# Explaining the decay of the distance of two kinks with linear dynamics and a sum of Gaussians

#twokinks #1D #linear_regime

# Motivation

If $C(t)<0$ for a long time (as it happens in the cases above) eventually $u(x) \ll C_0$ and we expect the non-linearity to play a negligible role in the dynamics. So it is **natural** to expect that **the LINEAR dynamics is SUFFICIENT** to describe the **steps** that we see in the decay of the distance.
Even if the initial state is, in principle, relevant for the decay $d(t)$, we build the initial state **by summing Gaussian** functions, for simplicity in the calculations.



# Estimating the duration of a step

Here I present a way for estimating the duration of a step, that is **independent on the model** adopted to predict the decay $d(t)$ along a step.

## At what time $t_0$ the step originates?

The **beginning of the step** corresponds **empirically** with the moment $t_0$ **when** $C(t_0) = 0; \dot{C}(t_0) < 0$ (when $C(t)$ becomes negative). For a quench experiment, then $t_0 = 0$.

## Duration of a step

**Assuming** we can neglect the non-linearity **from** the moment **when** $C(t)$ **becomes negative** (this is supported by the fact that we see a good fit if we start to compare model and simulation from $t_0 : C(t_0) = 0$ and $\dot{C}(t_0) < 0$). Then

$$u_{q=0}(t) = u_{q=0}(t_0)e^{B(t)} \quad \text{if } t > t_0$$

$$B(t) = \int_{t_0}^{t} dt' C(t')$$

so $u_{q=0}(t)$ initially decreases, but then it increases again, becoming bigger than the initial value $u_{q=0}(t_0)$ and then the non-linearity in **no more negligible**.
As a consequence, we estimate the time when the decay finishes $t_f$ as the time when $u_{q=0}(t) = u_{q=0}(t_0)$ **again** so

$$B(t_f) = B(t_0) = 0$$

$$\int_{t_0}^{t_f} dt' C(t') = 0$$

$$C(t') = \bar{C}[1 + \frac{A}{\bar{C}} \sin\left(\frac{2\pi t'}{T}\right)]$$

changing variable $\tau = \frac{t}{T}$ and integrating, we find

$$2\pi(\tau_f - \tau_0) = \frac{A}{\bar{C}}[\cos(2\pi\tau_f) - \cos(2\pi\tau_0)]$$

$$t_f = \tau_f T \quad t_0 = \tau_0 T \qquad \tau_0 = \frac{1}{2}\left[1 - \frac{1}{\pi}\arcsin\left(-\frac{\bar{C}}{A}\right)\right]$$

this means that $\tau_f, \tau_0$ do not depend on $T$ and so the **duration of the decay** (step) is

$$t_f - t_0 \propto T$$

in general

$$t_f - t_0 = f\left(\frac{\bar{C}}{A}\right)T$$

so it **does NOT depend on the initial distance**, as we see in simulations. The duration of the last step will follow a different rule, as the collapse time $t_c$ predicted with the model of the two gaussians will be smaller than $t_f - t_0$.
**Notice**: This estimate of the duration of the step does not depend on the model adopted to compute the decay $d(t)$.

## Expansion of the step's duration for large amplitude

We define

$$2\pi\frac{\bar{C}}{A} = \epsilon \ll 1$$

Then the equation for $\tau_f$ is

$$\epsilon(\tau_f - \tau_0) = \cos(2\pi\tau_f) - \cos(2\pi\tau_0)$$

$$\tau_0 = \frac{1}{2}\left[1 - \frac{1}{\pi} - \arcsin\left(-\frac{\epsilon}{2\pi}\right)\right] \simeq \frac{1}{2} + \frac{\epsilon}{(2\pi)^2} + O(\epsilon^2)$$

As $\tau_f \to \frac{3}{2}$ in the limit $\epsilon \to 0$ (think that this limit is achieved by $\bar{C} \to 0$ with $A$ finite. In this limit $\tau_0 = \frac{1}{2}$ and $\tau_f = \tau_0 + 1$) we estimate $\tau_f$ by expanding $\cos(2\pi\tau_f)$ close to $\tau_f \simeq \frac{3}{2}$

$$\cos(2\pi\tau_f) \simeq -1 + \frac{(2\pi)^2}{2}\left(\tau_f - \frac{3}{2}\right)^2 + \dots$$

using this in the first expression, and using

$$\sin(2\pi\tau_0) = -\frac{\epsilon}{2\pi} \implies \cos(2\pi\tau_0) = -\sqrt{1-\sin^2} \simeq -\left(1 - \frac{\epsilon^2}{2(2\pi)^2} + O(\epsilon^4)\right)$$

along with the estimate of $\tau_0$ written above, we find (neglecting $O(\epsilon^2)$)

$$\epsilon\left(\tau_f - \frac{1}{2}\right) \simeq \frac{(2\pi)^2}{2}\left(\tau_f - \frac{3}{2}\right)^2 + O(\epsilon^2)$$

considering the root $< \frac{3}{2}$ (as we expect $\tau_f$ to decrease if $A$ increases with fixed $\bar{C}$)

$$\tau_f = \frac{3}{2} - \frac{\sqrt{2}}{2\pi}\epsilon^{1/2} + \frac{\epsilon}{(2\pi)^2} + O(\epsilon^2)$$

