MATH 173 Problem Set 6

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Problem 1. TODO ◁

Solution.

(a) This problem is straightforward.

$$\begin{aligned}
\overline{\mathcal{F}(\phi)(y)} &= \overline{\int_{\mathbb{R}^n} e^{-ix\cdot y}\phi(x)dx} \\
&= \int_{\mathbb{R}^n} \overline{e^{-ix\cdot y}\phi(x)}dx \\
&= \int_{\mathbb{R}^n} e^{ix\cdot y}\overline{\phi(x)}dx \\
&= (2\pi)^n(2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot y}\overline{\phi(x)}dx \\
&= (2\pi)^n\mathcal{F}^{-1}\overline{\phi}(y)
\end{aligned}$$

This is what we wanted to show. □

(b) We know the Fourier inversion formula holds for Schwartz functions. By part (a) and the equation about interchanging fourier transform under the integral that we saw in class, we know

$$\begin{aligned}
(2\pi)^{-n}\int_{\mathbb{R}^n} \hat{\phi}\overline{\hat{\psi}} &= \int_{\mathbb{R}^n} \hat{\phi}\check{\overline{\psi}} \\
(2\pi)^{-n}\int_{\mathbb{R}^n} \hat{\phi}\overline{\hat{\psi}} &= \int_{\mathbb{R}^n} \check{\hat{\phi}}\hat{\check{\overline{\psi}}} \\
(2\pi)^{-n}\int_{\mathbb{R}^n} \hat{\phi}\overline{\hat{\psi}} &= \int_{\mathbb{R}^n} \phi\overline{\psi}.
\end{aligned}$$

Setting $\psi = \phi$, we see that

$$\int_{\mathbb{R}^n} \left|\hat{\phi}\right|^2 = (2\pi)^2\int_{\mathbb{R}^n} |\phi|^2,$$

as we wanted. □

Problem 2. TODO $\triangleleft$

Solution.

(a) Let $\tilde{u}(t,x)$ be defined as in the problem. On $(0,+\infty)\times(0,+\infty)$, $\tilde{u}$ is the same as u, so it satisfies the equation $\tilde{u}_t = \tilde{u}_{xx}$. On $(0,+\infty)\times(-\infty,0)$, we see that

$$\tilde{u}_t(t,x) = -u_t(t,-x) = -u_{xx}(t,-x) = \tilde{u}_{xx}(t,x).$$

So, $\tilde{u}_t = \tilde{u}_{xx}$ on all of $(0,+\infty)\times(0,+\infty)\cup(0,+\infty)\times(-\infty,0)$ and is C^2 there. Now, consider the points along $x=0$. We can define $\tilde{u}(t,0)=0$. We see that

$$\lim_{x\to 0} u(t,x) = u(t,0) = 0,$$

and thus $\lim_{x\to 0}\tilde{u}(t,x) = 0$ from both sides, and is equal to $\tilde{u}(t,0)$. So, $\tilde{u}$ is continuous in $[0,+\infty)\times\mathbb{R}$.

Now, let's consider differentiability. Let $t>0$. We see that

$$\begin{aligned}\lim_{h\to^+0}\frac{\tilde{u}(t,h)-\tilde{u}(t,0)}{h} &= \lim_{h\to^+0}\frac{\tilde{u}(t,h)}{h}\\ &= \lim_{h\to^+0}\frac{u(t,h)}{h}\\ &= \lim_{h\to^-0}\frac{u(t,-h)}{-h}\\ &= \lim_{h\to^-0}\frac{\tilde{u}(t,h)}{h}.\end{aligned}$$

Thus, the derivative from both sides matches up. We see that $\tilde{u}_t(t,0)=0$. Since u is continuously differentiable on the border, both components of the derivative are continuous, so u is differentiable on $(0,+\infty)\times\mathbb{R}$.

