MATH 173 Problem Set 7

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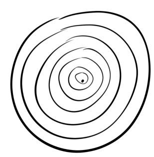
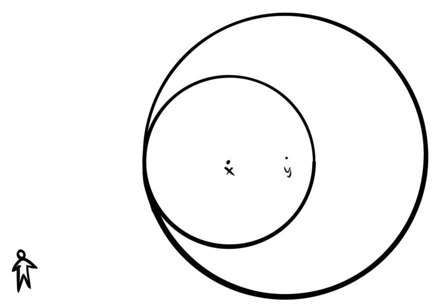
Problem 1. ◁

Solution. For a harmonic function, we know the average over the surface of any ball is equal to the value at the center. We also know that the average over the inside of the ball is equal to the average in the center because the average on the inside is simply an integral of the averages over many concentric balls.



Let x and y be two arbitrary points. Now, consider $B_x := B(x, R)$ and $B_y := B(y, R+d)$ where $d = |x-y|$, as the hint suggests. Then, we see that $B_x \subseteq B_y$.



Let $\text{Avg}(\Omega)$ denote the average of u over the region Ω. Let $|\Omega|$ denote the area of Ω. We see that

$$\begin{aligned}\text{Avg}(B_y) &= \text{Avg}(B_x)\frac{|B_x|}{|B_y|} + \text{Avg}(B_y \setminus B_x)\frac{|B_y \setminus B_x|}{|B_y|}\\ &\geq \text{Avg}(B_x)\frac{|B_x|}{|B_y|}.\end{aligned}$$

In the second step, we used that u is non-negative. Note that $\lim_{R\to\infty} \frac{|B_x|}{|B_y|} = 1$, so for any ε, we can find R large enough that

$$u(y) = \text{Avg}(B_y) \geq \text{Avg}(B_x)\frac{|B_x|}{|B_y|} \geq \text{Avg}(B_x)(1-\varepsilon) = u(x)(1-\varepsilon).$$

Thus, we can conclude that $u(y) \geq u(x)$. By an identical argument, we also know $u(y) \leq u(x)$. Since x and y were arbitrary, u must be the constant function. □

Problem 2. ◁

Solution.

(a) By the linearity properties of $\langle \cdot, \cdot \rangle$, we know

$$\begin{aligned} ||v+w||^2 + ||v-w||^2 &= \langle v+w, v+w\rangle + \langle v-w, v-w\rangle \\ &= \langle v,v\rangle + \langle v,w\rangle + \langle w,v\rangle + \langle w,w\rangle + \langle v,v\rangle - \langle v,w\rangle - \langle w,v\rangle + \langle w,w\rangle \\ &= 2\langle v,v\rangle + 2\langle w,w\rangle \\ &= 2(||v||^2 + ||w||^2) \end{aligned}$$

for any v, w. □

(b) Let $\Omega_1, \Omega_2 \subseteq \Omega$ be nonempty oben balls where $\Omega_1 \cap \Omega_2 = \varnothing$. Let f be a bump function such that $\operatorname{supp} f \subseteq \Omega_1$ and $f \geq 0$ and f achieves its maximum of 1 in Ω_1. Similarly, let g be a bump function such that $\operatorname{supp} g \subseteq \Omega_2$ and $g \geq 0$ and g achieves its maximum of 1 in Ω_2.

We see that

$$||f-g||^2 + ||f+g||^2 = 1 + 1 \neq 2(1+1) = 2(||f||^2 + ||g||^2).$$

So the parallelogram law does not hold. By part (a), this means this norm cannot be induced by an inner product. □

Problem 3. $\triangleleft$

Solution. Let $\langle f, g\rangle = \int_{-\pi}^{\pi} f\bar{g}$. We know from class that this is an inner product and we see that it induces the given norm. We saw that the functions $1, \cos(x), \sin(x), \cos(2x), \sin(2x)$ are orthogonal. So, by the best approximation lemma, we can take the inner products to find the coefficients. This gives

$$\begin{aligned}
a_0 &= \frac{\langle |x|, 1\rangle}{\langle 1, 1\rangle} = \frac{\pi}{2} \\
a_1 &= \frac{\langle |x|, \cos x\rangle}{\langle \cos x, \cos x\rangle} = \frac{2\int_0^\pi x\cos x dx}{\int_{-\pi}^{\pi}\cos^2 x dx} = -\frac{4}{\pi} \\
a_2 &= \frac{\langle |x|, \cos 2x\rangle}{\langle \cos 2x, \cos 2x\rangle} = \frac{2\int_0^\pi x\cos 2x dx}{\int_{-\pi}^{\pi}\cos^2 2x dx} = 0 \\
b_1 &= \frac{\langle |x|, \sin x\rangle}{\langle \sin x, \sin x\rangle} = 0 \\
b_2 &= \frac{\langle |x|, \sin 2x\rangle}{\langle \sin 2x, \sin 2x\rangle} = 0.
\end{aligned}$$

