

# MATH 173 PROBLEM SET 9

Stepan (Styopa) Zharkov

June 1, 2022

**Problem 1.**

◁

*Solution.*

**Problem 2.**

◁

***Solution.***

- (a) We need  $\int_0^1 |x^\alpha|^2 = \int_0^1 x^{2\alpha}$  to converge. This converges for  $\alpha > -1/2$  and diverges for  $\alpha \leq -1/2$ , so  $\phi_\alpha \in L^2((0, 1))$  for  $\alpha > -1/2$ .  $\square$
- (b) We need  $\phi_\alpha \in L^2((0, 1))$ , so  $\alpha > -1/2$ . But, since  $\phi_\alpha$  are smooth, we also need  $\int_0^1 |\phi'_\alpha|^2$  to converge. We see  $\phi'_\alpha = \alpha x^{\alpha-1}$ . and  $\int_0^1 |\alpha x^{\alpha-1}|^2 = |\alpha|^2 \int_0^1 x^{2(\alpha-1)}$  converges for  $\alpha > 1/2$  and diverges for  $\alpha \leq 1/2$ . So,  $\phi_\alpha \in H^1((0, 1))$  for  $\alpha > 1/2$ .  $\square$

**Problem 3.**

◁

**Solution.**

- (a) We know the statement is true for  $f \in C^1((a, b))$  by FTC. Now, let  $f_n \rightarrow f$  where  $f_n \in C^1((a, b))$ . By the continuity of the trace operator,

$$f(x) - f(y) = \lim_{n \rightarrow \infty} (f_n(x) - f_n(y)) = \lim_{n \rightarrow \infty} \int_x^y f'_n(t) dt$$

Now, since we are on a bounded interval, we can move the limit inside the derivative after applying dominated convergence to see that

$$\lim_{n \rightarrow \infty} \int_x^y f'_n(t) dt - \int_x^y f'(t) dt = \lim_{n \rightarrow \infty} \int_x^y (f_n(t) - f(t))' dt = 0$$

- (b) TODO

**Problem 4.**

◁

*Solution.*

**Problem 5.**

◁

**Solution.** Consider the dogbowl functions

$$f_n := 1 - \min(n \cdot d(x, \partial B), 1).$$



We see that  $Tf_n = 1$  for all  $n$ , so  $Tf_n \rightarrow 1 \neq 0$ . However,

$$\|f_n\|_{L^2}^2 = \int_B |f_n(x)|^2 dx = \int_{d(x, \partial B) < 1/n} |f_n(x)|^2 dx \leq \int_{d(x, \partial B) < 1/n} 1 dx = O(1/n) \rightarrow 0.$$

So,  $f_n \rightarrow 0$  in  $L^2$ .

□

**Problem 6.**

◁

**Solution.** Let  $u = \lim_{n \rightarrow \infty} u_n$  where  $u_n$  are compactly supported continuous functions. Note that we are given that  $u = \lim_{n \rightarrow \infty} -u_n(x^*)$ . This means that

$$u = \frac{\lim_{n \rightarrow \infty} u_n(x) + \lim_{n \rightarrow \infty} -u_n(x^*)}{2} = \lim_{n \rightarrow \infty} \frac{u_n(x) - u_n(x^*)}{2}.$$

Note that  $\frac{u_n(x) - u_n(x^*)}{2} = 0$  when  $x_n = 0$ , so

$$T_{B_+} \left( \frac{u_n(x) - u_n(x^*)}{2} \right) = \frac{u_n(x) - u_n(x^*)}{2} |_{\partial B_+} = 0.$$

We have shown in class that this is sufficient to say  $T_{B_+}(u|_{B_+}) = 0$ , so  $u|_{B_+} \in H_0^1(B_+)$ . □

**Problem 7.**

◁

**Solution.**

- (a) Let  $V_k = \{x : |x| < 1/k\}$  and let  $W_k = \{x : 1 - 1/k < |x| < 1\}$ . Then, consider lemonsqueezer functions  $f_k \in C_0^1(U)$  such that  $f \in C_0^1(U)$  and  $f_k|_{B(V_k \cap W_k)} = 1$  and  $f_k|_{V_{2k} \cup W_{2k}} = 0$ .

TODO: image

Note that  $u_k := uf_k \in C_0^1(U)$ . We claim  $u_k \rightarrow u$  in  $H^1(B)$ . Note that

$$\|u - u_k\|_{H^1}^2 = \int_B |u - u_k|^2 + \int_B |\nabla u - \nabla u_k|^2.$$

Now, since  $u$  is bounded

$$\begin{aligned} \int_B |u - u_k|^2 &= \int_B |u|^2 |1 - f_k|^2 \\ &= \int_B O(1) |1 - f_k|^2 \\ &= O(1) \int_B |1 - f_k|^2 \\ &= O(1) \int_{V_k \cup W_k} |1 - f_k|^2 \\ &= O(1) \int_{V_k \cup W_k} O(1) \\ &= O(1/k^n) = o(1) \end{aligned}$$

where the asymptotic notation is with respect to  $k$ . Also,

$$\int_B |\nabla u - \nabla u_k|^2 = \int_B |\nabla u - \nabla u f_k - u \nabla f_k|^2 \leq 2 \int_B |\nabla u|^2 |1 - f_k|^2 + 2 \int_B |u \nabla f_k|^2.$$

Note that since  $|\nabla u|$  is bounded, applying our above logic to the first part gives us

$$2 \int_B |\nabla u|^2 |1 - f_k|^2 = O(1) \int_B |1 - f_k|^2 = o(1).$$

So, we only need to deal with the second part. We see that since  $u$  is bounded and  $\nabla f_k$  is mostly 0,

$$\begin{aligned} 2 \int_B |u \nabla f_k|^2 &= 2 \int_B |u|^2 |\nabla f_k|^2 \\ &= O(1) \int_B |\nabla f_k|^2 \\ &= O(1) \int_{V_k \cup W_k} |\nabla f_k|^2 \\ &= O(1) \int_{V_k \cup W_k} |O(k)|^2 \\ &= O(1) O(1/k^n) O(k^2) \\ &= o(1) \end{aligned}$$

for  $n > 2$ . Thus, combining everything, we see that

$$\|u - u_k\|_{H^1}^2 = o(1),$$

which is what we needed to show that  $H_0^1(B) = H_0^1(U)$ . ◻

- (b) Consider  $u(x) = 1 - x^2 \in C^1((-1, 1))$ . Note that  $T_{(-1,1)}u = 0$ , so  $u \in H_0^1((-1, 1))$ . However,  $T_{(-1,0) \cup (0,1)}u \neq 0$  so  $u \notin H_0^1((-1, 0) \cup (0, 1))$ . Thus,

$$H_0^1((-1, 1)) \neq H_0^1((-1, 0) \cup (0, 1))$$

□

- (c) TODO