Fourier Series and the Discrete Fourier Transform: Quick Primer

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1. Introduction

The following material gives some of the mathematical background for two of the tools we use to determine the spectrum of a signal.

The Fourier Series (FS) and the Discrete Fourier Transform (DFT) should be thought of as playing similar roles for periodic signals in either continuous time (FS) or discrete time (DFT). Both *analyze* signals into amplitude, phases, and frequencies of complex exponentials; both *synthesize* signals by linearly combining complex exponentials with appropriate amplitude, phase, and frequency. Finally, both transforms have aspects that are extremely important to remember and other aspects that are important, but can be adjusted as necessary. As we work through some of the details, we'll identify these very important and the not so important aspects.

2. The Fourier Series

2.1. Synthesis: Building a periodic signal from a set of complex exponentials

Let $x(t)$ be a periodic continuous-time signal with period T_0. The Fourier series is a decomposition of such periodic signals into the sum of a (possibly infinite) number of complex exponentials whose frequencies are *harmonically* related. Specifically,

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j2\pi k f_0 t} \tag{1}$$

where $f_0 = 1/T_0$ is called the *fundamental* and the α_k are complex numbers.

We call Eq (1) a *synthesis* equation, because it tells us how to build, construct, or synthesize a periodic signal from other signals. In the present case, the building blocks are complex exponentials, $\{\exp(j2\pi k f_0 t)\}$, and the method of constructing $x(t)$ is to add *scaled* or *weighted* versions of each complex exponential according to the set of *expansion coefficients* $\{\alpha_k\}$. We also see that kf_0, the frequencies of the complex exponentials, are integer multiples of f_0, the fundamental.

According to the synthesis equation, we can distinguish between periodic signals in two ways. The first is by the period of the signal, T_0. Signals with different periods are built from different sets of complex exponentials, $\{\exp(j2\pi kf_0 t)\}$. But if two signals have the same period, how are we to differentiate between them? The synthesis equation suggests that two signals with the same period are different if their $\{\alpha_k\}$ sets differ. This, then, is the second way in which we can distinguish between two periodic signals.

2.1.1. But the textbook says...

You should compare Eq (1) with the synthesis equation found in section 3.4 of the *DSP First* textbook. At first blush, they look quite different. The text claims that all continuous-time signals with period T_0 can be expanded as

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi kf_0 t + \phi_k) \tag{2}$$

Examining Eq (2), we see that the set of building blocks are amplitude-scaled (A_k) phase-shifted (ϕ_k) cosines over the set of frequencies kf_0, where $k \geq 0$. However, Eq (1) states that the set of building blocks are complex amplitude-scaled (α_k) complex exponentials over the set of frequencies kf_0, where k is integer. Who's wrong?

It turns out that the apparent differences between Eq (1) and Eq (2) are one of those details of lesser importance in thinking about the Fourier Series. To see why, let's re-write Eq (2), the synthesis equation from the textbook, using Euler's identity for the cosine. Accordingly,

$$\begin{aligned} x(t) &= A_0 + \sum_{k=1}^{\infty} A_k\left(\frac{e^{j(2\pi kf_0 t + \phi_k)} + e^{-j(2\pi kf_0 t + \phi_k)}}{2}\right) \\ &= A_0 + \sum_{k=1}^{\infty} (A_k e^{j\phi_k}/2)e^{j2\pi kf_0 t} + (A_k e^{-j\phi_k}/2)e^{-j2\pi kf_0 t} \\ &= A_0 + \sum_{k=1}^{\infty} (A_k e^{j\phi_k}/2)e^{j2\pi kf_0 t} + \sum_{k=-\infty}^{-1} (A_{-k} e^{-j\phi_{-k}}/2)e^{j2\pi kf_0 t} \end{aligned} \tag{3}$$

Note that in arriving at the expression in the final line of Eq (3), we've inserted a negative on the indexing of the amplitude and phase terms, since Eq (2) refers to amplitudes over non-negative index values.

In comparing the final line of Eq (3) with Eq (1), we see that the two can be made identical if we let

$$\alpha_k = \begin{cases} \dfrac{A_k e^{j\phi_k}}{2} & k > 0 \\ \dfrac{A_{-k} e^{-j\phi_{-k}}}{2} & k < 0 \end{cases} \tag{4}$$

and recognize that the A_0 in Eq (3) is actually the amplitude term for a complex exponential with a frequency of 0 Hz, e.g., $k = 0$. Accordingly, if we let

$$\alpha_0 = A_0 \tag{5}$$

we have that

$$A_0 + \sum_{k=1}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j2\pi k f_0 t} \tag{6}$$

so that any periodic signal can be expanded using either equation, subject to the identities found in Eq (4) and Eq (5). *So the textbook and our original formulation above say exactly the same thing*.

