

MATH 173 PROBLEM SET 6

Stepan (Styopa) Zharkov

May 11, 2022

Problem 1.

◁

Solution.

(a) This problem is straightforward.

$$\begin{aligned}\overline{\mathcal{F}(\phi)(y)} &= \overline{\int_{\mathbb{R}^n} e^{-ix \cdot y} \phi(x) dx} \\ &= \int_{\mathbb{R}^n} \overline{e^{-ix \cdot y} \phi(x)} dx \\ &= \int_{\mathbb{R}^n} e^{ix \cdot y} \overline{\phi(x)} dx \\ &= (2\pi)^n (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot y} \overline{\phi(x)} dx \\ &= (2\pi)^n \mathcal{F}^{-1} \overline{\phi}(y)\end{aligned}$$

This is what we wanted to show.

□

(b) We know the Fourier inversion formula holds for Schwartz functions. By part (a) and the equation about interchanging fourier transform under the integral that we saw in class, we know

$$\begin{aligned}(2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\phi} \bar{\psi} &= \int_{\mathbb{R}^n} \hat{\phi} \check{\bar{\psi}} \\ (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\phi} \bar{\psi} &= \int_{\mathbb{R}^n} \check{\hat{\phi}} \hat{\bar{\psi}} \\ (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\phi} \bar{\psi} &= \int_{\mathbb{R}^n} \phi \bar{\psi}.\end{aligned}$$

Setting $\psi = \phi$, we see that

$$\int_{\mathbb{R}^n} |\hat{\phi}|^2 = (2\pi)^2 \int_{\mathbb{R}^n} |\phi|^2,$$

as we wanted.

□



Problem 2.

◁

Solution.

- (a) Let $\tilde{u}(t, x)$ be defined as in the problem. On $(0, +\infty) \times (0, +\infty)$, \tilde{u} is the same as u , so it satisfies the equation $\tilde{u}_t = \tilde{u}_{xx}$. On $(0, +\infty) \times (-\infty, 0)$, we see that

$$\tilde{u}_t(t, x) = -u_t(t, -x) = -u_{xx}(t, -x) = \tilde{u}_{xx}(t, x).$$

So, $\tilde{u}_t = \tilde{u}_{xx}$ on all of $(0, +\infty) \times (0, +\infty) \cup (0, +\infty) \times (-\infty, 0)$ and is C^2 there. Now, consider the points along $x = 0$. We can define $\tilde{u}(t, 0) = 0$. We see that

$$\lim_{x \rightarrow 0} u(t, x) = u(t, 0) = 0,$$

and thus $\lim_{x \rightarrow 0} \tilde{u}(t, x) = 0$ from both sides, and is equal to $\tilde{u}(t, 0)$. So, \tilde{u} is continuous in $[0, +\infty) \times \mathbb{R}$.

Now, let's consider differentiability. Let $t > 0$. We see that

$$\begin{aligned} \lim_{h \rightarrow +0} \frac{\tilde{u}(t, h) - \tilde{u}(t, 0)}{h} &= \lim_{h \rightarrow +0} \frac{\tilde{u}(t, h)}{h} \\ &= \lim_{h \rightarrow +0} \frac{u(t, h)}{h} \\ &= \lim_{h \rightarrow -0} \frac{u(t, -h)}{-h} \\ &= \lim_{h \rightarrow -0} \frac{\tilde{u}(t, h)}{h}. \end{aligned}$$

Thus, the derivative from both sides matches up. We see that $\tilde{u}_t(t, 0) = 0$. Since u is continuously differentiable on the border, both components of the derivative are continuous, so \tilde{u} is differentiable on $(0, +\infty) \times \mathbb{R}$.

We can now assume that $\tilde{u} = K_t * \tilde{g}$. Writing this out, we have

$$\begin{aligned} \tilde{u}(t, x) &= (K_t * \tilde{g})(x) \\ &= \int (4\pi t)^{-1/2} e^{-\frac{|x-y|^2}{4t}} \tilde{g}(y) dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{|x-y|^2}{4t}} \tilde{g}(y) dy + \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^0 e^{-\frac{|x-y|^2}{4t}} \tilde{g}(y) dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{(x-y)^2}{4t}} g(y) dy - \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{(x+y)^2}{4t}} g(y) dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^\infty g(y) \left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right) dy. \end{aligned}$$

Restricting to to half, we see that

$$u(t, x) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty g(y) \left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right) dy.$$

We can verify that the boundary conditions match. □

(b) This is very similar to part (a), but this time let us define

$$\tilde{u}(t, x) = \begin{cases} u(t, x) & \text{if } x \geq 0 \\ u(t, -x) & \text{if } x < 0 \end{cases}$$

to be the even extension.

