Explaining the decay of the distance of two kinks with linear dynamics and a sum of Gaussians

#twokinks #1D #linear_regime

Motivation

If $C(t) < 0$ for a long time (as it happens in the cases above) eventually $u(x) \ll C_0$ and we expect the non-linearity to play a negligible role in the dynamics. So it is **natural** to expect that **the LINEAR dynamics is SUFFICIENT** to describe the **steps** that we see in the decay of the distance.
Here we will approximate the kink's shape with an **Erf function**, in order to proceed analytically. The main difference between the Erf and the tanh is that the tails decay as $\sim e^{-x^2}$ and e^{-x} respectively.

Estimating the duration of a step

Here I present a way for estimating the duration of a step, that is **independent on the model** adopted to predict the decay $d(t)$ along a step.

At what time t_0 the step originates?

The **beginning of the step** corresponds **empirically** with the moment t_0 **when** $C(t_0) = 0; \dot{C}(t_0) < 0$ (when $C(t)$ becomes negative). For a quench experiment, then $t_0 = 0$.

Duration of a step

Assuming we can neglect the non-linearity **from** the moment **when** $C(t)$ **becomes negative** (this is supported by the fact that we see a good fit if we start to compare model and simulation from $t_0 : C(t_0) = 0$ and $\dot{C}(t_0) < 0$). Then

$$u_{q=0}(t) = u_{q=0}(t_0)e^{B(t)} \quad \text{if } t > t_0$$

$$B(t) = \int_{t_0}^{t} dt' C(t')$$

so $u_{q=0}(t)$ initially decreases, but then it increases again, becoming bigger than the initial value $u_{q=0}(t_0)$ and then the non-linearity in **no more negligible**.
As a consequence, we estimate the time when the decay finishes t_f as the time when $u_{q=0}(t) = u_{q=0}(t_0)$ **again** so

$$B(t_f) = B(t_0) = 0$$

$$\int_{t_0}^{t_f} dt' C(t') = 0$$

$$C(t') = \bar{C}[1 + \frac{A}{\bar{C}} \sin\left(\frac{2\pi t'}{T}\right)]$$

changing variable $\tau = \frac{t}{T}$ and integrating, we find

$$2\pi(\tau_f - \tau_0) = \frac{A}{\bar{C}}[\cos(2\pi\tau_f) - \cos(2\pi\tau_0)]$$

$$t_f = \tau_f T \quad t_0 = \tau_0 T \qquad \tau_0 = \frac{1}{2}\left[1 - \frac{1}{\pi}\arcsin\left(-\frac{\bar{C}}{A}\right)\right]$$

this means that τ_f, τ_0 do not depend on T and so the **duration of the decay** (step) is

$$t_f - t_0 \propto T$$

in general

$$t_f - t_0 = f\left(\frac{\bar{C}}{A}\right)T$$

so it **does NOT depend on the initial distance**, as we see in simulations. The duration of the last step will follow a different rule, as the collapse time t_c predicted with the model of the two gaussians will be smaller than $t_f - t_0$.

Notice: This estimate of the duration of the step does not depend on the model adopted to compute the decay $d(t)$.

Expansion of the step's duration for large amplitude

We define

$$2\pi\frac{\bar{C}}{A} = \epsilon \ll 1$$

Then the equation for τ_f is

$$\epsilon(\tau_f - \tau_0) = \cos(2\pi\tau_f) - \cos(2\pi\tau_0)$$

$$\tau_0 = \frac{1}{2}\left[1 - \frac{1}{\pi} - \arcsin\left(-\frac{\epsilon}{2\pi}\right)\right] \simeq \frac{1}{2} + \frac{\epsilon}{(2\pi)^2} + O(\epsilon^2)$$

As $\tau_f \to \frac{3}{2}$ in the limit $\epsilon \to 0$ (think that this limit is achieved by $\bar{C} \to 0$ with A finite. In this limit $\tau_0 = \frac{1}{2}$ and $\tau_f = \tau_0 + 1$) we estimate τ_f by expanding $\cos(2\pi\tau_f)$ close to $\tau_f \simeq \frac{3}{2}$

$$\cos(2\pi\tau_f) \simeq -1 + \frac{(2\pi)^2}{2}\left(\tau_f - \frac{3}{2}\right)^2 + \dots$$

using this in the first expression, and using

$$\sin(2\pi\tau_0) = -\frac{\epsilon}{2\pi} \implies \cos(2\pi\tau_0) = -\sqrt{1 - \sin^2} \simeq -\left(1 - \frac{\epsilon^2}{2(2\pi)^2} + O(\epsilon^4)\right)$$

along with the estimate of τ_0 written above, we find (neglecting $O(\epsilon^2)$)

$$\epsilon\left(\tau_f - \frac{1}{2}\right) \simeq \frac{(2\pi)^2}{2}\left(\tau_f - \frac{3}{2}\right)^2 + O(\epsilon^2)$$

considering the root $< \frac{3}{2}$ (as we expect τ_f to decrease if A increases with fixed $\bar{C}$)

$$\tau_f = \frac{3}{2} - \frac{\sqrt{2}}{2\pi}\epsilon^{1/2} + \frac{\epsilon}{(2\pi)^2} + O(\epsilon^2)$$

So, the estimated duration of the step is

$$\tau_f - \tau_0 \simeq 1 - \frac{\sqrt{2}}{2\pi}\epsilon^{1/2} + O(\epsilon^2)$$

remembering $\epsilon = \frac{2\pi\bar{C}}{A}$, then

$$t_f - t_0 = f\left(\frac{\bar{C}}{A}\right)T$$

$$f\left(\frac{\bar{C}}{A}\right) \simeq 1 - \sqrt{1/\pi}(\frac{\bar{C}}{A})^{1/2}$$

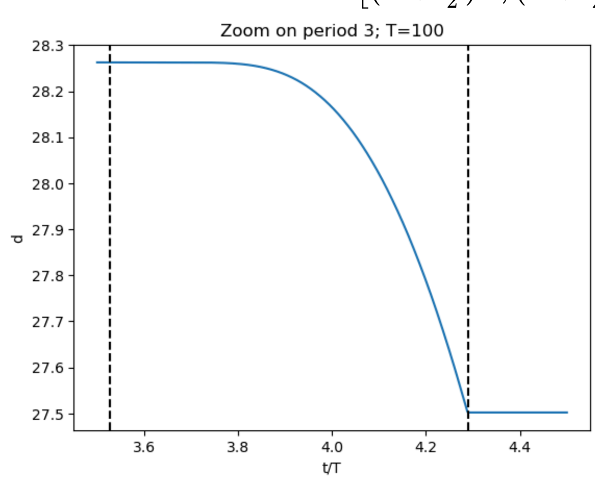
This means that, if $\frac{\bar{C}}{A} \ll \frac{1}{2\pi}$

$$\Delta t_{step} = t_f - t_0 = T\left(1 - 1/\sqrt{\pi}\left(\frac{\bar{C}}{A}\right)^{1/2}\right) \qquad \frac{\bar{C}}{A} \ll \frac{1}{2\pi}$$

Measuring the duration of the steps

I measure the duration of one step as $t_1 - t_0$ where

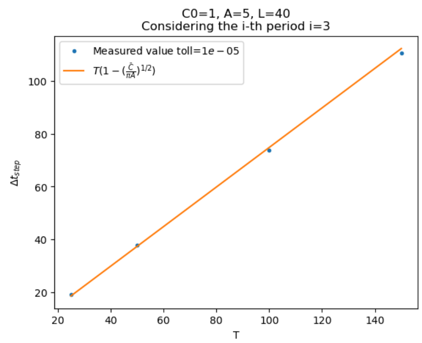
- I consider a time interval $\left[(n+\frac{1}{2})T, (n+\frac{1}{2}+1)T\right]$, so I can see the step



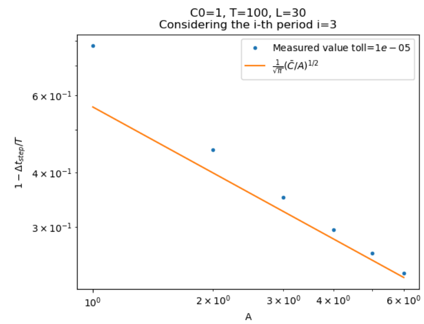
- t_0 is the instant when $C(t)$ becomes negative
- t_1 is estimated by strating from **the right** and lowerning t as soon as the derivative (numerical)

$$\frac{\delta d}{\delta t} > 10^{-5}$$

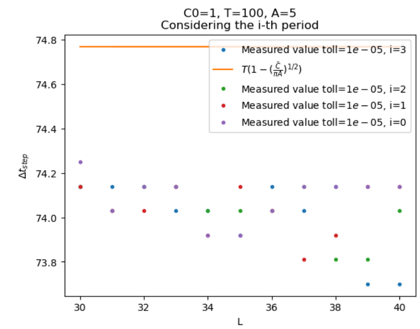
Linear dependence on T



Dependence on A



Dependence on L

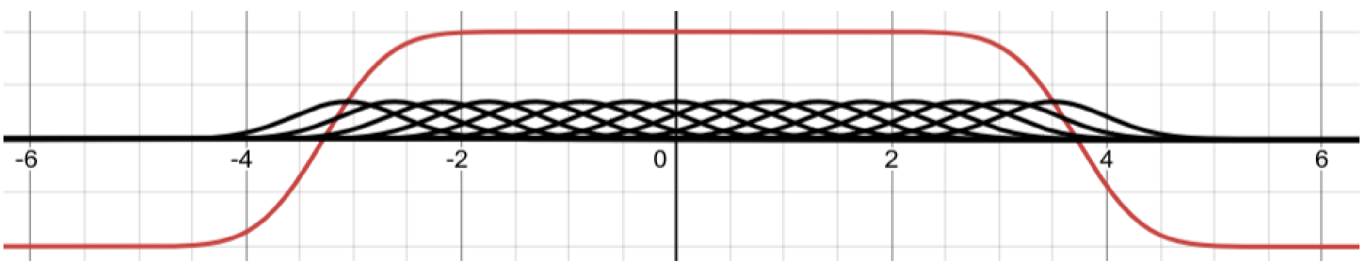


1D Model: Sum of INFINITE amount of Gaussians

Here we exploit the following results, about a sum of an infinite amount of Gaussian functions with the same σ

$$f(x) = \lim_{N\to\infty} \frac{2L}{N}\left(\sum_{n=1}^{N} g_n(x) - \frac{1}{2}\right) = \frac{1}{2}\left(\mathrm{erf}\left(\frac{x}{\sqrt{2}\sigma}\right) - \mathrm{erf}\left(\frac{x-L}{\sqrt{2}\sigma}\right) - 1\right)$$

$$g_n(x) = \frac{e^{-(x-nL/N)^2/2\sigma^2}}{\sqrt{2\pi}\sigma}$$



Proof

$$\frac{1}{\sigma\sqrt{2\pi}} \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \exp\left(-\frac{(x-\frac{i}{n})^2}{2\sigma^2}\right) = \frac{1}{\sigma\sqrt{2\pi}} \int_0^1 \exp\left(-\frac{(x-y)^2}{2\sigma^2}\right) dy =$$

$$= \frac{1}{2}\left(\mathrm{erf}\left(\frac{x}{\sqrt{2}\sigma}\right) - \mathrm{erf}\left(\frac{x-1}{\sqrt{2}\sigma}\right)\right)$$

Parameters of the model

The advantage of this profile is that the two parameters L, σ are **independent**!

