MATH 173 Problem Set 4

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Problem 1. $\triangleleft$

Solution. For this problem, we'll follow the first itinerary suggested by the hint. We know

$$f(x) = \int_0^x f'(t)dt.$$

Let $\chi_{[0,x]}$ be the characteristic function of $[0, x]$. We now see that by the Cauchy-Schwartz inequality,

$$\begin{aligned}
\int_0^1 f(x)^2 dx &= \int_0^1 \left(\int_0^x f'(t)dt \right) dx \\
&= \int_0^1 \left(\int_0^1 \chi_{[0,x]} f'(t)dt \right)^2 dx \\
&\leq \int_0^1 \left(\int_0^1 \chi_{[0,x]}^2 dt \right) \left(\int_0^1 f'(t)^2 dt \right) dx \\
&= \left(\int_0^1 f'(t)^2 dt \right) \int_0^1 \left(\int_0^1 \chi_{[0,x]}^2 dt \right) dx \\
&= \left(\int_0^1 f'(t)^2 dt \right) \int_0^1 x dx \\
&= \frac{1}{2} \int_0^1 f'(t)^2 dt.
\end{aligned}$$

So, an absolute constant of $C = 1/2$ works. $\square$

Problem 2. ◁

Solution.

(a) First, we see that for any $\varepsilon > 0$, there exists an R such that $|u(0, R)| \le \varepsilon$. So,

$$\sup_{x \in \mathbb{R}^n, t \in [0,t]} u(t, x) \ge -\varepsilon$$

and thus

$$\sup_{x \in \mathbb{R}^n, t \in [0,t]} u(t, x) \ge 0.$$

Also,

$$\sup_{x \in \mathbb{R}^n, t \in [0,t]} u(t, x) \ge \sup_{x \in \mathbb{R}^n} u(0, x)$$

so

$$\sup_{x \in \mathbb{R}^n, t \in [0,t]} u(t, x) \ge \max\{0, \sup_{x \in \mathbb{R}^n} u(0, x)\}.$$

The other direction is more involved. Let

$$C = \sup_{x \in \mathbb{R}^n, t \in [0,t]} u(t, x).$$

If $C = 0$, then we are done so let $C > 0$. For any $\varepsilon > 0$ such that $\varepsilon < C$, let x_0, t_0 be such that $u(x_0, t_0) > C - \varepsilon$. We know there exists an R such that

$$\sup_{|x| > R, t \in [0,t]} u(t, x) < C - \varepsilon.$$

Let $R_0 > \max t_0, R$. We then see by the maximum principle on the hypercylinder with the R_0-bal as the base that u achieves its supremum in the cylinder either on the base $\{(0, x) : |x| \le R_0\}$ or on the wall $\{(x, t) : t \in [0, T], |x| = R_0\}$. However we saw that the supremum on the wall is less than $C - \varepsilon < u(t_0, x_0)$, which is inside the cylinder. So, the supremum is attained on the base and there exits an x such that $u(0, x) \ge u(t_0, x_0) \ge C - \varepsilon$. Since ε was arbitrarily small, we have shown that RHS is at least C, so we can conclude that

$$\sup_{x \in \mathbb{R}^n, t \in [0,t]} u(t, x) = \max\{0, \sup_{x \in \mathbb{R}^n} u(0, x)\},$$

as we wanted. □

(b) If u and u' are solutions that go to 0 at infinity uniformly, then consider $v = u - u'$. We see that $v(0, x) = 0$ and that $v \to 0$ at infinity uniformly as well, so by part (a), we know $v \le 0$. By a similar reasoning $-v = u' - u \le 0$ as well. So, $v = 0$ and thus $u = u'$. Thus, the solution in the given class of functions must be unique. □

Problem 3. $\triangleleft$

Solution.

