Lecture 3: Nonlinear Dynamics and Instabilities

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Lecture 3

Linear Stability

Type III-s instability of flat state $u_0 = 0$

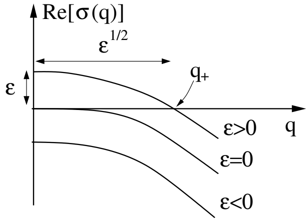
$$\mathrm{Re}[\sigma(q)] = L_0 - L_2 q^2$$

- Assume $x \to -x$ symmetry
- Only one slow lengthscale, no fast lengthscale
- Scalings $L_0 \sim \epsilon$

$$\mathrm{Re}[\sigma(q)] \sim \epsilon, \qquad q \sim \epsilon^{1/2}$$

- Slow time-scale $t \sim \epsilon^{-1}$
  large spatial "pattern" scale $\sim \epsilon^{-1/2}$
- Multi-scale analysis

$$t = \epsilon^{-1} T$$
$$x = \epsilon^{-1/2} X$$



Power-counting

$$t = \epsilon^{-1} T, \quad x = \epsilon^{-1/2} X, \quad u = \epsilon^{\alpha} U$$

Linear

$$\mathrm{Re}[\sigma(q)] = L_0 - L_2 q^2$$

General weakly-nonlinear expansion:

$$\partial_t u = \partial_{xx} u + \epsilon u + \epsilon^{\gamma} [\partial_x]^n [\partial_t]^{\ell} [u]^m$$

Examples

- $n = 0,\ m = 3,\ \ell = 0,\ \gamma = 0$: u^3
- $n = 2,\ m = 2,\ \ell = 0,\ \gamma = 0$: $(\partial_x u)^2,\ u\partial_{xx} u$
- $n = 1,\ m = 3,\ \ell = 1,\ \gamma = 1$: $\epsilon u \partial_x u \partial_t u,\ \epsilon u^2 \partial_{tx} u$

Constraints

- non-linear: $m > 1$
- non-singular: $\gamma, n, \ell, m \geq 0$
- $x \to -x$ symmetry: n even

Power-counting

$$\partial_t u = \partial_{xx} u + \epsilon u + \epsilon^\gamma [\partial_x]^n [\partial_t]^\ell [u]^m$$
$$\epsilon^{1+\alpha} \partial_T U = \epsilon^{1+\alpha} (\partial_{XX} U + U) + \epsilon^{\gamma + n/2 + \ell + m\alpha} [\partial_X]^n [\partial_T]^\ell [U]^m$$

$$\alpha = \frac{1 - n/2 - \gamma - \ell}{m - 1}$$

Smallest $u \sim \epsilon^\alpha$ for which nonlinearities matter
$\rightarrow$ largest possible value of α
First nonlinear term $\alpha = 1$, $m = 2$, $n = \ell = \gamma = 0$: u^2
Fisher Equation (also known as Fisher-Kolmogorov equation)

$$\partial_t u = \partial_{xx} u + \epsilon u + u^2$$

- associated to transcritical bifurcation
- stable solution $u = -\epsilon$, but divergences in finite time for $u \to +\infty$!
- u^2 forbidden in systems with field inversion $u \to -u$ symmetry

Power-counting

$$\partial_t u = \partial_{xx} u + \epsilon u + \epsilon^\gamma [\partial_x]^n [\partial_t]^\ell [u]^m$$

with $m > 1$

$$\alpha = \frac{1 - n/2 - \gamma - \ell}{m - 1}$$

Next nonlinear term $\alpha = 1/2$, $m = 3$, $n = \ell = \gamma = 0$: u^3
Cahn-Allen equation (Time-Dependent-Ginzburg-Landau equation, ϕ^4 model, model A)

