# MATH 173 Problem Set 9

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June 1, 2022

**Problem 1.** $\triangleleft$

***Solution.*** Recall that we defined $G(x,y) = E(x-y) - \phi^x(y)$ where $\Delta\phi^x = 0$ and $\phi^x|_{\partial\Omega} = E(x-\cdot)$. So, if $Q(x,y) := G(x,y) - E(x-y) = -\phi^x(y)$, then $Q|_{\partial(\Omega\times\Omega)} = -E(x-y)$ and $\Delta Q = 0$. So, by the maximum principle, the minimum is achieved on the boundary. But the boundary is nonnegative, so $Q > 0$, so $G(x,y) \geq E(x-y)$. $\square$

**Problem 2.** $\triangleleft$

***Solution.***

(a) We need $\int_0^1 |x^\alpha|^2 = \int_0^1 x^{2\alpha}$ to converge. This converges for $\alpha > -1/2$ and diverges for $\alpha \leq -1/2$, so $\phi_\alpha \in L^2((0,1))$ for $\alpha > -1/2$. $\square$

(b) We need $\phi_\alpha \in L^2((0,1))$, so $\alpha > -1/2$. But, since $\phi_\alpha$ are smooth, we also need $\int_0^1 |\phi_\alpha'|^2$ to converge. We see $\phi_\alpha' = \alpha x^{\alpha-1}$. and $\int_0^1 |\alpha x^{\alpha-1}|^2 = |\alpha|^2 \int_0^1 x^{2(\alpha-1)}$ converges for $\alpha > 1/2$ and diverges for $\alpha \leq 1/2$. So, $\phi_\alpha \in H^1((0,1))$ for $\alpha > 1/2$. $\square$

**Problem 3.** ◁

***Solution.***

(a) We know the statement is true for $f \in C^1((a,b))$ by FTC. Now, let $f_n \to f$ where $f_n \in C^1((a,b))$. By the continuity of the trace operator,

$$f(y) - f(x) = \lim_{n\to\infty} (f_n(y) - f_n(x)) = \lim_{n\to\infty} \int_x^y f_n'(t)dt$$

Now, since we are on a bounded interval, we can move the limit inside the derivative after applying dominated convergence to see that

$$\lim_{n\to\infty} \int_x^y f_n'(t)dt - \int_x^y f_n'(t)dt = \lim_{n\to\infty} \int_x^y (f_n(t) - f(t))'dt = 0.$$

So,

$$f(y) - f(x) = \int_x^y f'(t)dt$$

as we wanted. □

(b) By Cauchy-Schwartz and part (a) and that $|f'|$ is bounded by $C$,

$$\begin{aligned} |f(y) - f(x)|^2 &= |\int_x^y f'(t)dt|^2 \\ &\leq \int_x^y |f'(t)|^2 dt \\ &\leq \int_x^y C^2 dt \\ &= |y - x|C^2. \end{aligned}$$

So, $|f(x) - f(y)| \leq C|x - y|^{1/2}$. □

**Problem 4.** ◁

***Solution.*** First, note that for $f \in C^1((a,b))$ where $f(a) = f(b) = 0$, and any $x \in (a,b)$, we have by Cauchy Schwartz,

$$\begin{aligned}
2||f||_2||f'||_2 &= 2\left(\int_a^b |f(t)|^2\right)^{1/2}\left(\int_a^b |f'(t)|^2\right)^{1/2} \\
&\geq 2\int_a^b |f(t)f'(t)|dt \\
&\geq 2\int_a^x |f(t)f'(t)|dt \\
&\geq \left|\int_a^x 2f(t)f'(t)dt\right| \\
&= \left|\int_a^x (f(t)^2)'dt\right| \\
&= |f(x)^2 - f(a)^2| \\
&= |f(x)^2|.
\end{aligned}$$

Since $x$ was arbitrary,

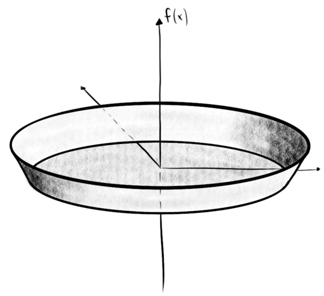
$$\sup_{(a,b)} |f(x)|^2 \leq 2||f||_2||f'||_2.$$

