# MATH 173 Problem Set 7

Stepan (Styopa) Zharkov

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**Problem 1.** ◁

***Solution.*** This problem is straightforward. Since $\overline{\Omega}$ is closed, and $c > 0$, there exists constants $C_1, C_2 > 0$ such that $C_1 < c(x) < C_2$. Assume $C_1 < 1$ and $C_2 > 1$. If not, we can always choose smaller $C_1$ and larger $C_2$. So, for any $u \in C^1(\overline{\Omega})$,

$$\begin{aligned} ||u||^2_{H^1_c(\Omega)} &= \int_\Omega |u(x)|^2 dx + \int_\Omega c(x)|\nabla u(x)|^2 dx \\ &\geq C_1^2 \int_\Omega |u(x)|^2 dx + \int_\Omega C_1^2 |\nabla u(x)|^2 dx \\ &= C_1^2 ||u||^2_{H^1(\Omega)}. \end{aligned}$$

Similarly,

$$\begin{aligned} ||u||^2_{H^1_c(\Omega)} &= \int_\Omega |u(x)|^2 dx + \int_\Omega c(x)|\nabla u(x)|^2 dx \\ &\leq C_2^2 \int_\Omega |u(x)|^2 dx + \int_\Omega C_2^2 |\nabla u(x)|^2 dx \\ &= C_2^2 ||u||^2_{H^1(\Omega)}. \end{aligned}$$

So, the statement follows for all $u \in C^1(\overline{\Omega})$. By continuity and density, it follows that the statement holds for $u \in H^1(\Omega)$. □

**Problem 2.** $\triangleleft$

***Solution.***

(a) Suppose $v = u + w = u' + w'$ where $u, u' \in M$, $w, w' \in M^\perp$. Then, we see

$$u' - u = u' + w - v = w - w'.$$

But $u' - u \in M$ and $w - w' \in M^\perp$ and $M \cap M^\perp = \{0\}$. So, $u' - u = w - w' = 0$. Thus, the decomposition is unique.

We see that $u = u + 0$. By uniqueness, $P(v) = u = P(u) = P(P(v))$, so $P = P^2$. $\square$

(b) Let $v = u + w$ and $v' = u' + w'$ with $u, u' \in M$, $w, w' \in M^\perp$. We see

$$\langle Pv, v' \rangle = \langle u, l' + w' \rangle = \langle u, u' \rangle + \langle u, w' \rangle = \langle u, u' \rangle = \langle u, u' \rangle + \langle w, u' \rangle = \langle u + w, u' \rangle = \langle v, Pv' \rangle,$$

So $P = P^*$ by definition. $\square$

(c) Since $T$ is linear, $T(H)$ is a subspace. For any sequence $v_j$ in $T(H)$ where $v_j \to v$ in $H$, we know $T(v_j) = v_j$ because $T^2 = T$. So, by continuity of $T$, we have $T(v) = v$, so $v \in T(H)$ and we can conclude $T(H)$ is closed.

Let $u = T(v)$ and let $w = v - u$. Note $u \in T(H)$. For any $y = T(x) \in T(H)$, we have

$$\langle y, w \rangle = \langle T(x), v - u \rangle = \langle x, T(v - u) \rangle = \langle x, T(v) - T(u) \rangle = \langle x, u - u \rangle = 0$$

because $T = T^2 = T^*$. So, $w \in T(H)^\perp$ and thus $T$ is the projection. $\square$

**Problem 3.** $\triangleleft$

***Solution.*** In this problem, we will use $H^1$ to denote $H^1((0,1))$ to reduce clutter.

(a) Consider the operator $T : H^1 \to H^1$ where $Tf = f - \int_0^1 f$. It's easy to see that $T$ is linear since integration is linear. We see that for any $f$,

$$\begin{aligned}
||Tf||_{H^1} &= \int_0^1 \left| f - \int_0^1 f \right|^2 + \int_0^1 |\nabla f|^2 \\
&\le \int_0^1 \left| |f| - \left| \int_0^1 f \right| \right|^2 + \int_0^1 |\nabla f|^2 \\
&\le \int_0^1 \left( 2|f|^2 + 2\left| \int_0^1 f \right|^2 \right) + \int_0^1 |\nabla f|^2 \\
&= 4\int_0^1 |f|^2 + \int_0^1 |\nabla f|^2 \\
&\le 4\int_0^1 |f|^2 + 4\int_0^1 |\nabla f|^2 \\
&= 4||f||_{H^1}.
\end{aligned}$$