$$\partial_t u = \partial_{xx} u + \epsilon u - u^3$$

adding term $\gamma = 1/2$ $m = 2$ breaks the $u \to -u$ symmetry

$$\partial_t u = \partial_{xx} u + \epsilon u + \epsilon^{1/2} u^2 - u^3$$

generic equation for Type III-s

normalization

Non-normalized equation (in physical units)

$$\partial_t u = D\partial_{xx} u + L_0 u + C_2 u^2 - C_3 u^3$$

with $D > 0$ and $C_3 > 0$
and with $L_0 > 0$ in the unstable regime

$$t = \frac{T}{L_0}$$

$$x = \left(\frac{D}{L_0}\right)^{1/2} X$$

$$u = \left(\frac{L_0}{C_3}\right)^{1/2} U$$

$$\gamma = \frac{C_2}{(C_3 L_0)^{1/2}}$$

One-parameter equation

$$\partial_T U = \partial_{XX} U + U + \gamma U^2 - U^3$$

Gradient dynamics

Generalized equation

$$\partial_t U = D\partial_{XX} U + g(U)$$

Double-well potential

$$g(U) = -V'(U)$$

We have

$$\partial_t U = -\frac{\delta \mathcal{F}}{\delta U}$$

Gradient dynamics / Lyapunov functional

$$\mathcal{F} = \int \mathrm{d}X \left(\frac{1}{2}(\partial_X U)^2 + V(U) \right) \geq 0$$

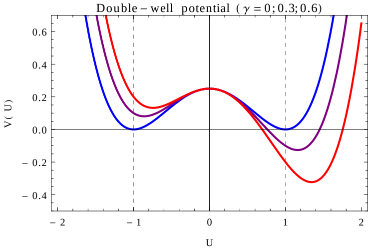
Then

$$\frac{\mathrm{d}}{\mathrm{d}T}\mathcal{F} = \int \mathrm{d}X\ \partial_T U \left(-\partial_{XX} U + V'(U) \right)$$

$$= -\int \mathrm{d}X\ (\partial_T U)^2 \leq 0$$

Special case: cubic $g(U) = U + \gamma U^2 - U^3$

$$V(U) = \frac{1}{4}(1-U^2)^2 - \frac{\gamma}{3} U^3$$



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Heteroclinic solution

Two heteroclinic solutions with velocity $\pm C$
Solution with velocity C

$$\partial_T U = -C\partial_X U \qquad \rightarrow -C\partial_X U = \partial_{XX} U + g(U)$$

Analogy with mechanics
inverted potential V, and friction C

$$g(U) = -V'(U)$$

Moving front $\rightarrow$ see lecture Alain Pumir

Cahn-Allen
Special case: cubic $g(U) = U + \gamma U^2 - U^3$
Quartic potential

$$V(U) = \frac{1}{4}(1 - U^2)^2 - \frac{\gamma}{3}U^3$$

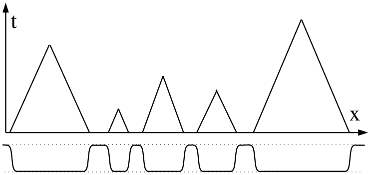
Solution

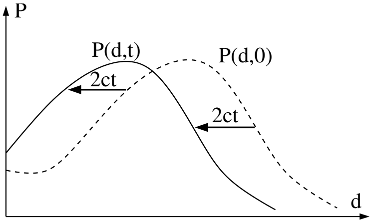
$$C = -2^{-1/2}\gamma$$

Heteroclinic solution

velocity v proportional to asymmetry γ

- Stable state invading meta-stable state
- Two types: "kinks" and anti-kinks with $C = \pm 2^{-1/2}\gamma$
- Kinks annihilate when they collide
- Advected kink-antikink distribution of distances in metastable phase
  $P(d, T) = P(d + 2CT, 0)$





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Kinks

$\gamma = 0$: $U \to -U$ symmetry / coexistence
Oscillatory solutions in X
(non-dissipative when $\gamma = 0$)
$V_0(U_+) = V_0(U_-)$
Max amplitude U_m

$$E = \frac{1}{2}(\partial_X U)^2 - V_0(U) = -V_0(U_m)$$
$$\partial_X U = 2^{1/2}[V_0(U) - V_0(U_m)]^{1/2}$$

