Lecture 2: Nonlinear Physics and Instabilities

Olivier Pierre-Louis

ILM-Lyon, France

4th October 2023

Lecture 2: Nonlinear Physics and Instabilities

Contents

Swift-Hohenberg equation

a simple nonlinear 1D model equation with I-s instability

$$\partial_t u = \epsilon u - (1 + \partial_{xx})^2 u + \alpha u^2 - u^3$$

Linear dispersion I-s

$$\sigma(q) = \epsilon - (1 - q^2)^2$$

Bifurcation amplitude equation

$$\partial_t u = \epsilon u + \alpha u^2 - u^3$$

SH model is a chimera "Chimera"

Multi-scale expansion

Linear dispersion I-s, $q_c = 1$

$$\sigma(q) = \epsilon - (1 - q^2)^2$$

Slow time-scale

$$t = T\epsilon^{-1}$$

Two length-scales

$$x \sim 1, \quad x = \epsilon^{-1/2} X \sim \epsilon^{-1/2}$$

Three independent variables x, X, T
Partial derivatives, $u(x,t) \to u(x,X,T)$:

$$\partial_t \to \epsilon \partial_T$$
$$\partial_x \to \partial_x + \epsilon^{1/2} \partial_X$$

Swift-Hohenberg equation

$$\epsilon \partial_T u = \epsilon u - (1 + \partial_{xx} + 2\epsilon^{1/2}\partial_{xX} + \epsilon \partial_{XX})^2 u + \alpha u^2 - u^3$$

Expansion of u

$$u(x,X,T) = u_0 + \epsilon^{1/2} u_{\frac{1}{2}} + \epsilon\, u_1 + \epsilon^{3/2} u_{\frac{3}{2}} + \ldots$$

Multi-scale expansion

order ϵ^0 Base state

$$u_0 = 0$$

Stationary homogeneous solution
order $\epsilon^{1/2}$ Linear order

$$(1+\partial_{xx})^2 u_{\frac{1}{2}} = 0$$

Solution ($x\mathrm{e}^{ix}$ unacceptable)

$$u_{\frac{1}{2}} = A(X,T)\mathrm{e}^{ix} + c.c.$$

order ϵ^1 Quadratic order

$$\begin{aligned}(1+\partial_{xx})^2 u_1 &= \alpha u_{\frac{1}{2}}^2 - 4(1+\partial_{xx})\partial_{xX} u_{\frac{1}{2}} \\ &= \alpha(A^2\mathrm{e}^{2ix} + AA^* + cc)\end{aligned}$$

Solution

$$u_1 = B(X,T)\mathrm{e}^{ix} + \alpha\left(\frac{A^2}{9}\mathrm{e}^{2ix} + AA^*\right) + cc$$

Multi-scale expansion

order $\epsilon^{3/2}$ Cubic order, where the time derivative is!

$$(1+\partial_{xx})^2 u_{\frac{3}{2}} = u_{\frac{1}{2}} - 4\partial_{xxXX} u_{\frac{1}{2}} - \partial_T u_{\frac{1}{2}} - (1+\partial_{xx})\partial_{xX} u_1 + 2\alpha u_{\frac{1}{2}} u_1 - u_{\frac{1}{2}}^3$$
$$= F_1(X,T)\mathrm{e}^{ix} + F_2(X,T)\mathrm{e}^{2ix} + F_3(X,T)\mathrm{e}^{3ix} + c.c.$$
$$F_1(X,T) = A + 4\partial_{XX} A - \partial_T A - (3 - \frac{38}{9}\alpha^2)A^2 A^*$$

term proportional to $F_1\mathrm{e}^{ix}$ resonant term $\rightarrow$ divergence $\rightarrow$ $F_1 = 0$: "Solvability condition"
Amplitude equation

$$\partial_T A = A + 4\partial_{XX} A - (3 - \frac{38}{9}\alpha^2)|A|^2 A$$

Remarks:

- $F_1\mathrm{e}^{ix}$ is periodic in x
- $F_1\mathrm{e}^{ix}$ is a solution of $(1+\partial_{xx})^2(F_1\mathrm{e}^{ix}) = 0$

$\rightarrow$ Frehdolm alternative

Fredholm alternative

Linear ordinary differential equation, variable x

$$Lu(x) = f(x)$$

u and f periodic functions of x

Scalar product of two functions of x

$$\langle u, v\rangle = \int_{\mathrm{period}} \mathrm{d}x \; u(x)v(x)$$

Adjoint operator

$$\langle Lu, v\rangle = \langle u, L^\dagger v\rangle$$

Integration by parts:
No boundary term if periodic!

Choose v such that

$$L^\dagger v = 0$$

Then

$$\langle f, v\rangle = \langle Lu, v\rangle = \langle u, L^\dagger v\rangle = 0$$

Thus

$$\langle f, v\rangle = 0$$

Amplitude equation

$$L = (1 + \partial_{xx})^2 = L^\dagger$$

$$v = \mathrm{e}^{ix} \qquad (L^\dagger v = Lv = 0)$$

$$f = F_1 \mathrm{e}^{ix} + F_1^* \mathrm{e}^{-ix} + F_2 \mathrm{e}^{2ix} + F_2^* \mathrm{e}^{-2ix} + F_3 \mathrm{e}^{3ix} + F_3^* \mathrm{e}^{-3ix}$$

Solvability condition / Fredholm alternative

$$\langle f, v \rangle = \int_0^{2\pi} \mathrm{d}x\ \mathrm{e}^{ix} (F_1 \mathrm{e}^{ix} + F_1^* \mathrm{e}^{-ix} + F_2 \mathrm{e}^{2ix} + F_2^* \mathrm{e}^{-2ix} + F_3 \mathrm{e}^{3ix} + F_3^* \mathrm{e}^{-3ix}) = 2\pi F_1^*$$

$$\langle f, v \rangle = 0 \Rightarrow F_1(X, T) = 0$$

Amplitude equation

$$\partial_T A = A + 4\partial_{XX} A - (3 - \frac{38}{9}\alpha^2)|A|^2 A$$

Remarks:

- $A \to -A$ symmetric !
- supercritical $\alpha^2 < 27/38$

Amplitude equation

Normalization $X \to 2X$, $A \to A/(3 - 38\alpha^2/9)^{1/2}$ $\to$ No-parameter equation

$$\partial_T A = A + \partial_{XX} A - |A|^2 A$$

Real Ginzburg-Landau equation

Linear dispersion

$$\Sigma(Q) = 1 - Q^2$$
$$q_x = 1 + \epsilon^{1/2} Q_X$$

Symmetries of the amplitude equation

$$u(x, X, T) = u_0 + u_{\frac{1}{2}}(x, X, T) = A(X, T)\,\mathrm{e}^{ix} + A(X, T)^*\,\mathrm{e}^{-ix}$$

Real Ginzburg Landau equation (RGL)

$$\partial_T A = A + \partial_{XX} A - |A|^2 A$$

- Translational invariance $x \to x + x_O$
  $A\mathrm{e}^{ix} \to A\mathrm{e}^{i(x+x_O)} \Rightarrow A \to A\mathrm{e}^{ix_O}$
  Arbitrary nonlinearity $A^n A^{*m}$

  $$\partial_T A = A + \partial_{XX} A + A^n A^{*m} \to \partial_T A\mathrm{e}^{ix_O} = A\mathrm{e}^{ix_O} + \partial_{XX} A\mathrm{e}^{ix_O} + A^n A^{*m}\mathrm{e}^{i(n-m)x_O} \Rightarrow n - m = 1$$

  lowest-order nonlinear term $n = 2$, $m = 1$, i.e. $A^2 A^* = A|A|^2$

- Space inversion symmetry $x \to -x$
  $A\mathrm{e}^{ix} + A^*\mathrm{e}^{-ix} \to A\mathrm{e}^{-ix} + A^*\mathrm{e}^{ix} \Rightarrow A \to A^*$

  $$(A \to A^*)\text{in}(\partial_T A = \alpha_1 A + \alpha_2 \partial_{XX} A - \alpha_3 |A|^2 A) \Rightarrow \partial_T A^* = \alpha_1 A^* + \alpha_2 \partial_{XX} A^* - \alpha_3 |A^*|^2 A^*$$

  $$(\partial_T A = \alpha_1 A + \alpha_2 \partial_{XX} A - \alpha_3 |A|^2 A)^* \Rightarrow \partial_T A^* = \alpha_1^* A^* + \alpha_2^* \partial_{XX} A^* - \alpha_3^* |A^*|^2 A^*$$

  $\alpha_1 = \alpha_1^*$, $\alpha_2 = \alpha_2^*$, $\alpha_3 = \alpha_3^* \Rightarrow \alpha_1, \alpha_2, \alpha_3$ real.

