MATH 173 Problem Set 6

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Problem 1. ◁

Solution.

(a) This problem is straightforward.

$$\begin{aligned}
\overline{\mathcal{F}(\phi)(y)} &= \overline{\int_{\mathbb{R}^n} e^{-ix\cdot y}\phi(x)dx} \\
&= \int_{\mathbb{R}^n} \overline{e^{-ix\cdot y}\phi(x)}dx \\
&= \int_{\mathbb{R}^n} e^{ix\cdot y}\overline{\phi(x)}dx \\
&= (2\pi)^n(2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot y}\overline{\phi(x)}dx \\
&= (2\pi)^n\mathcal{F}^{-1}\overline{\phi}(y)
\end{aligned}$$

This is what we wanted to show. □

(b) We know the Fourier inversion formula holds for Schwartz functions. By part (a) and the equation about interchanging fourier transform under the integral that we saw in class, we know

$$\begin{aligned}
(2\pi)^{-n}\int_{\mathbb{R}^n}\hat{\phi}\overline{\hat{\psi}} &= \int_{\mathbb{R}^n}\hat{\phi}\check{\bar{\psi}} \\
(2\pi)^{-n}\int_{\mathbb{R}^n}\hat{\phi}\overline{\hat{\psi}} &= \int_{\mathbb{R}^n}\check{\hat{\phi}}\hat{\check{\bar{\psi}}} \\
(2\pi)^{-n}\int_{\mathbb{R}^n}\hat{\phi}\overline{\hat{\psi}} &= \int_{\mathbb{R}^n}\phi\bar{\psi}.
\end{aligned}$$

Setting $\psi = \phi$, we see that

$$\int_{\mathbb{R}^n}\left|\hat{\phi}\right|^2 = (2\pi)^2\int_{\mathbb{R}^n}|\phi|^2,$$

as we wanted. □

Problem 2. $\triangleleft$

Solution.

(a) Let $\tilde{u}(t,x)$ be defined as in the problem. On $(0,+\infty)\times(0,+\infty)$, $\tilde{u}$ is the same as u, so it satisfies the equation $\tilde{u}_t = \tilde{u}_{xx}$. On $(0,+\infty)\times(-\infty,0)$, we see that

$$\tilde{u}_t(t,x) = -u_t(t,-x) = -u_{xx}(t,-x) = \tilde{u}_{xx}(t,x).$$

So, $\tilde{u}_t = \tilde{u}_{xx}$ on all of $(0,+\infty)\times(0,+\infty)\cup(0,+\infty)\times(-\infty,0)$ and is C^2 there. Now, consider the points along $x=0$. We can define $\tilde{u}(t,0)=0$. We see that

$$\lim_{x\to 0} u(t,x) = u(t,0) = 0,$$

and thus $\lim_{x\to 0}\tilde{u}(t,x) = 0$ from both sides, and is equal to $\tilde{u}(t,0)$. So, $\tilde{u}$ is continuous in $[0,+\infty)\times\mathbb{R}$.

Now, let's consider differentiability. Let $t>0$. We see that

$$\begin{aligned}
\lim_{h\to^+0}\frac{\tilde{u}(t,h)-\tilde{u}(t,0)}{h} &= \lim_{h\to^+0}\frac{\tilde{u}(t,h)}{h}\\
&= \lim_{h\to^+0}\frac{u(t,h)}{h}\\
&= \lim_{h\to^-0}\frac{u(t,-h)}{-h}\\
&= \lim_{h\to^-0}\frac{\tilde{u}(t,h)}{h}.
\end{aligned}$$

Thus, the derivative from both sides matches up. We see that $\tilde{u}_t(t,0)=0$. Since u is continuously differentiable on the border, both components of the derivative are continuous, so $\tilde{u}$ is differentiable on $(0,+\infty)\times\mathbb{R}$.