Kink solution $U_m = U_+$, and $V(U_+) = 0$
Kink profile, kink at $X = 0$

$$X = \int_0^X \mathrm{d}X' = \int_0^U \frac{\mathrm{d}U'}{\partial_X U}$$
$$= 2^{-1/2} \int_0^U \frac{\mathrm{d}U'}{V(U')^{1/2}}$$

Special case $V(U) = (1 - U^2)^2/4$

$$X = 2^{1/2} \int_0^U \frac{\mathrm{d}U'}{1 - U'^2} = 2^{-1/2}\mathrm{arctanh}(U)$$

Tanh profile kink

$$U(X) = \tanh \frac{X - X_k}{2^{1/2}}$$

...and antikink

$$U(X) = -\tanh \frac{X - X_a}{2^{1/2}}$$

Kink-antikink attraction and annihilation

Kink energy

$$
\begin{aligned}
\mathcal{F}_k &= \int \mathrm{d}X \left(\frac{1}{2} (\partial_X U)^2 + V(U) \right) \\
&= \int \mathrm{d}X \left(\frac{1}{2} (\partial_X U)^2 + \frac{1}{2} (\partial_X U)^2 \right) \\
&= \int \mathrm{d}X \partial_X U (\partial_X U) \\
&= 2^{1/2} \int \mathrm{d}U (V(U))^{1/2}
\end{aligned}
$$

For $V(U) = (1 - U^2)^2/4$,
one finds $\mathcal{F}_k = 2^{3/2}/3$

Exponentially decreasing tail $X \gg X_k$

$$U(X) \approx 1 - 2\mathrm{e}^{-2^{1/2}(X - X_k)}$$

leads to exponentially small interactions
Collapse time T_c of a domain of size L

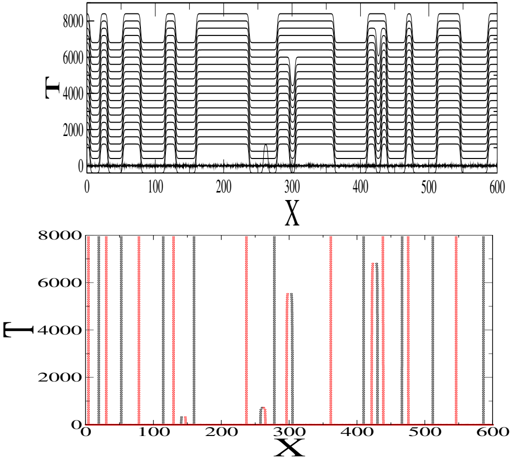
$$T_c = \frac{1}{24} \mathrm{e}^{2^{1/2} L}$$

exponentially slow!
(see TD)

Coarsening

Dynamics from random initial conditions

- Extremal dynamics: shortest domain collapses
- Coarsening $\lambda \sim \ln(T)$



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2D generic equation

Isotropic dynamics $\partial_{XX} \to \Delta = \partial_{XX} + \partial_{YY}$
Generic equation for III-s instability

$$\partial_T U = \Delta U + U + \gamma U^2 - U^3$$

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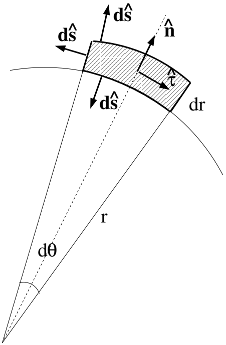
Differential geometry in the front vicinity

