MAT49?
Noncommutative Algebra

Personal Self-Study Notes

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A First Course in Noncommutative Rings, 2nd ed

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Contents

Preface

These notes were created during my own self-studying. They are likely filled with errors and/or mistakes, and should not be treated as study material for anyone but myself, unless specified by me. These notes were also written alongside a specific resource, usually a textbook, which can be found on this document's cover. These notes are formatted in a way which is catered to my own understanding; you may not learn topics or subjects in the same manner that I do, so please do not be surprised or feel bad if these notes do not make much sense.

If you are viewing this document in Obsidian, you may notice that the links in the pdf document do not work. This is intentional behaviour, as I currently do not have or know of a decent solution which allows them to behave well with the setup in Obsidian. However, below certain pages, there may be links to other documents - these are usually context-relevant links between notes of different areas of study. I created these links to point out potential similarities, or in case one area of study is borrowing a concept, definition, or theorem from another area of study, and you wish to see the full, original definition/derivation/proof or whatever it may be.

I Wedderburn-Artin Theory

1 Introduction

Week 1

Throughout this course, the term "ring" refers to an algebraic ring with multiplicative identity, usually denoted 1, which is not necessarily commutative.

Definition 1.1

Given a ring R, a **left ideal** $I \subseteq R$ satisfies for all $x, y \in I$ and $r \in R$:

$$x + y \in I \qquad -x \in I \qquad rx \in I$$

A **right ideal** is defined analogously.

When referring to an *ideal* $I \subseteq R$, we always mean a 2-sided ideal; that is, I is both a left and right ideal.

Recall that for any ideal, we can always form the quotient ring $\overline{R} = R/I$, which induces a natural surjection $R \twoheadrightarrow \overline{R}$ and satisfies the universal property of quotients.

Definition 1.2

A nonzero ring R is **simple** if (0) and R are the only ideals of R.

Evidently, a nonzero ring R is simple iff for all nonzero $a \in R$, $(a) = R$. Thus, a nonzero R is simple iff for all $a \neq 0$ in R, there exists $b_i, c_i \in R$ such that $\sum b_i a c_i = 1$. In particular, for commutative rings, R is simple iff R is a field.

Definition 1.3

A nonzero $a \in R$ is a **left (right) zero divisor** if for some nonzero $b \in R$:

$$ab = 0 \qquad (ba = 0)$$

Example 1.4

Source: Primary Source Material

Consider $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z} \end{pmatrix}$ with $a = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Clearly a is a left zero divisor, but:

$$0 = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2x & y \\ 0 & z \end{pmatrix}$$

implies $x, y, z = 0$. However, $b^2 = 0$, so b is both a left and right zero divisor.

Definition 1.5

A ring is a **domain** if $ab = 0$ implies $a = 0$ or $b = 0$. Commutative domains are typically known as **integral domains**.

Definition 1.6

A ring is **reduced** if it has no nonzero nilpotent elements, or equivalently, if $a^2 = 0$ implies $a = 0$.

As an example, the direct product of any family of domains is reduced.

Definition 1.7

We denote by $U(R)$ or sometimes R^* the set of all units of R. This is a group under multiplication in R, with identity as 1.

If $ab = 1$, then $a \in U(R)$ iff $ba = 1$. In literature, R is said to be *Dedekind-finite* (or *von-Neumann finite*) if $ab = 1 \implies ba = 1$. Many rings satisfying some "finiteness" property can be shown to be Dedekind-finite.

Consider the k-vector space V given as $ke_1 \oplus ke_2 \oplus \dots$ w countably inf basis $\{e_i\}$, and let $R = \mathrm{End}_k(V)$. Define $a, b \in R$ as:

$$b(e_i) = e_{i+1} \qquad a(e_i) = e_{i-1}$$

with $a(e_1) = 0$. Then $ab = 1 \neq ba$, so R is an example of a non-Dedekind-finite ring.

Definition 1.8

A ring is a **division ring** if $R \neq 0$ and $U(R) = R \setminus \{0\}$.

Notice that commutative division rings are just fields.

