MAT327
Introduction to Topology

Class Lecture Notes

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Preface

These notes were created during class lectures. As such, they may be incomplete or lacking in some detail at parts, and may contain confusing typos due to time-sensitivity. Additionally, these notes may not be comprehensive. Most statements in this document which are not Theorems, Problems, Lemmas, Corollaries, or similar, are likely paraphrased to a certain degree. Please do not treat any material in this document as the exact words of the original lecturer.

If you are viewing this document in Obsidian, you may notice that the links in the pdf document do not work. This is intentional behaviour, as I currently do not have or know of a decent solution which allows them to behave well with the setup in Obsidian. However, below certain pages, there may be links to other documents - these are usually context-relevant links between notes of different areas of study. I created these links to point out potential similarities, or in case one area of study is borrowing a concept, definition, or theorem from another area of study, and you wish to see the full, original definition/derivation/proof or whatever it may be.

I Point-Set Basics

1 Preliminaries and Initial Definitions

Lec 1 - May 7 (Week 1)

Definition 1.1

$f : \mathbb{R} \to \mathbb{R}$ cts iff $\forall\, x_0 \in \mathbb{R} \forall\, \varepsilon > 0 \exists\, \delta > 0 \forall\, x \in \mathbb{R}$ such that:

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$$

for metric spaces: replace $\mathbb{R}$ with X in the above, then:

$$d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon$$

damn that spacing sucks. anyway these are defns we're alr familiar with, i.e. metrics on a set (symmetry, posdef, triangle). note we call the 2-norm on $\mathbb{R}^n$ as d_2, or the euclidean metric

note homeos (as we know them intuitively; donut/mug) dont preserve distances. new goal: define continuity w/o using a metric.

Definition 1.2

$\mathcal{T}'$ is finer than $\mathcal{T}$ iff $\mathcal{T} \subseteq \mathcal{T}'$

no im not paying attn, how could u tell

Lec 3 - May 14 (Week 2)

lol. lmao.

Example 1.3: Sorgenfrey Line

Source: Primary Source Material

on $\mathbb{R}$, consider the collection:

$$\mathcal{B}_\ell := \{[a, b) \text{ bounded intervals}\}$$

$\mathcal{B}_\ell$ is a basis for the **lower limit topology** on $\mathbb{R}$. the space $\mathbb{R}_\ell := (\mathbb{R}, \mathcal{T}_\ell)$ is called the Sorgenfrey line.

checking basis is easy. $\mathcal{T}_\ell$ is strictly finer than the std topo, also easy to check. $\mathbb{R}_\ell$ is *not* 2nd ctbl; taking $\mathcal{B} = \{[a, b) : a, b \in \mathbb{Q}\}$ generates a different topology than $\mathcal{T}_\ell$.

Proposition 1.4

$\mathbb{R}_\ell$ is not 2nd ctbl

Proof.

Source: Primary Source Material

fix a basis $\mathcal{B}$. for any x, consider $U_x := [x, x+1)$. since $\mathcal{B}$ a basis, then there exists $x \in B_x \subseteq U_x$. we claim $x \mapsto B_x$ is inj.

if wlog $x < y$, then $x \notin U_y$, so $x \notin B_y$. since $x \in B_x$, $B_x \neq B_y$. ∎

defn: subsptop, prodtop

Lec 6 - May 28 (Week 4)

Title 1.5: Pasting Lemma

sps $X = A \sqcup B$, with A, B closed. let $f : A \to Y, g : B \to Y$ cts, and $f = g$ on $A \cap B$. then $h : X \to Y$ is cts, where:

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

2 Compactness stuff

Lec 8 - June 4 (week 5)

lets characterize cpt sets in terms of closed sets. given $\{U_\alpha : \alpha \in \Lambda\}$ open in X s.t.:

$$X = \bigcup_{\alpha \in \Lambda} U_\alpha$$

let $F_\alpha = X \setminus U_\alpha$ for all α. taking complements we get:

$$\bigcap_{\alpha \in \Lambda} F_\alpha = \varnothing$$

notice X is cpt:

- iff for all collections F_α of closed sets in X:

$$\bigcap_{\alpha} F_\alpha = \varnothing \implies \exists \Lambda_0 \subseteq \Lambda \text{ finite s.t. } \bigcap_{\alpha} F_\alpha = \varnothing$$

- iff for all collections F_α of closed sets in X:

$$\forall \Lambda_0 \subseteq \Lambda \text{ finite, } \bigcap_{\alpha \in \Lambda_0} F_\alpha \neq \varnothing \implies \bigcap_{\alpha} F_\alpha \neq \varnothing$$

Definition 2.1

a collection $\{F_\alpha\}$ of subsets of X has the **finite intersection property (fip)** if for all finite subcollections, their intersection is nonempty

Lec 10 - June 11 (Week 6)

Definition 2.2

let $(P, \leq)$ be a poset.

we say $C \subseteq P$ is a **chain** if $p \leq q$ or $q \leq p$ for all $p, q \in C$. we say $p \in P$ is **maximal** if $p \leq q \implies p = q$ for all $q \in P$.

Theorem 2.3: Zorn's Lemma

let P be a nonempty poset.

if every chain in P is bounded above, then P has a maximal element.

Theorem 2.4: Tychonoff's Theorem

(arbitrary) product of cpt spaces is cpt

proof is long, so we not puttin it in a box.

let $\{X_j : j \in J\}$ be a collection of cpt spaces, and set $X := \prod_j X_j$ with the product topo. let $\mathcal{F}$ be collection of closed sets in X with fip. we show $\bigcap_{\mathcal{F}} F \neq \varnothing$ in two parts.

Claim 1

there exists a collection $\mathcal{D}$ with fip and $\mathcal{F} \subseteq \mathcal{D}$ such that for all $A \subseteq X$ and $D \in \mathcal{D}$, if $A \cap D \neq \varnothing$ then $A \in \mathcal{D}$.

let $\mathbb{P}$ be the set of all collections $\mathcal{D} \supseteq \mathcal{F}$ with fip ordered by $\subseteq$. note $\mathbb{P} \neq \varnothing$ since $\mathcal{F} \in \mathbb{P}$. we claim that every chain is bounded above.

$\ulcorner$ fix $\mathbb{C} \subseteq \mathbb{P}$ a chain. if $\mathbb{C} = \varnothing$, then $\mathcal{F}$ an upper bound. otw, we define:

$$\mathbb{D} = \bigcup_{\mathcal{D} \in \mathbb{C}} \mathcal{D}$$

clearly it is an upper bound; we check $\mathbb{D} \in \mathbb{P}$, ie it has fip. let $D_1, \ldots, D_n \in \mathbb{D}$.

let $\mathcal{D}_1, \ldots, \mathcal{D}_n \in \mathbb{C}$ s.t. $D_i \in \mathcal{D}_i$ for all i. since $\mathbb{C}$ a chain, $\bigcup_i \mathcal{D}_i = \mathcal{D}_j$ for some $j = 1, \ldots, n$. then $D_1, \ldots, D_n \in \mathcal{D}_j$, so $\bigcap_i D_i \neq \varnothing$ since $\mathcal{D}_j$ has fip.

thus, every chain has an upper bound. by zorns lemma, let $\mathcal{D} \in \mathbb{P}$ be maximal. we claim that if $D_1, \ldots, D_n \in \mathcal{D}$, then $\bigcap_i D_i \in \mathcal{D}$ (subclaim 1).

given $D_1, \ldots, D_n \in \mathcal{D}$, note $\mathcal{D} \cup \bigcap_i D_i$ fip, since $\mathcal{D}$ fip. thus $\mathcal{D} \cup \bigcap_i D_i \in \mathbb{P}$, and by maximality, $\mathcal{D} = \mathcal{D} \cup \bigcap_i D_i$, so $\bigcap_i D_i \in \mathcal{D}$.

now, fix $A \subseteq X$, and suppose $A \cap D \neq \varnothing$ for all $D \in \mathcal{D}$. note $\mathcal{D} \cup \{A\}$ has fip. let $D_1, \ldots, D_n \in \mathcal{D}$; then $D := \bigcap_i D_i \in \mathcal{D}$ by subclaim 1. by assumption, $A \cap D \neq \varnothing$. by maximality, $\mathcal{D} = \mathcal{D} \cup \{A\}$, so $A \in \mathcal{D}$. this concludes the proof of claim 1.

note that $\mathcal{D}$ does not necessarily consist of closed sets. however, since $\mathcal{F} \subseteq \mathcal{D}$:

$$\bigcap_{D \in \mathcal{D}} \overline{D} \subseteq \bigcap_{F \in \mathcal{F}} F$$

Claim 2

the set $\bigcap_{D \in \mathcal{D}} \overline{D}$ is nonempty.

consider the projection map(s) $\pi_j : x \to x_j$, and let $\mathcal{D}_j := \{\pi_j(D) : D \in \mathcal{D}\}$. fix $D_1, \ldots, D_n \in \mathcal{D}$, and let $x \in \bigcap_i D_i$. then $\pi_j(x) \in \bigcap_i \pi_j(D_i)$, so $\mathcal{D}_j$ has fip. this also implies that $\left\{\overline{\pi_j(D)}\right\}$ has fip.