Extend the definition of $\hat{\mathbf{n}}, \hat{\tau}$ in the vicinity of a curve transported along $\hat{\mathbf{n}}$ radius of curvature r



$$\int \mathrm{d}A\, \nabla\cdot\hat{\mathbf{n}} = \oint \mathrm{d}\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}$$
$$= \mathrm{d}\theta(r+\mathrm{d}r) - \mathrm{d}\theta\, r = \mathrm{d}\theta\,\mathrm{d}r$$

and

$$\int \mathrm{d}A\, \nabla\cdot\hat{\mathbf{n}} = (\nabla\cdot\hat{\mathbf{n}})[\mathrm{d}\theta(r+\mathrm{d}r)^2/2 - \mathrm{d}\theta\, r^2/2]$$
$$= (\nabla\cdot\hat{\mathbf{n}})\mathrm{d}\theta\, r\mathrm{d}r$$

$\rightarrow \nabla\cdot\hat{\mathbf{n}} = 1/r$

$$\int \mathrm{d}A\, \nabla\cdot\hat{\tau} = \oint \mathrm{d}\hat{\mathbf{s}}\cdot\hat{\tau} = 0$$

and

$$\int \mathrm{d}A\, \nabla\cdot\hat{\tau} = (\nabla\cdot\hat{\tau})\mathrm{d}\theta\, r\mathrm{d}r$$

$\rightarrow \nabla\cdot\hat{\tau} = 0$

Differential geometry in the front vicinity

Aligned curvilinear coordinates (s,ξ)
ξ distance to the front along $\hat{\mathbf{n}}$
s arclength along the front, ℓ along the local $\hat{\tau}$
Radius of curvature at (ξ,s)

$$r=\frac{1}{\kappa}+\xi$$

interface arclength

$$\mathrm{d}s=\frac{1}{\kappa}\mathrm{d}\theta$$

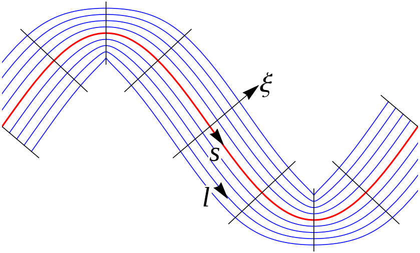
curvilinear arclength

$$\mathrm{d}\ell=rd\theta=(\frac{1}{\kappa}+\xi)\mathrm{d}\theta=(1+\xi\kappa)\mathrm{d}s$$

Summary

$$\nabla\cdot\hat{\mathbf{n}}=\frac{1}{\kappa^{-1}+\xi},\qquad \hat{\mathbf{n}}\cdot\nabla=\partial_\xi$$

$$\nabla\cdot\hat{\tau}=0,\qquad \hat{\tau}\cdot\nabla=\partial_\ell=\frac{1}{1+\xi\kappa}\partial_s$$



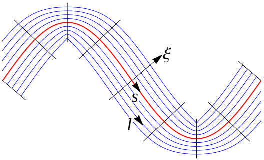
Laplacian

$$\begin{aligned}\Delta&=\nabla\cdot\nabla\\&=\nabla\cdot[\hat{\mathbf{n}}\,(\hat{\mathbf{n}}\cdot\nabla)+\hat{\tau}\,(\hat{\tau}\cdot\nabla)]\\&=\frac{\kappa}{1+\xi\kappa}\partial_\xi+\partial_{\xi\xi}+\frac{1}{1+\xi\kappa}\partial_s\left(\frac{1}{1+\xi\kappa}\partial_s\right)\end{aligned}$$

Differential geometry in the front vicinity

Aligned coordinates (s, ξ)
s arclength along the front
ξ distance to the front along $\mathbf{n}$
Orthogonal Curvilinear coordinates
length element

$$d\ell^2 = d\xi^2 + (1+\kappa\xi)^2 \mathrm{d}s^2$$



metrics

$$g = \begin{pmatrix} 1 & 0 \\ 0 & (1+\xi\kappa)^2 \end{pmatrix}$$

g_{ij} components of g
g^{ij} components of inverse g^{-1}

$$\nabla \cdot \mathbf{a} = \frac{1}{|g|^{1/2}} \partial_i (|g|^{1/2} a^i)$$
$$(\nabla u)^i = \partial^i u = g^{ij} \partial_j u$$