So, the estimated duration of the step is

$$\tau_f - \tau_0 \simeq 1 - \frac{\sqrt{2}}{2\pi}\epsilon^{1/2} + O(\epsilon^2)$$

remembering $\epsilon = \frac{2\pi\bar{C}}{A}$, then

$$t_f - t_0 = f\left(\frac{\bar{C}}{A}\right)T$$

$$f\left(\frac{\bar{C}}{A}\right) \simeq 1 - \sqrt{1/\pi}(\frac{\bar{C}}{A})^{1/2}$$

We can use this expression to estimate $A$ ($\bar{C} = 1$) from the measures of duration of the step $\Delta t_{step}$ by inverting

$$\Delta t_{step} = t_f - t_0 = T(1 - 1/\sqrt{\pi}A^{-1/2})$$

here we see the ratio of the estimated amplitude respect to the true value.

- It is expected that it is always underestimated as I expect the non-linearity do become negligible a little bit after C(t) crosses zero and not when it crosses it.
- It is expected that the last datapoint is not 1, because the last steps lasts a time $t_c < \Delta t_{step}$.

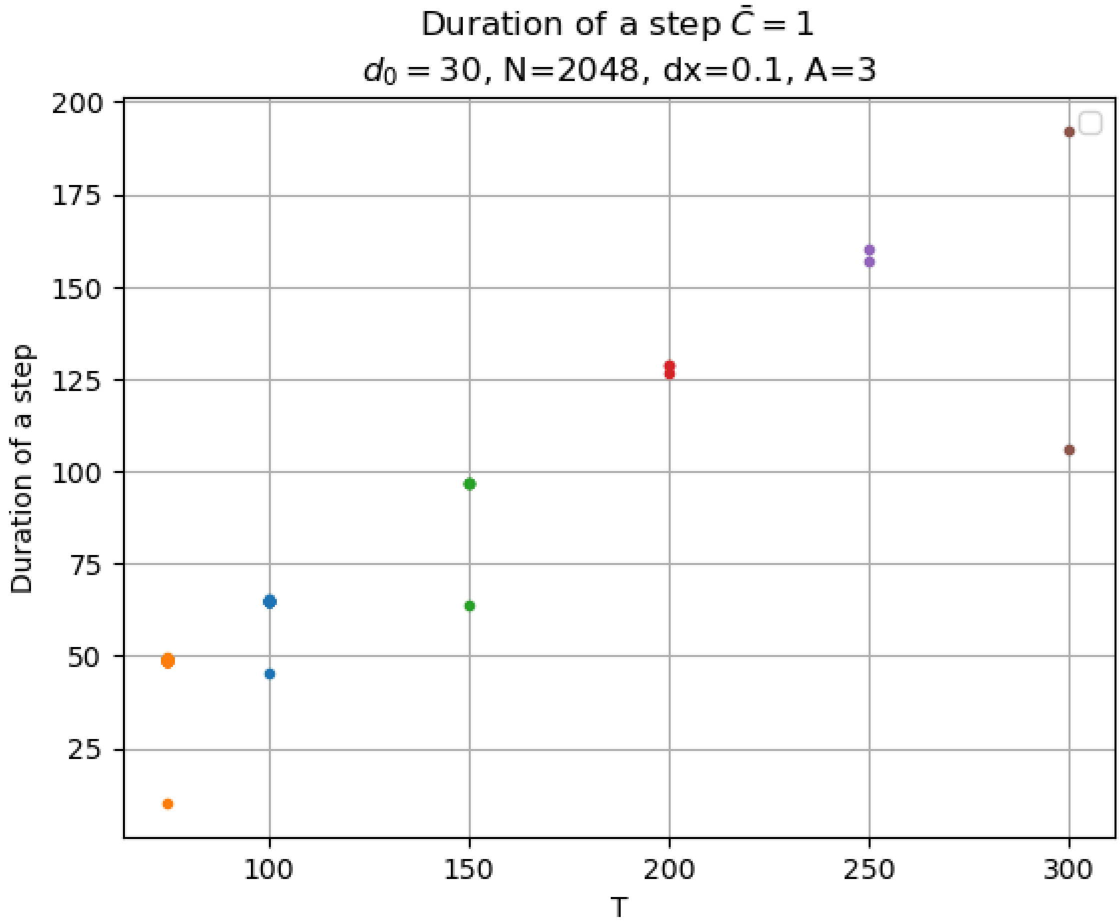
## Measuring the duration of the steps

I measure the duration of the steps $t_1 - t_0$ by considering

- $t_0$ as the instant when $C(t)$ becomes negative
- $t_1$ is estimated as the first time $t_1 > t_0$ where the derivative $\partial_t d$ becomes smaller than a tollerance $10^{-5}$

  **Linear dependence on T**

  The points far away represent the duration of the last step that is not expected to be linear in $T$.