since X_j cpt, choose $x_j \in \bigcap_D \overline{\pi_j(D)}$ for each j, using choice. let $x = (x_j)_{j \in J} \in X$. we verify that $x \in \overline{D}$ for all $D \in \mathcal{D}$. to do this, we first fix $j \in J$ and $U_j \subseteq X_j$ open. we claim that if $x \in \pi_j^{-1}(U_j)$, then

$$\pi_j^{-1}(U_j) \cap D \neq \varnothing$$

for all $D \in \mathcal{D}$ (subclaim 2).

fix j and U_j such that $x \in \pi_j^{-1}(U_j)$. fix $D \in \mathcal{D}$. since $x_j \in \overline{\pi_j(D)}$ and $x_j = \pi_j(x) \in U_j$, then $U_j \cap \pi_j(D) \neq \varnothing$. therefore, $\pi_j^{-1}(U_j) \cap D \neq \varnothing$.

by claim 1, this implies that $\pi_j^{-1}(U_j) \in \mathcal{D}$ for all j. now, fix $D \in \mathcal{D}$. we show that for every basic open $B \subseteq X$ such that $x \in B$, we have that $B \cap D \neq \varnothing$.

fix a basic open $B \subseteq X$. then:

$$B = \bigcap_{i=1}^{n} \pi_{j_i}^{-1}(U_{j_i})$$

for some $j_1, \ldots, j_n \in J$ and $U_{j_i} \subseteq X_{j_i}$ open. by subclaim 2, we have that $\pi_{j_i}^{-1}(U_{j_i}) \in \mathcal{D}$. by subclaim 1, we have:

$$B = \bigcap_{i=1}^{n} \pi_{j_i}^{-1}(U_{j_i}) \in \mathcal{D}$$

since $\mathcal{D}$ has fip, then $B \cap D \neq \varnothing$. since this is true for all basic open sets B containing x, this implies that $x \in \overline{D}$; furthermore, this holds for all $D \in \mathcal{D}$.

therefore, we have that $x \in \bigcap_{D \in \mathcal{D}} \overline{D}$ so it is nonempty, concluding the proof of claim 2.

3 One-Point Compactification

Lec 11 - June 13 (Week 6)

recall stereographic projection; we showed that we can embed $\mathbb{R}^n$ in a compact space, minus a point. tdy we analyze for which spaces we can do the same.

Definition 3.1

a **compactification** of a space X is a map $\varphi : X \to Y$ such that

- Y is cpt
- φ is an embedding
- $\varphi(X)$ is dense in Y

we say it is a **Hausdorff compactification** if Y is also Hausdorff. we say it is a (hausdorff) **one-point compactification** if $Y \setminus \varphi(X)$ is a singleton.

Proposition 3.2

(hausdorff) 1pt cptifications are unique up to homeo

box is killing me so f it

sps $\varphi_i : X \to Y_i$ are 1pt cptifications. define:

$$\varphi(y) = \begin{cases} \varphi_2(\varphi_1^{-1})(y) & y \in \varphi_1(X) \\ P_2 & y = P_1 \end{cases}$$

clearly φ is bij and maps cpt to hausdorff, so it suffices to check φ cts. fix $U \subseteq Y_2$ open. if $P_2 \notin U$, then $U \subseteq \varphi_2(X)$. since U open in Y_2, it is open in $\varphi_2(X)$. then

$$\varphi^{-1}(U) = \varphi_1(\varphi_2^{-1})(U)$$

open in $\varphi_1(X)$. since Y_1 hausdorff, $\{P_1\} = Y_1 \setminus \varphi_1(X)$ closed, so $\varphi_1(X)$ open in Y_1, so $\varphi^{-1}(U)$ open in Y_1.

now sps $P_2 \in U$. then $F = Y_2 \setminus U$ closed and $F \subseteq \varphi_2(X)$ so F cpt. then $\varphi^{-1}(F) = \varphi_1(\varphi_2^{-1}(F))$ cpt in Y_1. since Y_1 hausdorff, $\varphi^{-1}(F)$ closed in Y_1.

Proposition 3.3

sps $\varphi : X \to Y$ a hausdorff 1pt cptification. then

- X hausdorff
- X non-cpt (otw $\varphi(X)$ cpt so closed, contradicting dense)
- X [locally cpt]

Proof. Source: Primary Source Material

fix $x \in X$. since $\varphi(x) \neq P$, take open disjoint $U, V \subseteq Y$ with $\varphi(x) \in U, p \in V$. note $K = Y \setminus V$ closed in Y so cpt, and $\varphi(x) \in U \subseteq K$ so $x \in \varphi^{-1}(U) \subseteq \varphi^{-1}(K)$. ■

we know what locally cpt means.

Proposition 3.4

a space X has a hausdorff 1pt cptification iff it is loc cpt, non-cpt, hausdorff

Proof. Source: Primary Source Material

$(\implies)$ follows from above; it suffices to show converse.

let ∞ be a symbol for a pt not in X. define $X_\infty = X \cup \{\infty\}$ and a topology by $\mathcal{T}_\infty = \mathcal{T} \cup \{X_\infty \setminus U \text{ cpt in } X\}$. let $\varphi : X \to X_\infty$ be the inclusion map. we see φ is an embedding (iff X a subsp of X_∞).

if $U \subseteq X$ open in the subsptop, then $U = V \cap X$ where $V \in \mathcal{T}_\infty$. if V open in X then $U = V$ open in X.

otw, $X_\infty \setminus V \subseteq X$ cpt. then $X_\infty \setminus V$ closed in X since X hausdorff. thus $U = X \setminus (X_\infty \setminus V)$ open in X.

next, we see $\varphi(X)$ dense in X_∞. it suffices to show ∞ is a limpt of $\varphi(X) = X$.

sps otw, i.e. $\{\infty\}$ open in X_∞. then $\varphi(X) = \{\infty\}^c$ cpt, but X non-cpt.

(rest finished next lec)

we see X_∞ is cpt, as for any $\{U_\alpha\}$ open cvr, $\infty \in X_\infty$, so there is α_0 s.t. $\infty \in U_{\alpha_0}$. then $\{U_\infty \cap X\}$ is an open cover in X of $X_\infty \setminus U_{\alpha_0} \subseteq X$. by defn, it is cpt, so:

$$X_\infty \setminus U_{\alpha_0} \subseteq \bigcup_{i=1}^{n} (U_i \cap X)$$

then $X_\infty = \bigcup_i U_i \cup U_{\alpha_0}$.

we also see X_∞ is hausdorff. for fixed $x \neq y \in X_\infty$, sps $x \in X$ and $y = \infty$ (otw trivial). then by loc cpt, there is open $U \subseteq X$ and cpt $K \subseteq X$ s.t. $x \in U \subseteq K$. set $V := X_\infty \setminus K$; then V open in $X_\infty, y \in V$, and $U \cap V = \varnothing$. ■

4 Quotient Spaces

Lec 12 (Makeup) - Jun 17 (Week 7)

motivating example: the pacman game board is homeo to a torus.

Definition 4.1

let X, Y be spaces, $p : X \to Y$ be surj. we say p is a **quotient map** [qmap] if for any $U \subseteq Y$, U open iff $p^{-1}(U)$ open.

for any set Y and surj $p : X \to Y$, we define the **quotient topology** as subsets $U \subseteq Y$ with $p^{-1}(U)$ open.

exercise: check this is a topology

how do we characterize cts functions $f : X/\sim \to Y$?

Proposition 4.2

let X, Y be spaces, $\sim$ equiv rel on X, and p the qmap. if $g : X \to Y$ is a function that is constant on equiv classes, then g induces a map $f : X/\sim \to Y$ s.t. $g = f \circ p$.

in particular, f cts iff g cts, and f qmap iff g qmap.

Proof. Source: Primary Source Material

define f by $f([x]) = g(x)$. since g constant on classes, this is well-defined; clearly $g = f \circ p$.

clearly if f cts, g is also cts. if g cts, then for any open $U \subseteq Y$, $g^{-1}(U)$ open. but $g^{-1}(U) = p^{-1}(f^{-1}(U))$ so $f^{-1}(U)$ open by defn.

the fact about qmaps follows from p being a qmap. ■

Corollary 4.3

sps $g : X \to Y$ surj cts. define $\sim$ on X as in the canonical decomposition of g. then g induces a map $f : X/\sim \to Y$ and f homeo iff g qmap.

Proof. Source: Primary Source Material

clearly g constant on equiv classes, so f follows from the thm.

if f homeo, then f qmap, so g qmap; if g qmap, then f qmap, so f homeo since its bij. ■

II Other, New Spaces

5 separation axioms i guess

Lec 13 - Jul 2 (Week 8)

we start with "weaker" separation axioms.

