Q9

May 21, 2019

```
In [1]: import numpy as np
        import pandas as pd
        import matplotlib.pyplot as plt
        import scipy.stats as stats
        from scipy.special import digamma
        from scipy.stats import gamma,norm
```

0.1 prior setting

Let $\left(\sigma^2\right)^{-1} = \tau$ Then,

$$\begin{aligned} y_i|\mu,\tau &\sim N(\mu,\tau^{-1}) \\ \tau &\sim Gamma(a_0,b_0) \\ \mu|\tau &\sim N(0,(\tau/k)^{-1}) \end{aligned}$$

Thus

$$p(\mu,\tau|Y) = \frac{p(\mu,\tau)p(Y|\mu,\tau)}{p(Y)} = \frac{p(\mu|\tau)p(\tau)p(Y|\mu,\tau)}{p(Y)}$$

1 (a) Derive MF variational distribution of (μ,τ) and ELBO

Let $\theta = (a_0,b_0,k)$, $p = p(\mu,\tau|Y,\theta)$ and $q = q(\mu,\tau)$

$$p(\mu,\tau|Y,\theta) \approx q(\mu,\tau) = q_1(\mu)q_2(\tau)$$

Thus, mean-field variational distribution is

$$q(\mu,\tau) = q_1(\mu|t,u)q_2(\tau|v,w)$$

where $q_1 \sim N(t,u^{-1})$ and $q_2 \sim Gamma(v,w)$, variational parameters $\lambda = (t,u,v,w)$. ELBO is

$$\begin{aligned} ELBO(\lambda) =& E_q[\log p(\mu,\tau,Y)|\lambda] - E_q[\log q(\mu,\tau)|\lambda] \\ =& \sum_{i=1}^{n} E_q[\log p(y_i|\mu,\tau)|\lambda] + E_q[\log p(\mu|\tau)|t,u] + E_q[\log p(\tau)|w,v] \\ & - E_{q_1}[logq_1(\mu|t,u)|t,u] - E_{q_2}[logq_2(\tau)|v,w] \end{aligned}$$

2 (b) derive the coordinate descent algorithm

$$\begin{aligned}
\log q_1^*(\mu) &\propto E_{q_2}\left[\log p(\mu,\tau,Y)\right] \propto E_{q_2}\left[\log p(Y|\mu,\tau)+\log p(\mu|\tau)\right] \\
&\propto -\frac{1}{2}E_{q_2}[\tau]\cdot\sum_{i=1}^{n}(y_i-\mu)^2-\frac{E_{q_2}[\tau]}{2k}\mu^2 \\
&\propto -\frac{1}{2}\left[E_{q_2}[\tau]\left(n+\frac{1}{k}\right)\mu^2-2E_{q_2}[\tau]\mu\sum_{i=1}^{n}y_i\right]
\end{aligned}$$

Thus

$$t=\frac{k\sum_{i=1}^{n}y_i}{nk+1}, u=\frac{E_{q2}[\tau](nk+1)}{k}$$

as $q_1^*(\mu)\sim N(t,u^{-1})$

$$\begin{aligned}
E_{q_1}[\mu] &= t=\frac{k\sum_{i=1}^{n}y_i}{nk+1} \\
E_{q_1}[\mu^2] &= u^{-1}+t^2=\left[\frac{k}{E_{q2}[\tau](nk+1)}\right]+\left[\frac{k\sum_{i=1}^{n}y_i}{nk+1}\right]^2
\end{aligned}$$

where $E_{q2}[\tau]=v/w$

$$\begin{aligned}
\log q_2^*(\tau) &\propto E_{q_1}\left[\log p(\mu,\tau,Y)\right] \propto E_{q_1}\left[\log p(Y|\mu,\tau)+\log p(\mu|\tau)+\log p(\tau)\right] \\
&\propto (a_0-1)\log\tau-b_0\tau+\frac{n}{2}\log\tau-\frac{1}{2}\tau E_{q_1}\left[\sum_{i=1}^{n}(y_i-\mu)^2\right]+\frac{1}{2}\log\tau-\frac{\tau}{2k}E_{q_1}\left[\mu^2\right] \\
&\propto \log(\tau)\left(a_0-1+\frac{n+1}{2}\right)-\tau\left(b_0+\frac{1}{2}E_{q_1}\left[\sum_{i=1}^{n}(y_i-\mu)^2\right]+\frac{1}{2k}E_{q_1}\left[\mu^2\right]\right)
\end{aligned}$$