So, $T$ is bounded. It's easy to see that $T(f) \in M$ by definition, and if $f \in M$, then $T(f) = f$. So, $T = T^2$ and $T(H^1) = M$. Also,

$$\begin{aligned}
\langle Tf, g \rangle &= \langle f - \int_0^1 f, g \rangle \\
&= \int_0^1 \left( f\overline{g} + \nabla f \overline{\nabla g} \right) - \int_0^1 \left( \int_0^1 f \right) \overline{g} \\
&= \int_0^1 \left( f\overline{g} + \nabla f \overline{\nabla g} \right) - \left( \int_0^1 \overline{g} \right) \left( \int_0^1 f \right) \\
&= \int_0^1 \left( f\overline{g} + \nabla f \overline{\nabla g} \right) - \int_0^1 \left( \int_0^1 \overline{g} \right) f \\
&= \langle f, g - \int_0^1 g \rangle \\
&= \langle f, Tg \rangle.
\end{aligned}$$

So, $T = T^*$. Now, by problem 2c, $M = T(H)$ is closed.

Since $T$ is the projection onto $M$, we know $M^\perp = \ker T$. That is $f$ such that $f - \int_0^1 f = 0$. Thus, $f$ must be constant. We see that any constant function is in the kernel, so $M^\perp$ is the constant functions. $\square$

(b) First, let us derive a useful fact. Let $g$ be a function, $h, w$ be distributions. We know from homework that $w' = 0$ as distributions implies $w = c$ as functions. Now, we know $\left( \int_0^x \right)' = g$ by FTC, and the same is true as distributions. So, if $h' = g$ as distributions, then $\left( h - \int_0^x g \right)' = 0$ as distributions. Plugging in for $w$, we see $h = \int_0^x + C$. We will use this fact.

Let $g \in N^\perp$. We see that for any $f \in N$,

$$\int g'\overline{f}' + \int g\overline{f} = \langle g, f \rangle = 0,$$

som with $g$ ar a distribution,

$$g(\overline{f}) = \int g\overline{f} = -\int g'\overline{f}' = -g'(\overline{f}') = g''(\overline{f}).$$

Thus, $g = g''$ as distributions. By our useful fact,

$$g' = \int_0^x g + C$$

as functions, so $g'$ is continuous. Also $g''' = g'$ as distributions, so by our useful fact,

$$g'' = \int_0^x g' + C$$

as continuous functions. Thus, $g = g''$ as functions.

We see that for any $g$ such that $g = g''$, we have for any $f \in N$, integration by parts gives

$$\langle g, f \rangle = \int g\overline{f} + \int g'\overline{f}' = \int g\overline{f} - \int g''\overline{f} = 0.$$

So, $g \in N^\perp$. Thus, $N^\perp$ is all of the functions such that $g'' = g$. This is functions of the form

$$ae^x + be^{-x}.$$

We could have also done this by defining an operator and using problem 2c, but bounding that operator turned out to be painful. $\square$

**Problem 4.** ◁

***Solution.*** Let

$$J^*g = \int_0^x \int_0^t g(s)dsdt - x\int_0^1 \int_0^t g(s)dsdt.$$

It's easy to see that $J^*$ is linear and thi image is indeed in $H_0^1$ because it is differentiable and is 0 on the endpoints. Now, we show it is bounded. Note that

$$(J^*g)' = \int_0^x g(s)ds - \int_0^1 \int_0^t g(s)dsdt.$$

We see by Cauchy-Schwartz

$$\begin{aligned}
\int_0^1 \left| \int_0^x g(s)ds \right|^2 dx &\leq \int_0^1 \int_0^x \left|g(s)\right|^2 dsdx \\
&\leq \int_0^1 \int_0^1 \left|g(s)\right|^2 dsdx \\
&= \int_0^1 \left|g(s)\right|^2 ds \\
&= ||g||_{L^2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_0^1 \left| \int_0^1 \int_0^t g(s)ds \right|^2 dx &\leq \int_0^1 \int_0^1 \int_0^t \left|g(s)\right|^2 dsdx \\
&\leq \int_0^1 \int_0^1 \int_0^1 \left|g(s)\right|^2 dsdx \\
&= \int_0^1 \left|g(s)\right|^2 ds \\
&= ||g||_{L^2}.
\end{aligned}$$

