MATH 173 Problem Set 1

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Problem 1. Let $f \in C^2(\mathbb{R}^3)$. Define $F = \nabla f$. Show that $\nabla \times F = 0$ and that $\nabla \cdot F = \Delta f$. $\triangleleft$

Solution. First, we see that

$$\nabla \times F = \nabla \times \nabla f = (\partial_2 \partial_3 f - \partial_3 \partial_2 f, \partial_3 \partial_1 f - \partial_3 \partial_1 f, \partial_1 \partial_2 f - \partial_1 \partial_2 f) = (0, 0, 0)$$

because order of differentiation does not matter. Also,

$$\nabla \cdot F = \nabla \cdot \nabla f = \partial_1 \partial_1 f + \partial_2 \partial_2 f + \partial_3 \partial_3 f = \Delta f,$$

as we wanted. $\square$

Problem 2. Consider the following first order linear equation with constant coefficients $\partial_1 u + \partial_2 u = 0$, where $u = u(x_1, x_2) : \mathbb{R}^2 \to \mathbb{R}$.

(a) Give an example of a non-zero solution with satisfies $u(x, x) = 0$ for all $x \in \mathbb{R}$.

(b) Show that if u solves the equation and satisfies $u(x, -x) = 0$ for all x, then $u = 0$.

(c) Describe all solutions $u \in C^1(\mathbb{R}^2)$.

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Solution.

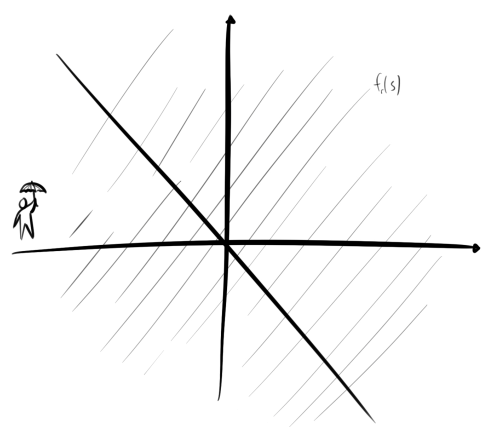
(a) Consider $u(x_1, x_2) = x_1 - x_2$. We see that $\partial_1 u + \partial_2 u = 1 - 1 = 0$ and $u(x, x) = 0$, but u is nonzero. □

(b) Suppose $u(\hat{x}_1, \hat{x}_2) \neq 0$ for some $\hat{x}_1, \hat{x}_2$. Consider the function $f(s) = u(\hat{x}_1 + s, \hat{x}_2 + s)$. We see that $f(0) \neq 0$ and $f((-\hat{x}_1 - \hat{x}_2)/2) = u((\hat{x}_1 - \hat{x}_2)/2, (-\hat{x}_1 + \hat{x}_2)/2) = 0$. By the mean value theorem, there is some point where $f' \neq 0$.

However, we see that $f'(s) = \partial_1 u(\hat{x}_1 + s, \hat{x}_2 + s) + \partial_2 u(\hat{x}_1 + s, \hat{x}_2 + s) = 0$. So, we have a contradiction and thus there is no such $\hat{x}_1, \hat{x}_2$ and $u = 0$. □

(c) Let $f_r(s) = u(r + s, -r + s)$. We $f'(s) = \partial_1 u(r + s, -r + s) + \partial_2 u(r + s, -r + s) = 0$, so f_r is constant. Thus, $u(r, -r)$ defines all of f_r. Note that any point (x_1, x_2) is expressed uniquely as $(r + s, -r + s)$, so the f_r cover the entire plane with no overlap.

In other words, Any solution can be described as $u(x_1, x_2) = g((x_1 - x_2)/2)$ where $g(r) : \mathbb{R} \to \mathbb{R}$ is a C^1 function. We also see that any choice of g gives a solution, so this characterizes all solutions. □



Problem 3. Let $p > 0$, the equation $\nabla \cdot (|\nabla u|^{p-2}\nabla u) = 0$ is called the p-Laplace equation.

