On the dynamics of the controlled Allen-Cahn equation

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We wish to describe the basic dynamical properties of the controlled Allen Cahn equation.

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I. DERIVATION OF THE CAC EQUATION

A. Definition of the problem

Le us consider an extended dynamical system with a scalar field $u(\mathbf{x},t)$ that depends on space coordinates $\mathbf{x}$ and time t. We assume that the system is isotropic and homogeneous in space, that the dynamics does not depend on time, and obeys field-inversion symmetry $u \to -u$. We then assume that the system undergoes a long-wavelength type-III instability upon the variation of a physical parameter c above a critical value c_c. Our goal is to describe the dynamical behavior of this system under a slow space-independent variation of c around c_c.

Our first result is that a self-consitant variation is that such system obeys the controlled-Allen-Cahn equation (or controlled time-dependent Ginzburg Landau equation), which reads in normalized variables

$$\partial_T U(\mathbf{X},T) = C(T)U(\mathbf{X},T) + \Delta U(\mathbf{X},T) - U(\mathbf{X},T)^3 \tag{1}$$

where capital letters indicate normalized variables, and $C(T)$ is a bounded function of time.

II. COMPARISON PRINCIPLE

The form of Eq.(1) is a special case of more general equations [**? ?**] that obey a comparison principle. As a consequence of this theorem, two solutions $u_+(\mathbf{x},t)$ and $u_-(\mathbf{x},t)$ of Eq.(1) that do not cross at $t=0$, i.e., that are such that $u_+(\mathbf{x},0) > u_-(\mathbf{x},0)$ for all x, will not cross later at $t=t_0>0$, i.e. $u_+(\mathbf{x},t_0) > u_-(\mathbf{x},t_0)$.

Intuitively, this result can be seen by taking the difference between the equation obeyed by u_+ and u_-. Defining $v = u_+ - u_-$, we have

$$\partial_t v = v[C - (u_+^2 + u_+u_- + u_-^2)] + \Delta v \tag{2}$$

Assume that $v>0$. Then, if $v(\mathbf{x}_0,t) \to 0$ around a point $\mathbf{x}_0$, the first term in the r.h.s. of Eq.(2) vanishes. Then, since we approach the axis $v=0$ from above, we expect that v is a convex function in the vicinity of $\mathbf{x}_0$, i.e., $\Delta v > 0$. As a consequence, we expect $\partial_t v(\mathbf{x}_0,t) > 0$. This argument does not hold rigorously because Δv could vanish for a convex function. However, the strong maximum principle [**?**] garantees that this statement is actually true.

As a first consequence, since $u_-(\mathbf{x},t)=0$ is a solution of Eq.(1), new zeros cannot appear via the the crossing of the minimum of u_+ and the $u=0$ axis. However, the opposite is true, as can be seen from simulations in Fig. XXXX. Indeed, when a minimum of u between two zeros crosses the axis, then the first term of the r.h.s. of Eq.(2) vanishes but the Laplacian is still positive so that $\partial_t u > 0$. Hence, the number of zeros can decrease, but not increase. This does not prove that the number of minimums will decrease. However, we will see below that is actually decreases in the course of time, giving rise to a coarsening process.

III. LINEAR DYNAMICS

We define the mean square wavenember as

$$\langle q^2\rangle(t) = \frac{1}{D}\frac{\int \mathrm{d}\mathbf{x}(\nabla u(x,t))^2}{\int \mathrm{d}\mathbf{x}\, u(x,t)^2} = \frac{1}{D}\frac{\int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^D}\, q^2|u_\mathbf{q}(t)|^2}{\int \frac{\mathrm{d}q}{(2\pi)^D}\,|u_\mathbf{q}(t)|^2} \tag{3}$$

and the related wavelength $\lambda = 2\pi/\langle q^2\rangle(t)^{1/2}$.

When $u(x,t)$ is small, the nonlinear term u^3 in Eq.(1) is negligible and we have

$$\partial_t u(x,t) = C(t)u(x,t) + \Delta u(x,t)\,. \tag{4}$$

As a consequence

$$\partial_t u_\mathbf{q}(t) = C(t)u_\mathbf{q}(t) - q^2 u_\mathbf{q}(t)\,, \tag{5}$$

which is solved as

$$u_\mathbf{q}(t) = u_\mathbf{q}(0)\mathrm{e}^{\int_0^t \mathrm{d}t'[C(t')-q^2]}\,. \tag{6}$$

Interestingly, since $C(t)$ does not depend on space, the resulting MSW does not depend on $C(t)$:

$$\langle q^2\rangle(t) = \frac{\int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^D}\, q^2|u_\mathbf{q}(0)|^2\mathrm{e}^{-2q^2t}}{D\int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^D}\,|u_\mathbf{q}(0)|^2\mathrm{e}^{-2q^2t}} \tag{7}$$

In the case of an initial flat (white) spectrum with a cutoff of at q_{min}, i.e. $-q_{min} \le q_i \le q_{min}$ for any component q_i of $\mathbf{q}$, we have

$$|u_\mathbf{q}(0)|^2 = \Theta(q<q_{min})\left(\frac{\pi L}{q_{min}}\right)^D W^2(0)\,, \tag{8}$$

where Θ is the Heaviside function and the amplitude is defined as a function of the initial square roughness $W^2(0)$. At all times, the mean square roughness reads

$$W^2(t) = \frac{1}{L^D}\int \mathrm{d}\mathbf{x}\, u(\mathbf{x},t)^2 = \frac{1}{L^D}\int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^D}\,|u_\mathbf{q}(t)|^2\,. \tag{9}$$