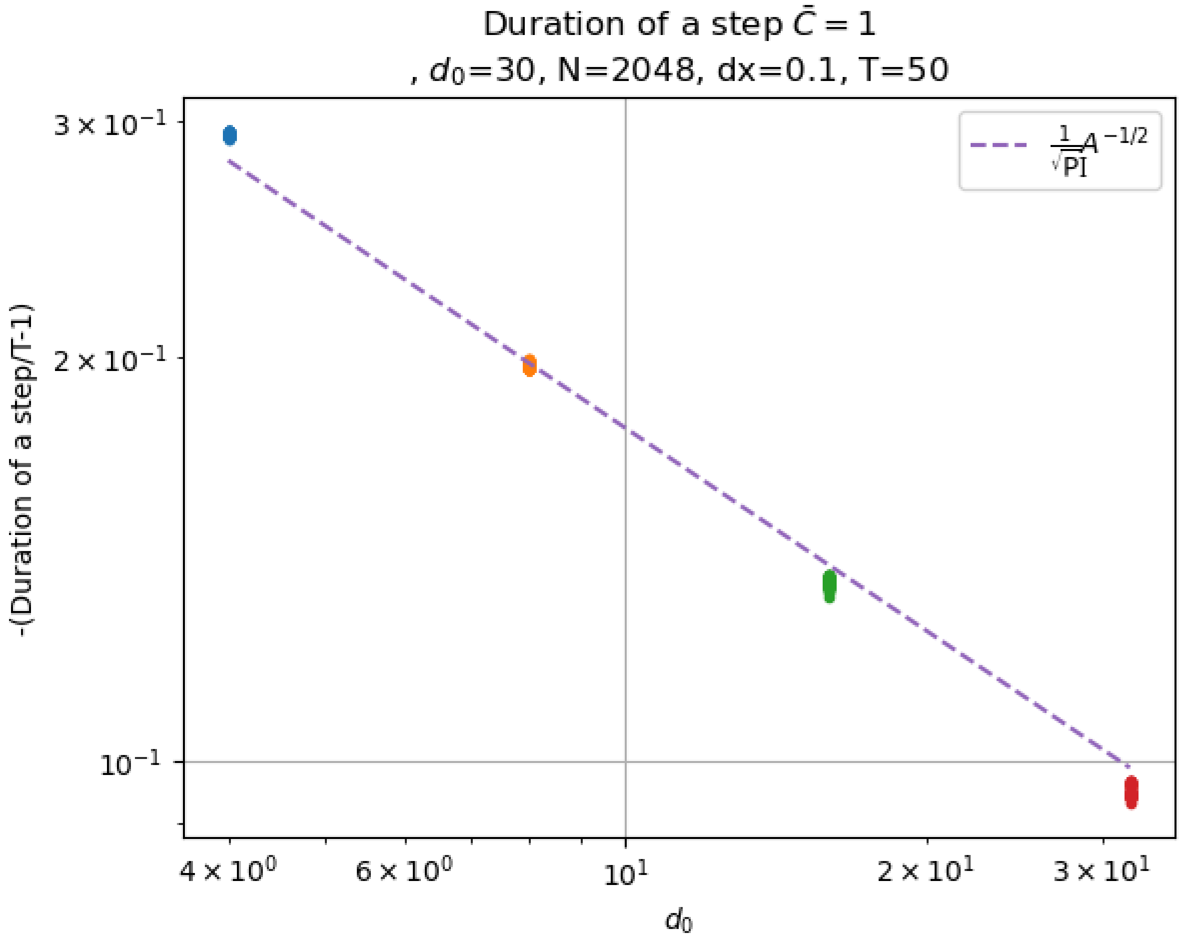



**Dependence on A**

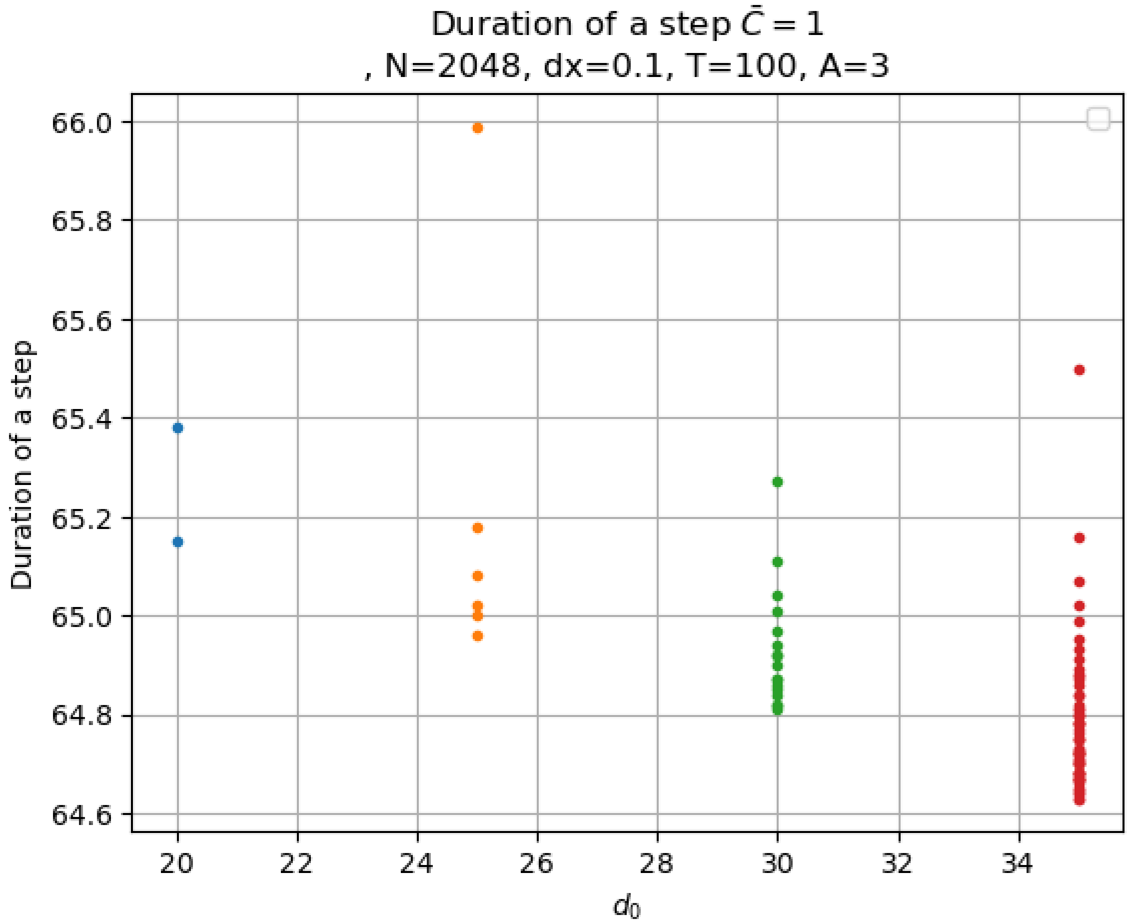
$$\Delta t_{step} \simeq T(1 - 1/\sqrt{\pi}A^{-1/2}) \implies -\left[\frac{\Delta t_{step}}{T} - 1\right] \simeq \frac{1}{\sqrt{\pi}}A^{-1/2}$$

Here we don't see any point far away, because the simulation ends before the collapse (notice here $T$ is smaller and I made this choice to lunch simulations with higher $A$ having at least a couple of steps before collapse).

Maybe considering higher order terms in the asymptotic expansion, we could get a better match