2.1.2. But other textbooks say...

Some of you have already noticed that Eq (2) in the *DSP First* textbook also differs from expressions for the Fourier Series found in some linear algebra books. In the latter case, we might find a synthesis equation of the form

$$x(t) = A_0 + \sum_{k=1}^{\infty} (\tilde{A}_k \cos(2\pi k f_0 t) + \tilde{B}_k \sin(2\pi k f_0 t)) \tag{7}$$

where the $\tilde{A}_k$ and $\tilde{B}_k$ are real. We'll leave it as an exercise for you to establish the appropriate relationships between $\{\tilde{A}_k\}$ and $\{\tilde{B}_k\}$, the cos and sin expansion coefficients, and the $\{\alpha_k\}$.

2.1.3. So what's important?

Whether it is a sum of cosines with different amplitude and phase, the sum of cosines and sines with different amplitudes, or the sum of complex exponentials with different complex amplitudes, the fact that Eq (1), Eq (2), and Eq (7) all involve sums of sinusoidal, oscillatory functions is the take-home message about the Fourier Series as we use it in signal processing. We build signals from functions that exhibit simple harmonic motion. The different forms require some book keeping to keep track of the coefficients, but they are interchangeable. Knowing one set of coefficients allows us to find either of the other two.

2.2. Analysis: Decomposing a given signal into a set of complex exponentials

2.2.1. Existence in theory doesn't always mean useful in practice

Some results in mathematics concern existence. You may have learned in algebra, for example, that there were up to n distinct (complex) numbers that satisfied the expression

$$a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x^1 + a_n x^0 = 0 \tag{8}$$

These numbers are called the roots of the n-th order (or degree) polynomial and is called the *Fundamental Theorem of Linear Algebra*. You may also recall that the quadratic formula gives you a method for finding the roots of a 2nd-degree (quadratic) polynomial. Ever wonder if formulas exist for arbitrary n? It turns out that they don't. Mathematicians can tell us that the roots exist, but they can't specify a formula for calculating the roots exactly for $n \geq 5$.

The name "Fourier" would never be mentioned in systems engineering if all we had were Eq (1), Eq (2), or Eq (7). Like the Fundamental Theorem of Linear Algebra above, the fact that Fourier coefficients *exist* is interesting, in theory, but remains of little use to us without having a method for discovering their value. In contrast to roots of polynomials, which exist but have no closed-form solution, a formula exists for calculating the Fourier coefficients $\{\alpha_k\}$. Being able to *analyze* a signal into its constituent parts and then *synthesize* the signal from these parts is one of the primary reasons why the name "Fourier" appears throughout communications, controls, signal processing, circuits, electromagnetics, optics, biological systems, econometrics, etc.

2.2.2. The Analysis Equation

The Fourier coefficients for Eq (1), the first version of the synthesis equation, can be determined by evaluating the following integral.

$$\alpha_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j2\pi k f_o t}\, dt \tag{9}$$

Note that this integral requires knowledge of T_0, the period of the signal, and a closed-form representation for the signal itself.

2.2.3. Analysis equation from the textbook

In Section 2.1, we established a relationship between the α_k 's and the A_k 's in the textbook. If we substitute this relationship into Eq (9), do we obtain the textbook's analysis equation? This question is important, since we hope that the pairing of analysis and synthesis equations lead us to the same results. Accordingly, from Eq (5)

$$\alpha_0 = A_0 \tag{10}$$

so that upon substituting into Eq (9), we have that

$$A_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\, dt \tag{11}$$

Similarly, from Eq (4)

$$\alpha_k = \begin{cases} \dfrac{A_k e^{j\phi_k}}{2} & k > 0 \\ \dfrac{A_{-k} e^{-j\phi_{-k}}}{2} & k < 0 \end{cases} \tag{12}$$

so that upon substituting into Eq (9), we have that

$$X_k = \frac{2}{T_0}\int_0^{T_0} x(t)e^{-j2\pi k f_o t}dt \qquad k > 0 \tag{13}$$

where $X_k = A_k e^{j\phi_k}$. Therefore, the pairs of analysis/synthesis equations are consistent with each other. Neither pair gives us more or less information about the signal.