Gradient Dynamics form of the amplitude equation

Real Ginzburg Landau equation (RGL) Generic amplitude equation

$$\partial_T A = A + \partial_{XX} A - |A|^2 A$$

RGL takes the form

$$\partial_T A = -\frac{\delta \mathcal{F}}{\delta A^*}, \qquad \partial_T A^* = -\frac{\delta \mathcal{F}}{\delta A}$$

with

$$\mathcal{F} = \int \mathrm{d}X \left(\frac{1}{2}(1 - |A|^2)^2 + |\partial_X A|^2 \right) \geq 0$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}T}\mathcal{F} = \int \mathrm{d}X \,(\partial_T A \,\frac{\delta \mathcal{F}}{\delta A} + \partial_T A^* \,\frac{\delta \mathcal{F}}{\delta A^*}) = -2 \int \mathrm{d}X \,\partial_T A \,\partial_T A^* = -2 \int \mathrm{d}X |\partial_T A|^2 \leq 0$$

RGL has a Lyapunov functional:

- decreasing functional $d\mathcal{F}/dT \leq 0$
- with a minimum $\mathcal{F} \geq 0$ (bounded from below)

Consequence of Lyapunov functional

- No travelling wave $A(X,T) = A(X-cT)$ with A not identically vanishing then $\mathcal{F}$ is time-independent

$$\begin{aligned}\mathcal{F} &= \int \mathrm{d}X \left(\frac{1}{2}(1-|A(X-cT)|^2)^2 + |\partial_X A(X-cT)|^2 \right) \\ &= \int \mathrm{d}X' \left(\frac{1}{2}(1-|A(X')|^2)^2 + |\partial_X A(X')|^2 \right)\end{aligned}$$

  so that $\mathrm{d}\mathcal{F}/\mathrm{d}T = 0$ (if $\mathcal{F}$ is finite)
  However

$$\frac{\mathrm{d}}{\mathrm{d}T}\mathcal{F} = -2\int \mathrm{d}X|\partial_T A|^2 = -2c^2\int \mathrm{d}X|\partial_X A|^2$$

  Therefore $c = 0$.
- No oscillation $A(X, T+T_0) = A(X,T)$
  then $\mathcal{F}(T+T_0) = \mathcal{F}(T)$
  However

$$\mathcal{F}(T+T_0) - \mathcal{F}(T) = \int_T^{T+T_0} \mathrm{d}T' \frac{\mathrm{d}}{\mathrm{d}T}\mathcal{F}(T') = -2\int_T^{T+T_0} \mathrm{d}T' \int \mathrm{d}X|\partial_T A(T')|^2 < 0$$

  $\Rightarrow \mathcal{F}(T+T_0) < \mathcal{F}(T)$ contradiction!
- If "suitable" boundary conditions $\rightarrow$ Dynamics leads to local minimum of $\mathcal{F}$

Contents

Linear and nonlinear steady-states

Real Ginzburg Landau equation (RGL)

$$\partial_T A = A + \partial_{XX} A - |A|^2 A$$

$A = \rho e^{i\theta}$ modulus $\rho = |A|$ and phase θ

$$\partial_T \rho = [1 - (\partial_X \theta)^2]\rho - \rho^3 + \partial_{XX}\rho$$

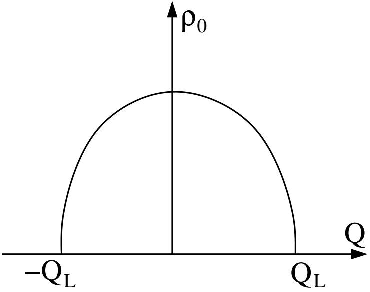
$$\partial_T \theta = \partial_{XX}\theta + 2\partial_X \theta \frac{\partial_X \rho}{\rho}$$

Linear Stationary solution $\theta = iQ_L X$, $\rho_L \ll 1$

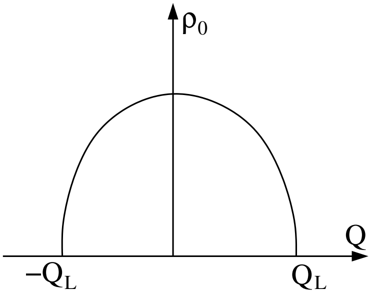
$$A_L(X) = \rho_L e^{\pm iQ_L X}$$
$$Q_L = 1$$

Nonlinear Steady-state solution
for any $Q_0 < Q_L = 1$:

$$A_0(X) = \rho_0 e^{iQ_0 X}$$
$$\rho_0 = (1 - Q_0^2)^{1/2}$$



Linear and nonlinear steady-states

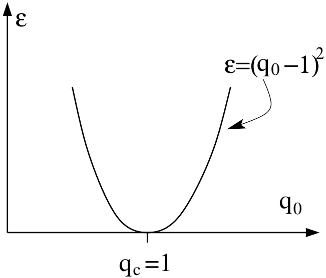


Back to variables of the Swift-Hohenberg eq.

$$\epsilon^{1/2} Q_0 = q_0 - 1, \qquad \epsilon = \frac{(q_0-1)^2}{Q_0^2}$$

Nonlinear steady-states $|Q_0| < |Q_L|$

$$\epsilon > \frac{(q_0-1)^2}{Q_L^2}$$



Contents

Eckhaus instability

Linear Stability analysis of nonlin. steady-state
Nonlinear Stationary solution for any $Q_0 < 1$:

$$A_0(X) = \rho_0 \mathrm{e}^{iQ_0 X}$$
$$\rho_0 = (1 - Q_0^2)^{1/2}$$

Perturbation of stationary solution

$$\rho = (1 - Q_0^2)^{1/2} + \delta\rho, \qquad \theta = Q_0 X + \delta\theta$$
$$\delta\rho = \delta\rho_{Q_\delta} \mathrm{e}^{iQ_\delta X + \sigma T}, \qquad \delta\theta = \delta\theta_{Q_\delta} \mathrm{e}^{iQ_\delta X + \sigma T}$$

Linear system

$$\begin{pmatrix} \sigma + 2\rho_0^2 + Q_\delta^2 & 2i\rho_0 Q_0 Q_\delta \\ -2iQ_0 Q_\delta/\rho_0 & \sigma + Q_\delta^2 \end{pmatrix} \begin{pmatrix} \delta\rho_{Q_\delta} \\ \delta\theta_{Q_\delta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Determinant must vansih

$$(\sigma + 2\rho_0^2 + Q_\delta^2)(\sigma + Q_\delta^2) - 4Q_0^2 Q_\delta^2 = 0$$
$$(\sigma + \rho_0^2 + Q_\delta^2)^2 - \rho_0^4 - 4Q_0^2 Q_\delta^2 = 0$$

Two solutions

$$\sigma_\pm = -(\rho_0^2 + Q_\delta^2) \pm (\rho_0^4 + 4Q_0^2 Q_\delta^2)^{1/2}$$

$\sigma_- < 0$

$$\sigma_+\sigma_- = (\rho_0^2 + Q_\delta^2)^2 - (\rho_0^4 + 4Q_0^2 Q_\delta^2)$$
$$= Q_\delta^2(2 - 6Q_0^2 + Q_\delta^2)$$

