MAT367

Intro to Differential Geometry

Class Lecture Notes

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Class Lectures

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Contents

Preface

These notes were created during class lectures. As such, they may be incomplete or lacking in some detail at parts, and may contain confusing typos due to time-sensitivity. Additionally, these notes may not be comprehensive. Most statements in this document which are not Theorems, Problems, Lemmas, Corollaries, or similar, are likely paraphrased to a certain degree. Please do not treat any material in this document as the exact words of the original lecturer.

If you are viewing this document in Obsidian, you may notice that the links in the pdf document do not work. This is intentional behaviour, as I currently do not have or know of a decent solution which allows them to behave well with the setup in Obsidian. However, below certain pages, there may be links to other documents - these are usually context-relevant links between notes of different areas of study. I created these links to point out potential similarities, or in case one area of study is borrowing a concept, definition, or theorem from another area of study, and you wish to see the full, original definition/derivation/proof or whatever it may be.

I Introduction and Manifolds

1 Preliminaries and Basics

Lec 1 - Jan 6 (Week 1)

we know what manifolds are. recall implicit thm:

Theorem 1.1: Implicit Function Thm

given eqn $f(x_1, \ldots, x_{n+1}) = 0$ for smooth f and a soln $p \in \mathbb{R}^{n+1}$ s.t. $\nabla f \neq 0$ at p, then $f(x_1, \ldots, x_{n+1})$ is the soln set near p.

furthermore, we can represent solns as $(x_1, \ldots, x_n, g(x_1, \ldots, x_n))$, where g also smooth.

note $f(x_1, \ldots, x_{n+1}) = 0$ is locally the graph of a fn. if 0 a reg pt of f, then we can cvr $\{x : f(x) = 0\}$ by graphs, i.e. charts. oh example spam sure

Theorem 1.2: Whitney Embedding Thm

every n-mfld has an embedding in $\mathbb{R}^{2n}$.

we wont get to this thm in this course, but its neat. or sth

Example 1.3

Source: Primary Source Material

let M be the set of all rots of a ball. this is a *configuration space* - pts represent ways to configure another object/space.

how do we put coords on (a piece of) this space? one way: identify w SO(3). another way: where N goes has 2 degs of freedom. where the vector $e_1 = (1, 0, 0)$ goes @ N is 1 deg of freedom. thus, M is 3-dim; this also gives a way of defining a chart near the id.

Example 1.4

Source: Primary Source Material

consider M as the config space of a *linkage* (picture a graph, but physical, where edges can rotate around vertices - saw a vid abt this i think). in this ex, sps it "looks like" a quadrilateral, and wlog one edge is fixed in space; all edges coplanar. what dim is M?

each vec is in $\mathbb{R}^2$, so we start w 8 variables. but since its a closed shape, $n_1 + n_2 + n_3 + n_4 = 0$, so -2 dims. furthermore, $\|n_i\|$ fixed, so -4 dim. since n_1 fixed, -1 dim; this leaves 1 dim, and indeed, M is 2-dim'l. it can in fact be parameterized by two adj angles θ, φ.

Fact 1.5

"closed" (ie cpt) surfaces are easy to enumerate (classify).

- orientable: sphere or conn sum of torii
- non-orientable: $\mathbb{R}P^2$, klein bottle, conn sum of those w torii (handles)

ok now for our proper defns.

Definition 1.6

a **coordinate chart** is an inj $\varphi : U \longrightarrow \mathbb{R}^n$ w open img for some $U \subseteq M$.

two charts $\varphi : U \longrightarrow \mathbb{R}^n$ and $\psi : V \longrightarrow \mathbb{R}^n$ are **compatible** if the transition function given by:

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \psi(U \cap V)$$

is a diffeo. we also enforce that if $U \cap V = \varnothing$, then they are compatible.

Lec 2 - Jan 8 (Week 1)

idea: charts give a coord system on $U \subseteq M$. in particular, coordinates are given by $\varphi(p) = (x^1, \ldots, x^n)$. then, transition maps represent coord *changes*.

this guy is doing weird inverse wizardry. ok whatever man.

Definition 1.7

a chart is compatible with an atlas (we know what this is) if it is compatible w every chart in the atlas.

Lemma 1.8

if two charts are compatible w the same atlas, they are compatible w each other.

Definition/Theorem 1.9

given an atlas A on M, theres a unique maximal atlas $\widetilde{A}$ which consists of every chart compatible w A. in particular, every chart compatible w $\widetilde{A}$ is alr in $\widetilde{A}$.

Proof.

Source: Primary Source Material

pf that $\widetilde{A}$ is an atlas: it cvrs M since $A \subseteq \widetilde{A}$, and is pairwise compatible by the lemma.

pf that all charts compat w $\widetilde{A}$ is in $\widetilde{A}$: compat w $\widetilde{A}$ implies compat w A which means its in $\widetilde{A}$. ■

Lec 3 - Jan 13 (Week 2)

still tryna define a mfld. maybe ill actually start takign notes. hmmmm

current draft: a mfld is a set M w a maximal atlas A. note that given any atlas, we can uniquely complete to a max atlas. furthermore, 2 charts gen the same atlas iff each chart in one is cptbl w each chart in the other.

possible problems: M is "too big". eg, $M = \mathbb{R}$ w singleton charts, long line, line w two origins. in particular, the line w two origins is not T2 at the origins.

we can now give the final defn for a mfld.