2.3. Doing the reality check: Are the analysis and synthesis equations consistent?

We can perform a reality check on Eq (1) by plugging it into Eq (9). If the two equations are consistent, then we should obtain an identity. Accordingly, in the sequence of steps below, we begin with Eq (9), insert the synthesis equation in place of $x(t)$, and carry out the expansion.

$$\begin{aligned} \alpha_k &= \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi k f_o t}dt \\ &= \frac{1}{T_0}\int_0^{T_0}\left(\sum_{l=-\infty}^{\infty}\alpha_l e^{j2\pi l f_0 t}\right)e^{-j2\pi k f_o t}dt \\ &= \frac{1}{T_0}\sum_{l=-\infty}^{\infty}\alpha_l\left(\int_0^{T_0} e^{j2\pi l f_0 t}e^{-j2\pi k f_o t}dt\right) \\ &= \frac{1}{T_0}\sum_{l=-\infty}^{\infty}\alpha_l\left(\int_0^{T_0} e^{j2\pi(l-k)f_0 t}dt\right) \end{aligned} \tag{14}$$

At this point, it looks like we have a mess. The left hand side is the single variable α_k , whereas the right hand side depends on an *infinite* collection of the α 's. Our best hope is to evaluate the integral on the right hand side and hope for some simplification. Recalling the general mathematical result

$$\begin{aligned} \int_0^{T_0} e^{at}dt &= \frac{1}{a}e^{at}\Big|_0^{T_0} \\ &= \frac{1}{a}(e^{aT_0} - 1) \end{aligned} \tag{15}$$

we can substitute $j2\pi(l-k)f_0$ for a to obtain

$$\begin{aligned} \int_0^{T_0} e^{j2\pi(l-k)f_0 t}dt &= \frac{1}{j2\pi(l-k)f_0}(e^{j2\pi(l-k)f_0 T_0} - 1) \\ &= \frac{1}{j2\pi(l-k)f_0}(e^{j2\pi(l-k)} - 1) \end{aligned} \tag{16}$$

When $l \neq k$, $l-k$ is an integer so that

$$
\begin{aligned}
e^{j2\pi(l-k)} &= \cos(2\pi(l-k)) + j\sin(2\pi(l-k)) \\
&= 1
\end{aligned}
\tag{17}
$$

Accordingly,

$$
\int_0^{T_0} e^{j2\pi(l-k)f_0 t}\,dt = 0 \qquad l \neq k \tag{18}
$$

When $l = k$, substitution into Eq (16) leads to a zero in the denominator - *not the best situation to find yourself in*! However, we wouldn't have found ourselves in this situation if we had noted in the original equation that

$$
\begin{aligned}
\int_0^{T_0} e^{j2\pi(l-k)f_0 t}\,dt &= \int_0^{T_0} 1\,dt \\
&= T_0
\end{aligned}
\tag{19}
$$

when $l = k$.

Using the results, we return to Eq (14) and break up the summation into two parts, $l \neq k$ and $l = k$ and arrive at *exactly* the result we had sought:

$$
\begin{aligned}
\frac{1}{T_0}\sum_{l=-\infty}^{\infty} \alpha_l \left(\int_0^{T_0} e^{j2\pi(l-k)f_0 t}\,dt \right) &= \frac{1}{T_0}\sum_{\substack{l=-\infty \\ l \neq k}}^{\infty} \alpha_l \left(\int_0^{T_0} e^{j2\pi(l-k)f_0 t}\,dt \right) + \frac{\alpha_k}{T_0}\int_0^{T_0} 1\,dt \\
&= 0 + \alpha_k
\end{aligned}
\tag{20}
$$

In conclusion, we see that the analysis and synthesis equations are consistent. The key trick to pushing through the result is Eq (17): this identity allowed us to achieve the important reduction in the expression of the right hand side from an *infinite* number of the α 's to the *single* number, α_k.

2.4. Additional observations

Note, our additional observations are organized as follows. Each subsection begins with a key result and a brief discussion of why the result is important. The "Mathematical Development" material that follows provides an argument for why the result holds. This material can be passed by on first reading. You should go back, however, and see how complex numbers, integration, and Fourier series are used to establish the results.

2.4.1. Conjugate symmetry

For real-valued signals, there's an important simplification in the Fourier coefficients. Specifically:

$$\alpha_{-k} = \alpha_k^* \tag{21}$$

This is a very useful result since it means we only have to compute the Fourier coefficients for $k \geq 0$, which is a 50% savings in work.

The conjugate symmetry also forces even and odd symmetry in the magnitude and phase spectra of the signal. To see this, let's write the Fourier coefficient in polar form

$$\alpha_k = |\alpha_k| e^{j\angle\alpha_k} \tag{22}$$

so that

$$\alpha_{-k} = (|\alpha_k| e^{j\angle\alpha_k})^* = |\alpha_k| e^{-j\angle\alpha_k} \tag{23}$$

Therefore, the magnitude spectrum has even symmetry about the origin, $k = 0$, whereas the phase spectrum has odd symmetry.