Changes sign when $Q_\delta \ll 1$ for $Q_0^2 = 1/3$
Eckhaus instability criterion $\sigma_+ > 0$

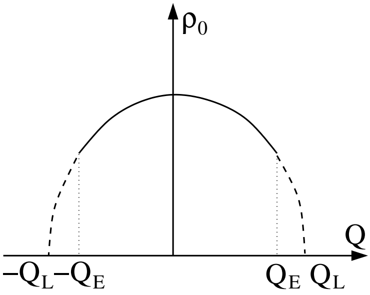
$$Q_E = 3^{-1/2} < |Q_0| < Q_L = 1$$

Long wavelength $Q_\delta \ll 1$

$$\sigma_+ = \frac{3Q_0^2 - 1}{\rho_0^2} Q_\delta^2 - 2\frac{Q_0^4}{\rho_0^6} Q_\delta^4$$

II-s instability

Eckhaus instability



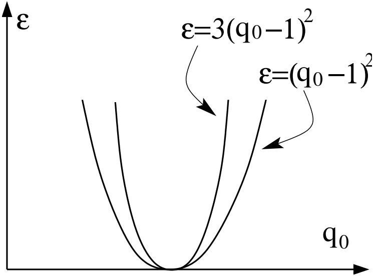
$$Q_E = 3^{-1/2} < |Q_0| < Q_L = 1$$

Back to variables of the Swift-Hohenberg eq.

$$\epsilon^{1/2} Q_0 = q_0 - 1, \qquad \epsilon = \frac{(q_0-1)^2}{Q_0^2}$$

Eckhaus Instability

$$\frac{(q_0-1)^2}{Q_L^2} < \epsilon < \frac{(q_0-1)^2}{Q_E^2}$$



Eckhaus instability

Linear Stability analysis of nonlin. steady-state

Nonlinear Stationary solution for any $Q_0 < 1$:

$$A_0(X) = \rho_0 \mathrm{e}^{iQ_0 X}$$
$$\rho_0 = (1 - Q_0^2)^{1/2}$$

Perturbation of stationary solution

$$\rho = (1 - Q_0^2)^{1/2} + \delta\rho, \qquad \theta = Q_0 X + \delta\theta$$
$$\delta\rho = \delta\rho_{Q_\delta} \mathrm{e}^{iQ_\delta X + \sigma T}, \qquad \delta\theta = \delta\theta_{Q_\delta} \mathrm{e}^{iQ_\delta X + \sigma T}$$

Linear system

$$\begin{pmatrix} \sigma + 2\rho_0^2 + Q_\delta^2 & 2i\rho_0 Q_0 Q_\delta \\ -2iQ_0 Q_\delta/\rho_0 & \sigma + Q_\delta^2 \end{pmatrix} \begin{pmatrix} \delta\rho_{Q_\delta} \\ \delta\theta_{Q_\delta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Long wavevelength limit $Q_\delta^2 \to 0$

$$\sigma_- = -2\rho_0^2$$
$$\sigma_+ = 0$$

Linear system

$$\begin{pmatrix} \sigma + 2\rho_0^2 & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} \delta\rho_{Q_\delta} \\ \delta\theta_{Q_\delta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Corresponding eigenvectors

$$\sigma_- = -2\rho_0^2 \qquad \leftrightarrow \delta\rho_{Q_\delta} \neq 0,\ \delta\theta_{Q_\delta} = 0$$
$$\sigma_+ = 0 \qquad \leftrightarrow \delta\rho_{Q_\delta} = 0,\ \delta\theta_{Q_\delta} \neq 0$$

Eckhaus instability is a phase instability

Eckhaus as a phase instability

Real Ginzburg Landau equation (RGL)

$$\partial_T A = A + \partial_{XX} A - |A|^2 A$$

$A = \rho e^{i\theta}$ modulus $\rho = |A|$ and phase θ

$$\partial_T \rho = [1 - (\partial_X \theta)^2]\rho - \rho^3 + \partial_{XX}\rho$$

$$\partial_T \theta = \partial_{XX}\theta + 2\partial_X\theta \frac{\partial_X \rho}{\rho}$$

X varies on large scales $\Rightarrow \partial_X$ small
$\partial_X \theta = K$ like a wavevector is of order one

$$\partial_T \rho = [1 - K^2]\rho - \rho^3 + \partial_{XX}\rho$$

$$\partial_T \theta = \partial_X K + 2K \frac{\partial_X \rho}{\rho}$$

ρ relaxes quickly $\partial_T \rho \approx 0$
if $K < 1$

$$\rho = [1 - K^2]^{1/2}$$

adiabatic control of ρ
θ evolves slowly

$$\partial_T \theta = (1 - 2\frac{K^2}{1 - K^2})\partial_X K = \frac{1 - 3K^2}{1 - K^2}\partial_{XX}\theta$$

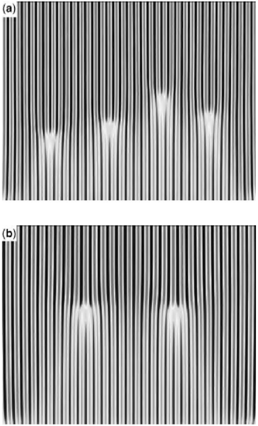
$$\partial_T \theta = D\partial_{XX}\theta$$

Phase diffusion equation
Stability $D > 0$
$\Rightarrow K^2 > 1/3$
Eckhaus instability is a phase instability

Eckhaus instability

1D Swift-Hohenberg numerical solution

$$\partial_t u = \epsilon u - (1 + \partial_{xx})^2 u + \alpha u^2 - u^3$$



(a) $q_0 = 0.88 q_c$, (b) $q_0 = 1.115 q_c$

Provatas, Elder 2010

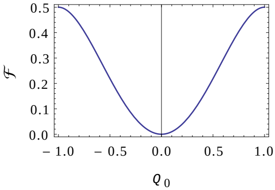
Eckhaus instability

Lyapunov functional

$$\mathcal{F} = \int \mathrm{d}X \left(\frac{1}{2}(1 - |A|^2)^2 + |\partial_x A|^2 \right)$$

Energy density of nonlinear steady-states

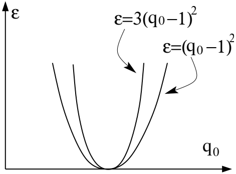
$$\frac{1}{2}(1 - \rho_0^2)^2 + Q_0^2 \rho_0^2 = Q_0^2 \left(1 - \frac{Q_0^2}{2} \right)$$



Back to variables of the Swift-Hohenberg eq.

$$\epsilon^{1/2} Q_0 = q_0 - 1, \qquad \epsilon = \frac{(q_0 - 1)^2}{Q_0^2}$$

$$Q_0 \to 0 \Rightarrow q_0 \to q_c = 1$$



Lecture 2: Nonlinear Physics and Instabilities

Contents

2D Swift-Hohenberg

2D S.H.

$$\partial_t u = \epsilon u - (1+\Delta)^2 u + \alpha u^2 - u^3$$

Laplacian $\Delta = \partial_{xx} + \partial_{yy}$
$q^2 = \mathbf{q}^2 = q_x^2 + q_y^2$
Linear dispersion I-s

$$\sigma(q) = \epsilon - (1-q^2)^2$$

In 1D: $q_x = 1 + \epsilon^{1/2} Q_X \Rightarrow q^2 = q_x^2 = 1 + 2\epsilon^{1/2} Q_X + \mathrm{h.o.t.}$
In 2D:

$$q^2 = q_x^2 + q_y^2 = (1+\epsilon^{1/2}Q_X)^2 + q_y^2 = 1 + 2\epsilon^{1/2}Q_X + \epsilon Q_X^2 + q_y^2$$

Leading order contributions comparable $\to q_y^2 \sim \epsilon^{1/2} \Rightarrow q_y \sim \epsilon^{1/4}$, or $q_y = \epsilon^{1/4} Q_Y$

$$q^2 = 1 + \epsilon^{1/2}(2Q_X + Q_Y^2) + \mathrm{h.o.t.}$$

1D to 2D: $Q_X \to Q_X + Q_Y^2/2$.

