MAT357
Foundations of Real Analysis

Class Lecture Notes

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Class Lectures
Prof. Benjamin Landon

Contents

Preface

These notes were created during class lectures. As such, they may be incomplete or lacking in some detail at parts, and may contain confusing typos due to time-sensitivity. Additionally, these notes may not be comprehensive. Most statements in this document which are not Theorems, Problems, Lemmas, Corollaries, or similar, are likely paraphrased to a certain degree. Please do not treat any material in this document as the exact words of the original lecturer.

If you are viewing this document in Obsidian, you may notice that the links in the pdf document do not work. This is intentional behaviour, as I currently do not have or know of a decent solution which allows them to behave well with the setup in Obsidian. However, below certain pages, there may be links to other documents - these are usually context-relevant links between notes of different areas of study. I created these links to point out potential similarities, or in case one area of study is borrowing a concept, definition, or theorem from another area of study, and you wish to see the full, original definition/derivation/proof or whatever it may be.

I Introduction and Preliminaries

1 Review, Metric, and Normed Vector Spaces

Lec 1 - Jan 6 (Week 1)

we start w some review, particularly of $\mathbb{R}$. recall the least upper bound property: for any $S \subseteq \mathbb{R}$, M is an upper bd for S if for all $x \in S$ we have $x \leq M$.

fact: any nonempty $S \subseteq \mathbb{R}$ bdd above has a least upper bd.

Theorem 1.1: Archimedean Property

If $a < b \in \mathbb{R}$ are distinct, then $\exists\, q \in \mathbb{Q}$ such that $a < q < b$.

Proof.

Source: Primary Source Material

sps wlog $0 < a < b$. let $M \in \mathbb{N}$ such that:

$$M > \frac{1}{b-a} \implies M(b-a) > 1$$

let $N \in \mathbb{N}$ be the largest s.t. $N \leq Ma$. then $q = \frac{N+1}{M}$ satisfies $a < q < b$. indeed:

$$N + 1 > Ma \implies a < \frac{N+1}{M} \qquad Mb > Ma + 1 \geq N + 1 \implies b > \frac{N+1}{M}$$

■

genuinely hes just reviewing 157. why. at least hes moved on to metric spaces at [checks watch] 9:47. its now 10:00 and hes defining a nvs.

Lec 2 - Jan 8 (Week 1)

ℓ^p spaces. we know these. note $p \in [1, \infty]$.

$$\ell^p = \left\{ (a_n) : \sum_n |a_n|^p \right\} < \infty, p < \infty \qquad \ell^\infty = \{(a_n) : \sup_n |a_n| < \infty\}$$

we know these norms. check ℓ^p is a vector space:

$$|a_n + b_n|^p \;\le\; (|a_n| + |b_n|)^p \;\le\; (2\max(|a_n|, b_n))^p \;\le\; 2^p(|a_n|^p + |b_n|^p)$$
$$\implies \sum |a_n + b_n|^p < \infty$$

now we claim p-norm is a norm. most important is triangle inequality.

Lemma 1.2: Young's Inequality

if $x, y \ge 0$, then:

$$xy \le \frac{x^p}{p} + \frac{y^q}{q}$$

for conjugate exponents $p, q \in (1, \infty)$.

Proof.

exercise; consequence of the fact that $s \mapsto \ell^s$ is cvx. apparently.

Theorem 1.4: Holder's Inequality

sps $p, q \in [1, \infty]$ s.t. $1/p + 1/q = 1$ (conjugate exponents). let $a \in \ell^p$, $b \in \ell^q$. then:

$$\sum |a_n b_n| \le \|a\|_p \|b\|_q$$

Proof.

Source: Primary Source Material

case $p = 1, q = \infty$. then:

$$\sum |a_n b_n| \le \sum |a_n| \text{sup}_n |b_n| = (\text{sup}_n |b_n|) \sum |a_n| = \|a\|_1 \|b\|_\infty$$

case $p, q \in (1, \infty)$: wlog, assume $\|a\|_p = \|b\|_q = 1$. apply young's:

$$|a_n b_n| = |a_n||b_n| \le \frac{|a_n|^p}{p} + \frac{|b_n|^q}{q}$$

$$\sum |a_n b_n| \le p^{-1} \sum |a_n|^p + q^{-1} \sum |b_n|^q = \frac{\|a\|_p^p}{p} + \frac{\|b\|_q^q}{q} = p^{-1} + q^{-1} = 1$$

■

Theorem 1.5: Minkowski's Inequality

let $p \in [1, \infty]$, $a, b \in \ell^p$. then $\|a + b\|_p \le \|a\|_p + \|b\|_p$.

