0725

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In [1]: import numpy as np
        import pandas as pd
```

$$
\begin{aligned}
Y - \bar{y}1_n &= b^*(X)\gamma + Z^*\beta + \epsilon \\
\tilde{Y} &= b^*(X)\gamma + Z^*\beta + \epsilon \\
\tilde{Y} &= W\Theta + \epsilon
\end{aligned}
$$

Where

$$\Theta = \{\gamma', \beta'\}' = \{\theta_{11}, \ldots, \theta_{Gm_G}\}' W = \{b^*(X), Z^*\} = \{W_{11}, \ldots W_{Gm_G}\} \epsilon \sim N(0, \sigma^2)$$

0.1 Prior

$$
\begin{aligned}
Y|W, \Theta &\sim N(W\Theta, \sigma^2 I) \\
\theta_{gj}|\pi_g, \sigma^2, v_{gj}^2 &\sim^{ind} \pi_g I(\theta_{gj} = 0) + (1 - \pi_g)N(0, \sigma^2 v_{gj}^2), \quad g = 1 \ldots, G \ \ , j = 1, \ldots, m_g \\
v_{gj}^2|\lambda^2 &\sim^{iid} Exp(\lambda^2/2) \\
\lambda^2 &\sim Gamma(a_{\lambda^2}, b_{\lambda^2}) \\
\pi_g &\sim^{ind} Beta(a_{\pi_g}, b_{\pi_g}) \\
\sigma^2 &\sim Inverse - Gamma(a, b), \ \ a = 0, \ b = 1
\end{aligned}
$$

0.2 Posterior

$$
\begin{aligned}
p(\Theta, V, \lambda^2, \pi, \sigma^2|\tilde{Y}) \propto & p(\tilde{Y}|\Theta, \sigma^2) \prod_{g=1}^{G} \prod_{j=1}^{m_g} p(\theta_{gj}|\pi_g, \sigma^2, v_{gj}^2) \\
& \prod_{g=1}^{G} \prod_{j=1}^{m_g} p(v_{gj}^2|\lambda^2) \prod_{g=1}^{G} p(\pi_g) p(\lambda^2) p(\sigma^2) \\
\propto & (\sigma^2)^{-n/2} \prod_{g=1}^{G} \prod_{j=1}^{m_g} exp\left(-\frac{\theta_{gj}^2}{2\sigma^2} W_{gj}' W_{gj} + \frac{\theta_{gj}}{\sigma^2} W_{gj}' (\tilde{Y} - W_{-gj}\Theta_{-gj})\right) \\
& \prod_{g=1}^{G} \prod_{j=1}^{m_g} \left[\pi_g I(\theta_{gj} = 0) + (1-\pi_g)(\sigma^2 v_{gj}^2)^{-1/2} exp\left(-\frac{1}{2\sigma^2 v_{gj}^2} \theta_{gj}^2\right)\right] \\
& \prod_{g=1}^{G} \prod_{j=1}^{m_g} v_{gj}^2 \frac{\lambda^2}{2} exp\left(-\frac{\lambda^2}{2} v_{gj}^2\right) \\
& \prod_{g=1}^{G} (\pi_g)^{a_{\pi_g}-1} (1-\pi_g)^{b_{\pi_g}-1} \\
& (\lambda^2)^{a_{\lambda^2}-1} exp(-b_{\lambda^2}\lambda^2) \\
& (\sigma^2)^{-1} exp(\frac{1}{\sigma^2})
\end{aligned}
$$

0.3 MFVB

$$
\begin{aligned}
p(\Theta, V, \lambda^2, \pi, \sigma^2|\tilde{Y}) &\approx q(\Theta, V, \lambda^2, \pi, \sigma^2) \\
&= \prod_{g=1}^{G} \prod_{j=1}^{m_g} q_1(\theta_{gj}) \prod_{g=1}^{G} \prod_{j=1}^{m_g} q_2(v_{gj}) \prod_{g=1}^{G} q_3(\pi_g) q_4(\lambda^2) q_5(\sigma^2)