(a) Rewrite the equation to show that it is quasilinear.

(b) Find the Euler-Lagrange equation for the functional $I(u) = \int_D F(x, u, \partial u)dx$, where $F(x, y, v) = |v|^p = \left(\sum_{j=1}^n v_j^2\right)^{p/2}$. Compare to the p-Laplace equation.

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Solution.

(a) This problem is solved by bashing. We see that

$$\begin{aligned}\nabla \cdot \left(|\nabla u|^{p-2}\, \nabla u\right) &= \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\, \frac{\partial u}{\partial x_i}\right)\\ &= \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i}|\nabla u|^{p-2}\right) + \sum_{i=1}^n |\nabla u|^{p-2}\, \frac{\partial^2 u}{\partial x_i^2}\\ &= \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i}|\nabla u|^{p-2}\right) + |\nabla u|^{p-2}\, \Delta u.\end{aligned}$$

Now, we can check that

$$\begin{aligned}\frac{\partial}{\partial x_j}|\nabla u|^{p-2} &= |\nabla u|^{p-3}\,(p-2)\frac{\partial}{\partial x_j}\left(\sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2\right)^{1/2}\\ &= |\nabla u|^{p-4}\,(p-2)\sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_i x_j}.\end{aligned}$$

Plugging this in to our first computation, we see that

$$\nabla \cdot \left(|\nabla u|^{p-2}\, \nabla u\right) = |\nabla u|^{p-4}\,(p-2)\sum_{j=1}^n\sum_{i=1}^n \left(\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_i x_j}\right) + |\nabla u|^{p-2}\, \Delta u.$$

We see that $|\nabla u|^{p-2}$ and $|\nabla u|^{p-4}$ are functions of the first partial derivatives of u, and Δu is a linear function of second derivatives, so this is indeed in the quasilinear form

$$\sum_{|\alpha|=2} a_\alpha(x, u, \partial u)(\partial^\alpha u)$$

and our expression is quasilinear. □

(b) This problem is also manipulations. First, note that for $F(x, y, v) = |v|^p$, we have $\partial_y F = 0$ and $\partial_{x_j} F = 0$. So.in the general form of the Euler-Lagrange equation, most terms are 0 and we are left with

$$\sum_{j=1}^n\sum_{k=1}^n \partial_{v_k}\partial_{v_j}|v|^p \partial_k\partial_j u = 0.$$

We can see that

$$\partial_{v_j}|v|^p = \partial_{v_j}\left(\sum_{i=1}^n v_i^2\right)^{p/2} = p v_j\left(\sum_{i=1}^n v_i^2\right)^{p/2-1}.$$

Continuing with the computation, we see that for $j \neq k$

$$\partial_{v_k}\partial_{v_j}|v|^p = v_k v_j p(p-2)\left(\sum_{i=1}^{n} v_i^2\right)^{p/2-2}$$

and for $j = k$,

$$\partial_{v_k}\partial_{v_j}|v|^p = v_k v_j p(p-2)\left(\sum_{i=1}^{n} v_i^2\right)^{p/2-2} + p\left(\sum_{i=1}^{n} v_i^2\right)^{p/2-1}.$$

Placing this in the original equation, we have

$$\begin{aligned}
0 &= \sum_{j=1}^{n}\sum_{k=1}^{n}\partial_{v_k}\partial_{v_j}|du|^p\partial_k\partial_j u \\
&= p\sum_{j,k=1}^{n}\left((\partial_k u)(\partial_j u)(p-2)\left(\sum_{i=1}^{n}(\partial_i u)^2\right)^{p/2-2}\partial_j\partial_k u\right) + p\sum_{j=1}^{n}\left(\left(\sum_{i=1}^{n}(\partial_i u)^2\right)^{p/2-1}\partial_j\partial_j u\right) \\
&= p\left((p-2)|\nabla u|^{p-4}\sum_{j,k=1}^{n}\left((\partial_j u)(\partial_k u)\partial_j\partial_k u\right) + |\nabla u|^{p-2}\sum_{j=1}^{n}(\partial_j\partial_j u)\right) \\
&= p\left(\nabla\cdot\left(|\nabla u|^{p-2}\, u\right)\right)
\end{aligned}$$