- T_0: for any distinct pts, $\exists$ open set containing one but not the other
  - also known as "Kolmogorov space"
- T_1: for distinct pts, $\exists$ open $U, V \subseteq X$ with $x \in U \setminus V, y \in V \setminus U$
  - also known as "Freéchet space"
- T_2: usual hausdorff

clearly $\mathsf{T}_2 \implies \mathsf{T}_1 \implies \mathsf{T}_0$. on $\mathbb{R}$, examples of T_1 but not T_2 include cofinite and coctbl. examples of T_0 but not T_1 include ray and particular pt.

Proposition 5.1

if X is T_1, constant seqs cvg to a unique pt.

Proof. Source: Primary Source Material

sps $a_n = x$ for all n and $a_n \to y \neq x$. by T_1, there are open $U, V \subseteq X$ with $x \in U \setminus V$ and $y \in V \setminus U$. but $a_n \notin V$, contradicting $a_n \to y$. ■

this is not true for T_0 spaces; for instance, in $(\mathbb{R}, \mathcal{T}_p)$, the seq $a_n = p$ cvgs to every pt.

Proposition 5.2

tfae

i) X is T_1

ii) singletons are closed

iii) finite sets are closed

iv) $\forall\, A \subseteq X, A = \bigcap\limits_{A \subseteq U} U$

exercise.

Proposition 5.3

if X is T_1, then $x \in A'$ iff every open $x \in U$ intersects A infinitely

see pp4 sth sth

we move on to strong separation axioms.

Definition 5.4

a space X is **regular** if for every closed $F \subseteq X$ and $x \notin F$, there are open sets $U, V \subseteq X$ s.t. $x \in U, F \subseteq V, U \cap V = \varnothing$.

we define a space as $\boldsymbol{\mathsf{T}_3}$ if it is reg and T_1.

example: indiscrete on $\mathbb{R}$ is (vacuously) regular but not hausdorff. note $\mathsf{T}_3 \implies \mathsf{T}_2$.

Example 5.5 Source: Primary Source Material

- discrete spaces
- metric spaces
- $\mathbb{R}_K$ is T_2 but not T_3

Proposition 5.6

if X reg, then the weak axioms are equivalent

Proof. Source: Primary Source Material

we show $\mathsf{T}_0 \implies \mathsf{T}_2$. let $x \neq y$; by T_0, there is open $x \in U, y \notin U$. then U^c closed; by reg, find U_0, V open with

$$x \in U_0, U^c \subseteq V, U_0 \cap V = \varnothing$$

then $x \in U_0, y \in V, U_0 \cap V = \varnothing$. ∎

note: authors don't agree on defns for T_3 and regular; munkres calls as "regular" our T_3.

Proposition 5.7

X reg iff for any open $x \in U \subseteq X$, there is open $V \subseteq X$ with $x \in V \subseteq \overline{V} \subseteq U$.

Proof. Source: Primary Source Material

fix $x \in U$. by reg, let $U^c \subseteq U_0$ and $x \in V$ open with $U_0 \cap V = \varnothing$. note $\overline{V} \subseteq U$ (or $\overline{V} \cap F = \varnothing$).

⌜ if $y \in \overline{V} \cap F$, then $y \in U_0$. but $U_0 \cap V = \varnothing$, contradicting $y \in \overline{V}$. ⌟

this proves the forward direction.

fix F closed and $x \notin F$. then $x \in F^c$. by assumption, there is open V with:

$$x \in V \subseteq \overline{V} \subseteq U$$

then, note $(\overline{V})^c \cap V = \varnothing$. since $F \cap \overline{V} = \varnothing$, then $F \subseteq U_0$. ■

Proposition 5.8

cpt hausdorff implies regular

Proof.

Source: Primary Source Material

fix closed F and $x \notin F$. then F cpt. for any $y \in F$, we can find U_y, V_y open s.t.:

$$x \in U_y \quad y \in V_y \quad U_y \cap V_y = \varnothing$$

by cpt, there is finite $F_0 \subseteq F$ s.t.:

$$F \subseteq \bigcup_{y \in F_0} V_y =: V$$

let $U = \bigcap_{F_0} U_y$. we claim $U \cap V = \varnothing$.

if $z \in U \cap V$ then $z \in V_y$ for some y. but then $z \in U_y \cap V_y$, contradicting $U_y \cap V_y = \varnothing$.

so $U \cap V = \varnothing$ as needed. ■

Definition 5.9

we say a space is **normal** if for every closed disjoint E, F there are open U, V with $E \subseteq U, F \subseteq V, U \cap V = \varnothing$.

we define a space as $\mathbf{T_4}$ if it is normal and T_1.

Example 5.10 Source: Primary Source Material

$X = \{0, 1\}$ with $\mathcal{T} = \{\varnothing, \{0\}, X\}$ is normal but not T_2. this is because given two disjoint closed sets, one must be empty.

note X is T_0, so it is not enough for the defn. also $\mathsf{T}_4 \implies \mathsf{T}_3$.

Example 5.11 Source: Primary Source Material

- discrete spaces are normal (thus T4)
- metric spaces are normal (thus T4)
- $\mathbb{R}_\ell$ is normal (thus T4)
- $\mathbb{R}_\ell^2$ is reg (reg is fin. productive) but not normal

Proposition 5.12

normall iff for every open U, closed V with $V \subseteq U$, there exists open $W \subseteq X$ s.t. $V \subseteq W \subseteq \overline{W} \subseteq U$.

pf: pp7

Proposition 5.13

cpt hausdorff implies normal

Proof. Source: Primary Source Material

fix E, F disj closed. since X reg, for every $x \in E$ we have open U_x, V_x s.t.:

$$x \in U_x \quad F \subseteq V_x \quad U_x \cap V_x = \varnothing \quad E \subseteq \bigcup_{x \in E} U_x$$

furthermore, E cpt. thus there is finite $E_0 \subseteq E$ s.t. bla bla. define:

$$U := \bigcup_{x \in E_0} U_x \qquad V := \bigcap_{x \in E_0} V_x$$

■

regular is hereditary and fin. productive, but normality is neither. SPACING AUGH

6 Urysohn's Metrization

Lec 15 - Jul 9 (Week 9)

Definition 6.1

two disjoint A, B are **separated by a cts function** if there is a cts $f : X \to [0, 1]$ s.t. $f|_A = 0$ and $f|_B = 1$.

notice that $[0, 1]$ can be replaced with any closed interval. furthermore, if two sets are ctsly separated, then they are separated by open sets $f^{-1}([0, 1/2))$ and $f^{-1}((1/2, 1])$.

Lemma 6.2: Urysohn's Lemma

X normal iff two disj closed sets are ctsly separated

Proof.

Source: Primary Source Material

($\Longleftarrow$) trivial / by defn

($\implies$) fix $E, F \subseteq X$ closed disjoint. first, construct a seq of open sets where:

$$\{U_p : p \in \mathbb{Q} \cap [0, 1]\} \qquad p < q \implies \overline{U_p} \subseteq U_q$$

enumerate $\mathbb{Q} \cap [0,1]$ as $\{p_n\}$. assume $p_0 = 1, p_1 = 0$. let $U_{p_0} = U_1 = F^c$. since $E \subseteq X \setminus F$, by normality, there exists $U_{p_1} = U_0$ such that:

$$E \subseteq U_0 \subseteq \overline{U_0} \subseteq X \setminus F = U_1$$

now sps $U_{p_0}, \ldots, U_{p_n}$ exist. let p be the largest p_i s.t. $p < p_{n+1}$, and let q be the smallest p_i s.t. $p_{n+1} < q$. since $\overline{U_p} \subseteq U_q$, by normality, there exists open $U_{p_{n+1}}$ s.t.:

$$\overline{U_p} \subseteq U_{p_{n+1}} \subseteq \overline{U_{p_{n+1}}} \subseteq U_q$$

note we can also define $U_p = \varnothing$ for $p < 0$ and $U_p = X$ for $p > 1$.

define $f(x) = \inf\{p \in \mathbb{Q} : x \in U_p\}$. note $0 \le f(x) \le 1$. then $f|_E = 0, f|_F = 1$.

$x \in E \implies x \in U_0$ by construction. then $f(x) \le 0 \implies f(x) = 0$.

similarly $x \in F \implies x \notin F^c = U_1$, so $x \notin U_p$ for all $p \le 1$. therefore, $x \in U_p \implies p > 1$ so $f(x) \ge 1 \implies f(x) = 1$.

$x \in \overline{U_p} \implies f(x) \le p$ and $x \notin U_p \implies f(x) \ge p$.

recall $f(x) = \inf A(x)$, where $A(x) = \{p \in \mathbb{Q} : x \in U_p\}$.

$x \in \overline{U_p} \implies x \in U_q$ for all $q > p$. then $(p, \infty) \subseteq A(x)$ so $f(x) \le p$.

