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1 Gaussian 2 changing point model

1.1 likelihood and prior

$$\beta_t \overset{\text{ind}}{\sim} \begin{cases} N(\mu_1, \tau_1^{-1}) & \text{for } t = 1, \dots, k \\ N(\mu_2, \tau_2^{-1}) & \text{for } t = k+1, \dots, T \end{cases}$$

Where ,

$$\begin{aligned} \mu_1 &\sim N(0, b_1^{-1}) \\ \mu_2 &\sim N(0, b_2^{-1}) \\ \tau_1 &\sim Gamma(c_1, d_1) \\ \tau_2 &\sim Gamma(c_2, d_2) \\ k &\sim unif\{1, T\} \end{aligned}$$

Then,

$$\begin{aligned} p(\mu_1, \mu_2, \tau_1, \tau_2, k|\boldsymbol{\beta}) \propto & p(\boldsymbol{\beta}|\mu_1, \mu_2, \tau_1, \tau_2, k)p(\mu_1)p(\mu_2)p(\tau_1)p(\tau_2) \\ \propto & (\tau_1)^{\frac{k}{2}} \exp\left(-\frac{1}{2}\tau_1 \sum_{t=1}^{k}(\beta_t - \mu_1)^2\right) \\ & \times (\tau_2)^{\frac{T-k}{2}} \exp\left(-\frac{1}{2}\tau_2 \sum_{t=k+1}^{T}(\beta_t - \mu_2)^2\right) \\ & \times b_1^{\frac{1}{2}} \exp\left(-\frac{1}{2}b_1\mu_1^2\right) \times b_2^{\frac{1}{2}} \exp\left(-\frac{1}{2}b_2\mu_2^2\right) \\ & \times \tau_1^{c_1-1} \exp(-d_1\tau_1) \times \tau_2^{c_2-1} \exp(-d_2\tau_2) \\ & \times 1 \end{aligned}$$

1.2 Gibbs sampler

$$
\begin{aligned}
\mu_1|\mu_2,\tau_1,\tau_2,k,\boldsymbol{\beta} &\sim N\left((k\cdot\tau_1+b_1)^{-1}\tau_1\sum_{t=1}^{k}\beta_t,(k\cdot\tau_1+b_1)^{-1}\right)\\
\mu_2|\mu_1,\tau_1,\tau_2,k,\boldsymbol{\beta} &\sim N\left(((T-k)\cdot\tau_2+b_2)^{-1}\tau_2\sum_{t=k+1}^{T}\beta_t,((T-k)\cdot\tau_2+b_2)^{-1}\right)\\
\tau_1|\mu_1,\mu_2,\tau_2,k,\boldsymbol{\beta} &\sim Gamma\left(c_1+\frac{k}{2},\frac{1}{2}\sum_{t=1}^{k}(\beta_t-\mu_1)^2+d_1\right)\\
\tau_2|\mu_1,\mu_2,\tau_1,k,\boldsymbol{\beta} &\sim Gamma\left(c_2+\frac{T-k}{2},\frac{1}{2}\sum_{t=k+1}^{T}(\beta_t-\mu_2)^2+d_2\right)\\
p(k|\mu_1,\mu_2,\tau_1,\tau_2,\boldsymbol{\beta}) &\propto \exp\left(\frac{1}{2}\sum_{t=1}^{k}\left[\tau_2(\beta_t-\mu_2)^2-\tau_1(\beta_t-\mu_1)^2\right]+\frac{k}{2}\log(\tau_1/\tau_2)\right)=H(k)\\
p(k|\mu_1,\mu_2,\tau_1,\tau_2,\boldsymbol{\beta}) &= \frac{H(k)}{\sum_{t=1}^{T}H(k)}
\end{aligned}
$$

1.3 Variational Bayes

Let the variational distribution is

$$
\begin{aligned}
q_1^*(\mu_1) &\sim N(m_1,s_1)\\
q_2^*(\mu_2) &\sim N(m_2,s_2)\\
q_3^*(\tau_1) &\sim Gamma(\gamma_1,\eta_1)\\
q_4^*(\tau_2) &\sim Gamma(\gamma_2,\eta_2)\\
q_5^*(k) &\sim Categorical
\end{aligned}
$$

