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1 Natural estimator

1.1 Two-Stage-Least-Squares

1. $\mathbf{x}_{1i}, \upsilon_i$ are observable and unobservable characteristics which effects on target variable $\mathbf{y}_N$.
2. $\mathbf{x}_{2i}, a_i$ are observable and unobservable characteristics which effects on link formation ($\mathbf{D}_N$ or $\mathbf{G}_N$)
3. $\mathbf{x}_i = \mathbf{x}_{1i} \cup \mathbf{x}_{2i}$

2SLS estimator is valid when $E[\mathbf{G}_N \boldsymbol{\upsilon}_N] = 0$. Specifically, the validity of the 2SLS estimator depends on the orthogonality condition $E[\boldsymbol{\upsilon}_N|\mathbf{Z}_N] = 0$ which is implied if $E[\boldsymbol{\upsilon}_N|\mathbf{X}_{1N}, \mathbf{D}_N] = 0$

$$\hat{\beta}_N^{2SLS} = (\mathbf{W}_N'\mathbf{Z}_N(\mathbf{Z}_N'\mathbf{Z}_N)^{-1}\mathbf{Z}_N'\mathbf{W}_N)^{-1}\mathbf{W}_N'\mathbf{Z}_N(\mathbf{Z}_N'\mathbf{Z}_N)^{-1}\mathbf{Z}_N'\mathbf{y}_N \tag{1}$$

where $\mathbf{W}_N = [\mathbf{G}_N\mathbf{y}_N, \mathbf{X}_{1N}, \mathbf{G}_N\mathbf{X}_{1N}]$ and $\mathbf{Z}_N = [\mathbf{X}_{1N}, \mathbf{G}_N\mathbf{X}_{1N}, \mathbf{G}_N^2\mathbf{X}_{1N}]$.

When the network matrix is endogenous, $E[\mathbf{G}_N \boldsymbol{\upsilon}_N] \neq 0$ and it may be that $E[\boldsymbol{\upsilon}_N|\mathbf{X}_{1N}, \mathbf{D}_N] \neq 0$

2 Identification of Peer Effect

2.1 Assumption 1

(i) $(\mathbf{x}_i, a_i, \upsilon_i)$ are i.i.d. for all i, i = 1, . . . , N

(ii) $\{u_{ij}\}_{i,j=1,\ldots,N}$ are independent of $(\mathbf{X}_N, \mathbf{a}_N, \boldsymbol{\upsilon}_N)$ and i.i.d across (i, j) with cdf $\Phi(\cdot)$

(iii) $E[\upsilon_i|\mathbf{x}_i, a_i] = E[\upsilon_i|a_i]$

Assumption 1(i) implies observable $(x)_i$ and unobservable a_i, υ_i are randomly drawn, which is standard assumption in the peer effects literature. Assumption1(ii) assumes that link formation error u_{ij} is orthogoanl th all other observables and unobservables in the model. It means that u_{ij} from the link formation process does not influence outcomes $y_1, \ldots, y_N$, However we allow dependence between unobserved components a_i and υ_i. Assumption 1(iii) assume that the dependence between $\mathbf{x}_i$ and υ_i exists only through a_i

2.2 Lemma 1 (Control Function of Peer Group of Endogeneity)

$$E[\upsilon_i|\mathbf{X}_N, \mathbf{D}_N, a_i] = E[\upsilon_i|a_i] \tag{2}$$

$$\begin{aligned}
E[(\mathbf{z}_i - E[\mathbf{z}_i|a_i])(\upsilon_i - E[\upsilon_i|a_i]) \,|\, a_i] &= E[\mathbf{z}_i\upsilon_i|a_i] - E[\mathbf{z}_i|a_i]E[\upsilon_i|a_i] \\
&= E[E[\mathbf{z}_i\upsilon_i|a_i, \mathbf{X}_{1N}, \mathbf{G}_N]|a_i] - E[\mathbf{z}_i|a_i]E[\upsilon_i|a_i] \\
&= E[\mathbf{z}_i E[\upsilon_i|a_i, \mathbf{X}_{1N}, \mathbf{G}_N]|a_i] - E[\mathbf{z}_i|a_i]E[\upsilon_i|a_i] \\
&= E[\mathbf{z}_i E[\upsilon_i|a_i]|a_i] - E[\mathbf{z}_i|a_i]E[\upsilon_i|a_i] \\
&= 0
\end{aligned} \tag{3}$$