2 Manifolds, Properly

Definition 2.1

a **manifold** (mfld) is a set M w max atlas A, s.t. A is 2nd ctbl and Hausdorff.

nows a good time as any to remind you* that charts need not be cts, since we havent defnd a topology on M yet. go look at the defn again.

a quick note on notation - we write $(x^0 : x^1 : \cdots : x^n)$ for the equivalence class of $(x^0, x^1, \ldots, x^n) \in \mathbb{R}^{n+1} \setminus \{0\}$.

Example 2.2

Source: Primary Source Material

1. S^n by stereo proj
2. $\mathbb{R}P^n$, real projective n-space $= \{$1-dim subspaces of $\mathbb{R}^n\}$. ok hes just spamming RPn defs now.

   as quot of S^n, we can choose a chart such as the proj map on the open upper hemisphere, or equivalently:

   $$(x^0 : \cdots : x^n) \mapsto \left(\frac{x^1}{x^0}, \ldots, \frac{x^n}{x^0}\right) \qquad (x^0 : \cdots : x^n) \in \left\{(x^0 : \cdots : x^n) : x^0 \neq 0\right\}$$

   note the maps are equiv in the sense that their transition fn is smooth. note we can also do the same for any coordinate.

   thus, we get a chart for each, i.e. $n+1$ charts, which cover $\mathbb{R}P^n$.

Lec 4 - Jan 15 (Week 2)

snow day looll

Lec 5 - Jan 20 (Week 3)

Definition 2.3

a subset $U \subseteq M$ is **open** if for all charts $\varphi_\alpha : U_\alpha \longrightarrow \mathbb{R}^n$, the image $\varphi_\alpha(U \cap U_\alpha)$ is open in $\mathbb{R}^n$.

Proposition 2.4

$U \subseteq M$ open iff $\varphi_\beta(U \cap U_\beta)$ open for some coll'n (U_β, φ_β) cvring U.

Proof.

Source: Primary Source Material

fix $\{(U_\beta, \varphi_\beta)\}$. let φ_α be another chart in the (max'l) atlas. then:

$$\varphi_\alpha(U \cap U_\alpha) = \bigcup_\beta \varphi_\alpha(U \cap U_\alpha \cap U_\beta)$$

$$\begin{aligned} \varphi_\alpha(U \cap U_\alpha \cap U_\beta) &= (\varphi_\alpha \circ \varphi_\beta^{-1}) \circ \varphi_\beta(U \cap U_\alpha \cap U_\beta) \\ &= (\varphi_\alpha \circ \varphi_\beta^{-1})\,(\varphi_\beta(U \cap U_\beta) \cap \varphi_\beta(U_\alpha \cap U_\beta)) \end{aligned}$$

notice $\varphi_\beta(U_\alpha \cap U_\beta)$ is open by compatibility and $\varphi_\beta(U \cap U_\beta)$ is open by assumption, so φ_α is open by defn of transition maps. ∎

fact: if A atlas on M, $U \subseteq M$ open, then:

$$A_U = \left\{(U \cap U_\alpha, \varphi_\alpha\big|_{U \cap U_\alpha})\right\}$$

is an atlas. moreover, if A max/2nd ctbl/T2, so is A_U. in particular, any open subset of a mfld is a mfld.

we also borrow certain terms from topology, namely nbhds, closed sets, connected.

note we define a nbhd of a pt as a (!) set containing an open set containing the pt. importantly, the nbhd need not be open itself. [verify]

Remark 2.5

U open iff $U = \bigcup U_\alpha$ for charts $(U_\alpha, \varphi_\alpha)$ in a max atlas of M

in other words, a max atlas is indeed a basis for the mfld topology.

Proof.

by propn 2.4, since U open, can take coll'n of charts $(U_\alpha, \varphi_\alpha)$ which cvr U. the propn then guarantees imgs of $U \cap U_\alpha$ are open. define:

$$\varphi'_\alpha = \varphi_\alpha\big|_{U \cap U_\alpha}$$

a calc verifies it is compat w the atlas. therefore we have a coll'n of charts given by $(U'_\alpha, \varphi'_\alpha)$. by maximality of the atlas, we're done. ■

had to dip (did above pf myself), so rest of this lec is going off of txtbk.

Exercise 2.6 Source: textbook

sps A maximal, (U, φ) a chart, $V \subseteq \mathbb{R}^m$ open. show $\varphi^{-1}(V)$ open.

Proof.

