MATH 173 Problem Set 6

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Problem 1. TODO ◁

Solution.

(a) This problem is straightforward.

$$
\begin{aligned}
\overline{\mathcal{F}(\phi)(y)} &= \overline{\int_{\mathbb{R}^n} e^{-ix\cdot y}\phi(x)dx} \\
&= \int_{\mathbb{R}^n} \overline{e^{-ix\cdot y}\phi(x)}dx \\
&= \int_{\mathbb{R}^n} e^{ix\cdot y}\overline{\phi(x)}dx \\
&= (2\pi)^n(2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot y}\overline{\phi(x)}dx \\
&= (2\pi)^n\mathcal{F}^{-1}\overline{\phi}(y)
\end{aligned}
$$

This is what we wanted to show. □

(b) We know the Fourier inversion formula holds for Schwartz functions. By part (a) and the equation about interchanging fourier transform under the integral that we saw in class, we know

$$
\begin{aligned}
(2\pi)^{-n}\int_{\mathbb{R}^n}\hat{\phi}\bar{\hat{\psi}} &= \int_{\mathbb{R}^n}\hat{\phi}\check{\bar{\psi}} \\
(2\pi)^{-n}\int_{\mathbb{R}^n}\hat{\phi}\bar{\hat{\psi}} &= \int_{\mathbb{R}^n}\check{\hat{\phi}}\hat{\check{\bar{\psi}}} \\
(2\pi)^{-n}\int_{\mathbb{R}^n}\hat{\phi}\bar{\hat{\psi}} &= \int_{\mathbb{R}^n}\phi\bar{\psi}.
\end{aligned}
$$

Setting $\psi = \phi$, we see that

$$
\int_{\mathbb{R}^n}\left|\hat{\phi}\right|^2 = (2\pi)^2\int_{\mathbb{R}^n}|\phi|^2,
$$

as we wanted. □

Problem 2. TODO ◁

Solution.

(a) Let $\tilde{u}(t,x)$ be defined as in the problem. On $(0,+\infty)\times(0,+\infty)$, $\tilde{u}$ is the same as u, so it satisfies the equation $\tilde{u}_t = \tilde{u}_{xx}$. On $(0,+\infty)\times(-\infty,0)$, we see that

$$\tilde{u}_t(t,x) = -u_t(t,-x) = -u_{xx}(t,-x) = \tilde{u}_{xx}(t,x).$$

So, $\tilde{u}_t = \tilde{u}_{xx}$ on all of $(0,+\infty)\times(0,+\infty)\cup(0,+\infty)\times(-\infty,0)$ and is C^2 there. Now, consider the points along $x=0$. We can define $\tilde{u}(t,0)=0$. We see that

$$\lim_{x\to 0} u(t,x) = u(t,0) = 0,$$

and thus $\lim_{x\to 0}\tilde{u}(t,x) = 0$ from both sides, and is equal to $\tilde{u}(t,0)$. So, $\tilde{u}$ is continuous in $[0,+\infty)\times\mathbb{R}$.

Now, let's consider differentiability. Let $t>0$. We see that

$$\begin{aligned}
\lim_{h\to^+0}\frac{\tilde{u}(t,h)-\tilde{u}(t,0)}{h} &= \lim_{h\to^+0}\frac{\tilde{u}(t,h)}{h}\\
&= \lim_{h\to^+0}\frac{u(t,h)}{h}\\
&= \lim_{h\to^-0}\frac{u(t,-h)}{-h}\\
&= \lim_{h\to^-0}\frac{\tilde{u}(t,h)}{h}.
\end{aligned}$$

Thus, the derivative from both sides matches up. We see that $\tilde{u}_t(t,0)=0$. Since u is continuously differentiable on the border, both components of the derivative are continuous, so u is differentiable on $(0,+\infty)\times\mathbb{R}$.

We can now assume that $\tilde{u} = K_t * \tilde{g}$. Writing this out, we have

$$\begin{aligned}
\tilde{u}(t,x) &= (K_t * \tilde{g})(x)\\
&= \int (4\pi t)^{1/2} e^{-\frac{|x-y|^2}{4t}}\tilde{g}(y)dy\\
&= \sqrt{4\pi t}\int_0^\infty e^{-\frac{|x-y|^2}{4t}}\tilde{g}(y)dy + \sqrt{4\pi t}\int_{-\infty}^0 e^{-\frac{|x-y|^2}{4t}}\tilde{g}(y)dy\\
&= \sqrt{4\pi t}\int_0^\infty e^{-\frac{(x-y)^2}{4t}} g(y)dy - \sqrt{4\pi t}\int_0^\infty e^{-\frac{(x+y)^2}{4t}} g(y)dy\\
&= \sqrt{4\pi t}\int_0^\infty g(y)\left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}}\right)dy.
\end{aligned}$$

Restricting to to half, we see that

$$u(t,x) = \sqrt{4\pi t}\int_0^\infty g(y)\left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}}\right)dy.$$

We can verify that the boundary conditions match. □

(b) This is very similar to part (a), but this time let us define

$$\tilde{u}(t,x) = \begin{cases} u(t,x) & \text{if } x \geq 0 \\ u(t,-x) & \text{if } x \geq 0 \end{cases}$$

to be the even extension.

We know $\tilde{u}$ is C^2 and satisfies the equation on $(0,+\infty) \times (0,+\infty) \cup (0,+\infty) \times (-\infty,0)$ for the same reason as in part (a).

We see that

$$\lim_{x \to 0} u(t,x) = u(t,0),$$

and thus $\lim_{x\to 0} \tilde{u}(t,x) = u(t,0)$ from both sides, and is equal to $\tilde{u}(t,0)$. So, $\tilde{u}$ is continuous in $[0,+\infty) \times \mathbb{R}$.

Now, we notice that

$$\begin{aligned} \lim_{h\to^+0} \frac{\tilde{u}(t,h) - \tilde{u}(t,0)}{h} &= \lim_{h\to^+0} \frac{u(t,h) - u(t,0)}{h} \\ &= 0 \\ &= \lim_{h\to^+0} -\frac{u(t,h) - u(t,0)}{h} \\ &= \lim_{h\to^-0} -\frac{u(t,-h) - u(t,0)}{h} \\ &= \lim_{h\to^-0} \frac{\tilde{u}(t,h) - \tilde{u}(t,0)}{h}. \end{aligned}$$

Thus, the derivative from both sides matches up. We see that $\tilde{u}_t(t,0) = 0$. Since u is continuously differentiable on the border, both components of the derivative are continuous, so u is differentiable on $(0,+\infty) \times \mathbb{R}$.

Problem 3. TODO $\triangleleft$

Solution. TODO

Problem 4. TODO $\triangleleft$

Solution. TODO

Problem 5. TODO $\triangleleft$

Solution. TODO

Problem 6. TODO $\triangleleft$

Solution. TODO

Problem 7. TODO ◁

Solution. TODO