Mathematical development. We show Eq (21) as follows. Take the conjugate of α_k

$$\alpha_k^* = \left(\frac{1}{T_0}\int_0^{T_0} x(t) e^{-j2\pi k f_o t} dt\right)^* \tag{24}$$

and use the fact that the conjugate of a product is the product of the conjugates

$$\left(\frac{1}{T_0}\int_0^{T_0} x(t) e^{-j2\pi k f_o t} dt\right)^* = \left(\frac{1}{T_0}\right)^* \int_0^{T_0} (x(t))^* (e^{-j2\pi k f_o t})^* dt \tag{25}$$

The conjugate of a real number is the number itself while the conjugate of e^{-ja} is e^{ja}. Accordingly,

$$\left(\frac{1}{T_0}\right)^* \int_0^{T_0} (x(t))^* (e^{-j2\pi k f_o t})^* dt = \frac{1}{T_0}\int_0^{T_0} x(t) e^{j2\pi k f_o t} dt \tag{26}$$

Finally, to get the right hand side into the legitimate form of the analysis equation, we rewrite it as

$$\begin{aligned} \frac{1}{T_0}\int_0^{T_0} x(t) e^{j2\pi k f_o t} dt &= \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j2\pi(-k) f_o t} dt \\ &= \alpha_{-k} \end{aligned} \tag{27}$$

2.4.2. Why the scalars?

As you work further with Fourier concepts, be they the Fourier Series or related transforms such as the Fourier Transform, the Discrete Fourier Transform, or the Discrete Time Fourier Transform, you will find that textbooks differ with respect to the scalars that appear in the analysis and synthesis equations. In our development, for example, the analysis equation has the scalar $1/T_0$ appearing in it, e.g,.

$$\alpha_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi k f_o t}dt \tag{28}$$

whereas the synthesis equation is scaled by 1, e.g.,

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j2\pi k f_0 t} \tag{29}$$

Other textbooks may scale the analysis equation by 1 and the synthesis equation by T_0. Still other textbooks may scale both the analysis and synthesis equations by $1/\sqrt{T_0}$. Who's right?

The answer is that all three options are correct, as long as we consider the analysis and synthesis equations as *pairs*. When used as a pair, you won't result in any mistakes. However, mixing and matching the two equations can lead to problems. Specifically, synthesizing a signal from the wrong pairing of analysis/synthesis equations will result in a signal that either has too little or too much power.

Mathematical development.

One constraint on the scaling factors is that the analysis-synthesis equations must be consistent, as argued in Section 2.3. That is, if we consider the pair

$$\begin{aligned} \alpha_k &= G\int_0^{T_0} x(t)e^{-j2\pi k f_o t}dt \\ x(t) &= H\sum_{k=-\infty}^{\infty} \alpha_k e^{j2\pi k f_0 t} \end{aligned} \tag{30}$$

then, beginning with Eq (14) and finishing with Eq (20)

$$\begin{aligned} \alpha_k &= GH\sum_{l=-\infty}^{\infty} \alpha_l\left(\int_0^{T_0} e^{j2\pi(l-k)f_0 t}dt\right) \\ &= \alpha_k GH\int_0^{T_0} 1\,dt \end{aligned} \tag{31}$$

we see that the scalars G and H must be chosen so that

$$GH\int_0^{T_0} 1\,dt = 1$$
$$\therefore GH = \frac{1}{T_0} \tag{32}$$

Accordingly, we see that any of the pairs

$$(G, H) = \begin{cases} (1, 1/T_0) \\ (1/T_0, 1) \\ (1/\sqrt{T_0}, 1/\sqrt{T_0}) \end{cases} \tag{33}$$

will work, along with an infinite number of other alternatives. The fact that one of these three options is generally found in the literature, rather than the infinite number of alternatives, is due to the linkage between the concepts of Fourier transforms, algebra, and the intrinsic measure of *length* provided by the variable T_0.

2.4.3. Conservation of power

We've seen that the power of a periodic signal is an important measure. One of the interesting features of the Fourier series is that we can compute the power of the signal directly from the Fourier coefficients. Specifically:

$$\frac{1}{T_0}\int_0^{T_0} x^2(t)dt = \sum_{k=-\infty}^{\infty} |\alpha_k|^2 \tag{34}$$

This result is known as *Parseval's Theorem.* The significance of this result is that the power of the signal is directly dependent upon the magnitude-square of its Fourier coefficients. You can't increase one without simultaneously increasing the other. Similarly, if you decrease the gain of some of the frequencies in the signal while keeping the gain of the other frequencies in the signal the same, you will always decrease the signal's power.