We know \tilde{u} is C^2 and satisfies the equation on $(0, +\infty) \times (0, +\infty) \cup (0, +\infty) \times (-\infty, 0)$ for the same reason as in part (a).

We see that

$$\lim_{x \rightarrow 0} u(t, x) = u(t, 0),$$

and thus $\lim_{x \rightarrow 0} \tilde{u}(t, x) = u(t, 0)$ from both sides, and is equal to $\tilde{u}(t, 0)$. So, \tilde{u} is continuous in $[0, +\infty) \times \mathbb{R}$.

Now, we notice that

$$\begin{aligned} \lim_{h \rightarrow +0} \frac{\tilde{u}(t, h) - \tilde{u}(t, 0)}{h} &= \lim_{h \rightarrow +0} \frac{u(t, h) - u(t, 0)}{h} \\ &= 0 \\ &= \lim_{h \rightarrow +0} -\frac{u(t, h) - u(t, 0)}{h} \\ &= \lim_{h \rightarrow -0} -\frac{u(t, -h) - u(t, 0)}{h} \\ &= \lim_{h \rightarrow -0} \frac{\tilde{u}(t, h) - \tilde{u}(t, 0)}{h}. \end{aligned}$$

Thus, the derivative from both sides matches up. We see that $\tilde{u}_t(t, 0) = 0$. Since u is continuously differentiable on the border, both components of the derivative are continuous, so \tilde{u} is differentiable on $(0, +\infty) \times \mathbb{R}$.

Making a similar assumption, we see that this time

$$\begin{aligned} \tilde{u}(t, x) &= (K_t * \tilde{g})(x) \\ &= \int (4\pi t)^{-1/2} e^{-\frac{|x-y|^2}{4t}} \tilde{g}(y) dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{|x-y|^2}{4t}} \tilde{g}(y) dy + \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^0 e^{-\frac{|x-y|^2}{4t}} \tilde{g}(y) dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{(x-y)^2}{4t}} g(y) dy + \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{(x+y)^2}{4t}} g(y) dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_0^\infty g(y) \left(e^{-\frac{(x-y)^2}{4t}} + e^{-\frac{(x+y)^2}{4t}} \right) dy. \end{aligned}$$

Restricting to to half, we see that

$$u(t, x) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty g(y) \left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right) dy.$$

We can verify that the boundary condition matches. □

Problem 3.

◁

Solution. This problem is full of tricks and surprises. First, consider $v(t, x) = u(t, x) - a(t)$. This means that

$$v_t = u_t - a'(t) = u_{xx} - a'(t) = v_{xx} - a'(t).$$

So,

$$v_t - v_{xx} = -a'(t)$$

with $v(t, 0) = 0$ and $v(0, x) = 0$. Also, $a(0) = 0$. So, we have an inhomogeneous heat equation.

Let $S_t(\phi)$ be the operator in problem 2. More precisely, let

$$S_t(\phi)(x) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty \phi(y) \left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right) dy.$$

Then, by Duhamel's principle,

$$v(t, x) = \int_0^t S_{t-s}(f(s, \cdot)) ds$$

where $f(t, x) = -a'(t)$. Expanding, and changing variables, we have

$$\begin{aligned} v(t, x) &= \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int_0^\infty -a'(s) \left(e^{-\frac{(x-y)^2}{4(t-s)}} - e^{-\frac{(x+y)^2}{4(t-s)}} \right) dy ds \\ &= \int_0^t \frac{-a'(s)}{\sqrt{4\pi(t-s)}} \left[\int_0^x e^{-\frac{(y)^2}{4(t-s)}} dy + \int_0^\infty e^{-\frac{(y)^2}{4(t-s)}} dy - \int_x^\infty e^{-\frac{(y)^2}{4(t-s)}} dy \right] ds \\ &= \int_0^t \frac{-a'(s)}{\sqrt{4\pi(t-s)}} \left[\int_0^x e^{-\frac{(y)^2}{4(t-s)}} dy + \int_0^x e^{-\frac{(y)^2}{4(t-s)}} dy \right] ds \\ &= \int_0^t \frac{-2a'(s)}{\sqrt{4\pi(t-s)}} \int_0^x e^{-\frac{(y)^2}{4(t-s)}} dy ds \end{aligned}$$

