MATH 173 PROBLEM SET 9

Stepan (Styopa) Zharkov June 1, 2022

Problem 1. \triangleleft Solution.

Problem 2.

Solution.

(a) We need $\int_0^1 |x^{\alpha}|^2 = \int_0^1 x^{2\alpha}$ to converge. This converges for $\alpha > -1/2$ and diverges for $\alpha \leq -1/2$, so $\phi_{\alpha} \in L^2((0,1))$ for $\alpha > -1/2$.

(b) We need $\phi_{\alpha} \in L^2((0,1))$, so $\alpha > -1/2$. But, since ϕ_{α} are smooth, we also need $\int_0^1 |\phi_{\alpha}'|^2$ to converge. We see $\phi_{\alpha}' = \alpha x^{\alpha-1}$. and $\int_0^1 |\alpha x^{\alpha-1}|^2 = |\alpha|^2 \int_0^1 x^{2(\alpha-1)}$ converges for $\alpha > 1/2$ and diverges for $\alpha \le 1/2$. So, $\phi_{\alpha} \in H^1((0,1))$ for $\alpha > 1/2$.

Problem 3.

Solution.

(a) We know the statement is true for $f \in C^1((a,b))$ by FTC. Now, let $f_n \to f$ where $f_n \in C^1((a,b))$. By the continuity of the trace operator,

$$f(x) - f(y) = \lim_{n \to \infty} (f_n(x) - f_n(y)) = \lim_{n \to \infty} \int_x^y f'_n(t)dt$$

TODO: finish this.

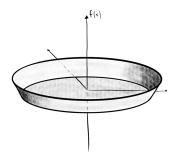
(b) TODO

Problem 4. \triangleleft Solution.

Problem 5.

Solution. Consider the dogbowl functions

$$f_n := 1 - \min(n \cdot d(x, \partial B), 1).$$



We see that $Tf_n = 1$ for all n4, so $Tf_n \to 1 \neq 0$. However,

$$||f_n||_{L_2}^2 = \int_B |f_n(x)|^2 dx = \int_{d(x,\partial B)<1/n} |f_n(x)|^2 dx \le \int_{d(x,\partial B)<1/n} 1 dx = O(1/n) \to 0.$$

So,
$$f_n \to 0$$
 in L^2 .

Problem 6.

Solution. Let $u = \lim_{n \to \infty} u_n$ where u_n are compactly supported continuous functions. Note that we are given that $u = \lim_{n \to \infty} -u_n(x^*)$. This means that

$$u = \frac{\lim_{n \to \infty} u_n(x) + \lim_{n \to \infty} - u_n(x^*)}{2} = \lim_{n \to \infty} \frac{u_n(x) - u_n(x^*)}{2}.$$

Note that $\frac{u_n(x)-u_n(x^*)}{2}=0$ when $x_n=0$, so

$$T_{B_+}\left(\frac{u_n(x)-u_n(x^*)}{2}\right) = \frac{u_n(x)-u_n(x^*)}{2}|_{\partial B_+} = 0.$$

We have shown in class that this is sufficient to say $T_{B_+}(u|_{B_+})=0$, so $u|_{B_+}\in H^1_0(B_+)$.

Problem 7.

Solution.

(a) Let $V_n=\{x:|x|<1/n\}$ and let $W_n=\{x:1-1/n<|x|<1\}$. Then, consider lemonsqueezer functions $f_n\in$ such that $f\in C^1_0(U)$ and $f_n|_{B(V_n\cap W_n)}=1$ and $f_n|_{V_{2n}\cup W_{2n}}=0$.

TODO: image

Note that $u_n := uf_n \in C_0^1(U)$. We claim $u_n \to u$ in $H^1(B)$. Note that

$$||u - u_n||_{H^1}^2 = \int_B |u - u_n|^2 + \int_B |\nabla u' - u_n'|^2.$$

Now, since u is bounded,

$$\begin{split} \int_{B} |u - u_{n}|^{2} &= \int_{B} |u|^{2} |1 - f_{n}|^{2} \\ &= \int_{B} O(1) |1 - f_{n}|^{2} \\ &= O(1) \int_{B} |1 - f_{n}|^{2} \\ &= O(1) \int_{V_{n} \cup W_{n}} |1 - f_{n}|^{2} \\ &= O(1) \int_{V_{n} \cup W_{n}} O(1) \\ &= O(1/n^{2}) \end{split}$$

- (b) TODO
- (c) TODO