MATH 173 Problem Set 4

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Problem 1. TODO ◁

Solution.

(a) This problem is a simple computation. We see that, with a change of variables $z = x - a$, we have

$$\begin{aligned}
\hat{f}_a(y) &= \mathcal{F}(f(x-a))(y) \\
&= \int e^{-ixy} f(x-a)dx \\
&= \int e^{-i(z+a)y} f(z)dx \\
&= e^{iay} \int e^{-izy} f(z)dx \\
&= e^{iay} \hat{f}(y),
\end{aligned}$$

as we wanted. □

(b) This problem is even simpler computation. We see that

$$\begin{aligned}
\hat{g}_a(y) &= \mathcal{F}(e^{ixa} f(x)) \\
&= \int e^{-ixy} e^{ixa} f(x)dx \\
&= \int e^{-ix(y-a)} f(x)dx \\
&= \hat{f}(y-a),
\end{aligned}$$

as we wanted. □

Problem 2. TODO ◁

Solution.

(a) This problem is also computation.

$$\begin{aligned}(\mathcal{F}\chi_{(-a,a)}(y)) &= \int_{-\infty}^{\infty} e^{-ixy}\chi_{(-a,a)}(x)dx \\ &= \int_{-a}^{a} e^{-ixy}dx \\ &= \begin{cases} -\frac{i}{y}\left(e^{-iay}-e^{iay}\right) & \text{if } y \neq 0 \\ 2a & \text{if } y = 0 \end{cases}\end{aligned}$$

□

(b) Note that since $y \in \mathbb{R}$ and $a > 0$, we know $iy - a \neq 0$ and $iy + a \neq 0$ so we can divide by them. So,

$$\begin{aligned}(\mathcal{F}(e^{-a|x|})(y) &= \int_{-\infty}^{\infty} e^{-ixy}e^{a|x|}dx \\ &= \int_{0}^{\infty} e^{-x(iy+a)}dx + \int_{-\infty}^{0} e^{-x(iy-a)}dx \\ &= \left[-\frac{1}{iy+a}e^{-x(iy+a)}\right]_0^\infty + \left[-\frac{1}{iy-a}e^{-x(iy-a)}\right]_{-\infty}^0 \\ &= \frac{1}{iy+a} - \frac{1}{iy-a}\end{aligned}$$

because $a > 0$. □

(c) In this problem, we will use repeated integration by parts. We see that executing integration by parts, we have

$$\begin{aligned}\mathcal{F}(|x|^n e^{-a|x|})(y) &= \int_{-\infty}^{\infty} |x|^n e^{-ixy-a|x|}dx \\ &= \int_{-\infty}^{0} (-x)^n e^{-ixy+ax}dx + \int_{0}^{\infty} x^n e^{-ixy-ax}dx \\ &= \int_{-\infty}^{0} (-x)^n e^{-ix(y-a)}dx + \int_{0}^{\infty} x^n e^{-ix(y+a)}dx \\ &= \frac{-n}{yi-a}\int_{-\infty}^{0} (-x)^{n-1} e^{-ix(y-a)}dx + \frac{n}{yi+a}\int_{0}^{\infty} x^{n-1} e^{-ix(y+a)}dx.\end{aligned}$$

Note that the boundary terms vanish in the integration by parts. Repeating integration by parts n times, we see that

$$\begin{aligned}\mathcal{F}(|x|^n e^{-a|x|})(y) &= (-1)^n \frac{n!}{(yi-a)^n}\int_{-\infty}^{0} e^{-ix(y-a)}dx + \frac{n!}{(yi+a)^n}\int_{0}^{\infty} e^{-ix(y+a)}dx \\ &= (-1)^{n+1}\frac{n!}{(yi-a)^{n+1}} + \frac{n!}{(yi+a)^{n+1}},\end{aligned}$$

and $yi \pm a$ does not vanish because $a > 0$. □

Problem 3. TODO $\triangleleft$

Solution.

1. The solution to this is straightforward. As Tadashi Tokieda would say, "follow your nose". First, let $f(x) = f(-x)$. Then, letting $z = -x$ power a change of variables, we see that

$$\begin{aligned}
\hat{f}(y) &= \int_{-\infty}^{\infty} e^{-ixy} f(x)dx \\
&= \int_{\infty}^{-\infty} -e^{izy} f(-z)dz \\
&= \int_{-\infty}^{\infty} e^{izy} f(-z)dz \\
&= \int_{-\infty}^{\infty} e^{-iz(-y)} f(z)dz \\
&= \hat{f}(-y).
\end{aligned}$$

Similarly, now let $f(x) = -f(-x)$. Then,

$$\begin{aligned}
\hat{f}(y) &= \int_{-\infty}^{\infty} e^{-ixy} f(x)dx \\
&= \int_{\infty}^{-\infty} -e^{izy} f(-z)dz \\
&= \int_{-\infty}^{\infty} e^{izy} f(-z)dz \\
&= \int_{-\infty}^{\infty} -e^{-iz(-y)} f(z)dz \\
&= -\hat{f}(-y).
\end{aligned}$$

So, the fourier transform preserves evenness and oddness. $\square$

Problem 4. TODO ◁

Solution. TODO

Problem 5. TODO $\triangleleft$

Solution. TODO

Problem 6. TODO $\triangleleft$

Solution. TODO

Problem 7. TODO $\triangleleft$

Solution. TODO