MATH 173 Problem Set 9

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Problem 1. ◁

Solution. Recall that we defined $G(x, y) = E(x - y) - \phi^x(y)$ where $\Delta\phi^x = 0$ and $\phi^x|_{\partial\Omega} = E(x - \cdot)$. So, if $Q(x, y) := G(x, y) - E(x - y) = -\phi^x(y)$, then $Q|_{\partial(\Omega\times\Omega)} = -E(x - y)$ and $\Delta Q = 0$. So, by the maximum principle, the minimum is achieved on the boundary. But the boundary is nonnegative, so $Q > 0$, so $G(x, y) \geq E(x - y)$. □

Problem 2. $\triangleleft$

Solution.

(a) We need $\int_0^1 |x^\alpha|^2 = \int_0^1 x^{2\alpha}$ to converge. This converges for $\alpha > -1/2$ and diverges for $\alpha \leq -1/2$, so $\phi_\alpha \in L^2((0,1))$ for $\alpha > -1/2$. $\square$

(b) We need $\phi_\alpha \in L^2((0,1))$, so $\alpha > -1/2$. But, since ϕ_α are smooth, we also need $\int_0^1 |\phi_\alpha'|^2$ to converge. We see $\phi_\alpha' = \alpha x^{\alpha-1}$. and $\int_0^1 |\alpha x^{\alpha-1}|^2 = |\alpha|^2 \int_0^1 x^{2(\alpha-1)}$ converges for $\alpha > 1/2$ and diverges for $\alpha \leq 1/2$. So, $\phi_\alpha \in H^1((0,1))$ for $\alpha > 1/2$. $\square$

Problem 3. ◁

Solution.

(a) We know the statement is true for $f \in C^1((a,b))$ by FTC. Now, let $f_n \to f$ where $f_n \in C^1((a,b))$. By the continuity of the trace operator,

$$f(y) - f(x) = \lim_{n\to\infty} (f_n(y) - f_n(x)) = \lim_{n\to\infty} \int_x^y f_n'(t)dt$$

Now, since we are on a bounded interval, we can move the limit inside the derivative after applying dominated convergence to see that

$$\lim_{n\to\infty} \int_x^y f_n'(t)dt - \int_x^y f_n'(t)dt = \lim_{n\to\infty} \int_x^y (f_n(t) - f(t))'dt = 0.$$

So,

$$f(y) - f(x) = \int_x^y f'(t)dt$$

as we wanted. □

(b) By Cauchy-Schwartz and part (a) and that $|f'|$ is bounded by C,

$$\begin{aligned} |f(y) - f(x)|^2 &= |\int_x^y f'(t)dt|^2 \\ &\le \int_x^y |f'(t)|^2 dt \\ &\le \int_x^y C^2 dt \\ &= |y - x|C^2. \end{aligned}$$

So, $|f(x) - f(y)| \le C|x - y|^{1/2}$. □

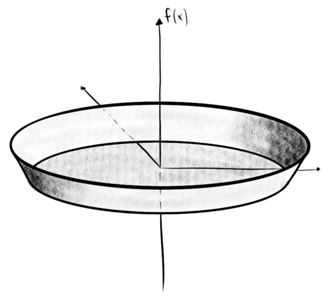
Problem 4. $\triangleleft$

Solution.

Problem 5. ◁

Solution. Consider the dogbowl functions

$$f_n := 1 - \min(n \cdot d(x, \partial B), 1).$$



We see that $Tf_n = 1$ for all n, so $Tf_n \to 1 \neq 0$. However,

$$||f_n||_{L_2}^2 = \int_B |f_n(x)|^2 dx = \int_{d(x,\partial B)<1/n} |f_n(x)|^2 dx \leq \int_{d(x,\partial B)<1/n} 1 dx = O(1/n) \to 0.$$

So, $f_n \to 0$ in L^2. □

Problem 6. ◁

Solution. Let $u = \lim_{n\to\infty} u_n$ where u_n are compactly supported continuous functions. Note that we are given that $u = \lim_{n\to\infty} -u_n(x^*)$. This means that

$$u = \frac{\lim_{n\to\infty} u_n(x) + \lim_{n\to\infty} -u_n(x^*)}{2} = \lim_{n\to\infty} \frac{u_n(x) - u_n(x^*)}{2}.$$

