MATH 173 Problem Set 3

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Problem 1. Show that the only solution $u \in \mathcal{D}'(\mathbb{R})$ of $u' = 0$ is $u = c$, where c is a constant function. $\triangleleft$

Solution. As the hint suggests, $u' = 0$ means by definition $u(\phi) = 0$ for any $\phi \in C_c^\infty(\mathbb{R})$. For any $\psi \in C_c^\infty(\mathbb{R})$, let $\phi_0 \in C_c^\infty(\mathbb{R})$ be a bump function such that $\int_\mathbb{R} \phi_0(x)dx = 1$. Let $\hat{\psi} = \psi - \phi_0 \int_\mathbb{R} \psi(x)dx$. We see that

$$\int_\mathbb{R} \hat{\psi}(x)dx = \int_\mathbb{R} \psi(x)dx - \int_R \psi(x)dx \cdot \int_\mathbb{R} \phi_0(x)dx = \int_\mathbb{R} \psi(x)dx - \int_\mathbb{R} \psi(x)dx = 0.$$

So, we can let

$$\phi(x) = \int_{-\infty}^x \hat{\psi}(x)dx.$$

We see $\hat{\psi}$ has compact support and is in $C_c^\infty(\mathbb{R})$ (since it is the sum of two compact support functions). Since $\int_\mathbb{R} \hat{\psi}(x)dx = 0$, we know ϕ must have compact support as well and be in $C_c^\infty(\mathbb{R})$. Now, let $c = u(\phi_0)$. We see that by linearity of u,

$$u(\psi) = u(\hat{\psi}) + u(\phi_0) \cdot \int_\mathbb{R} \psi(x)dx = u(\phi') + c\int_\mathbb{R} \psi(x)dx = 0 + c\int_\mathbb{R} \psi(x)dx = c\int_\mathbb{R} \psi(x)dx,$$

which is exactly what we wanted to prove. $\square$

Problem 2. Let $f \in \mathcal{D}'(\mathbb{R})$, define a solution $u \in \mathcal{D}'(\mathbb{R}^2)$ such that $u_t + cu_x = 0$ and $u(t,x) = f(x-ct)$ in the sense of distributions. $\triangleleft$

Solution. First, let us define $f(x-ct)$ in a way that aligns with the case that f is a nice function. We see that if f were nice, then

$$\begin{aligned}
f(x-ct)(\phi) &= \int_{\mathbb{R}^2} f(x-ct)\phi(t,x)dxdt \\
&= \int_{\mathbb{R}^2} f(z)\phi(s, z+cs)dzds \\
&= \int_{\mathbb{R}^2} f(z)\phi(s, z+cs)dsdz \\
&= \int_{z\in\mathbb{R}} f(z)\int_{s\in\mathbb{R}} \phi(s, z+cs)dsdz \\
&= f\left(\int_{s\in\mathbb{R}} \phi(s, z+cs)ds\right).
\end{aligned}$$

So, we see that $f(x-ct)(\phi) = f(\Phi)$ where $\Phi = \int_{\mathbb{R}} \phi(s, z+cs)ds$. Note that $\Phi \in C_c^\infty(\mathbb{R})$ because the integral of a smooth function is smooth and ϕ is compactly supported. So, we can define $u = f(x-ct)$.

Now, we must show that u satisfies the PDE. This is done by simply writing out our definition and using the linearity of f. More precisely,

$$\begin{aligned}
(u_t + cu_x)(\phi) &= -f\left(\int_s \phi_t(s, z+cs)ds\right) - cf\left(\int_s \phi_x(s, z+cs)ds\right) \\
&= -f\left(\int_s \left[\phi_t(s, z+cs) + s\phi_x(s, z+cs)\right] ds\right) \\
&= -f\left(\int_s \partial_s\phi(s, z+cs)ds)\right) \\
&= -f(0) \\
&= 0.
\end{aligned}$$

Note that we used the fundamental theorem of calculus, that ϕ has compact support, and that f is linear in the above computation.