Using eq. (8) in eq. (7), we obtain a result that is independent of the dimension D

$$\langle q^2\rangle(t) = \frac{1}{4t}\left\{1 - q_{min}\frac{2^{3/2}}{\pi^{1/2}}t^{1/2}\frac{\mathrm{e}^{-2tq_{min}^2}}{\mathrm{erf}[q_{min}(2t)^{1/2}]}\right\} \tag{10}$$

From a Taylor expansion at short times, we obtain

$$\langle q^2\rangle(t) = \frac{q_{min}^2}{3}\left(1 - \frac{8}{15}q_{min}^2 t\right) \tag{11}$$

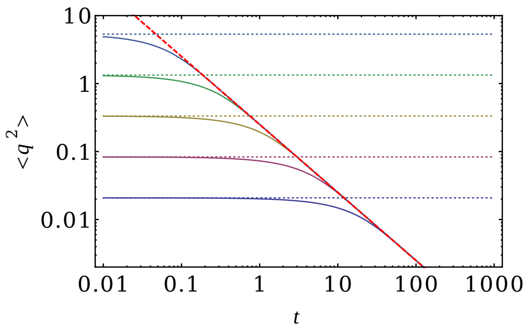

Figure 1. Evolution of $\langle q^2 \rangle$ from initial white noise with cutoff from the linear approximation with various values of $q_c = 10^n$ with $n = -2, -1, 0, 1, 2$. The dahsed line is $1/(4t)$ and the dotted lines correspond to $q_c^2/3$.

leading to

$$q_{min}^2 = 3\langle q^2 \rangle(0) \tag{12}$$

In contrast for large times

$$\langle q^2 \rangle(t) = \frac{1}{4t} \tag{13}$$

which leads to the scaling law

$$\lambda(t) = 4\pi t^{1/2} \tag{14}$$

Appendix A: Derivation of eq. (10)

We assume that $|u_{\mathbf{q}}(0)|^2 = b$ for $|q_i| < q_{min}$, and $|u_{\mathbf{q}}(0)|^2 = 0$ for $|q_i| > q_{min}$, where q_i are the components of $\mathbf{q}$. Then, starting with eq. (7), we have

$$
\begin{aligned}
\langle q^2\rangle(t) &= \frac{\int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^D}\, q^2 |u_{\mathbf{q}}(0)|^2 \mathrm{e}^{-2q^2 t}}{D\int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^D}\, |u_{\mathbf{q}}(0)|^2 \mathrm{e}^{-2q^2 t}} \\
&= -\frac{1}{2D}\partial_t \ln \int \frac{\mathrm{d}\mathbf{q}}{(2\pi)^D}\, |u_{\mathbf{q}}(0)|^2 \mathrm{e}^{-2q^2 t} \\
&= -\frac{1}{2D}\partial_t \ln \left\{ b \prod_{i=1}^{D} \int_{-q_{min}}^{q_{min}} \frac{\mathrm{d}q_i}{2\pi} \mathrm{e}^{-2q_i^2 t} \right\} \\
&= -\frac{1}{2D}\partial_t \left\{ \ln b + \sum_{i=1}^{D} \ln \int_{-q_{min}}^{q_{min}} \frac{\mathrm{d}q_i}{2\pi} \mathrm{e}^{-2q_i^2 t} \right\} \\
&= -\frac{1}{2D}\partial_t \sum_{i=1}^{D} \ln \left\{ \int_{-q_{min}}^{q_{min}} \frac{\mathrm{d}q_i}{2\pi} \mathrm{e}^{-2q_i^2 t} \right\} \\
&= -\frac{1}{2D}\partial_t \sum_{i=1}^{D} \ln \left\{ \int_{-q_{min}}^{q_{min}} \frac{\mathrm{d}q_i}{2\pi} \mathrm{e}^{-2q_i^2 t} \right\} \\
&= -\frac{1}{2}\partial_t \ln \left\{ \frac{1}{(2t)^{1/2}} \int_{-q_{min}(2t)^{1/2}}^{q_{min}(2t)^{1/2}} \frac{\mathrm{d}p}{2\pi} \mathrm{e}^{-p^2} \right\} \\
&= -\frac{1}{2}\partial_t \ln \left\{ \frac{1}{(2t)^{1/2}} \frac{\mathrm{erf}[q_{min}(2t)^{1/2}]}{2\pi} \right\} \\
&= \frac{1}{2} \left\{ \frac{1}{2t} - \frac{\partial_t \mathrm{erf}[q_{min}(2t)^{1/2}]}{\mathrm{erf}[q_{min}(2t)^{1/2}]} \right\} \\
&= \frac{1}{2} \left\{ \frac{1}{2t} - q_{min} \left(\frac{2}{\pi t}\right)^{1/2} \frac{\mathrm{e}^{-2tq_{min}^2}}{\mathrm{erf}[q_{min}(2t)^{1/2}]} \right\} \\
&= \frac{1}{4t} \left\{ 1 - q_{min} \frac{2^{3/2}}{\pi^{1/2}} t^{1/2} \frac{\mathrm{e}^{-2tq_{min}^2}}{\mathrm{erf}[q_{min}(2t)^{1/2}]} \right\} \qquad \text{(A1)}
\end{aligned}
$$