1D Slow oscillations (A<<C0) (Analytical)

- $C(t)$ is a **periodic** function:
  - **Average** $\bar{C}$
  - **Period** T
  - **Amplitude** A.
- **Slow** oscillations: the period T is **large** compared to the other timescale(s) of the system.
- The **average** $\bar{C}$ is **positive**
- **A<< C0**: the amplitude is small compared to its average, so $C(t)$ is
  - **Strictly positive** $C(t) > 0 \quad \forall t$
  - **Far from zero**.

If the average is negative, from the 0D analysis we expect all domains to disappear (exponentially fast in time). So we're not interested in this case.

Main results

- The time-scale of the system $\tau_{linear} \sim C^{-1}$ is **always of the same order**

$$\tau_{linear} \sim \bar{C}^{-1}$$

- Slow oscillations

$$T \gg \tau_{linear} \sim \bar{C}^{-1}$$

$$\epsilon \sim \frac{\tau_{linear}}{T} \ll 1$$

- The leading order correction to the shape of an isolated kink is

$$\delta u(x,t) = \epsilon \beta_1(t) u_{k_1}(\chi); \quad \chi = C^{1/2}(t)x$$

$$\beta_1(t) = \frac{1}{2} C^{-3/2}(t)(\epsilon^{-1}\partial_t C(t))$$

(where a term of order $\sim \epsilon$ comes from $\partial_t C(t)$ so $\beta_1 \sim 1$)

$$u_{k_0} + \chi \partial_\chi u_{k_0} = \partial_{\chi\chi} u_{k_1} + u_{k_1} - 3u_{k_0}^2 \partial_\chi u_{k_1}$$

while this is the equation determining the shape $u_{k_1}(\chi)$

$$u_{k_0}(\chi) \to \pm 1$$

$$u_{k_1}(\chi) \to \mp\frac{1}{2}$$

- Kinks dynamics is not affected by a time depending $C(t)$ to leading order (neglecting terms of order higher than $\epsilon\delta u_{k_0}$, $\delta u_{k_0}^2$)

$$\dot{x_n}(t) = 16C^{\frac{1}{2}}(t)\frac{[e^{-2^{\frac{1}{2}}C(t)^{\frac{1}{2}}l_n} - e^{-2^{\frac{1}{2}}C(t)^{\frac{1}{2}}l_{n+1}}]}{\int_{\chi_{n-0.5}}^{\chi_{n+0.5}} d\chi\partial_\chi u_p(\chi)}$$

that for two isolated kinks leads to a decay of the distance

$$\dot{d}(t) \simeq -24\sqrt{2}C^{\frac{1}{2}}(t)e^{-2^{\frac{1}{2}}C(t)^{\frac{1}{2}}d}$$

that is the formula found for constant C, with $C \to C(t)$.

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Multiple scale analysis

If $C(t)$ is a periodic function of time, there are **at least two time-scales** in the system

- τ_{linear} it is the time-scale associated to the growth of unstable modes in the linear regime.
- T is the period of $C(t)$.

Timescale of the system

The time-scale $\tau_{linear} \sim C^{-1}$ but now $C(t)$ is a function of time!
As the amplitude A of the oscillation is small compared to the average $\bar{C}$, **then $C^{-1}(t)$ is kept of the same order ($\bar{C}^{-1}$) at any time**. So

$$\tau_{linear} \sim \bar{C}^{-1}$$

As there are different time-scales in the system, then different processes of the dynamics may be characterized by different time-scales.
The **idea** of the multiple scale analysis is to introduce **new time variables**

$$t \to t_0, t_1, t_2, \dots$$

where each variable t_i is associated to a different time-scale τ_i, explicitly

$$\delta t_i \sim 1 \iff \delta t \sim \tau_i$$

and then we **hope** to **simplify** the calculations, by capturing processes characterized by different time-scales τ_i in **different equations** (each containing only derivatives respect to one variable t_i)