**Dependence on initial distance**



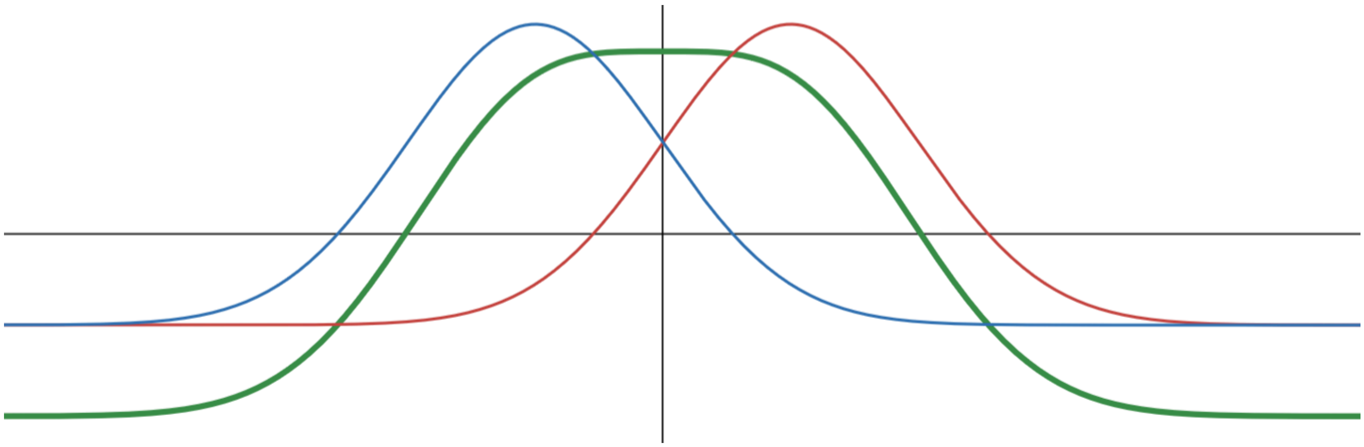

**Comment**: It seems there is a dependence of the duration of the step on the index of the step and on the initial distance. It could be related to **how I measure the step duration**. As I estimate the end of the decay by checking when $\partial_t d$ goes below a tollerance and not $\frac{\partial_t d}{d}$

# Model 1: Sum of TWO Gaussian with the same $\sigma$

We compute analytically the decay of the distance of this initial state, according to linear dynamics.

$$u(x, t_0) = \mathcal{N}\left[g_+(x) + g_-(x) - \frac{e^{-1/2}}{\sqrt{2\pi}\sigma}\right]$$

where $\mathcal{N}$ is a normalization, that sets the value of the green curve at $x = 0$ (and $x = \pm\infty$).



$$g_{\pm}(x) = \frac{e^{-(x-x_{\pm})^2/2\sigma^2}}{\sqrt{2\pi}\sigma}$$

As $\partial_{xx} g_{\pm}(x_{\pm}) = \frac{3e^{-2}}{\sigma^2}$, then we choose $L = 2\sigma$ (where $L = x_+ - x_-$) such that the second derivative is zero in the midpoint (**plateau**).

## Parameters of the model

There is **only one parameter** to be set in order to compute the prediction of $d(t)$, that is **the initial width** $\sigma$ of the kinks. That's because $x_+ - x_- = 2\sigma$ is the condition required to have a plateau between the gaussians (a property that is kept with time as the width of the two gaussian is the same).
The initial distance between kinks $d_0$, is related to the distance $L$ between the centers of the Gaussians ($x_{\pm}$) and their initial width $\sigma$ as

$$d_0 \simeq L + 2\sigma$$

then

$$\sigma \simeq \frac{d_0}{4}$$

When **approximating** the shape of $u(x)$ to a sum of two Gaussians, this is the **natural** way of determining the (only) parameter $\sigma$, by measuring the distance between the kinks.

## Decay of the distance

In [Linear dynamics twokinks with Gaussian profile] is calculated analytically the evolution of the above profile, under the linear dynamics

$$u(x,t) = e^{B(t)} \mathcal{N} \left[ \frac{e^{-(x-x_+)^2/2\sigma(t)^2}}{\sqrt{2\pi}\sigma(t)} + \frac{e^{-(x-x_-)^2/2\sigma(t)^2}}{\sqrt{2\pi}\sigma(t)} - \frac{e^{-1/2}}{\sqrt{2\pi}\sigma} \right]$$