Mathematical Development. Using our tools, we prove this relationship as follows. We first expand one of the $x(t)$ using the synthesis equation

$$\frac{1}{T_0}\int_0^{T_0} x^2(t)\,dt = \frac{1}{T_0}\int_0^{T_0} x(t)\left(\sum_{k=-\infty}^{\infty} \alpha_k e^{j2\pi k f_o t}\right)dt \tag{35}$$

We then interchange the summation and integration

$$\frac{1}{T_0}\int_0^{T_0} x(t)\left(\sum_{k=-\infty}^{\infty} \alpha_k e^{j2\pi k f_o t}\right)dt = \sum_{k=-\infty}^{\infty} \alpha_k\left(\frac{1}{T_0}\int_0^{T_0} x(t)e^{j2\pi k f_o t}\,dt\right) \tag{36}$$

and recognize, quite conveniently, that the integral on the right is the complex conjugate of the expression for α_k

$$\frac{1}{T_0}\int_0^{T_0} x(t)e^{j2\pi k f_o t}dt = \left(\frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi k f_o t}dt\right)^* \\ = \alpha_k^* \tag{37}$$

So that we have

$$\frac{1}{T_0}\int_0^{T_0} x^2(t)dt = \frac{1}{T_0}\int_0^{T_0} x(t)\left(\sum_{k=-\infty}^{\infty}\alpha_k e^{j2\pi k f_o t}\right)dt \\ = \sum_{k=-\infty}^{\infty}\alpha_k\alpha_k^* \tag{38}$$

our desired result.

2.4.4. Orthogonality

When α_k, the Fourier coefficient of a real-valued, periodic signal, is zero, the signal does not contain a component at the frequency $2\pi k f_0$. What if $\alpha_{25} = 5e^{j0.2}$? Does this tell you anything about the values of the other coefficients? The answer to this question is "it depends on the value of *k*". There are two cases to consider.

Case (i). Because the signal is real-valued, we know from Section 2.4.1, that α_{-k} is the complex conjugate of α_k. So, knowing that $\alpha_{25} = 5e^{j0.2}$, means that we also know the value of α_{-25}, that is, $\alpha_{-25} = 5e^{-j0.2}$.

Case (ii). Suppose $k \neq 25, -25$. Then by the synthesis equation, the signal $x(t)$

$$x(t) = 5e^{j0.2}e^{j2\pi 25 f_0 t} + \sum_{\substack{l=-\infty \\ l\neq 25}}^{\infty}\alpha_l e^{j2\pi l f_0 t} \tag{39}$$

but by the analysis equation, the coefficient α_k

$$\alpha_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi k f_o t}dt \\ = \frac{1}{T_0}\int_0^{T_0} 5e^{j0.2}e^{j2\pi 25 f_0 t}e^{-j2\pi k f_o t}dt + \frac{1}{T_0}\int_0^{T_0}\left(\sum_{\substack{l=-\infty \\ l\neq 25}}^{\infty}\alpha_l e^{j2\pi l f_0 t}\right)e^{-j2\pi k f_o t}dt \tag{40}$$

depends *only on the value of* α_k *, since, as we showed in Section 2.3,*

$$\int_0^{T_0} e^{j2\pi(l-k)f_0 t} dt = 0 \qquad l \neq k \tag{41}$$

This is an important enough point to break out as a separate definition. We say that two periodic signals, $s_1(t)$ and $s_2(t)$, with period T_0 are *orthogonal* if and only if

$$\int_0^{T_0} s_1(t) s_2^*(t) dt = 0 \tag{42}$$

In light of this definition, Eq (41) shows that the complex exponential functions $\{e^{j2\pi k f_0 t}\}$ are *orthogonal* over the period T_0. Besides the convenient 0's that result, the impact of orthogonality is that knowing how the periodic waveform is built at a particular complex exponential tells you nothing about how it is built at every other complex exponential function (up to the complex conjugate of the given complex exponential). Put in another way, you are free to choose the weights for each complex exponential when forming the signal, since the complex exponentials are orthogonal to each other.

Orthogonality turns out to be key in how we think about signals and in how we transmit information about signals. If you're used to thinking about the 90-degree angle between the *x*- and *y*- axis in a two-dimensional plot, you're already used to thinking about orthogonality. We say that the *two axes are orthogonal*, and note that knowing the x-coordinate of a point tells you nothing about its y-coordinate. Fourier series are just a generalization of this idea.

They are *efficient*, in that each coefficient carriers information about itself and no other coefficient. They are also *non-robust*, in that if you lose one of the coefficients, you can't recover it based on knowledge of the other coefficients. In communications engineering, we look trade-offs between efficient representations and robust representations when transmitting or storing information over noisy, lossy channels such as telephone lines or magnetic media.