$$\partial_X \sim_{\mathrm{FT}} iQ_X \to iQ_X + \frac{i}{2}Q_Y^2 \sim_{\mathrm{FT}^{-1}} \partial_X - \frac{i}{2}\partial_{YY}$$

$$\partial_T A = A + \partial_{XX} A - A|A|^2 \quad \to \quad \partial_T A = A + (\partial_X - \frac{i}{2}\partial_{YY})^2 A - A|A|^2$$

2D Expansion

Partial derivatives, $u(x,t) \to u(x,X,T)$:

$$\partial_t \to \epsilon\partial_T$$
$$\partial_x \to \partial_x + \epsilon^{1/2}\partial_X$$
$$\partial_y \to \epsilon^{1/4}\partial_Y$$

Swift-Hohenberg equation

$$\epsilon\partial_T u = \epsilon u - (1 + \partial_{xx} + 2\epsilon^{1/2}\partial_{xX} + \epsilon\partial_{XX} + \epsilon^{1/2}\partial_{YY})^2 u + \alpha u^2 - u^3$$

- Assume $\alpha = 0$

$$\epsilon\partial_T u = \epsilon u - (1+\Delta)^2 u - u^3$$

- Expansion of u

$$u(x,y,t) = u_0 + \epsilon^{1/2} u_{\frac{1}{2}}(x,X,Y,T) + \epsilon u_1(x,X,Y,T) + \epsilon^{3/2} u_{\frac{3}{2}}(x,X,Y,T) + \ldots$$

- Choose a direction $x \to$ break rotational invariance

2D Expansion

order ϵ^0 Base state

$$u_0 = 0$$

Stationary homogeneous solution
order $\epsilon^{1/2}$ Linear order

$$(1+\partial_{xx})^2 u_{\frac{1}{2}} = 0$$

Stipes solution

$$u_{\frac{1}{2}} = A(X,Y,T)\mathrm{e}^{ix} + c.c.$$

order ϵ

$$(1+\partial_{xx})^2 u_1 = -2(1+\partial_{xx})(2\partial_{xX} + \partial_{YY})u_{\frac{1}{2}} = 0$$

Solution

$$u_1 = B(X,Y,T)\mathrm{e}^{ix} + c.c.$$

2D Expansion

order $\epsilon^{3/2}$ Cubic order

$$(1+\partial_{xx})^2 u_{\frac{3}{2}} = -\partial_T u_{\frac{1}{2}} + u_{\frac{1}{2}} - u_{\frac{1}{2}}^3 - 2(2\partial_{xX} + \partial_{YY})^2 u_{\frac{1}{2}}$$

Resonnant terms proportional to e^{ix}

$$\begin{aligned}
\partial_T u_{\frac{1}{2}} &\to \partial_T A \\
u_{\frac{1}{2}} &\to A \\
u_{\frac{1}{2}}^3 &\to 3A^2A^* = 3A|A|^2 \\
(2\partial_{xX} + \partial_{YY})^2 u_{\frac{1}{2}} &\to (2i\partial_X + \partial_{YY})^2 A
\end{aligned}$$

Finally

$$\partial_T A = A - (2i\partial_X + \partial_{YY})^2 A - A|A|^2$$

2D Expansion

Normalization

$$\partial_T A = A + (2\partial_X + i\partial_{YY})^2 A - 3A|A|^2$$

$$X \to 2X$$
$$Y \to 2^{1/2} Y$$
$$A \to 3^{1/2} A$$

Canonical form

$$\partial_T A = A + (\partial_X - \frac{i}{2}\partial_{YY})^2 A - A|A|^2$$

Lost rotational invariance

Zigzag Instability

$$\partial_T A = A + (\partial_X - \frac{i}{2}\partial_{YY})^2 A - A|A|^2$$

Recall our 1D Nonlinear Steady-state solution
for any $Q_0 < Q_L = 1$:

$$A_0(X) = \rho_0 \mathrm{e}^{iQ_0 X}$$
$$\rho_0 = (1 - Q_0^2)^{1/2}$$

Linear perturbation $A(X, Y, T) = A_0(X) + \delta A(X, Y, T)$

$$\partial_T \delta A = \delta A + (\partial_X - \frac{i}{2}\partial_{YY})^2 \delta A - 2|A_0|^2 \delta A - A_0^2 \delta A^*$$

Zigzag Instability

"longitudinal" perturbations along X $\rightarrow$ Eckhaus
"transverse" perturbations along Y $\rightarrow$??
Assume for simplicity perturbation depend only on Y

$$
\begin{aligned}
A(X,Y,T) &= \rho \mathrm{e}^{i\theta} \\
\rho &= \rho_0 + \delta\rho(Y,T) \\
\theta &= \theta_0 + \delta\theta(Y,T) = Q_0 X + \delta\theta(Y,T) \\
A(X,Y,T) &= (\rho_0 + \delta\rho(Y,T))\mathrm{e}^{iQ_0X + i\delta\theta(Y,T)} \approx \rho_0 \mathrm{e}^{iQ_0X} + [\delta\rho(Y,T) + i\rho_0\delta\theta(Y,T)]\mathrm{e}^{iQ_0X} \\
\delta A(X,Y,T) &= [\delta\rho(Y,T) + i\rho_0\delta\theta(Y,T)]\mathrm{e}^{iQ_0X}
\end{aligned}
$$

Linearized equations

$$
\partial_T \delta A = \delta A + (\partial_X - \frac{i}{2}\partial_{YY})^2 \delta A - 2|A_0|^2 \delta A - A_0^2 \delta A^*
$$

leads to

$$
\partial_T \delta A = \delta A + (iQ_0 - \frac{i}{2}\partial_{YY})^2 \delta A - 2|A_0|^2 \delta A - A_0^2 \delta A^*
$$

Zigzag Instability

Linearized equations

$$\partial_T \delta A = \delta A - (Q_0 - \frac{1}{2}\partial_{YY})^2 \delta A - 2|A_0|^2 \delta A - A_0^2 \delta A^*$$
$$\delta A(X, Y, T) = [\delta\rho(Y, T) + i\rho_0 \delta\theta(Y, T)]\mathrm{e}^{iQ_0 X}$$

Substitution

$$\partial_T \delta\rho = -\left(2\rho_0^2 - Q_0 \partial_{YY} + \frac{1}{4}\partial_{YYYY}\right)\delta\rho$$
$$\partial_T \delta\theta = -\left(-Q_0 \partial_{YY} + \frac{1}{4}\partial_{YYYY}\right)\delta\theta$$

$\delta\rho(Y, T) = \delta\rho_{Q_Y}(T)\mathrm{e}^{iQ_Y Y}$ and $\delta\theta(Y, T) = \delta\theta_{Q_Y}(T)\mathrm{e}^{iQ_Y Y}$ leads to

$$\sigma_{\delta\rho} = -\left(2\rho_0^2 + Q_0 Q_Y^2 + \frac{1}{4}Q_Y^4\right)$$
$$\sigma_{\delta\theta} = -\left(Q_0 Q_Y^2 + \frac{1}{4}Q_Y^4\right)$$

Zigzag instability criterion

$$Q_0 < 0$$

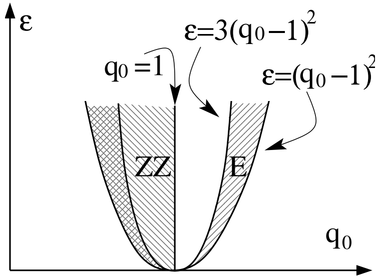
Phase Instability and Type II-s instability