Note that $\frac{u_n(x)-u_n(x^*)}{2} = 0$ when $x_n = 0$, so

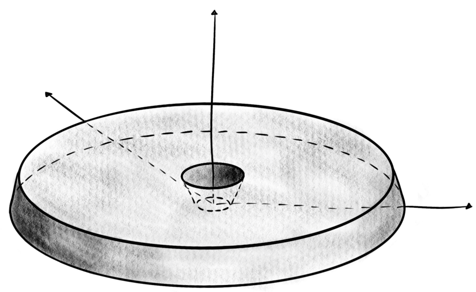
$$T_{B_+}\left(\frac{u_n(x) - u_n(x^*)}{2}\right) = \frac{u_n(x) - u_n(x^*)}{2}|_{\partial B_+} = 0.$$

We have shown in class that this is sufficient to say $T_{B_+}(u|_{B_+}) = 0$, so $u|_{B_+} \in H_0^1(B_+)$. □

Problem 7. ◁

Solution.

(a) Let $V_k = \{x : |x| < 1/k\}$ and let $W_k = \{x : 1 - 1/k < |x| < 1\}$. Then, consider lemonsqueezer functions $f_k \in$ such that $f \in C_0^1(U)$ and $f_k|_{B(V_k \cap W_k)} = 1$ and $f_k|_{V_{2k} \cup W_{2k}} = 0$ with $O(k)$ derivatives.



Note that $u_k := u f_k \in C_0^1(U)$. We claim $u_k \to u$ in $H^1(B)$. Note that

$$||u - u_k||_{H^1}^2 = \int_B |u - u_k|^2 + \int_B |\nabla u - \nabla u_k|^2.$$

Now, since u is bounded

$$\begin{aligned}
\int_B |u - u_k|^2 &= \int_B |u|^2 |1 - f_k|^2 \\
&= \int_B O(1)|1 - f_k|^2 \\
&= O(1) \int_B |1 - f_k|^2 \\
&= O(1) \int_{V_k \cup W_k} |1 - f_k|^2 \\
&= O(1) \int_{V_k \cup W_k} O(1) \\
&= O(1/k^n) = o(1)
\end{aligned}$$

where the asymptotic notation is with respect to k. Also,

$$\int_B |\nabla u - \nabla u_k|^2 = \int_B |\nabla u - \nabla u f_k - u \nabla f_k|^2 \leq 2 \int_B |\nabla u|^2 |1 - f_k|^2 + 2 \int_B |u \nabla f_k|^2.$$

Note that since $|\nabla u|$ is bounded, applying our above logic to the first part gives us

$$2 \int_B |\nabla u|^2 |1 - f_k|^2 = O(1) \int_B |1 - f_k|^2 = o(1).$$

So, we only need to deal with the second part. We see that since u is bounded and ∇f_k is mostly

0,

$$\begin{aligned}
2\int_B |u\nabla f_k|^2 &= 2\int_B |u|^2|\nabla f_k|^2 \\
&= O(1)\int_B |\nabla f_k|^2 \\
&= O(1)\int_{V_k\cup W_k} |\nabla f_k|^2 \\
&= O(1)\int_{V_k\cup W_k} |O(k)|^2 \\
&= O(1)O(1/k^n)O(k^2) \\
&= o(1)
\end{aligned}$$

for $n > 2$. Thus, combining everything, we see that

$$||u - u_k||_{H^1}^2 = o(1),$$

which is what we needed to show that $H_0^1(B) = H_0^1(U)$. $\square$

(b) Consider $u(x) = 1 - x^2 \in C^1((-1,1))$. Note that $T_{(-1,1)}u = 0$, so $u \in H_0^1((-1,1))$. However, $T_{(-1,0)\cup(0,1)}u \neq 0$ so $u \notin H_0^1((-1,0)\cup(0,1))$. Thus,

$$H_0^1((-1,1)) \neq H_0^1((-1,0)\cup(0,1))$$

$\square$

(c) TODO