We can now assume that $\tilde{u} = K_t * \tilde{g}$. Writing this out, we have

$$\begin{aligned}
\tilde{u}(t,x) &= (K_t * \tilde{g})(x)\\
&= \int (4\pi t)^{-1/2} e^{-\frac{|x-y|^2}{4t}}\tilde{g}(y)dy\\
&= \frac{1}{\sqrt{4\pi t}}\int_0^\infty e^{-\frac{|x-y|^2}{4t}}\tilde{g}(y)dy + \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^0 e^{-\frac{|x-y|^2}{4t}}\tilde{g}(y)dy\\
&= \frac{1}{\sqrt{4\pi t}}\int_0^\infty e^{-\frac{(x-y)^2}{4t}}g(y)dy - \frac{1}{\sqrt{4\pi t}}\int_0^\infty e^{-\frac{(x+y)^2}{4t}}g(y)dy\\
&= \frac{1}{\sqrt{4\pi t}}\int_0^\infty g(y)\left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}}\right)dy.
\end{aligned}$$

Restricting to to half, we see that

$$u(t,x) = \frac{1}{\sqrt{4\pi t}}\int_0^\infty g(y)\left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}}\right)dy.$$

We can verify that the boundary conditions match. $\square$

(b) This is very similar to part (a), but this time let us define

$$\tilde{u}(t,x) = \begin{cases} u(t,x) & \text{if } x \geq 0 \\ u(t,-x) & \text{if } x \geq 0 \end{cases}$$

to be the even extension.

We know $\tilde{u}$ is C^2 and satisfies the equation on $(0,+\infty) \times (0,+\infty) \cup (0,+\infty) \times (-\infty, 0)$ for the same reason as in part (a).

We see that

$$\lim_{x \to 0} u(t,x) = u(t,0),$$

and thus $\lim_{x\to 0} \tilde{u}(t,x) = u(t,0)$ from both sides, and is equal to $\tilde{u}(t,0)$. So, $\tilde{u}$ is continuous in $[0,+\infty) \times \mathbb{R}$.

Now, we notice that

$$\begin{aligned}
\lim_{h \to {}^+0} \frac{\tilde{u}(t,h) - \tilde{u}(t,0)}{h} &= \lim_{h \to {}^+0} \frac{u(t,h) - u(t,0)}{h} \\
&= 0 \\
&= \lim_{h \to {}^+0} -\frac{u(t,h) - u(t,0)}{h} \\
&= \lim_{h \to {}^-0} -\frac{u(t,-h) - u(t,0)}{h} \\
&= \lim_{h \to {}^-0} \frac{\tilde{u}(t,h) - \tilde{u}(t,0)}{h}.
\end{aligned}$$

Thus, the derivative from both sides matches up. We see that $\tilde{u}_t(t,0) = 0$. Since u is continuously differentiable on the border, both components of the derivative are continuous, so $\tilde{u}$ is differentiable on $(0,+\infty) \times \mathbb{R}$.

Making a similar assumption, we see that this time

$$\begin{aligned}
\tilde{u}(t,x) &= (K_t * \tilde{g})(x) \\
&= \int (4\pi t)^{-1/2} e^{-\frac{|x-y|^2}{4t}} \tilde{g}(y) dy \\
&= \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{|x-y|^2}{4t}} \tilde{g}(y) dy + \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^0 e^{-\frac{|x-y|^2}{4t}} \tilde{g}(y) dy \\
&= \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{(x-y)^2}{4t}} g(y) dy + \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{(x+y)^2}{4t}} g(y) dy \\
&= \frac{1}{\sqrt{4\pi t}} \int_0^\infty g(y) \left(e^{-\frac{(x-y)^2}{4t}} + e^{-\frac{(x+y)^2}{4t}} \right) dy.
\end{aligned}$$

Restricting to to half, we see that

$$u(t,x) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty g(y) \left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right) dy.$$

We can verify that the boundary condition matches. □

Problem 3. ◁

Solution. This problem is full of tricks and surprises. First, consider $v(t,x) = u(t,x) - a(t)$. This means that

$$v_t = u_t - a'(t) = u_{xx} - a'(t) = v_x x - a'(t).$$

So,

$$v_t - v_{xx} = -a'(t)$$

with $v(t,0) = 0$ and $v(0,x) = 0$. Also, $a(0) = 0$. So, we have an inhomogeneous heat equation.