$x \notin U_p \implies x \notin U_q$ for all $q \le p$. then $A(x) \subseteq (p, \infty)$ so $p \le f(x)$.

lastly, we show f cts. fix $U = (a,b) \cap [0,1]$. fix $x \in f^{-1}(U)$. we construct open $x \in W \subseteq f^{-1}(U)$. notice $a < f(x) < b$. choose $p, q \in \mathbb{Q}$ such that:

$$a < p < f(x) < q < b$$

then $x \in V := U_q \setminus \overline{U_p}$ open. we claim $V \subseteq f^{-1}(U)$. fix $y \in V$. then:

$$y \in U_q \implies f(y) \le q < b \qquad y \notin \overline{U_p} \implies a < p \le f(x)$$

■

question: what are the differences between "regular" vs "pt and closed set are ctsly separated", and "hausdorff" vs "distinct pts ctsly separated"?

Theorem 6.3: Urysohn's Metrization Theorem

X 2nd ctbl and T_3 implies metrizable

idea: embed $X \hookrightarrow \mathbb{R}^{\mathbb{N}}$.

Lemma 6.4

X 2nd ctbl T_3. there exists ctbl $f_n : X \to [0,1]$ cts s.t. for every open U and $x \in U$, there is f_n st $f_n(x) > 0$ and $f_n|_{U^c} = 0$.

Proof.

Source: Primary Source Material

fix ctbl basis $\mathcal{B}$ of X. let $A := (n,m) \in \mathbb{N}^2 : \overline{B_n} \subseteq B_m$. for each $(n,m) \in A$, we have $\overline{B_n} \cap B_m^c = \varnothing$.

since X is T_3 2nd ctbl, it is normal. by urysohn's lemma, find cts $f_{n,m}$ s.t.:

$$f_{n,m}|_{\overline{B_n}} = 1 \qquad f_{n,m}|_{B_m^c} = 0$$

then for any open U and $x_0 \in U$, $\exists\,(n,m) \in A$ s.t. $f_{n,m}(x_0) > 0$ and $f_{n,m}|_U = 0$.

⌜ fix open U and $x \in U$. let m be such that $x \in B_m \subseteq U$. by reg, $\exists\,V$ open with $x \in V \subseteq \overline{V} \subseteq B_m$.

find n s.t. $x \in B_n \subseteq V$. then $(n,m) \in A$ by construction. then, note $f_{n,m}(x_0) > 0$ and for $x \notin U$, we have $x \notin B_m$ and thus $f_{n,m}(x) = 0$ by construction. ⌟

■

we now prove urysohn's thm.

by the lemma, $\exists$ seq of cts f_n s.t.:

$$\forall\,U \text{ open } \exists\,n \text{ s.t. } f_n|_U > 0 \text{ and } f_n|_{U^c} = 0$$

define $F : X \to \mathbb{R}^{\mathbb{N}}$ by $F(x) = (f_1(x), f_2(x), \dots)$. we show this is the desired embedding.

first, F cts since each coord is cts. next, F is inj.

fix $x \neq y$, and let $U = \{y\}^c$. note U open and $x \in U$. thus, $\exists\, n$ s.t. $f_n(x) > 0$ but $f_n(y) = 0$, so $f(x) \neq f(y)$.

lastly, for any open U, $F(U)$ open in $F(X)$.

fix $y_0 \in F(U)$. let $x_0 \in U$ s.t. $y_0 = F(x_0)$. then $\exists\, n$ s.t. $f_n(x_0) > 0$ and $f_n|_{U^c} = 0$.

let $V := \pi_n^{-1}((0, \infty))$. this is open in $\mathbb{R}^{\mathbb{N}}$. let $W = V \cap F(x)$ which is open in $F(x)$. we claim $y_0 \in W \subseteq F(x)$. indeed, note $\pi_n(y_0) = f_n(x_0) > 0$, so $y_0 \in W$.

fix $y \in W$. then $y = F(x)$ for some x and $\pi_n(y) > 0$. then $f_n(x) > 0$ so $x \in U$, thus $y \in F(U)$.

so F is an embedding, and we are done.

another application of urysohn's lemma:

Theorem 6.5: Tietze's Extension Theorem

X normal, A closed.

(a) if $f : A \to [a, b]$ cts, then it has a cts ext to X.

(b) if $f : A \to \mathbb{R}$ cts, then it has a cts ext to X.

idea: construct seq $s_n : X \to \mathbb{R}$ cts st s_n cvgs uniformly to some f the lim of s_n.

Lemma 6.6

fix space X, metric space Y. suppose f_n is a seq of cts f s.t. $f_n \to f$ uniform. then f cts.

Proof. Source: Primary Source Material

fix open V. let $x_0 \in f^{-1}(V)$. then $\exists\, \varepsilon > 0$ s.t. $B(x_0, \varepsilon) \subseteq V$. we want open U s.t. $d(f(x_0), f(x)) < \varepsilon$ for all $x \in U$.

by uniform continuity, $\exists\, N$ s.t. $d(f_n(x), f(x)) < \varepsilon/3$ for all $n \geq N, x \in X$. since f_N cts, $U = f_N^{-1}(B(f(x_0), \varepsilon/3))$ open. then:

$$d(f(x_0), f(x)) \;\leq\; d(f(x_0), f_n(x_0)) + d(f_n(x_0), f_n(x)) + d(f_n(x), f(x)) \;\leq\; 3\varepsilon/3 \;=\; \varepsilon$$

■

Lec 16 - Jul 11 (Week 9)

today we prove tietze's ext thm.

first, consider the case where $f : A \to [-r, r]$ with $r > 0$. we want to construct cts $g : X \to \mathbb{R}$ s.t.:

(1) $|g(x)| \leq r/3$ for all x

(2) $|f(x) - g(x)| \leq 2r/3$ for all x

let $E = f^{-1}([r/3, r])$ and $F = f^{-1}([-r, -r/3])$. note E, F closed disj. by urysohn's lemma, there is cts $g : X \to [-r/3, r/3]$ s.t.:

$$g|_E = \frac{-r}{3} \qquad g|_F = \frac{r}{3}$$

now, we prove (a).

since $[a,b] \simeq [-1,1]$, sps wlog $f : X \to [-1,1]$. apply above construction to find cts $g_1 : X \to \mathbb{R}$ with:

$$|g_1| \leq \frac{1}{3} \qquad |(f - g_1)|_A| \leq \frac{2}{3}$$

apply again to $f - g_1 : A \to [-2/3, 2/3]$ to get cts $g_2 : X \to \mathbb{R}$ with:

$$|g_2| \leq \frac{1}{3}\left(\frac{2}{3}\right) \qquad |(f - g_1 - g_2)|_A| \leq \left(\frac{2}{3}\right)^2$$

continue inductively; given $g_1, \ldots, g_n : X \to \mathbb{R}$ cts with:

$$|g_i| \leq \frac{1}{3}\left(\frac{2}{3}\right)^{i-1} \qquad \left|\left(f - \sum_{i=1}^{n} g_i\right)\Big|_A\right| \leq \left(\frac{2}{3}\right)^n$$

apply construction to $f - \sum g_i : A \to [(-2/3)^n, (2/3)^n]$ to get cts $g_{n+1} : X \to \mathbb{R}$ with:

$$|g_{n+1}| \leq \frac{1}{3}\left(\frac{2}{3}\right)^n \qquad \left|\left(f - \sum_{i=1}^{n+1} g_i\right)\Big|_A\right| \leq \left(\frac{2}{3}\right)^{n+1}$$

we claim $\sum g_n(x)$ cvgs for all $x \in X$.

since

$$|g_n| \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} \qquad \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} < \infty$$

then by comparison test $\sum g_n$ cvgs.

define $g := \sum g_n$. denote by $s_n(x)$:

$$s_n(x) = \sum_{i=1}^{n} g_i(x)$$

then $s_n \to g$ ptwise for all x. we claim $s_n \to g$ uniformly.

for $k > n$:

$$|s_k - s_n| \leq \sum_{i=n+1}^{k} |g_i| < \sum_{i=n+1}^{\infty} |g_i| \leq \sum_{i=n+1}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{i-1} = \left(\frac{2}{3}\right)^{n}$$

taking $k \to \infty$, $|g - s_n| \leq (2/3)^n$ for all x.

thus we have that g cts. note $f|_A = g|_A$:

$$|f - s_n| \leq \left(\frac{2}{3}\right)^{n} \implies |f - g| = 0 \text{ as } n \to \infty$$

furthermore, $|g(x)| \leq 1$ for all x:

$$|g| \leq \sum_{i=1}^{\infty} |g_i| \leq \sum_{i=1}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{i-1} = 1$$

this proves (a). we now prove (b).

since $\mathbb{R} \simeq (-1,1)$ assume $f : A \to (-1,1)$. by (a), there exists cts $g : X \to [-1,1]$ extending f. define B as:

$$B := \{x \in X : |g(x)| = 1\} = g^{-1}(-1) \cup g^{-1}(1)$$

since A, B closed disj, by urysohn's lemma, let $\varphi : X \to [0,1]$ be cts s.t.:

$$\varphi|_B = 0 \qquad \varphi|_A = 1$$

consider $\widehat{f} = g\varphi$. then $\widehat{f}$ cts and:

$$\widehat{f} : X \to (-1,1) \qquad \widehat{f}|_A = f|_A$$

this proves (b).

question: does the converse hold? A: yes! note:

$$f(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \end{cases}$$

this is cts by pasting lemma; then extend.