Then $E_{q_5^*}(k) = \sum_{t=1}^{k} k \cdot q_5^*(k)$, and

$$
\begin{aligned}
q_{1^*}(\mu_1) \sim & N\left(\left[E_q^*[k]\frac{\gamma_1}{\eta_1}+b_1\right]^{-1}\frac{\gamma_1}{\eta_1}\sum_{t=1}^{E_q^*[k]}\beta_t, \left(E_q^*[k]\frac{\gamma_1}{\eta_1}+b_1\right)^{-1}\right)\\
q_{1^*}(\mu_2) \sim & N\left(\left(T-[E_q^*[k])\frac{\gamma_2}{\eta_2}+b_2\right]^{-1}\frac{\gamma_2}{\eta_2}\sum_{t=E_q^*[k]+1}^{T}\beta_t, \left((T-E_q^*[k])\frac{\gamma_2}{\eta_2}+b_2\right)^{-1}\right)\\
q_{1^*}(\tau_1) \sim & Gamma\left(c_1+\frac{E_q^*[k]}{2}, \frac{1}{2}\sum_{t=1}^{E_q^*[k]}\left\{\beta_t^2-2\beta_t m_1+m_1^2+s_1\right\}+d_1\right)\\
q_{1^*}(\tau_2) \sim & Gamma\left(c_2+\frac{T-E_q^*[k]}{2}, \frac{1}{2}\sum_{t=E_q^*[k]+1}^{T}\left\{\beta_t^2-2\beta_t m_2+m_2^2+s_2\right\}+d_2\right)\\
q_{1^*}(\mu_1) = & \frac{\mathcal{H}(k)}{\sum_{t=1}^{T}\mathcal{H}(k)}\\
\log(\mathcal{H}(k)) = & \frac{1}{2}\sum_{t=1}^{k}\left[\frac{\gamma_2}{\eta_2}(\beta_t^2-2\beta_t m_2+m_2^2+s_2)-\frac{\gamma_1}{\eta_1}(\beta_t^2-2\beta_t m_1+m_1^2+s_1)\right]\\
& + \frac{k}{2}(-\log\eta_1+\psi(\gamma_1)+\log\eta_2-\psi(\gamma_2)))
\end{aligned}
$$

We can use

$$
\begin{aligned}
X &\sim Gamma(\alpha,\beta)\\
E[\log X] &= -\log\beta+\psi(\alpha)
\end{aligned}
$$

where ψ means digamma function Prior and initial value are

2 Simulation

Make simulation data from

$$
\beta_t \overset{\text{iid}}{\sim} \begin{cases} N(1,1) & t=1,\dots,30 \\ N(0,1) & t=31,\dots,100 \end{cases}
$$

setting the prior as

$$
\begin{aligned}
b=1 \quad b=1\\
c=1 \quad c=1\\
d=1 \quad d=1
\end{aligned}
$$