by part (a). Since $p > 0$, we can divide both sides by p to get

$$0 = \left(\nabla\cdot\left(|\nabla u|^{p-2}\, u\right)\right),$$

which is precisely the p-Laplace equation. $\square$

Problem 4. Solve the equation $x_1\partial_1 u + x_2\partial_2 u + (x_1 - 1)u = 0$ with the condition $u(x, e^x) = x$. In which region is u uniquely determined? $\triangleleft$

Solution. We can rewrite the equation as

$$x_1\partial_1 u + x_2\partial_2 u = (2 - x_1)u$$

First, let's find the characteristic curves with starts on the curve $\Gamma : x_2 = e^{x_1}$. Any characteristic curve f has $f_1'(s) = s$ and $f_2'(s) = s$ with $f_2(0) = e^{f_1(0)}$.

Solving this, we have $f_1(s) = re^s$ and $f_2(s) = e^r e^s$ for some r. Let $f_r(s) = (re^s, e^r e^s)$ be the characteristic curves, then. Since e^s can be any positive number and the vector (r, e^r) can point in any direction above the x_1-axis and above the line of slope e, we see that our characteristic curves cover the plane above these two lines.

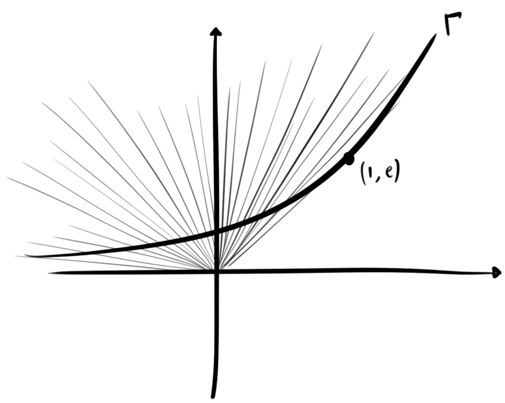
Let $y_r(s) = u(f_r(s))$. Since f_r are characteristic curves, we know $y_r'(s) = (2 - re^s)$ and $y_r(0) = re^0 = r$. Using our calculus methods, we have $dy/y = (2 - re^s)ds$, so $\ln y = 2s - re^s + c$. The initial condition gives us that

$$y_r(s) = e^{2s - re^s + r + \ln r} = re^{2s - re^s + r}.$$

Since we can express $x_1 = re^s$ and $x_2 = e^{r+s}$, we see that

$$u(x_1, x_2) = y_r(s) = re^{2s - re^s + r} = re^s \cdot e^{s+r} \cdot e^{-re^s} = x_1 x_2 e^{-x_1}$$

for all points (x_1, x_2) that are on some characteristic curve. We can confirm the solution by differentiating. So, we have found a solution u that is locally determined on the region above the x_1-axis and the line with slope e. $\square$



Problem 5.

(a) Solve the equation $x_1\partial_1 u + x_2\partial_2 u + x_1x_2\partial_3 u = 0$ with the condition $u(x1, x2, 0) = x_1^2 + x_2^2$.

(b) Compute $u(1, 1, 1)$ for the solution in part (a). Explain why $u(1, 1, 1)$ is negative while the initial condition is non-negative.

$\triangleleft$

Solution.

