MATH 173 Problem Set 3

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Problem 1. TODO ◁

Solution. As the hint suggests, $u' = 0$ means by definition $u(\phi) = 0$ for any $\phi \in C_c^\infty(\mathbb{R})$. For any $\psi \in C_c^\infty(\mathbb{R})$, let $\phi_0 \in C_c^\infty(\mathbb{R})$ be a bump function such that $\int_{\mathbb{R}} \phi_0(x)dx = 1$. Let $\hat{\psi} = \psi - \phi_0 \int_{\mathbb{R}} \psi(x)dx$. We see that

$$\int_{\mathbb{R}} \hat{\psi}(x)dx = \int_{\mathbb{R}} \psi(x)dx - \int_{R} \psi(x)dx \cdot \int_{\mathbb{R}} \phi_0(x)dx = \int_{\mathbb{R}} \psi(x)dx - \int_{\mathbb{R}} \psi(x)dx = 0.$$

So, we can let

$$\phi(x) = \int_0^x \hat{\psi}(x)dx.$$

We see $\hat{\psi}$ has compact support and is in $C_c^\infty(\mathbb{R})$ (since it is the sum of two compact support functions). Since $\int_{\mathbb{R}} \hat{\psi}(x)dx = 0$, we know ϕ must have compact support as well and be in $C_c^\infty(\mathbb{R})$. Now, let $c = u(\phi_0)$. We see that by linearity of u,

$$u(\psi) = u(\hat{\psi}) + u(\phi_0) \cdot \int_{\mathbb{R}} \psi(x)dx = u(\phi') + c\int_{\mathbb{R}} \psi(x)dx = 0 + c\int_{\mathbb{R}} \psi(x)dx = c\int_{\mathbb{R}} \psi(x)dx,$$

which is exactly what we wanted to prove. □

Problem 2. TODO ◁

Solution. TODO □

Problem 3. TODO ◁

Solution. TODO □

Problem 4. TODO ◁

Solution. TODO □

Problem 5. TODO ◁

Solution. We assume $t \geq 0$, as always. First, lets' consider the case $c = 0$. Then, we have $u_{tt} = 0$, so $u_t(x, t)$ is constant along the line $\{(t, x_0)\}$ for any x_0. So, $u(t, x)$ has is a line of constant slope along any $\{(t, x_0)\}$. The starting point and the slope are defined by the initial conditions, so we see that

$$u(t, x) = \phi(x) + t\psi(x).$$

This vanishes when $\phi(0) + t\psi = 0$. Since ϕ, ψ are positive, ϕ must be 0, so $|x| \geq 1$. Any point where $|x| \geq 1$ works.

We can also check that u is linear (and thus C^∞) everywhere except for the "creases" of ϕ and ψ. In other words u is C^∞ when $x \neq -1, 0, 1$.

Now, let's look at the nonzero case. We know from the notes that the solution is

$$u(t, x) = \frac{1}{2}(\phi(x + ct) - \phi(x - ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\sigma) d\sigma.$$

Notice that we can assume $c > 0$ because $c < 0$ would also flip the integral, leading to the same solution.

tt □

Problem 6. TODO $\triangleleft$

Solution. As the hint suggests, consider $v(x_0, x_n, t) = u(x_0, x_n, t) - u(x_0, -x_n, t)$. Notice that

$$v_{tt} - c^2\Delta_x v = u_{tt}(x_0, x_n, t) - u_{tt}(x_0, -x_n, t) - c^2\Delta_x u(x_0, x_n, t) + c^2\Delta_x u(x_0, -x_n, t) = f(x_0, x_n, t) - f(x_0, -x_n, t) = 0.$$

We also see that

$$v(x, x_n, 0) = u(x_0, x_n, 0) - u(x_0, -x_n, 0) = \phi(x_0, x_n) - \phi(x_0, -x_n)$$

and

$$v_t(x, x_n, 0) = u_t(x_0, x_n, 0) - u_t(x_0, -x_n, 0) = \psi(x_0, x_n) - \psi(x_0, -x_n).$$

So, v solves the equation $v_{tt} - c^2\Delta_x v = 0$ with 0 initial conditions. We know that the only solution to this is $v = 0$. Thus, we see that $u(x_0, x_n, t) - u(x_0, -x_n, t) = v(x_0, x_n, t) = 0$, so

$$u(x_0, x_n, t) = u(x_0, -x_n, t),$$

and u is even with respect to x_n, exactly as we wanted. $\square$ $\square$

Problem 7. TODO $\triangleleft$

Solution. TODO $\square$