# MATH 173 Problem Set 7

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**Problem 1.** $\triangleleft$

***Solution.*** This problem is straightforward. Since $\overline{\Omega}$ is closed, and $c > 0$, there exists constants $C_1, C_2 > 0$ such that $C_1 < c(x) < C_2$. Assume $C_1 < 1$ and $C_2 > 1$. If not, we can always choose smaller $C_1$ and larger $C_2$. So, for any $u \in C^1(\overline{\Omega})$,

$$\begin{aligned}
||u||^2_{H^1_c(\Omega)} &= \int_\Omega |u(x)|^2 dx + \int_\Omega c(x)|\nabla u(x)|^2 dx \\
&\geq C_1^2 \int_\Omega |u(x)|^2 dx + \int_\Omega C_1^2 |\nabla u(x)|^2 dx \\
&= C_1^2 ||u||^2_{H^1(\Omega)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
||u||^2_{H^1_c(\Omega)} &= \int_\Omega |u(x)|^2 dx + \int_\Omega c(x)|\nabla u(x)|^2 dx \\
&\leq C_2^2 \int_\Omega |u(x)|^2 dx + \int_\Omega C_2^2 |\nabla u(x)|^2 dx \\
&= C_2^2 ||u||^2_{H^1(\Omega)}.
\end{aligned}$$

So, the statement follows for all $u \in C^1(\overline{\Omega})$. By continuity and density, it follows that the statement holds for $u \in H^1(\Omega)$. $\square$

**Problem 2.** $\triangleleft$

***Solution.***

(a) Suppose $v = u + w = u' + w'$ where $u, u' \in M$, $w, w' \in M^\perp$. Then, we see

$$u' - u = u' + w - v = w - w'.$$

But $u' - u \in M$ and $w - w' \in M^\perp$ and $M \cap M^\perp = \{0\}$. So, $u' - u = w - w' = 0$. Thus, the decomposition is unique.

We see that $u = u + 0$. By uniqueness, $P(v) = u = P(u) = P(P(v))$, so $P = P^2$. $\square$

(b) Let $v = u + w$ and $v' = u' + w'$ with $u, u' \in M$, $w, w' \in M^\perp$. We see

$$\langle Pv, v' \rangle = \langle u, l' + w' \rangle = \langle u, u' \rangle + \langle u, w' \rangle = \langle u, u' \rangle = \langle u, u' \rangle + \langle w, u' \rangle = \langle u + w, u' \rangle = \langle v, Pv' \rangle,$$

So $P = P^*$ by definition. $\square$

(c) Since $T$ is linear, $T(H)$ is a subspace. For any sequence $v_j$ in $T(H)$ where $v_j \to v$ in $H$, we know $T(v_j) = v_j$ because $T^2 = T$. So, by continuity of $T$, we have $T(v) = v$, so $v \in T(H)$ and we can conclude $T(H)$ is closed.

Let $u = T(v)$ and let $w = v - u$. Note $u \in T(H)$. For any $y = T(x) \in T(H)$, we have

$$\langle y, w \rangle = \langle T(x), v - u \rangle = \langle x, T(v - u) \rangle = \langle x, T(v) - T(u) \rangle = \langle x, u - u \rangle = 0$$

because $T = T^2 = T^*$. So, $w \in T(H)^\perp$ and thus $T$ is the projection. $\square$

**Problem 3.** ◁

***Solution.*** In this problem, we will use $H^1$ to denote $H^1((0,1))$ to reduce clutter.

(a) Consider the operator $T : H^1 \to H^1$ where $Tf = f - \int_0^1 f$. It's easy to see that $T$ is linear since integration is linear. We see that for any $f \in C^1((0,1))$,

$$\begin{aligned}
||Tf||_{H^1} &= \int_0^1 \left| f - \int_0^1 f \right|^2 + \int_0^1 |\nabla f|^2 \\
&\leq \int_0^1 \left| |f| - \left| \int_0^1 f \right| \right|^2 + \int_0^1 |\nabla f|^2 \\
&\leq \int_0^1 \left( 2|f|^2 - \left| \int_0^1 f \right|^2 \right) + \int_0^1 |\nabla f|^2 \\
&= 4 \int_0^1 |f|^2 + \int_0^1 |\nabla f|^2 \\
&= 4 \int_0^1 |f|^2 + 4 \int_0^1 |\nabla f|^2 \\
&= ||f||_{H^1}.
\end{aligned}$$

By density, the argument extends to $f \in H^1((0,1))$

(b) TODO

**Problem 4.** $\triangleleft$

***Solution.*** TODO

**Problem 5.** $\triangleleft$

***Solution.*** TODO

**Problem 6.** $\triangleleft$

***Solution.*** TODO