MATH 173 Problem Set 3

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April 20, 2022

Problem 1. TODO $\triangleleft$

Solution. As the hint suggests, $u' = 0$ means by definition $u(\phi) = 0$ for any $\phi \in C_c^\infty(\mathbb{R})$. For any $\psi \in C_c^\infty(\mathbb{R})$, let $\phi_0 \in C_c^\infty(\mathbb{R})$ be a bump function such that $\int_{\mathbb{R}} \phi_0(x)dx = 1$. Let $\hat{\psi} = \psi - \phi_0 \int_{\mathbb{R}} \psi(x)dx$. We see that

$$\int_{\mathbb{R}} \hat{\psi}(x)dx = \int_{\mathbb{R}} \psi(x)dx - \int_R \psi(x)dx \cdot \int_{\mathbb{R}} \phi_0(x)dx = \int_{\mathbb{R}} \psi(x)dx - \int_{\mathbb{R}} \psi(x)dx = 0.$$

So, we can let

$$\phi(x) = \int_{-\infty}^{x} \hat{\psi}(x)dx.$$

We see $\hat{\psi}$ has compact support and is in $C_c^\infty(\mathbb{R})$ (since it is the sum of two compact support functions). Since $\int_{\mathbb{R}} \hat{\psi}(x)dx = 0$, we know ϕ must have compact support as well and be in $C_c^\infty(\mathbb{R})$. Now, let $c = u(\phi_0)$. We see that by linearity of u,

$$u(\psi) = u(\hat{\psi}) + u(\phi_0) \cdot \int_{\mathbb{R}} \psi(x)dx = u(\phi') + c\int_{\mathbb{R}} \psi(x)dx = 0 + c\int_{\mathbb{R}} \psi(x)dx = c\int_{\mathbb{R}} \psi(x)dx,$$

which is exactly what we wanted to prove. $\square$

Problem 2.TODO ◁

Solution. First, let us define $f(x-ct)$ in a way that aligns with the case that f is a nice function. We see that if f were nice, then

$$\begin{aligned} f(x-ct)(\phi) &= \int_{\mathbb{R}^2} f(x-ct)\phi(t,x)dxdt \\ &= \int_{\mathbb{R}^2} f(z)\phi(s,z+cs)dzds \\ &= \int_{\mathbb{R}^2} f(z)\phi(s,z+cs)dsdz \\ &= \int_{z\in\mathbb{R}} f(z) \int_{s\in\mathbb{R}} \phi(s,z+cs)dsdz \\ &= f\left(\int_{t\in\mathbb{R}} \phi(t,z+ct)dt\right). \end{aligned}$$

So, we see that $f(x-ct)(\phi) = f(\Phi)$ where $\Phi = \int_{\mathbb{R}} \phi(s,z+cs)ds$. Note that $\Phi \in C_c^\infty(\mathbb{R})$ because the integral of a smooth function is smooth and ϕ is compactly supported. So, we can define $u = f(x-ct)$.

Now, we must show that u satisfies the PDE. This is done by simply writing out our definition and using the linearity of f. More precisely,

$$\begin{aligned} (u_t + cu_x)(\phi) &= -f\left(\int_s \phi_t(s,z+cs)ds\right) - cf\left(\int_s \phi_x(s,z+cs)ds\right) \\ &= -f\left(\int_s \left[\phi_t(s,z+cs) + s\phi_x(s,z+cs)\right] ds\right) \\ &= -f\left(\int_s \partial_s\phi(s,z+cs)ds)\right) \\ &= -f(0) \\ &= 0. \end{aligned}$$

Note that we used the fundamental theorem of calculus, that ϕ has compact support, and that f is linear in the above computation.

Thus, $u = f(x-ct)$ by definition and u solves the PDE in the sense of distribution, as we wanted to show. □

Problem 3. TODO $\triangleleft$

Solution. TODO $\square$

Problem 4. TODO $\triangleleft$

Solution. TODO $\square$

Problem 5. TODO ◁

Solution. We assume $t \geq 0$, as always. First, lets' consider the case $c = 0$. Then, we have $u_{tt} = 0$, so $u_t(x, t)$ is constant along the line $\{(t, x_0)\}$ for any x_0. So, $u(t, x)$ has is a line of constant slope along any $\{(t, x_0)\}$. The starting point and the slope are defined by the initial conditions, so we see that

$$u(t, x) = \phi(x) + t\psi(x).$$

This vanishes when $\phi(0) + t\psi = 0$. Since ϕ, ψ are positive, ϕ must be 0, so $|x| \geq 1$. Any point where $|x| \geq 1$ works.

We can also check that u is linear (and thus C^∞) everywhere except for the places where of ϕ and ψ are not C^∞. In other words u is C^∞ when $x \neq -1, 0, 1$.

Now, let's look at the nonzero case. We know from the notes that the solution is

$$u(t, x) = \frac{1}{2}(\phi(x + ct) + \phi(x - ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\sigma)d\sigma.$$

Notice that we can assume $c > 0$ because $c < 0$ would also flip the integral, leading to the same solution.

We see that ϕ, ψ are non-negative, so this vanishes only when $\phi(x - ct) = 0, \phi(x + c) = 0$, and $\psi = 0$ on the interval $(x - ct, x + ct)$. For the analysis of when u vanishes, we will only consider $c = 1$. Instead of writing out the cases, consider the following picture of where this is true:
TODO: image
We see that u is C^∞ everywhere but the discontinuities of ϕ, ψ. In other words, u is C^∞ when $x + ct \neq -1, 0,$ or 1. ☐

Problem 6. TODO ◁

Solution. As the hint suggests, consider $v(x_0, x_n, t) = u(x_0, x_n, t) - u(x_0, -x_n, t)$. Notice that

$$v_{tt} - c^2\Delta_x v = u_{tt}(x_0, x_n, t) - u_{tt}(x_0, -x_n, t) - c^2\Delta_x u(x_0, x_n, t) + c^2\Delta_x u(x_0, -x_n, t) = f(x_0, x_n, t) - f(x_0, -x_n, t) = 0.$$

We also see that

$$v(x, x_n, 0) = u(x_0, x_n, 0) - u(x_0, -x_n, 0) = \phi(x_0, x_n) - \phi(x_0, -x_n)$$

and

$$v_t(x, x_n, 0) = u_t(x_0, x_n, 0) - u_t(x_0, -x_n, 0) = \psi(x_0, x_n) - \psi(x_0, -x_n).$$

So, v solves the equation $v_{tt} - c^2\Delta_x v = 0$ with 0 initial conditions. We know that the only solution to this is $v = 0$. Thus, we see that $u(x_0, x_n, t) - u(x_0, -x_n, t) = v(x_0, x_n, t) = 0$, so

$$u(x_0, x_n, t) = u(x_0, -x_n, t),$$

and u is even with respect to x_n, exactly as we wanted. □

Problem 7. TODO $\triangleleft$

Solution. TODO $\square$