7 Connectedness

Lec 17 - Jul 16 (Week 10)

Definition 7.1

X is **disconnected** if there are non-empty disjoint open U, V s.t. $U \cup V = X$. otw, it is **connected**.

Proposition 7.2

tfae

(i) X conn

(ii) no A, B non-empty s.t. $A \cup B = X$ and $\overline{A} \cap B = A \cap \overline{B} = \varnothing$

(iii) only clopen subsets are $\varnothing$ and X

(iv) for all non-empty proper $A \subseteq X$, $\partial A = \varnothing$

Proof. Source: Primary Source Material

(iii) $\iff$ (iv) by assignment 2.

(i) $\implies$ (ii) sps A, B satisfy (ii). then $U = X \setminus \overline{A}$ and $V = X \setminus \overline{B}$ disconn X.

(ii) $\implies$ (iii) A clopen $\implies$ $X \setminus A$ clopen; they satisfy (ii)

(iii) $\implies$ (i) if it has a clopen subset, it disconns X ■

some questions:

- is conn productive? [yes]
- is conn hereditary? no - $\{0, 1\}$ in $\mathbb{R}$
- is conn ctsly preserved? (yes)

Lemma 7.3

if U, V disconn X, conn components are either in U or V

not writing that

Proposition 7.4

$A \subseteq X$ conn. if $A \subseteq B \subseteq \overline{A}$ then B conn

Proof. Source: Primary Source Material

sps $C \cup D = B$ and $\overline{C} \cap D = C \cap \overline{D} = \varnothing$. then $A = (C \cap A) \cup (D \cap A)$, so wlog $C \subseteq A$. then $B \subseteq \overline{A} \subseteq \overline{C}$, but $\overline{C} \cap D = \varnothing$ so $B \cap D = \varnothing$. ■

Lec 18 - Jul 18 (Week 10)

Definition 7.5

a **path** from x to y is a cts $\gamma : [0, 1] \to X$ s.t. $\gamma(0) = x$ and $\gamma(1) = y$

properties:

- pathconn implies conn (we been knew)
- pathconn is *not* hereditary, but *is* productive
- in $\mathbb{R}$, conn iff pathconn

examples:

- cvx implies pathconn in [nvs]
- $\mathbb{R}^n$ pathconn for $n > 1$
- S^n pathconn
- topologists sine curve: closure is conn but not pathconn

oh we just spent the rest of the time proving the toposine lol

Proposition 7.6

in $\mathbb{R}^n$, open conn implies pathconn

Proof. Source: Primary Source Material

(sketch) fix x and consider the set A of pts in U with a path to x. then, use connectedness to show $A = U$; that is, nonempty and clopen in U. ■

III Algebraic Topology

8 Path Homotopy

Lec 19 - Jul 23 (Week 11)

ALGTOP RAAAHHHHHHHHHHHHH

Definition 8.1

given γ_0, γ_1 from x to y, a **path homotopy** from γ_0 to γ_1 is a cts $F : [0,1]^2 \to X$ such that the following holds:

$$F(s,0) = \gamma_0(s) \text{ and } F(s,1) = \gamma_1(s) \qquad F(0,t) = x \text{ and } F(1,t) = y$$

we say γ_0 is **path homotopic** to γ_1, or $\gamma_0 \simeq_p \gamma_1$, if a pathtopy exists

[pathtopy is certainly a. choice.] note that its important for F to be cts; this is strictly stronger than requiring F to be cts in each coordinate (beloved $xy/x^2 + y^2$).

Example 8.2 Source: Primary Source Material

$A \subseteq \mathbb{R}^n$ cvx $\implies$ any two paths w/ same endpoints are homotopic: for fixed s, take $F(s,t) = (1-t)\gamma_0(s) + t\gamma_1(s)$. check that this is a pathtopy.

we can generalize pathtopy to deformations of general cts functions.

Definition 8.3

sps $f, g : X \to Y$ cts. a **homotopy** from f to g is a cts $F : X \times [0,1] \to Y$ with

$$F(x,0) = f(x) \qquad F(x,1) = g(x)$$

we write $f \simeq g$ in this case.

clearly, both $\simeq, \simeq_p$ are equiv rels. note transitivity uses pasting lemma [technically. its like a single pt tho].

9 The Fundamental Group

[yeah this should go here tbh]

Definition 9.1

for a path γ_0 from x to y and γ_1 from y to z, define:

$$\gamma_0 * \gamma_1(s) = \begin{cases} \gamma_0(2s) & s \in [0, 1/2] \\ \gamma_1(2s-1) & s \in [1/2, 1] \end{cases}$$

this induces an operation $*$ on the set of equivalence classes.

Proposition 9.2

$*$ is well-defined on equivalence classes.

Proof.

Source: Primary Source Material

fix $\gamma_0 \simeq \gamma_0'$ and $\gamma_1 \simeq \gamma_1'$. let $F, G : [0,1]^2 \to X$ be pathtopies from γ_i to γ_i' respectively. consider:

$$H(s,t) = \begin{cases} F(2s, t) & (s,t) \in [0,1/2] \times [0,1] \\ G(2s-1, t) & (s,t) \in [1/2, 1] \times [0,1] \end{cases}$$

it is easy to see that H is then a homotopy from $\gamma_0 * \gamma_1$ to $\gamma_0' * \gamma_1'$. ∎

Definition 9.3

a **loop** is a path $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1)$. we say that γ is a loop at x_0 if $\gamma(0) = \gamma(1) = x_0$.

given a fixed x_0, we denote by $\pi_1(X, x_0)$ the set of all pathtopy equiv classes of loops at x_0.

we define by $e_x : [0,1] \to X$ and $\overline{\gamma} : [0,1] \to X$ as:

$$e_x(s) = x \qquad \overline{\gamma}(s) = \gamma(1-s)$$

these are the "constant" and "inverse" paths respectively.

Definition 9.4

given γ, let $\varphi : [0,1] \to [0,1]$ be cts with $\varphi(0) = 0$ and $\varphi(1) = 1$. we call $\gamma \circ \varphi$ a **reparametrization** of γ.

Lemma 9.5

$\gamma \simeq_p \gamma \circ \varphi$

Proof. Source: Primary Source Material

$F(s,t) = \gamma((1-t)s + t\varphi(s))$ is a pathtopy. ■

okay the rest is just proving that $\pi_1(X, x_0)$ is a grp under $*$. uhh the pfs look kinda annoying to write so im just not gonna. uses the reparametrization tho

Lec 20 - Jul 25 (Week 11)

from last time: given X and a basept $x_0 \in X$, we associated the group $\pi_1(X, x_0)$ to (X, x_0) called the fundamental grp of X w basept x_0.

π_1 is also known as a **functor**.

$$\begin{aligned} \text{Topological space} &\xrightarrow{\pi_1} \text{Group} \\ \text{Continuous map} &\longrightarrow \text{Homomorphism} \\ \text{Homeomorphism} &\longrightarrow \text{Isomorphism} \end{aligned}$$

today we will prove this! whatever that means. in the meantime: did you know the torus has fundamental group $\mathbb{Z}$?

Q: what happens to $\pi_1(X, x_0)$ if we change the basept?

Proposition 9.6

fix $x_0, x_1 \in X$. let α be a path from x_0 to x_1. define $\widehat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ as:

$$\widehat{\alpha}([\gamma]) = [\overline{\alpha} * \gamma * \alpha]$$

then $\widehat{\alpha}$ is well-defined, and $\widehat{\alpha}$ is an isomorphism.

Source: Primary Source Material

Proof.

well-definedness is an exercise. show:

$$\gamma_0 \simeq_p \gamma_1 \implies \overline{\alpha} * \gamma_0 * \alpha \simeq_p \overline{\alpha} * \gamma_1 * \alpha$$

it is a homomorphism because:

$$\begin{aligned} \widehat{\alpha}([\gamma_0] * [\gamma_1]) = \widehat{\alpha}([\gamma_0 * \gamma_1]) &= [\overline{\alpha} * \gamma_0 * \gamma_1 * \alpha] \\ &= [\overline{\alpha} * \gamma_0 * e_{x_0} * \gamma_1 * \alpha] \\ &= [\overline{\alpha} * \gamma_0 * \alpha * \overline{\alpha} * \gamma_1 * \alpha] \\ &= [\overline{\alpha} * \gamma_0 * \alpha] * [\overline{\alpha} * \gamma_1 * \alpha] \\ &= \widehat{\alpha}([\gamma_0]) * \widehat{\alpha}([\gamma_1]) \end{aligned}$$

it is bijective because:

$$\widehat{\alpha} \circ \widehat{\overline{\alpha}}([\gamma]) = \widehat{\alpha}([\alpha * \gamma * \overline{\alpha}]) = [\overline{\alpha} * \alpha * \gamma * \overline{\alpha} * \alpha] = [\gamma]$$

$\widehat{\overline{\alpha}} \circ \widehat{\alpha}$ is similarly id. ■

Corollary 9.7

X pathconn then $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$ for all x_0, x_1. in this case, fundgrp does not depend on basept, and we can denote it $\pi_1(X)$.

