

July 04, 2019

1 Variable Selection

Start with spline model

$$Y = b(X)\beta + \epsilon$$

Select variable? select basis?

- $Y = b(\Gamma X)\beta + \epsilon$
- $Y = b(X)\Gamma\beta + \epsilon$

2 Basis Selection Model

$$Y = b(X)\Gamma\beta + \epsilon$$

Let $Z = b(X)$

2.1 prior

$$Y|\beta, \sigma^2, \Gamma \sim N(Z\Gamma\beta, \sigma^2 \cdot I)$$

$$\sigma^2 \sim \text{Inverse} - \text{Gamma}(a, b)$$

$$\beta_j | \sigma_{\beta_j}^2 \stackrel{\text{iid}}{\sim} N(0, \sigma_{\beta_j}^2)$$

$$\sigma_{\beta_j}^2 \stackrel{\text{iid}}{\sim} \text{Inverse} - \text{Gamma}(c, d)$$

$$\gamma_j \stackrel{\text{iid}}{\sim} \text{Bernoulli}(\rho)$$

- Z is $n \times p$ design matrix
- $\Gamma = \text{diag}(\gamma_j)$ for $j = 1, \dots, p$
- a, b, c, d is flat prior

2.2 posterior

$$\begin{aligned}
p(\beta, \sigma^2, \sigma_\beta^2, \Gamma|Y) &\propto p(\beta, \sigma^2, \sigma_\beta^2, \Gamma, Y) \\
&\propto p(Y|\beta, \sigma^2, \Gamma) p(\beta|\sigma_\beta^2) p(\Gamma) p(\sigma^2) p(\sigma_\beta^2) \\
&\propto (\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} (Y - Z\Gamma\beta)' (Y - Z\Gamma\beta)\right) \\
&\quad \times \prod_{j=1}^p (\sigma_{\beta_j}^2)^{-1/2} \exp\left(-\frac{1}{2} \sum_{j=1}^p \frac{\beta_j^2}{\sigma_{\beta_j}^2}\right) \\
&\quad \times \prod_{j=1}^p \rho^{\gamma_j} (1 - \rho)^{-\gamma_j} \\
&\quad \times (\sigma^2)^{-a-1} \exp\left(-\frac{b}{\sigma^2}\right) \\
&\quad \times \prod_{j=1}^p (\sigma_{\beta_j}^2)^{-c-1} \exp\left(-\sum_{j=1}^p \frac{d}{\sigma_{\beta_j}^2}\right)
\end{aligned}$$

2.3 Variational Inference

$$p(\beta, \sigma^2, \sigma_\beta^2, \Gamma|Y) \approx q(\beta, \sigma^2, \sigma_\beta^2, \Gamma) = \prod_{j=1}^p q_1^*(\beta_j) q_2^*(\sigma^2) \prod_{j=1}^p q_3^*(\sigma_{\beta_j}^2) \prod_{j=1}^p q_4^*(\gamma_j)$$

Use coordinate ascent algorithm, q density of β is

$$\begin{aligned}
q_1^*(\beta) &\propto E_{-q_1} [p(\beta, \sigma^2, \sigma_\beta^2, \Gamma, Y)] \\
&\propto E_{-q_1} \left[\exp\left(-\frac{1}{2\sigma^2} (Y - Z\Gamma\beta)' (Y - Z\Gamma\beta) - \frac{1}{2} \sum_{j=1}^p \frac{\beta_j^2}{\sigma_{\beta_j}^2}\right) \right] \\
&\propto \exp\left(-\frac{1}{2}\beta' \langle D \rangle - \frac{1}{2} \left\langle \frac{1}{\sigma^2} \right\rangle (\beta' \langle \Gamma \rangle' Z' Z \langle \Gamma \rangle \beta - 2\beta' \langle \Gamma \rangle' Z' Y)\right) \\
&\propto \exp\left(-\frac{1}{2} \left[\beta' \left(\langle D \rangle + \left\langle \frac{1}{\sigma^2} \right\rangle \langle \Gamma \rangle' Z' Z \langle \Gamma \rangle \right) \beta - 2 \left\langle \frac{1}{\sigma^2} \right\rangle \beta' \langle \Gamma \rangle' Z' Y \right] \right) \\
&\sim N(\mu, \Sigma)
\end{aligned}$$