Note that the last two are 0 because $|x|$ is an even function and $sin(kx)$ is odd, so they are orthogonal. Thus, we have found the coefficients. $\square$

Problem 4. $\triangleleft$

Solution.

(a) Let $v \in \mathcal{D}$ where $v \neq 0$ be an eigenvector with an eigenvalue λ. So, $Av = \lambda v$. Then, we see

$$\lambda\langle v, v\rangle = \langle \lambda v, v\rangle = \langle Av, v\rangle \geq 0.$$

Since $\langle v, v\rangle > 0$ we can divide both sides by it to see that $\lambda \geq 0$. Thus, all eigenvalues are non-negative. $\square$

(b) We see that the boundary conditions give us a vanishing boundary term as

$$\begin{aligned}\langle Af, f\rangle &= \int_0^\ell -f''\bar{f} \\ &= \int_0^\ell f'\bar{f}' \\ &\geq 0.\end{aligned}$$

So, A is positive.

Now, we will find its eigenvalues. We know by part (a) that eigenvalues are non-negative. Constant functions give the 0 eigenvalues, so we can consider eigenvalues $\lambda^2 > 0$. Then, we have $-f'' = \lambda^2 f$. The general solution to this is

$$f(x) = a\sin(\lambda x) + b\cos(\lambda x).$$

As we've seen in class, the boundary conditions give us that f is of the form

$$f(x) = a\sin\left(\frac{2\pi}{\ell}nx\right) + b\cos\left(\frac{2\pi}{\ell}nx\right).$$

Thus the eigenvalues are

$$\left\{\frac{4\pi^2}{\ell^2}n^2 : n \in \mathbb{Z}\right\}.$$

$\square$

Problem 5. $\triangleleft$

Solution.

(a) We see that the boundary conditions give us

$$\begin{aligned}
\langle Af, g\rangle &= \int_0^\ell f^{(4)}\bar{g} \\
&= \left[f^{(3)}\bar{g}\right]_0^\ell - \int_0^\ell f^{(3)}\bar{g}' \\
&= -\int_0^\ell f^{(3)}\bar{g}' \\
&= \left[f''\bar{g}'\right]_0^\ell + \int_0^\ell f''\bar{g}'' \\
&= \int_0^\ell f''\bar{g}'' \\
&= \left[f'\bar{g}''\right]_0^\ell - \int_0^\ell f'\bar{g}^{(3)} \\
&= -\int_0^\ell f'\bar{g}^{(3)} \\
&= \left[f\bar{g}^{(3)}\right]_0^\ell + \int_0^\ell f\bar{g}^{(4)} \\
&= \int_0^\ell f\bar{g}^{(4)} \\
&= \langle f, Ag\rangle
\end{aligned}$$

So, A is symmetric. Now, we also see by the above computation that

$$\langle Af, f\rangle = \int_0^\ell f''\bar{f}'' \geq 0$$

So A is positive. $\square$

(b) By part (a) and by problem 4, we know all eigenvalues are nonnegative. We see that any constant function gives the 0 eigenvalue, so we can search for positive eigenvalues from now on. Let γ^4 be an eigenvalue (note any positive number can be written like this). So, $f^{(4)} = \gamma f$. As the hint says, the general solution to this has form

$$f(x) = a\cosh(\gamma x) + b\sinh(\gamma x) + c\cos(\gamma x) + d\sin(\gamma x).$$

This means

$$f'(x) = \gamma(a\sinh(\gamma x) + b\cosh(\gamma x) - c\sin(\gamma x) + d\cos(\gamma x)).$$