Let $S_t(\phi)$ be the operator in problem 2. More precisely, let

$$S_t(\phi)(x) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty \phi(y) \left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right) dy.$$

Then, by Duhamel's principle,

$$v(t,x) = \int_0^t S_{t-s}(f(s,\cdot))ds$$

where $f(t,x) = -a'(t)$. Expanding, and changing variables, we have

$$\begin{aligned}
v(t,x) &= \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int_0^\infty -a'(s) \left(e^{-\frac{(x-y)^2}{4(t-s)}} - e^{-\frac{(x+y)^2}{4(t-s)}} \right) dyds \\
&= \int_0^t \frac{-a'(s)}{\sqrt{4\pi(t-s)}} \left[\int_0^x e^{-\frac{(y)^2}{4(t-s)}} dy + \int_0^\infty e^{-\frac{(y)^2}{4(t-s)}} dy - \int_x^\infty e^{-\frac{(y)^2}{4(t-s)}} dy \right] ds \\
&= \int_0^t \frac{-a'(s)}{\sqrt{4\pi(t-s)}} \left[\int_0^x e^{-\frac{(y)^2}{4(t-s)}} dy + \int_0^x e^{-\frac{(y)^2}{4(t-s)}} dy \right] ds \\
&= \int_0^t \frac{-2a'(s)}{\sqrt{4\pi(t-s)}} \int_0^x e^{-\frac{(y)^2}{4(t-s)}} dyds
\end{aligned}$$

Now, using the hint, we can change variables and apply integration by parts to see that

$$\begin{aligned}
v(t,x) &= \int_0^t \frac{-2a'(s)}{\sqrt{\pi}} \int_0^{x(4(t-s))^{-1/2}} e^{-z^2} dzds \\
&= \frac{-2a(t)}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} + \int_0^t \frac{-2a(s)}{\sqrt{\pi}} e^{\frac{-x^2}{4(t-s)}} \left(-(t-s)^{-3/2} \cdot \frac{1}{4} \right) ds \\
&= -a(t) + \frac{x}{\sqrt{4\pi}} \int_0^t (t-s)^{-3/2} e^{\frac{-x^2}{4(t-s)}} a(s) ds \\
&= -a(t) + \frac{x}{\sqrt{4\pi}} \int_0^t s^{-3/2} e^{\frac{-x^2}{4(s)}} a(t-s) ds.
\end{aligned}$$

Thus, we can conclude that

$$u(t,x) = \frac{x}{\sqrt{4\pi}} \int_0^t s^{-3/2} e^{\frac{-x^2}{4(s)}} a(t-s) ds,$$

which is what we wanted. Note that the integral is only converges for $x > 0$, so we can add on that $u(t,0) = a(t)$. □

Problem 4. $\triangleleft$

Solution.

(a) By d'Alembert's formula, we know

$$\begin{aligned}
u(x,t) &= \frac{1}{2}(\phi(x+ct)+\phi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\sigma)d\sigma \\
&= \frac{1}{2}((x+ct)^2+(x-ct)^2) + \frac{1}{2c}\int_{x-ct}^{x+ct} 1 d\sigma \\
&= x^2+(ct)^2+t^2.
\end{aligned}$$