We can now assume that $\tilde{u} = K_t * \tilde{g}$. Writing this out, we have

$$\begin{aligned}\tilde{u}(t,x) &= (K_t * \tilde{g})(x)\\ &= \int (4\pi t)^{-1/2} e^{-\frac{|x-y|^2}{4t}}\tilde{g}(y)dy\\ &= \frac{1}{\sqrt{4\pi t}}\int_0^\infty e^{-\frac{|x-y|^2}{4t}}\tilde{g}(y)dy + \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^0 e^{-\frac{|x-y|^2}{4t}}\tilde{g}(y)dy\\ &= \frac{1}{\sqrt{4\pi t}}\int_0^\infty e^{-\frac{(x-y)^2}{4t}} g(y)dy - \frac{1}{\sqrt{4\pi t}}\int_0^\infty e^{-\frac{(x+y)^2}{4t}} g(y)dy\\ &= \frac{1}{\sqrt{4\pi t}}\int_0^\infty g(y)\left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}}\right)dy.\end{aligned}$$

Restricting to to half, we see that

$$u(t,x) = \frac{1}{\sqrt{4\pi t}}\int_0^\infty g(y)\left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}}\right)dy.$$

We can verify that the boundary conditions match. $\square$

(b) This is very similar to part (a), but this time let us define

$$\tilde{u}(t,x) = \begin{cases} u(t,x) & \text{if } x \geq 0 \\ u(t,-x) & \text{if } x \geq 0 \end{cases}$$

to be the even extension.

We know $\tilde{u}$ is C^2 and satisfies the equation on $(0,+\infty)\times(0,+\infty)\cup(0,+\infty)\times(-\infty,0)$ for the same reason as in part (a).

We see that

$$\lim_{x\to 0} u(t,x) = u(t,0),$$

and thus $\lim_{x\to 0}\tilde{u}(t,x) = u(t,0)$ from both sides, and is equal to $\tilde{u}(t,0)$. So, $\tilde{u}$ is continuous in $[0,+\infty)\times\mathbb{R}$.

Now, we notice that

$$\begin{aligned}
\lim_{h\to^+0}\frac{\tilde{u}(t,h)-\tilde{u}(t,0)}{h} &= \lim_{h\to^+0}\frac{u(t,h)-u(t,0)}{h} \\
&= 0 \\
&= \lim_{h\to^+0} -\frac{u(t,h)-u(t,0)}{h} \\
&= \lim_{h\to^-0} -\frac{u(t,-h)-u(t,0)}{h} \\
&= \lim_{h\to^-0}\frac{\tilde{u}(t,h)-\tilde{u}(t,0)}{h}.
\end{aligned}$$

Thus, the derivative from both sides matches up. We see that $\tilde{u}_t(t,0) = 0$. Since u is continuously differentiable on the border, both components of the derivative are continuous, so u is differentiable on $(0,+\infty)\times\mathbb{R}$.

Making a similar assumption, we see that this time

$$\begin{aligned}
\tilde{u}(t,x) &= (K_t * \tilde{g})(x) \\
&= \int (4\pi t)^{-1/2} e^{-\frac{|x-y|^2}{4t}}\tilde{g}(y)dy \\
&= \frac{1}{\sqrt{4\pi t}}\int_0^\infty e^{-\frac{|x-y|^2}{4t}}\tilde{g}(y)dy + \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^0 e^{-\frac{|x-y|^2}{4t}}\tilde{g}(y)dy \\
&= \frac{1}{\sqrt{4\pi t}}\int_0^\infty e^{-\frac{(x-y)^2}{4t}} g(y)dy + \frac{1}{\sqrt{4\pi t}}\int_0^\infty e^{-\frac{(x+y)^2}{4t}} g(y)dy \\
&= \frac{1}{\sqrt{4\pi t}}\int_0^\infty g(y)\left(e^{-\frac{(x-y)^2}{4t}} + e^{-\frac{(x+y)^2}{4t}}\right)dy.
\end{aligned}$$

