MATH 173 Problem Set 4

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Problem 1. Suppose that $f : \mathbb{R} \to \mathbb{C}$ is continuous with $(1 + |x|)^N f(x)$ bounded for some $N > 1$ (or indeed simply that $f \in L^1(\mathbb{R})$) and $a \in R$:

(a) Let $f_a(x) = f(x - a)$. Show that $\hat{f}_a(y) = e^{-iay}\hat{f}(y)$.

(b) Let $g_a(x) = e^{ixa}f(x)$. Show that $\hat{g}_a(y) = \hat{f}(y - a)$.

$\triangleleft$

Solution.

(a) This problem is a simple computation. We see that, with a change of variables $z = x - a$, we have

$$\begin{aligned}
\hat{f}_a(y) &= \mathcal{F}(f(x-a))(y) \\
&= \int e^{-ixy} f(x-a)dx \\
&= \int e^{-i(z+a)y} f(z)dx \\
&= e^{iay} \int e^{-izy} f(z)dx \\
&= e^{iay}\hat{f}(y),
\end{aligned}$$

as we wanted. $\square$

(b) This problem is even simpler computation. We see that

$$\begin{aligned}
\hat{g_a}(y) &= \mathcal{F}(e^{ixa}f(x)) \\
&= \int e^{-ixy} e^{ixa} f(x)dx \\
&= \int e^{-ix(y-a)} f(x)dx \\
&= \hat{f}(y-a),
\end{aligned}$$

as we wanted. $\square$

Problem 2. Compute the Fourier transform of the following functions:

(a) $\chi_{(-a,a)}$

(b) $f(x) = e^{a|x|}, a > 0$

(c) $g(x) = |x|^n e^{-a|x|}, a > 0, n \in \mathbb{N}$. $\triangleleft$

Solution.

(a) This problem is also computation.

$$\begin{aligned}(\mathcal{F}\chi_{(-a,a)}(y)) &= \int_{-\infty}^{\infty} e^{-ixy}\chi_{(-a,a)}(x)dx \\ &= \int_{-a}^{a} e^{-ixy}dx \\ &= \begin{cases} -\frac{i}{y}\left(e^{-iay} - e^{iay}\right) & \text{if } y \neq 0 \\ 2a & \text{if } y = 0 \end{cases}\end{aligned}$$

$\square$

(b) Note that since $y \in \mathbb{R}$ and $a > 0$, we know $iy - a \neq 0$ and $iy + a \neq 0$ so we can divide by them. So,

$$\begin{aligned}(\mathcal{F}(e^{-a|x|})(y) &= \int_{-\infty}^{\infty} e^{-ixy}e^{a|x|}dx \\ &= \int_{0}^{\infty} e^{-x(iy+a)}dx + \int_{-\infty}^{0} e^{-x(iy-a)}dx \\ &= \left[-\frac{1}{iy+a}e^{-x(iy+a)}\right]_0^{\infty} + \left[-\frac{1}{iy-a}e^{-x(iy-a)}\right]_{-\infty}^{0} \\ &= \frac{1}{iy+a} - \frac{1}{iy-a}\end{aligned}$$

because $a > 0$. $\square$

(c) In this problem, we will use repeated integration by parts. We see that executing integration by parts, we have

$$\begin{aligned}\mathcal{F}(|x|^n e^{-a|x|})(y) &= \int_{-\infty}^{\infty} |x|^n e^{-ixy-a|x|}dx \\ &= \int_{-\infty}^{0} (-x)^n e^{-ixy+ax}dx + \int_{0}^{\infty} x^n e^{-ixy-ax}dx \\ &= \int_{-\infty}^{0} (-x)^n e^{-ix(y-a)}dx + \int_{0}^{\infty} x^n e^{-ix(y+a)}dx \\ &= \frac{-n}{yi-a}\int_{-\infty}^{0} (-x)^{n-1}e^{-ix(y-a)}dx + \frac{n}{yi+a}\int_{0}^{\infty} x^{n-1}e^{-ix(y+a)}dx.\end{aligned}$$

Note that the boundary terms vanish in the integration by parts. Repeating integration by parts n times, we see that

$$\begin{aligned}\mathcal{F}(|x|^n e^{-a|x|})(y) &= (-1)^n \frac{n!}{(yi-a)^n}\int_{-\infty}^{0} e^{-ix(y-a)}dx + \frac{n!}{(yi+a)^n}\int_{0}^{\infty} e^{-ix(y+a)}dx \\ &= (-1)^{n+1}\frac{n!}{(yi-a)^{n+1}} + \frac{n!}{(yi+a)^{n+1}},\end{aligned}$$

and $yi \pm a$ does not vanish because $a > 0$. $\square$

Problem 3.