$$\sigma(t)^2 = \sigma_0^2 + 2(t - t_0)$$

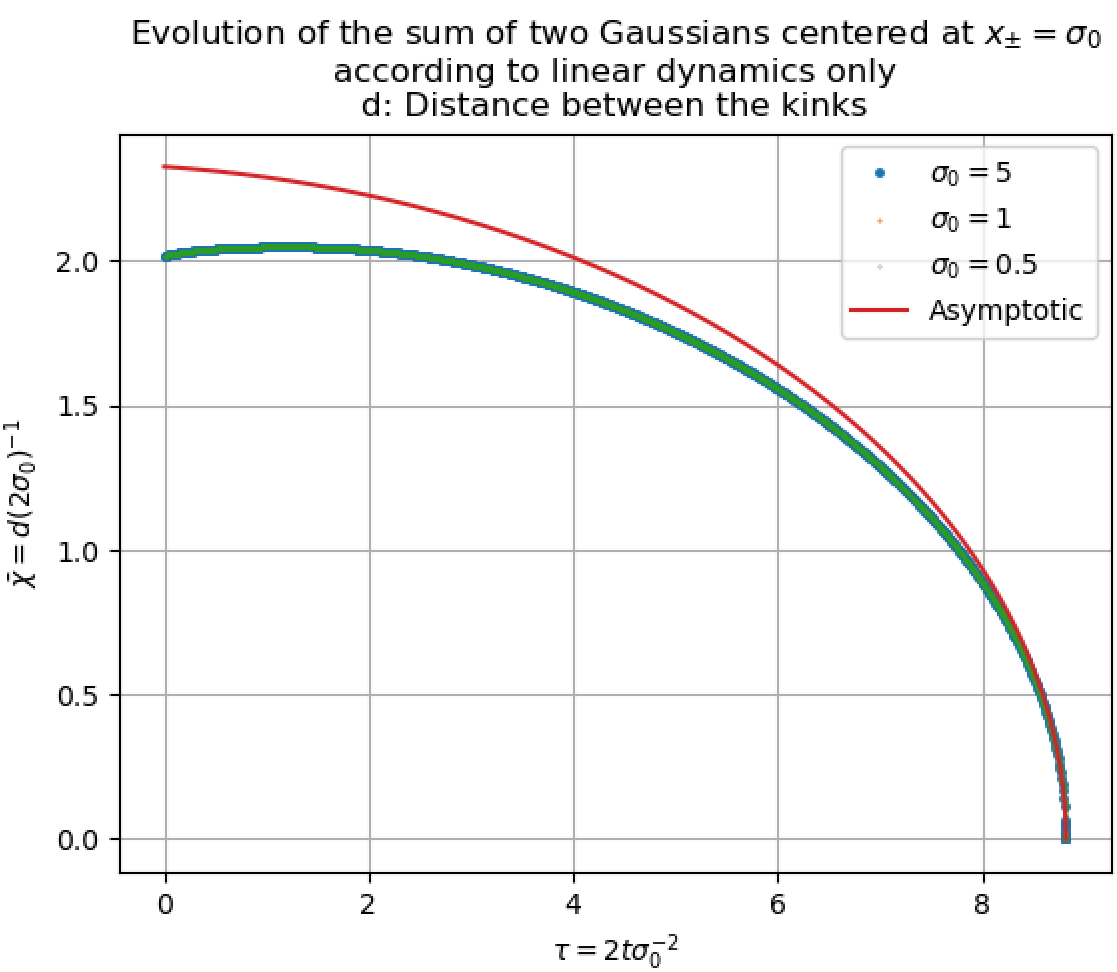
**Notice**: the positions of the zeros (kinks), and so their distance, does NOT depend on $C(t)$ and on the amplitude $\mathcal{N}$ of the initial profile.
Properly **rescaling** the axis (Linear dynamics twokinks with Gaussian profile), I find a profile whose shape is not dependent on the only parameter $\sigma$.

- $d \to \frac{d}{\sigma_0}$ and $\bar{\chi} = \frac{d}{2}$ is the position of the positive kink.
- $t \to \tau = 2\frac{t-t_0}{\sigma_0^2}$
  I cannot write a formula for $d(t)$, but I can find it numerically with the **Newton's algorithm**.
  And I can find an asymptotic expansion, close to the collapse time $t_c$

$$\bar{\chi} \simeq \frac{1+\tau_c}{\tau_c^{1/2}}\left[\frac{\tau_c-\tau}{1+\tau_c}+\frac{(\tau_c-\tau)^2}{2(1+\tau_c)^2}\right]^{1/2} \quad \text{if } \frac{\tau_c-\tau}{\tau_c} \ll 1$$