Restricting to to half, we see that

$$u(t,x) = \frac{1}{\sqrt{4\pi t}}\int_0^\infty g(y)\left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}}\right)dy.$$

We can verify that the boundary condition matches. $\square$

Problem 3. TODO ◁

Solution. This problem is full of tricks and surprises. First, consider $v(t,x) = u(t,x) - a(t)$. This means that

$$v_t = u_t - a'(t) = u_{xx} - a'(t) = v_x x - a'(t).$$

So,

$$v_t - v_{xx} = -a'(t)$$

with $v(t,0) = 0$ and $v(0,x) = 0$. Also, $a(0) = 0$. So, we have an inhomogeneous heat equation.

Let $S_t(\phi)$ be the operator in problem 2. More precisely, let

$$S_t(\phi)(x) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty \phi(y) \left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right) dy.$$

Then, by Duhamel's principle,

$$v(t,x) = \int_0^t S_{t-s}(f(s,\cdot))ds$$

where $f(t,x) = -a'(t)$. Expanding, and changing variables, we have

$$\begin{aligned}
v(t,x) &= \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int_0^\infty -a'(s) \left(e^{-\frac{(x-y)^2}{4(t-s)}} - e^{-\frac{(x+y)^2}{4(t-s)}} \right) dyds \\
&= \int_0^t \frac{-a'(s)}{\sqrt{4\pi(t-s)}} \left[\int_0^x e^{-\frac{(y)^2}{4(t-s)}} dy + \int_0^\infty e^{-\frac{(y)^2}{4(t-s)}} dy - \int_x^\infty e^{-\frac{(y)^2}{4(t-s)}} dy \right] ds \\
&= \int_0^t \frac{-a'(s)}{\sqrt{4\pi(t-s)}} \left[\int_0^x e^{-\frac{(y)^2}{4(t-s)}} dy + \int_0^x e^{-\frac{(y)^2}{4(t-s)}} dy \right] ds \\
&= \int_0^t \frac{-2a'(s)}{\sqrt{4\pi(t-s)}} \int_0^x e^{-\frac{(y)^2}{4(t-s)}} dyds
\end{aligned}$$

Now, using the hint, we can change variables and apply integration by parts to see that

$$\begin{aligned}
v(t,x) &= \int_0^t \frac{-2a'(s)}{\sqrt{4\pi(t-s)}} \int_0^{x(4(t-s))^{-1/2}} e^{-z^2} dzds \\
&= \int_0^t \frac{-2a(s)}{\sqrt{\pi}} e^{\frac{-x^2}{4(t-s)}} \left(-(t-s)^{-3/2} \cdot \frac{1}{4} \right) ds \\
&= \frac{x^2}{\sqrt{4\pi}} \int_0^t (t-s)^{-3/2} e^{\frac{-x^2}{4(t-s)}} a(s)ds \\
&= \frac{x^2}{\sqrt{4\pi}} \int_0^t (s)^{-3/2} e^{\frac{-x^2}{4(s)}} a(t-s)ds.
\end{aligned}$$

Thus, we can conclude that

$$u(t,x) = a(t) + \frac{x^2}{\sqrt{4\pi}} \int_0^t (s)^{-3/2} e^{\frac{-x^2}{4(s)}} a(t-s)ds,$$

which is not exactly what we wanted, but I suspect there is a typo in the problem. □

Problem 4. TODO ◁

Solution.

