September 19, 2019

1 Empirical Bayes

$$
\begin{aligned}
Y|\beta, \sigma^2, \gamma &\sim N(Z\Gamma\beta, \sigma^2 I) \\
\sigma^2 &\sim Inverse - Gamma(A, B) \quad A = 0,\ B = 0 \\
\beta_j &\sim N(0, \sigma_\beta^2) \\
\gamma_j &\sim Bernoulli(\rho)
\end{aligned}
$$

We need to calculate the likelihood $p(y|\rho)$ and select the hyper parameter ρ which maximize the likelihood

by varational bayes approach we can find lower bound of $p(y|\rho)$

$$
\log p(y|\rho) \geq \sigma_\gamma \int q(\beta, \sigma^2, \gamma) \log \left\{ \frac{p(y, \beta, \sigma^2, \gamma)}{q(\beta, \sigma^2, \gamma)} \right\} d\beta d\sigma^2 = ELBO(\rho)
$$

$$
\begin{aligned}
ELBO(\rho) =& c - (A + \frac{n}{2}) \log(s) + \frac{1}{2}|\Sigma| - \frac{1}{\sigma_\beta^2} tr(\mu\mu' + \Sigma) \\
& + \sum_{j=1}^{p} \left[w_j \log \left(\frac{\rho}{w_j} + (1 - w_j) \log \left(\frac{1-\rho}{1-w_j} \right) \right) \right]
\end{aligned}
$$

when variation distribution is

$$
\begin{aligned}
q(\beta) &\sim N(\mu, \Sigma) \\
q(\sigma^2) &\sim Inverse - Gamma(A + n/2, s) \\
q(\gamma_j) &\sim Bern(w_j) \quad for \ \ j = 1, \ldots, p
\end{aligned}
$$