(a) This problem is just computation. By the product of sines formula,

$$\sin(n\pi x)\sin(n\pi x) = \frac{1}{2}(\cos((n-m)\pi x) - \cos((n+m)\pi x)).$$

So, if $m \neq n$, then

$$\begin{aligned}\int_0^1 \sin n\pi x \sin m\pi x dx &= \frac{1}{2}\left[\frac{1}{(n-m)\pi}\sin((n-m)\pi x)\right]_0^1 + \frac{1}{2}\left[\frac{1}{(n+m)\pi}\sin((n+m)\pi x)\right]_0^1 \\ &= \frac{1}{2}(0+0) \\ &= 0.\end{aligned}$$

For the other case, if $m = n$, then

$$\begin{aligned}\int_0^1 \sin n\pi x \sin m\pi x dx &= \frac{1}{2}\int_0^1 (\cos((0)\pi x) - \cos((2n)\pi x))dx \\ &= \frac{1}{2}\left[\cos(0) - \frac{1}{2n\pi}\cos(2n\pi x)\right]_0^1 \\ &= \frac{1}{2}(1-0) \\ &= \frac{1}{2}.\end{aligned}$$

So, we have shown the equality we wanted to show. $\square$

(b) This problem can be solved by heavy computation and using part (a). Instead, we'll use what we have seen in class. Fix s and y. We know

$$u(t,x) = \int_0^1 K(t,x,r)K(s,r,y)dr$$

gives a solution for $u_t = u_{xx}$ with initial conditions $u(0,x) = K(s,x,y)$. We also know that the heat kernel satisfies the heat equation for $t > 0$. Note that this means

$$u'(t,x) = K(t+s,x,y)$$

satisfies $u'_t = u'_{yy}$. Also note that we can check that $u(0,x) = K(s,x,y)$, so the initial conditions are the same as above. Since the solution with the same initial conditions is unique, our two solutions must be the same, so

$$\int_0^1 K(t,x,r)K(s,r,y)dr = K(t+s,x,y),$$

as we wanted. $\square$

Problem 4. ◁

Solution. This problem is similar to the use of Poincaré's inequality in class, but with a positive-definite matrix in place of the c^2 constant.

Let us define

$$E = \int_\Omega (A(x)\nabla u \cdot \nabla u + qu^2)dx.$$

Notice that all parts in the integral are non-negative, so E is also non-negative. By the same integration by parts that we've done in class, we see that the dirichlet boundary conditions give us

$$\begin{aligned} E &= 0 - \int_\Omega (\nabla \cdot (A(x)\nabla u) - qu)udx \\ &= -\int_\Omega fudx. \end{aligned}$$

Applying the Cauchy-Schwartz inequality, then the AMGM inequality, then the Poincaré inequality, we know there exists a $C > 0$ such that

$$\begin{aligned} E = |E| &= \left| \int_\Omega fudx \right| \\ &\le \left(\int_\Omega f^2 dx \right)^{1/2} \left(\int_\Omega u^2 dx \right)^{1/2} \\ &= \left(\frac{C}{c_o} \int_\Omega f^2 dx \right)^{1/2} \left(\frac{c_0}{C} \int_\Omega u^2 dx \right)^{1/2} \\ &\le \frac{1}{2}\frac{C}{c_0} \int_\Omega f^2 dx + \frac{1}{2}\frac{c_0}{C} \int_\Omega u^2 dx \\ &\le \frac{C}{2c_0} \int_\Omega f^2 dx + \frac{C}{2}\frac{c_0}{C} \int_\Omega |\nabla u|^2 \, dx \\ &= \frac{C}{2c_0} \int_\Omega f^2 dx + \frac{c_0}{2} \int_\Omega |\nabla u|^2 \, dx. \end{aligned}$$

We also know that

$$\begin{aligned} E &= \int_\Omega (A(x)\nabla u \cdot \nabla u + qu^2)dx \\ &\ge \int_\Omega (A(x)\nabla u \cdot \nabla u)dx \\ &\ge \int_\Omega (c_0 |\nabla u|^2 \cdot \nabla u)dx. \end{aligned}$$

Thus,

$$\int_\Omega (c_0 |\nabla u|^2 \cdot \nabla u)dx \le \frac{C}{2c_0} \int_\Omega f^2 dx + \frac{c_0}{2} \int_\Omega |\nabla u|^2 \, dx,$$

so, subtracting, we get

$$\int_\Omega \frac{c_0}{2} |\nabla u|^2 \, dx \le \frac{C}{2c_0} \int_\Omega f^2 dx.$$

Also, by Poincaré's inequality,

$$\int_\Omega \frac{c_0}{2} u^2 dx \le C \int_\Omega \frac{c_0}{2} |\nabla u|^2 \, dx \le \frac{C^2}{2c_0} \int_\Omega f^2 dx.$$