Therefore

$$\Delta u = \nabla \cdot \nabla u = \frac{1}{|g|^{1/2}} \partial_i \left(|g|^{1/2} g^{ij} \partial_j u \right)$$

Differential geometry in the front vicinity

Aligned coordinates (s, ξ)
Laplacian

$$\Delta = \frac{\kappa}{1+\xi\kappa}\partial_\xi + \partial_{\xi\xi} + \frac{1}{1+\xi\kappa}\partial_s\left(\frac{1}{1+\xi\kappa}\partial_s\right)$$

Small curvature expansion

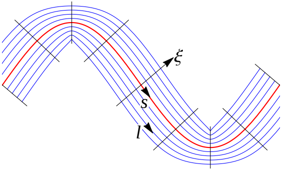
$$\kappa = \epsilon\kappa_1, \qquad s = \epsilon^{-1}S$$

Laplacian to linear order

$$\Delta U = \partial_{\xi\xi}U + \epsilon\kappa_1\partial_\xi U + \ldots$$

Differential geometry in the front vicinity

Aligned coordinates (s, ξ)



Approximated locally by polar coordinates (r, θ)

$$\begin{aligned}\Delta u &\approx \frac{1}{r}\partial_r\left(r\partial_r u\right) + \frac{1}{r^2}\partial_{\theta\theta} u + \mathrm{h.o.t.}\\ &\approx \partial_{rr} u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_{\theta\theta} u + \mathrm{h.o.t.}\\ &\approx \partial_{\xi\xi} u + \kappa\partial_r u + \kappa^2\partial_{\theta\theta} u + \mathrm{h.o.t.}\end{aligned}$$

where $\partial_r = \partial_\xi$ and $r \approx 1/\kappa$

Small curvature expansion

$$\kappa = \epsilon\kappa_1, \qquad s = \epsilon^{-1} S$$

Laplacian to linear order

$$\Delta U = \partial_{\xi\xi} U + \epsilon\kappa_1 \partial_\xi U + \dots$$

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Weakly driven front

$$\partial_T U = \Delta U + g(U)$$

Expansion close to coexistence
slow evolution $\partial_T = -\epsilon C_1 \partial_\xi$

$$-\epsilon C_1 \partial_\xi U = \partial_{\xi\xi} U + \epsilon \kappa_1 \partial_\xi U + \dots$$
$$- V_0'(U) - \epsilon V_1'(U)$$

Expansion of U

$$U = U_0 + \epsilon U_1 + \dots$$

Order 0

$$\partial_{\xi\xi} U_0 - V_0'(U_0) = 0$$

$U_0(\xi)$ is the equilibrium 1D kink solution independent of S

Order 1

$$\partial_{\xi\xi} U_1 - U_1 V_0''(U_0)$$
$$= -\kappa_1 \partial_\xi U_0 - C_1 \partial_\xi U_0 + V_1'(U_0)$$

Since $\partial_\xi U_0$ solution of homogeneous equation

$$\partial_{\xi\xi}(\partial_\xi U_0) - \partial_\xi U_0 \, V_0''(U_0) = 0$$

Fredholm alternative

$$C_1 \int_-^+ \mathrm{d}\xi \, (\partial_\xi U_0)^2 = \int_-^+ \mathrm{d}\xi \, \partial_\xi U_0 V_1'(U_0)$$
$$- \kappa_1 \int_-^+ \mathrm{d}\xi \, (\partial_\xi U_0)^2$$

finally

$$C_1 = \frac{V_1(U_{0+}) - V_1(U_{0-})}{\int \mathrm{d}\xi \, (\partial_\xi U_0)^2} - \kappa_1$$

Eikonal equation

Eikonal equation

$$C = C_* - \kappa$$

Two terms:

- constant term C_* can be > 0 or < 0

$$C_* = \frac{V(U_{0+}) - V(U_{0-})}{\int \mathrm{d}\xi \, (\partial_\xi U_0)^2}$$

  Normal propagation (Eikonal) like in ray physics
- curvature term has fixed sign: always stabilizing

Eikonal equation

Back to physical variables I

$$t = \frac{T}{L_0} \qquad\qquad x = \left(\frac{D}{L_0}\right)^{1/2} X$$

$C_* = (DL_0)^{-1/2} c_*$ and $\kappa = (D/L_0)^{1/2} \varkappa$

$$c = c_* - D\varkappa$$

For $V(U) = (1 - U^2)^2/4 - \gamma U^3/3$, we have $C_* = \pm 2^{-1/2}\gamma$

$$c_* = \pm 2^{-1/2}\gamma\,(DL_0)^{1/2} = \pm 2^{-1/2} C_2 \left(\frac{D}{C_3}\right)^{1/2}$$

Lyapunov functional for the Eikonal equation

Eikonal equation

$$C = \frac{1}{\mathcal{I}_0}\left(V(U_{0+}) - V(U_{0-}) - \mathcal{I}_0\kappa\right)$$

where

$$\mathcal{I}_0 = \int \mathrm{d}\xi\, (\partial_\xi U_0)^2 = 2^{1/2}\int \mathrm{d}U\, V(U)^{1/2}$$

Two global geometric observables

- Area both sides $\mathcal{A}_+, \mathcal{A}_-$
- perimeter $\mathcal{L}$

Functional

$$\mathcal{F} = V(U_{0+})\mathcal{A}_+ + V(U_{0-})\mathcal{A}_- + \mathcal{I}_0\mathcal{L}$$

kink energy = Line tension (Landau Models)

$$\mathcal{I}_0 = \mathcal{F}_k$$

Variation $\delta\mathbf{r}(s)$

$$\mathrm{d}\mathcal{F} = V(U_{0+})\mathrm{d}\mathcal{A}_+ + V(U_{0-})\mathrm{d}\mathcal{A}_- + \mathcal{I}_0\mathrm{d}\mathcal{L}$$

$$\mathrm{d}\mathcal{A}_+ = -\oint \mathrm{d}s\,(\delta\mathbf{r}(s)\cdot\hat{\mathbf{n}})$$

$$\mathrm{d}\mathcal{A}_- = +\oint \mathrm{d}s\,(\delta\mathbf{r}(s)\cdot\hat{\mathbf{n}})$$

$$\mathrm{d}\mathcal{L} = \oint \mathrm{d}s\,(\delta\mathbf{r}(s)\cdot\hat{\mathbf{n}})\kappa$$

$$\delta\mathbf{r}(s)\cdot\hat{\mathbf{n}} = C\,\mathrm{d}t$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} = \oint \mathrm{d}s\, C\left(-V(U_{0+}) + V(U_{0-}) + \mathcal{I}_0\kappa\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} = -\mathcal{I}_0\oint \mathrm{d}s\, C^2 \leq 0$$

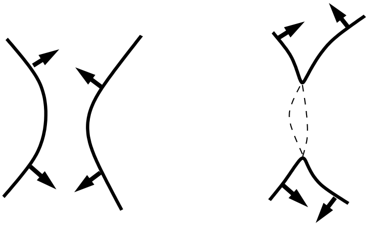
Global minimum: whole system in low-potential phase with no boundary
→ $\mathcal{F}$ is a Lyapunov functional (coarse grained)

Eikonal dynamics

Eikonal equation

$$c = c_* - D\kappa$$

- Several interfaces: collisions of interfaces $\rightarrow$ topological changes
- "quick" invasion of the system by low-energy domains



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Coarsening dynamics at coexistence

