Chapter 3

Controlling the linear dynamics

In the previous chapter, we described the dynamics of the Time-Dependent Ginzburg-Landau (TDGL) equation with a fixed, positive control parameter C. In the following chapters, we will explore the possibility of **controlling** the properties of the system's state, such as domain **size** and **position**, by varying the control parameter over time: $= C(t)$. Our analysis will be divided into two separate chapters, each focusing on one of the two distinct regimes of the dynamics. The first chapter will examine how the initial linear dynamics, during which domains are forming, is influenced by variations in $C(t)$. The second chapter will investigate the effects of $C(t)$ on the asymptotic dynamics, when domains coarsen.
In this chapter, we will present that the dynamics of the system is not influenced by the time variations of the control parameter C during the linear dynamics. If $C(t)$ is strictly positive, the dynamic is linear during the initial stages, when domains are forming. However, if $C(t)$ can be negative, the TDGL equation may still be approximated by a linear equation even long after $C(t)$ starts varying. While this chapter places no constraints on $C(t)$, the $C > 0$ case is extensively explored in later chapters, while the other case is under ongoing study (see Appendix H for some numerical results).
Our analysis will show that, during the linear regime, the characteristic length of the system $\ell(t) \sim t^{\frac{1}{2}}$, at any dimension and *independently* on the specific *shape of* $C(t)$. However, the duration of the linear regime can be controlled by the average value $\bar{C}$ around which $C(t)$ oscillates, such that $\tau_{\text{linear}} \sim \bar{C}$. This control over τ_{linear} allows us to influence the value of the characteristic length scale ℓ at the end of the linear regime. In 1D systems, where the subsequent coarsening dynamics is slow, the size of the domains formed during the linear phase can be considered effectively frozen. This enables the control of the 1D domain sizes during their formation. Conversely, in two-dimensional (2D) systems, domains growth cannot be neglected in the asymptotic regime, not giving the possibility of controlling domain sizes through the same mechanism.

3.1 Effects of a time dependent C on the formation of domains

Let's consider an initial condition that is a small and **random** perturbation of zero, this means that the amplitude of each Fourier component $M(\mathbf{q})$ is small and equal to each other

$$M(\mathbf{q},0) = \delta \quad \forall \mathbf{q} \qquad \text{where: } \delta << 1$$

In this case, the initial dynamics will be the ruled by the linear part of the TDGL and will lead to the formation of domains. In this section, we will show that the rules describing the dynamics in this regime are not affected by how $C(t)$ varies in time.

To do so, we define a characteristic length of the system ℓ that represents the typical size of the features appearing in the state of the system $m(\mathbf{x})$. To define ℓ, we follow the idea that the wave-length associated to each Fourier mode is $\lambda_{\mathbf{q}} = \frac{2\pi}{\sqrt{q^2}}$, where $\mathbf{q}$ is the momentum of the mode. Then we can define a typical wave-length of the system, by calculating the average

$$< q^2 >= \frac{\int q^2 |M(\mathbf{q})|^2 d\mathbf{q}}{D \int |M(\mathbf{q})|^2 d\mathbf{q}} \tag{3.1.1}$$

where $M(\mathbf{q})$ is the Fourier transform of the state $m(\mathbf{x})$, or equivalently the amplitude of the Fourier mode with momentum $\mathbf{q}$. While D is the dimension of the system. Then a typical wave-length of the system will be

$$\ell = \frac{2\pi}{\sqrt{< q^2 >}} \tag{3.1.2}$$

We say that ℓ is the typical size of the features of the state, as the presence of a feature of order λ is described by the amplitude of the Fourier mode with wave-length λ.

In the linear regime, the length ℓ increases in time as $\ell \sim t^{\frac{1}{2}}$, independently on the choice of the function $C(t)$. We can see this, noticing that in the linear regime

$$\partial_t M \simeq [-q^2 + C(t)]M$$

so it follows that

$$M(\mathbf{q},t) = M(\mathbf{q},0) e^{-q^2 t + \int_0^t C(t')dt'} \tag{3.1.3}$$

Using this property in the integral

$$< q^2 >= \frac{\int q^2 M(\mathbf{q},0)^2 e^{-2q^2 t} d\mathbf{q} * e^{2\int_0^t C(t')dt'}}{D \int M(\mathbf{q},0)^2 e^{-2q^2 t} d\mathbf{q} * e^{2\int_0^t C(t')dt'}} = \frac{\int q^2 M(\mathbf{q},0)^2 e^{-2q^2 t} d\mathbf{q}}{D \int M(\mathbf{q},0)^2 e^{-2q^2 t} d\mathbf{q}}$$

As the terms containing $C(t)$ got cancelled, it means that $< q^2 >$, and so ℓ, does not depend on the shape of $C(t)$. It is also possible to calculate explicitly the ratio between the integrals, exploiting that the initial state is a random perturbation of zero, so $M(\mathbf{q},0) = \delta \quad \forall \mathbf{q}$ and we can cancel the terms $M(\mathbf{q},0)$ out of the integral.