(a) By d'Alembert's formula, we know

$$\begin{aligned} u(x,t) &= \frac{1}{2}(\phi(x+ct)+\phi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(\sigma)d\sigma \\ &= \frac{1}{2}((x+ct)^2+(x-ct)^2) + \frac{1}{2c}\int_{x-ct}^{x+ct} 1 d\sigma \\ &= x^2+(ct)^2+t^2. \end{aligned}$$

□

(b) For this problem, we can repeatedly use the fundamental theorem of calculus. We assume that u_x vanishes at infinity. We see

$$\begin{aligned} \int u(t,x)dx &= \int \left(u(0,x) + \int_0^t u_t(\tau,x)d\tau \right) dx \\ &= \int \left(u(0,x) + \int_0^t \left(u_t(0,x) + \int_0^\tau u_{tt}(s,x)ds \right) d\tau \right) dx \\ &= \int \left(u(0,x) + \int_0^t \left(u_t(0,x) + \int_0^\tau c^2 u_{xx}(s,x)ds \right) d\tau \right) dx \\ &= \int u(0,x)dx + \int\int_0^t u_t(0,x)d\tau dx + \int\int_0^t\int_0^\tau c^2u_{xx}(s,x)dsd\tau dx \\ &= \int \phi(x)dx + \int\int_0^t \psi(x)d\tau dx + \int\int_0^t\int_0^\tau c^2u_{xx}(s,x)dsd\tau dx \\ &= \int \phi(x)dx + t\int \psi(x)d\tau dx + \int_0^t\int_0^\tau\int c^2u_{xx}(s,x)dxdsd\tau \\ &= \int \phi(x)dx + t\int \psi(x)d\tau dx + \int_0^t\int_0^\tau 0 \; dsd\tau \\ &= \int \phi(x)dx + t\int \psi(x)d\tau dx, \end{aligned}$$

exactly as we wanted. □

Problem 5. TODO ◁

Solution. To solve this, we take the partial Fourier transform of

$$u_{tt} = u_{xx} - m^2 u$$

to get

$$\hat{u}_t t = -y^2 \hat{u} - m^2 \hat{u}$$

with conditions

$$\hat{u}(0, y) = \hat{g}(y), \;\; \hat{u}_t(0, y) = \hat{h}(y).$$

Solving this ODE, we have that

$$\hat{u} = \cos \sqrt{y^2 + m^2} y t \hat{g}(y) + \frac{\sin(\sqrt{y^2 + m^2} y t)}{\sqrt{y^2 + m^2}} \hat{h}(y).$$

This is the solution we were looking for. □

Problem 6. TODO ◁

Solution. The hint unlocks the secret to this problem. Suppose no such C, m exist. In particular, no constant $C = m = j \in \mathbb{N}$ works. Let $\phi_j \in \mathcal{S}(\mathbb{R}^n)$ be the counterexample that gives $\left|U(\phi_j)\right| > j||\phi_j||_j$. Now, let's define

$$\psi_j = j^{-1}||\phi_j||_j{}^{-1}\phi_j.$$

Since ϕ_j are Schwartz, for any multi-indices α, β, $x^\alpha \partial^\beta \phi_j$ are bounded. For any $\varepsilon > 0$, select $j > \max(\varepsilon^{-1}, |\alpha|, |\beta|)$. Then,

$$\begin{aligned}
\sup_x \left| x^\alpha \partial^\beta \psi_j \right| &= \sup_x \left| x^\alpha \partial^\beta j^{-1} \left(\sum_{|\mathsf{a}|,|\mathsf{b}|<j} \sup_y \left| y^{\mathsf{a}} \partial^{\mathsf{b}} \phi_j(y) \right| \right)^{-1} \phi_j(x) \right| \\
&= j^{-1} \frac{\sup_x \left| x^\alpha \partial^{\beta-1} \phi_j(x) \right|}{\sum_{|\mathsf{a}|,|\mathsf{b}|<j} \sup_y \left| y^{\mathsf{a}} \partial^{\mathsf{b}} \phi_j(y) \right|} \\
&\le j^{-1} \\
&< \varepsilon.
\end{aligned}$$

Thus, by definition, $\psi_j \to 0$ as a Schwartz function.

However, we see that by linearity of U,

$$\left|U(\psi_j)\right| = \left|U(j^{-1}||\phi_j||_j{}^{-1}\phi_j\right|$$

Problem 7. TODO $\triangleleft$

Solution. TODO