Definition 9.8

X is **simply connected** iff X pathconn and fundgrp is trivial.

for instance, any cvx subset of $\mathbb{R}^n$ is simply conn.

notation: we write $\varphi : (X, x_0) \to (Y, y_0)$ to mean that φ cts, $\varphi(x_0) = x_1$.

any map $\varphi : (X, x_0) \to (X, x_1)$ induces a homomorphism $\varphi_* : \pi_1(X, x_0) \to \pi_1(X, x_1)$ as:

$$\varphi_*([\gamma]) = [\varphi \circ \gamma]$$

this is known as the **induced map** of φ. exercise: check this is well-defined, that is $\gamma_0 \simeq_p \gamma_1 \implies \varphi \circ \gamma_0 \simeq_p \varphi \circ \gamma_1$. idea: if $F : [0,1]^2 \to X$ is path homotopy from γ_0 to γ_1, then $G = \varphi \circ F$ is a path homotopy from $\varphi \circ \gamma_0$ to $\varphi \circ \gamma_1$. (check this!)

Proposition 9.9

let $\varphi : (X, x_0) \to (Y, y_0)$. then the induced map is a homomorphism.

Proof. Source: Primary Source Material

we check $\varphi_*([\gamma_0] * [\gamma_1]) = \varphi_*([\gamma_0]) * \varphi_*([\gamma_1])$.

$$\begin{aligned} \varphi_*([\gamma_0] * [\gamma_1]) &= \varphi_*([\gamma_0 * \gamma_1]) = [\varphi \circ (\gamma_0 * \gamma_1)] \\ &\color{red}{=} [(\varphi \circ \gamma_0) * (\varphi \circ \gamma_1)] = \varphi_*([\gamma_0]) * \varphi_*([\gamma_1]) \end{aligned}$$

to see red equality, note that $\varphi \circ (\gamma_0 * \gamma_1) = (\varphi \circ \gamma_0) * (\varphi \circ \gamma_1)$. check this! ∎

some properties:

(i) if $\varphi : (X, x_0) \to (Y, y_0)$ and $\psi : (Y, y_0) \to (Z, z_0)$, then $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.

(ii) if $\iota : (X, x_0) \to (X, x_0)$ is id, then ι_* is id.

(iii) if $\varphi : (X, x_0) \to (Y, y_0)$ is homeo, then φ_* is iso.

proofs:

(i) $(\varphi \circ \psi)_*([\gamma]) = [\varphi \circ \psi \circ \gamma] = \varphi_*([\psi \circ \gamma]) = \varphi_*(\psi_*([\gamma]))$

(ii) $\iota_*([\gamma]) = [\iota \circ \gamma] = [\gamma]$

(iii) by (i) and (ii), $\varphi_* \circ (\varphi^{-1})_* = \iota_*$ and $(\varphi^{-1})_* \circ \varphi_* = \iota_*$. this also shows $(\varphi_*)^{-1} = (\varphi^{-1})_*$.

summary: given the following:

$$(X, x_0) \xrightarrow{\quad\varphi\quad} (Y, y_0)$$

we can apply π_1 to transform this into:

$$\pi_1(X, x_0) \xrightarrow{\quad\varphi_*\quad} \pi_1(Y, y_0)$$

10 Covering Spaces

Lec 21 - Jul 30 (Week 12)

today's mission: prove $\pi_1(S^1) = \mathbb{Z}$. we notate $I = [0, 1]$.

proof sketch: let $\omega_n(s) = (\cos(2\pi sn), \sin(2\pi sn))$. this is a loop in S^1 at $x_0 = (1, 0)$ that does n revolutions. moves ccw if $n > 0$, cw if $n < 0$. we want to show any loop in S^1 is pathtopic to some ω_n.

idea: show every $\gamma : I \to S^1$ can be "uniquely lifted" to a path $\widetilde{\gamma} : I \to \mathbb{R}$ from 0 to some n. embed $\mathbb{R} \hookrightarrow \mathbb{R}^3$ as the "helix":

$$s \to (\cos(2\pi s), \sin(2\pi s), s)$$

let $P : \mathbb{R} \to S^1$ be the projection of the helix onto the xy-plane. we want to show two things:

(a) for any loop $\gamma : I \to S^1$, there is a unique $\widetilde{\gamma} : I \to \mathbb{R}$ starting at 0 s.t.:

[commutative diagram]

(b) for every pathtopy $F : I^2 \to S^1$ s.t. $F(0,0) = x_0$, there is a unique pathtopy $\widetilde{F} : I^2 \to \mathbb{R}$ s.t. $\widetilde{F}(0,0) = 0$ and:

[commutative diagram]

Definition 10.1

given spaces X, E, we say $P : E \to X$ is a **covering map** if for every $x_0 \in X$ there is an open $x_0 \in U$ which is "evenly covered by P", i.e.:

$$P^{-1}(U) = \bigsqcup_{\alpha \in \Lambda} V_\alpha$$

is a union of pairwise disjoint open sets in E such that the map $P|_{V_\alpha} : V_\alpha \to U$ is a homomorphism.

in this context, we say E is a **covering space** of X. each V_α is also known as a **sheet** or **slice**.

some examples:

- $\mathrm{id} : X \to X$ is a cvring space (1-sheeted)
- $P : \mathbb{R} \to S^1$ given by $P(s) = (\cos(2\pi s), \sin(2\pi s))$ (countably many sheets)
- $P_n : S^1 \to S^1$ given by $P_n(z) = z^n$, where $S^1 \subseteq \mathbb{C}$ (n-sheeted)

properties of covering maps:

- $\forall\, x \in X$, $P^{-1}(x)$ is a discrete subspace of E. that is, every $e \in P^{-1}(x)$ has an open $V \subseteq E$ s.t. $V \cap P^{-1}(x) = \{e\}$
- cvring maps are open maps (exercise)
- cvring maps are **local homeos**. that is, for all $e \in E$, there is an open $e \in V \subseteq E$ s.t. $P|_V : V \to P(V)$ is a homeo

Definition 10.2

let $P : E \to X$ be a cvring map, $f : X \to Y$ be cts. a **lifting of f** is a cts $\widetilde{f} : Y \to E$ s.t. $f = P \circ \widetilde{f}$

[diagram]

Lemma 10.3: Path-lifting property

let $P : E \to X$ be a cvring map and $x = P(e)$. if $\gamma : I \to X$ is a path starting at x, then there is a unique lifting to a path $\widetilde{\gamma} : I \to E$ starting at e.

Proof.

Source: Primary Source Material

step 1: find a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ s.t. $\gamma([t_{i-1}, t_i])$ is contained in some evenly covered open U_i.

for all $t \in I$, $\gamma(t) \in X$. then $\exists$ open nbhd U_t of $\gamma(t)$ s.t. each U_t evenly cvred:

$$I = \bigcup_{t \in I} \gamma^{-1}(U)$$

by lebesgue number lemma, let $\delta > 0$ s.t. for any $\operatorname{diam}(A) < \delta$, there is some $t \in I$ s.t. $A \subseteq \gamma^{-1}(U_t)$. choose a partition $P = \{t_0, \ldots, t_n\}$ s.t. $\|P\| = \max |t_i - t_{i-1}| < \delta$. then we have that $[t_{i-1}, t_i] \subseteq \gamma^{-1}(U)$ for some open U evenly cvred.

step 2: we prove existence of $\widetilde{\gamma}$. we construct $\widetilde{\gamma}$ inductively on each subinterval of the partition.