- If $L \gg \sigma$, then $d \simeq L$ is the (initial) **distance** between kinks;
- while σ describes the **width** of the kinks $W = \sqrt{2}\sigma$.

Decay of the distance

Under **linear dynamics only**, each gaussian of the sum evoves trivially and so this profile has a trivial evolution at time $t > 0$:

$$\sigma_0^2 \to \sigma(t)^2 = \sigma_0^2 + 2t$$

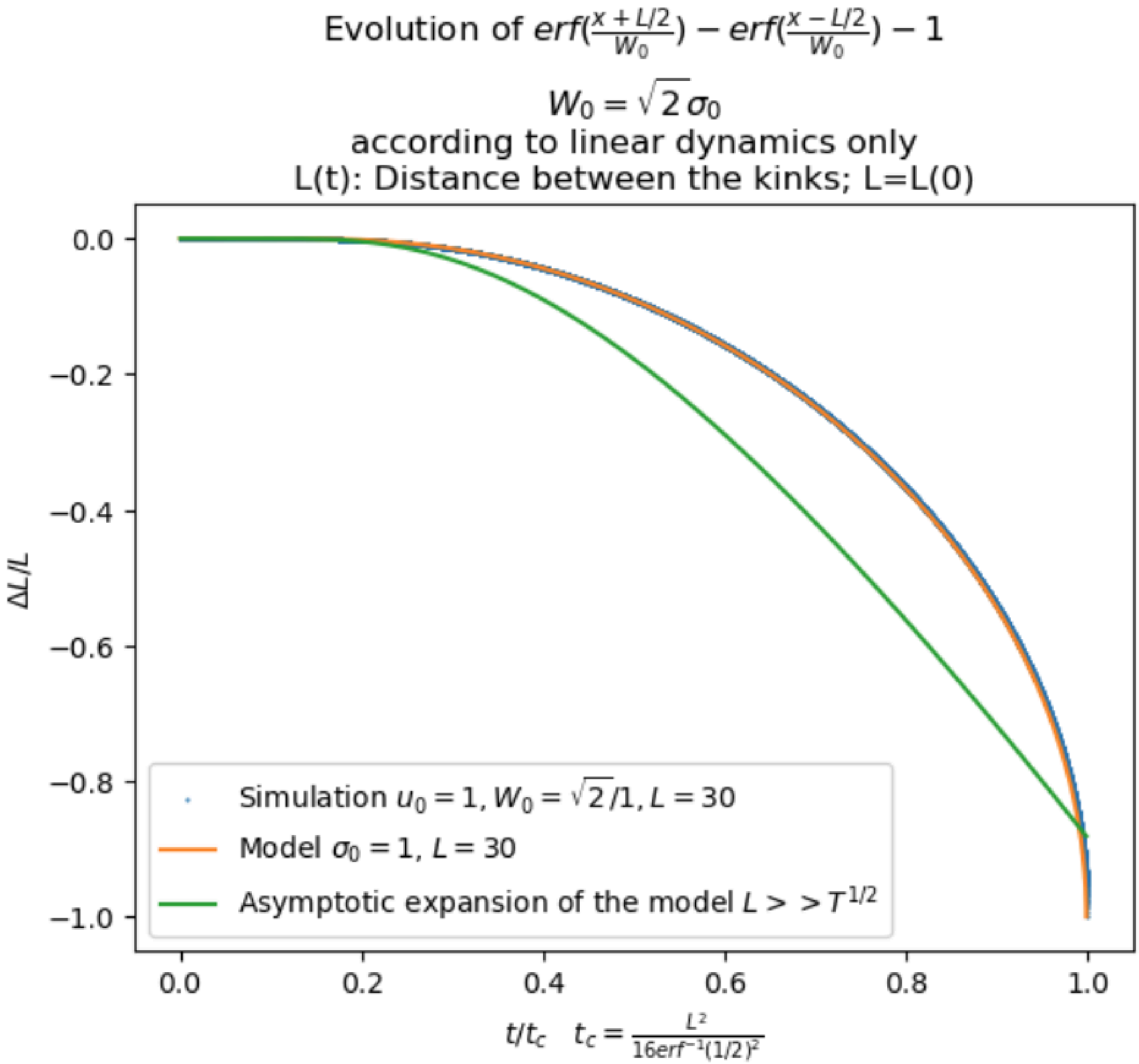
$$f(x) = \frac{1}{2}\left(\mathrm{erf}\left(\frac{x}{\sqrt{2}\sigma(t)}\right) - \mathrm{erf}\left(\frac{x-L}{\sqrt{2}\sigma(t)}\right) - 1\right)$$

To find the distance as a function of time, we look for the zeros of $f(x)$: $x^*_\pm$

$$d(t) = x^*_+(t) - x^*_-(t) = 2x^*$$

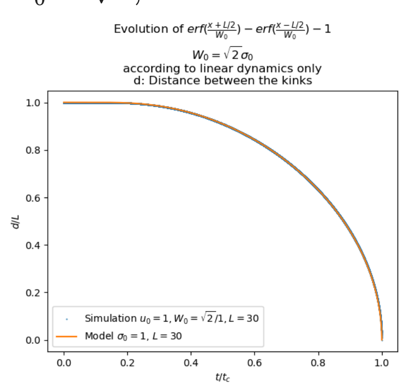
$$\mathrm{erf}\left(\frac{x^* + \frac{L}{2}}{\sqrt{2}\sigma(t)}\right) - \mathrm{erf}\left(\frac{x^* - \frac{L}{2}}{\sqrt{2}\sigma(t)}\right) = 1$$

Computing this with the **Newton's method** .

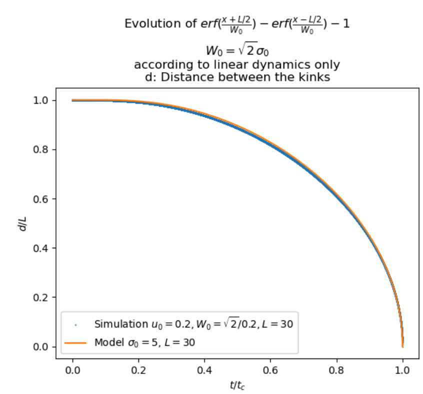


Simulations of **quench** experiments collapse on the same curve predicted by the model, except when there is overlap.

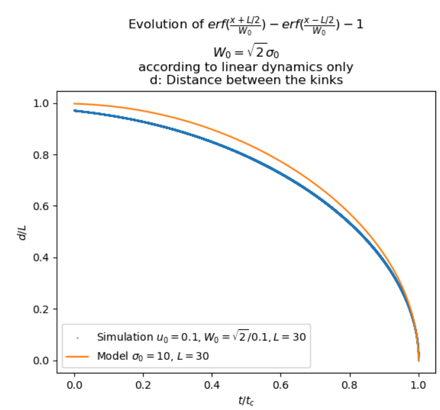
$W_0 = \sqrt{2}, L = 30$



$W_0 = 5\sqrt{2}, L = 30$



$W_0 = 10\sqrt{2}, L = 30$



Notice: The prediction of $d(t)$ is always decreasing, also at the beginning: as in simulations.

Inital distance d_0 and collapse time t_c

If in the **initial state** the kinks are overlapping ($2\sigma \sim L$), then L does not represent anymore the initial distance.
It is possible to calculate the deviation of d_0 from L by expanding the equation for $d(t)$ in powers of $\alpha = \frac{L}{\sigma_0} \gg 1$ and for $t = 0$:

$$d(t=0) = 2x^*(t=0) = \simeq L - \frac{4\sigma_0}{L} e^{-(\sigma_0/L)^2/2}$$

While the collapse time can be found by requiring $x^*(t_c) = 0$

$$t_c = \frac{L^2}{16\left(erf^{-1}\left(\frac{1}{2}\right)\right)^2} - \frac{\sigma_0^2}{2}$$

Rescaling

If there is no overlap ($L \gg 2\sigma_0$) in the initial state, the previous formulas simplify to

- $d_0 \simeq L$
- $t_c \simeq \frac{L^2}{16\left(erf^{-1}\left(\frac{1}{2}\right)\right)^2}$
  And rescaling $\Delta d(t) = d(t) - d_0$ and t as
- $\frac{\Delta d(t)}{L}$
- $\frac{t}{t_c}$
  we see that both simulations and numerical solutions $x^*(t)$ of

$$\text{erf}\left(\frac{x^* + \frac{L}{2}}{\sqrt{2}\sigma(t)}\right) - \text{erf}\left(\frac{x^* - \frac{L}{2}}{\sqrt{2}\sigma(t)}\right) = 1$$

collapse on the same curve, independently on the choice of σ_0, L (if $L \gg 2\sigma_0$).
It means that, if $L \gg 2\sigma_0$

$$\frac{\Delta d(t)}{L} = 1 - \frac{d(t)}{L} = f\left(\frac{t}{t_c(L)}\right)$$

so we can **predict** the **depth** of the step, knowing the duration of the step

$$\Delta d_{step} = Lf\left(\frac{\Delta t_{step}}{t_c(L)}\right)$$

where L is the distance at the beginning of the step.

