Golden Rule

$\hat{H} = \hat{H}_0 + \Delta\hat{H}$ con $\Delta\hat{H} << \hat{H}_0$

$\hat{H}_0 |m\rangle = E_m |m\rangle$; $\langle n|m\rangle = \delta_{nm}$

$|\psi(t)\rangle = \sum_m C_m(t)\,|m\rangle$ DOVE, se $\Delta H = \phi \Rightarrow C_m(t) = e^{-\frac{i}{\hbar}E_m t}\, C_m(\phi)$

ALTRIMENTI posso scrivere $C_m(t) = e^{-\frac{i}{\hbar}E_m t}\, a_m(t)$ con $a_m(\phi) = C_m(\phi)$

NOTA: NON POSSO SCRIVERE $a_m(t)C_m(\phi)$ con $a_m(\phi) = 1$, perché SE $C_m(\phi) = \phi \Rightarrow C_m(t) = \phi\ \forall t$ che è FALSO siccome $H \neq H_0$!!!

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = (\hat{H}_0 + \Delta\hat{H})\,|\psi(t)\rangle$$

$$\hookrightarrow i\hbar \sum_n \frac{\partial}{\partial t}(C_n(t))\,|n\rangle = \sum_n C_n \hat{H}_0 |n\rangle + \sum_n C_n \Delta\hat{H}\,|n\rangle$$

$$i\hbar \sum_m \left(-\frac{i}{\hbar}E_m a_m(t) + \frac{\partial a_m(t)}{\partial t}\right) e^{-\frac{i}{\hbar}E_m t}|m\rangle = \sum_m C_m E_m |m\rangle + \sum_m C_m \Delta\hat{H}\,|m\rangle$$

$\langle J|$:

$$i\hbar \left(-\frac{i}{\hbar}E_J a_J(t) + \frac{\partial a_J(t)}{\partial t}\right) e^{-\frac{i}{\hbar}E_J t} = C_J(t) E_J + \sum_m C_m(t) \langle J|\Delta\hat{H}|m\rangle$$

$$i\hbar \left(\cancel{-\frac{i}{\hbar}E_J a_J(t)} + \frac{\partial a_J(t)}{\partial t}\right) e^{-\frac{i}{\hbar}E_J t} = \cancel{e^{-\frac{i}{\hbar}E_J t} a_J(t) E_J} + \sum_m e^{-\frac{i}{\hbar}E_m t} a_m(t) \langle J|\Delta\hat{H}|m\rangle$$

$$\frac{\partial a_J(t)}{\partial t} = \frac{1}{i\hbar}\sum_m e^{-\frac{i}{\hbar}(E_m - E_J)t} a_m(t) \langle J|\Delta\hat{H}|m\rangle$$

$$\dot{a}_J = -\frac{i}{\hbar}\sum_m e^{+i\omega_{Jm}t} a_m(t) \langle J|\Delta\hat{H}|m\rangle \quad ; \quad \omega_{Jm} = \frac{E_J - E_m}{\hbar}$$

NO APPROX UNTIL HERE!!!

• Per trovare $a_J(t)$ $\forall J$ ho (sistema di) eqs (1 eq. $\forall J$)

DEVO FARE UNA SEMPLIFICAZIONE

UNICA APPROX: IDEA: SE $\Delta H << H_0 \Rightarrow \langle\psi(t)|\hat{O}|\psi(t)\rangle$ DEVE ESSERE CIRCA UGUALE AL VALORE CHE AVREBBE SE $\Delta H = \phi$, PER t PICCOLO $\forall$ OSSERVABILE $\hat{O}$

Scelgo $\hat{O} = \hat{H}_0$

• Se $\Delta H \neq 0$: $\langle\psi(t)|\hat{H}_0|\psi(t)\rangle = \sum_{n,m} e^{+\frac{i}{\hbar}E_n t} a_n^*(t) \underbrace{\langle n|\hat{H}_0|m\rangle}_{E_m \delta_{nm}} a_m(t) e^{-\frac{i}{\hbar}E_m t} =$

$$= \sum_m |a_m(t)|^2 E_m$$

• Se $\Delta H = \phi \Rightarrow a_m(t) = C_m(\phi)\ \forall m \Rightarrow \langle\psi(t)|\hat{H}_0|\psi(t)\rangle = \sum_m E_m |C_m(\phi)|^2$

