MATH 173 Problem Set 1

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Problem 1. TODO $\triangleleft$

Solution. First, we see that

$$\nabla \times F = \nabla \times \nabla f = (\delta_2\delta_3 f - \delta_3\delta_2 f, \delta_3\delta_1 f - \delta_3\delta_1 f, \delta_1\delta_2 f - \delta_1\delta_2 f) = (0, 0, 0)$$

because order of differentiation does not matter. Also,

$$\nabla \cdot F = \nabla \cdot \nabla f = \delta_1\delta_1 f + \delta_2\delta_2 f + \delta_3\delta_3 f = \Delta f,$$

as we wanted. $\square$

Problem 2. TODO ◁

Solution.

(a) Consider $u(x_1, x_2) = x_1 - x_2$. We see that $\delta_1 u + \delta_2 u = 1 - 1 = 0$ and $u(x, x) = 0$, but u is nonzero. □

(b) Suppose $u(\hat{x}_1, \hat{x}_2) \neq 0$ for some $\hat{x}_1, \hat{x}_2$. Consider the function $f(s) = u(\hat{x}_1 + s, \hat{x}_2 + s)$. We see that $f(0) \neq 0$ and $f((-\hat{x}_1 - \hat{x}_2)/2) = u((\hat{x}_1 - \hat{x}_2)/2, (-\hat{x}_1 + \hat{x}_2)/2) = 0$. By the mean value theorem, there is some point where $f' \neq 0$.

However, we see that $f'(s) = \delta_1 u(\hat{x}_1 + s, \hat{x}_2 + s) + \delta_2 u(\hat{x}_1 + s, \hat{x}_2 + s) = 0$. So, we have a contradiction and thus there is no such $\hat{x}_1, \hat{x}_2$ and $u = 0$. □

(c) Let $f_r(s) = u(r + s, -r + s)$. We $f'(s) = \delta_1 u(r + s, -r + s) + \delta_2 u(r + s, -r + s) = 0$, so f_r is constant. Thus, $u(r, -r)$ defines all of f_r. Note that any point (x_1, x_2) is expressed uniquely as $(r + s, -r + s)$, so the f_r cover the entire plane with no overlap.

In other words, Any solution can be described as $u(x_1, x_2) = g((x_1 - x_2)/2)$ where $g(r) : \mathbb{R} \to \mathbb{R}$ is a C^1 function. We also see that any choice of g gives a solution, so this characterizes all solutions. □

Problem 3. TODO $\triangleleft$

Solution.

(a) TODO

(b) TODO

Problem 4. TODO ◁

Solution. We can rewrite the equation as

$$x_1\delta_1 u + x_2\delta_2 u = (2 - x_1)u$$

First, let's find the characteristic curves with starts on the curve $\Gamma : x_2 = e^{x_1}$. Any characteristic curve f has $f_1'(s) = s$ and $f_2'(s) = s$ with $f_2(0) = e^{f_1(0)}$.

Solving this, we have $f_1(s) = re^s$ and $f_2(s) = e^r e^s$ for some r. Let $f_r(s) = (re^s, e^r e^s)$ be the characteristic curves, then. Since e^s can be any positive number and the vector (r, e^r) can point in any direction above the x_1-axis and above the line of slope e, we see that our characteristic curves cover the plane above these two lines.

Let $y_r(s) = u(f_r(s))$. Since f_r are characteristic curves, we know $y_r'(s) = (2-re^s)$ and $y_r(0) = re^0 = r$. Using our calculus methods, we have $dy/y = (2 - re^s)ds$, so $\ln y = 22s - re^s + c$. The initial condition gives us that

$$y_r(s) = e^{2s-re^s+r+\ln r} = re^{2s-re^s+r}.$$

Since we can express $x_1 = re^s$ and $x_2 = e^{r+s}$, we see that

$$u(x_1, x_2) = y_r(s) = re^{2s-re^s+r} = re^s \cdot e^{s+r} \cdot e^{re^s} = x_1 x_2 e^{x_1}$$

for all points (x_1, x_2) that are on some characteristic curve. So, we have found a solution u that is uniquely determined on the region above the x_1-axis and the line with slope e. □

Problem 5. TODO ◁

Solution.

(a) This problem is solved using the same idea as problem 4, but with three variables.

(b) TODO

Problem 6. TODO $\triangleleft$

Solution. TODO

Problem 7. TODO $\triangleleft$

Solution.

(a) TODO

(b) TODO