2.4.5. What does the Fourier series say about non-periodic waveforms?

When a signal is non-periodic, what prevents us from using the Fourier series to represent it? Nothing, as long as we properly interpret what the representation says. Given any arbitrary signal, $x(t)$, we can take a chunk of that signal of length T_0 beginning with some starting time t_{start}

$$x_{CHUNK}(t) = x(t_{start} + t) \qquad 0 \leq t \leq T_0 \tag{43}$$

and find the Fourier coefficients of $x_{CHUNK}(t)$. The Fourier coefficients indicate that $x_{CHUNK}(t)$ can be synthesized by summing a possibly infinite set of sinusoids. Does this mean that $x(t)$ is so constructed? The answer is definitely "No". The signal that is constructed under Fourier synthesis is $x_{CHUNK-FOREVER}(t)$ which is defined for all time by the periodicity property

$$x_{CHUNK-FOREVER}(t + nT_0) = x_{CHUNK}(t) \qquad 0 \leq t \leq T_0, n \in \{\text{Integers}\} \tag{44}$$

So, unless $x_{CHUNK-FOREVER}(t)$ is the same as $x(t)$, in which case we had a periodic signal to begin with, the two signals are different and the Fourier coefficients say little about the frequency content of $x(t)$ over its entire length. However, the coefficients *do say something* about the frequency content of $x(t)$ over the chunk of time we've taken. This idea of looking at the frequency content of a signal around local chunks of time is fundamental to *time-frequency analysis*. The textbook introduces the spectrogram as a type of time-frequency analysis.

2.5. Terminology: Fourier Series and Inverse Fourier Series

Consistent with the textbook, we've talked about the Fourier series as consisting of two components: synthesis and analysis. The broader community uses a slightly different terminology in which the *Fourier analysis* of a periodic waveform is accomplished through the Fourier Series and the Inverse Fourier Series. In this case, the "Fourier Series" refers to what we have called the *analysis* equation and the "Inverse Fourier Series" refers to what we have called the *synthesis* equation.

3. The Discrete Fourier Transform

The Discrete Fourier Transform serves the same purposes for discrete-time signals as the Fourier Series does for continuous-time signals. It decomposes (analyzes) signals into complex exponentials in the discrete-time domain and it composes (synthesizes) discrete-time signals from linear combinations of such complex exponentials.

3.1. Circular Structure of Frequency in Discrete Time

Time marches on. We never get younger; we only get older. Yet, superimposed on our concept of linear time is "circular" time. Measured in hours of the day, time increases, starting with midnight, until 12 (or 24 hours) later, depending on how you keep time, when time becomes small again. We're accustomed to speaking of *linear* time in this circular fashion, and we never confuse the two ideas.

When we encounter frequency in the discrete-time domain, we also find a circular structure. Consider the two complex exponentials with frequencies θ_0 and $\theta_0 + 2\pi$:

$$e^{j\theta_0 n}, e^{j(\theta_0 + 2\pi)n} \tag{45}$$

Clearly, their frequencies differ by 2π, but

$$\begin{aligned} e^{j(\theta_0 + 2\pi)n} &= e^{j\theta_0 n} e^{j2\pi n} \\ &= e^{j\theta_0 n} \end{aligned} \tag{46}$$

since

$$\begin{aligned} e^{j2\pi n} &= \cos(2\pi n) + j\sin(2\pi n) \\ &= 1 + j \cdot 0 \end{aligned} \tag{47}$$

so that the two *signals* are identical. Like time on a clock, the frequency of complex exponentials in the discrete-time domain moves around a circle, from 0 to 2π. Once it reaches 2π, like time on a 24-hr. clock, increases in frequency merely change where the "hand" points on our 2π -radians clock.

In continuous time, we built cosines and sines from pairs of complex exponentials. We can do so as well in discrete time, e.g.,

$$\begin{aligned} \cos\theta n &= \frac{e^{j\theta n} + e^{-j\theta n}}{2} \\ \sin\theta n &= \frac{e^{j\theta n} - e^{-j\theta n}}{2j} \end{aligned} \tag{48}$$

only how are we to interpret *negative* frequency, when positive frequency in discrete time is circular? The answer is that we treat *negative* frequency in exactly the same way that we treat time on the clock. If we add hours, we move clockwise; if we subtract hours, we move counterclockwise. Accordingly, negative frequencies operate "in the opposite direction" to positive frequencies, ranging from 0 to -2π.

The consequence of building our cosines from complex exponentials in the discrete-time domain is that we don't have as many unique frequencies as we might have thought. The problem is that as we let the positive frequency go from 0 to 2π, the negative frequency goes from 0 to -2π. Where do we get into trouble? Let's return to the clock analogy. 2 AM is two hours added to midnight or 22 hours taken away from midnight. Thus, subtracting 22 hours is the same as adding two hours. In our 2π-radians clock, we have that

$$e^{-j\theta n} = e^{j(2\pi - \theta)n} \tag{49}$$

To prevent the *negative-frequency* complex exponential from becoming the *positive-frequency* one, and vice versa, we need the two complex exponentials to stay within a range of π radians, rather than 2π. Therefore, the range of *frequencies* for real-valued trigonometric functions, e.g., sines and cosines. is 0 to π in the discrete-time domain. In terms of our 24-hr. clock, the same restriction limit hours *before* and hours *after* midnight to 12, instead of 24. In this manner, hours before midnight are the PM prior to midnight, while hours after midnight are the AM following midnight.