Now, employing the boundary conditions, and assuming $\gamma > 0$,

$$\begin{aligned}
f(0) = 0 &\implies & c + a &= 0 \\
f'(0) = 0 &\implies & b + d &= 0 \\
f(\ell) = 0 &\implies & a(\cosh(\gamma\ell) - \cos(\gamma\ell)) + b(\sinh(\gamma\ell) - \sin(\gamma\ell)) &= 0 \\
f'(\ell) = 0 &\implies & a(\sinh(\gamma\ell) + \sin(\gamma\ell)) + b(\cosh(\gamma\ell) - \cos(\gamma\ell)) &= 0
\end{aligned}$$

So,

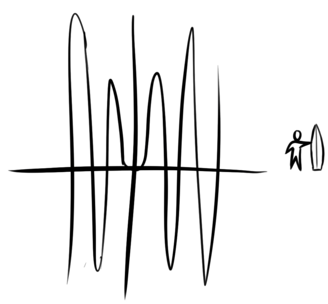
$$
\begin{aligned}
&\frac{\sinh(\gamma\ell)-\sin(\gamma\ell)}{\cosh(\gamma\ell)-\cos(\gamma\ell)} = -\frac{a}{b} = \frac{\cosh(\gamma\ell)-\cos(\gamma\ell)}{\sinh(\gamma\ell)+\sin(\gamma\ell)} \\
\implies &\sinh(\gamma\ell)^2 - \sin(\gamma\ell)^2 = \cosh(\gamma\ell)^2 - 2\cos(\gamma\ell)\cosh(\gamma\ell) + \cos(\gamma\ell)^2 \\
\implies &0 = 2 - 2\cos(\gamma\ell)\cosh(\gamma\ell) \\
\implies &\cos(\gamma\ell)\cosh(\gamma\ell) - 1 = 0.
\end{aligned}
$$

Note that if $b = 0$, we can express $-b/a$ in two ways instead. We can't have both a and b zero since f is nonzero.

We see that $\gamma = 0$ also fits this equation, so we can say eigenvalues are

$$\{\gamma^4 : \cos(\gamma\ell)\cosh(\gamma\ell) - 1 = 0\}.$$

Note that cosh is non-negative and grows in both direction and cos oscillates, so $\cos(\gamma\ell)\cosh(\gamma\ell) - 1$ must have infinite roots by the intermediate value theorem. Thus, there are indeed infinite eigenvalues. $\square$



Problem 6. ◁

Solution. First, we show that the sequence is Cauchy. For any ε, let $N > 1/\varepsilon$. We see that for any $n, m > N$, we have

$$\begin{aligned} ||f_n - f_m|| &= \int_0^1 |f_n - f_m|^2 \\ &= \int_{1/2}^{1/2+1/N} |f_n - f_m|^2 \\ &\leq \int_{1/2}^{1/2+1/N} 1 \\ &= \frac{1}{N} \\ &\leq \varepsilon. \end{aligned}$$

So, the sequence is Cauchy.

Now let's show that there is no limit. Suppose for contradiction that there is some f such that $f_n \to f$. Let $x > 1/2$. Now, let $\varepsilon = x - 1/2 > 0$. Suppose $f(x) \neq 1$. Then, define

$$c = \int_x^1 |f - 1|^2.$$

Note that $c > 0$. We see that for $n > 1/\varepsilon$, we have

$$||f_n - f|| = \int_0^1 |f - f_n|^2 \geq \int_x^1 |f - f_n|^2 = \int_x^1 |f - 1|^2 = c.$$

Since $||f_n - f|| \to 0$, this is not possible. So, it must be that $f(x) = 1$.

By a similar argument, let $x \leq 1/2$. Suppose $f(x) \neq 0$. Define

$$c = \int_0^x |f|^2.$$

We see that for any n,

$$||f_n - f|| = \int_0^1 |f - f_n|^2 \geq \int_0^x |f - f_n|^2 = \int_0^x |f|^2 = c.$$

This also can't be. Thus, $f(x) = 0$. So, it must be that

$$f(x) = \begin{cases} 1 & \text{if } x > 1/2 \\ 0 & \text{if } x \leq 1/2 \end{cases}.$$

However, this isn't continuous, so we have a contradiction. So, this sequence is Cauchy but not convergent and our space is not complete. □

Problem 7. ◁

Solution.