as $y_i = \mathbf{w}_i'\beta^0 + \upsilon_i$

$$\begin{aligned}
0 &= E[(\mathbf{z}_i - E[\mathbf{z}_i|a_i])(y_i - \mathbf{w}_i'\beta) - E[y_i - \mathbf{w}_i'\beta|a_i]] \\
&= E[(\mathbf{z}_i - E[\mathbf{z}_i|a_i])(\upsilon - E[\upsilon_i|a_i])'](\beta - \beta^0) + E[(\mathbf{z}_i - E[\mathbf{z}_i|a_i])(\upsilon_i - E[\upsilon_i|a_i])] \\
&= E[(\mathbf{z}_i - E[\mathbf{z}_i|a_i])(\upsilon - E[\upsilon_i|a_i])'](\beta - \beta^0) \\
&\Leftrightarrow \beta = \beta^0
\end{aligned} \tag{4}$$

2.3 Assumption 2 (Rank condition)

$$E[(\mathbf{z}_i - E[\mathbf{z}_i|a_i])(\mathbf{w}_i - E[\mathbf{w}_i|a_i])'] \, has \, full \, rank \tag{5}$$

2.4 Theorem 3.1 (identification)

β^0 is identified by moment condition

$$E[(\mathbf{z}_i - E[\mathbf{z}_i|a_i])(y_i - E[y_i|a_i] - (\mathbf{w}_i - E[\mathbf{w}_i|a_i])'\beta)] = 0 \Leftrightarrow \beta = \beta^0 \tag{6}$$

2.5 Assumption 3

$$\mathbf{x}_{1i} \cap \mathbf{x}_{2i} = \emptyset$$

Under these assumption

$$E[\upsilon_i|\mathbf{X}_N, \mathbf{D}_N, a_i] = E[\upsilon_i|a_i] = E[\upsilon_i|\mathbf{x}_{2i}, a_i] \tag{7}$$

$$
\begin{aligned}
&E[(\mathbf{z}_i - E[\mathbf{z}_i|\mathbf{x}_{2i}, a_i])(\upsilon_i - E[\upsilon_i|\mathbf{x}_{2i}, a_i]) \,|\, \mathbf{x}_{2i}, a_i] \\
&= E[\mathbf{z}_i\upsilon_i|\mathbf{x}_{2i}, a_i] - E[\mathbf{z}_i|\mathbf{x}_{2i}, a_i]E[\upsilon_i|\mathbf{x}_{2i}, a_i] \\
&= E[E[\mathbf{z}_i\upsilon_i|a_i, \mathbf{X}_{1N}, \mathbf{G}_N]|\mathbf{x}_{2i}, a_i] - E[\mathbf{z}_i|\mathbf{x}_{2i}, a_i]E[\upsilon_i|\mathbf{x}_{2i}, a_i] \\
&= E[\mathbf{z}_i E[\upsilon_i|\mathbf{x}_{2i}, a_i]|\mathbf{x}_{2i}, a_i] - E[\mathbf{z}_i|\mathbf{x}_{2i}, a_i]E[\upsilon_i|\mathbf{x}_{2i}, a_i] \\
&= 0
\end{aligned} \tag{8}
$$

2.6 Assumption 4 (Rank condition)

$$
E[(\mathbf{z}_i - E[\mathbf{z}_i|\mathbf{x}_{2i}, a_i])(\mathbf{w}_i - E[\mathbf{w}_i|\mathbf{x}_{2i}, a_i])'] \, has \, full \, rank \tag{9}
$$