Zigzag Instability

$Q_0 < 0 \rightarrow q_0 < q_c = 1$

Properties of the zigzag instability

- phase instability
- Type II-s instability

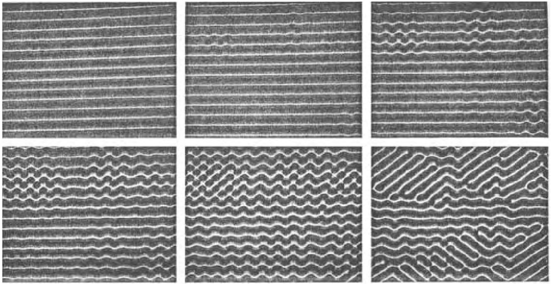


Experiments

Rayleigh-Bénard

- Prandt number $\nu/\kappa_T \approx 10^2$
  cell aspect ratio $\Gamma \approx 160$
  $Ra = 3600$
  initial $q_0/q_c \approx 0.72$
- times 9 to 72 minutes
- zigzag instability

Busse Whitehead 1971

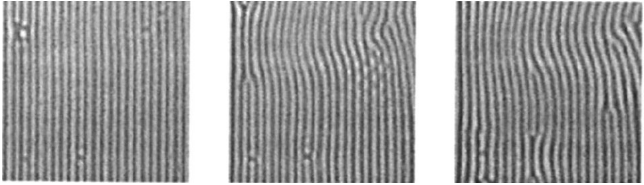


Experiments

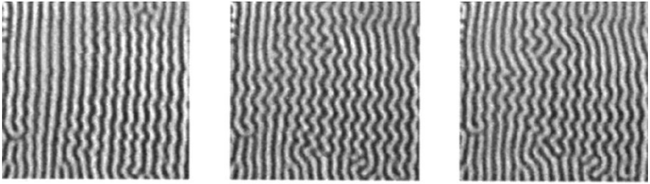
CDIMA reaction

- wavelengths $\sim 0.5\ mm$
- times $\sim$ hour
- Eckhaus and zigzag instability

Pena 2003



Pena 2003



Goldstone modes

Goldstone modes

Translation and rotational invariance
$\rightarrow$ no relaxation of perturbations
$\rightarrow$ slow relaxation at long wavelength
$\rightarrow$ long-wavelength and slow phase dynamics

Generic instabilitis of stripes

- Eckhaus $\leftrightarrow$ translational invariance
- Zigzag $\leftrightarrow$ rotational invariance

Contents

2D lattices

One-mode: Stripes (a)

$$u = a_1 \mathrm{e}^{i\mathbf{q}_1\cdot\mathbf{r}} + cc$$

with $|\mathbf{q}_1| = q_c = 1$

Superposition of two arrays of stripes

$$u = a_1 \mathrm{e}^{i\mathbf{q}_1\cdot\mathbf{r}} + a_2 \mathrm{e}^{i\mathbf{q}_2\cdot\mathbf{r}} + cc$$

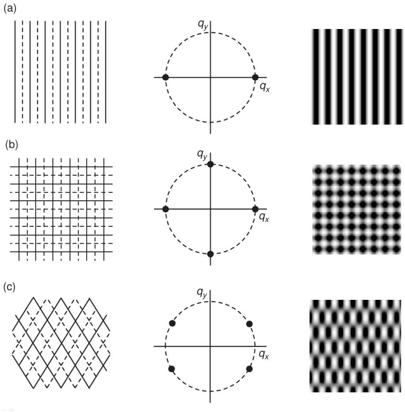
Condition

- $|\mathbf{q}_n| = q_c = 1$
- $|a_1| = |a_2|$

ϕ angle between stripes: $\mathbf{q}_1 \cdot \mathbf{q}_2 = q_1 q_2 \cos\phi$

not general 2D Bravais Lattices

- (b) $\phi = \pi/2 \rightarrow \mathbf{q}_1 \cdot \mathbf{q}_2 = 0$ square symmetry
- (c) $\phi \neq \pi/2 \rightarrow \mathbf{q}_1 \cdot \mathbf{q}_2 \neq 0$ rectangles $\rightarrow$ Orthorombic



Cross and Greenside p.154

2D lattices

Amplitudes:

$$a_n = \rho_0 \mathrm{e}^{i\theta_n}$$

Shift of the origin $\mathbf{r} \to \mathbf{r} + \mathbf{r}_O$, with $\mathbf{r}_O = (x_O, y_O)$

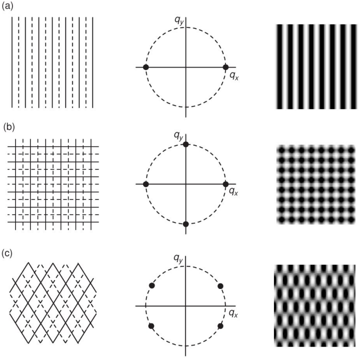
$$a_n \mathrm{e}^{i\mathbf{q}_n\cdot\mathbf{r}} = \rho_0 \mathrm{e}^{i\mathbf{q}_n\cdot\mathbf{r}+i\theta_n} \to \rho_0 \mathrm{e}^{i\mathbf{q}_n\cdot\mathbf{r}+i(\mathbf{q}_n\cdot\mathbf{r}_O+\theta_n)}$$

- orthorombic lattices:
  Choose $\mathbf{r}_O$ such that

$$\mathbf{q}_1 \cdot \mathbf{r}_O + \theta_1 = 0$$
$$\mathbf{q}_2 \cdot \mathbf{r}_O + \theta_2 = 0$$

  two equations determine x_O, y_O

$$u = \rho_0(\mathrm{e}^{i\mathbf{q}_1\cdot\mathbf{r}} + \mathrm{e}^{i\mathbf{q}_2\cdot\mathbf{r}}) + cc$$



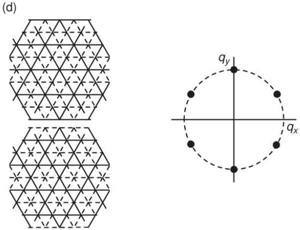
2D lattices

Hexagonal pattern
we can add a third set of stripes!
$\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ such that $\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 = \mathbf{0}$
3 Amplitudes:

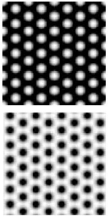
$$u = a_1 \mathrm{e}^{i\mathbf{q}_1 \cdot \mathbf{r}} + a_2 \mathrm{e}^{i\mathbf{q}_2 \cdot \mathbf{r}} + a_3 \mathrm{e}^{i\mathbf{q}_3 \cdot \mathbf{r}} + cc$$

with

- $|\mathbf{q}_n| = q_c = 1$
- $|a_1| = |a_2| = |a_3| = \rho_0$



Cross and Greenside p.154



2D lattices

Amplitudes:

$$a_n = \rho_0 \mathrm{e}^{i\theta_n}$$

Shift of the origin $\mathbf{r} \to \mathbf{r} + \mathbf{r}_O$, with $\mathbf{r}_O = (x_O, y_O)$

$$a_n \mathrm{e}^{i\mathbf{q}_n \cdot \mathbf{r}} = \rho_0 \mathrm{e}^{i\mathbf{q}_n \cdot \mathbf{r} + i\theta_n} \to \rho_0 \mathrm{e}^{i\mathbf{q}_n \cdot \mathbf{r} + i(\mathbf{q}_n \cdot \mathbf{r}_O + \theta_n)}$$

Find $\mathbf{r}_O$ such that:

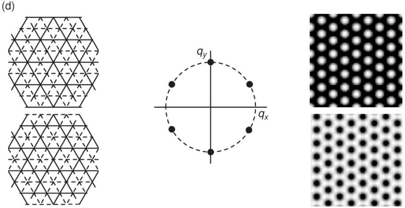
$$\mathbf{q}_1 \cdot \mathbf{r}_O + \theta_1 = 0$$
$$\mathbf{q}_2 \cdot \mathbf{r}_O + \theta_2 = 0$$

$$\mathbf{q}_3 \cdot \mathbf{r}_O + \theta_3 = \theta_1 + \theta_2 + \theta_3 = \psi$$

third phase cannot be set to zero

$$u = \rho_0 (\mathrm{e}^{i\mathbf{q}_1 \cdot \mathbf{r}} + \mathrm{e}^{i\mathbf{q}_2 \cdot \mathbf{r}} + \mathrm{e}^{i\mathbf{q}_3 \cdot \mathbf{r} + i\psi}) + cc$$