Adding the previous two results together and dividing by $c_0/2$, we have

$$\int_\Omega \left(|\nabla u|^2 + u^2 \right) dx \leq \frac{C + C^2}{c_0^2} \int_\Omega f^2 dx.$$

So, a constant of $\frac{C+C^2}{c_0^2}$ works, and we are done. $\square$

Problem 5. ◁

Solution. This problem is solved the same as our familiar quasilinear PDEs, but the system of ODEs is a little more annoying than usual. Let $f(s) = (x(s), y(s), z(s))$ be a characteristic curve. The initial conditions give us that $y(0) = 1$ and $z(0) = 1 + x(0)$. Let $r = x(0)$ be the variable parameter for the characteristic curve.

We see that

$$\begin{aligned} x' &= y + z \\ y' &= y \\ z' &= x - y. \end{aligned}$$

So, we can immediately deduce from the initial conditions that $y(s) = e^s$. We also see that $(x + z)' = x' + z' = x + z$, and we know $x(0) + z(0) = 2r + 1$, so

$$x + z = (2r + 1)e^s.$$

Plugging this back in, we have

$$\begin{aligned} & x' = y + z = e^s + (2r + 1)e^s - x \\ \implies & x' + x = e^s + (2r + 1)e^s \\ \implies & (xe^s)' = e^s(x' + x) = e^{2s} + (2r + 1)e^{2s} = 2(r + 1)e^{2s} \\ \implies & xe^s = (r + 1)e^{2s} + C \end{aligned}$$

for some constant C. Since $x(0) = r$, we know $x(0)e^0 = r$, so $C = -1$. Thus,

$$\begin{aligned} x &= (r + 1)e^s - e^{-s} \\ z &= re^s + e^{-s}. \end{aligned}$$

So, we can see that for $y > 0$, we can pick r such that

$$u(x, y) = z(s) = x - y + \frac{2}{y}.$$

This is the solution we were after. □

Problem 6. ◁

Solution. We can factor the equation to see that

$$(\partial_x + 8\partial_y)(\partial_x - 2\partial_y)u = 0.$$

So, we know the solution is of the form

$$u(x, y) = f(8x - y) + g(2x + y).$$

Now, we can use the initial conditions to solve for f and g. We know

$$\begin{aligned} f(-10x) + g(0) &= x \\ f(8x) + g(2x) &= \sin x. \end{aligned}$$

So,

$$\begin{aligned} f(40x) + g(0) &= -4x \\ g(10x) - g(0) &= 4x + \sin 5x. \end{aligned}$$

We can conclude that

$$\begin{aligned} u(x, y) &= f(8x - y) + g(2x + y) \\ &= \frac{8x - y}{-10} - g(0) + 4\left(\frac{2x + y}{10}\right) + \sin\left(\frac{2x + y}{2}\right) + g(0) \\ &= \frac{y}{2} + \sin\left(x + \frac{y}{2}\right) \end{aligned}$$

is our solution. □

Problem 7. ◁

Solution.

(a) For this problem, we can follow the standard uniqueness proof using the maximum principle. Suppose u and u' are two solutions to the equation. $v = u - u'$. We see that v solves $v_t - v_{xx} = 0$ and $v(x, 0) = 0$ and $v(0, t) = v(1, t) = 0$. Thus, by the maximum principle on inhomogenous heat equations, $v \leq 0$. We see that $-v = u' - u$ solves the same equation with the same properties, so $-v \leq 0$. Thus, $v = 0$. This means that $u = u'$ and the solution is unique. □

(b) We know from lecture notes that the solution in

$$\begin{aligned} u(t,x) &= \int_0^1 K(t,x,y)f(y)dy + \int_0^t \int_0^1 K(t-s,x,y)e^{-t}\sin 3\pi y dyds \\ &= \int_0^1 K(t,x,y)y\sin \pi x dy + \int_0^t \int_0^1 K(t-s,x,y)e^{-t}\sin 3\pi y dyds \end{aligned}$$

where

$$K(t,x,y) = 2\sum_{n=1}^{\infty} e^{-(\pi n)^2 t}\sin(n\pi x)\sin(n\pi y)$$

is the heat kernel. It is possible to simplify the latter part of the expression by writing out K and using the ideas in problem 3. However, this can only get us so far. The first part of the expression is not nice because of that extra x in the definition of f. □