M^\perp$ . We see

$$\langle Pv, v' \rangle = \langle u, u' + w' \rangle = \langle u, u' \rangle + \langle u, w' \rangle = \langle u, u' \rangle = \langle u, u' \rangle + \langle w, u' \rangle = \langle u + w, u' \rangle = \langle v, Pv' \rangle,$$

So  $P = P^*$  by definition.  $\square$

- (c) Since  $T$  is linear,  $T(H)$  is a subspace. For any sequence  $v_j$  in  $T(H)$  where  $v_j \rightarrow v$  in  $H$ , we know  $T(v_j) = v_j$  because  $T^2 = T$ . So, by continuity of  $T$ , we have  $T(v) = v$ , so  $v \in T(H)$  and we can conclude  $T(H)$  is closed.

Let  $u = T(v)$  and let  $w = v - u$ . Note  $u \in T(H)$ . For any  $y = T(x) \in T(H)$ , we have

$$\langle y, w \rangle = \langle T(x), v - u \rangle = \langle x, T(v - u) \rangle = \langle x, T(v) - T(u) \rangle = \langle x, u - u \rangle = 0$$

because  $T = T^2 = T^*$ . So,  $w \in T(H)^\perp$  and thus  $T$  is the projection.  $\square$

**Problem 3.**

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**Solution.** In this problem, we will use  $H^1$  to denote  $H^1((0, 1))$  to reduce clutter.

- (a) Consider the operator  $T : H^1 \rightarrow H^1$  where  $Tf = f - \int_0^1 f$ . It's easy to see that  $T$  is linear since integration is linear. We see that for any  $f$ ,

$$\begin{aligned} \|Tf\|_{H^1} &= \int_0^1 \left| f - \int_0^1 f \right|^2 + \int_0^1 |\nabla f|^2 \\ &\leq \int_0^1 \left| f - \int_0^1 f \right|^2 + \int_0^1 |\nabla f|^2 \\ &\leq \int_0^1 \left( 2|f|^2 + 2 \left| \int_0^1 f \right|^2 \right) + \int_0^1 |\nabla f|^2 \\ &= 4 \int_0^1 |f|^2 + \int_0^1 |\nabla f|^2 \\ &\leq 4 \int_0^1 |f|^2 + 4 \int_0^1 |\nabla f|^2 \\ &= 4 \|f\|_{H^1}^2. \end{aligned}$$

So,  $T$  is bounded. It's easy to see that  $T(f) \in M$  by definition, and if  $f \in M$ , then  $T(f) = f$ . So,  $T = T^2$  and  $T(H^1) = M$ . Also,

$$\begin{aligned} \langle Tf, g \rangle &= \langle f - \int_0^1 f, g \rangle \\ &= \int_0^1 (f\bar{g} + \nabla f \overline{\nabla g}) - \int_0^1 \left( \int_0^1 f \right) \bar{g} \\ &= \int_0^1 (f\bar{g} + \nabla f \overline{\nabla g}) - \left( \int_0^1 \bar{g} \right) \left( \int_0^1 f \right) \\ &= \int_0^1 (f\bar{g} + \nabla f \overline{\nabla g}) - \int_0^1 \left( \int_0^1 \bar{g} \right) f \\ &= \langle f, g - \int_0^1 g \rangle \\ &= \langle f, Tg \rangle. \end{aligned}$$

So,  $T = T^*$ . Now, by problem 2c,  $M = T(H)$  is closed.

Since  $T$  is the projection onto  $M$ , we know  $M^\perp = \ker T$ . That is  $f$  such that  $f - \int_0^1 f = 0$ . Thus,  $f$  must be constant. We see that any constant function is in the kernel, so  $M^\perp$  is the constant functions.  $\square$

- (b) First, let us derive a useful fact. Let  $g$  be a function,  $h, w$  be distributions. We know from homework that  $w' = 0$  as distributions implies  $w = c$  as functions. Now, we know  $\left( \int_0^x \right)' = g$  by FTC, and the same is true as distributions. So, if  $h' = g$  as distributions, then  $\left( h - \int_0^x g \right)' = 0$  as distributions. Plugging in for  $w$ , we see  $h = \int_0^x g + C$ . We will use this fact.

Let  $g \in N^\perp$ . We see that for any  $f \in N$ ,

$$\int g' \bar{f}' + \int g \bar{f} = \langle g, f \rangle = 0,$$

so with  $g$  as a distribution,

$$g(\bar{f}) = \int g \bar{f} = - \int g' \bar{f}' = -g'(\bar{f}') = g''(\bar{f}).$$

Thus,  $g = g''$  as distributions. By our useful fact,

$$g' = \int_0^x g + C$$

as functions, so  $g'$  is continuous. Also  $g''' = g'$  as distributions, so by our useful fact,

$$g'' = \int_0^x g' + C$$

as continuous functions. Thus,  $g = g''$  as functions.