Where $D = \text{diag}(\frac{1}{\sigma_{\beta_j}^2})$, $\langle \cdot \rangle$ means expectation under q functions and

$$\Sigma = \left(\langle D \rangle + \beta' \left\langle \frac{1}{\sigma^2} \right\rangle \langle \Gamma \rangle' Z' Z \langle \Gamma \rangle \right)^{-1}, \quad \mu = \left\langle \frac{1}{\sigma^2} \right\rangle \Sigma \langle \Gamma \rangle' Z' Y$$

q density of σ^2 is

$$\begin{aligned} q_2^*(\sigma^2) &\propto E_{-q_2} [p(\beta, \sigma^2, \sigma_\beta^2, \Gamma, Y)] \\ &\propto E_{-q_2} \left[(\sigma^2)^{-\left(\frac{n}{2}+a\right)-1} \exp \left(-\frac{1}{\sigma^2} (b + (Y - Z\Gamma\beta)'(Y - Z\Gamma\beta)) \right) \right] \\ &\propto \text{Inverse} - \text{Gamma} \left(a + \frac{n}{2}, b + \frac{1}{2} (Y'Y - 2\langle\beta\rangle' \langle\Gamma\rangle' Z'Y + \text{tr}((Z'Z \odot \Omega)(\mu\mu' + \Sigma))) \right) \end{aligned}$$

Where \odot is hadamard product and

- $\gamma = (\gamma_1, \dots, \gamma_p)$
- $\Omega = \langle\gamma\rangle \langle\gamma\rangle' + \langle\Gamma\rangle \odot (I - \langle\Gamma\rangle)$

q density of $\sigma_{\beta_j}^2$ is

$$\begin{aligned} q_3^*(\sigma_{\beta_j}^2) &\propto E_{-q_3} [p(\beta, \sigma^2, \sigma_\beta^2, \Gamma, Y)] \\ &\propto \prod_{j=1}^p \left[(\sigma_{\beta_j}^2)^{-\left(\frac{1}{2}+c\right)-1} \exp \left(-\frac{1}{\sigma_{\beta_j}^2} \left(d + \frac{1}{2} \langle\beta_j^2\rangle \right) \right) \right] \\ q_3^*(\sigma_{\beta_j}^2) &\sim \text{Inverse} - \text{Gamma} \left(c + \frac{1}{2}, d + \frac{1}{2} \langle\beta_j^2\rangle \right) \end{aligned}$$

q density of γ is

$$\begin{aligned} q_4^*(\gamma) E_{-q_4} &\left[\propto \prod_{j=1}^p \rho^{\gamma_j} (1 - \rho)^{-\gamma_j} \exp \left(-\frac{1}{2} \frac{1}{\sigma^2} (\beta' \Gamma' Z' Z \Gamma \beta - 2\beta' \Gamma' Z' y) \right) \right] \\ &\propto \exp \left[\gamma_j \left(\log(\rho/(1 - \rho)) - \frac{1}{2} \left\langle \frac{1}{\sigma^2} \right\rangle \langle\beta_j^2\rangle Z_j' Z_j + \left\langle \frac{1}{\sigma^2} \right\rangle Z_j' [Y\mu_j - X_{-j} \langle\Gamma_{-j}\rangle (\mu_{-j}\mu_j + \Sigma_{-j,j})] \right) \right] \end{aligned}$$

Where

- X_j means j th coloumn of X
- X_{-j} means without j th column
- $X_{-i,j}$ means j th column without i th component
- μ_j is j th component of vector and μ_{-j} means without j th component