Thus

$$v=a_0+\frac{n+1}{2}, w=b_0+\frac{1}{2}E_{q_1}\left[\sum_{i=1}^{n}(y_i-\mu)^2\right]+\frac{1}{2k}E_{q_1}\left[\mu^2\right]$$

where,

$$E_{q_1}\left[\sum_{i=1}^{n}(y_i-\mu)^2\right]=\sum y_i^2-2\sum y_iE[\mu]+E[\mu^2]$$

$q_2^*(\tau)\sim Gamma(v,w)$

3 (c) write down python code

input data

```
In [2]: Y = np.array([1.64,1.70,1.72,1.74,1.82,1.82,1.82,
                      1.90,2.08,1.78,1.86,1.96,1.96,2.00,2.00])
```

setting the initail value

```
In [45]: def mfvb(Y,maxiter = 100):
             a_0=2
             b_0=60
             k=2
             n= len(Y)
             expected_mu = 0
             expected_mu2 = 1

             v = a_0 +(n+1)/2
             param_out=[]
             i=0
             while i<maxiter:
                 w = b_0 + 0.5*((Y**2).sum() -2 *Y.sum() * expected_mu
                                + expected_mu2) + expected_mu2/(2*k)
                 expected_tau = v/w
                 t = (k*Y.sum())/(n*k+1)
                 u = (expected_tau * (n*k+1))/k
                 expected_mu = t
                 expected_mu2 = 1/u + t**2
                 param_out.append([v,w,t,u])
                 i= i+1
                 if i>1:
                     if param_out[-1] == param_out[-2]:
                         break
             out = pd.DataFrame(param_out)
             out.columns = ['v','w','t','u']
             return(out)

In [46]: mfvb(Y)
```

Out[46]:

	v	w	t	u
0	10.0	86.625200	1.793548	1.789318
1	10.0	38.846321	1.793548	3.990082
2	10.0	38.615133	1.793548	4.013970
3	10.0	38.614014	1.793548	4.014087
4	10.0	38.614009	1.793548	4.014087
5	10.0	38.614009	1.793548	4.014087
6	10.0	38.614009	1.793548	4.014087
7	10.0	38.614009	1.793548	4.014087
8	10.0	38.614009	1.793548	4.014087

```
In [13]: ksigma2 = 1/4.014087
         sigma2 = ksigma2/2
         print(ksigma2,sigma2)
```

```
0.24912265229926506 0.12456132614963253
```

