I Wedderburn-Artin Theory

1 Terminology and Examples

Remark 1.19

M noeth and artin iff M has a comp series.

Proof.

Suppose M is both noetherian and artinian. Then, M has a maximal submodule N_1, by considering all proper submodules of M under inclusion. Similarly, N_1 has such a maximal submodule N_2.

Indeed, for any k, let N_k be the maximal submodule of N_{k-1}. Since any submodule if M is noetherian, this is well-defined. Thus, we have a chain of submodules:

$$\cdots < N_3 < N_2 < N_1 < M$$

Since M is artinian, there is some k such that $N_k = N_{k+1} = \dots$. In other words, the chain terminates:

$$\{0\} = N_k < N_{k-1} < \cdots < N_2 < N_1 < M$$

Since each N_k is maximal in N_{k-1} by construction, then N_k/N_{k-1} is simple for all k. Thus, M has a composition series as needed. ∎

Remark 1.20

N submod of M. then M noeth iff $N, M/N$ noeth

Proof.

First, notice that if N and A/N are f.g., then so is A. Indeed, suppose:

$$N = \langle x_1, \ldots, x_n \rangle \qquad A/N = \langle \overline{y}_1, \ldots, \overline{y}_m \rangle$$

Let $a \in A$. Then:

$$\overline{a} = \sum_{i=1}^{m} s_i \overline{y}_i = \overline{\sum_{i=1}^{m} s_i y_i} \implies a - \sum_{i=1}^{m} s_i y_i = \sum_{j=1}^{n} r_j x_j$$

Thus, we have that $A = \langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle$.

Now, suppose N and M/N are noetherian. Let $A \leq M$ be any submodule. By the Second Isomorphism Theorem, we have:

$$\frac{A}{A \cap N} \simeq \frac{A+N}{N} \leq \frac{M}{N}$$

Since $A \cap N \leq N$ is f.g. and $A/(A \cap N) \leq M/N$ is f.g., by the above, it follows that A is f.g., and thus M is noetherian.

Next, suppose M is noetherian. Then N is clearly noetherian, so let A be any submodule of M/N. By the Fourth(*) Isomorphism Theorem, $A \simeq L/N$ for some submodule $N \leq L \leq M$.

But since M noetherian, N and L are finitely generated, so it follows that A is finitely generated. $\blacksquare$

Proposition 1.21(b)

R right noeth, M_R f.g.. then M_R noeth.

Proof.

Suppose $M = \langle m_1, \ldots, m_n \rangle$ for some n. Then:

$$m = \sum_{i=1}^{n} r_i m_i$$

for all $m \in M$.

Consider $R^n = R \times R \times \cdots \times R$. We have a map $\varphi : R^n \longrightarrow M$ given by:

$$\varphi(r_1, \ldots, r_n) = \sum_{i=1}^{n} r_i m_i$$

Clearly, φ is surjective. Furthermore, since R is noetherian, R^n is as well. Therefore, by the First Isomorphism Theorem, we have that

$$\frac{R^n}{\ker(\varphi)} \simeq M$$

is noetherian as needed. ∎

Exercise 1.7a Source: Primary Source Material

sps $R = I_1 \oplus \cdots \oplus I_n$ a dirsum of left ideals. show that $I_k = Re_k$ for e_k idempotent elems of R satisfying $e_1 + \cdots + e_n = 1$ and $e_i e_j = 0$ for $i \neq j$.

Proof.