Now, problem 3(b) and the fact that $H_0^1((a,b)) \subseteq H^1((a,b)) \subseteq C([a,b])$ gives us the same inclusion argument as in chapter 9.1, leading us to conclude the statement for all $f \in H^1((a,b))$. □

**Problem 5.** $\triangleleft$

***Solution.*** Consider the dogbowl functions

$$f_n := 1 - \min(n \cdot d(x, \partial B), 1).$$




We see that $Tf_n = 1$ for all $n$, so $Tf_n \to 1 \neq 0$. However,

$$||f_n||_{L_2}^2 = \int_B |f_n(x)|^2 dx = \int_{d(x,\partial B)<1/n} |f_n(x)|^2 dx \leq \int_{d(x,\partial B)<1/n} 1 dx = O(1/n) \to 0.$$

So, $f_n \to 0$ in $L^2$. $\square$

**Problem 6.** ◁

***Solution.*** Let $u = \lim_{n\to\infty} u_n$ where $u_n$ are compactly supported continuous functions. Note that we are given that $u = \lim_{n\to\infty} -u_n(x^*)$. This means that

$$u = \frac{\lim_{n\to\infty} u_n(x) + \lim_{n\to\infty} - u_n(x^*)}{2} = \lim_{n\to\infty} \frac{u_n(x) - u_n(x^*)}{2}.$$

Note that $\frac{u_n(x)-u_n(x^*)}{2} = 0$ when $x_n = 0$, so

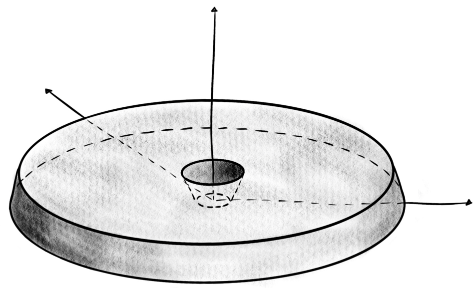
$$T_{B_+}\left(\frac{u_n(x) - u_n(x^*)}{2}\right) = \frac{u_n(x) - u_n(x^*)}{2}|_{\partial B_+} = 0.$$

We have shown in class that this is sufficient to say $T_{B_+}(u|_{B_+}) = 0$, so $u|_{B_+} \in H_0^1(B_+)$. □

**Problem 7.** ◁

***Solution.***

(a) Let $V_k = \{x : |x| < 1/k\}$ and let $W_k = \{x : 1 - 1/k < |x| < 1\}$. Then, consider lemonsqueezer functions $f_k \in$ such that $f \in C_0^1(U)$ and $f_k|_{B(V_k \cap W_k)} = 1$ and $f_k|_{V_{2k} \cup W_{2k}} = 0$ with $O(k)$ derivatives.



Note that $u_k := uf_k \in C_0^1(U)$. We claim $u_k \to u$ in $H^1(B)$. Note that

$$||u - u_k||_{H^1}^2 = \int_B |u - u_k|^2 + \int_B |\nabla u - \nabla u_k|^2.$$

Now, since $u$ is bounded

$$\begin{aligned}
\int_B |u - u_k|^2 &= \int_B |u|^2 |1 - f_k|^2 \\
&= \int_B O(1)|1 - f_k|^2 \\
&= O(1) \int_B |1 - f_k|^2 \\
&= O(1) \int_{V_k \cup W_k} |1 - f_k|^2 \\
&= O(1) \int_{V_k \cup W_k} O(1) \\
&= O(1/k^n) = o(1)
\end{aligned}$$

where the asymptotic notation is with respect to $k$. Also,

$$\int_B |\nabla u - \nabla u_k|^2 = \int_B |\nabla u - \nabla u f_k - u \nabla f_k|^2 \leq 2 \int_B |\nabla u|^2 |1 - f_k|^2 + 2 \int_B |u \nabla f_k|^2.$$

Note that since $|\nabla u|$ is bounded, applying our above logic to the first part gives us

$$2 \int_B |\nabla u|^2 |1 - f_k|^2 = O(1) \int_B |1 - f_k|^2 = o(1).$$

So, we only need to deal with the second part. We see that since $u$ is bounded and $\nabla f_k$ is mostly