$$q_2(\tau) \sim Gamma(10, 38.614009)$$
$$q_1(\mu) \sim N(1.793548, 0.249123)$$

4 (d) draw contour of the target posterior distribution and compare it with MFVB

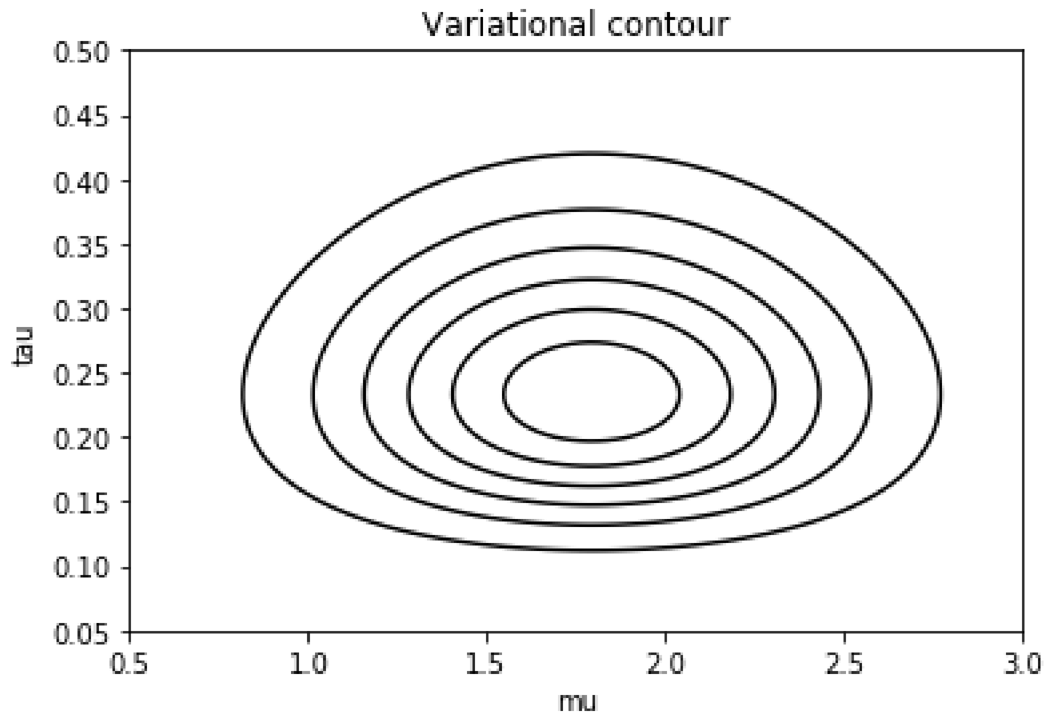
4.1 VI Contour

$$q_2(\tau) \sim Gamma(10, 38.614009)$$
$$q_1(\mu) \sim N(1.793548, 0.249123)$$

```
In [5]: def q(mu,tau):
            out = gamma.pdf(tau,a = 10,scale = 1/38.614009)*\
                  norm.pdf(mu,loc = 1.793548,
                           scale = np.sqrt(0.249123))
            return(out)
```

```
In [28]: tau = np.linspace(0.05, 0.5, 100)
         mu = np.linspace(0.5, 3, 100)

         Mu, Tau = np.meshgrid(mu, tau)
         vi = q(Mu, Tau)
         plt.contour( Mu,Tau, vi, colors='black')
         plt.title('Variational contour')
         plt.xlabel('mu')
         plt.ylabel('tau')
         plt.show()
```



4.2 Posterior contour

$$p(\mu, \tau|Y) \propto p(\mu, \tau, Y) \propto p(Y|\mu, \tau)p(\mu|\tau)p(\tau)$$