We see that for any  $g$  such that  $g = g''$ , we have for any  $f \in N$ , integration by parts gives

$$\langle g, f \rangle = \int g \bar{f} + \int g' \bar{f}' = \int g \bar{f} - \int g'' \bar{f} = 0.$$

So,  $g \in N^\perp$ . Thus,  $N^\perp$  is all of the functions such that  $g'' = g$ . This is functions of the form

$$ae^x + be^{-x}.$$

We could have also done this by defining an operator and using problem 2c, but bounding that operator turned out to be painful.  $\square$

**Problem 4.**

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**Solution.** Let

$$J^*g = \int_0^x \int_0^t g(s) ds dt - x \int_0^1 \int_0^t g(s) ds dt.$$

It's easy to see that  $J^*$  is linear and thi image is indeed in  $H_0^1$  because it is differentiable and is 0 on the endpoints. Now, we show it is bounded. Note that

$$(J^*g)' = \int_0^x g(s) ds - \int_0^1 \int_0^t g(s) ds dt.$$

We see by Cauchy-Schwartz

$$\begin{aligned} \int_0^1 \left| \int_0^x g(s) ds \right|^2 dx &\leq \int_0^1 \int_0^x |g(s)|^2 ds dx \\ &\leq \int_0^1 \int_0^1 |g(s)|^2 ds dx \\ &= \int_0^1 |g(s)|^2 ds \\ &= \|g\|_{L^2}^2. \end{aligned}$$

Similarly,

$$\begin{aligned} \int_0^1 \left| \int_0^1 \int_0^t g(s) ds \right|^2 dx &\leq \int_0^1 \int_0^1 \int_0^t |g(s)|^2 ds dx \\ &\leq \int_0^1 \int_0^1 \int_0^1 |g(s)|^2 ds dx \\ &= \int_0^1 |g(s)|^2 ds \\ &= \|g\|_{L^2}^2. \end{aligned}$$

So, by the triangle inequality on the  $L^2$  norm,

$$\|J^*g\|_{H_0^1} = \|(J^*g)'\|_{L^2} \leq 2\|g\|_{L^2}.$$

Thus,  $J^*$  is bounded. The last thing to check is that  $J^*$  is actually the adjoint. For that, we importantly note that  $(J^*g)'' = -g$ . Integrating by parts,

$$\begin{aligned} \langle Jf, g \rangle_{L^2} &= \langle f, g \rangle_{L^2} \\ &= \int f \bar{g} \\ &= - \int f \overline{(J^*g)''} \\ &= \int f' \overline{(J^*g)'} \\ &= \langle f, J^*g \rangle_{H_0^1}. \end{aligned}$$

So,  $J^*$  is the adjoint. □

**Problem 5.**

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**Solution.**

- (a) This problem is a computation. Since  $F_\varepsilon$  is rotationally symmetric, we can use the given fact to see that

$$\Delta F_\varepsilon = h''(|x|) + (n-2)|x|^{-1}h'(x)$$

where

$$h(y) = c_n(y^2 + \varepsilon)^{(2-n)/2}.$$

We compute that

$$h'(y) = yc_n(2-n)(y^2 + \varepsilon^2)^{-n/2}.$$

Also,

$$h''(y) = c_n(2-n)(y^2 + \varepsilon^2)^{-n/2} - y^2 c_n(2-n)n(y^2 + \varepsilon^2)^{-(n+2)/2}.$$

So, plugging this in, we have

$$\begin{aligned} \Delta F_\varepsilon &= c_n(2-n)(|x|^2 + \varepsilon^2)^{-n/2} - |x|^2 c_n(2-n)n(|x|^2 + \varepsilon^2)^{-(n+2)/2} - (n-1)(2-n)c_n(|x|^2 + \varepsilon^2)^{-n/2} \\ &= \varepsilon^2 c_n(2-n)(|x|^2 + \varepsilon^2)^{-(n+2)/2} \\ &= -\varepsilon^{-n} c_n(n-2)(|x/\varepsilon|^2 + 1)^{-(n+2)/2} \\ &= \varepsilon^{-n} g(x/\varepsilon), \end{aligned}$$

as we wanted. □

- (b) Since  $F_\varepsilon$  is bounded by  $F$ , for any test function  $\phi$ , the dominated convergence theorem gives us

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} |(F - F_\varepsilon)(\phi)| &= \lim_{\varepsilon \rightarrow 0} \left| \int c_n \left( |x|^{2-n} - (|x|^2 + \varepsilon^2)^{(2-n)/2} \right) \phi(x) dx \right| \\ &= \left| \int c_n \left( \lim_{\varepsilon \rightarrow 0} \left( |x|^{2-n} - (|x|^2 + \varepsilon^2)^{(2-n)/2} \right) \right) \phi(x) dx \right| \\ &= \left| \int c_n(0) \phi(x) dx \right| \\ &= 0. \end{aligned}$$

So,  $F_\varepsilon \rightarrow F$  in the sense of distributions indeed.

Now we see that from part (a) and page 59 of chapter 5,

$$\begin{aligned} \Delta F(\phi) &= F(\Delta \phi) \\ &= \lim_{\varepsilon \rightarrow 0} F_\varepsilon(\Delta \phi) \\ &= \lim_{\varepsilon \rightarrow 0} \int \varepsilon^{-n} g(x/\varepsilon) \phi(x) dx \\ &= \delta(\phi) \\ &= \phi(0). \end{aligned}$$

□

**Problem 6.**

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**Solution.**

- (a) From problem 5(b), and what we did in class, we see since  $F$  is compactly supported,

$$\Delta u = \Delta F * f = \delta * f = f.$$

□

- (b) We have seen that the Fourier transform of  $\delta$  is 1, but let us confirm this. We see that for any test function  $\phi$ ,

$$\begin{aligned}\mathcal{F}(\delta)(\phi) &= \delta(\mathcal{F}(\phi)) \\ &= \delta\left(\int \phi(x)e^{-ix \cdot y} dx\right) \\ &= \int \phi(x) dx.\end{aligned}$$

So, by definition,  $\mathcal{F}(\delta) = 1$ .

Now, note that using our Fourier transform of derivative rules, we have

$$\mathcal{F}(\Delta F) = \sum_{j=1}^n (-iy_j)^2 \mathcal{F}(F) = -|y|^2 \mathcal{F}(F).$$

Since  $\mathcal{F}(\Delta F) = \mathcal{F}(\delta) = 1$ , we can say

$$\mathcal{F}(F) = -\frac{1}{|y|^2}.$$

□