

# X Power X: A Journey from Real to Complex

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## Abstract

This report explores the fascinating world of complex exponentiation, starting from the real plane and extending into the complex plane. The goal is to delve into the mathematical intricacies of functions like  $(-1)^x$  and  $x^x$ , and how their behaviors change when we consider complex solutions. I will also investigate the role of the parameter  $k$  in determining the number of revolutions in the complex plane. Feel free to skip parts that are familiar to you and enjoy the ride.

## 1 The Real Part of the graph $y = x^x$

We will start with something familiar: the real part of the function. In real analysis, the domain of the function is often  $x > 0$  since the function is well defined and obtains real outputs.

For the negative part, things get complicated. For example, the following is real:

$$(-2)^{-2} = \frac{1}{(-2)^2} = \frac{1}{4}$$

But the following is complex:

$$\left(-\frac{1}{2}\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{-\frac{1}{2}}} = \frac{1}{i\sqrt{2}} = -\frac{i}{\sqrt{2}}$$

Don't worry if you are confused, we will elaborate them in later sections.

Although  $0^0$  is considered to be 1 for convenience in some contexts like combinatorics, in the context of this report, we will consider it indeterminate. However, we can still find the limit behavior near 0 from the right hand side.

### 1.1 The behaviors of the graph as $x \rightarrow 0^+$

Before we do any algebra manipulation, it is worth pointing out that  $x^x$  is equivalent to  $e^{\ln(x^x)} = e^{x\ln(x)}$  (recall that  $a = e^{\ln(a)}$ ). We can then derive the following limit:

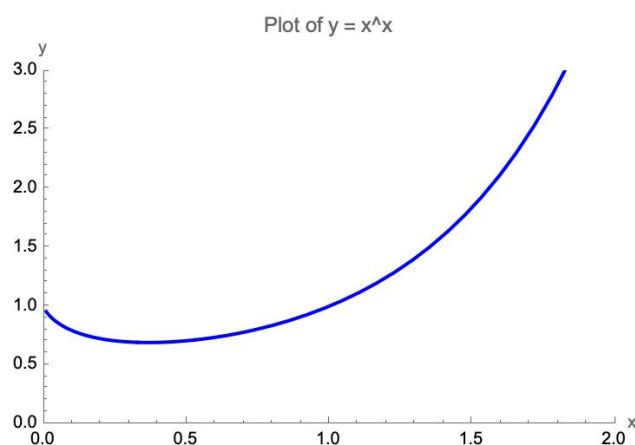
$$\lim_{x \rightarrow 0^+} x^x = \lim_{x \rightarrow 0^+} e^{x\ln(x)} = \lim_{x \rightarrow 0^+} e^{\frac{\ln(x)}{\frac{1}{x}}} = e^{\lim_{x \rightarrow 0^+} \frac{\ln(x)}{\frac{1}{x}}} = e^{\lim_{x \rightarrow 0^+} -\frac{\frac{1}{x}}{\frac{1}{x^2}}} = e^{\lim_{x \rightarrow 0^+} -x} = 1$$

## 1.2 The critical points

First,

## 1.3 Summary

With all the derived information from Calculus, we can gain a sense of the key properties of the graph in the real plane. Here is the graph plotted by Mathematica for  $x > 0$ :



## 2 Complex Exponentiation

## 3 Understanding Graphs with Complex Solutions

## 4 The graph $y = (-1)^x$

## 5 The graph $y = x^x$

## 6 Miscellaneous