I Wedderburn-Artin Theory

1 Terminology and Examples

Remark 1.19

M noeth and artin iff M has a comp series.

Proof.

Suppose M is both noetherian and artinian. Then, M has a maximal submodule N_1, by considering all proper submodules of M under inclusion. Similarly, N_1 has such a maximal submodule N_2.

Indeed, for any k, let N_k be the maximal submodule of N_{k-1}. Since any submodule if M is noetherian, this is well-defined. Thus, we have a chain of submodules:

$$\cdots < N_3 < N_2 < N_1 < M$$

Since M is artinian, there is some k such that $N_k = N_{k+1} = \dots$. In other words, the chain terminates:

$$\{0\} = N_k < N_{k-1} < \cdots < N_2 < N_1 < M$$

Since each N_k is maximal in N_{k-1} by construction, then N_k/N_{k-1} is simple for all k. Thus, M has a composition series as needed. ■

Remark 1.20

N submod of M. then M noeth iff $N, M/N$ noeth

Proof.

First, notice that if N and A/N are f.g., then so is A. Indeed, suppose:

$$N = \langle x_1, \ldots, x_n \rangle \qquad A/N = \langle \overline{y}_1, \ldots, \overline{y}_m \rangle$$

Let $a \in A$. Then:

$$\overline{a} = \sum_{i=1}^{m} s_i \overline{y}_i = \overline{\sum_{i=1}^{m} s_i y_i} \implies a - \sum_{i=1}^{m} s_i y_i = \sum_{j=1}^{n} r_j x_j$$

Thus, we have that $A = \langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle$.

Now, suppose N and M/N are noetherian. Let $A \leq M$ be any submodule. By the Second Isomorphism Theorem, we have:

$$\frac{A}{A \cap N} \simeq \frac{A+N}{N} \leq \frac{M}{N}$$

Since $A \cap N \leq N$ is f.g. and $A/(A \cap N) \leq M/N$ is f.g., by the above, it follows that A is f.g., and thus M is noetherian.

Next, suppose M is noetherian. Then N is clearly noetherian, so let A be any submodule of M/N. By the Fourth(*) Isomorphism Theorem, $A \simeq L/N$ for some submodule $N \leq L \leq M$.

But since M noetherian, N and L are finitely generated, so it follows that A is finitely generated. ∎

Proposition 1.21(b)

R right noeth, M_R f.g.. then M_R noeth.

Proof.

Suppose $M = \langle m_1, \ldots, m_n \rangle$ for some n. Then:

$$m = \sum_{i=1}^{n} r_i m_i$$

for all $m \in M$.

Consider $R^n = R \times R \times \cdots \times R$. We have a map $\varphi : R^n \longrightarrow M$ given by:

$$\varphi(r_1, \ldots, r_n) = \sum_{i=1}^{n} r_i m_i$$

Clearly, φ is surjective. Furthermore, since R is noetherian, R^n is as well. Therefore, by the First Isomorphism Theorem, we have that

$$\frac{R^n}{\ker(\varphi)} \simeq M$$

is noetherian as needed. ■

Exercise 1.12a

Source: Primary Source Material

${}_RM$ is called *Hopfian* if every surj endo is inj. show every noeth ${}_RM$ is hopfian.

Proof.

Let φ be a surjective endomorphism of M. Recall that $\ker(\varphi)$ is a submodule of M. Consider $\varphi^{(2)} := \varphi \circ \varphi$. Since φ is surjective, then there exists $m \in M$ such that $\varphi(m) \in \ker(\varphi)$. Thus, we have that $\ker(\varphi) \leq \ker(\varphi^{(2)})$.

In general, we can consider $\ker(\varphi^{(k)})$ in the same way. Since φ is surjective, we have that $im(\varphi^{(k)}) = M$, so we can construct the following chain:

$$\ker(\varphi) \leq \ker(\varphi^{(2)}) \leq \ker(\varphi^{(3)}) \leq \cdots$$

Since M is noetherian, this chain must terminate; if $\ker(\varphi)$ is non-trivial, then the above construction holds. Therefore $\ker(\varphi) = \{0\}$, or in other words, φ is injective. ∎

Exercise 1.12b

Source: Primary Source Material

prove ${}_RR$ is hopfian iff dedekind-finite

Proof.

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First, suppose ${}_RR$ is Hopfian and fix $x \in R$ such that $xy = 1$ for some $y \in R$. It suffices to show that x is left-invertible.

For any $r \in R$, let $\varphi_r(s) = rs$. This is evidently an endomorphism. Consider the map φ_x; note that this is surjective, as it has a right-inverse:

$$(\varphi_x \circ \varphi_y)(s) \;=\; (xy)s \;=\; s \quad\Longrightarrow\quad \varphi_x \circ \varphi_y \;=\; \varphi_1 \;=\; \mathrm{id}$$

Since R is Hopfian, then φ_x is injective. By uniqueness of inverses, we have:

$$(\varphi_y \circ \varphi_x)(s) \;=\; (yx)s \;=\; s$$

Since this is true of all $s \in R$, then $yx = 1$ as needed.

Now, suppose R is Dedekind-finite. Let $\varphi : R \longrightarrow R$ be a surjective endomorphism. Notice:

$$\varphi(r) \;=\; \varphi(1r) \;=\; \varphi(1)r$$

Thus $\varphi = \varphi_{\varphi(1)}$ is given by scaling. Fix $\varphi(1) = r$. Since φ is surjective, then there exists some $s \in R$ such that:

$$\varphi(s) = rs = 1$$

Similarly to above, we therefore have that:

$$\varphi \circ \varphi_s \;=\; \varphi_r \circ \varphi_s \;=\; \varphi_1 \;=\; \mathrm{id}$$

Since R is Dedekind-finite, then:

$$\varphi_s \circ \varphi \;=\; \varphi_s \circ \varphi_r \;=\; \varphi_1 \;=\; \mathrm{id}$$

It follows that φ is injective, and so R is indeed Hopfian. ■