

MATH 173 PROBLEM SET 9

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Problem 1.

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Solution. Recall that we defined $G(x, y) = E(x - y) - \phi^x(y)$ where $\Delta\phi^x = 0$ and $\phi^x|_{\partial\Omega} = E(x - \cdot)$. So, if $Q(x, y) := G(x, y) - E(x - y) = -\phi^x(y)$, then $Q|_{\partial(\Omega \times \Omega)} = -E(x - y)$ and $\Delta Q = 0$. So, by the maximum principle, the minimum is achieved on the boundary. But the boundary is nonnegative, so $Q > 0$, so $G(x, y) \geq E(x - y)$. \square

Problem 2.

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Solution.

- (a) We need $\int_0^1 |x^\alpha|^2 = \int_0^1 x^{2\alpha}$ to converge. This converges for $\alpha > -1/2$ and diverges for $\alpha \leq -1/2$, so $\phi_\alpha \in L^2((0, 1))$ for $\alpha > -1/2$. \square
- (b) We need $\phi_\alpha \in L^2((0, 1))$, so $\alpha > -1/2$. But, since ϕ_α are smooth, we also need $\int_0^1 |\phi'_\alpha|^2$ to converge. We see $\phi'_\alpha = \alpha x^{\alpha-1}$. and $\int_0^1 |\alpha x^{\alpha-1}|^2 = |\alpha|^2 \int_0^1 x^{2(\alpha-1)}$ converges for $\alpha > 1/2$ and diverges for $\alpha \leq 1/2$. So, $\phi_\alpha \in H^1((0, 1))$ for $\alpha > 1/2$. \square

Problem 3.

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Solution.

- (a) We know the statement is true for $f \in C^1((a, b))$ by FTC. Now, let $f_n \rightarrow f$ where $f_n \in C^1((a, b))$. By the continuity of the trace operator,

$$f(x) - f(y) = \lim_{n \rightarrow \infty} (f_n(x) - f_n(y)) = \lim_{n \rightarrow \infty} \int_x^y f'_n(t) dt$$

Now, since we are on a bounded interval, we can move the limit inside the derivative after applying dominated convergence to see that

$$\lim_{n \rightarrow \infty} \int_x^y f'_n(t) dt - \int_x^y f'(t) dt = \lim_{n \rightarrow \infty} \int_x^y (f'_n(t) - f'(t)) dt = 0.$$

So,

$$f(x) - f(y) = \int_x^y f'(t) dt$$

as we wanted.

□

- (b) TODO

Problem 4.

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Solution.

Problem 5.

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Solution. Consider the dogbowl functions

$$f_n := 1 - \min(n \cdot d(x, \partial B), 1).$$



We see that $Tf_n = 1$ for all n , so $Tf_n \rightarrow 1 \neq 0$. However,

$$\|f_n\|_{L^2}^2 = \int_B |f_n(x)|^2 dx = \int_{d(x, \partial B) < 1/n} |f_n(x)|^2 dx \leq \int_{d(x, \partial B) < 1/n} 1 dx = O(1/n) \rightarrow 0.$$

So, $f_n \rightarrow 0$ in L^2 .

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Problem 6.

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Solution. Let $u = \lim_{n \rightarrow \infty} u_n$ where u_n are compactly supported continuous functions. Note that we are given that $u = \lim_{n \rightarrow \infty} -u_n(x^*)$. This means that

$$u = \frac{\lim_{n \rightarrow \infty} u_n(x) + \lim_{n \rightarrow \infty} -u_n(x^*)}{2} = \lim_{n \rightarrow \infty} \frac{u_n(x) - u_n(x^*)}{2}.$$

Note that $\frac{u_n(x) - u_n(x^*)}{2} = 0$ when $x_n = 0$, so

$$T_{B_+} \left(\frac{u_n(x) - u_n(x^*)}{2} \right) = \frac{u_n(x) - u_n(x^*)}{2} |_{\partial B_+} = 0.$$

We have shown in class that this is sufficient to say $T_{B_+}(u|_{B_+}) = 0$, so $u|_{B_+} \in H_0^1(B_+)$. □

Problem 7.

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Solution.

- (a) Let $V_k = \{x : |x| < 1/k\}$ and let $W_k = \{x : 1 - 1/k < |x| < 1\}$. Then, consider lemonsqueezer functions $f_k \in C_0^1(U)$ such that $f_k|_{B(V_k \cap W_k)} = 1$ and $f_k|_{V_{2k} \cup W_{2k}} = 0$ with $O(k)$ derivatives.



Note that $u_k := uf_k \in C_0^1(U)$. We claim $u_k \rightarrow u$ in $H^1(B)$. Note that

$$\|u - u_k\|_{H^1}^2 = \int_B |u - u_k|^2 + \int_B |\nabla u - \nabla u_k|^2.$$

Now, since u is bounded

$$\begin{aligned} \int_B |u - u_k|^2 &= \int_B |u|^2 |1 - f_k|^2 \\ &= \int_B O(1) |1 - f_k|^2 \\ &= O(1) \int_B |1 - f_k|^2 \\ &= O(1) \int_{V_k \cup W_k} |1 - f_k|^2 \\ &= O(1) \int_{V_k \cup W_k} O(1) \\ &= O(1/k^n) = o(1) \end{aligned}$$

where the asymptotic notation is with respect to k . Also,

$$\int_B |\nabla u - \nabla u_k|^2 = \int_B |\nabla u - \nabla u f_k - u \nabla f_k|^2 \leq 2 \int_B |\nabla u|^2 |1 - f_k|^2 + 2 \int_B |u \nabla f_k|^2.$$

Note that since $|\nabla u|$ is bounded, applying our above logic to the first part gives us

$$2 \int_B |\nabla u|^2 |1 - f_k|^2 = O(1) \int_B |1 - f_k|^2 = o(1).$$

So, we only need to deal with the second part. We see that since u is bounded and ∇f_k is mostly

0,

$$\begin{aligned}
2 \int_B |u \nabla f_k|^2 &= 2 \int_B |u|^2 |\nabla f_k|^2 \\
&= O(1) \int_B |\nabla f_k|^2 \\
&= O(1) \int_{V_k \cup W_k} |\nabla f_k|^2 \\
&= O(1) \int_{V_k \cup W_k} |O(k)|^2 \\
&= O(1) O(1/k^n) O(k^2) \\
&= o(1)
\end{aligned}$$

for $n > 2$. Thus, combining everything, we see that

$$\|u - u_k\|_{H^1}^2 = o(1),$$

which is what we needed to show that $H_0^1(B) = H_0^1(U)$. □

- (b) Consider $u(x) = 1 - x^2 \in C^1((-1, 1))$. Note that $T_{(-1, 1)}u = 0$, so $u \in H_0^1((-1, 1))$. However, $T_{(-1, 0) \cup (0, 1)}u \neq 0$ so $u \notin H_0^1((-1, 0) \cup (0, 1))$. Thus,

$$H_0^1((-1, 1)) \neq H_0^1((-1, 0) \cup (0, 1))$$

□

- (c) TODO