To check if a ring is a division ring, it suffices to verify every nonzero $a \in R$ is right-invertible(?). It follows that $R \neq 0$ is a division ring iff $(0), R$ are the only right ideals. Note we can replace "right" with "left" analogously, so we can also freely use these results.

In connection to the above comment, it is useful to consider the following operation on a ring.

Definition 1.9

The **opposite ring** R^{op} consists of the elements of R, w multiplication given as:

$$a^{\text{op}} \cdot b^{\text{op}} = (ba)^{\text{op}}$$

This construction lets us obtain analogous results "on the left" when used appropriately.

Consider the following construction: sps R, S are rings and M an (R, S)-bimodule, that is, M is a left R-module and right S-module s.t. $(rm)s = r(ms)$. We can then form:

$$A = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$$

Under the usual matrix multiplication, this forms a ring. Rings constructed in this way are known as **triangular rings**. By varying choices of R, S, and M, we can get many examples and counterexamples; this is mainly due to the (sided) ideals in A.

First, we can identify R, S, and M as subgroups of A in the obvious? way, writing $A = R \oplus M \oplus S$. In terms of this decomposition, multiplication in A can be described by this chart:

	R	M	S
R	R	M	0
M	0	0	M
S	0	0	S

From this, we can see that R is a left ideal, S is a right ideal, and M is a (square zero)(?) ideal in A. Furthermore, $R \oplus M$ and $M \oplus S$ are both ideals of A, with $R \oplus S$ a subring of A, and:

$$S \simeq A/(R \oplus M) \qquad R \simeq A/(M \oplus S)$$

Proposition 1.10

1. Left ideals of A are $I_1 \oplus I_2$, where I_1 a left R-submodule of $R \oplus M$ containing MI_2, I_2 a left ideal of S.
2. Right ideals of A are $J_1 \oplus J_2$, where J_1 a right ideal of R, J_2 a right S-submodule of $M \oplus S$ containing $J_2 M$.
3. Ideals of A are $K_1 \oplus K_0 \oplus K_2$, where K_1, K_2 ideals in R, S resp., K_0 an (R, S)-subbimodule of M containing $K_1 M + M K_2$.

Proof. Source: Primary Source Material

The fact that such $I_1 \oplus I_2, J_1 \oplus J_2, K_1 \oplus K_0 \oplus K_2$ are their corresp. ideals is clear from the mult table above(?).

Let I be a left ideal of A. If $\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \in I$, then so are:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} r & m \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix}$$

Therefore, $I = I_1 \oplus I_2$, where $I_1 = I \cap (R \oplus M)$ and $I_2 = I \cap S$. Clearly, I_1 is a left R-submodule of $R \oplus M$ and I_2 a left ideal of S.

Furthermore:

$$MI_2 = M(I \cap S) \subseteq I \cap M \subseteq I \cap (R \oplus M) = I_1$$

The second is proved similarly.

Finally, if K an ideal of A and $\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \in K$, then so are:

$$\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix}$$

and hence also $\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$. This shows that $K = K_1 \oplus K_0 \oplus K_2$, where:

$$K_1 = K \cap R \qquad K_0 = K \cap M \qquad K_2 = K \cap S$$

Since K, M ideals, we have $K_1 M + M K_2 \subseteq K \cap M = K_0$; the rest is clear. ■

The above tells us that the sided ideals in A are closely tied to the corresp. left R-module or right S-module structures on M. Often, these modules can be quite different, and so the sided ideals of A will appropriately exhibit drastically different behaviour.

To illustrate this point, we use this construction for a ring which is left but not right Noetherian (resp. Artinian). First, however, we will quickly review these topics.

Definition 1.11

A **left Noetherian ring** is a ring such that its left ideals satisfy the *ascending chain condition* (ACC): that is, there does not exist an infinite sequence of left ideals such that:

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \dots$$

A **left Artinian ring** is a ring such that its left ideals satisfy the corresponding *descending chain condition* (DCC).

Let M be either a left or right R-module. We say M is **Noetherian** (**Artinian**) if its submodules satisfy ACC (DCC).

