MATH 173 Problem Set 2

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Problem 1. TODO

Solution. This is a semilinear PDE and can be solved with the same methods as last week. Let Let $(x(s), y(s))$ be a characteristic curve. We see that $x'(s) = 1, y'(s) = \cos$ with $x(0) = 0$.

Solving this, we get that $x(s) = s, y(s) = \sin(s) + a$. Let $f_a(s) = (s, sin(s) + a)$ be the characteristic curve. Also, let $\omega_a(s) = u(f_a(s))$. We know $\omega'_a(s) = y(s) = \sin(s) + a$ and $\omega_a(0) = a$. So, we see that $\omega_a(s) = -\cos(s) + as + a + 1$. We see that $x = s$, $y = \sin(s) + a$. Solving for a, s, we have that $s = x$ and $a = y - \sin(x) + y$. Thus,

$$u(x, y) = \omega_a(s) = -\cos(s) + as + a + 1 = -\cos(x) + xy - x\sin(x) + y - \sin(x) + 1.$$

We can check that this solution fits $u_x + \cos(x)u_y = y$ indeed.

TODO: insert image here $\square$

Problem 2. TODO ◁

For this question, we will answer both parts (a) and (b) together because we must go through the ideas to solve part (b) to be able to find the solution in (a) anyway. This is a quasilinear PDE that can be solved by finding the characteristic curves in 3-dimensional space, as discussed in chapters 3 and 4.

Let $(t(s), x(s), z(s)$ be the characteristic curve on the graph of $u(t,x)$. We see that $t'(s) = 1, x'(s) = \sqrt{z(s)}, z'(s) = 0$ and that the initial conditions give us $t(0) = 0, z(0) = x(0)^2$. Solving this, we have that $t(s) = s, z(s) = a$ and $x(s) = s\sqrt{a} \pm \sqrt{a}$ for some constant $a \geq 0$. Let $\omega_a^+(s) = (s, s\sqrt{a} + \sqrt{a}, a)$ and $\omega_a^-(s) = (s, s\sqrt{a} - \sqrt{a}, a)$ be the characteristic curves. Note that ω_0^+ and ω_0^- are the same curve. When projected onto the (x,y) plane, the curves look as follows:

The value of u is constant and equal to a along each curve. We see that in the region where $|t| > 1$, the curves either intersect or do no pass through at all. For the region $|t| < 1$, we can find a solution. Namely, we see that $t = s$ and $x = s\sqrt{a} \pm \sqrt{a}$, so we can solve to see that any (x,t) where $t < 1$ and $x \neq 0$ can be uniquely expressed by setting

$$s = t, a = \begin{cases} \left(\frac{x}{t+1}\right)^2 & \text{if } x \geq 0 \\ \left(\frac{x}{t-1}\right)^2 & \text{if } x \leq 0. \end{cases}$$

Note that since ω_0^+ and ω_0^- are the same curve with $a = 0$, we do not have any problems at 0. Since $u(t,x) = z(s) = a$, we can see that

$$u(t,x) = \begin{cases} \left(\frac{x}{t+1}\right)^2 & \text{if } x \geq 0 \\ \left(\frac{x}{t-1}\right)^2 & \text{if } x \leq 0. \end{cases}$$

is our solution fo $|t| < 1$. To finish answering part (b), we see that for $T = 1$, our expression is continuously differentiable on $[0,T] \times \mathbb{R}$. The only thing we have to check for this is that the derivatives near $x = 0$ match up, but we see that u_x approaches 0 from both sides, so we can continue.

Finally, we can check that for any $T > 1$, we encounter problems. The characteristic curves ω_0^- and ω_1^- pass through the same point $1, 0$, but with different values of a, so any solution must be equal to both 0 and 1 at that point, which is impossible. Thus, $T = 1$ is the largest choice we could have made. $\square$

Problem 3. TODO

Solution. TODO $\square$

Problem 4. TODO

Solution.

1. We can approach this as we approached conservation laws in class. As always, let $(t(s), x(s), z(s))$ be a characteristic curve on the graph of u. From our derivation in class, we know that $t(s) = s, x(s) = F'(g(a))s + s$, and $z(s) = g(a)$ for the parameter a. So, all the characteristics are straight lines and u is constant along them. Rearranging our expressions, we see that $s = t$ and $a = x - F'(g(a))t$. Plugging this in, we see that

$$u(t, x) = z(s) = g(a) = g(x - F'(g(a))t) = g(x - F'(u(t, x))t),$$

as we wanted. □

2. TODO

Problem 5. TODO

Solution.

1. TODO
2. TODO

Problem 6. TODO

Solution.
TODO

Problem 7. TODO

Solution.

1. TODO
2. TODO