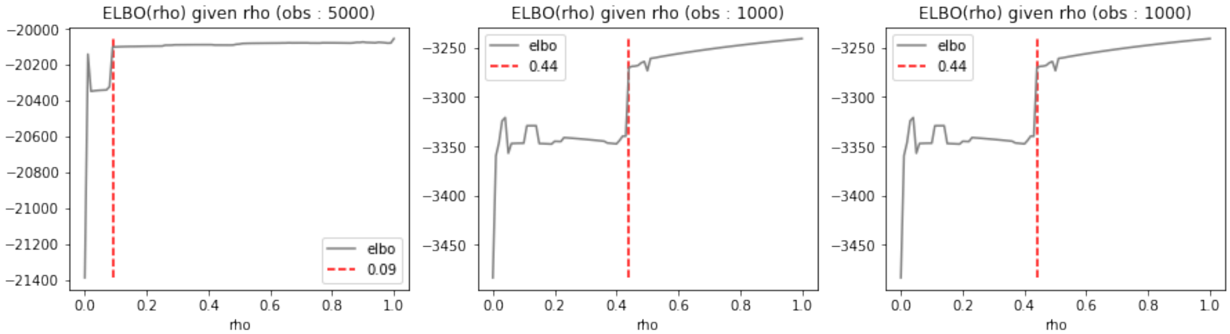

Figure 1: ELBO(ρ) plot by observation counts

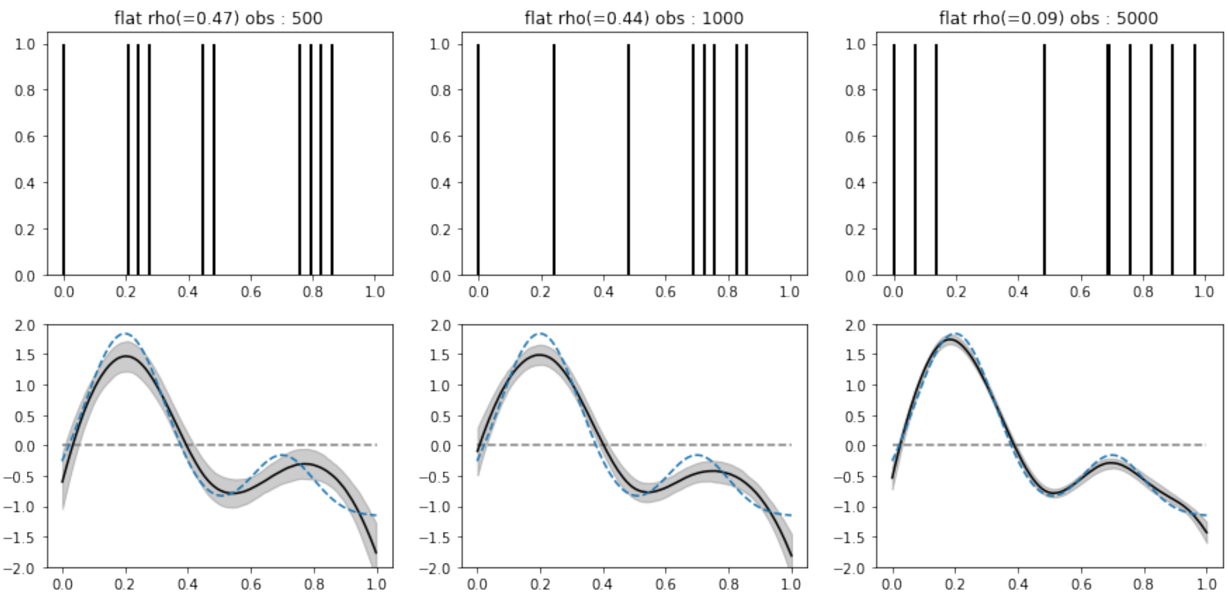

Figure 2: Empirical Bayes with optiaml ρ

2 Hierarchical Bayes

$$
\begin{aligned}
Y|\beta, \sigma^2, \gamma &\sim N(Z\Gamma\beta, \sigma^2 I) \\
\sigma^2 &\sim Inverse - Gamma(A, B) \quad A = 0,\ B = 0 \\
\beta_j &\sim N(0, \sigma_\beta^2) \\
\gamma_j &\sim Bernoulli(\rho) \\
\rho &\sim Beta(C, D)
\end{aligned}
$$