Predicting the depth of the steps

If $C(t)$ is oscillating, we can use the last formula, along with the formula for estimating Δt_{step} to predict the depth of each step.

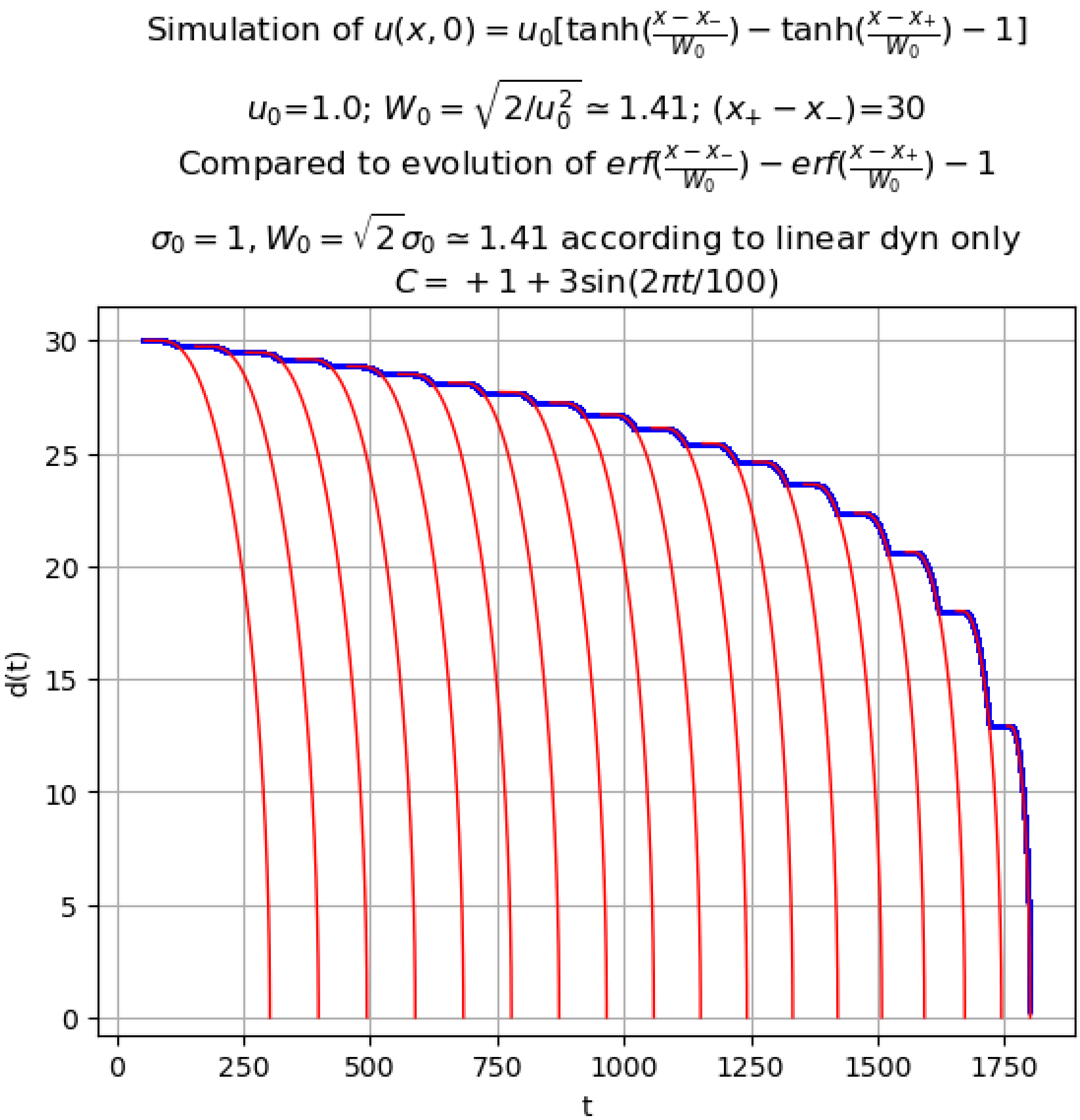
$$\Delta d_{step} = Lf\left(\frac{\Delta t_{step}}{t_c(L)}\right)$$

where

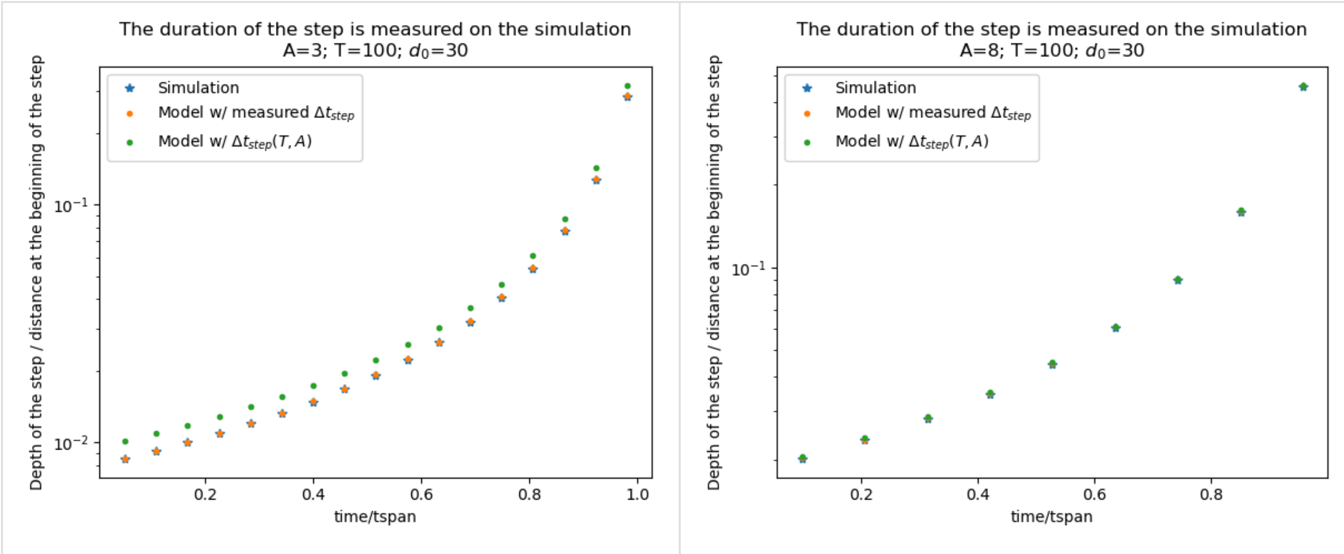
$$\Delta t_{step} = t_f - t_0 \simeq T\left(1 - 1/\sqrt{\pi}\left(\frac{\bar{C}}{A}\right)^{1/2}\right) \quad \frac{\bar{C}}{A} \ll \frac{1}{2\pi}$$

$$t_c(L) \simeq \frac{L^2}{16\left(erf^{-1}\left(\frac{1}{2}\right)\right)^2}$$

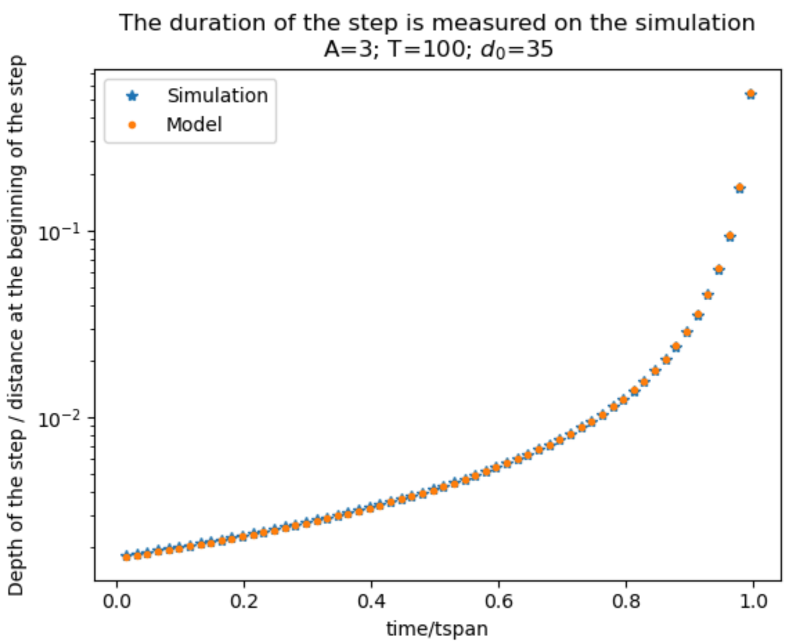
and the step starts when $C(t)$ becomes negative (while it finishes after Δt_{span}).



The estimation of Δt_{step} is **better when A is larger**, as we are using an asymptotic expansion in $A \gg \bar{C}$.



Also for the first steps it works really well.



