

# MATH 173 PROBLEM SET 9

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## Problem 1.

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**Solution.** Recall that we defined  $G(x, y) = E(x - y) - \phi^x(y)$  where  $\Delta\phi^x = 0$  and  $\phi^x|_{\partial\Omega} = E(x - \cdot)$ . So, if  $Q(x, y) := G(x, y) - E(x - y) = -\phi^x(y)$ , then  $Q|_{\partial(\Omega \times \Omega)} = -E(x - y)$  and  $\Delta Q = 0$ . So, by the maximum principle, the minimum is achieved on the boundary. But the boundary is nonnegative, so  $Q > 0$ , so  $G(x, y) \geq E(x - y)$ .  $\square$

**Problem 2.**

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***Solution.***

- (a) We need  $\int_0^1 |x^\alpha|^2 = \int_0^1 x^{2\alpha}$  to converge. This converges for  $\alpha > -1/2$  and diverges for  $\alpha \leq -1/2$ , so  $\phi_\alpha \in L^2((0, 1))$  for  $\alpha > -1/2$ .  $\square$
- (b) We need  $\phi_\alpha \in L^2((0, 1))$ , so  $\alpha > -1/2$ . But, since  $\phi_\alpha$  are smooth, we also need  $\int_0^1 |\phi'_\alpha|^2$  to converge. We see  $\phi'_\alpha = \alpha x^{\alpha-1}$ . and  $\int_0^1 |\alpha x^{\alpha-1}|^2 = |\alpha|^2 \int_0^1 x^{2(\alpha-1)}$  converges for  $\alpha > 1/2$  and diverges for  $\alpha \leq 1/2$ . So,  $\phi_\alpha \in H^1((0, 1))$  for  $\alpha > 1/2$ .  $\square$

**Problem 3.**

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**Solution.**

- (a) We know the statement is true for  $f \in C^1((a, b))$  by FTC. Now, let  $f_n \rightarrow f$  where  $f_n \in C^1((a, b))$ . By the continuity of the trace operator,

$$f(y) - f(x) = \lim_{n \rightarrow \infty} (f_n(y) - f_n(x)) = \lim_{n \rightarrow \infty} \int_x^y f'_n(t) dt$$

Now, since we are on a bounded interval, we can move the limit inside the derivative after applying dominated convergence to see that

$$\lim_{n \rightarrow \infty} \int_x^y f'_n(t) dt - \int_x^y f'_n(t) dt = \lim_{n \rightarrow \infty} \int_x^y (f_n(t) - f(t))' dt = 0.$$

So,

$$f(y) - f(x) = \int_x^y f'(t) dt$$

as we wanted. □

- (b) By Cauchy-Schwartz and part (a) and that  $|f'|$  is bounded by  $C$ ,

$$\begin{aligned} |f(y) - f(x)|^2 &= \left| \int_x^y f'(t) dt \right|^2 \\ &\leq \int_x^y |f'(t)|^2 dt \\ &\leq \int_x^y C^2 dt \\ &= |y - x| C^2. \end{aligned}$$

So,  $|f(x) - f(y)| \leq C|x - y|^{1/2}$ . □

**Problem 4.**

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**Solution.** First, note that for  $f \in C^1((a, b))$  where  $f(a) = f(b) = 0$ , and any  $x \in (a, b)$ , we have by Cauchy Schwartz,

$$\begin{aligned}
 2\|f\|_2\|f'\|_2 &= 2 \left( \int_a^b |f(t)|^2 \right)^{1/2} \left( \int_a^b |f'(t)|^2 \right)^{1/2} \\
 &\geq 2 \int_a^b |f(t)f'(t)| dt \\
 &\geq 2 \int_a^x |f(t)f'(t)| dt \\
 &\geq \left| \int_a^x 2f(t)f'(t) dt \right| \\
 &= \left| \int_a^x (f(t)^2)' dt \right| \\
 &= |f(x)^2 - f(a)^2| \\
 &= |f(x)^2|.
 \end{aligned}$$