Thus, $u = f(x-ct)$ by definition and u solves the PDE in the sense of distribution, as we wanted to show. $\square$

Problem 3.

(a) Find the general C^2 solution of the PDE $u_{xx} - u_{xt} - 6u_{tt} = 0$ by reducing it to a system of first order PDEs.

(b) Show that if $f, g \in \mathcal{D}'(RR)$, and we define new distributions $v, w \in \mathcal{D}'(\mathbb{R}^2)$ similar to Problem 2, such that formally $v(x,t) = f(3x+t), w(x,t) = g(2x-t)$, then $u = v + w$ solves the PDE in part (a).

$\triangleleft$

Solution.

(a) This part follows directly from what we have done in class. We can factor

$$u_{xx} - u_{xt} - 6u_{tt} = (\partial_x - 3\partial_t)(\partial_t + 2\partial_t)u,$$

so the general C^2 solution is

$$u(x,t) = g(2x-t) + f(3x+t)$$

for some C^2 functions f, g. $\square$

(b) This problem is similar to problem 2, but here we should apply a trick instead of mindlessly taking derivatives. We will split up $(\partial_x - 3\partial_t)(\partial_t + 2\partial_t)$ and apply them to the parts of u in different orders. More precisely, we see that

$$\begin{aligned}(\partial_x - 3\partial_t)(\partial_t + 2\partial_t)u &= (\partial_x - 3\partial_t)(\partial_t + 2\partial_t)(g(2x-t) + f(3x+t)) \\ &= (\partial_t + 2\partial_t)(\partial_x - 3\partial_t)f(3x+t) + (\partial_x - 3\partial_t)(\partial_t + 2\partial_t)g(2x-t).\end{aligned}$$

By a similar derivation to that in problem 2, we can define

$$f(3x+t)(\phi) = f\left(\frac{1}{3}\int_s \phi(s,(z-s)/3)ds\right)$$

and

$$g(2x-t)(\phi) = g\left(\frac{1}{2}\int_s \phi(s,(z+s)/2)ds\right)$$

Now, we see that

$$\begin{aligned}(\partial_t + 2\partial_t)(\partial_x - 3\partial_t)f(3x+t)(\phi) &= (\partial_t + 2\partial_t)(\partial_x - 3\partial_t)f\left(\frac{1}{3}\int_s \phi(s,(z-s)/3)ds\right) \\ &= -(\partial_t + 2\partial_t)f\left(\int_s \left[\frac{1}{3}\phi_x(s,(z-s)/3) - \phi_t(s,(z-s)/3)\right]ds\right) \\ &= -(\partial_t + 2\partial_t)f\left(\int_s \partial_s\phi(s,(z-s)/3)ds\right) \\ &= -(\partial_t + 2\partial_t)f(0) \\ &= -(\partial_t + 2\partial_t)0 \\ &= 0,\end{aligned}$$

where we used the fundamental theorem of calculus along with the fact that ϕ must have compact support.

Similarly,

$$\begin{aligned}
(\partial_x - 3\partial_t)(\partial_t + 2\partial_t)g(2x-t)(\phi) &= (\partial_x - 3\partial_t)(\partial_t + 2\partial_t)g\left(\frac{1}{2}\int_s \phi(s,(z+s)/2)ds\right) \\
&= -(\partial_t - 3\partial_t)g\left(\int_s \left[\frac{1}{2}\phi_x(s,(z+s)/2) + \phi_t(s,(z-s)/2)\right] ds\right) \\
&= -(\partial_t - 3\partial_t)g\left(\int_s \partial_s\phi(s,(z+s)/2)ds\right) \\
&= -(\partial_t - 3\partial_t)f(0) \\
&= -(\partial_t - 3\partial_t)0 \\
&= 0,
\end{aligned}$$

where, again, we used the fundamental theorem of calculus along with the fact that ϕ must have compact support.