Now, using the hint, we can change variables and apply integration by parts to see that

$$\begin{aligned} v(t, x) &= \int_0^t \frac{-2a'(s)}{\sqrt{\pi}} \int_0^{x(4(t-s))^{-1/2}} e^{-z^2} dz ds \\ &= \frac{-2a(t)}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} + \int_0^t \frac{-2a(s)}{\sqrt{\pi}} e^{\frac{-x^2}{4(t-s)}} \left(-(t-s)^{-3/2} \cdot \frac{1}{4} \right) ds \\ &= -a(t) + \frac{x}{\sqrt{4\pi}} \int_0^t (t-s)^{-3/2} e^{\frac{-x^2}{4(t-s)}} a(s) ds \\ &= -a(t) + \frac{x}{\sqrt{4\pi}} \int_0^t s^{-3/2} e^{\frac{-x^2}{4(s)}} a(t-s) ds. \end{aligned}$$

Thus, we can conclude that

$$u(t, x) = \frac{x}{\sqrt{4\pi}} \int_0^t s^{-3/2} e^{\frac{-x^2}{4(s)}} a(t-s) ds,$$

which is what we wanted. Note that the integral is only converges for $x > 0$, so we can add on that $u(t, 0) = a(t)$. \square

Problem 4.

<

Solution.

(a) By d'Alembert's formula, we know

$$\begin{aligned}
 u(x, t) &= \frac{1}{2}(\phi(x + ct) + \phi(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(\sigma) d\sigma \\
 &= \frac{1}{2}((x + ct)^2 + (x - ct)^2) + \frac{1}{2c} \int_{x-ct}^{x+ct} 1 d\sigma \\
 &= x^2 + (ct)^2 + t^2.
 \end{aligned}$$

□

(b) For this problem, we can repeatedly use the fundamental theorem of calculus. We assume that u_x vanishes at infinity. We see

$$\begin{aligned}
 \int u(t, x) dx &= \int \left(u(0, x) + \int_0^t u_t(\tau, x) d\tau \right) dx \\
 &= \int \left(u(0, x) + \int_0^t \left(u_t(0, x) + \int_0^\tau u_{tt}(s, x) ds \right) d\tau \right) dx \\
 &= \int \left(u(0, x) + \int_0^t \left(u_t(0, x) + \int_0^\tau c^2 u_{xx}(s, x) ds \right) d\tau \right) dx \\
 &= \int u(0, x) dx + \int \int_0^t u_t(0, x) d\tau dx + \int \int_0^t \int_0^\tau c^2 u_{xx}(s, x) ds d\tau dx \\
 &= \int \phi(x) dx + \int \int_0^t \psi(x) d\tau dx + \int \int_0^t \int_0^\tau c^2 u_{xx}(s, x) ds d\tau dx \\
 &= \int \phi(x) dx + t \int \psi(x) d\tau dx + \int_0^t \int_0^\tau \int c^2 u_{xx}(s, x) dx ds d\tau \\
 &= \int \phi(x) dx + t \int \psi(x) d\tau dx + \int_0^t \int_0^\tau 0 ds d\tau \\
 &= \int \phi(x) dx + t \int \psi(x) d\tau dx,
 \end{aligned}$$

exactly as we wanted.

□

Problem 5.

◁

Solution. To solve this, we take the partial Fourier transform of

$$u_{tt} = u_{xx} - m^2 u$$

to get

$$\hat{u}_t t = -y^2 \hat{u} - m^2 \hat{u}$$

with conditions

$$\hat{u}(0, y) = \hat{g}(y), \quad \hat{u}_t(0, y) = \hat{h}(y).$$

Solving this ODE, we have that

$$\hat{u} = \cos \sqrt{y^2 + m^2} y t \hat{g}(y) + \frac{\sin(\sqrt{y^2 + m^2} y t)}{\sqrt{y^2 + m^2}} \hat{h}(y).$$

This is the solution we were looking for.

□

Problem 6.