$\psi \to$ family of inequivalent lattices

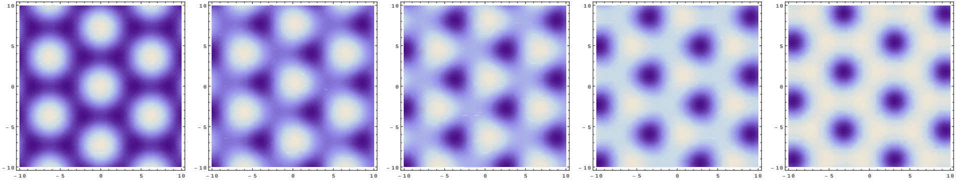


phase shift $\psi \to \psi + \pi$
with new change of origin $\mathbf{r} \to \mathbf{r} + \mathbf{r}_{O'}$
such that $\mathbf{q}_1 \cdot \mathbf{r}_{O'} = \mathbf{q}_2 \cdot \mathbf{r}_{O'} = \pi$
then $\mathbf{q}_3 \cdot \mathbf{r}_{O'} + \psi + \pi = \psi - \pi$
Hence $u \to -u$
π shift of phase $\psi \leftrightarrow$ field invesrion symmetry

3-stripes Lattices

$$u = \rho_0(\mathrm{e}^{i\mathbf{q}_1\cdot\mathbf{r}} + \mathrm{e}^{i\mathbf{q}_2\cdot\mathbf{r}} + \mathrm{e}^{i\mathbf{q}_3\cdot\mathbf{r}+i\psi}) + cc$$

$\psi = 0$, $\psi = \pi/4$, $\psi = \pi/2$, $\psi = 3\pi/4$, $\psi = \pi$.



Amplitude equation for 2D lattices

2D S.H.

$$\partial_t u = \epsilon u - (1+\Delta)^2 u + \alpha u^2 - u^3$$

Laplacian $\Delta = \partial_{xx} + \partial_{yy}$
$q^2 = \mathbf{q}^2 = q_x^2 + q_y^2$
Linear dispersion I-s

$$\sigma(q) = \epsilon - (1 - q^2)^2$$

For simplicity

- Expansion of u assume no dependence on X, Y for simplicity

$$u(x,y,X,Y,T) = u_0 + \epsilon^{1/2} u_{\frac{1}{2}}(x,y,T) + \epsilon u_1(x,y,T) + \epsilon^{3/2} u_{\frac{3}{2}}(x,y,T) + ...$$

- Assume $\alpha = \epsilon^{1/2}\alpha_1$ (meaning of-the-order-of, not proportional-to)

$$\epsilon \partial_T u = \epsilon u - (1+\Delta)^2 u + \epsilon^{1/2}\alpha_1 u^2 - u^3$$

- 3 set of stripes with $|\mathbf{q}_n| = 1$, and amplitudes:

$$A_1(T)\mathrm{e}^{i\mathbf{q}_1\cdot\mathbf{r}} + A_2(T)\mathrm{e}^{i\mathbf{q}_2\cdot\mathbf{r}} + A_3(T)\mathrm{e}^{i\mathbf{q}_3\cdot\mathbf{r}}$$

  Non-hexagonal: $A_3 = 0$, Hexagonal: $A_3 \neq 0$ and $\mathbf{q}_3 = -\mathbf{q}_1 - \mathbf{q}_2$

Expansion for lattices

order ϵ^0 Base state

$$u_0 = 0$$

Stationary homogeneous solution
order $\epsilon^{1/2}$ Linear order

$$(1+\Delta)^2 u_{\frac{1}{2}} = 0$$

Solution superposition of stripes

$$u_{\frac{1}{2}} = \sum_{n=1}^{3} A_n(T)\mathrm{e}^{i\mathbf{q}_n\cdot\mathbf{r}} + c.c.$$

order ϵ

$$(1+\Delta)^2 u_1 = 0$$

Solution

$$u_1 = \sum_{n=1}^{3} B_n(T)\mathrm{e}^{i\mathbf{q}_n\cdot\mathbf{r}} + c.c.$$

Amplitude equation for 2D lattices

order $\epsilon^{3/2}$ Cubic order

$$(1+\Delta)^2 u_{\frac{3}{2}} = -\partial_T u_{\frac{1}{2}} + u_{\frac{1}{2}} + \alpha_1 u_{\frac{1}{2}}^2 - u_{\frac{1}{2}}^3$$

3 Resonant terms (Fredholm) along $\mathbf{q}_n$, $n = 1, 2, 3$
Case of $\mathbf{q}_1$ resonant contributions:

$$\begin{aligned}
\partial_T u_{\frac{1}{2}} &\rightarrow \partial_T A_1 \\
u_{\frac{1}{2}} &\rightarrow A_1 \\
\alpha_1 u_{\frac{1}{2}}^2 &\rightarrow 2\alpha_1 A_2^* A_3^* \\
u_{\frac{1}{2}}^3 &\rightarrow 3A_1^2 A_1^* + 6A_1(A_2 A_2^* + A_3 A_3^*)
\end{aligned}$$

Collecting the terms, amplitude equation

$$\begin{aligned}
\partial_T A_1 &= A_1 + 2\alpha_1 A_2^* A_3^* - 3A_1|A_1|^2 - 6A_1(|A_2|^2 + |A_3|^2) \\
\partial_T A_2 &= A_2 + 2\alpha_1 A_1^* A_3^* - 3A_2|A_2|^2 - 6A_2(|A_1|^2 + |A_3|^2) \\
\partial_T A_3 &= A_3 + 2\alpha_1 A_2^* A_1^* - 3A_3|A_3|^2 - 6A_3(|A_2|^2 + |A_1|^2)
\end{aligned}$$

Remarks

- $A_1(|A_2|^2 + |A_3|^2)$ cubic coupling terms
- Hexagons only : $\alpha_1 A_2^* A_3^*$ quadratic coupling terms
  break the $u \rightarrow -u$ symmetry

Amplitude equation for 2D lattices

normalized Amplitude equations for lattices non-hexagonal

$$\partial_T A_1 = A_1 - A_1|A_1|^2 - \nu A_1|A_2|^2$$
$$\partial_T A_2 = A_2 - A_2|A_2|^2 - \nu A_2|A_1|^2$$

normalized Amplitude equations for hexagons

$$\partial_T A_1 = A_1 + \gamma_1 A_2^* A_3^* - A_1|A_1|^2 - \nu A_1(|A_2|^2 + |A_3|^2)$$
$$\partial_T A_2 = A_2 + \gamma_1 A_1^* A_3^* - A_2|A_2|^2 - \nu A_2(|A_1|^2 + |A_3|^2)$$
$$\partial_T A_3 = A_3 + \gamma_1 A_2^* A_1^* - A_3|A_3|^2 - \nu A_3(|A_2|^2 + |A_1|^2)$$

$\gamma_1 > 0$ (if not $A \to -A$)

- Correspondence with Swift-Hohenberg:
  $A_n \to A_n/3^{1/2}$
  $\gamma_1 = 2\alpha_1 3^{1/2}$
  $\nu = 2$
- In general angle-dependence $\nu(\phi)$ with ϕ angle between stripes
  $\nu(\phi) = \nu(\pi - \phi)$
  Isotropy (non-chiral) $\nu(\phi) = \nu(-\phi)$

Contents

Lattices non-hexagonal

Amplitude equation

$$\partial_T A_1 = A_1 - A_1|A_1|^2 - \nu A_1|A_2|^2$$
$$\partial_T A_2 = A_2 - A_2|A_2|^2 - \nu A_2|A_1|^2$$

Non-scaled: $a_n = \epsilon^{1/2} A_n$, and $\gamma = \epsilon^{1/2}\gamma_1$

$$\partial_t a_1 = \epsilon a_1 - a_1|a_1|^2 - \nu a_1|a_2|^2$$
$$\partial_t a_2 = \epsilon a_2 - a_2|a_2|^2 - \nu a_2|a_1|^2$$