$\widetilde{\gamma}(0) = e_0$. sps $\widetilde{\gamma}$ defined on $[0, t_{i-1}]$. we extend to $[0, t_i]$ by defining $\widetilde{\gamma}$ on the next subinterval $[t_{i-1}, t_i]$.

by step 1, there is an open U evenly covered s.t. $\gamma([t_{i-1}, t_i]) \subseteq U$. notice that $\widetilde{\gamma}(t_{i-1}) \in P^{-1}(U)$ since:

$$P \circ \widetilde{\gamma}(t_{i-1}) = \gamma(t_{i-1}) \in U$$

then $\widetilde{\gamma}(t_{i-1}) \in V_\alpha$ where V_α is a sheet of $P^{-1}(U)$.

recall $P|_{V_\alpha} : V_\alpha \to U$ is homeo. define:

$$\widetilde{\gamma}(s) = \left(P^{-1}|_{V_\alpha}\right)(\gamma(s)) \qquad s \in [t_{i-1}, t_i]$$

note $\widetilde{\gamma}|_{[0,t_i]}$ cts by pasting lemma, and $\gamma = P \circ \widetilde{\gamma}$.

step 3: we prove uniqueness of $\widetilde{\gamma}$.

sps $\widehat{\gamma}$ is another lifting of γ with $\widehat{\gamma}(0) = e_0$. we show $\widehat{\gamma}(s) = \widetilde{\gamma}(s)$ for all $s \in [t_{i-1}, t_i]$ inductively.

sps $\widehat{\gamma}|_{[0,t_{i-1}]} = \widetilde{\gamma}|_{[0,t_{i-1}]}$. note $\widehat{\gamma}([t_{i-1}, t_i])$ conn and $\widehat{\gamma}(t_{i-1}) \in V_\alpha$ where V_α is the sheet we used to define $\widetilde{\gamma}$ on $[t_{i-1}, t_i]$ in step 2.

it then follows that $\widehat{\gamma}([t_{i-1}, t_i]) \subseteq V_\alpha$ by conn. since $\gamma = P \circ \widehat{\gamma}$:

$$\widehat{\gamma}(s) = \left(P^{-1}|_{V_\alpha}\right)(\gamma(s)) = \widetilde{\gamma}(s)$$

for all $s \in [t_{i-1}, t_i]$.

■

Lemma 10.4: Path-homotopy lifting property

sps $P : E \to X$ cvring map, $x = P(e)$. if $F : I^2 \to X$ is a pathtopy with $F(0,0) = x$, then there is a unique lifting $\widetilde{F} : I^2 \to E$ which is a **pathtopy** in E s.t. $\widetilde{F}(0,0) = e$.

pf: very similar to the one above. not writing allat

Corollary 10.5

let $P : E \to X$ be a cvring map and $x = P(e)$. if $\gamma_0 \simeq_p \gamma_1$, then $\widetilde{\gamma}_0 \simeq_p \widetilde{\gamma}_1$.

Theorem 10.6

$\pi_1(S^1) = \mathbb{Z}$.

dont rly wanna put this in a box

let $x_0 = (1,0) \in S^1$ and $P : \mathbb{R} \to S^1$ be the cvring map given by:

$$P(s) = (\cos(2\pi s), \sin(2\pi s))$$

given $[\gamma] \in \pi_1(S^1, x_0)$, let $\widetilde{\gamma} : I \to \mathbb{R}$ be the unique lifting of γ s.t. $\widetilde{\gamma}(0) = 0$. define $\varphi : \pi_1(S^1, x_0) \to \mathbb{Z}$ given by:

$$\varphi([\gamma]) = \widetilde{\gamma}(1) \in P^{-1}(x_0) = \mathbb{Z}$$

first, we show φ is well-defined.

by prev crll, if $\gamma_0 \simeq_p \gamma_1$, then $\widetilde{\gamma}_0 \simeq_p \widetilde{\gamma}_1$. in particular, $\widetilde{\gamma}_0(1) = \widetilde{\gamma}_1(1)$.

next, φ is surjective.

fix $n \in \mathbb{Z}$. consider $\omega_n(s) = (\cos(2\pi sn), \sin(2\pi sn))$. then $\varphi([\omega_n]) = \widetilde{\omega_n}(1) = n$.

next, φ is injective.

sps $[\gamma_0], [\gamma_1] \in \pi_1(S^1, x_0)$ s.t. $\varphi([\gamma_0]) = \varphi([\gamma_1])$. note $\widetilde{\gamma_0}, \widetilde{\gamma_1} : I \to \mathbb{R}$ are paths in $\mathbb{R}$. since $\mathbb{R}$ cvx, then $\widetilde{\gamma_0} \simeq_p \widetilde{\gamma_1}$ via pathtopy $\widetilde{F} : I^2 \to \mathbb{R}$. then, $P \circ F : I^2 \to S^1$ is a pathtopy from $P \circ \widetilde{\gamma_0} = \gamma_0$ to $P \circ \widetilde{\gamma_1} = \gamma_1$, so $[\gamma_0] = [\gamma_1]$.

finally, φ is homo.

given $[\gamma_0], [\gamma_1] \in \pi_1(S^1, x_0)$, let:

$$\widetilde{\gamma_0}(1) = n \qquad \widetilde{\gamma_1}(1) = m$$

then $\gamma_0 \simeq_p \omega_n$ and $\gamma_1 \simeq_p \omega_m$. we show $\varphi([\omega_n] * [\omega_m]) = n + m$.

let $\widetilde{\omega_n}$ be the lifting starting at 0 and ending at n. let $\widetilde{\omega_m}$ be the lifting starting at n. then, we have that:

$$\varphi([\omega_n] * [\omega_m]) = (\widetilde{\omega_n} * \widetilde{\omega_m})(1) = n + m$$

note: check that $\widetilde{\omega_n} * \widetilde{\omega_m}$ is indeed a lifting of $\omega_n * \omega_m$.

11 Retractions

Lec 22 - Aug 6 (Week 13)

last time, we showed that $\pi_1(S^1) = \mathbb{Z}$. today we examine some applications.

Definition 11.1

let $A \subseteq X$. we say A is [a] **retract** of X if there is a cts $r : X \to A$ s.t. $r(a) = a$ for all $a \in A$. we call the map r a **retraction**.

Proposition 11.2

if A a retract of X, then the homo given as

$$\iota_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$$

induced by the inclusion $\iota : A \to X$ is injective.

Proof. Source: Primary Source Material

let $r : X \to A$ be a retraction. note $r \circ \iota : A \to A$ is the identity, so $r_* \circ \iota_*$ is the trivial homo. since r_* is a left-inv of ι_*, we are then done. ■

Example 11.3 Source: Primary Source Material

S^1 is *not* a retract of $D^2 = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$.

by the prev thm, $\iota_* : \pi_1(S^1) \to \pi_1(D^2)$ would be inj, but:

$$\pi_1(S^1) = \mathbb{Z} \qquad \pi_1(D^2) = \{e\}$$

so this is not possible.

Example 11.4 Source: Primary Source Material

S^1 is a retract of the "figure 8" space (i.e. $S^1 \vee S^1$).

[label each copy of S^1 as A, B resp. and the base pt as x_0.] then the map r given by

$$r(x) = \begin{cases} x & x \in A \\ x_0 & x \in B \end{cases}$$

is a retraction.

Example 11.5 Source: Primary Source Material

$S^1 \vee S^1$ is *not* a retract of $D^2 \vee D^2$.

by contra, sps $r : D^2 \vee D^2 \to S^1 \vee S^1$ is a retraction. then:

$$D^2 \hookrightarrow D^2 \vee D^2 \longrightarrow S^1 \vee S^1 \longrightarrow S^1$$

would be a retraction $D^2 \to S^1$, a contradiction.

Definition 11.6

let $A \subseteq X$. we say A is a **deformation retract** if id : $X \to X$ is homotopic to a retraction via $F : X \times I \to X$ s.t. $F(a, t) = a$ for all $t \in I$ and $a \in A$.

the homotopy F is called a **deformation retraction**.

Example 11.7

Source: Primary Source Material

S^1 is a deform retract of $\mathbb{R}^2 \setminus \{0\}$.

take $F : \mathbb{R}^2 \setminus \{0\} \times I \to \mathbb{R}^2 \setminus \{0\}$ as:

$$F(x, t) = (1 - t)x + \frac{tx}{\|x\|}$$

this is a deform retraction.

Example 11.8

Source: Primary Source Material

consider $X = \mathbb{R}^3 \setminus \{\lambda e_3\}$, or $\mathbb{R}^3$ without the z-axis. then, $\mathbb{R}^2 \setminus \{0\}$ is a deform retract of X.

take $F((x, y, z), t) = (x, y, (1 - t)z)$.

Example 11.9

Source: Primary Source Material

let X be $\mathbb{R}^2$ minus two pts. then $S^1 \vee S^1$ is a deform retract of X.

[u can just visualize this one tbh.]

Proposition 11.10

if A deform retract of X and $a_0 \in A$, then the homo $\iota_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$ induced by $\iota : A \to X$ is iso.

Proof. Source: Primary Source Material

it suffices to show ι_* surj.

fix $F : X \times I \to X$ deform retraction of X onto A, and $[\gamma] \in \pi_1(X, a_0)$. consider $G : I \times I \to X$ as:

$$G(s,t) = F(\gamma(s), t)$$

note G is a pathtopy from γ to some loop α in A:

$$G(0,t) = F(\gamma(0), t) = F(a_0, t) = a_0$$

for all $t \in I$. then:

$$\iota_*([\alpha]) = [\alpha] = [\gamma]$$

■

Corollary 11.11

$\pi_1(\mathbb{R}^2 \setminus \{0\}) = \mathbb{Z}$.

ok lets prove brower fixed pt (for D^2) using algtop. step one:

Definition 11.12

a cts $f : X \to Y$ is **nullhomotopic** if $f \simeq e_{x_0}$, i.e. f homotopic to a constant map.