◁

Solution. The hint unlocks the secret to this problem. Suppose no such C, m exist. In particular, no constant $C = m = j \in \mathbb{N}$ works. Let $\phi_j \in \mathcal{S}(\mathbb{R}^n)$ be the counterexample that gives $|U(\phi_j)| > j \|\phi_j\|_j$. Now, let's define

$$\psi_j = j^{-1} \|\phi_j\|_j^{-1} \phi_j.$$

Since ϕ_j are Schwartz, for any multi-indices α, β , $x^\alpha \partial^\beta \phi_j$ are bounded. For any $\varepsilon > 0$, select $j > \max(\varepsilon^{-1}, |\alpha|, |\beta|)$. Then,

$$\begin{aligned} \sup_x |x^\alpha \partial^\beta \psi_j| &= \sup_x \left| x^\alpha \partial^\beta j^{-1} \left(\sum_{|\mathbf{a}|, |\mathbf{b}| < j} \sup_y |y^\mathbf{a} \partial^\mathbf{b} \phi_j(y)| \right)^{-1} \phi_j(x) \right| \\ &= j^{-1} \frac{\sup_x |x^\alpha \partial^{\beta-1} \phi_j(x)|}{\sum_{|\mathbf{a}|, |\mathbf{b}| < j} \sup_y |y^\mathbf{a} \partial^\mathbf{b} \phi_j(y)|} \\ &\leq j^{-1} \\ &< \varepsilon. \end{aligned}$$

Thus, by definition, $\psi_j \rightarrow 0$ as a Schwartz function.

However, we see that by linearity of U ,

$$\begin{aligned} |U(\psi_j)| &= \left| U(j^{-1} \|\phi_j\|_j^{-1} \phi_j) \right| \\ &= |U(\phi_j)| j^{-1} \|\phi_j\|_j^{-1} \\ &> 1 \end{aligned}$$

by our choice of ϕ_j . But $U(0) = 0$, so U is not continuous and we have a contradiction. Thus, the constants C and m must exist. \square

Problem 7.

◁

Solution.

- (a) For this problem, we must go back to the definition of an integral through Riemann sums. First, notice that

$$\begin{aligned} \int_{\mathbb{R}^n} g(y)\phi(y)dy &= \int_{\mathbb{R}^n} U(f(x)e^{-ix \cdot y})\phi(y)dy \\ &= \int_{\mathbb{R}^n} U(f(x)e^{-ix \cdot y}\phi(y))dy \\ &= \lim_{N \rightarrow \infty} \sum_{k=1}^N |I_k| U(f(x)e^{-ix \cdot y_k}\phi(y_k)) \\ &= \lim_{N \rightarrow \infty} U \left(\sum_{k=1}^N |I_k| f(x)e^{-ix \cdot y_k}\phi(y_k) \right) \end{aligned}$$

where y_k is some point in the Riemann cube I_k and the limit notation denotes taking smaller and smaller intervals and covering more and more of space.

We want to be able to move the integral inside of U , but we can't do that directly. We can move the finite sum inside by linearity of U , and continuity gives us that we can move the limit inside as long as the inside converges as Schwartz functions. So, let us show that

$$\sum_{k=1}^N |I_k| f(x)e^{-ix \cdot y_k}\phi(y_k) \rightarrow \int f(x)e^{-ix \cdot y}\phi(y)dy$$

in the Schwartz sense. First, we know f has compact support, so we can assume x is bounded by some constant R . Let α, β be multi-indices. We need to check that

$$\sup_x \left| x^\alpha \partial^\beta \left(\sum_{k=1}^N |I_k| f(x)e^{-ix \cdot y_k}\phi(y_k) - \int f(x)e^{-ix \cdot y}\phi(y)dy \right) \right|$$

converges to 0 as $N \rightarrow \infty$. When we expand out the derivatives, each term in the massive sum will be of the form

$$x^\alpha \left(\sum_{k=1}^N |I_k| \partial^\gamma f(x)(-iy_k)^\omega e^{-ix \cdot y_k}\phi(y_k) - \int \partial^\gamma f(x)(-iy)^\omega e^{-ix \cdot y}\phi(y)dy \right)$$

where γ and ω are some multi-indices such that $|\gamma|, |\omega| \leq |\beta|$. Since ϕ is a Schwartz function, $(iy)^\omega \phi(y)$ is bounded. By definition of an integral, the expression inside the parentheses converges to 0 pointwise (for a set x). The expression is continuous and has compact support (because ∂^γ has compact support). We know that pointwise convergence of continuous functions on compact support implies uniform convergence. So, there is some sequence M_N such that $M_N \rightarrow 0$ and

$$\sum_{k=1}^N |I_k| \partial^\gamma f(x)(-iy_k)^\omega e^{-ix \cdot y_k}\phi(y_k) - \int \partial^\gamma f(x)(-iy)^\omega e^{-ix \cdot y}\phi(y)dy < M_N.$$