Then, we notice that if the system has a finite size L^D, Fourier modes that are larger than the system size are not available. So the integration must be restricted to the intervals

$$q_i \in [-q_{min}, +q_{min}]$$

where $q_{min} = \frac{2\pi}{L}$, so

$$< q^2 >= \frac{\int_{-q_{min}}^{+q_{min}} q^2 e^{-2q^2 t} d\mathbf{q}}{D \int_{-q_{min}}^{+q_{min}} e^{-2q^2 t} d\mathbf{q}} \tag{3.1.4}$$

In Appendix C an approximation of the integral is calculated, that is valid in the limit where $t >> \frac{1}{2} q_{min}^{-2}$, but not so large such that the dynamics is still governed by the linear part of the TDGL equation. So, during the initial dynamics, at **any dimension**, the characteristic lenght $\ell(t)$ evolves *independently on the shape of* $C(t)$ and according to the law

$$< q^2 > \simeq \frac{t^{-1}}{4} \implies \ell(t) = 4\pi t^{\frac{1}{2}} \tag{3.1.5}$$

However, even if the linear dynamics does not depend on the control parameter $C(t)$, the time it takes for the non-linear effects to become relevant (the duration of the linear regime) τ_{linear} depends on it. To estimate this characteristic time, we consider the fastest growing mode. As the *growth rate* of each Fourier mode is $-q^2 + \frac{1}{t} \int_0^t C(t')dt'$ (see Eq. 3.1.3), then the one with momentum $\mathbf{q} = \mathbf{0}$ is the one we consider

$$M(0,t) = M(0,0) e^{+\int_0^t C(t')dt'}$$

If we require $C(t)$ to be a *periodic oscillation* with period T and around the average value $\bar{C} = \frac{1}{T} \int_0^T C(t')dt'$, it follows that, if $t >> T$

$$M(0,t) \simeq M(0,0) e^{\bar{C} t}$$

and so the characteristic time for the growth of $M(0,t)$ is

$$\tau_{\text{linear}} \sim \bar{C}^{-1}$$

Inserting this result in Eq. (3.1.5) the typical size of the features of the system, at the end of the linear regime, is

$$\ell(\tau_{\text{linear}}) \sim 4\pi \bar{C}^{-\frac{1}{2}}$$

In a 1D system, it is possible to take advantage of this result, to control the typical size of the domains. In fact, during the asymptotic dynamics, the size of the domains is effectively frozen in a 1D, so the sizes they have at the end of the linear regime, do not change anymore, as shown in Figure 3.1.

In the same figure, one can see that a different choice of C (that is equal to $\bar{C}$ as $C(t)$ is kept constant in time) determines a different position for the peak of ℓ_{DW}. This behaviour is expected, as the position of the peak describes the duration of the linear regime $\tau_{\text{linear}} \sim C^{-1}$.

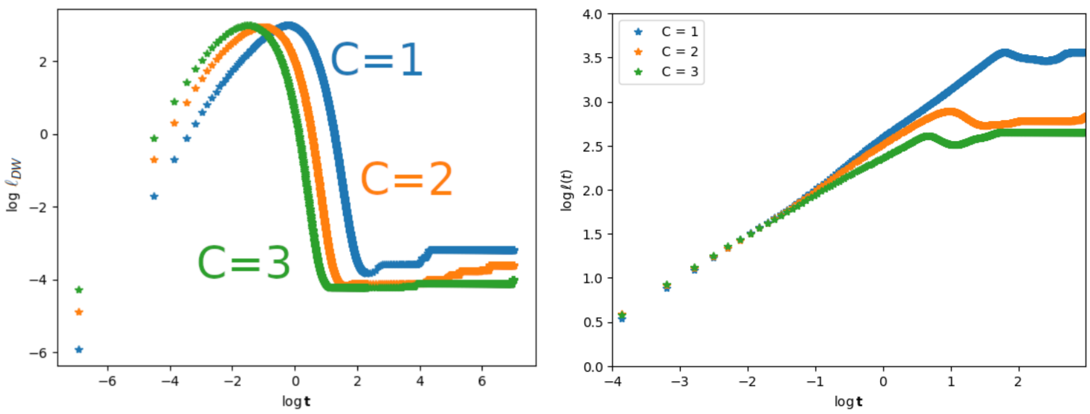

Figure 3.1: Evolution of the characteristic lenghts ℓ_{DW}(average domain size) and ℓ in a simulation of the 1D TDGL equation with different values of C (constant). The initial state is the same and it is a random perturbation of zero. In the asymptotic regime, λ_{DW} is about constant in time (precisely $\lambda_{DW} \sim \log t$, but a longer simulation is needed to see this time dependence) and takes different "final" values depending on the choice of C.