Asymptotics for $L \gg T^{1/2}$

We can evaluate numerically $f(\xi)$, but we lack of an analytical expression. Although, it is possible to make an expansion of

$$\operatorname{erf}\left(\frac{x^*(t)+\frac{L}{2}}{\sqrt{2}\sigma(t)}\right)-\operatorname{erf}\left(\frac{x^*(t)-\frac{L}{2}}{\sqrt{2}\sigma(t)}\right)=1$$

around $x^*(t)=\frac{L}{2}$, remembering that $d(t)=2x(t)$

$$x^*=\frac{L}{2}(1-\epsilon) \quad 0<\epsilon\ll 1$$

about the time, we can write $\sigma(t) = \sigma_0(1+\tau)^{1/2}$ where $\tau = \frac{2t}{\sigma_0^2}$ (where $\sigma_0 \sim W_0 \sim \bar{C}^{-1/2}$).
We will be interested, to estimate the depth of a step, in evaluating $L(t)$ at $t = \Delta t_{step} \sim T$, so

$$\tau \sim T\bar{C}$$

and in the **slow oscillation limit** $T \gg \bar{C}$ and so $\tau \gg 1$ and $(1+\tau) \simeq \tau$.

$$erf(2\alpha) - erf(-\epsilon\alpha) = 1$$

where, evaluating $\tau = \Delta\tau_{step} = \frac{2\Delta t_{step}}{\sigma_0^2}$

$$\alpha = \frac{L}{4\Delta t_{step}^{1/2}} \gg 1 \quad \text{as } \Delta t_{step} \sim T \text{ and } L \gg T^{1/2}$$

we use the asymptotic expansion of erf for the first term and the taylor expansion at zero for the second.

$$\epsilon \simeq \frac{e^{-4\alpha^2}}{2\sqrt{\pi}\alpha^2}$$

To put back Δt_{step}, we define

$$\xi = \frac{\Delta t_{step}}{t_c(L)} \quad \gamma = (erf^{-1}\left(\frac{1}{2}\right))^2$$

and we recognize that

$$\alpha^2 = \frac{t_c\gamma}{\Delta t_{span}} = \frac{\gamma}{\xi}$$

then

$$\epsilon = \frac{\xi}{2\sqrt{\pi}\gamma} e^{-4\gamma/\xi} \equiv f(\xi)$$

and remembering $\frac{\Delta L(\Delta t_{step})}{L} = f\left(\frac{\Delta t_{step}}{t_c(L)}\right)$, we conclude that

$$\frac{\Delta L(\Delta t_{step})}{L} = f\left(\frac{\Delta t_{step}}{t_c(L)}\right) \simeq \frac{\frac{\Delta t_{step}}{t_c(L)}}{2\sqrt{\pi}\gamma} e^{-4\gamma/\frac{\Delta t_{step}}{t_c(L)}} = \frac{8}{\sqrt{\pi}} \Delta t_{step} L^{-2} e^{-L^2/4\Delta t} \quad \text{where } \frac{\Delta t_{step}}{t_c(L)} \sim \frac{T}{L^2} \ll 1$$

Note: Here $t_c(L)$ is the collapse time of the Infinite Gaussian packet under linear dynamics, when the initial distance is L. In a simulation of the TDGL, the collapse time is indicated by T_c and is way larger!
Note: We used **other** assumptions **over** $L \gg T^{1/2}$ to find this result:

- $L \gg \sigma_0$ to approximate $t_c \simeq \frac{L^2}{16\gamma}$ (neglecting the $\sim \sigma_0^2$ term)

- $T \gg \sigma_0^2$ to approximate $(\tau+1)^{1/2} \simeq \tau^{1/2}$.
  Intuitively, as σ_0 is the order of the width ($W \sim \sqrt{2}\sigma_0$) of the kinks when $C(t)$ crosses zero, and in the profile $\tanh\left(x\sqrt{\frac{\bar{C}}{2}}\right)$ the width is $W \sim \sqrt{2}\bar{C}^{-1/2}$, then we state $\sigma_0 \sim \bar{C}^{-1/2}$ and so the **assumptions** are
- $L \gg \bar{C}^{-1/2}$
- $T \gg \bar{C}^{-1}$
- $L \gg L_1^* \sim T^{1/2}$
  As the last two imply the first one, we conclude that the result

$$\frac{\Delta L(\Delta t_{step})}{L} = f\left(\frac{\Delta t_{step}}{t_c(L)}\right) \simeq \frac{\frac{\Delta t_{step}}{t_c(L)}}{2\sqrt{\pi}\gamma} e^{-4\gamma/\frac{\Delta t_{step}}{t_c(L)}}$$

holds if

$$L \gg L_1^* = 4T^{1/2}\left(1 - \frac{1}{\sqrt{\pi}}\left(\frac{\bar{C}}{A}\right)^{1/2}\right)^{1/2} \quad \text{and} \quad T \gg \bar{C}^{-1}$$

Attraction in the $L \gg T^{1/2}$ limit

We will call the collapse time $T_c(L, T, A)$ to distinguish it from the collapse time of the Infinite gaussian $\tau_c(L)$.
If we are interested in variations of $L(t)$ **along a timescale** $\gg T$**, then we can define a "macroscopic derivative"**

$$\frac{\Delta L_{step}}{T} \simeq \partial_t L$$

If $L \gg T^{1/2}$ and we approximate $\Delta t_{step} = T$ then

$$\partial_t L \simeq f(\xi)\frac{L}{T} = -\frac{8}{\sqrt{\pi}} L^{-1} e^{-L^2/4T}$$

more generally, if $L \gg T^{1/2}$

$$\partial_t L \simeq -\frac{8}{\sqrt{\pi}} L^{-1}\left[1 - \frac{1}{\sqrt{\pi}}\left(\frac{A}{\bar{C}}\right)^{-1/2}\right] e^{-L^2/4T[1-1/\sqrt{\pi}(A/\bar{C})^{-1/2}]}$$

intuitively we will say that this **attraction** is negligible respect to the kink dynamics (that we expected still to have, this is ad **additional** effect **we expect**)

$$\partial_t L|_{Ccost} = -24\sqrt{2C}e^{-\sqrt{2C}L}$$

but let's be **quantitative**!

Notice: The "macroscopic derivative" is defined by considering a time interval $\Delta t \gg T$ and measuring the ΔL along this time interval. So to **verify experimentally** the law, we have to measure ΔL along $\Delta t \gg T$.

3 Regimes

When this effect **dominates on the other**? When the following inequality holds:

$$-24\sqrt{2C}e^{-\sqrt{2C}L} \ll \frac{8}{\sqrt{\pi}}L^{-1}e^{-L^2/4T}$$

if we apply the log at both sides, in the limit $L \gg max(192\bar{C}, 1)$ we can make an approximation

$$\frac{L^2}{4T} \ll (2\bar{C})^{1/2}L$$

$$L \ll 4\sqrt{2}\bar{C}^{1/2}T\left(1 - \frac{1}{\sqrt{\pi}}\left(\frac{\bar{C}}{A}\right)^{1/2}\right)$$

So there are **three regimes**

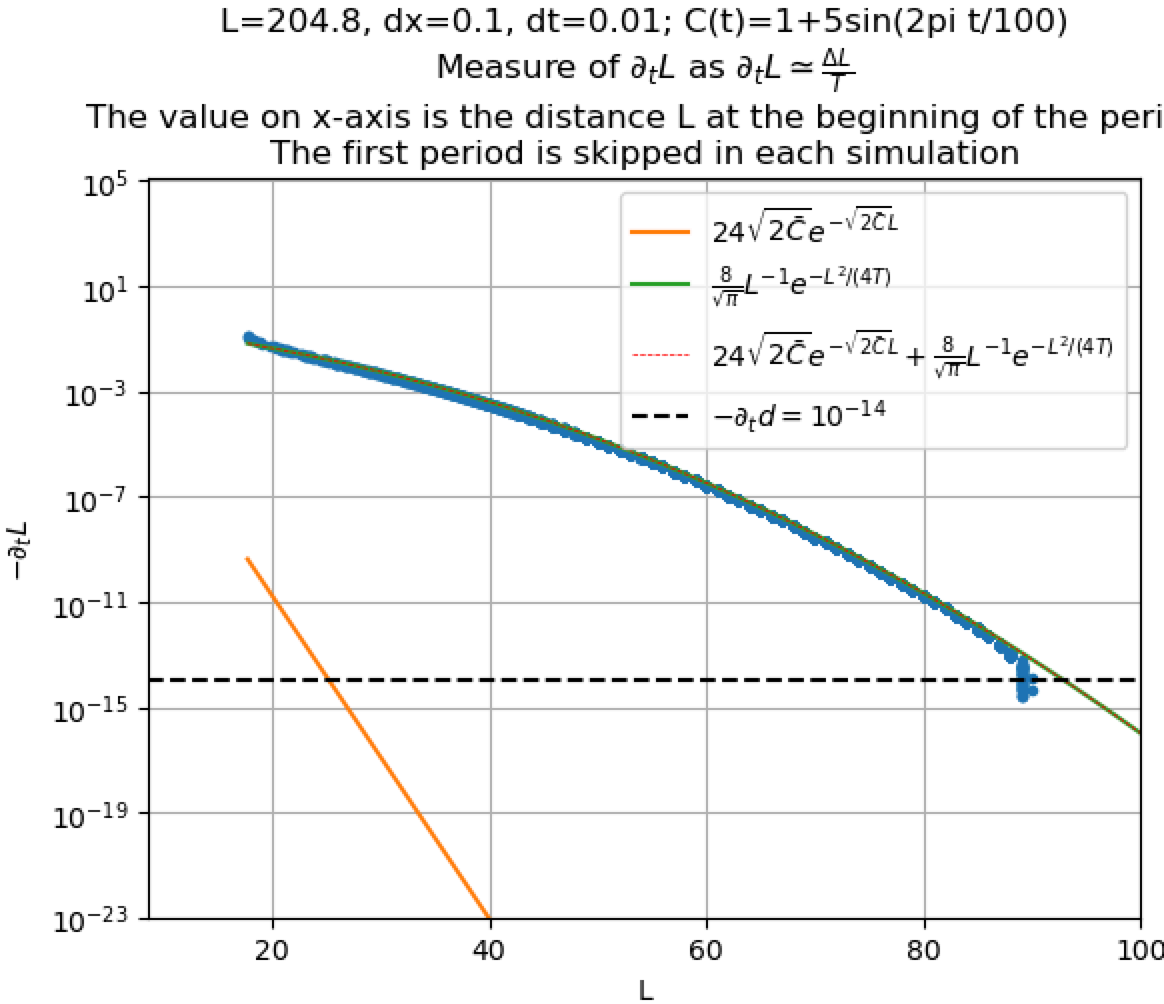
$$L_1^* = 4T^{1/2}\left(1 - \frac{1}{\sqrt{\pi}}\left(\frac{\bar{C}}{A}\right)^{1/2}\right)^{1/2}, T \gg \bar{C}^{-1} \qquad L_2^* = 4\sqrt{2}\bar{C}^{1/2}T\left(1 - \frac{1}{\sqrt{\pi}}\left(\frac{\bar{C}}{A}\right)^{1/2}\right)$$

