MATH 173 Problem Set 7

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Problem 1. ◁

Solution. This problem is straightforward. Since $\overline{\Omega}$ is closed, and $c > 0$, there exists constants $C_1, C_2 > 0$ such that $C_1 < c(x) < C_2$. Assume $C_1 < 1$ and $C_2 > 1$. If not, we can always choose smaller C_1 and larger C_2. So, for any $u \in C^1(\overline{\Omega})$,

$$\begin{aligned}
||u||^2_{H^1_c(\Omega)} &= \int_\Omega |u(x)|^2 dx + \int_\Omega c(x)|\nabla u(x)|^2 dx \\
&\geq C_1^2 \int_\Omega |u(x)|^2 dx + \int_\Omega C_1^2 |\nabla u(x)|^2 dx \\
&= C_1^2 ||u||^2_{H^1(\Omega)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
||u||^2_{H^1_c(\Omega)} &= \int_\Omega |u(x)|^2 dx + \int_\Omega c(x)|\nabla u(x)|^2 dx \\
&\leq C_2^2 \int_\Omega |u(x)|^2 dx + \int_\Omega C_2^2 |\nabla u(x)|^2 dx \\
&= C_2^2 ||u||^2_{H^1(\Omega)}.
\end{aligned}$$

So, the statement follows for all $u \in C^1(\overline{\Omega})$. By continuity and density, it follows that the statement holds for $u \in H^1(\Omega)$. □

Problem 2. ◁

Solution.

1. Suppose $v = u + w = u' + w'$ where $u, u' \in M$, $w, w' \in M^\perp$. Then, we see

   $$u' - u = u' + w - v = w - w'.$$

   But $u' - u \in M$ and $w - w' \in M^\perp$ and $M \cap M^\perp = \{0\}$. So, $u' - u = w - w' = 0$. Thus, the decomposition is unique.

   We see that $u = u + 0$. By uniqueness, $P(v) = u = P(u) = P(P(v))$, so $P = P^2$. □

2. Let $v = u + w$ and $v' = u' + w'$ with $u, u' \in M$, $w, w' \in M^\perp$. We see

   $$\langle Pv, v' \rangle = \langle u, l' + w' \rangle = \langle u, u' \rangle + \langle u, w' \rangle = \langle u, u' \rangle = \langle u, u' \rangle + \langle w, u' \rangle = \langle u + w, u' \rangle = \langle v, Pv' \rangle$$

3. TODO

Problem 3. $\triangleleft$

Solution. TODO

Problem 4. $\triangleleft$

Solution. TODO

Problem 5. $\triangleleft$

Solution. TODO

Problem 6. $\triangleleft$

Solution. TODO