$\square$

(b) For this problem, we can repeatedly use the fundamental theorem of calculus. We assume that u_x vanishes at infinity. We see

$$\begin{aligned}
\int u(t,x)dx &= \int \left(u(0,x) + \int_0^t u_t(\tau,x)d\tau \right) dx \\
&= \int \left(u(0,x) + \int_0^t \left(u_t(0,x) + \int_0^\tau u_{tt}(s,x)ds \right) d\tau \right) dx \\
&= \int \left(u(0,x) + \int_0^t \left(u_t(0,x) + \int_0^\tau c^2 u_{xx}(s,x)ds \right) d\tau \right) dx \\
&= \int u(0,x)dx + \int \int_0^t u_t(0,x)d\tau dx + \int \int_0^t \int_0^\tau c^2 u_{xx}(s,x) ds d\tau dx \\
&= \int \phi(x)dx + \int \int_0^t \psi(x)d\tau dx + \int \int_0^t \int_0^\tau c^2 u_{xx}(s,x) ds d\tau dx \\
&= \int \phi(x)dx + t\int \psi(x)d\tau dx + \int_0^t \int_0^\tau \int c^2 u_{xx}(s,x) dx ds d\tau \\
&= \int \phi(x)dx + t\int \psi(x)d\tau dx + \int_0^t \int_0^\tau 0 \ ds d\tau \\
&= \int \phi(x)dx + t\int \psi(x)d\tau dx,
\end{aligned}$$

exactly as we wanted. $\square$

Problem 5. ◁

Solution. To solve this, we take the partial Fourier transform of

$$u_{tt} = u_{xx} - m^2 u$$

to get

$$\hat{u}_t t = -y^2 \hat{u} - m^2 \hat{u}$$

with conditions

$$\hat{u}(0, y) = \hat{g}(y),\ \ \hat{u}_t(0, y) = \hat{h}(y).$$

Solving this ODE, we have that

$$\hat{u} = \cos\sqrt{y^2 + m^2} y t \hat{g}(y) + \frac{\sin(\sqrt{y^2 + m^2} y t)}{\sqrt{y^2 + m^2}} \hat{h}(y).$$

This is the solution we were looking for. □

Problem 6. $\triangleleft$

Solution. The hint unlocks the secret to this problem. Suppose no such C, m exist. In particular, no constant $C = m = j \in \mathbb{N}$ works. Let $\phi_j \in \mathcal{S}(\mathbb{R}^n)$ be the counterexample that gives $\left|U(\phi_j)\right| > j||\phi_j||_j$. Now, let's define

$$\psi_j = j^{-1}||\phi_j||_j{}^{-1}\phi_j.$$

Since ϕ_j are Schwartz, for any multi-indices α, β, $x^\alpha \partial^\beta \phi_j$ are bounded. For any $\varepsilon > 0$, select $j > \max(\varepsilon^{-1}, |\alpha|, |\beta|)$. Then,

$$\begin{aligned}
\sup_x \left|x^\alpha \partial^\beta \psi_j\right| &= \sup_x \left|x^\alpha \partial^\beta j^{-1} \left(\sum_{|\mathsf{a}|,|\mathsf{b}|<j} \sup_y \left|y^{\mathsf{a}} \partial^{\mathsf{b}} \phi_j(y)\right|\right)^{-1} \phi_j(x)\right| \\
&= j^{-1} \frac{\sup_x \left|x^\alpha \partial^{\beta-1} \phi_j(x)\right|}{\sum_{|\mathsf{a}|,|\mathsf{b}|<j} \sup_y \left|y^{\mathsf{a}} \partial^{\mathsf{b}} \phi_j(y)\right|} \\
&\leq j^{-1} \\
&< \varepsilon.
\end{aligned}$$

Thus, by definition, $\psi_j \to 0$ as a Schwartz function.