Corollary 1.6

ℓ^p is a nvs.

Proof. Source: Primary Source Material

$p = 1, \infty$ easy, so sps $p \in (1, \infty)$. then:

$$\sum |a_n + b_n|^p = \sum |a_n + b_n||a_n + b_n|^{p-1} \le \sum |a_n||a_n + b_n|^{p-1} + \sum |b_n||a_n + b_n|^{p-1}$$

note:

$$p^{-1} + q^{-1} = 1 \implies q^{-1} = 1 - p^{-1} = \frac{p-1}{p}$$

apply holder's to first sum:

$$\sum |a_n||a_n + b_n|^{p-1} \le \|a\|_p \left(\sum |a_n + b_n|^{(p-1)q}\right)^{1/q} = \|a\|_p \left(\sum |a_n + b_n|^p\right)^{1-p^{-1}}$$

applying to both sums gives:

$$\sum |a_n + b_n|^p \ \leq \ (\|a\|_p + \|b\|_p) \left(\sum |a_n + b_n|^p\right)^{1-p^{-1}} \ = \ \left(\|a\|_p + \|b\|_p\right) \|a+b\|_p^{p-1}$$

$$\|a+b\|_p^p \leq (\|a\|_p + \|b\|_p) \|a+b\|_p^{p-1} \implies \|a+b\|_p \leq \|a\|_p + \|b\|_p$$

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Exercise 1.7 Source: Primary Source Material

prove that if $p < q$ then $\ell^p \subsetneq \ell^q$. hint: consider $\sum 1/n^s$ for some good s.

Example 1.8 Source: Primary Source Material

$C[0,1]$, the nvs of cts $f : [0,1] \longrightarrow \mathbb{R}$. p-norm for $p \in [1, \infty]$ entirely analogous.

Exercise 1.9: HW 1.4 Source: Primary Source Material

repeat holder, minkowski pfs to show:

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \qquad \|f+g\|_p \leq \|f\|_p + \|g\|_p$$

"sequences and convergence" are we just doing topology again. well now we're doing continuity, specifically seq continuity. so yes, we're just doing topology again.

hint (idea?) for hw3, basically versions of 1-x but seq → more curve.

Lec 3 - Jan 13 (Week 2)

some equivs btwn continuity, bdries, i forgot what else. evidently im not rly paying attn

oh equiv of metrics/norms. note that if $p < q$, then $\ell^p \subsetneq \ell^q$ so $\|\cdot\|_p$ and $\|\cdot\|_q$ not equiv.

Lec 4 - Jan 15 (Week 2)

tdy was a snow day so going off posted notes but i still think nothings happened.

Example 1.10

Source: Primary Source Material

$C([0,1])$ is complete under sup norm.

let f_n be cauchy. for all $x \in [0,1]$, we have:

$$|f_n(x) - f_m(x)| \leq \sup_x |f_n(x) - f_m(x)| = \|f_n - f_m\|_\infty$$

thus, $(f_n(x))$ is cauchy for fixed x, so $f_n(x) \to f(x)$ as $n \to \infty$ for fixed x. want to show that $f \in C([0,1])$ and $\|f - f_n\|_\infty \to \infty$ as $n \to \infty$.

let $\varepsilon > 0$. then there exists $N > 0$ s.t. for all $n, m \geq N$, $\|f_n - f_m\|_\infty < \varepsilon$. since f_n is cauchy, for all x, we have for $n \geq N$:

$$|f(x) - f_n(x)| = \lim_{m\to\infty} |f_m(x) - f_n(x)| \leq \lim_{m\to\infty} \left(\sup_x |f_m(x) - f_n(x)| \right) < \varepsilon$$

thus for all $n \geq N$, $\sup_x |f(x) - f_n(x)| < \varepsilon$, so f_n cvgs uniformly to f.

now we show f cts. let $\varepsilon > 0$ and $x \in [0,1]$. then there exists N such that $\sup_y |f(y) - f_N(y)| < \varepsilon$. since f_N cts, $\exists \delta > 0$ s.t. $\forall y \in [0,1]$ w $|x - y| < \delta$, we have $|f_N(x) - f_N(y)| < \varepsilon$. thus:

$$|f(x) - f(y)| \;\leq\; |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \;\leq\; 3\varepsilon$$

this concludes the proof.

lol ok thats it.