- **Small distances**: Here it is not clear what happens.
- **Intermediate regime**: If $L_1^* \ll L \ll L_2^*$ and $T \gg \bar{C}^{-1}$ we can use the asymptotic expansion of $f(\xi)$ and we know how $L(t)$ scales, as

  $$\partial_t L \simeq= -\frac{8}{\sqrt{\pi}}L^{-1}e^{-L^2/4T}$$

  Notice that to see an effect due to oscillations, we need that the time where $L_1^* \ll L \ll L_2^*$ is of the order of many periods T.
  Example: $L_1^* = 40, L_2^* = 40\sqrt{2}, T = \bar{C}^{-1}$ (they do not estimate correctly the intersection of

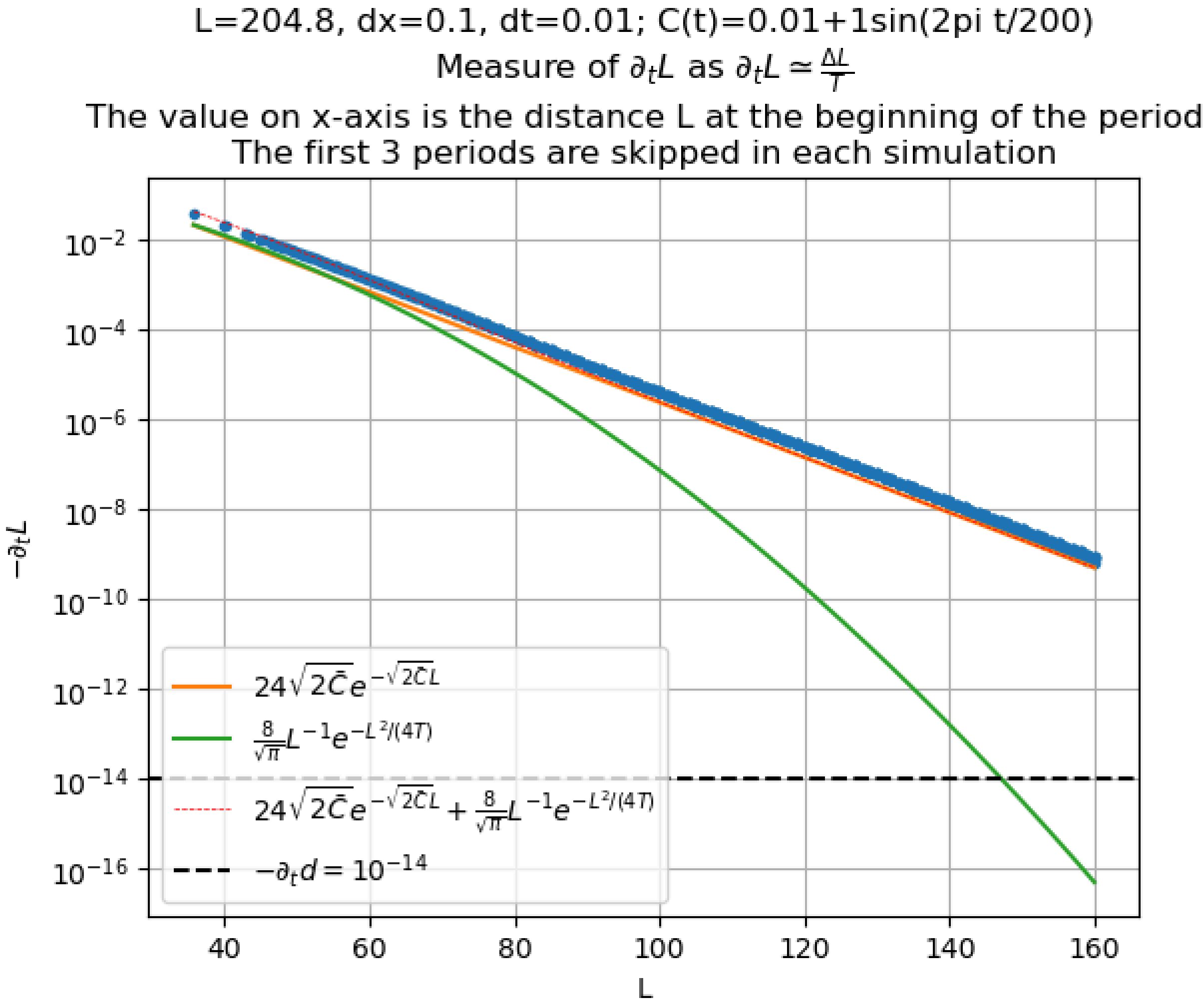
green and orange)



- **Asymptotic regime**: If $L \gg L_2^*$, then the effect due to the large oscillations disappears as it becomes negligible respect to the kinks dynamics. So we expect to see the well-known behaviour

$$\partial_t L = -(48\bar{C})^{-1}e^{-\sqrt{2\bar{C}}L}$$

Example: $L_1^* = 40\sqrt{2}, L_2^* = 120\sqrt{2}, T = 2\bar{C}^{-1}$ (they do not estimate correctly the intersection of green and orange)



If L is smaller, there are not enough periods to calculate $\frac{\Delta L}{10T}$.

How to see BOTH regimes in the SAME simulation

To see both regimes we need that

- $L \sim L_2^*$, **intersection between the two curves**, is simulable, in the sense that we can measure the steps depth $\partial_t L|_{L_2^*} T$ (this is large enough to be catched by the simulation)

$$\partial_t L|_{L_2^*} T \gg \epsilon \qquad \epsilon = 10^{-14}$$

- $L_1^* \ll L_2^*$ so we can see the intermediate regime when $L_1^* \ll L \ll L_2^*$. And we need also $T \gg \bar{C}^{-1}$ so the intermediate regime law is expected (in the intermediate regime).
  By using that $k = \left(1 - \frac{1}{\sqrt{\pi}}\left(\frac{\bar{C}}{A}\right)^{1/2}\right) \simeq 1$ and so $k > \frac{1}{2}$, then we find the **requirement**

