

MATH 173 PROBLEM SET 4

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Problem 1. Suppose that $f : \mathbb{R} \rightarrow \mathbb{C}$ is continuous with $(1 + |x|)^N f(x)$ bounded for some $N > 1$ (or indeed simply that $f \in L^1(\mathbb{R})$) and $a \in \mathbb{R}$:

(a) Let $f_a(x) = f(x - a)$. Show that $\hat{f}_a(y) = e^{-ia y} \hat{f}(y)$.

(b) Let $g_a(x) = e^{ixa} f(x)$. Show that $\hat{g}_a(y) = \hat{f}(y - a)$.

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Solution.

(a) This problem is a simple computation. We see that, with a change of variables $z = x - a$, we have

$$\begin{aligned}\hat{f}_a(y) &= \mathcal{F}(f(x - a))(y) \\ &= \int e^{-ixy} f(x - a) dx \\ &= \int e^{-i(z+a)y} f(z) dz \\ &= e^{ia y} \int e^{-izy} f(z) dz \\ &= e^{ia y} \hat{f}(y),\end{aligned}$$

as we wanted. □

(b) This problem is even simpler computation. We see that

$$\begin{aligned}\hat{g}_a(y) &= \mathcal{F}(e^{ixa} f(x)) \\ &= \int e^{-ixy} e^{ixa} f(x) dx \\ &= \int e^{-ix(y-a)} f(x) dx \\ &= \hat{f}(y - a),\end{aligned}$$

as we wanted. □

Problem 2. Compute the Fourier transform of the following functions:

- (a) $\chi_{(-a,a)}$
- (b) $f(x) = e^{a|x|}, a > 0$
- (c) $g(x) = |x|^n e^{-a|x|}, a > 0, n \in \mathbb{N}$.

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Solution.

- (a) This problem is also computation.

$$\begin{aligned} (\mathcal{F}\chi_{(-a,a)})(y) &= \int_{-\infty}^{\infty} e^{-ixy} \chi_{(-a,a)}(x) dx \\ &= \int_{-a}^a e^{-ixy} dx \\ &= \begin{cases} -\frac{i}{y} (e^{-ia y} - e^{ia y}) & \text{if } y \neq 0 \\ 2a & \text{if } y = 0 \end{cases} \end{aligned}$$

□

- (b) Note that since $y \in \mathbb{R}$ and $a > 0$, we know $iy - a \neq 0$ and $iy + a \neq 0$ so we can divide by them. So,

$$\begin{aligned} (\mathcal{F}(e^{-a|x|}))(y) &= \int_{-\infty}^{\infty} e^{-ixy} e^{-a|x|} dx \\ &= \int_0^{\infty} e^{-x(iy+a)} dx + \int_{-\infty}^0 e^{-x(iy-a)} dx \\ &= \left[-\frac{1}{iy+a} e^{-x(iy+a)} \right]_0^{\infty} + \left[-\frac{1}{iy-a} e^{-x(iy-a)} \right]_{-\infty}^0 \\ &= \frac{1}{iy+a} - \frac{1}{iy-a} \end{aligned}$$

because $a > 0$.

□

- (c) In this problem, we will use repeated integration by parts. We see that executing integration by parts, we have

$$\begin{aligned} \mathcal{F}(|x|^n e^{-a|x|})(y) &= \int_{-\infty}^{\infty} |x|^n e^{-ixy-a|x|} dx \\ &= \int_{-\infty}^0 (-x)^n e^{-ixy+ax} dx + \int_0^{\infty} x^n e^{-ixy-ax} dx \\ &= \int_{-\infty}^0 (-x)^n e^{-ix(y-a)} dx + \int_0^{\infty} x^n e^{-ix(y+a)} dx \\ &= \frac{-n}{yi-a} \int_{-\infty}^0 (-x)^{n-1} e^{-ix(y-a)} dx + \frac{n}{yi+a} \int_0^{\infty} x^{n-1} e^{-ix(y+a)} dx. \end{aligned}$$

Note that the boundary terms vanish in the integration by parts. Repeating integration by parts n times, we see that

$$\begin{aligned} \mathcal{F}(|x|^n e^{-a|x|})(y) &= (-1)^n \frac{n!}{(yi-a)^n} \int_{-\infty}^0 e^{-ix(y-a)} dx + \frac{n!}{(yi+a)^n} \int_0^{\infty} e^{-ix(y+a)} dx \\ &= (-1)^{n+1} \frac{n!}{(yi-a)^{n+1}} + \frac{n!}{(yi+a)^{n+1}}, \end{aligned}$$

and $yi \pm a$ does not vanish because $a > 0$.