However, we see that by linearity of U,

$$\begin{aligned}
\left|U(\psi_j)\right| &= \left|U(j^{-1}||\phi_j||_j{}^{-1}\phi_j)\right| \\
&= \left|U(\phi_j)\right| j^{-1}||\phi_j||_j{}^{-1} \\
&> 1
\end{aligned}$$

by our choice of ϕ_j. But $U(0) = 0$, so U is not continuous and we have a contradiction. Thus, the constants C and m must exist. $\square$

Problem 7. ◁

Solution.

(a) For this problem, we must go back to the definition of an integral through Riemann sums. First, notice that

$$\begin{aligned}
\int_{\mathbb{R}^n} g(y)\phi(y)dy &= \int_{\mathbb{R}^n} U(f(x)e^{-ix\cdot y})\phi(y)dy \\
&= \int_{\mathbb{R}^n} U(f(x)e^{-ix\cdot y}\phi(y))dy \\
&= \lim_{N\to\infty} \sum_{k=1}^{N} |I_k| U(f(x)e^{-ix\cdot y_k}\phi(y_k)) \\
&= \lim_{N\to\infty} U\left(\sum_{k=1}^{N} |I_k| f(x)e^{-ix\cdot y_k}\phi(y_k)\right)
\end{aligned}$$

where y_k is some point in the Riemann cube I_k and the limit notation denotes taking smaller and smaller intervals and covering more and more of space.

We want to be able to move the integral inside of U, but we can't do that directly. We can move the finite sum inside by linearity of U, and continuity gives us that we can move the limit inside as long as the inside converges as Schwartz functions. So, let us show that

$$\sum_{k=1}^{N} |I_k| f(x)e^{-ix\cdot y_k}\phi(y_k) \to \int f(x)e^{-ix\cdot y}\phi(y)dy$$

in the Schwartz sense. First, we know f has compact support, so we can assume x is bounded by some constant R. Let α, β be multi-indices. We need to check that

$$\sup_x \left| x^\alpha \partial^\beta \left(\sum_{k=1}^{N} |I_k| f(x)e^{-ix\cdot y_k}\phi(y_k) - \int f(x)e^{-ix\cdot y}\phi(y)dy\right)\right|$$

converges to 0 as $N \to \infty$. When we expand out the derivatives, each term in the massive sum will be of the form

$$x^\alpha \left(\sum_{k=1}^{N} |I_k| \partial^\gamma f(x)(-iy_k)^\omega e^{-ix\cdot y_k}\phi(y_k) - \int \partial^\gamma f(x)(-iy)^\omega e^{-ix\cdot y}\phi(y)dy\right)$$

where γ and ω are some multi-indices such that $|\gamma|, |\omega| \leq |\beta|$. Since ϕ is a Schwartz function, $(iy)^\omega\phi(y)$ is bounded. By definition of an integral, the expression inside the parentheses converges to 0 pointwise (for a set x). The expression is continuous and has compact support (because ∂^γ has compact support). We know that poinwise convergence of continuous functions on compact support implies uniform convergence. So, there is some sequence M_N such that $M_N \to 0$ and

$$\sum_{k=1}^{N} |I_k| \partial^\gamma f(x)(-iy_k)^\omega e^{-ix\cdot y_k}\phi(y_k) - \int \partial^\gamma f(x)(-iy)^\omega e^{-ix\cdot y}\phi(y)dy < M_N.$$

Thus we see that

$$\begin{aligned}
&\sup_x\left|x^\alpha\left(\sum_{k=1}^N |I_k|\partial^\gamma f(x)(-iy_k)^\omega e^{-ix\cdot y_k}\phi(y_k) - \int \partial^\gamma f(x)(-iy)^\omega e^{-ix\cdot y}\phi(y)dy\right)\right| \\
&= \sup_{x<R}\left|x^\alpha\left(\sum_{k=1}^N |I_k|\partial^\gamma f(x)(-iy_k)^\omega e^{-ix\cdot y_k}\phi(y_k) - \int \partial^\gamma f(x)(-iy)^\omega e^{-ix\cdot y}\phi(y)dy\right)\right| \\
&\leq \sup_{x<R}|x^\alpha M_N| \\
&\leq R^{|\alpha|}M_N
\end{aligned}$$