The circularity of frequency in the discrete-time domain is a concept that plays out repeatedly in computer engineering. For example, the representation of numbers on a computer is achieved using a sequence of bits, that is, a *bit string*. Since the number of bits is finite, the collection of numbers represented by a bit string must have a smallest and a largest. Once we exceed the largest and attempt to become smaller than the smallest, we find ourselves back in the pack of our valid numbers. *Modular arithmetic* is the name we assign to how addition, subtraction, multiplication and division are performed over such finite sets of numbers. As fancy a name it might be, modular

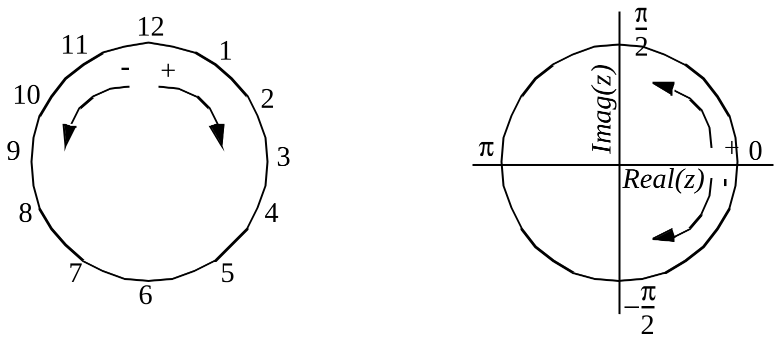

Figure 1. Circular time and circular frequency. Oriented with respect to the standard 12-hr. clock, positive increments in time are clockwise in motion, whereas negative increments in time are counter-clockwise. Oriented with respect to the real and imaginary axes, positive increments in the frequency of a complex exponential are *counter-clockwise*, whereas negative increments in frequency are *clockwise*.

arithmetic is simply the mathematics of time on the clock; if you're ever stuck, just translate your problem to the clock, and you should gain some insight into how your problem should be solved.

We draw this discussion to a close by noting the orientation of the two "clocks" shown in Figure 1. Convention has it (*Alice in Wonderland* notwithstanding) that time advances in a clockwise direction (rotation to the right). For the case of complex exponentials, we're free to choose whatever coordinate system we wish. A useful convention is to represent the frequency of the complex exponential by

$$z = e^{j\theta} \tag{50}$$

in which case, "12" rotates to the "3 o'clock" position to represent the frequency "0", since

$$e^{j0} = \cos(0) + j\sin(0) \tag{51}$$

and positive increments in frequency rotate to the left (counter-clockwise) whereas negative increments in frequency rotate to the right (clockwise). Examining the figure, we can see the problem when the rotation is greater than π, since the negative and positive frequencies encroach on each other's domain.

3.2. The Synthesis Equation

The brief discussion of the structure of frequency in discrete time helps us understand the synthesis equation for the Discrete Fourier Transform. Specifically, any discrete-time signal with period N_0 can be constructed according to

$$x(n) = \sum_{k=0}^{N_0-1} X(k)e^{j2\pi nk/N_0} \tag{52}$$

Without some knowledge about frequency in discrete time, Eq (52) looks to be wrong when compared with Eq (1) for continuous time. First, the bounds of summation differ. Contrary to Eq (52), continuous-time periodic signals are constructed from up to an infinite number of harmonically-related complex exponentials. Discrete-time periodic signals appear to require only a finite number of such components. Similarly, continuous-time periodic signals are constructed from positive- as well as negative-frequency complex exponentials. Discrete-time periodic signals appear to require only *non-negative* frequencies! Have things changed that much when going from continuous to discrete?

Based on Section 3.1, the answer is “No”. We see that the circular nature of frequency in the discrete-time domain means that we can have at most a finite number of harmonically-related frequencies. If we keep multiplying the fundamental frequency $2\pi/N_0$ by integers, we’re going to eventually exceed 2π and find ourselves completely around the circle, starting again at small values of frequency. In addition, negative frequencies do appear in Eq (52), again, because of the circular nature of frequency in the discrete-time domain. If we adopt the convention that negative frequencies occupy the region from 2π to π, whereas positive frequencies occupy the region from 0 to π, we can use the identity in Eq (49) to rewrite Eq (52) as

$$x(n) = \sum_{k=-\lfloor N_0/2 \rfloor}^{\lfloor N_0/2 \rfloor} X(k) e^{j2\pi nk/N_0} \tag{53}$$

where we need to throw in the floor function to make sure our bounds of summation (which must be integer after all) make sense. That is, in the present case

$$\lfloor N_0/2 \rfloor = \begin{cases} \frac{N_0}{2} & N_0 \text{ even} \\ (N_0 - 1)/2 & N_0 \text{ odd} \end{cases} \tag{54}$$