Since  $x$  was arbitrary,

$$\sup_{(a,b)} |f(x)|^2 \leq 2\|f\|_2\|f'\|_2.$$

Now, problem 3(b) and the fact that  $H_0^1((a, b)) \subseteq H^1((a, b)) \subseteq C([a, b])$  gives us the same inclusion argument as in chapter 9.1, leading us to conclude the statement for all  $f \in H^1((a, b))$ .  $\square$

**Problem 5.**

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**Solution.** Consider the dogbowl functions

$$f_n := 1 - \min(n \cdot d(x, \partial B), 1).$$



We see that  $Tf_n = 1$  for all  $n$ , so  $Tf_n \rightarrow 1 \neq 0$ . However,

$$\|f_n\|_{L_2}^2 = \int_B |f_n(x)|^2 dx = \int_{d(x, \partial B) < 1/n} |f_n(x)|^2 dx \leq \int_{d(x, \partial B) < 1/n} 1 dx = O(1/n) \rightarrow 0.$$

So,  $f_n \rightarrow 0$  in  $L^2$ .

□

**Problem 6.**

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**Solution.** Let  $u = \lim_{n \rightarrow \infty} u_n$  where  $u_n$  are compactly supported continuous functions. Note that we are given that  $u = \lim_{n \rightarrow \infty} -u_n(x^*)$ . This means that

$$u = \frac{\lim_{n \rightarrow \infty} u_n(x) + \lim_{n \rightarrow \infty} -u_n(x^*)}{2} = \lim_{n \rightarrow \infty} \frac{u_n(x) - u_n(x^*)}{2}.$$

Note that  $\frac{u_n(x) - u_n(x^*)}{2} = 0$  when  $x_n = 0$ , so

$$T_{B_+} \left( \frac{u_n(x) - u_n(x^*)}{2} \right) = \frac{u_n(x) - u_n(x^*)}{2} |_{\partial B_+} = 0.$$

We have shown in class that this is sufficient to say  $T_{B_+}(u|_{B_+}) = 0$ , so  $u|_{B_+} \in H_0^1(B_+)$ . □

**Problem 7.**

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**Solution.**

- (a) Let  $V_k = \{x : |x| < 1/k\}$  and let  $W_k = \{x : 1 - 1/k < |x| < 1\}$ . Then, consider lemonsqueezer functions  $f_k \in C_0^1(U)$  such that  $f_k|_{B(V_k \cap W_k)} = 1$  and  $f_k|_{V_{2k} \cup W_{2k}} = 0$  with  $O(k)$  derivatives.



Note that  $u_k := uf_k \in C_0^1(U)$ . We claim  $u_k \rightarrow u$  in  $H^1(B)$ . Note that

$$\|u - u_k\|_{H^1}^2 = \int_B |u - u_k|^2 + \int_B |\nabla u - \nabla u_k|^2.$$

Now, since  $u$  is bounded

$$\begin{aligned} \int_B |u - u_k|^2 &= \int_B |u|^2 |1 - f_k|^2 \\ &= \int_B O(1) |1 - f_k|^2 \\ &= O(1) \int_B |1 - f_k|^2 \\ &= O(1) \int_{V_k \cup W_k} |1 - f_k|^2 \\ &= O(1) \int_{V_k \cup W_k} O(1) \\ &= O(1/k^n) = o(1) \end{aligned}$$

where the asymptotic notation is with respect to  $k$ . Also,

$$\int_B |\nabla u - \nabla u_k|^2 = \int_B |\nabla u - \nabla u f_k - u \nabla f_k|^2 \leq 2 \int_B |\nabla u|^2 |1 - f_k|^2 + 2 \int_B |u \nabla f_k|^2.$$

Note that since  $|\nabla u|$  is bounded, applying our above logic to the first part gives us

$$2 \int_B |\nabla u|^2 |1 - f_k|^2 = O(1) \int_B |1 - f_k|^2 = o(1).$$

So, we only need to deal with the second part. We see that since  $u$  is bounded and  $\nabla f_k$  is mostly