So, by the triangle inequality on the $L^2$ norm,

$$||J^*g||_{H_0^1} = ||(J^*g)'||_{L^2} \leq 2||g||_{L^2}.$$

Thus, $J^*$ is bounded. The last thing to check is that $J^*$ is actually the adjoint. For that, we importantly note that $(J^*g)'' = -g$. Integrating by parts,

$$\begin{aligned}
\langle Jf, g\rangle_{L^2} &= \langle f, g\rangle_{L^2} \\
&= \int f\overline{g} \\
&= -\int f\overline{(J^*g)}'' \\
&= \int f'\overline{(J^*g)}' \\
&= \langle f, J^*g\rangle_{H_0^1}.
\end{aligned}$$

So, $J^*$ is the adjoint. □

**Problem 5.** $\triangleleft$

***Solution.***

(a) This problem is a computation. Since $F_\varepsilon$ is rotationally symmetric, we can use the given fact to see that

$$\Delta F_\varepsilon = h''(|x|) + (n-2)|x|^{-1}h'(x)$$

where

$$h(y) = c_n(y^2+\varepsilon)^{(2-n)/2}.$$

We compute that

$$h'(y) = yc_n(2-n)(y^2+\varepsilon^2)^{-n/2}.$$

Also,

$$h''(y) = c_n(2-n)(y^2+\varepsilon^2)^{-n/2} - y^2c_n(2-n)n(y^2+\varepsilon^2)^{-(n+2)/2}.$$

So, plugging this in, we have

$$\begin{aligned}
\Delta F_\varepsilon &= c_n(2-n)(|x|^2+\varepsilon^2)^{-n/2} - |x|^2c_n(2-n)n(|x|^2+\varepsilon^2)^{-(n-2)/2} - (n-1)(2-n)c_n(|x|^2+\varepsilon^2)^{-n/2}\\
&= \varepsilon^2c_n(2-n)(|x|^2+\varepsilon^2)^{-(n+2)/2}\\
&= -\varepsilon^{-n}c_n(n-2)(|x/\varepsilon|^2+1)^{-(n+2)/2}\\
&= \varepsilon^{-n}g(x/\varepsilon),
\end{aligned}$$

as we wanted. $\square$

(b) Since $F_\varepsilon$ is bounded by $F$, for any test function $\phi$, tho dominated convergence theorem gives us

$$\begin{aligned}
\lim_{\varepsilon\to 0}|(F-F_\varepsilon)(\phi)| &= \lim_{\varepsilon\to 0}\left|\int c_n\left(|x|^{2-n} - (|x|2+\varepsilon^2)^{(2-n)/2}\right)\phi(x)dx\right|\\
&= \left|\int c_n\left(\lim_{\varepsilon\to 0}\left(|x|^{2-n} - (|x|2+\varepsilon^2)^{(2-n)/2}\right)\right)\phi(x)dx\right|\\
&= \left|\int c_n\,(0)\,\phi(x)dx\right|\\
&= 0.
\end{aligned}$$

So, $F_\varepsilon \to F$ in the sense of distributions indeed.

Now we see that from part (a) and page 59 of chapter 5,

$$\begin{aligned}
\Delta F(\phi) &= F(\Delta\phi)\\
&= \lim_{\varepsilon\to 0} F_\varepsilon(\Delta\phi)\\
&= \lim_{\varepsilon\to 0}\int \varepsilon^{-n}g(x/\varepsilon)\phi(x)dx\\
&= \delta(\phi)\\
&= \phi(0).
\end{aligned}$$

$\square$

**Problem 6.** $\triangleleft$

***Solution.***

(a) From problem 5(b), and what we did in class, we see since $F$ is compactly supported,

$$\Delta u = \Delta F * f = \delta * f = f.$$

$\square$

(b) We have seen that the Fourier transform of $\delta$ is 1, but let us confirm this. We see that for any test function $\phi$,

$$\begin{aligned}
\mathcal{F}(\delta)(\phi) &= \delta(\mathcal{F}(\phi)) \\
&= \delta\left(\int \phi(x) e^{-ix\cdot y} dx\right) \\
&= \int \phi(x) dx.
\end{aligned}$$

So, by definition, $\mathcal{F}(\delta) = 1$.

Now, note that using our Fourier transform of derivative rules, we have

$$\mathcal{F}(\Delta F) = \sum_{j=1}^{n} (-iy_j)^2 \mathcal{F}(F) = -|y|^2 \mathcal{F}(F).$$

Since $\mathcal{F}(\Delta F) = \mathcal{F}(\delta) = 1$, we can say

$$\mathcal{F}(F) = -\frac{1}{|y|^2}.$$

$\square$