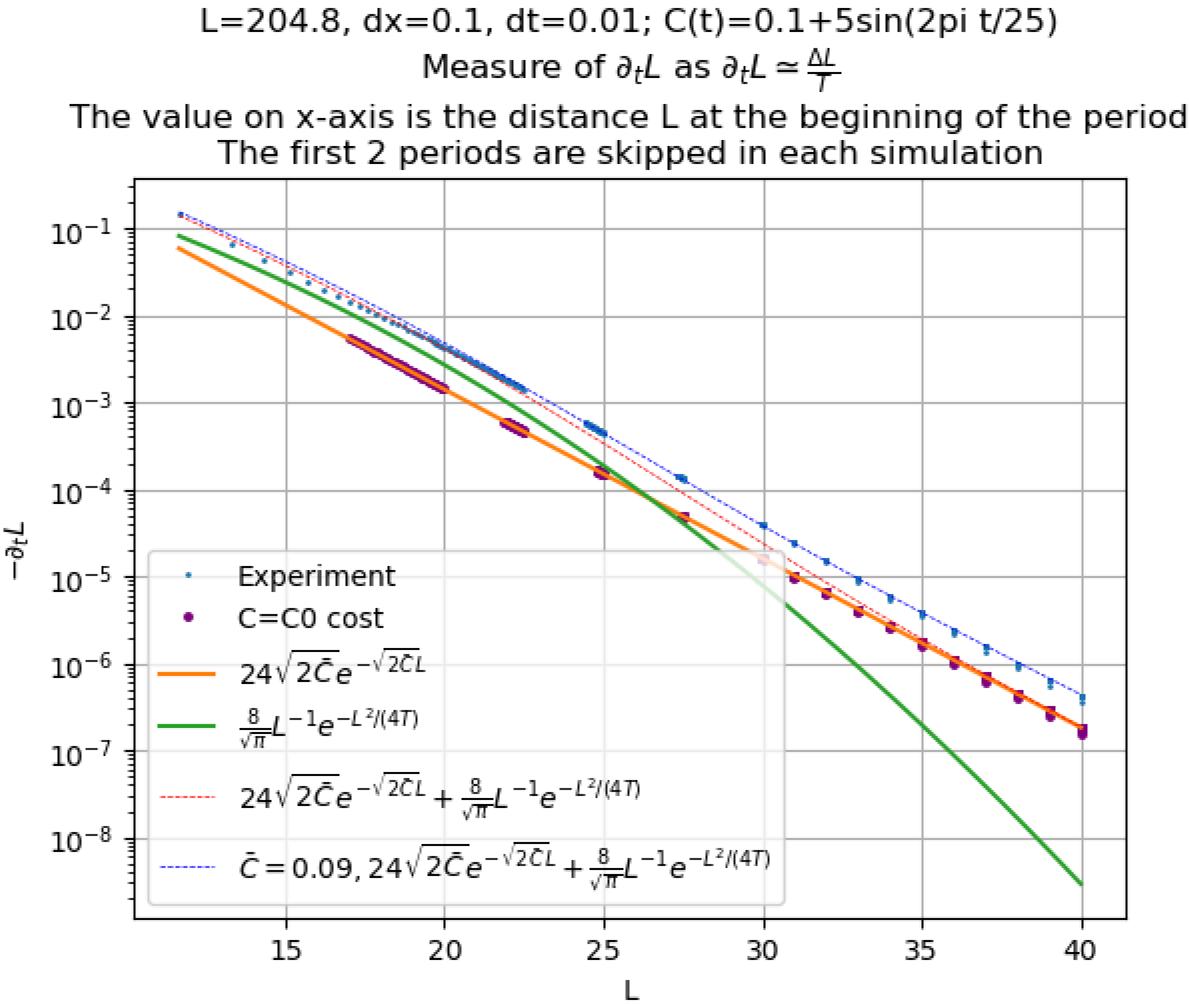
$$T \gg \bar{C}^{-1}$$

$$T \ll \frac{1}{8\bar{C}k}(\log \epsilon - \log(24\sqrt{2\bar{C}}) - \log T)$$

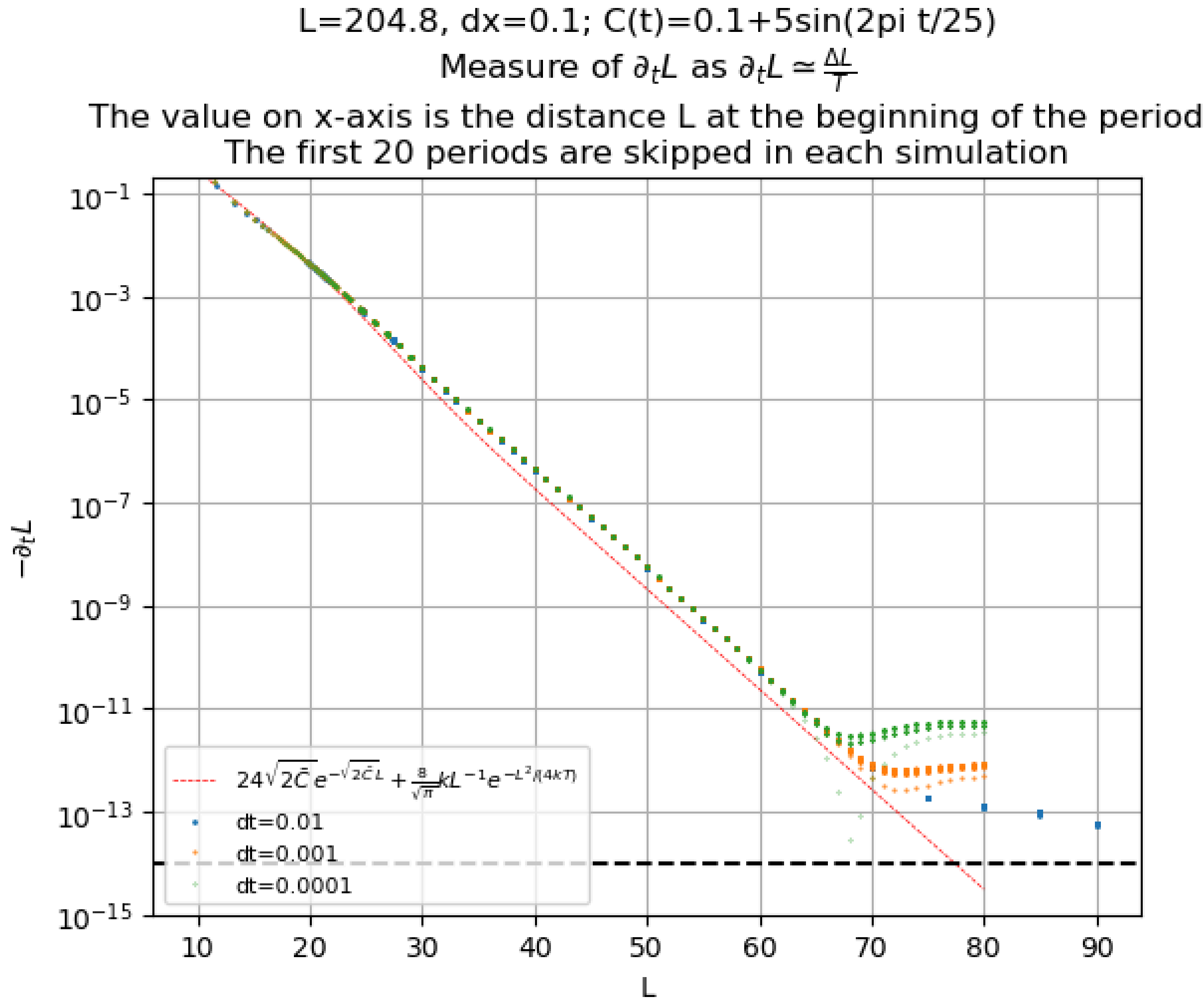
That is satisfied for **$\bar{C}$**=0.1, T=25, A=5** (so $k \simeq 0.8$). Where we have

$$L_1^* \simeq 19, T = 2.5\bar{C}^{-1} \gg \bar{C}$$

Maybe is not the average value of C that we should consider in the asymptotic law.

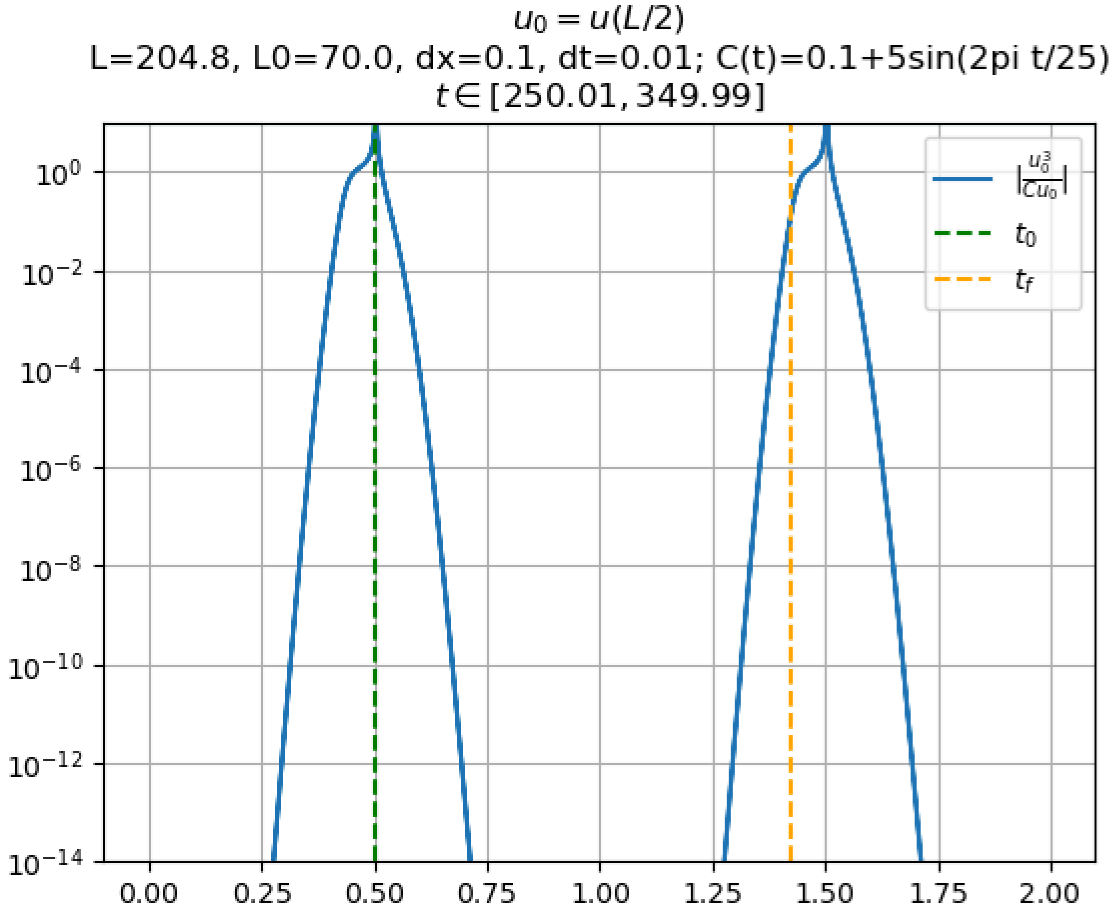


The fact that asymptotically the points are not on the orange line, it is **not** a numerical effect, as we can show here



The linear approximation is valid almost for the whole period

Here below we can see that the non-linear term can be neglected respect to the linear one, not only in the region t_0, t_f (that is the duration of the step), but for almost the whole period!



this suggests that we can try to do the math for kink dynamics neglecting the non-linearity, and then considering what kind of correction this small time interval where it is relevant apport.
Then, as we monitored the ratio between linear and non-linear term, **we should also monitor the shape of the kinks'tail.**

2D Circular domain

We can follow a similar approach to describe what happens to a 2D circular domain when $C(t)$ spends much time (per period) in the negative semiaxis.
We sum an infinite amount of 2D Gaussian centered inside a circle of radius R and with width σ

$$G(r_0) = \int_0^R rdr \int_0^{2\pi} d\theta g(\mathbf{r}, \mathbf{r_0}, \sigma) - \frac{1}{2}$$

$$g(\mathbf{r}, \mathbf{r_0}, \sigma) = \frac{1}{2\pi\sigma^2} e^{-(\mathbf{r}-\mathbf{r_0})^2/2\sigma^2}$$

It follows

$$G(r_0) = \frac{1}{\sigma} \int_0^{R/\sqrt{2}\sigma} re^{-(r^2+r_0^2)/2\sigma^2} B_I\left(0, \frac{2rr_0}{2\sigma^2}\right) dr - \frac{1}{2}$$

and in the limit where $r_0 \gg \sqrt{2}\sigma$, then we can asymptotically expand the Bessel function. And using that $re^{-(r-r_0)^2}$ is significatively different from zero only when $r \simeq r_0$ then

$$G(r_0) \simeq \frac{1}{2}\left[erf\left(\frac{r_0}{\sqrt{2}\sigma}\right) - erf\left(\frac{r_0 - R}{\sqrt{2}\sigma}\right) - 1\right] \quad \text{if } r_0 \gg \sqrt{2}\sigma$$

$r_0 \gg \sqrt{2}\sigma$, because this is the width of the left kink, centered at $r_0 = 0$, that is a feature of the approximation and **not** of $G(r_0)$.

Validity of the new law

We can estimate the depth of the step ΔR with the **same formula** we used in the 1D case for two kinks. This means that

$$\Delta R_{step} = -\frac{8}{\sqrt{\pi}} TkR^{-1}e^{-R^2/4kT}$$

where $k = \left(1 - \frac{1}{\sqrt{\pi}}\left(\frac{\bar{C}}{A}\right)^{1/2}\right)$ if $\bar{C} \ll \pi A$ and R is the radius at the beginning of the step.

Inferior extreme

If $R \gg \sqrt{2}\sigma$ then this approximation is good around the zero of the function. It means that, until $\sigma(t) = \sigma_0\sqrt{1 + \frac{2t}{\sigma_0^2}} \ll \frac{R}{\sqrt{2}}$ (where R is the initial radius), then we can estimate the position of the zero of $G(r_0)$ by estimating the position of the zero **of the approximation**.