However, it is important to note that the characteristic length ℓ is not specifically defined to measure the average size of the domains. While $C(t)$ influences the typical size of the domains, this size will not necessarily correspond to $\ell(\tau_{\text{linear}})$.
In a 2D system, the characteristic size ℓ, along with the size of the domains (denoted as ℓ_{DW}), continues to increase even after the initial dynamics (as shown in Figure 3.2). This makes not possible to control the size of the domains.
It is worth to notice that, during the asymptotic dynamic, the quantities ℓ and ℓ_{DW} are not independent. In fact in Appendix G it is proved an equation linking the two.

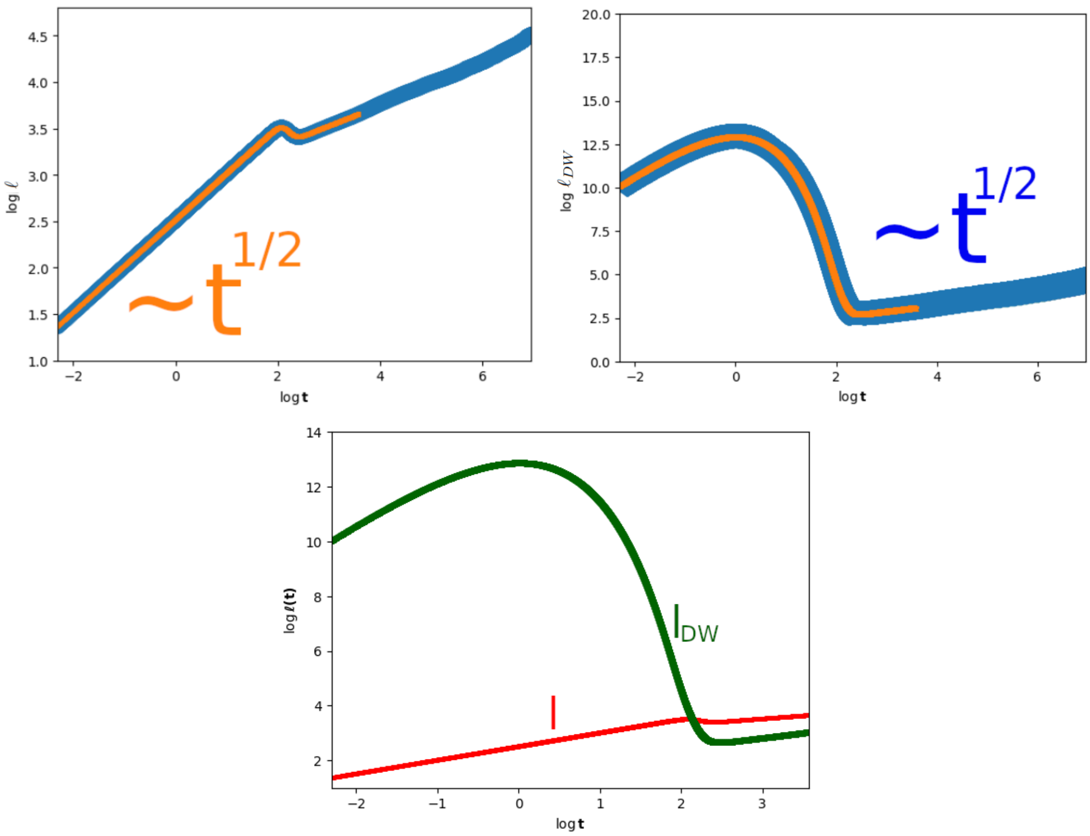

Figure 3.2: Evolution of the characteristic lengths of the system $\ell(t)$ (left) and ℓ_{DW} (right) measured during a simulation of the 2D TDGL equation with $C = 1$ (blue curve) and $C = 1 + \frac{1}{4}\sin\left(2\pi\frac{t}{T}\right); T = 0.025$ (orange curve). After the linear regime, $\ell(t)$ still grows in time as $\ell \sim t^{\frac{1}{4}}$. In the last plot both $\ell(t)$ (red) and $\ell_{DW}(t)$ (dark green) are represented on the same canvas. Notice the two function overlap at the end of the linear regime, suggesting the possibility of estimating the initial domain size as $l(t)$ evaluated at the end of the linear regime.