(a) This problem is solved using the same idea as problem 4, but with three variables. First, let's find the characteristic curves starting on the surface $\Gamma : x_3 = 0$. Any characteristic curve f has $f_1'(s) = f_1(s)$, $f_2'(s) = f_2(s)$, and $f_3'(s) = f_1(s)f_2(s)$ with $f_3(0) = 0$.

Solving this, we have $f_1(s) = ae^s$, $f_2(s) = be^s$, and $f_3(s) = \frac{1}{2}abe^{2s} - \frac{1}{2}ab$. So, let

$$f_{a,b}(s) = \left(ae^s, be^s, \frac{1}{2}abe^{2s} - \frac{1}{2}ab\right)$$

be the characteristic curves. Now, along the curve $f_{a,b}$, we can define $y_{a,b}(s) = u(f_{a,b}(s))$ and we know $y_{a,b}'(s) = 0$ as well as $y_{a,b}(0) = f_1(0)^2 + f_2(0)^2 = a^2 + b^2$. So, $y_{a,b}$ is the constant function with a value of $a^2 + b^2$.

We can write $x_1 = ae^s$, $x_2 = be^s$ and $x_3 = \frac{1}{2}abe^{2s} - \frac{1}{2}ab$. So,

$$s = \frac{1}{2}\ln\left(\frac{\frac{1}{2}x_1x_2}{\frac{1}{2}x_1x_2 - x_3}\right), a = \frac{x_1}{\sqrt{\frac{\frac{1}{2}x_1x_2}{\frac{1}{2}x_1x_2 - x_3}}}, b = \frac{x_2}{\sqrt{\frac{\frac{1}{2}x_1x_2}{\frac{1}{2}x_1x_2 - x_3}}}$$

for all x_1, x_2, x_3 such that

$$\frac{\frac{1}{2}x_1x_2}{\frac{1}{2}x_1x_2 - x_3} > 0.$$

Plugging this in, we have that

$$u(x_1, x_2, x_3) = y_{a,b}(s) = a^2 + b^2 = \frac{x_1^2 + x_2^2}{\frac{\frac{1}{2}x_1x_2}{\frac{1}{2}x_1x_2 - x_3}} = \frac{(x_1^2 + x_2^2)(x_1x_2 - 2x_3)}{x_1x_2}$$

for all such x_1, x_2, x_3. We can confirm the solution by differentiating. So, we have solved the equation on part of the space. $\square$

(b) We see that $u(1, 1, 1) = -2$ even though initial conditions are non-negative and u is constant along any characteristic curve. This is possible because the point $(1, 1, 1)$ is not within the boundary of where our solution works. No characteristic curve goes through $(1, 1, 1)$. We needed

$$\frac{\frac{1}{2}x_1x_2}{\frac{1}{2}x_1x_2 - x_3} > 0$$

and this isn't true for $(1, 1, 1)$. $\square$

Problem 6. Find general solution to the equation $x_1\partial_1 u + ... + x_n\partial_n u = cu$. $\triangleleft$

Solution. Let Γ be the unit sphere $\mathbb{S}^{n-1}$. Let u restricted to the unit sphere be $g : \mathbb{S}^{n-1} \to \mathbb{R}$. If f is a characteristic curve with a start on Γ, then $f_i'(s) = f_i(s)$, so $f_i(s) = a_i e^s$ and we can define

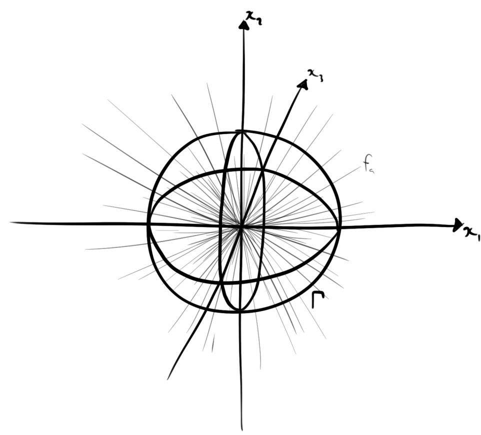
$$f_a(s) = (a_i e^s, \ldots, a_n e^s)$$

as the characteristic curve, where $a = (a_1, \ldots, a_n) \in \mathbb{S}^{n-1}$. Intuitively, these are rays pointing out of the origin. Let $y_a(s) = u(f_a(s))$. We see that $y_a'(s) = cy_a(s)$ and $y_a(0) = g(a)$, so $y_a(s) = g(a)e^{cs}$.

