# Q8

## (a) Complete data log likelihood under under expanded parameter space $\Theta = (\theta^*, \alpha)$

$$
\begin{aligned}
L(\Theta|S,Y) &= \prod_{i=1}^{n} P(S,Y|\Theta) \\
&= \prod_{i=1}^{n} P(Y|S,\Theta)P(S|\Theta) \\
&= \prod_{i=1}^{n} \frac{e^{-(1-\alpha)}(1-\alpha)^{y_i-s_i}}{(y_i-s_i)!} \frac{\theta^{*s_i}e^{-\theta^*}}{s_i!} \\
&= e^{-n(1-\alpha+\theta^*)} \cdot \theta^{*\sum_{i=1}^{n} s_i} \cdot \prod_{i=1}^{n} \frac{(1-\alpha)^{y_i-s_i}}{(y_i-s_i)! \cdot s_i!} \\
l(\Theta|S,Y) &= -n(1-\alpha) - n\theta^* + log(1-\alpha)\sum_{i=1}^{n}(y_i-s_i) + log(\theta^*)\sum_{i=1}^{n} s_i - \sum_{i=1}^{n} log\left[(y_i-s_i)!s_i!\right]
\end{aligned}
$$

## (b) show that observed data and complete data model preserved under the expanded parameter space

Model preserved when $\theta = \theta^* - \alpha$. Then,

$$R(\Theta) = \theta^* - \alpha$$

Observed-data model is preserved

$$Y_{obs}|\Theta \sim P(Y_{obs}|\theta = R(\Theta))$$

Complete data model is preserved at $\alpha = \alpha_0$

$$
\begin{aligned}
P_x(Y_{com}|\Theta = (\theta^*, \alpha_0)) &= e^{-n(1-\alpha_0+\theta^*)} \cdot \theta^{*\sum_{i=1}^{n} s_i} \cdot \prod_{i=1}^{n} \frac{(1-\alpha_0)^{y_i-s_i}}{(y_i-s_i)! \cdot s_i!} \\
P(Y_{com}|\theta = \theta^*) &= e^{-n(1+\theta^*)} \cdot \theta^{*\sum_{i=1}^{n} s_i} \cdot \prod_{i=1}^{n} \frac{1}{(y_i-s_i)! \cdot s_i!}
\end{aligned}
$$

$P_x(Y_{com}|\Theta = (\theta^*, \alpha_0)) = P(Y_{com}|\theta = \theta^*)$ when $\alpha_0 = 0$, Complete data model is preserved at $\alpha_0 = 0$

## (c) Derive PX-EM and show that converges in one iteration

### 0.1 E-step

$$\begin{aligned}
Q(\Theta|\Theta^{(t)}) &= E[l(\Theta|Y_{com})|Y_{obs}, \Theta^{(t)}] \\
&= -n(1-\alpha) - n\theta^* + log(1-\alpha)\sum_{i=1}^{n}(y_i - E\left[s_i|Y, \Theta^{(t)}\right]) + log(\theta^*)\sum_{i=1}^{n} E\left[s_i|Y, \Theta^{(t)}\right]
\end{aligned}$$

need to compute pmf of $s_i|Y_{obs}, \Theta^{(t)}$ is

$$\begin{aligned}
P(s_i|Y_{obs} = y_i, \Theta^{(t)}) &= \frac{P(s_i, y_i|\Theta^{(t)})}{P(y_i|\Theta^{(t)})} \\
&= \frac{{\theta^{*(t)}}^{s_i} e^{-\theta^{*(t)}}}{s_i!} \cdot \frac{(1-\alpha)^{y_i - s_i} e^{-(1-\alpha)}}{(y_i - s_i)!} \cdot \left[\frac{(\theta^{*(t)} + 1 - \alpha)^{y_i} e^{-(\theta^{*(t)}+1-\alpha)}}{y_i!}\right]^{-1} \\
&= \frac{y_i!}{(y_i - s_i)! s_i!}\left(\frac{\theta^{*(t)}}{\theta^{*(t)} + 1 - \alpha}\right)^{s_i}\left(\frac{1}{\theta^{*(t)} + 1 - \alpha}\right)^{y_i - s_i}
\end{aligned}$$

Thus $s_i|Y_{obs} = y_i, \Theta^{(t)} \sim Binomial\left(y_i, \frac{\theta^{*(t)}}{\theta^{*(t)}+1-\alpha}\right)$, Then $E[s_i|Y_{obs}, \theta^{(t)}] = y_i \cdot \frac{\theta^{*(t)}}{\theta^{*(t)}+1-\alpha}$

### 0.2 M-step

$$Q(\Theta|\Theta^{(t)}) = -n(1-\alpha) - n\theta^* + log(1-\alpha)\sum_{i=1}^{n}(y_i - y_i \cdot \frac{\theta^{*(t)}}{\theta^{*(t)} + 1 - \alpha}) + log\theta^* \cdot \sum_{i=1}^{n} y_i \cdot \frac{\theta^{*(t)}}{\theta^{*(t)} + 1 - \alpha}$$

Maximize the $Q$function by letting the first derivative 0

$$\begin{aligned}
\frac{\partial Q(\Theta|\Theta^{(t)})}{\partial \theta^*} &=^{let} 0 \\
&= -n + \frac{\sum_{i=1}^{n} y_i \cdot \frac{\theta^{*(t)}}{\theta^{*(t)}+1-\alpha}}{\theta^*} \\
\theta^{*(t+1)} &= \frac{1}{n} \cdot \sum_{i=1}^{n} y_i \cdot \frac{\theta^{*(t)}}{\theta^{*(t)} + 1 - \alpha} \\
\frac{\partial Q(\Theta|\Theta^{(t)})}{\partial \alpha} &=^{let} 0 \\
&= n - \frac{\sum_{i=1}^{n}\left(y_i - y_i \cdot \frac{\theta^{*(t)}}{\theta^{*(t)}+1-\alpha}\right)}{1-\alpha} \\
&= n - \frac{\sum_{i=1}^{n} y_i\left(1 - \frac{\theta^{*(t)}}{\theta^{*(t)}+1-\alpha}\right)}{1-\alpha} \\
\hat{\alpha} &= 1 - \frac{\sum_{i=1}^{n} y_i\left(1 - \frac{\theta^{*(t)}}{\theta^{*(t)}+1-\alpha}\right)}{n}
\end{aligned}$$

First iteration is,

$$
\begin{aligned}
\theta^{(t+1)} &= R(\Theta^{(t+1)}) \\
&= \hat{\theta}^{*} - \hat{\alpha} \\
&= \frac{1}{n} \cdot \sum_{i=1}^{n} y_i \cdot \frac{\theta^{*(t)}}{\theta^{*(t)} + 1 - \alpha} - 1 + \frac{\sum_{i=1}^{n} y_i \left(1 - \frac{\theta^{*(t)}}{\theta^{*(t)}+1-\alpha}\right)}{n} \\
&= \frac{\sum_{i=1}^{n} y_i}{n} - 1
\end{aligned}
$$

Which is same value in Q7 (unique solution of EM-algorithm), it converges in first iteration.