Since we can write $R = I_1 \oplus \cdots \oplus I_n$, then for all $r \in R$, there exists a unique linear combination such that:

$$r \ = \ \sum_{k=1}^{n} r_k \qquad r_k \in I_k$$

In particular, there is a unique linear combination to write $1 = e_1 + \cdots + e_n$. Now, note that each $r_k \in I_k$ can be written as:

$$r_k \ = \ \sum_{j=1}^{n} r_k \delta_{jk}$$

where δ_{jk} is the Kronecker delta. In particular, this holds for each e_k; it immediately follows that they are idempotent. We also see that for any $r_k \in I_k$:

$$r_k \ = \ r_k 1 \ = \ \left(\sum_{j=1}^{n} r_k \delta_{jk} \right) \left(\sum_{k=1}^{n} e_k \right) \ = \ \sum_{j=1}^{n} r_k \delta_{jk} e_k \ = \ r_k e_k$$

So $I_k = Re_k$ as needed. Finally, we have that $e_j e_k = 0$ for $j \neq k$, since:

$$e_j e_k \ = \ \left(\sum_{i=1}^{n} e_j \delta_{ij} \right) \left(\sum_{i=1}^{n} e_k \delta_{ik} \right) \ = \ \sum_{i=1}^{n} e_j \delta_{ij} e_k \delta_{ik} \ = \ 0$$

since $j \neq k$. ∎

Exercise 1.7b

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show I_k is a ring w identity e_k, and $R \simeq I_1 \times \cdots \times I_n$.

Proof.

From part a), we saw that $r_k = r_k e_k$ for all $r_k \in I_k$. The proof that $r_k = e_k r_k$ is symmetric, so it remains to show that $R \simeq I_1 \times \cdots \times I_n$.

Indeed, suppose S is any ring with $\varphi_k : S \longrightarrow I_k$ ring homomorphisms. Define:

$$\varphi : S \longrightarrow I_1 \oplus \cdots \oplus I_n \qquad \varphi(s) \;:=\; \varphi_1(s) + \cdots + \varphi_n(s)$$

We have $\varphi(1) = 1$ since necessarily $\varphi_k(1) = e_k$; clearly $\varphi(a+b) = \varphi(a) + \varphi(b)$. Now:

$$\varphi_i(a)\varphi_j(b) \;=\; \left(\sum_{k=1}^{n} \varphi_i(a)\delta_{ik}\right)\left(\sum_{k=1}^{n} \varphi_j(b)\delta_{jk}\right) \;=\; \varphi_i(a)\varphi_j(b)\delta_{ij}$$

where the Kronecker delta commutes, since it is either 0 or 1. This implies:

$$\begin{aligned} \varphi(ab) \;&=\; \sum_{k=1}^{n} \varphi_k(ab) \;=\; \sum_{k=1}^{n} \varphi_k(a)\varphi_k(b) \;=\; \sum_{i=1}^{n}\sum_{j=1}^{n} \varphi_i(a)\varphi_j(b) \\ &=\; \left(\sum_{i=1}^{n} \varphi_i(a)\right)\left(\sum_{j=1}^{n} \varphi_j(b)\right) \\ &=\; \varphi(a)\varphi(b) \end{aligned}$$

Therefore, φ is indeed a ring homomorphism. Define $\pi_k : I_1 \oplus \cdots \oplus I_n \longrightarrow I_k$ as $\pi_k(r) = r_k$, where $r = \sum_{k=1}^{n} r_k$. This is well-defined by the uniqueness of the decomposition of r, and is clearly a ring homomorphism.

Finally, notice $\varphi_k = \pi_k \circ \varphi$ for all k. Thus, by the universal property of products in the category Ring, we are done. ■

Exercise 1.7c

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show if I_k are two-sided ideals, then $e_k \in Z(R)$.

Proof.

First, recall that if $j \neq k$, then $I_j \cap I_k = \{0\}$ must be trivial by uniqueness of decomposition. Then, it simply suffices to note that for any $r \in R$:

$$re_k = r_k e_k = r_k = e_k r_k = e_k r$$

by the above fact and work shown in previous parts. ∎

Exercise 1.8

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sps $R = I \oplus J$ for ideals I, J. show every ideal of R is of the form $I' \oplus J'$, where I', J' ideals in I, J resp.

Proof.

Suppose $S \subseteq R$ is an ideal. By Exercise 1.7, there exists $e_i \in I$ and $e_j \in J$ such that $I = Re_i, J = Re_j, e_i + e_j = 1$, and e_i, e_j is the identity of I, J resp. as a ring. Define $I' := Se_i$ and $J' := Se_j$. We want to show that $S = I' \oplus J'$.