Proposition 11.13

let $h : S^1 \to X$. tfae:

(i) h nulltopic

(ii) there exists cts ext $k : D^2 \to X$ of h

(iii) h_* is trivial homo

Proof. Source: Primary Source Material

(i) $\implies$ (ii)

sps $h \simeq e_{x_0}$. let $F : S^1 \times I \to X$ be homotopy from h to e_{x_0}.

note $D^2 \cong (S^1 \times I)/(S^1 \times \{1\})$. consider $p : S^1 \times I \to D^2$ given as $p(x,t) = (1-t)x$. this is a qmap.

F constant on $S^1 \times \{1\}$. by properties of quotients, there exists some cts $k : D^2 \to X$ s.t. $F = k \circ p$. then for $x \in S^1$, $k(x) = F(x, 0) = h(x)$.

(ii) $\implies$ (iii)

sps $k : D^2 \to X$ cts ext of h. let $\iota : S^1 \to D^2$ be inclusion. note $h = k \circ \iota$.

then $h_* = k_* \circ \iota_*$. note:

$$\iota_* : \pi_1(S^1) \to \pi_1(D^2) \qquad \pi_1(S^1) = \mathbb{Z} \qquad \pi_1(D^2) = \{e\}$$

(iii) $\implies$ (i)

sps h_* trivial. note S^2 is a quotient of I, since $S^1 = I / \{0, 1\}$ with:

$$x_0 = (1, 0) \qquad p(s) = (\cos(2\pi s), \sin(2\pi s))$$

let $[p] \in \pi_1(S^1, x_0)$ and $h_*[p] = [e_{h(x_0)}]$. note $h_*([p]) = [h \circ p] =: [f]$.

fix pathtopy $F : I^2 \to X$ from f to $e_{h(x_0)}$. let $q(s,t) = (p(s), t)$; this is qmap from I^2 to $S^1 \times I$.

since F pathtopy, it is constant on pts identified by q. then, there exists some cts $G : S^1 \times I \to X$ s.t. $F = G \circ q$. to check homotopy, note that:

$$G(x, 0) = F(s, 0) = f(s) = h(p(s)) = h(x)$$

■

Example 11.14 Source: Primary Source Material

$\iota : S^1 \to \mathbb{R}^2 \setminus \{0\}$, the inclusion map, is not nulltopic since $\iota_* : \mathbb{Z} \to \mathbb{Z}$ is the id, so nontrivial. for the same reason, $\mathrm{id} : S^1 \to S^1$ is not nulltopic.

ok, back to fixed points. step two: vector fields on D^2.. what.

Definition 11.15

a **vector field** on D^2 is a cts $\mathcal{V} : D^2 \to \mathbb{R}^2$. we say $\mathcal{V}$ is **non-vanishing** if we have that $\mathcal{V}(x) \neq 0$ for all $x \in D^2$.

Proposition 11.16

if $\mathcal{V} : D^2 \to \mathbb{R}^2 \setminus \{0\}$ non-vanishing vecfield, then:

(1) there exists $x \in S^1$ s.t. $\mathcal{V}(x) = \alpha x$ for some $\alpha < 0$. that is, $\mathcal{V}(x)$ points directly inwards

(2) there exists $x \in S^1$ s.t. $\mathcal{V}(x) = \alpha x$ for some $\alpha > 0$, that is, $\mathcal{V}(x)$ points directly outwards

Proof. Source: Primary Source Material

(2) follows from applying (1) to $-\mathcal{V}$.

by contradiction, sps no $\mathcal{V}(x)$ point directly inwards. consider the map given by $h = \mathcal{V}|_{S^1} : S^1 \to \mathbb{R}^2 \setminus \{0\}$. then $\mathcal{V}$ is a cts ext of h; by prev. prop, h is nulltopic. we claim h homotopic to inclusion $\iota : S^1 \to \mathbb{R}^2 \setminus \{0\}$.

consider $F(s,t) = (1-t)h(s) + ts$. we check $F : S^1 \times I \to \mathbb{R}^2 \setminus \{0\}$, i.e., $F(s,t) \neq 0$. indeed, sps $F(s,t) = 0$. then $(1-t)h(s) = -ts$. so $\mathcal{V}(s) = h(s) = -ts/(1-t)$, so $\mathcal{V}(s)$ points inwards, contradiction.

but $\iota : S^1 \to \mathbb{R}^2 \setminus \{0\}$ not nulltopic, so contradiction (again). ∎

Theorem 11.17: Brouwer's Fixed Point Theorem (for the 2D disc)

if $f : D^2 \to D^2$ cts, then there is $x \in D^2$ s.t. $f(x) = x$.

Proof. Source: Primary Source Material

by contra, sps $f(x) \neq x$ for all $x \in X$. let $\mathcal{V}(x) = f(x) - x$, so $\mathcal{V}$ is non-vanishing vecfield. by prev thm, there is $\alpha > 0$ and $x \in S^1$ s.t. $\mathcal{V}(x) = \alpha x$. , but $f(x) = (\alpha + 1)x \notin D^2$, a contradiction. ∎

is brouwer fixed pt true for D^n? yes [duh], but it requires more advanced algtop [x doubt]: homotopy theory and homotopy groups $\pi_n(X)$.

Lec 23 - Aug 8 (Week 13)

we now move on to borsuk ulam.

Definition 11.18

fix $x \in S^n$. we say $-x$ is the **antipode** of x.

a map $f : S^n \to S^n$ is **antipode-preserving** if for all x, $f(-x) = -f(x)$.

Proposition 11.19

if $h : S^1 \to S^1$ cts and antipode preserving, then h *not* nulltopic.

Proof. Source: Primary Source Material

consider the cvring map $q : S^1 \to S^1$ given by:

$$q(\cos\theta, \sin\theta) = (\cos 2\theta, \sin 2\theta) \qquad q(z) = z^2$$

note $q(z) = q(-z)$ and is a qmap. consider $q \circ h : S^1 \to S^1$. this has the property:

$$(q \circ h)(-z) = q(-h(z)) = q(h(z)) = (q \circ h)(z)$$

note there exists cts k s.t. $q \circ h = k \circ q$. let $x_0 = (1, 0)$. we claim $k_* : \pi_1(S^1, x_0) \to \pi_1(S^1, x_0)$ is *not* trivial.

consider the path $\widetilde{\gamma}$ from x_0 to $-x_0$. then $\gamma = q \circ \widetilde{\gamma}$ is a loop at x_0, and:

$$k_*([\gamma]) = [k \circ \gamma] = [k \circ q \circ \widetilde{\gamma}] = [q \circ h \circ \widetilde{\gamma}]$$

it suffices to show $[q \circ h \circ \widetilde{\gamma}] \neq [e_{x_0}]$.

sps otw, i.e. $q \circ h \circ \widetilde{\gamma} \simeq_p e_{x_0}$. note $h \circ \widetilde{\gamma}$ is the lifting of $q \circ h\widetilde{\gamma}$ starting at x_0. thus, the pathtopy lifts to a pathtopy from $h \circ \widetilde{\gamma}$ to e_{x_0}. but:

$$h \circ \widetilde{\gamma}(1) \;=\; h(-x_0) \;=\; -h(x_0) = -x_0 \neq x_0.$$

note we can assume $h(x_0) = x_0$. otw, consider the rotation $\rho : S^1 \to S^1$ with $\rho(h(x_0)) = x_0$. apply the result to $\rho \circ h$. thus $\rho \circ h$ not nulltopic, so h not nulltopic.

now, we claim $h_* : \pi_1(S^1, x_0) \to \pi_1(S^1, x_0)$ is not trivial.

we know $k_* : \mathbb{Z} \to \mathbb{Z}$ nontrivial, thus must be inj. also, $q_* : \mathbb{Z} \to \mathbb{Z}$ given as $q_*(a) = 2a$, so q_* inj. therefore, $q_* \circ h_* = k_* \circ q_*$ must be inj. in particular, this means h_* cannot be trivial.

■

Proposition 11.20

there is no cts antipode-preserving map $g : S^2 \to S^1$.

Proof. Source: Primary Source Material

sps by contradiction g is such a map. let $E \subseteq S^2$ be an equator of S^2. consider $h := g|_E : S^1 \to S^1$. then h is cts and antipode-preserving.

by the prev thm, h is not nulltopic. however, h has cts ext to the upper hemisphere of S^2, which is homeo to D^2, a contradiction. ■

Theorem 11.21: Borsuk-Ulam

if $f : S^2 \to \mathbb{R}^2$ cts, there exists $x \in S^2$ s.t. $f(x) = f(-x)$.

Proof. Source: Primary Source Material

sps otw, i.e. $f(x) \neq f(-x)$ for all x. consider $g : S^2 \to S^1$ given by:

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$$

then, notice that:

$$g(-x) = \frac{f(-x) - f(x)}{\|f(x) - f(-x)\|} = -g(x)$$

this contradicts the prev thm. ∎

notice that it is not true for the torus - simply project onto $\mathbb{R}^2$.