(a) We can directly compute the geometric series. We see that

$$P_r(\theta) = \sum_{n=0}^{\infty} r^n e^{in\theta} + \sum_{n=1}^{\infty} r^n e^{-in\theta}.$$

Now,

$$\sum_{n=0}^{\infty} r^n e^{in\theta} = \sum_{n=0}^{\infty} (re^{i\theta})^n = \frac{1}{1-re^{i\theta}}$$

and

$$\sum_{n=1}^{\infty} r^n e^{-in\theta} = \sum_{n=1}^{\infty} (re^{-i\theta})^n = \frac{re^{-i\theta}}{1-re^{-i\theta}}.$$

So,

$$\begin{aligned} P_r(\theta) &= \frac{1}{1-re^{i\theta}} + \frac{re^{-i\theta}}{1-re^{-i\theta}} \\ &= \frac{1-re^{-i\theta}+re^{-i\theta}-r^2}{1-r(e^{i\theta}+e^{-i\theta})+r^2} \\ &= \frac{1-r^2}{1-2r\cos(\theta)+r^2}, \end{aligned}$$

as we wanted. □

(b) For this problem, I will add a factor of $1/(2\pi)$ to the expression because otherwise I couldn't get it to work. First, we know P_r is a C^2 function in the disk. We know convolution preserves this property, so u is C^2 on the disk. Thus, (formally, by the inverse function theorem), v is also C^2 on the disk.

Now, let's check the Laplacian. Let $f(r\theta) = r^{|n|}e^{in\theta}$. For $|n| \geq 2$, we have $f_{rr} = r^{|n|-2}(|n|-1)|n|e^{in\theta}$ and $f_{\theta\theta} = -r^{|n|}|n|^2 e^{in\theta}$ and $f_r = |n|r^{|n|-1}e^{in\theta}$. We can check that

$$f_{rr} + \frac{1}{r^2} f_{\theta\theta} + \frac{1}{r} f_r = 0.$$

The $n = 0$ and combined $|n| = 1$ cases are similarly easy to check. Thus, f is harmonic. So, P_r is the sum of harmonic functions, so it is harmonic. Thus,

$$\mathring{\Delta} u = \mathring{\Delta} \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta-\phi)h(\phi)d\phi = \frac{1}{2\pi}\int_0^{2\pi} \mathring{\Delta} P_r(\theta-\phi)h(\phi)d\phi = 0,$$

where $\mathring{\Delta}$ denotes the polar Laplacian. So, we can conclude that $\Delta v = 0$. The origin does not give us problems, as said in chapter 14 of the notes.

A nicer way to show the above is to notice that setting $z = re^{i\theta}$, we have $P_r(\theta) = \mathrm{Re}\left(\frac{1+z}{1-z}\right)$ and then use the fact that the real part of a holomorphic function is harmonic. With this method, we do not need to worry about the origin. However, we do not know this complex analysis fact in this class.

For the last part, we must show continuity on the border. Let h_n be the fourier coefficients of h.

We see

$$\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-\phi)\left(\sum_{n\in\mathbb{Z}} h_n e^{in\phi}\right) d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{n\in\mathbb{Z}} r^{|n|} e^{in(\theta-\phi)}\right)\left(\sum_{n\in\mathbb{Z}} h_n e^{in\phi}\right) d\phi \\
&= \sum_{n,m\in\mathbb{Z}} \frac{1}{2\pi}\int_0^{2\pi} \left(r^{|n|} e^{in(\theta-\phi)}\right)\left(h_m e^{im\phi}\right) d\phi \\
&= \sum_{n,m\in\mathbb{Z}} r^{|n|} h_m e^{in\theta} \frac{1}{2\pi}\int_0^{2\pi} e^{i\phi(m-n)} d\phi.
\end{aligned}$$

For $n \neq m$, the term is 0, so

$$u(r,\theta) = \sum_{n\in\mathbb{Z}} r^{|n|} h_n e^{in\theta} \frac{1}{2\pi}\int_0^{2\pi} 1 d\phi = \sum_{n\in\mathbb{Z}} r^{|n|} h_n e^{in\theta}.$$

We see that

$$\lim_{r\to 1} \sum_{n\in\mathbb{Z}} r^{|n|} h_n e^{in\theta} = \sum_{n\in\mathbb{Z}} h_n e^{in\theta} = h(\theta),$$

so continuity on the border holds. $\square$