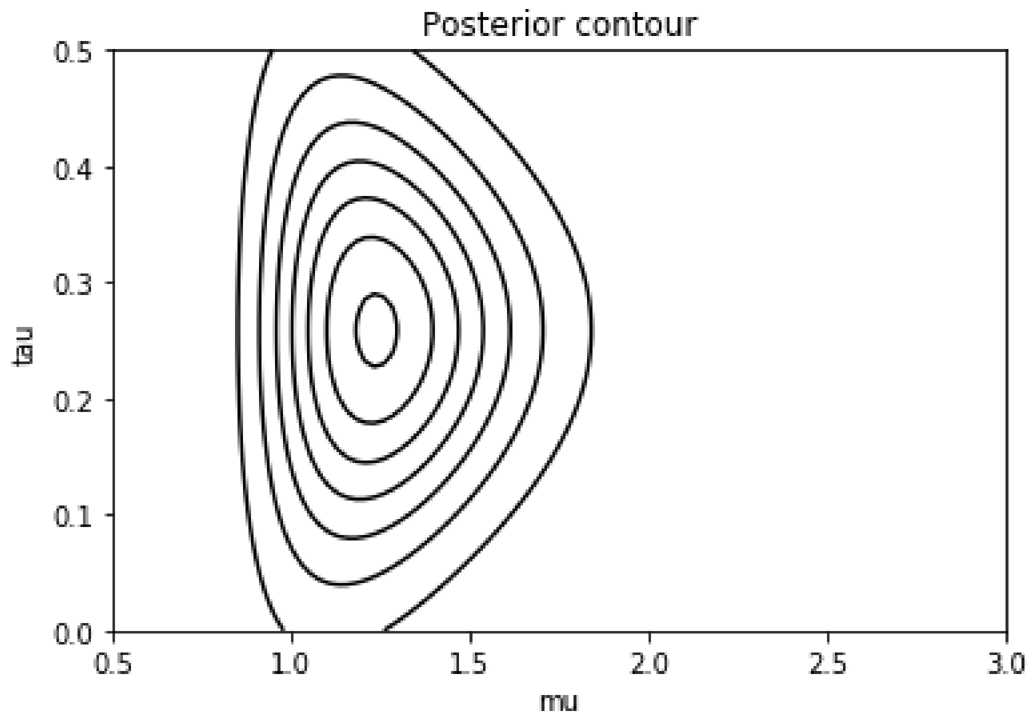
Thus

where $a_0 = 2, b_0 = 60$ and $k = 2$

```
In [8]: def p(mu,tau,Y):
            out = np.prod(norm.pdf(Y,loc = mu,
                          scale = np.sqrt(tau**(-1))))*\
            norm.pdf(mu,loc = 0,
                     scale = np.sqrt(2*(tau**(-1))))*\
            gamma.pdf(tau,a = 2,scale = 1/60)
            return(out)
```

```
In [29]: post = np.zeros([100,100])
         for i in range(100):
             for j in range(100):
                 post[i,j] = p(mu[i], tau[j],Y)
```

```
In [27]: plt.contour(Mu, Tau, post, colors='black')
         plt.title('Posterior contour')
         plt.xlabel('mu')
         plt.ylabel('tau')
         plt.show()
```

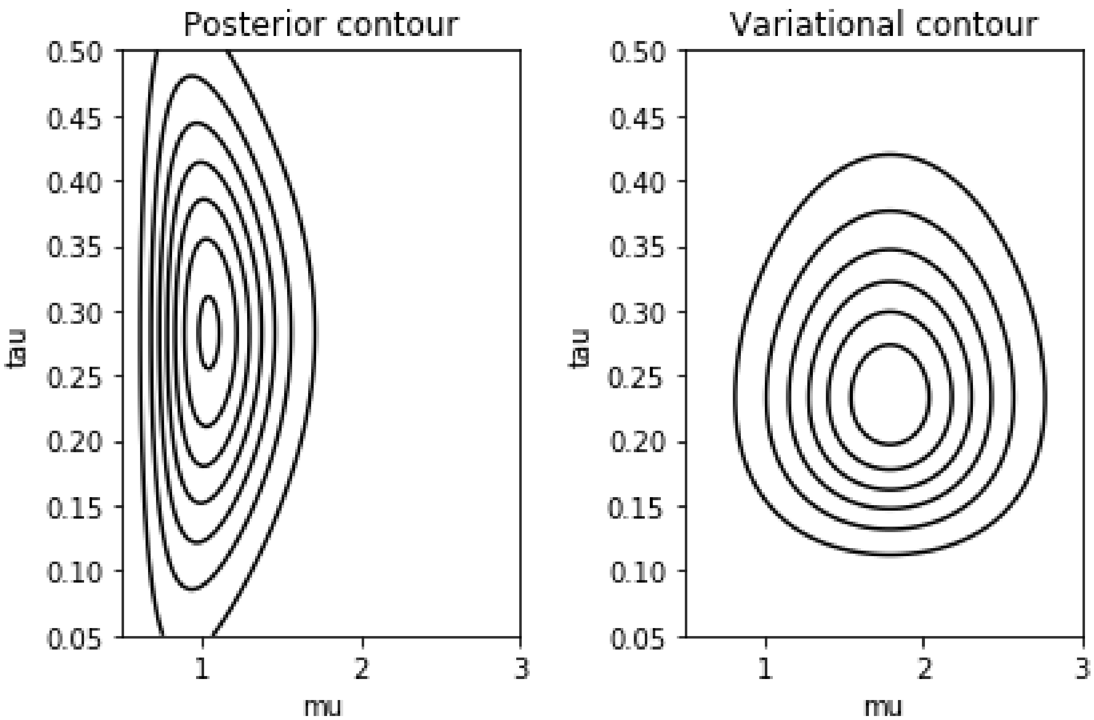


4.3 Compare

```
In [40]: def posteriorplot(ax):
              ax.contour(Mu, Tau, post, colors='black')
              ax.set_xlabel('mu')
              ax.set_ylabel('tau')
              ax.set_title('Posterior contour')
```

```
In [41]: def viplot(ax):
              ax.contour( Mu,Tau, vi, colors='black')
              ax.set_xlabel('mu')
              ax.set_ylabel('tau')
              ax.set_title('Variational contour')
```

```
In [42]: fig, ((ax1), (ax4)) = plt.subplots(nrows=1, ncols=2)
         posteriorplot(ax1)
         viplot(ax4)
         plt.tight_layout()
```



It seems that mean-field vairational distribution overestimate variance of μ and underestimate variance of τ