\end{aligned}
$$

0.3.1 Variational Distribution of Θ

$$
\begin{aligned}
q^*(\theta_{gj}) \propto & E_{-\theta_{gj}}\left[p(\Theta, V, \lambda^2, \pi, \sigma^2|\tilde{Y})\right] \\
\propto & l_{gj} I(\theta_{gj} = 0) + (1 - l_{gj}) N(\mu_{gj}, \Sigma_{gj})
\end{aligned}
$$

where

$$
\Sigma_{gj} = \langle\sigma^2\rangle \left\langle \frac{1}{W_{gj}' W_{gj} + \frac{1}{v_{gj}^2}} \right\rangle \mu_{gj} = \left\langle \frac{1}{\sigma^2} \right\rangle \Sigma_{gj} (\tilde{Y} - W_{-gj} \langle\Theta - gj\rangle)' W_{gj} l_{gj} = \left\langle \frac{\pi_g}{\pi_g + (1-\pi_g)(1 + v_{gj} W_{gj}' W_{gj})^{-1/2} exp
$$

and

$$
E[\theta_{gj}] = (1 - l_{gj})\mu_{gj} E[\theta_{gj}^2] = (1 - l_{gj})(\Sigma_{gj} + \mu_{gj}) E[I(\theta_{gj}) = 0] = l_{gj}
$$

0.3.2 Variational Distribution of π_g

$$
\begin{aligned}
q^*(\pi_g) \propto & E_{-pi_g}\left[p(\Theta, V, \lambda^2, \pi, \sigma^2|\tilde{Y})\right] \\
\sim & Beta(a_{\pi_g} + \sum_{j=1}^{m_g} < I(\theta_{gj}) = 0 >), b_{\pi_g} + m_g - \sum_{j=1}^{m_g} < I(\theta_{gj}) = 0 >))
\end{aligned}
$$

0.3.3 Variational Distribution of σ^2

$$
\begin{aligned}
q^*(\sigma^2) \propto & E_{-\sigma^2}\left[p(\Theta, V, \lambda^2, \pi, \sigma^2|\tilde{Y})\right] \\
\sim & IG\left(\frac{n + \sum_{g=1}^{G} m_g - \sum_{g=1}^{G}\sum_{j=1}^{m_g} < I(\theta_{gj}=0) > -1}{2}, \frac{< \|(\tilde{Y} - W\Theta\|_2^2 > + < \Theta >' D^{-1} < \Theta >}{2}\right)
\end{aligned}
$$

where

$$
D = diag(< v_{11}^2 >, \ldots, < v_{Gm_G}^2 >)
$$

0.3.4 Variational Distribution of λ^2

$$
\begin{aligned}
q^*(\lambda^2) \propto & E_{-\lambda^2}\left[p(\Theta, V, \lambda^2, \pi, \sigma^2|\tilde{Y})\right] \\
\sim & Gamma\left(\sum_{g=1}^{G} m_g + a_{\lambda^2}, \frac{\sum_{g=1}^{G}\sum_{j=1}^{m_g} v_{gj}^2}{2} + b_{\lambda^2}\right)
\end{aligned}
$$

0.3.5 Variational Distribution of $(v_{gj}^2)^{-1}$

$$
\begin{aligned}
q^*((v_{gj}^2)^{-1}) \propto & E_{-v_{gj}^2}\left[p(\Theta, V, \lambda^2, \pi, \sigma^2|\tilde{Y})\right] \\
& \sim < I(\theta_{gj}=0) > Inverse - Gamma(1, \lambda^2/2) \\
& \quad + < I(\theta_{gj} \neq 0) > Inverse - Gaussian(< \theta_{gj}^{-1} >< \sqrt{\lambda^2} >< \sqrt{\sigma^2} >, < \lambda^2 >)
\end{aligned}
$$

1 Simulation when G=1