Inverting kink motion with initial condition

#twokinks #1D

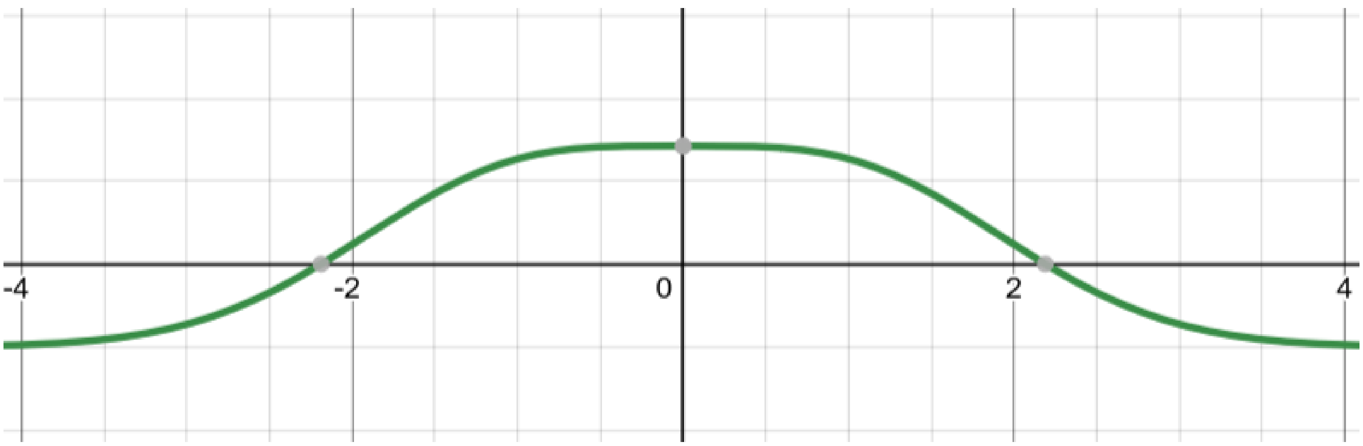
Examples

(1) Gaussian profile

We know that a Gaussian will broader in the linear regime, but what happens in general?

$$u(x, t_0) = u_0 \mathcal{N} \left[g_+(x) + g_-(x) - \frac{1}{2\sqrt{2\pi}\sigma} \right]$$

$$g_\pm(x) = \frac{e^{-(x-x_\pm)^2/2\sigma^2}}{\sqrt{2\pi}\sigma}$$



with $\sigma = 2(x_+ - x_-)$ such that the second derivative in $x = \frac{x_+ + x_-}{2}$ is zero and so $u(x, t_0)$ is **almost flat between** the kinks.

Second derivative at the positive kink x_{0_+}

Plots of this function and its derivatives, show that it is possible to **approximate** the second derivative at the right kink of $u(x, t_0)$ as the second derivative of the gaussian centered at x_+.

$$\partial_{xx} u \simeq g_+(x) \left[-\frac{1}{\sigma^2}(x - x_+)^2 + 1 \right]$$

Plots also show that $\exists x^* : \partial_{xx} u > 0 \quad \forall x > x^*$, so we can estimate x^* by looking at the zero of the above function

$$x^* = \sigma + x_+$$

Estimating the right kink

For the same reason, we can approximate the position of the positive kink x_{0_+} as the zero of $g_+(x) - \frac{1}{2\sqrt{2\pi}\sigma}$