From this, we can conclude that

$$u_{xx} - u_{xt} - 6u_{tt} = (\partial_t + 2\partial_t)(\partial_x - 3\partial_t)f(3x+t) + (\partial_x - 3\partial_t)(\partial_t + 2\partial_t)g(2x-t) = 0 + 0 = 0.$$

So $v + w$ does indeed solve the PDE in part (a). $\square$

Problem 4. Solve the equation $u_{xx} + 3u_{xy} - 4u_{yy} = 0, u(x,x) = \sin x, u_x(x,x) = 0$. $\triangleleft$

Solution. This problem can be solved using the standard method for second order hyperbolic PDEs of two variables with constant coefficients that we saw in class.

First, we can factor

$$u_{xx} + 3u_{xy} - 4u_{yy} = (\partial_x - \partial_y)(\partial_x + 4\partial_y)u.$$

So, as we know, the general solution has the form

$$u(t,x) = f(4x - y) + g(x + y).$$

Now, we just need to use the initial conditions to find the specific f and g. We see that

$$f(3x) + g(2x) = \sin x$$

and that

$$4f'(3x) + g'(2x) = 0.$$

Taking the derivative of the former gives us that

$$3f'(3x) + 2g'(2x) = \cos x.$$

Now, we have a system of linear equations, so we can solve to see that

$$3f'(3x) = -\frac{3}{5}\cos x \text{ and } g'(2x) = \frac{8}{5}\cos x.$$

Since $f(3x) + g(2x) = \sin x$, the constant during integration must be the same, so we can assume it to vanish and

$$f(x) = -\frac{3}{5}\sin(x/3) \text{ and } g(x) = \frac{8}{5}\sin(x/2).$$

Plugging this back in, we see that

$$u(x,t) = -\frac{3}{5}\sin\left(\frac{4x - y}{3}\right) + \frac{8}{5}\sin\left(\frac{x + y}{2}\right),$$

which is the solution we were after. $\square$

Problem 5. Solve the wave equation $u_{tt} - c^2 u_{xx} = 0, u(x, 0) = \phi(x), u_t(x, 0) = \psi(x)$,

$$\text{with } \phi(x) = \begin{cases} 0, x < -1, \\ 1 + x, -1 < x < 0, \\ 1 - x, 0 < x < 1, \\ 0, x > 1, \end{cases} \qquad \text{and } \psi(x) = \begin{cases} 0, x < -1, \\ 2, -1 < x < 1, \\ 0, x > 1. \end{cases}$$

Also describe where the solution vanishes and where it is C^∞. You can assume $c = 1$ for this. ◁

Solution. First, lets' consider the case $c = 0$. Then, we have $u_{tt} = 0$, so $u_t(x, t)$ is constant along the line $\{(t, x_0)\}$ for any x_0. So, $u(t, x)$ has is a line of constant slope along any $\{(t, x_0)\}$. The starting point and the slope are defined by the initial conditions, so we see that

$$u(t, x) = \phi(x) + t\psi(x).$$

This vanishes when $\phi(0) + t\psi = 0$. Since ϕ, ψ are positive, ϕ must be 0, so $|x| \geq 1$. Any point where $|x| \geq 1$ works.

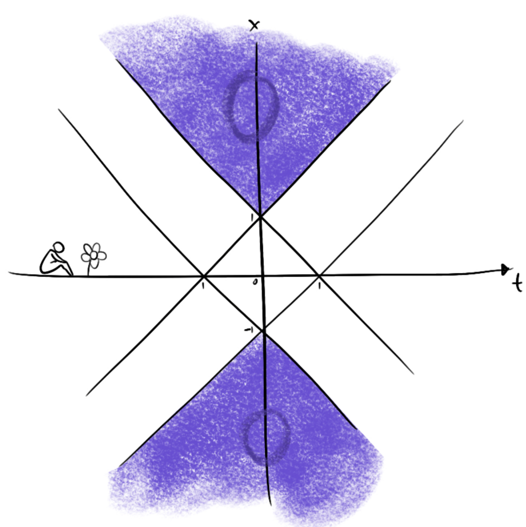
We can also check that u is linear (and thus C^∞) everywhere except for the places where of ϕ and ψ are not C^∞. In other words u is C^∞ when $x \neq -1, 0, 1$.