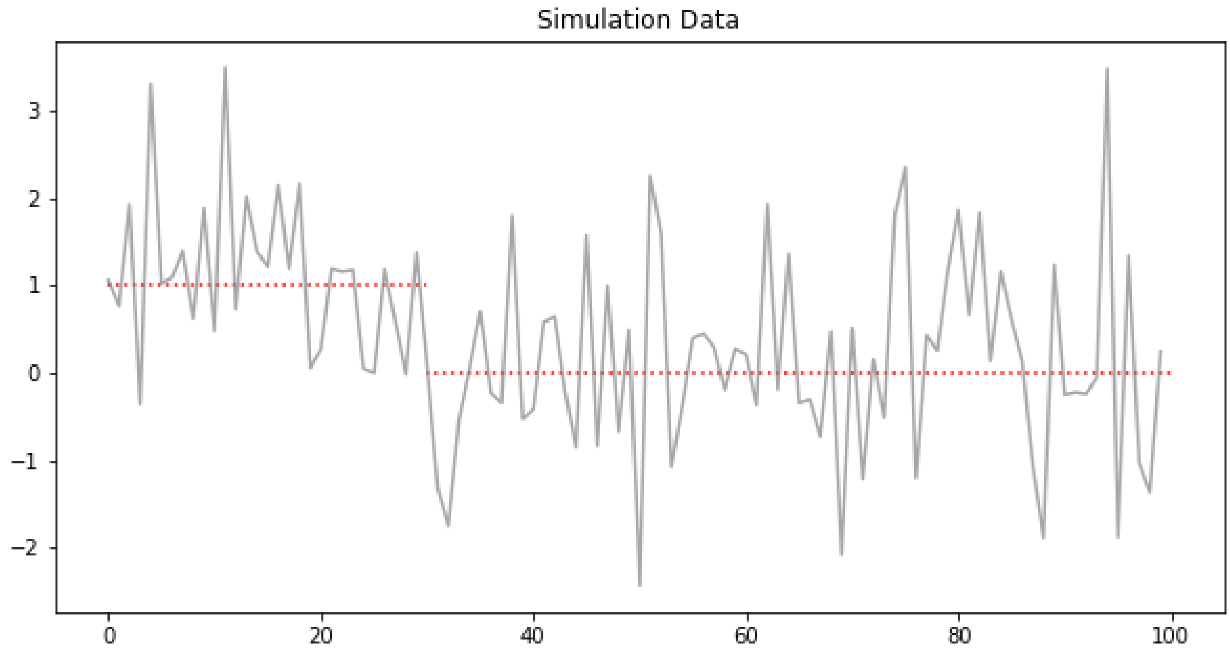

Figure 1: Simulation data of β_t

and initail values as

$$\gamma_1 = 1 \quad \eta_1 = 1$$

$$\gamma_2 = 1 \quad \eta_2 = 1$$

$$k = 1$$

Variational distribution is

$$q_1^*(\mu_1) \sim N(1.16, 0.03)$$

$$q_2^*(\mu_2) \sim N(0.13, 0.03)$$

$$q_3^*(\tau_1) \sim Gamma(14.82, 13.61)$$

$$q_4^*(\tau_2) \sim Gamma(37.17, 86.89)$$

and q_5^* is

$$E_{q_3}^*[k] = 27.65$$

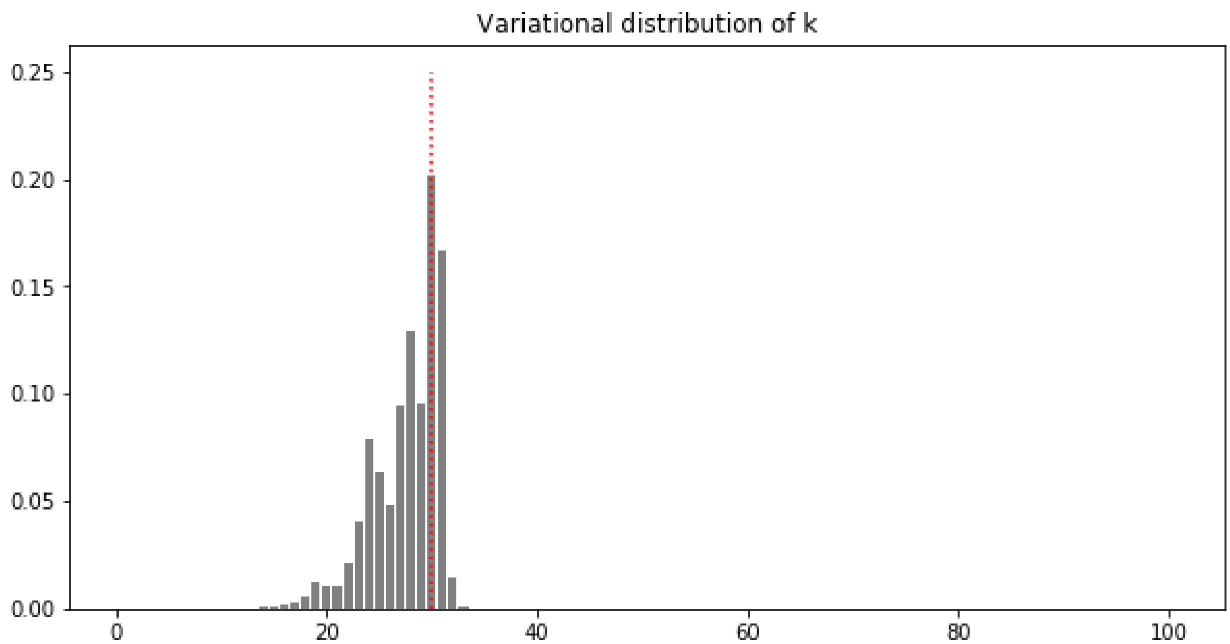

Figure 2: Variational distribution of k