(a) Show that if f is even (or odd) then so is its Fourier transform.

(b) Suppose that f is even and real valued. Show that $\hat{f}$ is also real valued. What can you say about $\hat{f}$ when f is odd and real valued.

$\triangleleft$

Solution.

(a) The solution to this is straightforward. As Tadashi Tokieda would say, "follow your nose". First, let $f(x) = f(-x)$. Then, letting $z = -x$ power a change of variables, we see that

$$\begin{aligned}
\hat{f}(y) &= \int_{-\infty}^{\infty} e^{-ixy} f(x)dx \\
&= \int_{\infty}^{-\infty} -e^{izy} f(-z)dz \\
&= \int_{-\infty}^{\infty} e^{izy} f(-z)dz \\
&= \int_{-\infty}^{\infty} e^{-iz(-y)} f(z)dz \\
&= \hat{f}(-y).
\end{aligned}$$

Similarly, now let $f(x) = -f(-x)$. Then,

$$\begin{aligned}
\hat{f}(y) &= \int_{-\infty}^{\infty} e^{-ixy} f(x)dx \\
&= \int_{\infty}^{-\infty} -e^{izy} f(-z)dz \\
&= \int_{-\infty}^{\infty} e^{izy} f(-z)dz \\
&= \int_{-\infty}^{\infty} -e^{-iz(-y)} f(z)dz \\
&= -\hat{f}(-y).
\end{aligned}$$

So, the fourier transform preserves evenness and oddness. $\square$

(b) Let f be even. Then,

$$\begin{aligned}
\hat{f}(y) &= \int_{-\infty}^{\infty} e^{-ixy} f(x)dx \\
&= \int_{-\infty}^{0} e^{-ixy} f(x)dx + \int_{0}^{\infty} e^{-ixy} f(x)dx \\
&= \int_{0}^{\infty} e^{ixy} f(-x)dx + \int_{0}^{\infty} e^{-ixy} f(x)dx \\
&= \int_{0}^{\infty} e^{ixy} f(x)dx + \int_{0}^{\infty} e^{-ixy} f(x)dx \\
&= \int_{0}^{\infty} \left(e^{ixy} + e^{-ixy}\right) f(x)dx
\end{aligned}$$

This is real valued because $e^{ixy} + e^{-ixy} \in \mathbb{R}$ and $f(x) \in \mathbb{R}$.

The idea is the same when f is odd. By a similar computation,

$$\begin{aligned}
\hat{f}(y) &= \int_{-\infty}^{\infty} e^{-ixy} f(x)dx \\
&= \int_{-\infty}^{0} e^{-ixy} f(x)dx + \int_{0}^{\infty} e^{-ixy} f(x)dx \\
&= \int_{0}^{\infty} e^{ixy} f(-x)dx + \int_{0}^{\infty} e^{-ixy} f(x)dx \\
&= \int_{0}^{\infty} -e^{ixy} f(x)dx + \int_{0}^{\infty} e^{-ixy} f(x)dx \\
&= \int_{0}^{\infty} \left(-e^{ixy} + e^{-ixy}\right) f(x)dx
\end{aligned}$$

We see that this integral has no real part because $-e^{ixy}+e^{-ixy}$ has no real part and $f(x) \in \mathbb{R}$. $\square$

Problem 4. Define the Hermite functions by $h_n = (x - ddx)^n e^{-x^2/2}$, such that $h_0 = e^{-x^2/2}$ and $h_{n+1}(x) = xh_n(x) - h_n'(x)$. Show that $\hat{h}_n = (-i)^n \sqrt{2\pi} h_n$. ◁