0,

$$\begin{aligned}
2\int_B |u\nabla f_k|^2 &= 2\int_B |u|^2|\nabla f_k|^2 \\
&= O(1)\int_B |\nabla f_k|^2 \\
&= O(1)\int_{V_k\cup W_k} |\nabla f_k|^2 \\
&= O(1)\int_{V_k\cup W_k} |O(k)|^2 \\
&= O(1)O(1/k^n)O(k^2) \\
&= o(1)
\end{aligned}$$

for $n > 2$. Thus, combining everything, we see that

$$||u - u_k||_{H^1}^2 = o(1),$$

which is what we needed to show that $H_0^1(B) = H_0^1(U)$. □

(b) Consider $u(x) = 1 - x^2 \in C^1((-1,1))$. Note that $T_{(-1,1)}u = 0$, so $u \in H_0^1((-1,1))$. However, $T_{(-1,0)\cup(0,1)}u \neq 0$ so $u \notin H_0^1((-1,0)\cup(0,1))$. Thus,

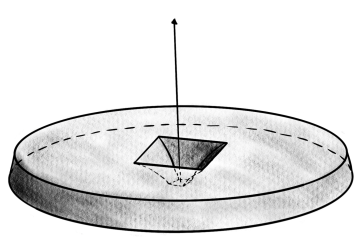
$$H_0^1((-1,1)) \neq H_0^1((-1,0)\cup(0,1))$$

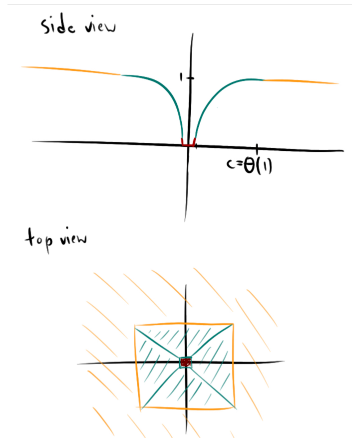
□

(c) Note that the logic in part (a) does not go through with $n = 2$ because we get $O(1)$ where we need $o(1)$. To fix this, it suffices to pick a different set of $f_k$ and it turns out that $H_0^1(B) = H_0^1(U)$ for two-dimensional balls. The problem is that our $f_k$ were interesting (nonflat) on a quickly shrinking region, so the derivative was forced to be large.

So, instead we will make the $f_k$ interesting on a nonshrinking region. Keeping it linear will not work because we need it to still approach 1 quickly. So, consider the following set of lemonsqueezer functions:

Let $c$ be a constant less than $1/\sqrt{2}$. Let $a_k$ be a quickly decreasing sequence. Now, let $f_k$ be constant 0 inside the square of sidelength $2a_k$ around the origin. Let $f_k$ be constant 1 outside the square of sidelength $2c$ but outside of $W_k$. Let the outer edge be smoothed out to 0, as in part (a). For the square in the middle, let each side follow the function $1-(1-r/c)^k$ where $r := \min(|x-a_k|, |y-a_k|)$. We can smooth out the edges on arbitrarily small areas later. So, our picture looks something like this:






Note that all of the steps in part (a) go through the same for $n = 2$ except for bounding

$$\int_B |u \nabla f_k|^2.$$

Since $u$ approaches 0, on the edges, we only have to worry about what happens in the center.

$$\int_{x,y:r<c-a_k} |u \nabla f_k|^2.$$

We see that since $u = O(1)$ and $|\nabla f_k| = O(k)(1-r)^{k-1}$, we can split the area into 8 triangles to see

$$\begin{aligned}
\int_{x,y:r<c-a_k} |u \nabla f_k|^2 &= O(1) \int_{x,y:r<c-a_k} |\nabla f_k|^2 \\
&= O(1) \int_{x,y:r<c-a_k} O(k)(1-r)^{k-1} \\
&= O(1) \int_0^c \int_0^x O(k^2)(1-r)^{2k-1} dydx \\
&= O(k) \int_0^c \int_0^x (1-r)^{2k-2} drdx \\
&= O(k) \int_0^c \int_0^x (1-r)^{\Theta(k)} drdx \\
&= O(k) \int_0^c \int_0^x (1-r)^{\Theta(k)} drdx \\
&= O(k)O(k^{-1}) \int_0^c (1-x)^{\Theta(k)} dx \\
&= O(k)O(k^{-1})O(k^{-1}) \\
&= o(1).
\end{aligned}$$

Thus, combining this with the rest of the reasoning from part (a), we have that

$$||u - u_k||_{H^1}^2 = o(1),$$

completing the proof. $\square$