MATH 173 Problem Set 7

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May 18, 2022

Problem 1. TODO ◁

Solution. For a harmonic function, we know the average over the surface of any ball is equal to the value at the center. We also know that the average over the inside of the ball is equal to the average in the center because the average on the inside is simply an integral of the averages over many concentric balls.
TODO: img
Let x and y be two arbitrary points. Now, consider $B_x := B(x, R)$ and $B_y := B(y, R + d)$ where $d = |x - y|$, as the hint suggests. Then, we see that $B_x \subseteq B_y$.
TODO: img
Let $\mathrm{Avg}(\Omega)$ denote the average of u over the region Ω. Let $|\Omega|$ denote the area of Ω. We see that

$$\begin{aligned}\mathrm{Avg}(B_y) &= \mathrm{Avg}(B_x)\frac{|B_x|}{|B_y|} + \mathrm{Avg}(B_y \setminus B_x)\frac{|B_y \setminus B_x|}{|B_y|} \\ &\geq \mathrm{Avg}(B_x)\frac{|B_x|}{|B_y|}.\end{aligned}$$

In the second step, we used that u is non-negative. Note that $\lim_{R\to\infty} \frac{|B_x|}{|B_y|} = 1$, so for any ε, we can find R large enough that

$$u(y) = \mathrm{Avg}(B_y) \geq \mathrm{Avg}(B_x)\frac{|B_x|}{|B_y|} \geq \mathrm{Avg}(B_x)(1 - \varepsilon) = u(x)(1 - \varepsilon).$$

Thus, we can conclude that $u(y) \geq u(x)$. By an identical argument, we also know $u(y) \leq u(x)$. Since x and y were arbitrary, u must be the constant function. □

Problem 2. TODO ◁

Solution.

(a) By the linearity properties of $\langle \cdot, \cdot \rangle$, we know

$$\begin{aligned} ||v+w||^2 + ||v-w||^2 &= \langle v+w, v+w\rangle + \langle v-w, v-w\rangle \\ &= \langle v,v\rangle + \langle v,w\rangle + \langle w,v\rangle + \langle w,w\rangle + \langle v,v\rangle - \langle v,w\rangle - \langle w,v\rangle + \langle w,w\rangle \\ &= 2\langle v,v\rangle + 2\langle w,w\rangle \\ &= 2(||v||^2 + ||w||^2) \end{aligned}$$

for any v, w. □

(b) Let $\Omega_1, \Omega_2 \subseteq \Omega$ be nonempty oben balls where $\Omega_1 \cap \Omega_2 = \varnothing$. Let f be a bump function such that $\text{supp} f \subseteq \Omega_1$ and $f \geq 0$ and f achieves its maximum of 1 in Ω_1. Similarly, let g be a bump function such that $\text{supp} g \subseteq \Omega_2$ and $g \geq 0$ and g achieves its maximum of 1 in Ω_2.

We see that

$$||f-g||^2 + ||f+g||^2 = 1 + 1 \neq 2(1+1) = 2(||f||^2 + ||g||^2).$$

So the parallelogram law does not hold. By part (a), this means this norm cannot be induced by an inner product. □

Problem 3. TODO $\triangleleft$

Solution. TODO

Problem 4. TODO $\triangleleft$

Solution. TODO

Problem 5. TODO $\triangleleft$

Solution. TODO

Problem 6. TODO $\triangleleft$

Solution. TODO

Problem 7. TODO $\triangleleft$

Solution. TODO