$$
\begin{aligned}
0 &= E[(\mathbf{z}_i - E[\mathbf{z}_i|\mathbf{x}_{2i}, a_i])(y_i - \mathbf{w}_i'\beta) - E[y_i - \mathbf{w}_i'\beta|\mathbf{x}_{2i}, a_i]] \\
&= E[(\mathbf{z}_i - E[\mathbf{z}_i|\mathbf{x}_{2i}, a_i])(\upsilon - E[\upsilon_i|\mathbf{x}_{2i}, a_i])'](\beta - \beta^0) + E[(\mathbf{z}_i - E[\mathbf{z}_i|\mathbf{x}_{2i}, a_i])(\upsilon_i - E[\upsilon_i|\mathbf{x}_{2i}, a_i])] \\
&= E[(\mathbf{z}_i - E[\mathbf{z}_i|\mathbf{x}_{2i}, a_i])(\upsilon - E[\upsilon_i|\mathbf{x}_{2i}, a_i])'](\beta - \beta^0) \\
&\Leftrightarrow \beta = \beta^0
\end{aligned} \tag{10}
$$

2.7 Theorem 3.2

β^0 is identified by moment condition

$$
E[(\mathbf{z}_i - E[\mathbf{z}_i|\mathbf{x}_{2i}, a_i])(y_i - E[y_i|\mathbf{x}_{2i}, a_i] - (\mathbf{w}_i - E[\mathbf{w}_i|\mathbf{x}_{2i}, a_i])'\beta)] = 0 \Leftrightarrow \beta = \beta^0 \tag{11}
$$

2.8 General Case

So far $\mathbf{x}_{1i} \cap \mathbf{x}_{2i} = \emptyset$, A more general case is when the regressor $\mathbf{x}_{1i}$ is consist of two component $\mathbf{x}_{1i} = (\mathbf{x}_{11i}, \mathbf{x}_{12i})$, where $\mathbf{x}_{11i}$ does not share any elements with $\mathbf{x}_{2i}$ and $\mathbf{x}_{12i} \subset \mathbf{x}_{2i}$

3 Estimation

3.1 with a_i as control function

$$
y_i - E[y_i|a_i] = (\mathbf{w}_i - E[\mathbf{w}_i|a_i])'\beta^0 + \upsilon_i - E[\upsilon_i|a_i] \tag{12}
$$

Let $h(a_i) = (h^y(a_i), \mathbf{h}^w(a_i), \mathbf{h}^z(a_i)) := (E[y_i|a_i], E[\mathbf{w}_i|a_i], E[\mathbf{z}_i|a_i])$ and $\tilde{\mathbf{W}}_N = (\mathbf{w}_1 - \mathbf{h}^w(a_1), \ldots, \mathbf{w}_N - \mathbf{h}^w(a_N))$, similarly define $\tilde{\mathbf{Z}}_N$, $\tilde{\mathbf{y}}_N$.

Suppose that we observe $\mathbf{h}(a_i)$ as $E[(\mathbf{z}_i - E[\mathbf{z}_i|a_i])(\upsilon_i - E[\upsilon_i|a_i])|a_i] = 0$,

$$\hat{\beta}_{2SLS}^{inf} = (\tilde{\mathbf{W}}_N'\tilde{\mathbf{Z}}_N(\tilde{\mathbf{Z}}_N'\tilde{\mathbf{Z}}_N)^{-1}\tilde{\mathbf{Z}}_N'\tilde{\mathbf{W}}_N)^{-1}\tilde{\mathbf{W}}_N'\tilde{\mathbf{Z}}_N(\tilde{\mathbf{Z}}_N'\tilde{\mathbf{Z}}_N)^{-1}\tilde{\mathbf{Z}}_N'\tilde{\mathbf{y}}_N \tag{13}$$

as a_i is not observed and the function $\mathbf{h}(\cdot)$ are not known. A natural implementation of the infeasible estimator $\hat{\beta}_{2SLS}^{inf}$ is to replace $\mathbf{h}(a_i)$ in $\tilde{\mathbf{W}}_N, \tilde{\mathbf{Z}}_N$ and $\tilde{\mathbf{y}}_N$ with its estimate, say $\hat{\mathbf{h}}(\hat{a}_i)$

(i) $h^l(a)$ is the l^{th} element in $\mathbf{h}(a)$ for $l = 1, \ldots, L$ where L is the dimension of $(y_i, \mathbf{w}_i', \mathbf{z}_i')'$

