## MATH 173 PROBLEM SET 9

Stepan (Styopa) Zharkov June 1, 2022

Problem 1.  $\triangleleft$  Solution.

Problem 2.

## Solution.

(a) We need  $\int_0^1 |x^{\alpha}|^2 = \int_0^1 x^{2\alpha}$  to converge. This converges for  $\alpha > -1/2$  and diverges for  $\alpha \leq -1/2$ , so  $\phi_{\alpha} \in L^2((0,1))$  for  $\alpha > -1/2$ .

(b) We need  $\phi_{\alpha} \in L^{2}((0,1))$ , so  $\alpha > -1/2$ . But, since  $\phi_{\alpha}$  are smooth, we also need  $\int_{0}^{1} |\phi_{\alpha}'|^{2}$  to converge. We see  $\phi_{\alpha}' = \alpha x^{\alpha-1}$ . and  $\int_{0}^{1} |\alpha x^{\alpha-1}|^{2} = |\alpha|^{2} \int_{0}^{1} x^{2(\alpha-1)}$  converges for  $\alpha > 1/2$  and diverges for  $\alpha \leq 1/2$ . So,  $\phi_{\alpha} \in H^{1}((0,1))$  for  $\alpha > 1/2$ .

Problem 3.

Solution.

(a) We know the statement is true for  $f \in C^1((a,b))$  by FTC. Now, let  $f_n \to f$  where  $f_n \in C^1((a,b))$ . By the continuity of the trace operator,

$$f(x) - f(y) = \lim_{n \to \infty} (f_n(x) - f_n(y)) = \lim_{n \to \infty} \int_x^y f'_n(t)dt$$

TODO: finish this.

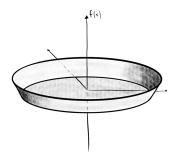
(b) TODO

Problem 4.  $\triangleleft$  Solution.

Problem 5.

**Solution.** Consider the dogbowl functions

$$f_n := 1 - \min(n \cdot d(x, \partial B), 1).$$



We see that  $Tf_n = 1$  for all n4, so  $Tf_n \to 1 \neq 0$ . However,

$$||f_n||_{L_2}^2 = \int_B |f_n(x)|^2 dx = \int_{d(x,\partial B)<1/n} |f_n(x)|^2 dx \le \int_{d(x,\partial B)<1/n} 1 dx = O(1/n) \to 0.$$

So, 
$$f_n \to 0$$
 in  $L^2$ .

Problem 6.

**Solution.** Let  $u = \lim_{n \to \infty} u_n$  where  $u_n$  are compactly supported continuous functions. Note that we are given that  $u = \lim_{n \to \infty} -u_n(x^*)$ . This means that

$$u = \frac{\lim_{n \to \infty} u_n(x) + \lim_{n \to \infty} - u_n(x^*)}{2} = \lim_{n \to \infty} \frac{u_n(x) - u_n(x^*)}{2}.$$

Note that  $\frac{u_n(x)-u_n(x^*)}{2}=0$  when  $x_n=0$ , so

$$T_{B_+}\left(\frac{u_n(x)-u_n(x^*)}{2}\right) = \frac{u_n(x)-u_n(x^*)}{2}|_{\partial B_+} = 0.$$

We have shown in class that this is sufficient to say  $T_{B_+}(u|_{B_+})=0$ , so  $u|_{B_+}\in H^1_0(B_+)$ .

Problem 7.

## Solution.

(a) Let  $V_k = \{x: |x| < 1/k\}$  and let  $W_k = \{x: 1-1/k < |x| < 1\}$ . Then, consider lemonsqueezer functions  $f_k \in$  such that  $f \in C^1_0(U)$  and  $f_k|_{B(V_k \cap W_k)} = 1$  and  $f_k|_{V_{2k} \cup W_{2k}} = 0$ .

TODO: image

Note that  $u_k := uf_k \in C_0^1(U)$ . We claim  $u_k \to u$  in  $H^1(B)$ . Note that

$$||u - u_k||_{H^1}^2 = \int_B |u - u_k|^2 + \int_B |\nabla u - \nabla u_k|^2.$$

Now, since u is bounded

$$\int_{B} |u - u_{k}|^{2} = \int_{B} |u|^{2} |1 - f_{k}|^{2}$$

$$= \int_{B} O(1) |1 - f_{k}|^{2}$$

$$= O(1) \int_{B} |1 - f_{k}|^{2}$$

$$= O(1) \int_{V_{k} \cup W_{k}} |1 - f_{k}|^{2}$$

$$= O(1) \int_{V_{k} \cup W_{k}} O(1)$$

$$= O(1/k^{2}) = o(1)$$

where the asymptotic notation is with respect to k. Also,

$$\int_{B} |\nabla u - \nabla u_{k}|^{2} = \int_{B} |\nabla u - \nabla u f_{k} - u \nabla f_{k}|^{2} \le 2 \int_{B} |\nabla u|^{2} |1 - f_{k}|^{2} + 2 \int_{B} |u \nabla f_{k}|^{2}.$$

Note that since  $|\nabla u|$  is bounded, applying our above logic to the first part gives us

$$2\int_{B} |\nabla u|^{2} |1 - f_{k}|^{2} = O(1) \int_{B} |1 - f_{k}|^{2} = o(1).$$

- (b) TODO
- (c) TODO