Visto che, per t piccolo, chiedo che siano $\simeq$ $\Rightarrow |a_m(t)| \simeq |C_m(t)|\ \forall m$ e t piccolo

Se non ti piace, puoi prendere $\hat{O} = |l\rangle\langle l|$ così non hai $\sum_m$ e trovi $|a_l(t)| \simeq |C_l(t)|$

FACCIO QUESTA APPROX SOLO AL MEMBRO DESTRA

Uso $a_m(t) \simeq \phi$ $\forall m$ e t piccolo al membro dx

ed integro da $\phi \to t$ con t piccolo:

$$a_J(t) - a_J(\phi) \simeq -\frac{i}{\hbar}\sum_n a_n(\phi)\int_\phi^t d\tau \left(e^{+i\omega_{Jn}\tau}\langle J|\Delta H|n\rangle\right)$$

• Se preparo $|\psi(\phi)\rangle = |\ell\rangle$ (in un autostato di H_0)

$$\hookrightarrow \begin{cases} c_\ell(\phi) = 1 \\ c_n(\phi) = \phi \quad \forall n \neq \ell \end{cases}$$

Siccome $a_n(\phi) = c_n(\phi)$ $\forall n$

$$\Rightarrow a_J(t) - c_J(\phi) \simeq -\frac{i}{\hbar}\sum_n c_n(\phi)\int_0^t d\tau\, e^{i\omega_{Jn}\tau}\langle J|\Delta H|n\rangle$$

ed usando che $|\psi(\phi)\rangle = |\ell\rangle$ | $\boxed{\text{SE } J \neq \ell}$

$$\hookrightarrow a_J(t) - \phi \simeq -\frac{i}{\hbar}\sum_n \delta_{n,\ell}\,(\cdots)$$

$$\Rightarrow \boxed{a_J(t) \simeq -\frac{i}{\hbar}\int_\phi^t d\tau\, e^{i\omega_{J\ell}\tau}\langle J|\Delta H|\ell\rangle}$$

si può fare così

Nota: $(\cdots)$ è sempre $(<\infty)$ a meno che $\langle J|\Delta H|n\rangle \sim \frac{1}{\tau^p}$ con $p>1$

xché $\int_0^t d\tau\, e^{i\omega\tau} < t$ $\forall\omega$

(x $\omega = \phi$ ho apparente ϕ/ϕ ma l'integrale fa t)

Se ΔH non dipende esplicitamente dal tempo

$$\Rightarrow a_J(t) \simeq -\frac{i}{\hbar}\langle J|\Delta H|\ell\rangle\int_0^t e^{i\omega_{J\ell}\tau}d\tau = -\frac{i}{\hbar}\langle J|\Delta H|\ell\rangle\frac{\left(e^{i\omega_{J\ell}t} - 1\right)}{i\omega_{J\ell}}$$

se scrivo $e^{i\omega_{J\ell}t} - 1 = e^{i\frac{\omega_{J\ell}t}{2}}\left(e^{i\frac{\omega_{J\ell}t}{2}} - e^{-i\frac{\omega_{J\ell}t}{2}}\right)$

$$\Rightarrow a_J(t) \simeq \frac{-1}{\hbar}\langle J|\Delta H|\ell\rangle\, 2\,\frac{\sin(\omega_{J\ell}t/2)}{\omega_{J\ell}}\, e^{i\frac{\omega_{J\ell}t}{2}}$$

$$\Rightarrow |a_J(t)|^2 \simeq \frac{4}{\hbar^2}\langle J|\Delta H|\ell\rangle\left(\frac{\sin\left(\frac{\omega_{J\ell}t}{2}\right)}{\omega_{J\ell}}\right)^2 \xrightarrow[t\to\phi]{} \frac{2\pi}{\hbar}\langle J|\Delta H|\ell\rangle\,\delta(\omega_{J\ell})$$

Try to plot this as a function of w, you'll see something picked in w=0.
The width rises with t (parameter in the plot)