(ii) Sieve estimator $h^l(a) = \sum_{k=1}^{K_N} q_k(a)\alpha_k^l$

(iii) $\mathbf{q}^K(a) = (q^1(a), \ldots, q^{K_N}(a))$

(iv) $\mathbf{Q}_N := \mathbf{Q}_n(\mathbf{a}_N) = (\mathbf{q}^K(a_1), \ldots, \mathbf{q}^K(a_n))$

(v) $\mathbf{h}^l(\mathbf{a}_N) = (h^l(a_1), \ldots, h^l(a_N))$

(vi) $\boldsymbol{\alpha}_N^l = (\alpha_1^l, \ldots, \alpha_{K_N}^l)$

(vii) b_i^l be the l^{th} element in $(y_i, \mathbf{w}_i', \mathbf{z}_i')'$ and $\mathbf{b}_N^l = (b_1^l, \ldots, b_N^l)$

If $\mathbf{a}_N = (a_1, \ldots, a_N)'$ is observed, we can estimate the unknown function $\mathbf{h}^l((a_N))$ by OLS of b_i^l on $\mathbf{q}^K(a_i)$ for $l = 1, \ldots, L$

$$\hat{\mathbf{h}}^l(\mathbf{a}_N) = \mathbf{P}_{\mathbf{Q}_N}\mathbf{b}_N^l \tag{14}$$

where $\mathbf{P}_{\mathbf{Q}_N} = \mathbf{Q}_N(\mathbf{Q}_N'\mathbf{Q}_N)^{-}\mathbf{Q}_N'$

Suppose $\hat{\mathbf{a}}_N = (\hat{a}_1, \ldots, \hat{a}_N)'$ is the estimator of $\mathbf{a}_N = (a_1, \ldots, a_N)'$. Denote $\hat{\mathbf{Q}}_N := \mathbf{Q}_n(\hat{\mathbf{a}}_N) = (\mathbf{q}^K(\hat{a}_1), \ldots, \mathbf{q}^K(\hat{a}_N))$. Then the estimator of $\mathbf{h}^l((a_N))$ is defined by

$$\hat{\mathbf{h}}^l := \hat{\mathbf{h}}^l(\hat{\mathbf{a}}_N) = \mathbf{P}_{\hat{\mathbf{Q}}_N}\mathbf{b}_N^l \tag{15}$$

for $l = 1, \ldots, L$, and the estimator of β^0 is

$$\begin{aligned}\hat{\beta}_{2SLS} =& (\mathbf{W}_N'\mathbf{M}_{\hat{\mathbf{Q}}_N}\mathbf{Z}_N(\mathbf{Z}_N'\mathbf{M}_{\hat{\mathbf{Q}}_N}\mathbf{Z}_N)^{-1}\mathbf{Z}_N'\mathbf{M}_{\hat{\mathbf{Q}}_N}\mathbf{W}_N)^{-1} \\ &\times \mathbf{W}_N'\mathbf{M}_{\hat{\mathbf{Q}}_N}\mathbf{Z}_N(\mathbf{Z}_N'\mathbf{M}_{\hat{\mathbf{Q}}_N}\mathbf{Z}_N)^{-1}\mathbf{Z}_N'\mathbf{M}_{\hat{\mathbf{Q}}_N}\mathbf{y}_N \end{aligned}\tag{16}$$

where $\mathbf{M}_{\hat{\mathbf{Q}}_N} = I_N - \mathbf{P}_{\hat{\mathbf{Q}}_N}$

3.2 with $(\mathbf{x}_{2i}, a_i)$ as control function

Let $\hat{deg}_i$ be the degree of node i scaled by the Network size

$$\hat{deg}_i := \frac{1}{N-1} \sum_{j=1,\neq j}^{N} d_{ij,N} \tag{17}$$

Let $\phi * (\cdot)$ be the cdf of u_{ij}. Also let $\phi(\mathbf{x}_2, a)$ be the joint density function of $(\mathbf{x}_{2i}, a_i)$. Then for $(\mathbf{x}_{2i}, a_i)$, by the WLLN conditioning on $(\mathbf{x}_{2i}, a_i)$

$$\hat{deg}_i := \frac{1}{N-1} \sum_{j=1,\neq j}^{N} I(g(t(\mathbf{x}_{2i}, \mathbf{x}_{2j}), a_i, a_j)) \tag{18}$$