$a_n = \rho_n \mathrm{e}^{\imath\theta_n}$

$$\begin{aligned}
\partial_t \rho_1 &= \epsilon\rho_1 - \rho_1[\rho_1^2 + \nu\rho_2^2] \\
\partial_t \theta_1 &= 0 \\
\partial_t \rho_2 &= \epsilon\rho_2 - \rho_2[\rho_2^2 + \nu\rho_1^2] \\
\partial_t \theta_2 &= 0
\end{aligned}$$

Frozen phases θ_1, θ_2

Stability of Stripes

$$\partial_t \rho_1 = \epsilon \rho_1 - \rho_1[\rho_1^2 + \nu \rho_2^2]$$
$$\partial_t \rho_2 = \epsilon \rho_2 - \rho_2[\rho_2^2 + \nu \rho_1^2]$$

Stripe base state :

$$\rho_{10} = \epsilon^{1/2}, \qquad \rho_{20} = 0$$

Linear stability analysis $\rho_n = \rho_{n0} + \delta\rho_n$

$$\partial_t \delta\rho_1 = -2\epsilon\delta\rho_1$$
$$\partial_t \delta\rho_2 = (1-\nu)\epsilon\delta\rho_2$$

$\delta\rho_1$ stable
$\delta\rho_2$: stable if $\nu > 1$, unstable if $\nu < 1$

Stability of lattices non-hexagons

$$\partial_t \rho_1 = \epsilon\rho_1 - \rho_1[\rho_1^2 + \nu\rho_2^2]$$
$$\partial_t \rho_2 = \epsilon\rho_2 - \rho_2[\rho_2^2 + \nu\rho_1^2]$$

Base lattice state:

$$\rho_{10} = \rho_{20} = \rho_0$$
$$\rho_0^2 = \frac{\epsilon}{1+\nu}$$

Supercritical solution for $\nu > -1$
Linear perturbations $\rho_n = \rho_{n0} + \delta\rho_n$
Linearized equations

$$\partial_t \delta\rho_1 = -2\frac{\epsilon}{1+\nu}\delta\rho_1 - 2\frac{\epsilon\nu}{1+\nu}\delta\rho_2$$
$$\partial_t \delta\rho_2 = -2\frac{\epsilon}{1+\nu}\delta\rho_2 - 2\frac{\epsilon\nu}{1+\nu}\delta\rho_1$$

- sum mode

$$\rho_\Sigma = \delta\rho_1 + \delta\rho_2$$
$$\partial_t \rho_\Sigma = -2\epsilon\rho_\Sigma$$
$$\sigma_\Sigma = -2\epsilon$$

  sum mode is stable

- difference mode

$$\rho_\Delta = \delta\rho_1 - \delta\rho_2$$
$$\partial_t \rho_\Delta = -2\epsilon\frac{1-\nu}{1+\nu}\rho_\Delta$$
$$\sigma_\Delta = -2\epsilon\frac{1-\nu}{1+\nu}$$

  diff mode: $\nu < 1$ stable, $\nu > 1$ unstable

Summary: $\nu < 1 \rightarrow$ Lattices and $\nu > 1 \rightarrow$ stripes

Recall: Swift-Hohenberg $\nu = 2$

Amplitude equation for hexagonal patterns

Amplitude equation

$$\partial_T A_1 = A_1 + \gamma_1 A_2^* A_3^* - A_1|A_1|^2 - \nu A_1(|A_2|^2 + |A_3|^2)$$

$\gamma > 0$ (if not $A \to -A$)
Non-scaled: $a_n = \epsilon^{1/2} A_n$, and $\gamma = \epsilon^{1/2}\gamma_1$

$$\partial_t a_1 = \epsilon a_1 + \gamma a_2^* a_3^* - a_1|a_1|^2 - \nu a_1(|a_2|^2 + |a_3|^2)$$

$a_n = \rho_n \mathrm{e}^{\imath\theta_n}$

$$\partial_t \rho_1 = \epsilon\rho_1 + \gamma\rho_2\rho_3\cos(\psi) - \rho_1[\rho_1^2 + \nu(\rho_2^2 + \rho_3^2)]$$
$$\partial_t \theta_1 = -\gamma\frac{\rho_2\rho_3}{\rho_1}\sin(\psi)$$

$\psi = \theta_1 + \theta_2 + \theta_3$

Phase selection

Phase relaxation:
$\psi = \theta_1 + \theta_2 + \theta_3$

$$\partial_t \psi = -p \sin(\psi), \qquad p = \gamma \left(\frac{\rho_2 \rho_3}{\rho_1} + \frac{\rho_1 \rho_3}{\rho_2} + \frac{\rho_1 \rho_2}{\rho_3} \right) > 0$$

Stable Steady-states: $\psi = 0$ and Unstable steady-state $\psi = \pi$
stable steady-state $\psi = 0$
$\rightarrow \partial_t \theta_n = 0$

- one can choose the origin such that $\theta_1 = \theta_2 = 0$ then additional relation
  $\psi = \theta_1 + \theta_2 + \theta_3 = 0$
  implies $\theta_3 = 0$
  sign of γ selects one Hexagonal lattice (dots)

Stability of Stripes

Stripe base state:

$$\rho_{10} = \epsilon^{1/2}, \qquad \rho_{20} = 0, \qquad \rho_{30} = 0$$

Linear stability analysis $\rho_n = \rho_{n0} + \delta\rho_n$

$$\partial_t \delta\rho_1 = -2\epsilon\delta\rho_1$$
$$\partial_t \delta\rho_2 = (1-\nu)\epsilon\delta\rho_2 + \gamma\epsilon^{1/2}\delta\rho_3$$
$$\partial_t \delta\rho_3 = (1-\nu)\epsilon\delta\rho_3 + \gamma\epsilon^{1/2}\delta\rho_2$$

$\delta\rho_1$ stable

Sum mode $\rho_\Sigma = \delta\rho_2 + \delta\rho_3$,
Difference mode $\rho_\Delta = \delta\rho_2 - \delta\rho_3$

$$\partial_t \rho_\Sigma = [(1-\nu)\epsilon + \gamma\epsilon^{1/2}]\rho_\Sigma$$
$$\partial_t \rho_\Delta = [(1-\nu)\epsilon - \gamma\epsilon^{1/2}]\rho_\Delta$$

Necessary condition for stability $\Rightarrow \nu > 1$
Strongest condition on sum $(1-\nu)\epsilon + \gamma\epsilon^{1/2} < 0$

Two conditions for stability:

$$\nu > 1$$
$$\epsilon > \epsilon_S = \frac{\gamma^2}{(\nu-1)^2}$$

Since we want to consider competition between stripes and hexagons, we will assume $\nu > 1$ in the following.

Hexagon Steady-states

Base Hexagon state:

$$\rho_{10} = \rho_{20} = \rho_{30} = \rho_0$$
$$\epsilon\rho_0 + \gamma\rho_0^2 - (1+2\nu)\rho_0^3 = 0$$

Trivial solutions

$$\rho_0 = 0$$

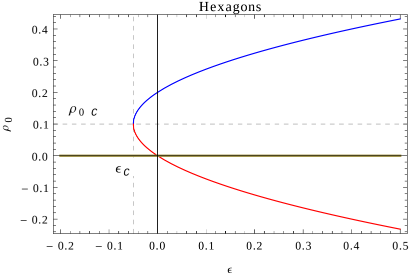
Non-trivial solutions

$$\epsilon = \rho_0[(1+2\nu)\rho_0 - \gamma]$$

$$\rho_{0\pm} = \frac{\gamma}{2(1+2\nu)} \pm \left[\left(\frac{\gamma}{2(1+2\nu)}\right)^2 + \frac{\epsilon}{1+2\nu}\right]^{1/2}$$

- Only consider "Supercritical" case $\nu > -1/2$
- Meaniful solution $\rho_{0\pm} \geq 0$

$(\nu = 2,\ \gamma = 1)$



Critical values

$$\epsilon_c = -\frac{\gamma^2}{4(1+2\nu)}$$
$$\rho_{0C} = \frac{\gamma}{2(1+2\nu)}$$

Stability of Hexagon

Assume $\nu > 1$ (case of Swift-Hohenberg Eq.)

