

# MATH 173 PROBLEM SET 7

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## Problem 1.

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**Solution.** This problem is straightforward. Since  $\overline{\Omega}$  is closed, and  $c > 0$ , there exists constants  $C_1, C_2 > 0$  such that  $C_1 < c(x) < C_2$ . Assume  $C_1 < 1$  and  $C_2 > 1$ . If not, we can always choose smaller  $C_1$  and larger  $C_2$ . So, for any  $u \in C^1(\overline{\Omega})$ ,

$$\begin{aligned} \|u\|_{H_c^1(\Omega)}^2 &= \int_{\Omega} |u(x)|^2 dx + \int_{\Omega} c(x) |\nabla u(x)|^2 dx \\ &\geq C_1^2 \int_{\Omega} |u(x)|^2 dx + \int_{\Omega} C_1^2 |\nabla u(x)|^2 dx \\ &= C_1^2 \|u\|_{H^1(\Omega)}^2. \end{aligned}$$

Similarly,

$$\begin{aligned} \|u\|_{H_c^1(\Omega)}^2 &= \int_{\Omega} |u(x)|^2 dx + \int_{\Omega} c(x) |\nabla u(x)|^2 dx \\ &\leq C_2^2 \int_{\Omega} |u(x)|^2 dx + \int_{\Omega} C_2^2 |\nabla u(x)|^2 dx \\ &= C_2^2 \|u\|_{H^1(\Omega)}^2. \end{aligned}$$

So, the statement follows for all  $u \in C^1(\overline{\Omega})$ . By continuity and density, it follows that the statement holds for  $u \in H^1(\Omega)$ .  $\square$

**Problem 2.**

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***Solution.***

1. Suppose  $v = u + w = u' + w'$  where  $u, u' \in M$ ,  $w, w' \in M^\perp$ . Then, we see

$$u' - u = u' + w - v = w - w'.$$

But  $u' - u \in M$  and  $w - w' \in M^\perp$  and  $M \cap M^\perp = \{0\}$ . So,  $u' - u = w - w' = 0$ . Thus, the decomposition is unique.

We see that  $u = u + 0$ . By uniqueness,  $P(v) = u = P(u) = P(P(v))$ , so  $P = P^2$ . □

2. TODO

3. TODO

**Problem 3.**

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*Solution.* TODO

**Problem 4.**

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*Solution.* TODO

**Problem 5.**

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*Solution.* TODO

**Problem 6.**

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*Solution.* TODO