Lec 5 - Jan 20 (Week 3)

still more cpt stuff ...

Definition 1.11

we say $f : M \longrightarrow N$ is **uniformly cts** if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x, y \in M$, w/

$$d_M(x,y) < \delta \implies d_N(f(x), f(y)) < \varepsilon$$

notice the distinction btwn this and regular continuity.

Theorem 1.12

if $f : M \longrightarrow N$ cts and M cpt, then f uni. cts.

Proof. Source: Primary Source Material

pick a seq $(x_n, y_n) \in M^2$ s.t.

$$d_M(x_n, y_n) \to 0$$

since M^2 cpt, $\exists (x_{n_k}, y_{n_k}) \to (x,y)$. then:

$$d(x,y) \leq d(x, x_{n_k}) + d(x_{n_k}, y_{n_k}) + d(y_{n_k}, y) \implies d(x,y) = 0$$

so $x = y$. therefore:

$$d(f(x_{n_k}), f(y_{n_k})) \leq d(f(x_{n_k}), f(x)) + d(f(y), f(y_{n_k})) \implies d(f(x_{n_k}), f(y_{n_k})) = 0$$

since f cts. ■

oh hey connectedness

Lec 6 - Jan 22 (Week 3)

more connectedness ig

Theorem 1.13

$\mathbb{R}$ is conn

Proof. Source: Primary Source Material

let $U \subseteq \mathbb{R}$ be non-empty clopen. wts $U = \mathbb{R}$.

let $p \in U$ and defn:

$$X := \{x \in U : (p, x) \subseteq U\}$$

X non-empty since U open, $p \in U$. claim: $\sup X = \infty$.

sps otw, let $s := \sup X < \infty$. we show $s \in X$.

by defn, $\exists x_n \to s$. then $\forall n, (p, x_n) \subseteq U$, so:

$$(p, s) = \bigcup_n (p, x_n) \subseteq U$$

since $x_n \in U$ and U closed, $s \in U$ so $s \in X$. since U open, $s + \varepsilon \in X$ for all $\varepsilon > 0$.

thus, we must have $(p, \infty) \subseteq U$. by a symmetric argument, $(-\infty, p) \subseteq U$. we have $(-\infty, p) \cup \{p\} \cup (p, \infty) \subseteq U$, so $U = \mathbb{R}$. ∎

"for culture".

one place conn is used: pdes, known as *method of continuity.*

generally, want to find soln to eq:

$$Lu := \sum a_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} u = f$$

given $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, find u s.t. $Lu = f$. for(?) $a = (a_{ij})$ is elliptic, $\lambda^{-1} \leq ? \leq \lambda$ for some $\lambda > 0$. when $a_{ij} = \delta_{ij}$, ie $a = \text{id}$, this gives poissons eq'n.

for $t \in [0, 1]$, define:

$$L_t = t \sum_i \frac{\partial^2}{\partial x_i^2} + (1-t) \sum_{i,j} a_{ij} \frac{p}{\partial x_i} \frac{\partial}{\partial x_j}$$

$$I = \{t \in [0, 1] : L_t u = f \text{ has soln}\}$$

wts $I = [0, 1]$. let ?? $\in I$, try to show ??? clopen.

anyway.

Definition 1.14

let $S \subseteq M$. $p \in M$ is a **cluster pt** of S if $\forall r > 0$, $|B_p(r) \cap S| = \infty$.

Theorem 1.15

tfae

1. $\exists (x_n) \in S$ of distinct pts cvg to p
2. $\forall r > 0$, $|B_p(r) \cap S| = \infty$
3. $\forall r > 0$, $|B_p(r) \cap S| \geq 2$
4. $\forall r > 0$, $B_p(r) \cap S$ contains a pt $\neq p$

Proof.