So, if Δt_{step} is such that

$$\sigma_0\sqrt{2}\Delta\tau_{step}^{1/2} \ll \frac{R}{\sqrt{2}} \implies R \gg R_1^* = 2T^{1/2}k^{1/2}\bar{C}^{-1/2}$$

where I considered that $\sigma_0 \sim \bar{C}^{1/2}$.

Superior extreme

The new effect **will compete** with motion by curvature. We say that both effect are present, because at large R the new law will vanish, but we see experimentally that MBC holds.

$$\Delta R_{MBC}(R) = R(t) - R = R\left(\sqrt{1 - \frac{2T}{R^2}} - 1\right)$$

and will be relevant when $\Delta R_{steps} \gg \Delta R_{MBC}$ where $R \ll R_2^*$

$$R_2^*\left(\sqrt{1 - \frac{2T}{{R_2^*}^2}} - 1\right) \ll \frac{8}{\sqrt{\pi}} kTR_2^* e^{-{R_2^*}^2/4kT}$$

Additional constrains

So we expect the new law to be valid and dominant (over MBC) if $R_1^* \ll R \ll R_2^*$. But actually there are more constrains on this region

R (the radius at the beginning of the step) must be big enough to see a full step before the collapse

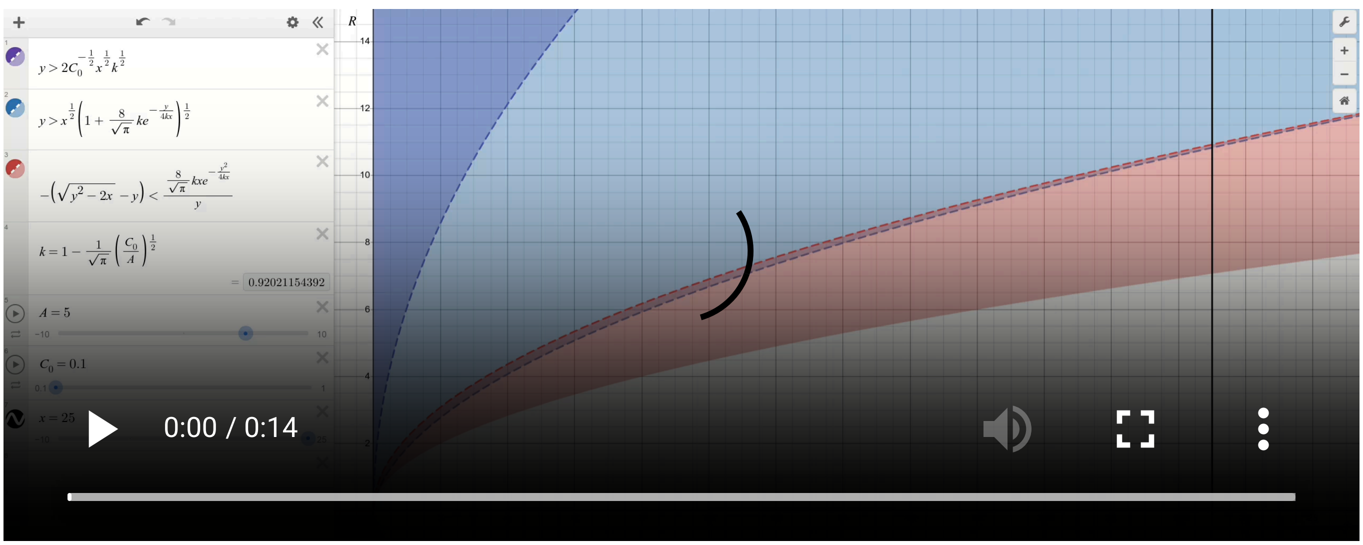
The condition for this, is that the step's depth must be smaller than the radius at the beginning of the step

$$-\Delta R_{step} < R$$

considering **both** the step-like and the MBC contributions

$$R^2 > T\left(1 + \frac{8}{\sqrt{\pi}} k e^{-R^2/4kT}\right)$$

if we consider togheter this condition along with $R_2^* > R > R_1^*$, we find that they are **never satisfied at the same time** , for any values of $\bar{C}, T$.
Here we plot R v.s. T and there is a parameter $\bar{C}$. Here A is fixed.



Even if we could find a set of parameters to satisfy the 3 conditions, only a few steps will be in the region of validity of the new law

The collapse time, considering just MBC (in reality is smaller)

$$T_c \sim \frac{R_0^2}{2}$$

and if we ask that the simulation lasts many periods $T_c = NT$, then

$$R_0 = \sqrt{2} N^{1/2} T^{1/2}$$

and requiring that R_0 is between R_1^*, R_2^* leads N to be of the order of **just a few steps**

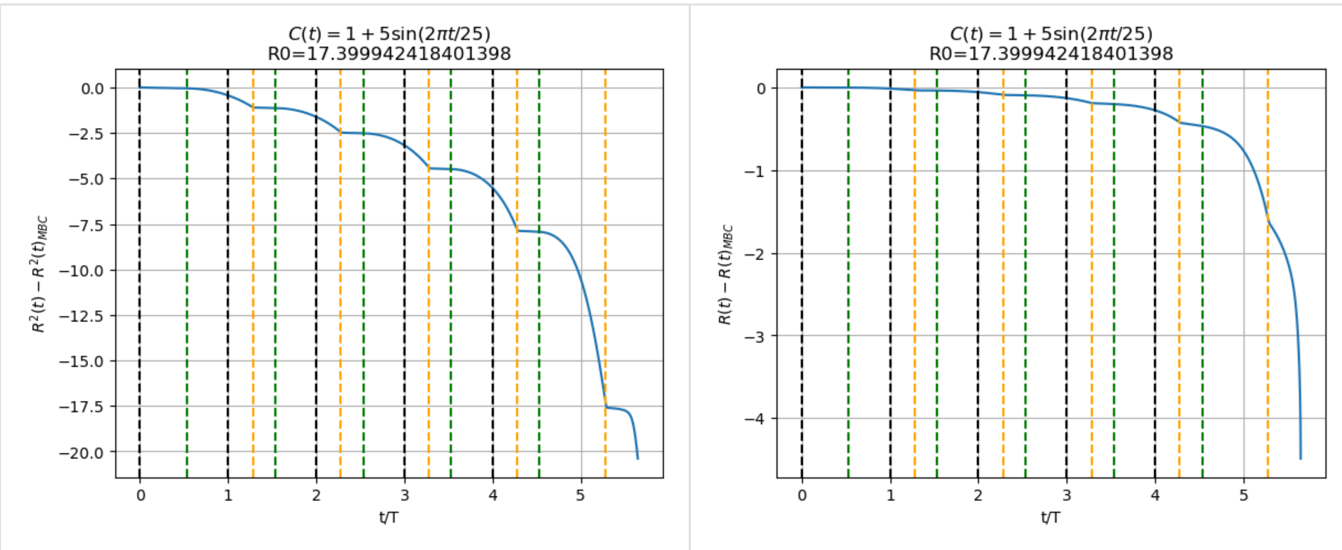
$$N < 2k\left(\log\left(\frac{8k}{\sqrt{\pi}}\right)\right)^2 \to_{k=1} 2\left(\log\left(\frac{8}{\sqrt{\pi}}\right)\right)^2 \simeq 4.5$$

$$N^{1/2} > \sqrt{2}k^{1/2}\bar{C}^{-1/2} \implies N > 2k\bar{C}^{-1/2}$$

This, we will see, forbids defining a macroscopic derivative as it would be bad defined in the region were it is supposed to hold.

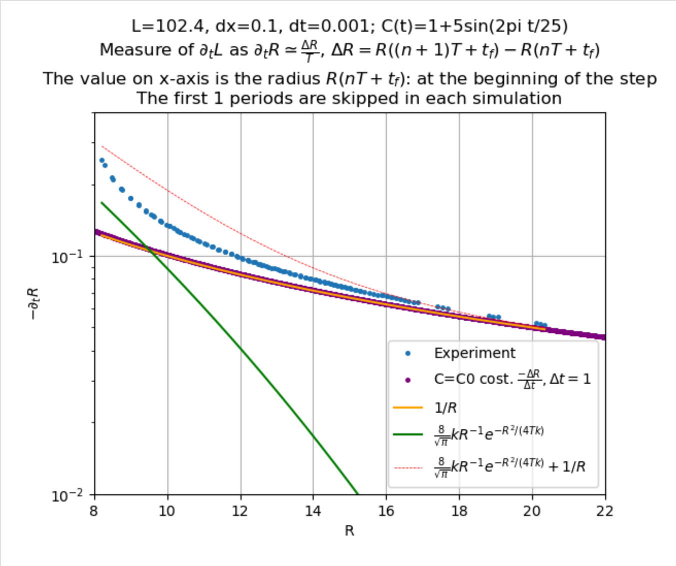
Steps

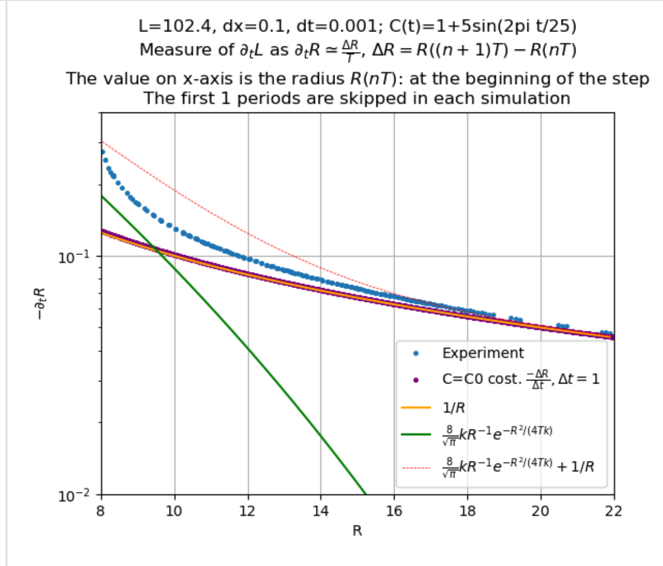
We can see clearly the steps if we subtract the MBC effect