□

Problem 3.

- (a) Show that if f is even (or odd) then so is its Fourier transform.
- (b) Suppose that f is even and real valued. Show that \hat{f} is also real valued. What can you say about \hat{f} when f is odd and real valued.

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Solution.

- (a) The solution to this is straightforward. As Tadashi Tokieda would say, “follow your nose”. First, let $f(x) = f(-x)$. Then, letting $z = -x$ power a change of variables, we see that

$$\begin{aligned}
 \hat{f}(y) &= \int_{-\infty}^{\infty} e^{-ixy} f(x) dx \\
 &= \int_{\infty}^{-\infty} -e^{izy} f(-z) dz \\
 &= \int_{-\infty}^{\infty} e^{izy} f(-z) dz \\
 &= \int_{-\infty}^{\infty} e^{-iz(-y)} f(z) dz \\
 &= \hat{f}(-y).
 \end{aligned}$$

Similarly, now let $f(x) = -f(-x)$. Then,

$$\begin{aligned}
 \hat{f}(y) &= \int_{-\infty}^{\infty} e^{-ixy} f(x) dx \\
 &= \int_{\infty}^{-\infty} -e^{izy} f(-z) dz \\
 &= \int_{-\infty}^{\infty} e^{izy} f(-z) dz \\
 &= \int_{-\infty}^{\infty} -e^{-iz(-y)} f(z) dz \\
 &= -\hat{f}(-y).
 \end{aligned}$$

So, the fourier transform preserves evenness and oddness. □

- (b) Let f be even. Then,

$$\begin{aligned}
 \hat{f}(y) &= \int_{-\infty}^{\infty} e^{-ixy} f(x) dx \\
 &= \int_{-\infty}^0 e^{-ixy} f(x) dx + \int_0^{\infty} e^{-ixy} f(x) dx \\
 &= \int_0^{\infty} e^{ixy} f(-x) dx + \int_0^{\infty} e^{-ixy} f(x) dx \\
 &= \int_0^{\infty} e^{ixy} f(x) dx + \int_0^{\infty} e^{-ixy} f(x) dx \\
 &= \int_0^{\infty} (e^{ixy} + e^{-ixy}) f(x) dx
 \end{aligned}$$

This is real valued because $e^{ixy} + e^{-ixy} \in \mathbb{R}$ and $f(x) \in \mathbb{R}$.

The idea is the same when f is odd. By a similar computation,

$$\begin{aligned}\hat{f}(y) &= \int_{-\infty}^{\infty} e^{-ixy} f(x) dx \\ &= \int_{-\infty}^0 e^{-ixy} f(x) dx + \int_0^{\infty} e^{-ixy} f(x) dx \\ &= \int_0^{\infty} e^{ixy} f(-x) dx + \int_0^{\infty} e^{-ixy} f(x) dx \\ &= \int_0^{\infty} -e^{ixy} f(x) dx + \int_0^{\infty} e^{-ixy} f(x) dx \\ &= \int_0^{\infty} (-e^{ixy} + e^{-ixy}) f(x) dx\end{aligned}$$

We see that this integral has no real part because $-e^{ixy} + e^{-ixy}$ has no real part and $f(x) \in \mathbb{R}$. \square

Problem 4. Define the Hermite functions by $h_n = (x - dd_x)^n e^{-x^2/2}$, such that $h_0 = e^{-x^2/2}$ and $h_{n+1}(x) = xh_n(x) - h'_n(x)$. Show that $\hat{h}_n = (-i)^n \sqrt{2\pi} h_n$. \triangleleft

Solution. This proof follows by induction. From class we know that

$$\hat{h}_0(y) = \sqrt{2\pi} e^{-y^2/2} = \sqrt{2\pi} h_0(y).$$

So, the base case holds. Now, suppose the statement is true up to h_n . Then, we see that by linearity of \mathcal{F} and the properties we have seen in class,

$$\begin{aligned} \mathcal{F}(h_{n+1}) &= \mathcal{F}(xh_n) - \mathcal{F}(h'_n) \\ &= i(\mathcal{F}h_n)' - iy(\mathcal{F}h_n) \\ &= -i(y\sqrt{2\pi}(-i)^n h_n - (\sqrt{2\pi}(-i)^n h_n)') \\ &= (-i)^{n+1} \sqrt{2\pi} (yh_n - h'_n) \\ &= (-i)^{n+1} \sqrt{2\pi} h_{n+1}. \end{aligned}$$