Linear stability analysis of Hexagons

$\rho_n = \rho_0 + \delta\rho_n$

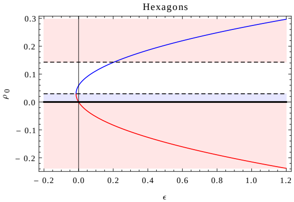
$$\partial_t \delta\rho_1 = [\epsilon - (3+2\nu)\rho_0^2]\delta\rho_1 + (\gamma\rho_0 - 2\nu\rho_0^2)(\delta\rho_2 + \delta\rho_3)$$

difference mode

$$\rho_\Delta = \delta\rho_1 - \delta\rho_2$$
$$\partial_t \rho_\Delta = [\epsilon - 3\rho_0^2 - \gamma\rho_0]\rho_\Delta$$
$$\sigma_\Delta = \epsilon - 3\rho_0^2 - \gamma\rho_0$$

sum mode

$$\rho_\Sigma = \delta\rho_1 + \delta\rho_2 + \delta\rho_3$$
$$\partial_t \rho_\Sigma = [\epsilon - 3(1+2\nu)\rho_0^2 + 2\gamma\rho_0]\rho_\Sigma$$
$$\sigma_\Sigma = \epsilon - 3(1+2\nu)\rho_0^2 + 2\gamma\rho_0$$



Replacing $\epsilon = \rho_0[(1+2\nu)\rho_0 - \gamma]$

$$\sigma_\Delta = [2(\nu-1)\rho_0 - 2\gamma]\rho_0$$
$$= 2(\nu-1)[\rho_0 - \rho_{0H}]\rho_0$$

$\rightarrow$ diff. mode unstable in pink zone

$$\sigma_\Sigma = [\gamma - 2(1+2\nu)\rho_0]\rho_0$$
$$= 2(1+2\nu)[\rho_{0C} - \rho_0]\rho_0$$

$\rightarrow$ sum mode unstable in blue zone

$$\rho_{0H} = \frac{\gamma}{\nu - 1}$$

Assume $\nu > 1$ then $\rho_{0c} < \rho_0 < \rho_{0H}$

Stripes vs Hexagons

Assume $\nu > 1$
Summary for hexagons:
Hexagon steady-state branches for

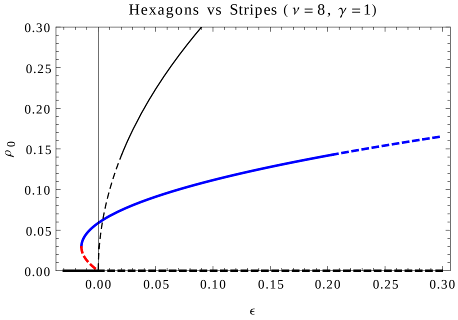
$$\epsilon_c = -\frac{\gamma^2}{4(1+2\nu)}$$

Hexagons stable in upper branch with

$$\epsilon < \epsilon_H = \gamma^2 \frac{2+\nu}{(\nu-1)^2}$$

Stripe stability:

$$\epsilon > \epsilon_S = \frac{\gamma^2}{(\nu-1)^2}$$



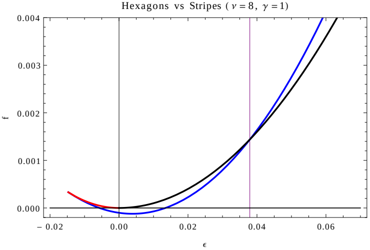
Lyapunov functional

Lyapunov functional

$$\mathcal{F} = \int \mathrm{d}X \int \mathrm{d}Y\ f$$

$$f = \frac{1}{2}\sum_{n=1}^{3}(\epsilon - |a_n|^2)^2 + \frac{\nu}{2}\sum_{n\neq m}|a_n|^2|a_m|^2$$

$$-\gamma\prod_{n=1}^{3}a_n - \gamma\prod_{n=1}^{3}a_n^*$$



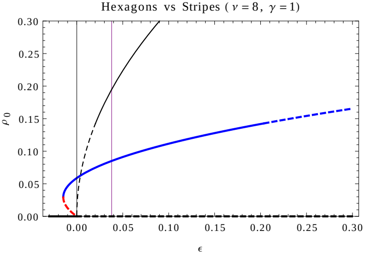
Lyapunov functional density,
Stripes and Hexagons

$$f_S = \epsilon^2$$

$$f_H = \frac{3}{2}(\epsilon - \rho_0^2)^2 + 3\nu\rho_0^4 - 2\gamma\rho_0^3$$

with $\epsilon = \rho_0((1+2\nu)\rho_0 - \gamma)$
Coexistence:

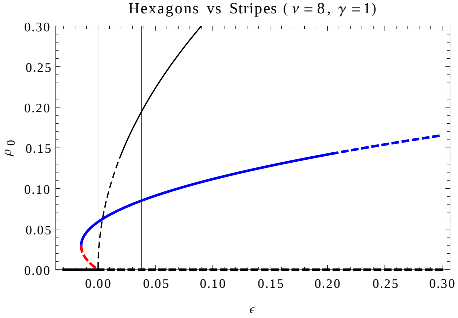
$$\epsilon_P = \frac{1}{[2(1+\nu)]^{3/2} - 2(1+3\nu)}$$



Summary

Summary

- ν too negative $\rightarrow$ subcritical ($\nu < -1$ for non-hex, $\nu < -1/2$ for hex.)
- $\nu < 1$ lattices stable, stripes unstable
- $\nu > 1$ Hexagons and Stripes compete

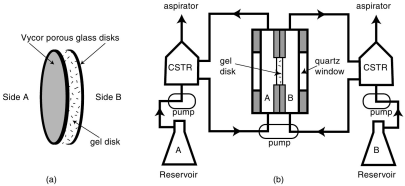


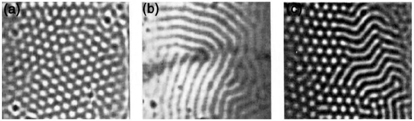
Experiments

Ouyang and Swinney (1992)

CIMA Reaction

- chlorite, iodide, malonic acid, starch
- Turing patterns





- wavelength ~ 0.2mm
- Malonic acid: 21 mM in (a), (b)
  Multistability
- Malonic acid: 14 mM in (c)
  Hexagons and Stripes
  Hexagons invade, 2 lattice sites/day

Lecture 2: Nonlinear Physics and Instabilities

Summary

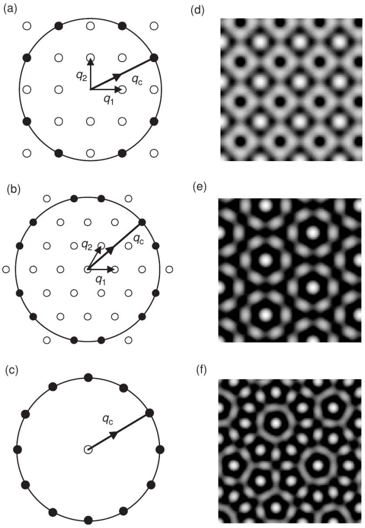
- Multi-scale expansion
  - l-s: slow and fast space variables (x, X and T)
  - complex amplitude A
  - Fredholm alternative
- Amplitude equations l-s: RGL equation
  - Lyapunov functional
  - Nonlinear steady-states
- Instab. of steady-states: *secondary instabilities*
  - Phase instability (Eckhaus, Zigzag), vs amplitude instability (Stripes, Lattices)
  - Multistability (Stripes-vs-Hexagons)

2D lattices

superlattices and ...

... a tribute to Roger Penrose
(Physics Nobel Prize 2020)

quasi-periodic patterns!



Cross and Greenside p.157