This certainly converges to 0. There are only a finite number of terms like this (because there are only so many options for what derivatives to take given that $|\gamma|, |\omega| < |\beta|$). So, we can conclude that

$$\sup_x\left|x^\alpha\partial^\beta\left(\sum_{k=1}^N |I_k| f(x)e^{-ix\cdot y_k}\phi(y_k) - \int f(x)e^{-ix\cdot y}\phi(y)dy\right)\right| \to 0.$$

This means that the Riemann sum does indeed converge to the integral as Schwartz functions. So, by continuity of U,

$$\begin{aligned}
\int_{\mathbb{R}^n} g(y)\phi(y)dy &= \lim_{N\to\infty} U\left(\sum_{k=1}^N |I_k| f(x)e^{-ix\cdot y_k}\phi(y_k)\right) \\
&= U\left(\lim_{N\to\infty}\sum_{k=1}^N |I_k| f(x)e^{-ix\cdot y_k}\phi(y_k)\right) \\
&= U\left(\int f(x)e^{-ix\cdot y}\phi(y)dy\right) \\
&= U(\mathcal{F}(\phi)) \\
&= \mathcal{F}(U)(\phi).
\end{aligned}$$

With this, we are done.

(b) Since U is linear and continuous,

$$U(\partial^\alpha f(x)e^{-ix\cdot y}) = \partial^\alpha U(f(x)e^{-ix\cdot y})$$

where the derivative is with respect to y as long as $\partial^\alpha f(x)e^{-ix\cdot y}$ is Schwartz in x.

We see that

$$\partial^\alpha f(x)e^{-ix\cdot y} = f(x)\partial^\alpha e^{ix\cdot y} = f(x)(ix)^\alpha e^{ix\cdot y}$$

is indeed Schwartz because f has compact support and is C^∞. Thus, we can conclude that $g \in C^\infty$.

To show the second part, we use problem 6. We know there exist C_0, m such that $|U(\phi)| \leq C_0||\phi||_m$ for all ϕ. This means that

$$\begin{aligned}
|g(y)| &= |U(f(x)e^{-ix\cdot y})| \\
&\leq C_0||f(x)e^{ix\cdot y}||_m \\
&= C_0\sum_{|\alpha|,|\beta|<m}\sup_x\left|x^\alpha\partial^\beta f(x)e^{ix\cdot y}\right|.
\end{aligned}$$

When we expand out the derivatives, each term in the sum inside the sup will be of the form

$$x^\alpha (iy)^\gamma e^{ix\cdot y} \partial^\omega f(x)$$

where $|\omega|, |\gamma| < m$ Since f has compact support, this is only nonzero for $x < R$ for some constant $R > 1$. Also, this means each derivative of f is bounded. Let F be a constant that bounds all derivatives of f up to m. Thus,

$$\begin{aligned} \left| x^\alpha (iy)^\gamma e^{ix\cdot y} \partial^\omega f(x) \right| &\leq R^m \left| (iy)^\gamma e^{ix\cdot y} \partial^\omega f(x) \right| \\ &\leq R^m \left| (iy)^\gamma \partial^\omega f(x) \right| \\ &\leq R^m F \left| (iy)^\gamma \right| \\ &\leq R^m F (|y| + 1)^m. \end{aligned}$$

Now, there is a finite number of terms in that sup. Call that number S. Then, by triangle inequality,

$$\sup_x \left| x^\alpha \partial^\beta f(x) e^{ix\cdot y} \right| \leq S R^m F (|y| + 1)^m.$$

There are also finitely many options for α, β. Call the number of options M. Then, we see that

$$|g(y)| \leq C_0 M S R^m F (|y| + 1)^m.$$

So, letting $C = C_0 M S R^m F$, we get the desired bound. □