So, by induction, the statement holds for all n . \square

Problem 5. Prove the Poisson summation formula

$$\sum_{n \in \mathbb{Z}} f(n) e^{-i\pi n} = \sum_{m \in \mathbb{Z}} \hat{f}(x + 2\pi m),$$

where f satisfies $|f(x)| + |\hat{f}(x)| \leq C(1 + |x|)^{-a}$ for some $a > 1$. \triangleleft

Solution. We will follow the hint. Let us compute the Fourier coefficients of the RHS. We get that with a change of variables,

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} \sum_{m \in \mathbb{Z}} \hat{f}(x + 2\pi m) dx \\ &= \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \int_{-\pi}^{\pi} e^{inx} \hat{f}(x + 2\pi m) dx \\ &= \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \int_{-\pi+2\pi m}^{\pi+2\pi m} e^{in(z-2\pi m)} \hat{f}(z) dz \\ &= \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \int_{-\pi+2\pi m}^{\pi+2\pi m} e^{inz} \hat{f}(z) dz \\ &= \frac{1}{2\pi} \int e^{inz} \hat{f}(z) dz \\ &= \mathcal{F}^{-1}(\hat{f})(n) \\ &= f(n). \end{aligned}$$

So, assuming that the Fourier series converges to the function,

$$\begin{aligned} \sum_{m \in \mathbb{Z}} \hat{f}(x + 2\pi m) &= \sum_{n \in \mathbb{Z}} c_n e^{-inx} \\ &= \sum_{n \in \mathbb{Z}} f(n) e^{-inx}, \end{aligned}$$

which is what we wanted to prove. \square

Problem 6. Prove the Whittaker–Shannon interpolation formula

$$f(x) = \sum_{n \in \mathbb{Z}} f(n) \text{sinc}(\pi(x - n))$$

provided $\text{supp}(\hat{f}) \subseteq [-\pi, \pi]$ and $|f(x)| \leq C(1 + |x|)^{-a}$. Here $\text{sinc}(x) = \sin(x)/x$. \triangleleft

Solution. Since $\text{supp}(\hat{f}) \subseteq [-\pi, \pi]$, we see that for $x \in (-\pi, \pi)$, by problem 5,

$$\sum_{n \in \mathbb{Z}} f(n) e^{-inx} = \sum_{m \in \mathbb{Z}} \hat{f}(x + 2\pi m) = \hat{f}(x).$$

Note that we saw that a finite number of points do not change the Fourier transform, so the points $-\pi, \pi$ are not a problem. Assuming that the inverse of the Fourier transform reverts the action of the Fourier transform, as in problem 5, we see that

$$\begin{aligned} f(y) &= \mathcal{F}^{-1}(\hat{f})(y) \\ &= \frac{1}{2\pi} \int e^{inx} \sum_{n \in \mathbb{Z}} f(n) e^{-ixy} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ixy} \sum_{n \in \mathbb{Z}} f(n) e^{-inx} dx \\ &= \sum_{n \in \mathbb{Z}} f(n) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ixy} e^{-inx} dx \\ &= \sum_{n \in \mathbb{Z}} f(n) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ix(y-n)} dx \\ &= \sum_{n \in \mathbb{Z}} f(n) \frac{1}{2\pi} \left[\frac{1}{i(y-n)} e^{ix(y-n)} \right]_{-\pi}^{\pi} dx \\ &= \sum_{n \in \mathbb{Z}} f(n) \left(\frac{e^{i\pi(y-n)} - e^{-i\pi(y-n)}}{2\pi i(y-n)} \right) dx \\ &= \sum_{n \in \mathbb{Z}} f(n) \left(\frac{\sin(\pi(y-n))}{\pi(y-n)} \right) dx \\ &= \sum_{n \in \mathbb{Z}} f(n) \text{sinc}(\pi(y-n)), \end{aligned}$$

exactly as we wanted. \square

Problem 7.

- (a) Use the Fourier transform to solve the following free Schrödinger equation $iut = -uxx$, with the initial condition $u(0, x) = g(x)$, where $u : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$ and $g \in \mathcal{S}(\mathbb{R})$ is a given Schwarz function. You may leave your solution as the inverse Fourier transform of a Schwartz function (you do not need to evaluate it explicitly).
- (b) Evaluate the solution in part (a) explicitly if $g(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, and show that

$$\int_{\mathbb{R}} |u(x, t)|^2 dx = 1$$

for all t .

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Solution. TODO