Clearly $I', J' \subseteq S$ since S is an ideal, thus $I' + J' \subseteq S$. On the other hand, for any $s \in S$, let:

$$s_i \in I', s_j \in J' \qquad s_i := se_i \qquad s_j := se_j$$

Then $s_i + s_j = se_i + se_j = s(e_i + e_j) = s$, so we see that $S = I' + J'$. Finally, since $s_i \in I$ and $s_j \in J$, then $s = s_i + s_j$ is the unique decomposition of s, and it follows that $S = I' \oplus J'$ as needed. ∎

Exercise 1.12a

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${}_RM$ is called *Hopfian* if every surj endo is inj. show every noeth ${}_RM$ is hopfian.

Proof.

Let φ be a surjective endomorphism of M. Recall that $\ker(\varphi)$ is a submodule of M. Consider $\varphi^{(2)} := \varphi \circ \varphi$. Since φ is surjective, then there exists $m \in M$ such that $\varphi(m) \in \ker(\varphi)$. Thus, we have that $\ker(\varphi) \leq \ker(\varphi^{(2)})$.

In general, we can consider $\ker(\varphi^{(k)})$ in the same way. Since φ is surjective, we have that $im(\varphi^{(k)}) = M$, so we can construct the following chain:

$$\ker(\varphi) \leq \ker(\varphi^{(2)}) \leq \ker(\varphi^{(3)}) \leq \cdots$$

Since M is noetherian, this chain must terminate; if $\ker(\varphi)$ is non-trivial, then the above construction holds. Therefore $\ker(\varphi) = \{0\}$, or in other words, φ is injective. ∎

Exercise 1.12b

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prove ${}_RR$ is hopfian iff dedekind-finite

Proof.

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First, suppose ${}_RR$ is Hopfian and fix $x \in R$ such that $xy = 1$ for some $y \in R$. It suffices to show that x is left-invertible.

For any $r \in R$, let $\varphi_r(s) = rs$. This is evidently an endomorphism. Consider the map φ_x; note that this is surjective, as it has a right-inverse:

$$(\varphi_x \circ \varphi_y)(s) \ = \ (xy)s \ = \ s \quad \Longrightarrow \quad \varphi_x \circ \varphi_y \ = \ \varphi_1 \ = \ \mathrm{id}$$

Since R is Hopfian, then φ_x is injective. By uniqueness of inverses, we have:

$$(\varphi_y \circ \varphi_x)(s) \ = \ (yx)s \ = \ s$$

Since this is true of all $s \in R$, then $yx = 1$ as needed.

Now, suppose R is Dedekind-finite. Let $\varphi : R \longrightarrow R$ be a surjective endomorphism. Notice:

$$\varphi(r) \;=\; \varphi(1r) \;=\; \varphi(1)r$$

Thus $\varphi = \varphi_{\varphi(1)}$ is given by scaling. Fix $\varphi(1) = r$. Since φ is surjective, then there exists some $s \in R$ such that:

$$\varphi(s) = rs = 1$$

Similarly to above, we therefore have that:

$$\varphi \circ \varphi_s \;=\; \varphi_r \circ \varphi_s \;=\; \varphi_1 \;=\; \mathrm{id}$$

Since R is Dedekind-finite, then:

$$\varphi_s \circ \varphi \;=\; \varphi_s \circ \varphi_r \;=\; \varphi_1 \;=\; \mathrm{id}$$

It follows that φ is injective, and so R is indeed Hopfian. ∎

Exercise 1.13a

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Show that if R is finite-dimensional as a k-vector space, R is algebraic. Is the converse true?

Proof.

Suppose R is finite-dimensional as a k-vector space, and fix $r \in R$. Let $s \in R$ be any element, and consider the endomorphism φ_r given as $\varphi_r(s) = rs$.