$$x_{0_+} = \sqrt{\ln 4}\sigma + \frac{L}{2}$$

We see that

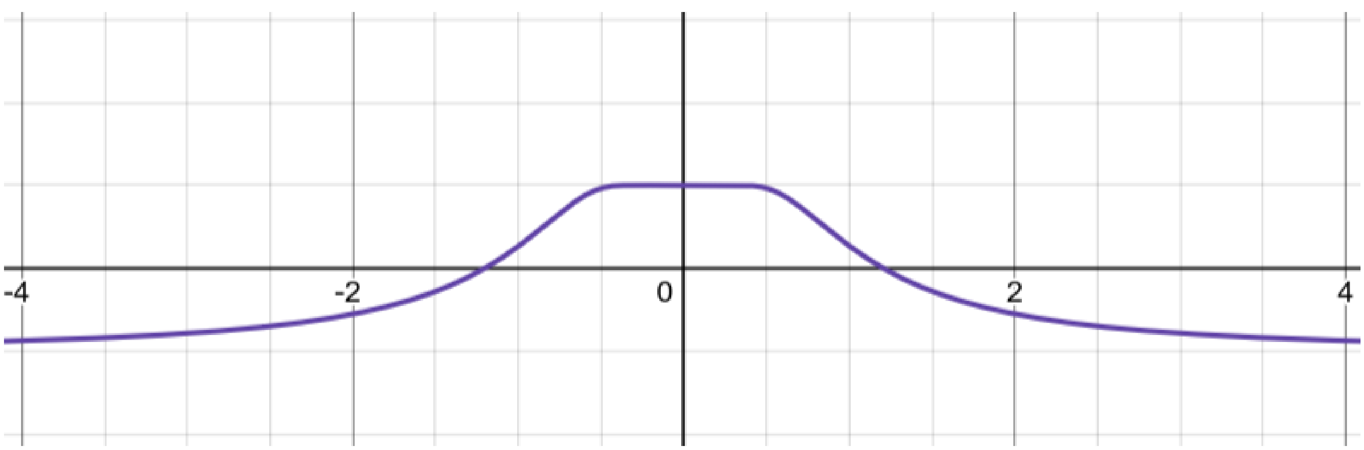
$$x_{0_+} > x^*$$

so the kink always moves towards right (see example below) and so the kinks **repulse each other** for any distance and also in the non-linear regime.
We expect the interface's shape to **relax** towards a tanh, so eventually the kinks will attract.

(2) Exponential profile

Let's consider a system with two **kinks** (a kink is **defined as a zero** of $u(x)$) at positions $x = \pm\frac{L}{2}$. It is possible to engine the shape of the state $u(x, t=0)$ in proximity of the kinks such that the velocity of each kink at time $t = 0$ correspods to a repulsive interaction, instead of an acctractive one.
To achieve this, you prepare the initial state as

$$u(x, t=0) = 2u_0 \left[\frac{1}{2} - e^{-1/(\alpha x)^2}\right]$$



here α is related to the distance L between the kinks (that are the zeroes of $u(x)$) as $\alpha = \frac{2}{L}\sqrt{\frac{1}{\log 2}}$.
To calculate the kink's velocity at time $t = 0$ we consider the TDGL equation

$$\partial_t u = \partial_{xx} u + C(t)u - u^3 \quad \forall x, t$$

now, if you assume that, close enough to the right kink $x_+ = +\frac{L}{2}$, you can assume that the kink propagates wthout changing shape, then

$$\partial_t u \simeq -\dot{x_+}\partial_x u \quad \text{if } x \simeq x_+$$

then, if you put this in the previous equation and you evaluate it at $x = x_+$ and $t = t_0 = 0$

$$-\dot{x_+}\partial_x u|_{x_+,t_0} = \partial_{xx} u|_{x_+,t_0}$$

this equation is so simple, because we used $u|_{x_+,t_0} = u^3|_{x_+,t_0} = 0$.
From the last equation, we can compute the **sign** of the initial velocity of the right kink $\dot{x_+}$ by computing the sign of the first and the second derivative of the initial state at $x = x_+$.

First derivative

The sign of the first derivative is trivial, as the function $u(x, t = 0)$ is decreasing at the position of the right kink, so $\partial_x u|_{x_+,t_0} < 0$.

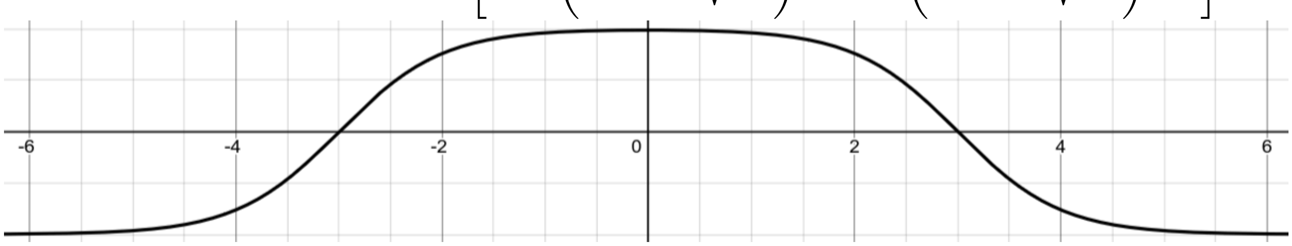
Second derivative

Here we should look at the concavity of the function $e^{-1/(\alpha x)^2}$.

- $\partial_x u(x, t = 0) = -4\alpha^{-2} u_0 x^{-3} e^{-1/(\alpha x)^2}$
- $\partial_{xx} u(x, t = 0)|_{x_+,t_0} = -2\alpha^{-2} u_0 x_+^{-4} \left[-\frac{3}{2} - \alpha^{-2} x_+^{-2}\right] > 0 \quad \forall x_+, \alpha$
  where in the last expression I used that $u(x_+, t = 0) = 0$.
  So we find that
- $\partial_{xx} u(x, t_0)|_{x_+,t_0} > 0$
- $\partial_x u(x, t_0)|_{x_+,t_0} < 0$
  and this means that $\dot{x_+} > 0$, so **the two kinks get far apart** instead of attracting. At least **at the beginning**, because then we expect the shape of the kinks to relax towoards the steady state profile ($\sim \tanh x$).

Examples where the kinks attract

- Hyperbolic tangent: $u(x, t_0) = u_0 \left[\tanh\left((x - x_-)\sqrt{\frac{C_0}{2}}\right) - \tanh\left((x - x_+)\sqrt{\frac{C_0}{2}}\right) - 1\right]$



In all these cases, the kink always attract since $t = t_0$, for any value of the parameters!