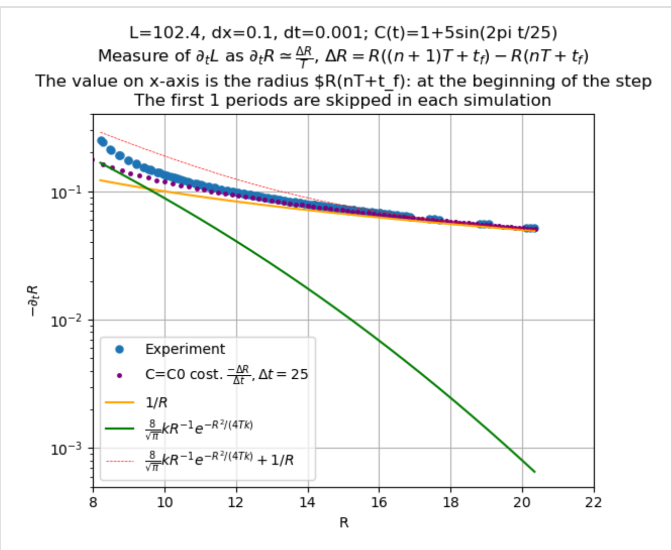
The macroscopic derivative $\partial_t R$ is BAD defined in the regime of validity of the new law

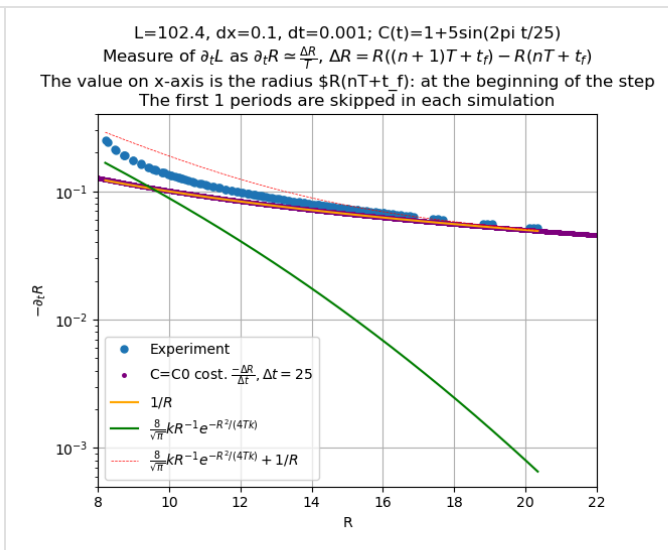
In the following picture, it does not look so bad defined though.





And the deviation of the constant C data from MBC is due to the finite size of the window of time along which the ΔR is calculated. In fact it disappears if it is heavily reduced in size

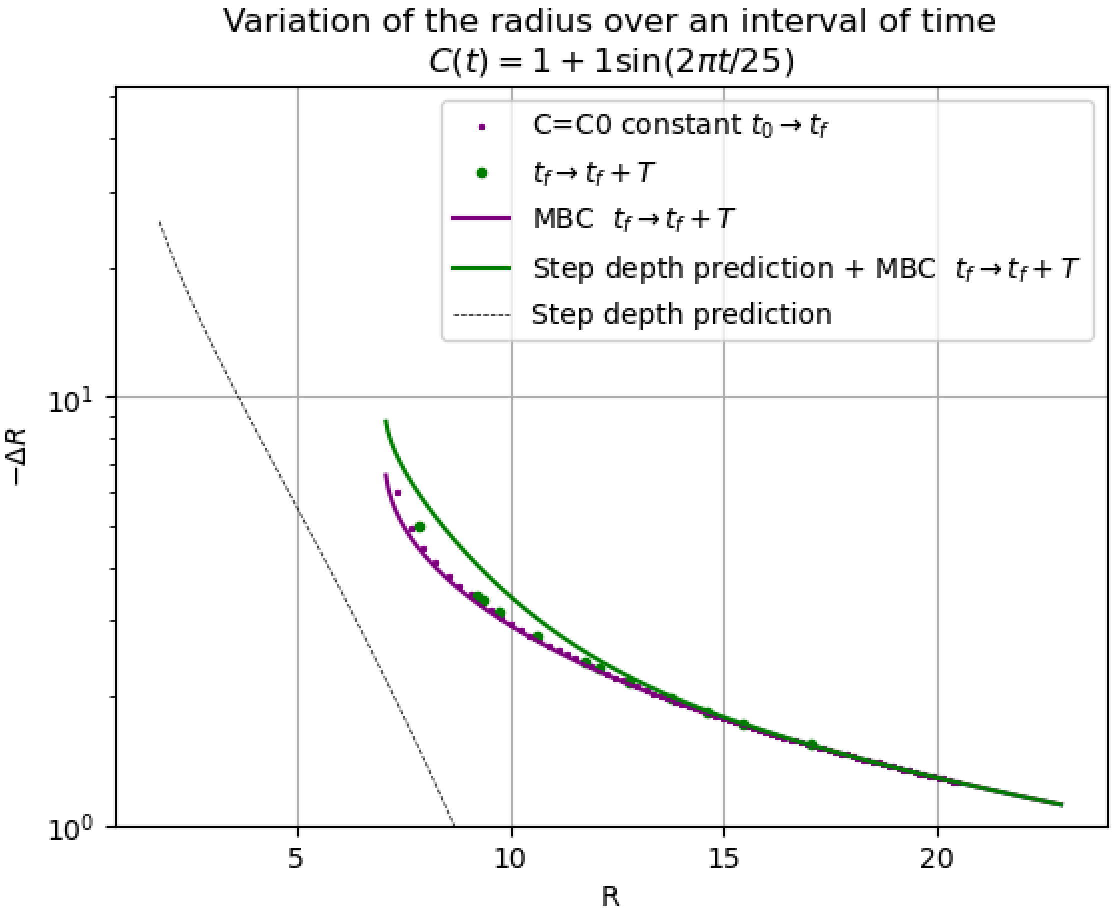


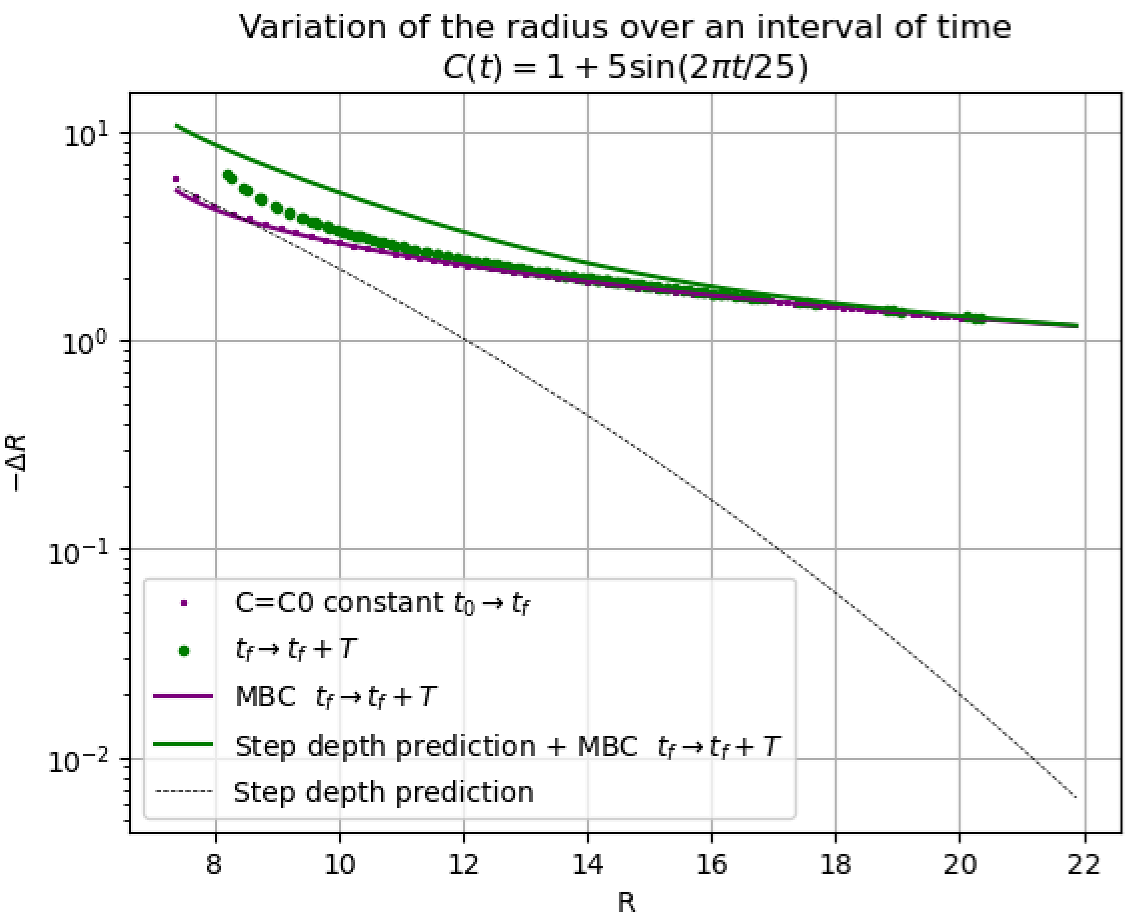


There is no dependance on the **window of time** we consider to write the derivative, **at large values of L**. Because when R is large, the **depth of consecutive steps** is very similar. But there is at small values of R!!!

Variation of R along a step

Instead of measuring the variation of R between t and $t + T$, we consider the variation between t_f and $t_f + T$, because this interval contains a **FULL step**.


Variation of the radius over an interval of time
$C(t) = 1 + 1\sin(2\pi t/25)$
C=C0 constant $t_0 \to t_f$
$t_f \to t_f + T$
MBC $t_f \to t_f + T$
Step depth prediction + MBC $t_f \to t_f + T$
Step depth prediction
10^1
10^0
$-\Delta R$
5
10
15
20
R


Variation of the radius over an interval of time
$C(t) = 1 + 5\sin(2\pi t/25)$
C=C0 constant $t_0 \to t_f$
$t_f \to t_f + T$
MBC $t_f \to t_f + T$
Step depth prediction + MBC $t_f \to t_f + T$
Step depth prediction
10^1
10^0
10^{-1}
10^{-2}
$-\Delta R$
8
10
12
14
16
18
20
22
R