MATH 173 Problem Set 2

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Problem 1. TODO

Solution. This is a semilinear PDE and can be solved with the same methods as last week. Let Let $(x(s), y(s))$ be a characteristic curve. We see that $x'(s) = 1, y'(s) = \cos$ with $x(0) = 0$.

Solving this, we get that $x(s) = s, y(s) = \sin(s) + a$. Let $f_a(s) = (s, sin(s) + a)$ be the characteristic curve. Also, let $\omega_a(s) = u(f_a(s))$. We know $\omega_a'(s) = y(s) = \sin(s) + a$ and $\omega_a(0) = a$. So, we see that $\omega_a(s) = -\cos(s) + as + a + 1$. We see that $x = s$, $y = \sin(s) + a$. Solving for a, s, we have that $s = x$ and $a = y - \sin(x) + y$. Thus,

$$u(x, y) = \omega_a(s) = -\cos(s) + as + a + 1 = -\cos(x) + xy - x\sin(x) + y - \sin(x) + 1.$$

We can check that this solution fits $u_x + \cos(x)u_y = y$ indeed.

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Problem 2.TODO ◁

For this question, we will answer both parts (a) and (b) together because we must go through the ideas to solve part (b) to be able to find the solution in (a) anyway. This is a quasilinear PDE that can be solved by finding the characteristic curves in 3-dimensional space, as discussed in chapters 3 and 4.

Let $(t(s), x(s), z(s)$ be the characteristic curve on the graph of $u(t, x)$. We see that $t'(s) = 1, x'(s) = \sqrt{z(s)}, z'(s) = 0$ and that the initial conditions give us $t(0) = 0, z(0) = x(0)^2$. Solving this, we have that $t(s) = s, z(s) = a$ and $x(s) = s\sqrt{a} \pm \sqrt{a}$ for some constant $a \geq 0$. Let $\omega_a^+(s) = (s, s\sqrt{a} + \sqrt{a}, a)$ and $\omega_a^-(s) = (s, s\sqrt{a} - \sqrt{a}, a)$ be the characteristic curves. Note that ω_0^+ and ω_0^- are the same curve. When projected onto the (x, y) plane, the curves look as follows:

The value of u is constant and equal to a along each curve. We see that in the region where $|t| > 1$, the curves either intersect or do no pass through at all. For the region $|t| < 1$, we can find a solution. Namely, we see that $t = s$ and $x = s\sqrt{a} \pm \sqrt{a}$, so we can solve to see that any (x, t) where $t < 1$ and $x \neq 0$ can be uniquely expressed by setting

$$s = t, a = \begin{cases} \left(\frac{x}{t+1}\right)^2 & \text{if } x \geq 0 \\ \left(\frac{x}{t-1}\right)^2 & \text{if } x \leq 0. \end{cases}$$

Note that since ω_0^+ and ω_0^- are the same curve with $a = 0$, we do not have any problems at 0. Since $u(t, x) = z(s) = a$, we can see that

$$u(t, x) = \begin{cases} \left(\frac{x}{t+1}\right)^2 & \text{if } x \geq 0 \\ \left(\frac{x}{t-1}\right)^2 & \text{if } x \leq 0. \end{cases}$$

is our solution fo $|t| < 1$. To finish answering part (b), we see that for $T = 1$, our expression is continuously differentiable on $[0, T] \times \mathbb{R}$. The only thing we have to check for this is that the derivatives near $x = 0$ match up, but we see that u_x approaches 0 from both sides, so we can continue.

Finally, we can check that for any $T > 1$, we encounter problems. The characteristic curves ω_0^- and ω_1^- pass through the same point $1, 0$, but with different values of a, so any solution must be equal to both 0 and 1 at that point, which is impossible. Thus, $T = 1$ is the largest choice we could have made. □

Problem 3. TODO

Solution. TODO $\square$

Problem 4. TODO

Solution.

1. We can approach this as we approached conservation laws in class. As always, let $(t(s), x(s), z(s))$ be a characteristic curve on the graph of u. From our derivation in class, we know that $t(s) = s, x(s) = F'(g(a))s + s$, and $z(s) = g(a)$ for the parameter a. So, all the characteristics are straight lines and u is constant along them. Rearranging our expressions, we see that $s = t$ and $a = x - F'(g(a))t$. Plugging this in, we see that

$$u(t,x) = z(s) = g(a) = g(x - F'(g(a))t) = g(x - F'(u(t,x))t),$$

   as we wanted. $\square$

2. TODO

Problem 5. TODO

Solution.

