

MATH 173 PROBLEM SET 9

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Problem 1.

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Solution.

Problem 2.

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Solution.

- (a) We need $\int_0^1 |x^\alpha|^2 = \int_0^1 x^{2\alpha}$ to converge. This converges for $\alpha > -1/2$ and diverges for $\alpha \leq -1/2$, so $\phi_\alpha \in L^2((0, 1))$ for $\alpha > -1/2$. \square
- (b) We need $\phi_\alpha \in L^2((0, 1))$, so $\alpha > -1/2$. But, since ϕ_α are smooth, we also need $\int_0^1 |\phi'_\alpha|^2$ to converge. We see $\phi'_\alpha = \alpha x^{\alpha-1}$. and $\int_0^1 |\alpha x^{\alpha-1}|^2 = |\alpha|^2 \int_0^1 x^{2(\alpha-1)}$ converges for $\alpha > 1/2$ and diverges for $\alpha \leq 1/2$. So, $\phi_\alpha \in H^1((0, 1))$ for $\alpha > 1/2$. \square

Problem 3.

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Solution.

- (a) We know the statement is true for $f \in C^1((a, b))$ by FTC. Now, let $f_n \rightarrow f$ where $f_n \in C^1((a, b))$. By the continuity of the trace operator,

$$f(x) - f(y) = \lim_{n \rightarrow \infty} (f_n(x) - f_n(y)) = \lim_{n \rightarrow \infty} \int_x^y f'_n(t) dt$$

TODO: finish this.

- (b) TODO

Problem 4.

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Solution.

Problem 5.

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Solution. Consider the dogbowl functions

$$f_n := 1 - \max(n \cdot d(x, \partial B), 1).$$

TODO: image

We see that $Tf_n = 1$ for all $n \geq 4$, so $Tf_n \rightarrow 1 \neq 0$. However,

$$\|f_n\|_{L^2}^2 = \int_B |f_n(x)|^2 dx = \int_{d(x, \partial B) < 1/n} |f_n(x)|^2 dx \leq \int_{d(x, \partial B) < 1/n} 1 dx = O(1/n) \rightarrow 0.$$

So, $f_n \rightarrow 0$ in L^2 .

□

Problem 6.

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Solution. Let $u = \lim_{n \rightarrow \infty} u_n$ where u_n are compactly supported continuous functions. Note that we are given that $u = \lim_{n \rightarrow \infty} -u_n(x^*)$. This means that

$$u = \frac{\lim_{n \rightarrow \infty} u_n(x) + \lim_{n \rightarrow \infty} -u_n(x^*)}{2} = \lim_{n \rightarrow \infty} \frac{u_n(x) - u_n(x^*)}{2}.$$

Note that $\frac{u_n(x) - u_n(x^*)}{2} = 0$ when $x_n = 0$, so

$$T_{B_+} \left(\frac{u_n(x) - u_n(x^*)}{2} \right) = \frac{u_n(x) - u_n(x^*)}{2} |_{\partial B_+} = 0.$$

We have shown in class that this is sufficient to say $T_{B_+}(u|_{B_+}) = 0$, so $u|_{B_+} \in H_0^1(B_+)$. □

Problem 7.

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Solution.