Since R is a k-algebra and φ_r is an endomorphism, we must have that φ_r is given by some square matrix A. Let $p \in k[x]$ be the characteristic polynomial of A. We claim that $p(r) = 0$.

Write the characteristic polynomial $p(x)$ of A as:

$$p(x) \;:=\; a_n x^n + \cdots + a_1 x + a_0$$

where each $a_i \in k$. Then:

$$\begin{aligned} p(r) \;=\; p(r) \cdot 1 \;=\; (a_n r^n + \cdots + a_1 r + a_0) \cdot 1 \;&=\; a_n r^n \cdot 1 + \cdots + a_1 r \cdot 1 + a_0 \cdot 1 \\ &=\; a_n A^n \cdot 1 + \cdots + a_1 A \cdot 1 + a_0 I_n \cdot 1 \\ &=\; a_n A^n + \cdots + a_1 A + a_0 I_n \\ &=\; 0 \end{aligned}$$

by the Cayley-Hamilton theorem. Therefore we have that $p(r) = 0$ as needed.

NOTE: idk for converse ■

Exercise 1.13b

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Henceforth, let R be an algebraic k-algebra. Show R is Dedekind-finite.

Proof.

Suppose $r, s \in R$ such that $rs = 1$. Since R is algebraic, there exists $p \in k[x]$ such that $p(r) = 0$. Write:

$$p(x) \ := \ a_n x^n + \cdots + a_1 x + a_0$$

Suppose m is the lowest index such that $a_m \neq 0$. Then:

$$p(r) \cdot s^m = (a_n r^n + \cdots + a_m r^m) s^m = a_n r^{n-m} + \cdots + a_{m+1} r + a_m = 0$$

Thus, suppose WLOG that $a_0 \neq 0$. Then:

$$\begin{aligned} a_n r^n + \cdots + a_1 r + a_0 = 0 \quad &\implies \quad a_n r^n + \cdots + a_1 r = -a_0 \\ &\implies \quad (a_n r^{n-1} + \cdots + a_2 r + a_1) r = -a_0 \\ &\implies \quad -(a_0^{-1}) \left(a_n r^{n-1} + \cdots + a_2 r + a_1 \right) r = 1 \end{aligned}$$

since $a_0 \in k$, and k is a field. Thus, r has a left-inverse, so by uniqueness of inverses, we have $sr = 1$. ■

Exercise 1.13c

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Show that every left zero divisor is also a right zero divisor.

Proof.

incomplete ■

Exercise 1.13d

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Suppose $r \in R$ is non-zero. Show that r is a zero divisor iff it is not a unit.

Proof.

If r is a unit, then r cannot be a (left) zero divisor as:

$$rs = 0 \implies s = r^{-1}rs = r^{-1} \cdot 0 = 0 \qquad sr = 0 \implies s = srr^{-1} = 0 \cdot r^{-1} = 0$$

On the other hand, if r is not a (left) zero divisor, then the map $\varphi_r(s) = rs$ is injective. Thus, it has a (left-)inverse φ. Since φ is necessarily surjective, $\varphi = \varphi_a$ for some $a \in R$. Then:

$$(\varphi_a \circ \varphi_r)(s) = ars = s \implies ar = 1$$

So r has a (left-)inverse.

Since the sided-ness does not affect either argument, we are done. ∎

Exercise 1.13e

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Let B be a k-subalgebra of R, $b \in B$. Show that if b is a unit in R, then $b^{-1} \in B$.

Proof.

Since R is algebraic, there exists non-zero $p \in k[x]$ such that $p(b) = 0$. Note that:

$$p(b) = a_n b^n + \cdots + a_1 b + a_0 = 0 \implies b \cdot \left(\frac{-1}{a_0} (a_n b^{n-1} + \cdots + a_2 b + a_1) \right) = b \cdot b^{-1} = 1$$

by the same argument as in part b). By uniqueness of inverses, $b^{-1} \in B$ as needed.
NOTE: a little unsure about this one ∎