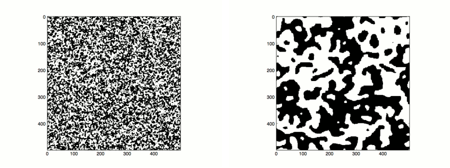
$c_* = 0$, i.e., $\gamma = 0$
Coexistence / Motion by curvature

$$c = -D\kappa$$

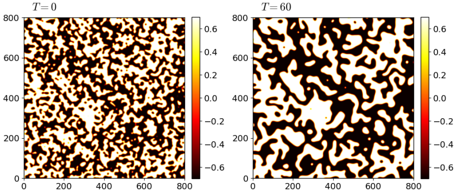
Heuristic scaling approach
Dynamics dominated by one lengthscale L
Domain size L, curvature $\kappa \sim 1/L$
$\partial_t L \sim c \sim D\kappa \sim D/L$
$\Rightarrow \partial_t L^2 \sim D \Rightarrow L \sim (Dt)^{1/2}$

scalings
$\kappa \sim 1/L \sim \epsilon$
$c \sim \partial_t L \sim D/L \sim \epsilon$
$t \sim L^2 \sim \epsilon^{-2}$
$\rightarrow$ derivation of Eikonal equation self-consistent

Ising model (500×500, inv temp 10^2);



Cahn-Allen (800×800)



Quench Ising

Coarsening dynamics

Closed hull of area $\mathcal{A}_H$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}_H = \oint \mathrm{d}s\ c = -D \oint \mathrm{d}s\ \kappa$$

Gauss-Bonnet $\oint \mathrm{d}s\ \kappa = 2\pi$

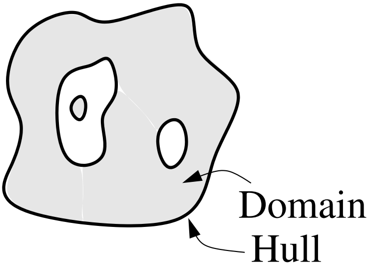
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}_H = -2\pi D$$

Hull area dynamics

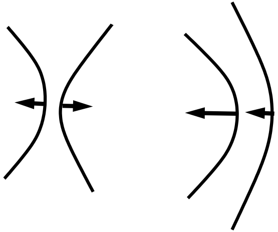
$$\mathcal{A}_H(t) = \mathcal{A}_H(0) - 2\pi D t$$

Collapse in finite time $\mathcal{A}_H(0)/(2\pi D)$
Advection of the probability distribution

$$P(\mathcal{A}_H, t) = P(\mathcal{A}_H + 2\pi D t, 0)$$



No collisions!



Coarsening dynamics

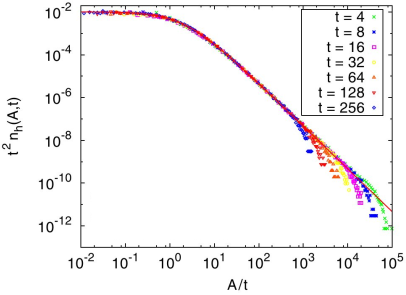
Example of application
Conformal field theory,
quenching from ∞ temperature

$$P(\mathcal{A}_H, 0) = \frac{2c_c}{\mathcal{A}_H^2}$$

c_c is called the central charge ($c_c = 1/(8\pi 3^{1/2})$)
finite time distribution

$$P(\mathcal{A}_H, t) = P(\mathcal{A}_H + 2\pi Dt, 0) = \frac{2c_c}{(\mathcal{A}_H + 2\pi Dt)^2}$$

$$t^2 P(\mathcal{A}_H; t) = f(\mathcal{A}_H/t) = \frac{2c_c}{[(\mathcal{A}_H/t) + 2\pi D]^2}$$



Arenzon et al (2007)

$D \approx 0.33$

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Conclusion

- III-s: Cahn-Allen equation
- 1D kink dynamics, Coarsening (coexistence $L \sim \ln t$)
- 2D Eikonal equation
- 2D Coarsening (coexistence $L \sim t^{1/2}$)