Now, let's look at the nonzero case. We know from the notes that the solution is

$$u(t, x) = \frac{1}{2}(\phi(x + ct) + \phi(x - ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\sigma) d\sigma.$$

Notice that we can assume $c > 0$ because $c < 0$ would also flip the integral, leading to the same solution.

We see that ϕ, ψ are non-negative, so this vanishes only when $\phi(x - ct) = 0, \phi(x + ct) = 0$, and $\psi = 0$ on the interval $(x - ct, x + ct)$. For the analysis of when u vanishes, we will only consider $c = 1$. We see that this is true when $[x - ct, x + ct] \cap (-1, 1) = \varnothing$. Instead of writing out the cases, consider the following picture of where this is true:



We see that u is C^∞ everywhere but the discontinuities of ϕ, ψ. In other words, u is C^∞ when $x + ct \neq -1, 0$, or 1. □

Problem 6. Consider the wave equation on $\mathbb{R}^n : u_{tt} - c^2 \nabla_x u = f, u(x,0) = \phi(x), u_t(x,0) = \psi(x)$, and write $x = (x_0, x_n)$ where $x_0 = (x_1, \ldots, x_{n-1})$. Show that if $f(x_0, x_n, t) = f(x_0, -x_n, t), \phi(x_0, -x_n) = \phi(x_0, x_n)$, and $\psi(x_0, -x_n) = \psi(x_0, x_n)$ for all x and t, then $u(x_0, x_n, t) = u(x_0, -x_n, t)$.

i.e. if f, ϕ and ψ, are all even functions of x_n, then u is an even function of x_n as well. ◁

Solution. As the hint suggests, consider $v(x_0, x_n, t) = u(x_0, x_n, t) - u(x_0, -x_n, t)$. Notice that

$$\begin{aligned} v_{tt} - c^2 \Delta_x v &= u_{tt}(x_0, x_n, t) - u_{tt}(x_0, -x_n, t) - c^2 \Delta_x u(x_0, x_n, t) + c^2 \Delta_x u(x_0, -x_n, t) \\ &= f(x_0, x_n, t) - f(x_0, -x_n, t) \\ &= 0. \end{aligned}$$

We also see that

$$v(x, x_n, 0) = u(x_0, x_n, 0) - u(x_0, -x_n, 0) = \phi(x_0, x_n) - \phi(x_0, -x_n) = 0$$

and

$$v_t(x, x_n, 0) = u_t(x_0, x_n, 0) - u_t(x_0, -x_n, 0) = \psi(x_0, x_n) - \psi(x_0, -x_n) = 0.$$

So, v solves the equation $v_{tt} - c^2 \Delta_x v = 0$ with 0 initial conditions. By the uniqueness in the notes, we know that the only solution to this is $v = 0$. Thus, we see that $u(x_0, x_n, t) - u(x_0, -x_n, t) = v(x_0, x_n, t) = 0$, so

$$u(x_0, x_n, t) = u(x_0, -x_n, t),$$

and u is even with respect to x_n, exactly as we wanted. □

Problem 7. In this problem we prove the finite speed of propagation for solutions of variable coefficient wave equation. Consider the PDE $u_{tt} - \nabla\cdot(c^2\nabla u) + qu = 0, u(x,0) = \phi(x), u_t(x,0) = \psi(x)$, where $c > 0$ and $q \geq 0$, depend on x only, and c is bounded between two positive constants, i.e., for some $c_1, c_2 > 0, c_1 \leq c(x) \leq c_2$ for all $x \in \mathbb{R}^n$. Assume that u is a C^2 solution. You can assume $n = 1$ for this problem.

(a) Fix $x_0 \in \mathbb{R}^n$ and $R_0 > 0$, and for $t < R_0/c_2$ let

$$E(t) = \int_{\Omega(t)} (u_t^2 + c(x)^2|\nabla u|^2 + q(x)u^2)dx$$

Show that E is non-increasing in t.

