

# MATH 173 PROBLEM SET 7

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## Problem 1.

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**Solution.** This problem is straightforward. Since  $\overline{\Omega}$  is closed, and  $c > 0$ , there exists constants  $C_1, C_2 > 0$  such that  $C_1 < c(x) < C_2$ . Assume  $C_1 < 1$  and  $C_2 > 1$ . If not, we can always choose smaller  $C_1$  and larger  $C_2$ . So, for any  $u \in C^1(\overline{\Omega})$ ,

$$\begin{aligned} \|u\|_{H_c^1(\Omega)}^2 &= \int_{\Omega} |u(x)|^2 dx + \int_{\Omega} c(x) |\nabla u(x)|^2 dx \\ &\geq C_1^2 \int_{\Omega} |u(x)|^2 dx + \int_{\Omega} C_1^2 |\nabla u(x)|^2 dx \\ &= C_1^2 \|u\|_{H^1(\Omega)}^2. \end{aligned}$$

Similarly,

$$\begin{aligned} \|u\|_{H_c^1(\Omega)}^2 &= \int_{\Omega} |u(x)|^2 dx + \int_{\Omega} c(x) |\nabla u(x)|^2 dx \\ &\leq C_2^2 \int_{\Omega} |u(x)|^2 dx + \int_{\Omega} C_2^2 |\nabla u(x)|^2 dx \\ &= C_2^2 \|u\|_{H^1(\Omega)}^2. \end{aligned}$$

So, the statement follows for all  $u \in C^1(\overline{\Omega})$ . By continuity and density, it follows that the statement holds for  $u \in H^1(\Omega)$ .  $\square$

**Problem 2.**

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**Solution.**

- (a) Suppose  $v = u + w = u' + w'$  where  $u, u' \in M$ ,  $w, w' \in M^\perp$ . Then, we see

$$u' - u = u' + w - v = w - w'.$$

But  $u' - u \in M$  and  $w - w' \in M^\perp$  and  $M \cap M^\perp = \{0\}$ . So,  $u' - u = w - w' = 0$ . Thus, the decomposition is unique.

We see that  $u = u + 0$ . By uniqueness,  $P(v) = u = P(u) = P(P(v))$ , so  $P = P^2$ . □

- (b) Let  $v = u + w$  and  $v' = u' + w'$  with  $u, u' \in M$ ,  $w, w' \in M^\perp$ . We see

$$\langle Pv, v' \rangle = \langle u, u' + w' \rangle = \langle u, u' \rangle + \langle u, w' \rangle = \langle u, u' \rangle = \langle u, u' \rangle + \langle w, u' \rangle = \langle u + w, u' \rangle = \langle v, Pv' \rangle,$$

So  $P = P^*$  by definition. □

- (c) Since  $T$  is linear,  $T(H)$  is a subspace. For any sequence  $v_j$  in  $T(H)$  where  $v_j \rightarrow v$  in  $H$ , we know  $T(v_j) = v_j$  because  $T^2 = T$ . So, by continuity of  $T$ , we have  $T(v) = v$ , so  $v \in T(H)$  and we can conclude  $T(H)$  is closed.

Let  $u = T(v)$  and let  $w = v - u$ . Note  $u \in T(H)$ . For any  $y = T(x) \in T(H)$ , we have

$$\langle y, w \rangle = \langle T(x), v - u \rangle = \langle x, T(v - u) \rangle = \langle x, T(v) - T(u) \rangle = \langle x, u - u \rangle = 0$$

because  $T = T^2 = T^*$ . So,  $w \in T(H)^\perp$  and thus  $T$  is the projection. □