3.3. Analysis Equation

To build the periodic signal, $x(n)$, requires knowledge of N_0 and the Fourier coefficients $\{X(k)\}$. The analysis equation provides the means to determine the Fourier coefficients. Specifically,

$$X(k) = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x(n) e^{-j2\pi nk/N_0} \tag{55}$$

3.4. Putting the two together

As we did with the Fourier series, a good check of the pair of equations is to make sure they yield consistent results. Accordingly, we begin with the analysis equation, insert the synthesis equation, and then work towards an identity:

$$\begin{aligned} X(k) &= \frac{1}{N_0}\sum_{n=0}^{N_0-1} x(n)e^{-j2\pi nk/N_0} \\ &= \frac{1}{N_0}\sum_{n=0}^{N_0-1}\left(\sum_{m=0}^{N_0-1} X(m)e^{j2\pi nm/N_0}\right)e^{-j2\pi nk/N_0} \end{aligned} \tag{56}$$

Re-arranging the order of summation, we have

$$X(k) = \frac{1}{N_0}\sum_{m=0}^{N_0-1} X(m)\sum_{n=0}^{N_0-1} e^{j2\pi nm/N_0}e^{-j2\pi nk/N_0} \tag{57}$$

The right hand side will equal the left hand side if we can eliminate all of the $X(m)$ save one, the $X(k)$ we're interested in. Considering

$$\frac{1}{N_0}\sum_{n=0}^{N_0-1} e^{j2\pi nm/N_0}e^{-j2\pi nk/N_0} = \frac{1}{N_0}\sum_{n=0}^{N_0-1} e^{j2\pi n(m-k)/N_0} \tag{58}$$

we see that the right hand side can be written more compactly as

$$\frac{1}{N_0}\sum_{n=0}^{N_0-1}\alpha^n \qquad \text{where} \qquad \alpha = e^{j2\pi(m-k)/N_0} \tag{59}$$

for which the partial sum evaluates to

$$\frac{1}{N_0}\sum_{n=0}^{N_0-1}\alpha^n = \frac{1}{N_0}\left(\frac{1-\alpha^{N_0}}{1-\alpha}\right) \tag{60}$$

When $m \neq k$,

$$\begin{aligned} \alpha^{N_0} &= \left(e^{j2\pi(m-k)/N_0}\right)^{N_0} \\ &= e^{j2\pi(m-k)} \\ &= 1 \end{aligned} \tag{61}$$

since $m-k$ is integer. The result is that

$$\frac{1}{N_0}\sum_{n=0}^{N_0-1}\alpha^n = 0 \tag{62}$$

Proceeding, for the case $m = k$, $\alpha = 1$, so we skip the partial sum formula and evaluate directly to obtain

$$\frac{1}{N_0}\sum_{n=0}^{N_0-1} 1 = 1 \tag{63}$$

Like the development of the Fourier series, we see from the above arguments that

$$\begin{aligned} X(k) &= \frac{1}{N_0}\sum_{n=0}^{N_0-1}\left(\sum_{m=0}^{N_0-1} X(m)e^{j2\pi nm/N_0}\right)e^{-j2\pi nk/N_0} \\ &= X(k) \end{aligned} \tag{64}$$

which is exactly the result we desired. The two equations do not lead to inconsistent results.

3.5. Additional remarks

A number of the properties of the Fourier Series have parallels in the Discrete Fourier Transform. A good exercise is to follow the outline in Section 2, substitute the DFT, and determine which of the results generalize. In particular, note how the scalar generalizes. As you might discover in the textbook, the placement of $1/N_0$ differs with what we presented here. Again, *it is a matter of convention*, you should get used to switching between such conventions, but you should always make sure you are using the appropriate pair of analysis and synthesis equations. If you don't, you're signals with either start blowing up or will fade away to zero! Finally, many textbooks will refer to the analysis equation as the Discrete Fourier Transform of the signal. In this case, the synthesis equation is the Inverse Discrete Fourier Transform.

3.6. What about the Fast Fourier Transform (FFT)?

Often, one reads about the FFT instead of the DFT. The FFT has a rich history in its own right, but for our purposes, we can summarize the FFT as simply a DFT in which the period N_0 is restricted to a certain set of values, such as powers of 2. The reason it is called "Fast" reflects a very interesting way of efficiently evaluating the Fourier coefficients. In the case of the DFT, N_0 multiplies are needed for each of the N_0 Fourier coefficients, so that we say the "order" of the algorithm's complexity is N_0^2. The FFT cleverly takes advantage of what is called the butterfly structure of the DFT when N_0 is a power of two. This results in a smaller number of multiplies proportional to $N_0 \log_2 N_0$.