MAT367
Intro to Differential Geometry

Class Lecture Notes

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Class Lectures

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Preface

These notes were created during class lectures. As such, they may be incomplete or lacking in some detail at parts, and may contain confusing typos due to time-sensitivity. Additionally, these notes may not be comprehensive. Most statements in this document which are not Theorems, Problems, Lemmas, Corollaries, or similar, are likely paraphrased to a certain degree. Please do not treat any material in this document as the exact words of the original lecturer.

If you are viewing this document in Obsidian, you may notice that the links in the pdf document do not work. This is intentional behaviour, as I currently do not have or know of a decent solution which allows them to behave well with the setup in Obsidian. However, below certain pages, there may be links to other documents - these are usually context-relevant links between notes of different areas of study. I created these links to point out potential similarities, or in case one area of study is borrowing a concept, definition, or theorem from another area of study, and you wish to see the full, original definition/derivation/proof or whatever it may be.

II Smooth Maps

4 Smooth Functions on Manifolds

principles:

1. charts and their inverses are smooth (by defn)
2. compositions of smooth fns are smooth

the [properties and behaviour we want] follow largely from these principles.

Definition 4.1

a fn $f : M \longrightarrow \mathbb{R}$ is **smooth at $\boldsymbol{p}$** if for some chart (U, φ) with $p \in U$, the map $f \circ \varphi^{-1} : \varphi(U) \longrightarrow \mathbb{R}$ is smooth (i.e. inf-diff'ble) at $\varphi(p)$.

we say f is **smooth** if it is smooth at every pt. equivalently, $f \circ \varphi^{-1}$ is smooth for every φ, or simply some φ which cover M (easiest to check).

remark: it doesnt depend on what chart you pick, since transition maps are smooth. notably, we can replace "some" w "every" and nothing changes.

Example 4.2

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1) if $f : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ smooth, then $f\big|_{S^1} : S^1 \subseteq \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ smooth.

   indeed, take stereo proj. then $\varphi_N^{-1}, \varphi_S^{-1}$ smooth, and the rest follows.

2) $f : S^2 \longrightarrow \mathbb{R}$, given as $f(x, y, z) = \sqrt{1 - z^2} = \sqrt{x^2 + y^2}$. using the chart

   $\varphi : \{z > 0\} \longrightarrow B(0, 1)$ as $\varphi(x, y, z) = (x, y)$, want $(f \circ \varphi^{-1})(x, y) = \sqrt{x^2 + y^2}$ smooth. but $f \circ \varphi^{-1}$ not smooth at $(0, 0)$, so f not smooth.

Definition 4.3

we define $C^\infty(M) = \{\text{smooth fn's } M \longrightarrow \mathbb{R}\}$.

some facts:

- if $f, g : M \longrightarrow \mathbb{R}$ smooth, then $af + bg$ smooth where $a, b \in \mathbb{R}$.
- products of smooth fn's are smooth
- the constant fn $1 \in C^\infty(M)$ is smooth

in other words, $C^\infty(M)$ is an $\mathbb{R}$-algebra (lmao?).

Definition 4.4

cts fn's on M are defn'd similarly: $f : M \longrightarrow \mathbb{R}$ cts if $f \circ \varphi^{-1}$ cts for all (equivalently, enough) charts φ. in particular, smooth fn's are cts.

Proposition 4.5

$f : M \longrightarrow \mathbb{R}$ is cts iff $f^{-1}(J)$ open for all open $J \subseteq \mathbb{R}$.

Proof.

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by the same fact for $U \subseteq \mathbb{R}^n$, we have:

$$(f \circ \varphi_\alpha^{-1})^{-1}(a, b) \subseteq \varphi_\alpha(U_\alpha)$$

open for every chart φ_α. since $V \subseteq M$ open iff it's a union of $\varphi_\alpha^{-1}(V_\alpha)$ for open $V_\alpha \subseteq \varphi_\alpha(U_\alpha)$, then:

$$f^{-1}(a, b) = \bigcup_\alpha \varphi_\alpha^{-1}((f \circ \varphi_\alpha^{-1})(a, b))$$

is open. $\blacksquare$

Definition 4.6

the **support** of $f : M \longrightarrow \mathbb{R}$ is the set:

$$\operatorname{supp}(f) \;:=\; \overline{f^{-1}(\mathbb{R} \setminus \{0\})}$$

equivalently, if $p \notin \operatorname{supp}(f)$, then f is 0 in a nbhd of p.

Lemma 4.7

sps $U \subseteq M$ open, $g : U \longrightarrow \mathbb{R}$ smooth, $\operatorname{supp}(g) \subseteq U$ closed in M. then g extends to a smooth $f : M \longrightarrow \mathbb{R}$ where $f\big|_U = g$ and $f(M \setminus U) = 0$.

Proof.

Source: Primary Source Material

let $V = M \setminus \operatorname{supp}(g)$ open in M. then f is smooth at every pt $p \in U$ and V, since we can choose chart containing p w domain in U or V resp., and $f \equiv 0$ in the case of V. since $M = U \cup V$, we're done. ∎