(U, φ) is a chart and clearly cvrs $\varphi^{-1}(V)$. then:

$$\varphi(\varphi^{-1}(V)) = \varphi(\varphi^{-1}(V) \cap U) = V \cap \varphi(U)$$

is open since $V \subseteq \mathbb{R}^m$ open and $\varphi(U)$ open. so by propn 2.4, we're done. ■

next up: cptness. besides topo defn, we also have:

Definition 2.7

$K \subseteq M$ is **cpt** if $K = K_1 \cup \cdots \cup K_n$ for some $K_i \subseteq U_i$, where $\varphi_i(K_i)$ cpt in $\mathbb{R}^m$.

Lemma 2.8

closed subset of cpt is cpt and finite union of cpt is cpt

Lemma 2.9

under the open cvr defn, cpt implies closed and cpt subset of cpt is cpt.

dont feel like proving those rn

Theorem 2.10

the above defn and the open cvr defn are equivalent

Proof. Source: textbook

defn by c-cpt a set cpt by the defn, and t-cpt a set cpt w the open cvr defn. note $K \subseteq U$ t-cpt iff $\varphi(K)$ cpt.

if $K = K_1 \cup \cdots \cup K_n$, then each K_i t-cpt, so K is too.

if K t-cpt, then for each pt, choose a chart φ_p and a closed ball $\overline{B_p}$ arnd $\varphi_p(p)$ in $\varphi_p(U_p)$. then, $\varphi_p^{-1}(\overline{B_p})$ t-cpt.

the $\varphi_p^{-1}(B_p)$ are an open cvr, so we have a finite subcvr:

$$K \subseteq \varphi_{p_1}^{-1}(B_{p_1}) \cup \cdots \cup \varphi_{p_n}^{-1}(B_{p_n}) \qquad K_i := K \cap \varphi_{p_i}^{-1}(\overline{B_{p_i}}) \text{ t-cpt}$$

since $K \cap \varphi_{p_i}^{-1}(B_{p_i}) \subseteq K_i$, $K = K_1 \cup \cdots \cup K_n$ c-cpt. ■

3 Orientability

Lec 6 - Jan 22 (Week 3)

finally... this time... this time for sure...

usual non-orientability example: mobius strip. if we were to place tangent vectors, perhaps completing each to a tangent space, then we would - in theory - want them to satisfy the "right-hand rule", or equivalently, that $\det(v_1\ v_2\ v_3) > 0$.

he drew hermit crabs on a mobius strip omg

Definition 3.1

a lin transformation is **orientation-preserving** if $\det > 0$.

a transition map τ btwn 2 charts is **orientation-preserving** if $\det D\tau > 0$ everywhere. in this case, we say the charts are **orientation-compatible**.

an atlas is **oriented** if all charts are orientation-compatible, a **maximal oriented atlas** is an oriented atlas which is maximal (and not vice versa), and a mfld is **orientable** if it has an oriented atlas.

Exercise 3.2

Source: Primary Source Material

given a max orient atlas and two charts orient compat w the atlas, show the charts are orient compat w each other.

Example 3.3

Source: Primary Source Material

sphere is orientable. note our usual atlas of double stereo proj is *not* oriented, but can be made oriented:

$$(\varphi_N, (-x_1, x_2, \ldots, x_n) \circ \varphi_S)$$

claim: this is oriented. pf: do a computation. alternatively, for $n \geq 2$:

note $U_N \cap U_S = S^n \setminus \{N, S\}$ is conn. therefore, $\det(D\tau) \neq 0$ must be the same on the entire img.

Lemma 3.4

if X conn and A some set, then any "locally constant" $f : X \longrightarrow A$ is constant.

f loc constant if every $p \in X$ has nbhd U s.t. $f(U) = f(p)$.

Proof. Source: Primary Source Material

$f^{-1}(a) = \bigcup_{p \in f^{-1}(a)} U_p$ open. then $X = \bigsqcup_A f^{-1}(a)$, so only one can be non-empty. ■

from book: converse is also true, ie if all loc const fn's are const, then X conn.

pf (orig): indeed, sps not conn. then defn f on $X = U \sqcup V$ as 0 on U, 1 on V. then f loc const but not const.

Exercise 3.5 Source: Primary Source Material

$\mathbb{R}P^2$ is not orientable. proof by hermit crabs :3 (exercise)

Definition 3.6

given orient mfld (M, A), the **opposite orientation** is given by the atlas:

$$\widetilde{A} := \{(F(U), F \circ \varphi) : (U, \varphi) \in A\} \qquad F(x) = (-x_1, x_2, \dots, x_n)$$

where F is a "flip" of the first coord.

Proposition 3.7

sps M orient mfld. then every conn chart (ie U conn) compat w α is orient compat w *either* A or $\widetilde{A}$.

Proof.

let $A = \{(U_\alpha, \varphi_\alpha)\}$. then:

$$\Sigma_p := \operatorname{sgn}(\det D(\varphi_\alpha \circ \varphi^{-1})|_{\varphi(p)})$$

is indep of α. we claim Σ_p also indep of p.

indeed, $\Sigma_p : U \longrightarrow \{\pm\}$ is loc const, since Σ_p const on $U_\alpha \cap U$ $\forall \alpha$. by earlier lemma, its constant. ■

Corollary 3.8

if M orient conn, there are exactly 2 max orient atlases.