(b) Suppose that ϕ and ψ satisfy $\phi(x) = \psi(x) = 0$ when $|x| > R$. Show that $u(t,x) = 0$ when $|x| > R + c_2t$.

(c) Prove that there is at most one C^2 solution in $[0,t_0]\times\mathbb{R}^n$.

◁

Solution.

(a) For this problem, we will show that the derivative of E is never positive. Let $\Omega(t)$ be the region where $|x - x_0| < R_0 - c_t$. We see that the product rule gives us that

$$\begin{aligned} E'(t) &= \partial_t \int_{\Omega(t)} (u_t^2 + c(x)^2|\nabla u|^2 + q(x)u^2)dx \\ &= \int_{\Omega(t)} \partial_t(u_t^2 + c(x)^2|\nabla u|^2 + q(x)u^2)dx + (-c_2)\int_{\partial\Omega(t)} (u_t^2 + c(x)^2|\nabla u|^2 + q(x)u^2)dS(x). \end{aligned}$$

Applying integration by parts, as we did in class, we see that

$$\begin{aligned} \int_{\Omega(t)} \partial_t(u_t^2 + c(x)^2|\nabla u|^2 + q(x)u^2)dx &= \int_{\Omega(t)} (u_{tt} - \nabla\cdot(c(x)^2\nabla u) + q(x)u)u_t dx \\ &\quad + \int_{\partial\Omega(t)} u_t c(x)^2 \nabla_x u \cdot \hat{n} dS(x) \\ &= 0 + \int_{\partial\Omega(t)} u_t c(x)^2 \nabla_x u \cdot \hat{n} dS(x) \end{aligned}$$

where we used that u is a solution to our PDE in the last step. Now, we can start bounding this value. We see that

$$\begin{aligned} \int_{\Omega(t)} \partial_t(u_t^2 + c(x)^2|\nabla u|^2 + q(x)u^2)dx &= \int_{\partial\Omega(t)} u_t c(x)^2 \nabla_x u \cdot \hat{n} dS(x) \\ &\leq \int_{\partial\Omega(t)} |u_t| c(x)^2 |\nabla_x u||\hat{n}| dS(x) \\ &= \int_{\partial\Omega(t)} |u_t| c(x)^2 |\nabla_x u| dS(x) \\ &\leq c_2 \int_{\partial\Omega(t)} |u_t||c(x)||\nabla_x u| dS(x). \end{aligned}$$

In the last step, we used that $c(x) \leq c_2$. Now, note that for non-negative real numbers a, b, we have

$$ab \leq 2ab \leq a^2 + b^2.$$

With this fact, we see that

$$\begin{aligned}
\int_{\Omega(t)} \partial_t(u_t^2 + c(x)^2|\nabla u|^2 + q(x)u^2)dx &\le c_2 \int_{\partial\Omega(t)} |u_t||c(x)||\nabla_x u| dS(x) \\
&\le c_2 \int_{\partial\Omega(t)} (|u_t|^2 + |c(x)|^2|\nabla u|^2) dS(x) \\
&\le c_2 \int_{\partial\Omega(t)} (u_t^2 + c(x)^2|\nabla u|^2) dS(x) \\
&\le c_2 \int_{\partial\Omega(t)} (u_t^2 + c(x)^2|\nabla u|^2 + qu^2) dS(x)
\end{aligned}$$

by the non-negativity of q. Now, plugging this back into the expression for $E'(t)$, we see that

$$\begin{aligned}
E'(t) &= \int_{\Omega(t)} \partial_t(u_t^2 + c(x)^2|\nabla u|^2 + q(x)u^2)dx - c_2 \int_{\partial\Omega(t)} (u_t^2 + c(x)^2|\nabla u|^2 + q(x)u^2) dS(x) \\
&\le c_2 \int_{\partial\Omega(t)} (u_t^2 + c(x)^2|\nabla u|^2 + q(x)u^2) dS(x) - c_2 \int_{\partial\Omega(t)} (u_t^2 + c(x)^2|\nabla u|^2 + q(x)u^2) dS(x) \\
&= 0.
\end{aligned}$$