  where $\tau_c \simeq 8.82$ is determined by

$$2\frac{e^{-1/2(1+\tau_c)}}{(1+\tau_c)^{1/2}} = e^{-1/2}$$

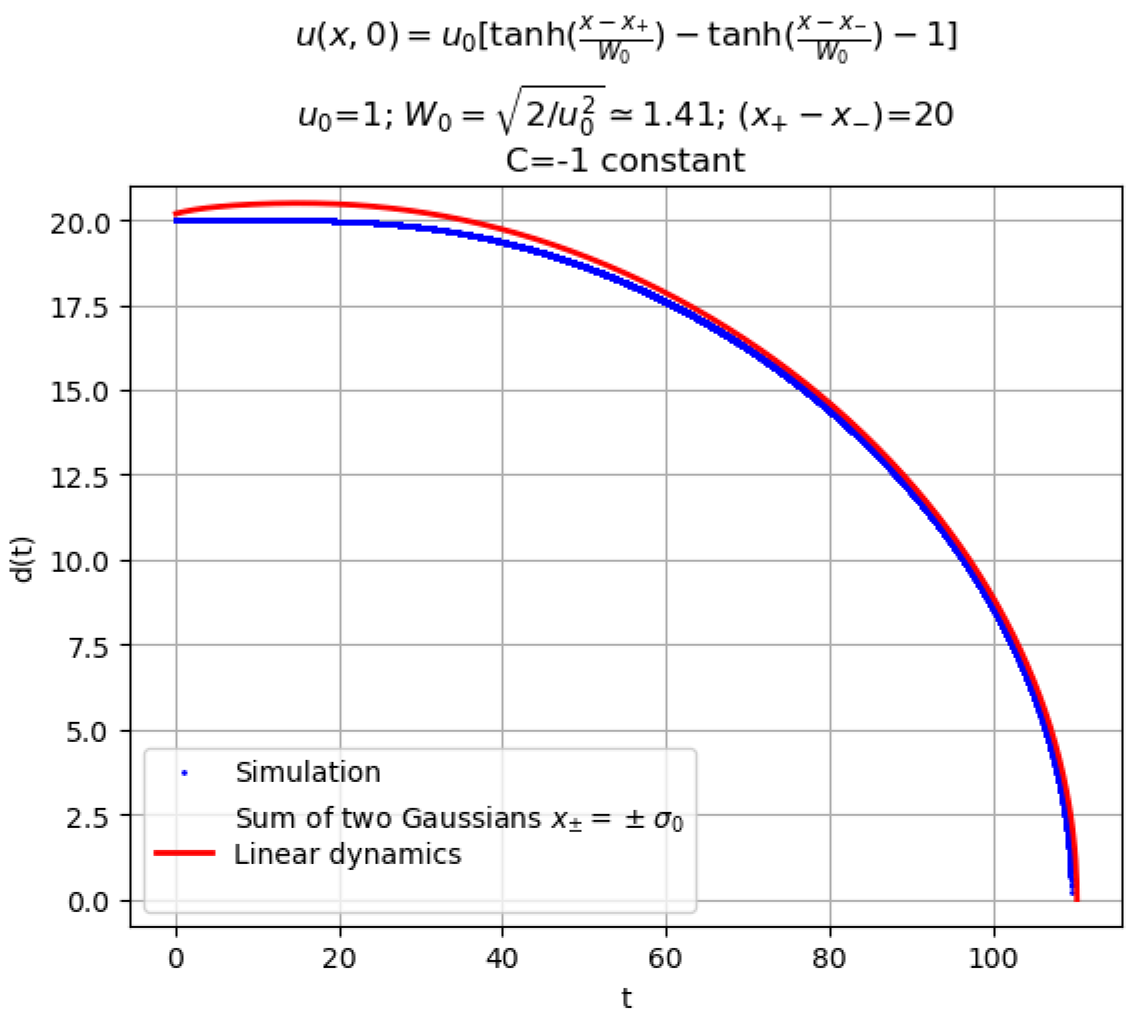



## Experiments

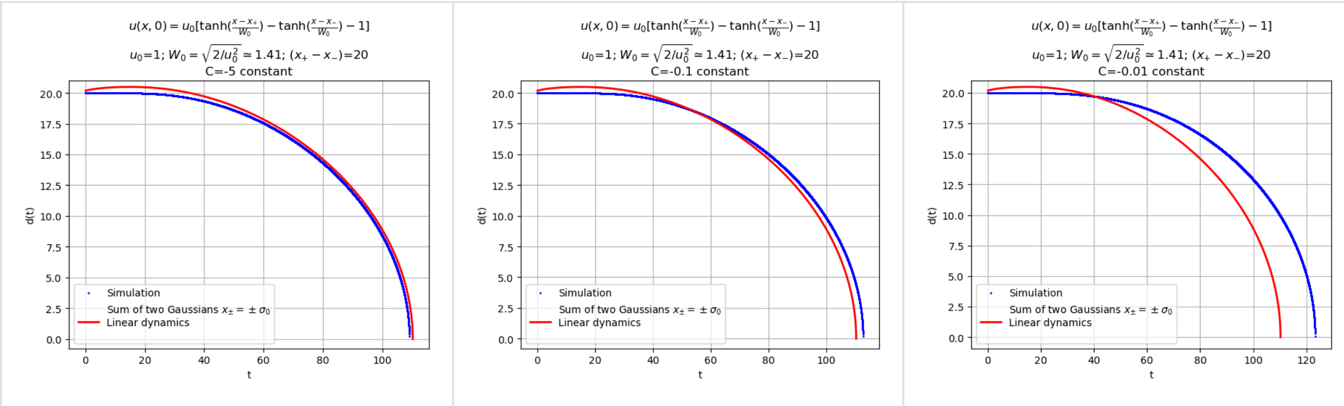
## 1) Quench to $C < 0$

**Paramters (for the SIMULATION)**:

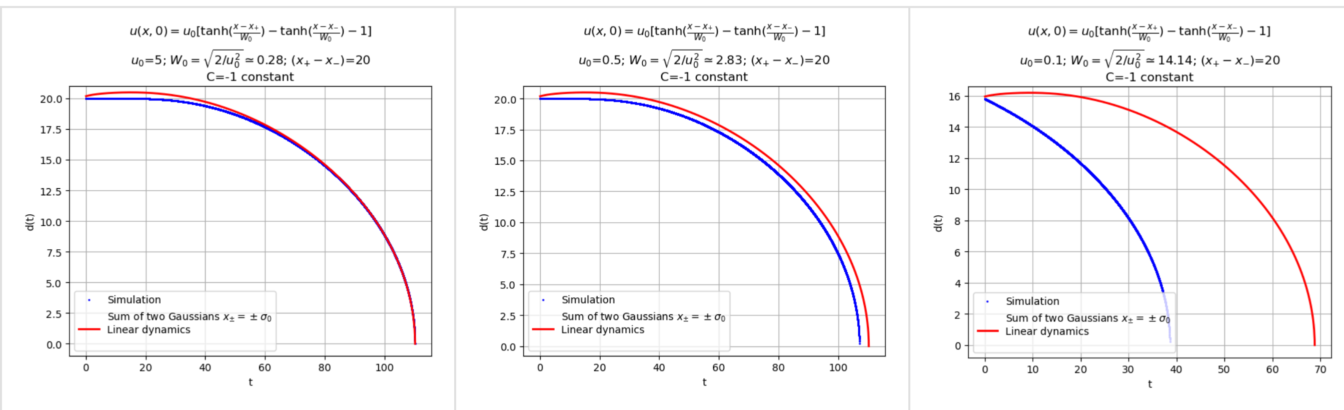
- $u_0$: amplitude of the initial state **and** $w_0 = u_0^{-1/2}$ initial width of the kinks
- $d_0$: initial distance beetween kinks
- $C < 0$: constant value of C