Source: Primary Source Material

clearly $1 \implies 2 \implies 3 \implies 4$. we show $4 \implies 1$.

let $x_1 \in B_p(1) \cap S$ s.t. $x_1 \neq p$. then $r_1 < d(x_1, p)$. let $x_2 \in B_p(r_1 \cap S)$ s.t. $x_2 \neq p$. note $x_2 \neq x_1$ since $r_1 < d(x_1, p)$. take $r_2 < \min(d(x_2, p), 1/2)$, induct. ∎

Definition 1.16

a metric space is **perfect** if every pt is a cluster pt.

Theorem 1.17

complete non-empty perf metric sp is unctbl. (crll: $\mathbb{R}$ is unctbl)

Proof.

Source: Primary Source Material

sps $M = \{x_1, x_2, \ldots\}$ ctbl. we construct a seq of closed sets of the form:

$$Y_i := \overline{B_{y_i}}(r_i)$$

s.t. $\forall n$:

1. $x_n \notin Y_n$
2. $Y_{n+1} \subseteq Y_n$
3. $r_n \leq 1/n$

assuming the construction, $(y_n)_n$ is cauchy. sps $y_n \to y$. by 2, since we have that $y_m \in Y_n$ $\forall m \geq n$, then $y \in Y_n$ $\forall n$. but $x_n \notin Y_n$ so $y \neq x_n$ $\forall n$, contradiction.

we now construct such a seq of sets. let $y_1 \in B(x_1, 1)$ s.t. $y_1 \neq x_1$. next, define $r_1 < \min(d(y, x_1), 1)$. then $Y_1 = \overline{B_{y_1}}(r_1)$ satisfies $x_1 \notin Y_1$.

for any k, choose $y_k \in B(y_{k-1}, r_{k-1})$ s.t. $y_k \neq x_k$. choose $r_k > 0$ s.t.:

$$r_k < d(y_k, x_k) \qquad r_k \leq 1/k \qquad B(y_k, r_k) \subseteq B(y_{k-1}, r_{k-1})$$

then $x_k \notin \overline{B}(y_k, r_k)$. ∎

Example 1.18: Cantor Set

see below

as a fun ex of a perf space sth sth containing no interval, we construct the std middle thirds cantor set.

oh he drew some pictures

$$A_n := [0,1] \setminus \left(\bigcup_{j=0}^{n-1} \left(\frac{1+3j}{3^n}, \frac{2+3j}{3^n} \right) \right) \qquad C := \bigcap_n A_n$$

properties of C:

1. closed + cpt
2. "totally disconn"
3. non-empty
4. perf but contains no interval
5. unctbl

"totally disconn" means $\forall r > 0, p \in C$, $\exists$ set U clopen in C s.t. $U \subseteq B(p,r) \cap C$.

n.b.: "proper" defn is that only singletons are conn.

pf that C non-empty: consider endpts.

pf that C unctbl: closed bdd subset of $[0,1]$, so complete + perf thus unctbl.

pf that C contains no interval: sps $(\alpha, \beta) \subseteq C$. then if $1/3^n < |\beta - \alpha|$, C_n is intervals of len $1/3^n$, so $(\alpha,\beta) \subsetneq C_n$. since $C \subseteq C_n$, $(\alpha, \beta) \subsetneq C$.

pf that C totally disconn: let $r > 0, p \in C$. choose n s.t. $1/3^n < r$. then $p \in C_n$ implies $\exists$ interval I of len $1/3^n$ in C_n s.t. $p \in I$. wts $U = I \cap C$ satisfies:

- $U \subseteq B(p,r) \cap C$
- U clopen in C

since $U = I \cap C$, I closed interval, U is thus closed in C. write $I = [a,b]$ and choose $\varepsilon < 1/3^n$. then:

$$I \cap C \;=\; [a,b] \cap C \;=\; (a - \varepsilon, b + \varepsilon) \cap C \;=\; U$$

so U open in C.

pf that C perf: let $p \in C$ and $r > 0$. wts $|B(p,r) \cap C| \geq 2$. if n s.t. $1/3^n < r$, then $p \in I$ is a closed interval of len $1/3^n$ in C_n. [take endpts]