1. We see that u is constant along the characteristic curves, so the value of u is 0 along any curve that starts at $(0, x)$ for $|x| > R$. We see that the slope of the projection onto the (t, x)-plane is $\frac{F'(g(a))}{1} = F'(g(a))$ for the curve that starts at $(0, a, g(a))$. Since $F'(g(a))$ is continuous on $[-R, R]$, we can let $C > \sup\{F'(g(a)) : |a| \leq R\}$. Then, the projections of the characteristics starting at $(0, a)$ for $|a| \leq R$ are limited to the region $\{(t, x) : |x| \leq R + Ct\}$. So, any point outside this region must be on a characteristic starting at $(0, a)$ for $|a| > R$ and thus u has a value of 0 there, which is what we wanted to prove. $\square$

2. TODO

Problem 6. TODO

Solution. This problem is similar to the example with only 1 and 0 in the range that we did in class, but we have to insert $1/2$ in between the two regions.

Let $h(t)$ be a border between two constant value regions and a_+, a_- be the values in the two regions. If u is a weak solution, then by the Rankine-Hugoniot condition, we have $(a_+ - a_-)h'(t) = a_+^2/2 - a_-^2/2$. Since we assume that the values are different, we can divide by $a_+ - a_- \neq 0$, so $h'(t) = (a_+ + a_-)/2$. Thus, we see that $h(t) = (a_+ + a_-)/2 \cdot t$. For (a_+, a_-) set to $(0, 1), (1/2, 1)$, and $(0, 1/2)$, we see $h(t)$ must be a line with a slope of $1/2, 3/4$, and $1/4$ respectively. Note that the order of a_+, a_- does not change the slope. Since the problem asks that all 3 values are in the range, and we know that $u(0, x)$ is 0 below $x = 0$ and 1 above, we can conclude that the only possible option satisfying the R-H condition. Namely,

$$u(t,x) = \begin{cases} 0 & \text{if } x < \frac{1}{4}y \\ 1/2 & \text{if } \frac{1}{4}y \leq x \leq \frac{3}{4}y \\ 1/2 & \text{if } x > \frac{3}{4}y. \end{cases}$$

We see that constant functions satisfy $u_t + uu_x = 0$ and our function satisfies the jump condition, so it is indeed a weak solution. $\square$

Problem 7. TODO

Solution.

(a) Let $v(t,x) = F'(u(t,x))$. We see that since $u_t(t,x) + F'(u(t,x))u_x(t,x) = 0$, we know

$$\begin{aligned} v_t + vv_x &= \partial_t F'(u(t,x)) + F'(u(t,x))\partial_x F'(u(t,x)) \\ &= F''(u(t,x))u_t(t,x) + F'(u(t,x))F''(u(t,x))u_x(t,x) \\ &= F''(u(t,x))(u_t(t,x) + F'(u(t,x))u_x(t,x)) \\ &= 0. \end{aligned}$$

So, by definition, v satisfies Burgers' equation. $\square$

(b) No, the implication doesn't hold. Let $F(x) = x^3$. Then, the equation becomes $u_t + 3u^2 u_x = 0$. Consider the function

$$u(t,x) = \begin{cases} 0 \text{ if } x \le t \\ 1 \text{ if } x > t. \end{cases}$$

We see that the constants satisfy $u_t + 3u^2 u_x = 0$. Also, on the boundary $h(t) = t$, we see that $(1-0)h'(t) = 1^3 - 0^3$, so the Rankine-Hugoniot condition is satisfied. Thus, u is a weak solution. However, we see that

$$v(x,t) = 3u^2(t,x) = \begin{cases} 0 \text{ if } x \le t \\ 3 \text{ if } x > t. \end{cases},$$

which does not satisfy the Rankine-Hugoniot condition for Burgers' function because for the boundary $h(t) = t$, we see $h'(t) = 1 \neq 3/2 = 3/2 + 0/2$. $\square$