Solution. This proof follows by induction. From class we know that

$$\hat{h_0}(y) = \sqrt{2\pi} e^{-y^2/2} = \sqrt{2\pi} h_0(y).$$

So, the base case holds. Now, suppose the statement is true up to h_n. Then, we see that by linearity of $\mathcal{F}$ and the properties we have seen in class,

$$\begin{aligned}
\mathcal{F}(h_{n+1}) &= \mathcal{F}(xh_n) - \mathcal{F}(h_n') \\
&= i(\mathcal{F}h_n)' - iy(\mathcal{F}h_n) \\
&= -i(y\sqrt{2\pi}(-i)^n h_n - (\sqrt{2\pi}(-i)^n h_n)') \\
&= (-i)^{n+1}\sqrt{2\pi}(yh_n - h_n') \\
&= (-i)^{n+1}\sqrt{2\pi} h_{n+1}.
\end{aligned}$$

So, by induction, the statement holds for all n. □

Problem 5. Prove the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} f(n)e^{-i\pi n} = \sum_{m\in\mathbb{Z}} \hat{f}(x+2\pi m),$$

where f satisfies $|f(x)| + |\hat{f}(x)| \le C(1+|x|)^{-a}$ for some $a > 1$. $\triangleleft$

Solution. We will follow the hint. Let us compute the Fourier coefficients of the RHS. We get that with a change of variables,

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx} \sum_{m\in\mathbb{Z}} \hat{f}(x+2\pi m)dx \\
&= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int_{-\pi}^{\pi} e^{inx}\hat{f}(x+2\pi m)dx \\
&= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int_{-\pi+2\pi m}^{\pi+2\pi m} e^{in(z-2\pi m)}\hat{f}(z)dz \\
&= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\int_{-\pi+2\pi m}^{\pi+2\pi m} e^{inz}\hat{f}(z)dz \\
&= \frac{1}{2\pi}\int e^{inz}\hat{f}(z)dz \\
&= \mathcal{F}^{-1}(\hat{f})(n) \\
&= f(n).
\end{aligned}$$

So, assuming that the Fourier series converges to the function,

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} \hat{f}(x+2\pi m) &= \sum_{n\in\mathbb{Z}} c_n e^{-inx} \\
&= \sum_{n\in\mathbb{Z}} f(n)e^{-inx},
\end{aligned}$$

which is what we wanted to prove. $\square$

Problem 6. Prove the Whittaker–Shannon interpolation formula

$$f(x) = \sum_{n\in\mathbb{Z}} f(n)\text{sinc}(\pi(x-n))$$

provided $\text{supp}(\hat{f}) \subseteq [-\pi,\pi]$ and $|f(x)| \le C(1+|x|)^{-a}$. Here $\text{sinc}(x) = \sin(x)/x$. ◁

Solution. Since $\text{supp}(\hat{f}) \subseteq [-\pi,\pi]$, we see that for $x \in (-\pi,\pi)$, by problem 5,

$$\sum_{n\in\mathbb{Z}} f(n)e^{-inx} = \sum_{m\in\mathbb{Z}} \hat{f}(x+2\pi m) = \hat{f}(x).$$

Note that we saw that a finite number of points do not change the fourier transform, so the points $-\pi,\pi$ are not a problem. Assuming that the inverse of the Fourier transform reverts the action of the Fourier transform, as in problem 5, we see that

$$\begin{aligned}
f(y) &= \mathcal{F}^{-1}(\hat{f})(y) \\
&= \frac{1}{2\pi}\int e^{inx}\sum_{n\in\mathbb{Z}} f(n)e^{-ixy}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ixy}\sum_{n\in\mathbb{Z}} f(n)e^{-inx}dx \\
&= \sum_{n\in\mathbb{Z}} f(n)\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ixy}e^{-inx}dx \\
&= \sum_{n\in\mathbb{Z}} f(n)\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix(y-n)}dx \\
&= \sum_{n\in\mathbb{Z}} f(n)\frac{1}{2\pi}\left[\frac{1}{i(y-n)}e^{ix(y-n)}\right]_{-\pi}^{\pi}dx \\
&= \sum_{n\in\mathbb{Z}} f(n)\left(\frac{e^{i\pi(y-n)}-e^{-i\pi(y-n)}}{2\pi i(y-n)}\right)dx \\
&= \sum_{n\in\mathbb{Z}} f(n)\left(\frac{\sin(\pi(y-n))}{\pi(y-n)}\right)dx \\
&= \sum_{n\in\mathbb{Z}} f(n)\text{sinc}(\pi(y-n)),
\end{aligned}$$

exactly as we wanted. □

Problem 7. TODO $\triangleleft$

Solution. TODO