Lemma 1.12

1. A ring R is Noetherian (Artinian) if it is Noetherian (Artinian) as an R-module.
2. M is Noetherian iff every submodule is finitely generated.
3. M is both Noetherian and Artinian iff M has a (finite) composition series.
4. For a submodule N, M is Noetherian (Artinian) iff N and M/N are both Noetherian (Artinian). Notably, the property is maintained by direct sums.
5. If M is a finitely generated left module over a left Noetherian (u get it) ring, then M is Noetherian (...).

The above (except possibly 5) are assumed from a prerequisite algebra course.

Proposition 1.13

Let $A = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ be as above. Then A is left noeth iff R, S are left noeth, and M as a left R-module is noeth. We can replace left with right (R-module to S-module) and noeth with artin.

Proof.

Source: Primary Source Material

It suffices to prove the left noeth case.

First, sps A is left noeth. Since R, S quot rings of A, they're also left noeth. If M_i is an asc. chain of left R-submods of M, by passing to $\begin{pmatrix} 0 & M_i \\ 0 & 0 \end{pmatrix}$, we get an asc. chain of left ideals of A. Thus M_i terminates, so M as a left R-mod is noeth.

Conversely, sps R, S left noeth and M as a left R-mod is noeth. Let I_j be an asc. chain of left ideals in A. The contraction of I_j to S must terminate, and similarly for $R \oplus M$, by the prior lemma. Recalling that:

$$I_j = (I_j \cap S) \oplus (I_j \cap (R \oplus M))$$

we see that I_j becomes stationary, so A is left noeth. ■

Corollary 1.14

Let S be a comm noeth domain not equal to R, its field of fractions. Then $A = \begin{pmatrix} R & R \\ 0 & S \end{pmatrix}$ is left noeth and not right noeth, and neither left nor right artin.

Proof.

Source: Primary Source Material

It suffices to show S is not artin and R as a right S-mod is not noeth.

First, simply note that for a nonunit $0 \neq s \in S$, the sequence of ideals $(s_k) = (s^k)$ does not satisfy DCC.

Now, assume instead R is a noeth S-mod. Then R is a finitely generated S-mod, so there exists a common denom $s \in S$ for all fractions in R. But then $1/s^2 = s'/s$ for some $s' \in S$, which means $s \in U(S)$, contradiction $S \neq R$. ■

Corollary 1.15

Let $S \subseteq R$ be fields s.t. $\dim_S R = \infty$. Then $A = \begin{pmatrix} R & R \\ 0 & S \end{pmatrix}$ is left noeth artin, but not right noeth nor artin.

There are two more useful observations of A constructed immediately above.

First, as a left A-mod, it has a comp. series of len 3:

$$A \supsetneq \begin{pmatrix} 0 & R \\ 0 & S \end{pmatrix} \supsetneq \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix} \supsetneq (0)$$

The fact this cannot be refined follows from the characterization of ideals of A, and directly shows A is left noeth artin.

Secondly, since $\dim_S R = \infty$, we can easily construct an inf dirsum $\bigoplus_{\mathbb{N}} M_i$ of nonzero right S-subspaces(?) in R. By passing to the corresp. ideals in A, we obtain an inf dirsum of nonzero right ideals in A. However, we cannot have such an inf dirsum of *left* ideals in A. Using future terminology, A is left Goldie but not right Goldie.

There are, ofc, other ways to construct such rings. We end with two more constructions.

Example 1.16

Source: Primary Source Material

Let σ be an endo of a divring k which is not an auto. Then $R = k[x;\sigma]$ is left noeth but not right noeth.

Indeed, if I is a nonzero left ideal of R, choose a monic left poly $f \in I$ of the least degree. The usual euclidean algo implies $I = Rf$. Thus, every left ideal is principal (ie R is a left PID, or principal left ideal domain); in particular, R left noeth.

On the other hand, fix $b \in k \setminus \sigma(k)$. We claim $\sum_{\mathbb{N}} x^i bxR$ is a dirsum of right ideals, which implies R not right noeth. Assume there exists an equation

$$x^n bx f_n(x) + \cdots + x^{n+m} bx f_{n+m}(x) = 0$$

where the first and last terms are nonzero. Since R dom, this gives $bxf_n(x) = xg(x)$ for some $g(x) \in R$. If f_n has highest deg term $c_r x^r$ with $c_r \neq 0$ and $g(x) = \sum a_i x^i$, a comparison of coeffs of x^{r+1} shows $b\sigma(c_r) = \sigma(a_r)$, contradicting $b \notin \sigma(k)$.