0,

$$\begin{aligned}
2 \int_B |u \nabla f_k|^2 &= 2 \int_B |u|^2 |\nabla f_k|^2 \\
&= O(1) \int_B |\nabla f_k|^2 \\
&= O(1) \int_{V_k \cup W_k} |\nabla f_k|^2 \\
&= O(1) \int_{V_k \cup W_k} |O(k)|^2 \\
&= O(1) O(1/k^n) O(k^2) \\
&= o(1)
\end{aligned}$$

for  $n > 2$ . Thus, combining everything, we see that

$$\|u - u_k\|_{H^1}^2 = o(1),$$

which is what we needed to show that  $H_0^1(B) = H_0^1(U)$ .  $\square$

- (b) Consider  $u(x) = 1 - x^2 \in C^1((-1, 1))$ . Note that  $T_{(-1, 1)}u = 0$ , so  $u \in H_0^1((-1, 1))$ . However,  $T_{(-1, 0) \cup (0, 1)}u \neq 0$  so  $u \notin H_0^1((-1, 0) \cup (0, 1))$ . Thus,

$$H_0^1((-1, 1)) \neq H_0^1((-1, 0) \cup (0, 1))$$

$\square$

- (c) Note that the logic in part (a) does not go through with  $n = 2$  because we get  $O(1)$  where we need  $o(1)$ . To fix this, it suffices to pick a different set of  $f_k$  and it turns out that  $H_0^1(B) = H_0^1(U)$  for two-dimensional balls. The problem is that our  $f_k$  were interesting (nonflat) on a quickly shrinking region, so the derivative was forced to be large.

So, instead we will make the  $f_k$  interesting on a nonshrinking region. Keeping it linear will not work because we need it to still approach 1 quickly. So, consider the following set of lemonsqueezer functions:

Let  $c$  be a constant less than  $1/\sqrt{2}$ . Let  $a_k$  be a quickly decreasing sequence. Now, let  $f_k$  be constant 0 inside the square of sidelength  $2a_k$  around the origin. Let  $f_k$  be constant 1 outside the square of sidelength  $2c$  but outside of  $W_k$ . Let the outer edge be smoothed out to 0, as in part (a). For the square in the middle, let each side follow the function  $1 - (1 - r/c)^k$  where  $r := \min(|x - a_k|, |y - a_k|)$ . We can smooth out the edges on arbitrarily small areas later. So, our picture looks something like this:

TODO: images

Note that all of the steps in part (a) go through the same for  $n = 2$  except for bounding

$$\int_B |u \nabla f_k|^2.$$

Since  $u$  approaches 0, on the edges, we only have to worry about what happens in the center.

$$\int_{x, y: r < c - a_k} |u \nabla f_k|^2.$$



We see that since  $u = O(1)$  and  $|\nabla f_k| = O(k)(1-r)^{k-1}$ , we can split the area into 8 triangles to see

$$\begin{aligned}
\int_{x,y:r < c-a_k} |u \nabla f_k|^2 &= O(1) \int_{x,y:r < c-a_k} |\nabla f_k|^2 \\
&= O(1) \int_{x,y:r < c-a_k} O(k)(1-r)^{k-1} \\
&= O(1) \int_0^c \int_0^x O(k^2)(1-r)^{2k-1} dy dx \\
&= O(k) \int_0^c \int_0^x (1-r)^{2k-2} dr dx \\
&= O(k) \int_0^c \int_0^x (1-r)^{\Theta(k)} dr dx \\
&= O(k) \int_0^c \int_0^x (1-r)^{\Theta(k)} dr dx \\
&= O(k) O(k^{-1}) \int_0^c (1-x)^{\Theta(k)} dx \\
&= O(k) O(k^{-1}) O(k^{-1}) \\
&= o(1).
\end{aligned}$$

Thus, combining this with the rest of the reasoning from part (a), we have that

$$\|u - u_k\|_{H^1}^2 = o(1),$$

completing the proof. □