Thus, we can conclude that $E(t)$ is non-increasing in t. $\square$

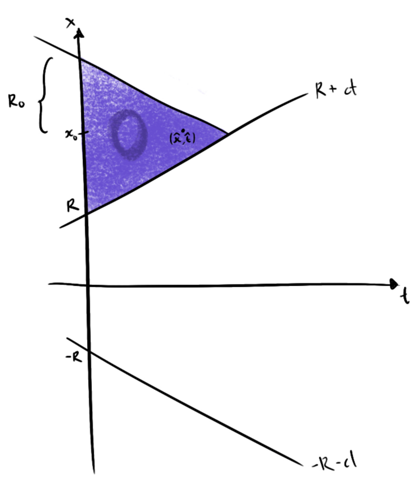
(b) For this problem, we will use part (a). We mimic the derivation in the notes.

Let $(\hat{x}, \hat{t})$ be a point such that $|\hat{x}| > R + c_2\hat{t}$. Let $x_0 = \hat{x}$ and let $R_0 = |x_0| - R$. We see that for $|x - x_0| < R_0$, we have $u(x, 0) = \phi(x) = 0$. Also, $u_t(x, 0) = \psi(x) = 0$. So, this means that $E(0) = 0$ with the definition we had in part (a).

Since $E(t)$ is non-increasing and non-negative, we know $E(t) = 0$ for any $t < R_0/c_2$. Since all the terms in $E(t)$ are non-negative, and we know $c(x) > 0$, we see $E(t) = 0$ implies that $u_t^2 = 0$ and $|\nabla u|^2 = 0$ where $|x_0 - x| < R_0 - c_2 t$. Thus, $u_t = \nabla u = 0$ on $|x - x_0| < R_0 - c_2 t$. This means that in fact $u = 0$ on $|x - x_0| < R_0 - c_2 t$.

In a sense, we have shown that u is 0 on a triangle outside of the region $|x| \le R + ct$. With our choice of x_0, R_0, this triangle covers the original arbitrary point.

More precisely, we see that $\hat{t} < (|\hat{x}| - R)/c_2$ and $|\hat{x} - x_0| < R$, so we can conclude that $u(\hat{x}, \hat{t})$. Since $(\hat{x}, \hat{t})$ was arbitrary, we have shown what we wanted. $\square$



(c) We have done this in class, but we can repeat the derivation. Let $t_0 < R_0/c_2$. Suppose u, u' are two solutions to the PDE in $[0, t_0] \times \mathbb{R}$. Note that the notation u' is not to be confused with a derivative of u. Consider $v = u - u'$. We see that

$$v_{tt} - \nabla \cdot (c^2 \nabla v) + qv = u_{tt} - \nabla \cdot (c^2 \nabla u) + qu - u'_{tt} + \nabla \cdot (c^2 \nabla u') - qu' = 0 - 0 = 0.$$

Moreover,

$$v(x, 0) = u(x, 0) - u'(x, 0) = \phi(x) - \phi(x) = 0$$

and

$$v_t(x, 0) = u_t(x, 0) - u'_t(x, 0) = \psi(x) - \psi(x) = 0.$$

So, we see that v satisfies the same equation with 0 initial conditions. If we define E as in part (a), but for v, we see that for any x_0, the initial conditions give us

$$E(0) = 0.$$

Since $E(t)$ is non-increasing and non-negative, we know that

$$E(t) = 0$$

for all $t \in [0, t_0]$. As in part (b), this means that v_x and v_t vanish where $|x - x_0| < R_0 - c_t$. So, v vanishes on this region as well. For any $(t, x) \in [0, t_0] \times \mathbb{R}$, we can select $x_0 = x$ and see that the point is in the region where $v = 0$. Thus, $v = 0$ on all of $[0, t_0] \times \mathbb{R}$.

So, we can conclude that $u = u'$, and the solution is unique. $\square$