Incidentally, R is not artin, since there are inf descending chains Rx^k and $x^k R$.

Example 1.17: Dieudonne

Source: Primary Source Material

consider $R = \mathbb{Z} < x, y > /(y^2, yx)$. we claim R left but not right noeth.

note R generated by x, y w rels $y^2 = yx = 0$. then, R has dirsum decomposition given by

$$R \;=\; \mathbb{Z}[x] \;\oplus\; \mathbb{Z}[x]y$$

here, $\mathbb{Z}[x]$ a subring, $\mathbb{Z}[x]y$ an ideal. assuming the Hilbert Basis Theorem, we get $\mathbb{Z}[x]$ is noeth. by lemma 1.12(5), R is noeth as a left $\mathbb{Z}[x]$-mod, and hence as an R-mod. thus R left noeth.

to show R not right noeth, we show $I = \mathbb{Z}[x]y$ not finitely gen'd as a right R-mod. since both x, y act trivially on the right of I, if I were finitely gen'd as a right R-mod, it would be finitely gen'd as an abel grp. but this is not the case, as

$$I \;=\; \mathbb{Z}[x]y \;=\; \bigoplus_{i=0}^{\infty} \mathbb{Z} \cdot x^i y$$

incidentally, R not artin, since I an ideal in R and $R/I \simeq \mathbb{Z}[x]$ not artin.

Week 1 - Meeting

Definition 1.18

Let k be a ring. A ring R is a k**-algebra** if there is a ring hom $m : k \longrightarrow Z(R)$:

$$m(a+b) = m(a) + m(b) \qquad m(ab) = m(a)m(b) \qquad m(1) = 1$$

This defines scaling in R via $ar = m(a)r$, where R is a k-mod.

2 Semisimplicity

Week 2

throughout this chapter (and possibly the book), we write ${}_RM$ or M_R to indicate whether M is a left or right R-mod. if M is an (R, S)-bimod, we may write M as ${}_RM_S$.

Definition 2.1

let R be a ring, M a (left) R-mod.

M is called **simple** (or **irreducible**) if $M \neq 0$ and has no R-submods other than (0) and M.

M is called **semisimple** (or **completely reducible**) if every R-submod is an R-mod direct summand of M; that is, for any submod N, there exists a complement P s.t. $M = N \oplus P$.

note the zero module is semisimple, but not simple. furthermore, evident from the following is:

Corollary 2.2

any submod (resp. qmod)(?) of a semisimp R-mod is semisimp.

we also clearly have that simple $\implies$ semisimp. we examine the relationship between the two more closely, but first prove an intermediate fact.

Lemma 2.3

any semisimp ${}_RM \neq 0$ contains a simple submod.

Proof.

Source: Primary Source Material

fix nonzero $m \in M$. by 2.2, it suffices to consider when $M = Rm$.

by zorns, there exists submod N of M maximal wrt the property $m \notin N$. take a (necessarily nonzero) submod N' s.t. $M = N \oplus N'$. we finish by showing N' simple.

indeed, if $N'' \neq 0$ a submod of N', $m \in N \oplus N''$ by maximality, so $N \oplus N'' = M$. ■

as a result, we get two other characterizations of semisimp mods, which can be useful as alt defns.

Theorem 2.4

tfae:

1. ${}_RM$ semisimp.
2. ${}_RM$ a dirsum of a family of simple submods.
3. ${}_RM$ a sum of a family of simple submods.

note that by convention, we take the empty (dir)sum to be the zero module.

Proof.

Source: Primary Source Material

(1 $\implies$ 3) let M_1 be the sum of all simple submods, and write $M = M_1 \oplus M_2$. if $M_2 \neq 0$, by 2.3, M_2 contains a simple submod, a contradiction.