Note that we can place any nonzero $x \in \mathbb{R}^n$ on a characteristic curve uniquely with $a = \frac{x}{|x|}$ and $e^s = |x|$. So, we can write

$$u(x) = y_a(s) = g(a)e^{cs} = g\left(\frac{x}{|x|}\right)|x|^c$$

as the general solution defined everywhere but the origin, where g is some C^1 function on the sphere. $\square$



Problem 7.

(a) Solve $u_t + u_x = u^2, u(0, x) = e^{-x^2}$.

(b) Show that there is $T > 0$ such that u blows up at time T , i.e. u is continuously differentiable for $t \in [0, T)$, and x arbitrary, but for some $x_0, \lim_{t \to T^-} |u(x_0, t)| = \infty$. What is T ?

$\triangleleft$

Solution.

(a) This problem is solved similar to problem 4 and 5. First, let's find the characteristic curves starting on the surface $\Gamma : t = 0$. We see that if f is a characteristic curve, then $f_t'(s) = 1$, $f_x(s)' = 1$ and $f_t(0) = 0$.

Solving this, we have $f_t(s) = s$ and $f_x(s) = s + r$. So, we can let

$$f_r(s) = (s, s + r)$$

be the characteristic curves. We can define $y_r(s) = u(f_r(s))$ along the curves, and we know $y_r'(s) = y_r(s)^2$ with $y_r(0) = e^{-f_x(0)^2} = e^{-r^2}$.

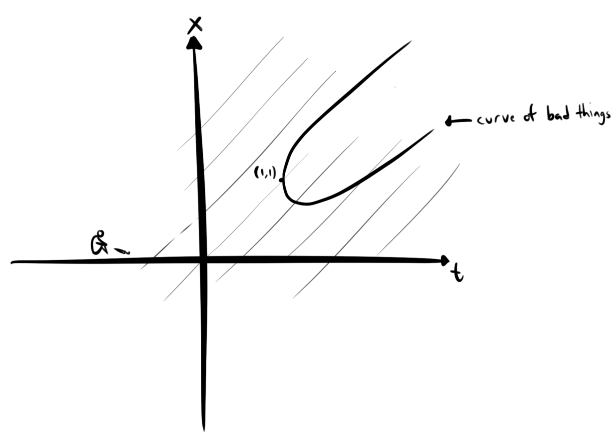
We can solve for y in the exact same way as Example 3.3 in the lecture notes to get that

$$y_r(s) = \frac{1}{e^{r^2} - s}$$

when $s \neq e^{r^2}$. Since we can write $s = t, r = x - t$, we see that

$$u(x, t) = \frac{1}{e^{(x-t)^2} - t}$$

when $e^{(x-t)^2} \neq t$. So, we have a solution that blows up on the curve $e^{(x-t)^2} = t$. $\square$



(b) The intuition for this problem is that we must pick a T such that the vertical line of $t = T$ just barely touches the blow up curve in the picture above.

Let $T = 1, x_0 = 1$. We see that for $t < T$, we have $e^{(x-t)^2} \geq 1 > t$, so $\frac{1}{e^{(x-t)^2} - t} = u(x, t)$ is continuously differentiable. However,

$$\lim_{t \to 1^-} \frac{1}{e^{(1-t)^2} - t} = \infty$$

because both $e^{(1-t)^2}$ and t approach 1 as $t \to 1^-$. So, we have found the desired point. $\square$