Thus we see that

$$\begin{aligned}
& \sup_x \left| x^\alpha \left(\sum_{k=1}^N |I_k| \partial^\gamma f(x) (-iy_k)^\omega e^{-ix \cdot y_k} \phi(y_k) - \int \partial^\gamma f(x) (-iy)^\omega e^{-ix \cdot y} \phi(y) dy \right) \right| \\
&= \sup_{x < R} \left| x^\alpha \left(\sum_{k=1}^N |I_k| \partial^\gamma f(x) (-iy_k)^\omega e^{-ix \cdot y_k} \phi(y_k) - \int \partial^\gamma f(x) (-iy)^\omega e^{-ix \cdot y} \phi(y) dy \right) \right| \\
&\leq \sup_{x < R} |x^\alpha M_N| \\
&\leq R^{|\alpha|} M_N
\end{aligned}$$

This certainly converges to 0. There are only a finite number of terms like this (because there are only so many options for what derivatives to take given that $|\gamma|, |\omega| < |\beta|$). So, we can conclude that

$$\sup_x \left| x^\alpha \partial^\beta \left(\sum_{k=1}^N |I_k| f(x) e^{-ix \cdot y_k} \phi(y_k) - \int f(x) e^{-ix \cdot y} \phi(y) dy \right) \right| \rightarrow 0.$$

This means that the Riemann sum does indeed converge to the integral as Schwartz functions. So, by continuity of U ,

$$\begin{aligned}
\int_{\mathbb{R}^n} g(y) \phi(y) dy &= \lim_{N \rightarrow \infty} U \left(\sum_{k=1}^N |I_k| f(x) e^{-ix \cdot y_k} \phi(y_k) \right) \\
&= U \left(\lim_{N \rightarrow \infty} \sum_{k=1}^N |I_k| f(x) e^{-ix \cdot y_k} \phi(y_k) \right) \\
&= U \left(\int f(x) e^{-ix \cdot y} \phi(y) dy \right) \\
&= U(\mathcal{F}(\phi)) \\
&= \mathcal{F}(U)(\phi).
\end{aligned}$$

With this, we are done.

(b) Since U is linear and continuous,

$$U(\partial^\alpha f(x) e^{-ix \cdot y}) = \partial^\alpha U(f(x) e^{-ix \cdot y})$$

where the derivative is with respect to y as long as $\partial^\alpha f(x) e^{-ix \cdot y}$ is Schwartz in x .

We see that

$$\partial^\alpha f(x) e^{-ix \cdot y} = f(x) \partial^\alpha e^{ix \cdot y} = f(x) (ix)^\alpha e^{ix \cdot y}$$

is indeed Schwartz because f has compact support and is C^∞ . Thus, we can conclude that $g \in C^\infty$.

To show the second part, we use problem 6. We know there exist C_0, m such that $|U(\phi)| \leq C_0 \|\phi\|_m$ for all ϕ . This means that

$$\begin{aligned}
|g(y)| &= |U(f(x) e^{-ix \cdot y})| \\
&\leq C_0 \|f(x) e^{ix \cdot y}\|_m \\
&= C_0 \sum_{|\alpha|, |\beta| < m} \sup_x \left| x^\alpha \partial^\beta f(x) e^{ix \cdot y} \right|.
\end{aligned}$$

When we expand out the derivatives, each term in the sum inside the sup will be of the form

$$x^\alpha (iy)^\gamma e^{ix \cdot y} \partial^\omega f(x)$$

where $|\omega|, |\gamma| < m$. Since f has compact support, this is only nonzero for $x < R$ for some constant $R > 1$. Also, this means each derivative of f is bounded. Let F be a constant that bounds all derivatives of f up to m . Thus,

$$\begin{aligned} \left| x^\alpha (iy)^\gamma e^{ix \cdot y} \partial^\omega f(x) \right| &\leq R^m \left| (iy)^\gamma e^{ix \cdot y} \partial^\omega f(x) \right| \\ &\leq R^m \left| (iy)^\gamma \partial^\omega f(x) \right| \\ &\leq R^m F \left| (iy)^\gamma \right| \\ &\leq R^m F (\lceil y \rceil + 1)^m. \end{aligned}$$

Now, there is a finite number of terms in that sup. Call that number S . Then, by triangle inequality,

$$\sup_x \left| x^\alpha \partial^\beta f(x) e^{ix \cdot y} \right| \leq SR^m F (\lceil y \rceil + 1)^m.$$

There are also finitely many options for α, β . Call the number of options M . Then, we see that

$$|g(y)| \leq C_0 MSR^m F (\lceil y \rceil + 1)^m.$$

So, letting $C = C_0 MSR^m F$, we get the desired bound. □

