MATH 173 Problem Set 3

Stepan (Styopa) Zharkov

April 20, 2022

Problem 1. TODO $\triangleleft$

Solution. As the hint suggests, $u' = 0$ means by definition $u(\phi) = 0$ for any $\phi \in C_c^\infty(\mathbb{R})$. For any $\psi \in C_c^\infty(\mathbb{R})$, let $\phi_0 \in C_c^\infty(\mathbb{R})$ be a bump function such that $\int_{\mathbb{R}} \phi_0(x)dx = 1$. Let $\hat{\psi} = \psi - \phi_0 \int_{\mathbb{R}} \psi(x)dx$. We see that

$$\int_{\mathbb{R}} \hat{\psi}(x)dx = \int_{\mathbb{R}} \psi(x)dx - \int_R \psi(x)dx \cdot \int_{\mathbb{R}} \phi_0(x)dx = \int_{\mathbb{R}} \psi(x)dx - \int_{\mathbb{R}} \psi(x)dx = 0.$$

So, we can let

$$\phi(x) = \int_0^x \hat{\psi}(x)dx.$$

We see $\hat{\psi}$ has compact support and is in $C_c^\infty(\mathbb{R})$ (since it is the sum of two compact support functions). Since $\int_{\mathbb{R}} \hat{\psi}(x)dx = 0$, we know ϕ must have compact support as well and be in $C_c^\infty(\mathbb{R})$. Now, let $c = u(\phi_0)$. We see that by linearity of u,

$$u(\psi) = u(\hat{\psi}) + u(\phi_0) \cdot \int_{\mathbb{R}} \psi(x)dx = u(\phi') + c\int_{\mathbb{R}} \psi(x)dx = 0 + c\int_{\mathbb{R}} \psi(x)dx = c\int_{\mathbb{R}} \psi(x)dx,$$

which is exactly what we wanted to prove. $\square$

Problem 2. TODO $\triangleleft$

Solution. TODO $\square$

Problem 3. TODO $\triangleleft$

Solution. TODO $\square$

Problem 4. TODO $\triangleleft$

Solution. TODO $\square$

Problem 5. TODO ◁

Solution. TODO □

Problem 6. TODO ◁

Solution. As the hint suggests, consider $v(x_0, x_n, t) = u(x_0, x_n, t) - u(x_0, -x_n, t)$. Notice that

$$v_{tt} - c^2 \Delta_x v = u_{tt}(x_0, x_n, t) - u_{tt}(x_0, -x_n, t) - c^2 \Delta_x u(x_0, x_n, t) + c^2 \Delta_x u(x_0, -x_n, t) = f(x_0, x_n, t) - f(x_0, -x_n, t) = 0.$$

We also see that

$$v(x, x_n, 0) = u(x_0, x_n, 0) - u(x_0, -x_n, 0) = \phi(x_0, x_n) - \phi(x_0, -x_n)$$

and

$$v_t(x, x_n, 0) = u_t(x_0, x_n, 0) - u_t(x_0, -x_n, 0) = \psi(x_0, x_n) - \psi(x_0, -x_n).$$

So, v solves the equation $v_{tt} - c^2 \Delta_x v = 0$ with 0 initial conditions. We know that the only solution to this is $v = 0$. Thus, we see that $u(x_0, x_n, t) - u(x_0, -x_n, t) = v(x_0, x_n, t) = 0$, so

$$u(x_0, x_n, t) = u(x_0, -x_n, t),$$

and u is even with respect to x_n, exactly as we wanted. □ □

Problem 7. TODO $\triangleleft$

Solution. TODO $\square$