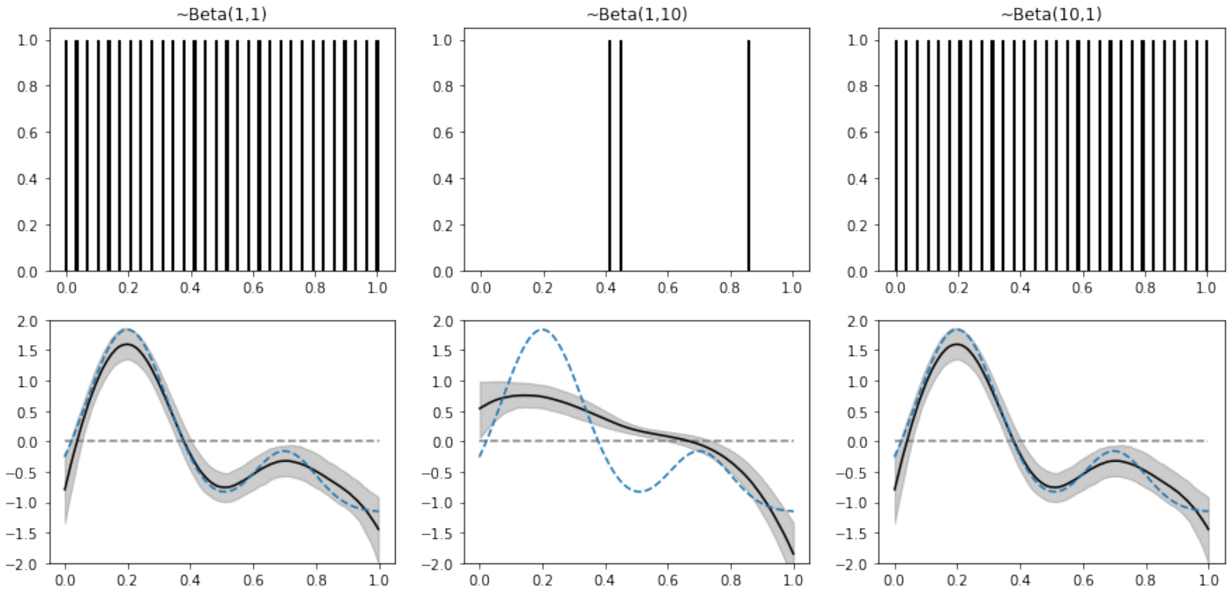

Figure 3: Hierarchical model with obs=500, posterior mean is 0.72, 0.22, 0.78

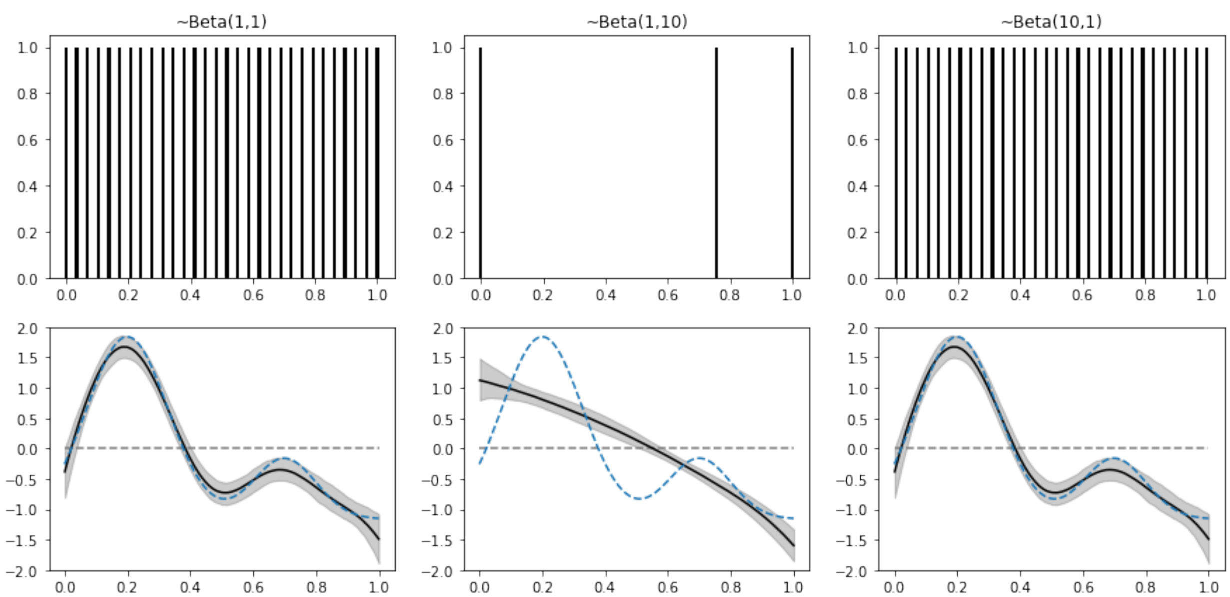

Figure 4: Hierarchical model with obs=1000, posterior mean is 0.72, 0.22, 0.78

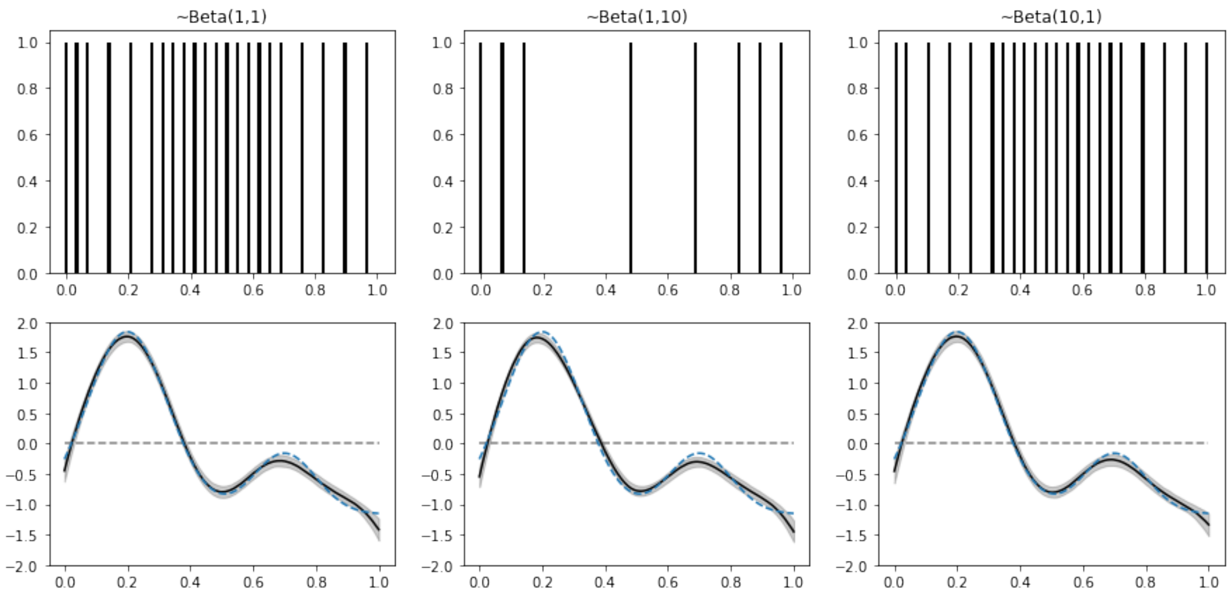

Figure 5: Hierarchical model with obs=5000, posterior mean is 0.72, 0.22, 0.78