Introducing new time variables

As we are considering T large compared to the other timescale(s) of the system

$$T \gg \tau_{linear} \sim \bar{C}^{-1}$$

As we identify only two time-scales, we **naturally** introduce **just two** new time-variables t_0, t_1 associated respectively to τ_{linear} and T.
This **change of variables** $t \to t_0, t_1$
implies in general

$$\partial_t = (\partial_t t_0)\partial_{t_0} + (\partial_t t_1)\partial_{t_1}$$

but, as t_1 describes the variation at the timescale T of $C(t)$

$$\partial_{t_0} C(t) = 0$$

this is where we use the idea of the multiple scale analysis.

Time derivative

If we consider the average value of the oscillation $\bar{C}$ to be **of order 1**, then $\tau_{linear} \sim 1$.
It follows that

$$\delta t_0 \sim 1 \iff \delta t \sim \tau_{linear} \sim 1$$

is true if $t_0 = t$
Instead, requiring

$$\delta t_1 \sim 1 \iff \delta t \sim T$$

is true if $t_1 = \frac{t}{T} = t\frac{\tau_{linear}}{T}$ (as $\tau_{linear} \sim 1$)
Now, as we are analyzing the **case where T is large**, then**

$$T \gg \tau_{linear} \implies \epsilon = \frac{\tau_{linear}}{T} \ll 1$$

$$t_0 = t; \quad t_1 = \epsilon t$$

$$\partial_t = \partial_{t_0} + \epsilon \partial_{t_1}$$

where ϵ is a small parameter**.

Kink shape correction

Here we look for a correction to the kink's shape (deviation from the conventional shape $u_{k_0}(\chi) \sim \tanh(2^{1/2}\chi)$) **to leading order** in ϵ.

Anstatz

$$u(x,t) = u_0(x,t) + \epsilon u_1(x,t) + O(\epsilon^2)$$

inside the TDGL eq. ($\partial_t u = \partial_{xx} u + Cu - u^3$) and

- considering only terms of order zero
- using $\partial_t = \partial_{t_0} + \epsilon \partial_{t_1}$

  $$\partial_{t_0} u_0 = \partial_{xx} u_0 + C(t) u_0 - u_0^3$$

  that is the stationary TDGL equation, that has single-kink solution

  $$u_0(x,t) = \beta(t) u_{k_0}(\chi) \quad \chi = C(t)^{1/2} x \quad \beta(t) = C(t)^{1/2}$$

  $$u_k(\chi) = \tanh(2^{1/2}\chi)$$

  Ansatz

  $$u_1(x,t) = \beta_1(t) u_{k_1}(\chi)$$

  using this inside the TDGL equation and
- using the result on u_0
- considering ony terms of order one

  $$\partial_{t_1} C^{1/2}(t)(u_{k_0} + \chi \partial_\chi u_{k_0}) = C(t)\beta_1(\partial_{\chi\chi} u_{k_1} + u_{k_1} - 3u_{k_0}^2 \partial_\chi u_{k_1})$$

  Solvability condition
  We require this condition in order to get rid ∂_{t_1} derivatives and leave an equation with only ∂_χ derivatives

  $$C(t)\beta_1 = \partial_{t_1} C^{1/2}(t)$$

  That determines the amplitude of the first order correction to the kink's shape

  $$\beta_1(t) = \frac{1}{2} C^{-3/2}(t)(\partial_{t_1} C(t))$$

  and we're left with the equation

$$u_{k_0} + \chi\partial_\chi u_{k_0} = \partial_{\chi\chi} u_{k_1} + u_{k_1} - 3u_{k_0}^2 \partial_\chi u_{k_1}$$

that can be solved numerically.
It's interesting to look at the limits ($\chi \to \pm\infty$):

$$u_{k_0}(\chi) \to \pm 1$$

$$u_{k_1}(\chi) \to \mp \frac{1}{2}$$

so the sign is opposite.

Kink dynamics