(3 $\implies$ 1) write $M = \sum_{i \in I} M_i$ and let $N \subseteq M$ be a given submod. consider subsets $J \subseteq I$ with the following properties:

- $\sum_{j \in J} M_j$ is a *direct* sum.
- $N \cap \sum_j M_j = 0$.

an easy check shows zorns applies to the family of all such J's under set inclusion. pick a maximal J and let:

$$M' \;:=\; N + \sum_j M_j \;=\; N \oplus \bigoplus_j M_j$$

we finish by showing $M' = M$, for which it suffices to show $M_i \subseteq M' \; \forall \, i$. if some $M_i \nsubseteq M'$, simplicity of M_i implies $M' \cap M_i = 0$, so:

$$M' + M_i = N \oplus \left(\bigoplus_j M_j \right) \oplus M_i$$

but this contradicts maximality of J.

($3 \implies 2$) follows from the above on $N = 0$, and ($2 \implies 3$) is a tautology. ■

we can now define a (left) semisimp ring.

Definition 2.5

a ring R is (left) **semisimple** if ${}_RR$ is semisimp.

Theorem 2.6

tfae:

1. all short exact seqs of left R-mods split.
2. all left R-mods are semisimp.
3. all finitegen left R-mods are semisimp.
4. all cyclic left R-mods are semisimp.
5. ${}_RR$ is semisimp.

Proof.

Source: Primary Source Material

note that clearly(?):

$$(1) \iff (2) \implies (3) \implies (4) \implies (5)$$

thus it suffices to prove $(5) \implies (2)$.

fix ${}_RM$ where R satisfies (5). by 2.2, (5) implies any cyclic submod Rm of M is semisimp. since $M = \sum_{m \in M} Rm$, it follows from 2.4(3) that M is semisimp. ■

let R be left semisimp. using 2.6(5), we have $R = \bigoplus_i U_i$ for simple left R-mods U_i, which are just minimal left ideals in R. since $1 \in R$, this dirsum is in fact *finite*. thus, we can write a composition series for ${}_RR$ w comp factors $\{{}_RU_i\}$. by 1.12(3), ${}_RR$ satisfies ACC and DCC for R-submods.

Corollary 2.7

a left semisimp ring is both left noeth and left artin.

the characterization 2.6(1) gives us a homological interpretation of left semisimplicity; this is done using the notion of a *projective module*.

Definition 2.8

${}_RP$ is called ***R*-projective** (or **projective**) if for any surj R-hom $f : {}_RA \longrightarrow {}_RB$ and any R-hom $g : {}_RP \longrightarrow {}_RB$ there exists an R-hom $h : {}_RP \longrightarrow {}_RA$ s.t. $f \circ h = g$.

the following propn from homalg offers two alt characterizations of proj mods.

Theorem 2.9

a (left) R-mod P is proj iff P is (iso to) a dirsum of a left free R-mod, iff any surj R-hom from any left R-mod onto P splits.

Proof.

exercise :) ■

we can now state the homological characterization of the class of (left) semisimp rings.

Theorem 2.10

tfae:

1. R is left semisimp.
2. all left R-mods are proj.
3. all finitegen left R-mods are proj.
4. all cyclic left R-mods are proj.

Proof. Source: Primary Source Material

we have:

$$(1) \iff (2) \implies (3) \implies (4)$$

we show $(4) \implies (1)$ by verifying ${}_RR$ semisimp.

consider any left ideal $U \subseteq R$. by our assumption, the left R-mod R/U is proj, so the following s.e.seq splits:

$$0 \longrightarrow U \longrightarrow R \longrightarrow R/U \longrightarrow 0$$

this implies U an R-mod direct summand of ${}_RR$ as needed. ∎

there is also the injective mod, directly dual to the proj mod. literally u can figure out the defn, its a dual. the 2nd part of 2.9 (?) admits the following dual: I is inj iff any inj R-hom to any left R-mod splits. we can thus deduce an analogous characterization:

1. R is left semisimp.
2. all left R-mods are inj.
3. all finitegen left R-mods are inj.
4. all cyclic left R-mods are inj.

note that despite its dual nature, the characterization $(4) \implies (1)$ is much harder, and is due to B. Osofsky.

there are also many more characterizations that have not been included here. Vol I of Rowen pp. 496 has an exhaustive(x) list of 23 characterizations.