TD 3

Ecole Normale Supérieure de Lyon — Université Claude Bernard Lyon 1

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Physique Nonlinéaire et Instabilités
OPL
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I. COLLAPSE OF A 1D DOMAIN IN THE CAHN-ALLEN EQUATION

1. Consider a 1D field $u(x,t)$ obeying the Cahn-Allen equation

$$\partial_t u = u + \partial_{xx} u - u^3.$$

Check that profiles of the form $u(x,t) = +u_k(x)$ and $u(x,t) = -u_k(x)$, where

$$u_k(x) = \tanh \frac{x}{2^{1/2}}$$

are steady solutions of the equation.

Since $\tanh' x = 1 - \tanh^2 x$, we have

$$u_k''(x) = -\tanh \frac{x}{2^{1/2}}(1 - \tanh \frac{x}{2^{1/2}})$$

as a consequence

$$u_k''(x) + u_k(x)(1 - u_k(x)^2) = 0$$

Hence $u_k(x)$ is a steady-state.

2. We wish to analyse the evolution of the size of an isolated domain, bounded by a kink and an antikink. We use an approximate profile based on the following double-kink ansatz for $x_0(t) \gg 1$

$$u(x,t) = u_k(x + x_0(t)) - u_k(x - x_0(t)) - 1 = u_+ - u_- - 1.$$

where

$$u_- = u_k(x - x_0(t)) = \tanh[2^{-1/2}(x - x_0(t))]$$
$$u_+ = u_k(x + x_0(t)) = \tanh[2^{-1/2}(x + x_0(t))].$$

Show that

$$\dot{x}_0(t)(1 - u_-^2 + 1 - u_+^2) = 3 \times 2^{1/2}(u_- - u_+)(1 - u_+)(1 + u_-), \tag{1}$$

where $\dot{x}_0$ denotes the derivative of x_0 with respect to time.

This is obtained by substitution of the double-kink ansatz into the Cahn-Allen equation.

$$\partial_t u(x,t) = \dot{x}_0(t)(u_k'(x + x_0(t)) + u_k'(x - x_0(t))) = \dot{x}_0(t)(1 - u_-^2 + 1 - u_+^2)$$
$$\partial_{xx} u(x,t) = u_-(1 - u_-^2) - u_+(1 - u_+^2) = (u_+ - u_-)(-1 + u_+^2 + u_-^2 + u_+ u_-)$$
$$u - u^3 = u(1 - u)(1 + u) = (-u_- + u_+ - 1)(2 + u_- - u_+)(-u_- + u_+)$$

Substituting these relations in the Cahn-Allen equation leads to Eq.(1).

3. Evaluate Eq.(1) at $x = +x_0(t)$ and obtain the following asymptotic evolution equation for $x_0(t) \gg 1$

$$\dot{x}_0(t) \approx -3 \times 2^{3/2} \mathrm{e}^{-2^{3/2} x_0(t)}.$$

Choosing $x = +x_0(t)$, we have

$$u_+ = \tanh[2^{1/2}x_0(t)]$$
$$u_- = 0$$

Hence, Eq.(1) leads to

$$\dot{x}_0(t)(2 - u_+^2) = -3 \times 2^{1/2}u_+(1 - u_+),$$

Then, from $x_0(t) \gg 1$ we have

$$u_+ \approx 1 - 2\mathrm{e}^{-2^{3/2}x_0(t)}$$

leading to

$$\dot{x}_0(t) \approx -3 \times 2^{3/2}\mathrm{e}^{-2^{3/2}x_0(t)},$$

4. Solve this equation and show that the collapse time T_c of the domain is

$$T_c \approx \frac{1}{24}\mathrm{e}^{2^{3/2}x_0(0)}, \tag{2}$$

where $2x_0(0)$ is the initial distance between the kink and the antikink.

The solution of the differential equation for $x_0(t)$ is

$$x_0(t) = 2^{-3/2}\ln\left[\mathrm{e}^{2^{3/2}x_0(0)} - 24t\right]$$

From the condition $x_0(T_c) = 0$, we find

$$1 = \mathrm{e}^{2^{3/2}x_0(0)} - 24T_c$$

Since $x_0(0) \gg 1$, the left hand side is negligible and we find Eq.(2).

II. GROWTH OR COLLAPSE OF A 2D CIRCULAR DOMAIN

1. Consider a circular domain of radius $r(t)$ in 2D. The edge of the domain obeys the Eikonal equation, i.e. the normal velocity c towards the exterior of the circular domain reads

$$c = c_* - D\kappa$$

where $D > 0$, κ is the (positive) local curvature, and c_* is a positive or negative constant. Write the equation obeyed by $r(t)$.

The radius $r(t)$ obeys

$$\dot{r} = c_* - \frac{D}{r}$$

2. Find the conditions under which (i) the domain radius r is growing indefinitely with time; (ii) the radius r is constant and equal to the critical radius r_*; and (iii) the radius r is decreasing.

Is r_* a stable or an unstable fixed point?

From the sign of the right hand side of the evolution equation for $r(t)$, we see that the critical radius is $r_* = D/c_*$: (i) r grows with time when $c_* > 0$ and $r > r_*$; (ii) r reaches a fixed point at $r = r_*$ when $c_* > 0$; (iii) r decreases when $c_* < 0$ or when $c_* > 0$ and $r < r_*$.

When $c_* > 0$, the right hand side of the evolution equation for $r(t)$ is positive for $r > r_*$ and negative for $r < r_*$. As a consequence, this is an unstable fixed point.

3. Consider the case where $c_* > 0$ and $r(0) < r_*$. Solve the equation for $r(t)$ in an implicit form, i.e., find t as a function of r. Calculate the collapse time t_c where r reaches 0.

   The solution reads

$$\frac{c_*^2 t}{D} = \ln\left[\frac{1 - r(t)/r_*}{1 - r(0)/r_*}\right] + \frac{r(t) - r(0)}{r_*}$$

   The collapse time t_c is obtained from the condition $r(t_c) = 0$. This leads to

$$\frac{c_*^2 t_c}{D} = \ln\left[\frac{1}{1 - r(0)/r_*}\right] - \frac{r(0)}{r_*}$$

4. Evaluate t_c the limit $c_* \to 0$ (use the relation $\ln[1/(1-x)] - x \approx x^2/2$ for $x \ll 1$). Compare this result with Eq.(2).

   In the limit $c_* \to 0$, from the series expansion $\ln[1/(1-x)] - x \approx x^2/2$ for small x, we obtain

$$t_c = \frac{r(0)^2}{2D}$$

   (this latter relation can also be be derived directly from the integration of the evolution of $r(t)$ when $c_* = 0$, which reads $\dot{r} = -D/r$.)

   The collapse of a 2D circular domain occurs in a time that is $\sim r(0)^2$, while the collapse time grows exponentially with the domain size in 1D. As a consequence, collapse is always slower in 1D for large domains.