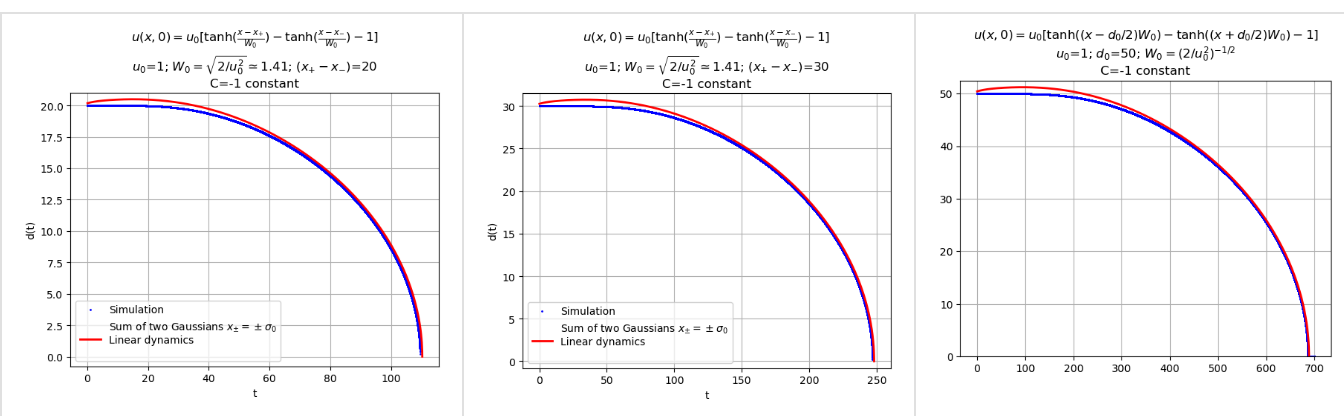

When $|C| \sim 0.1$ the fit is not good, not even at late times. I guess this is due to the relevance of the non linearity up to longer times, as $u$ **decays slower** to zero in this case.




- **Surprisingly** if the initial amplitude is small $u_0 \sim 0.1$, the fit is bad also at late times. I would expect it to be better, as the non-linearity is less important!




While the goodness of the fit does not depend on the initial distance $d_0$.



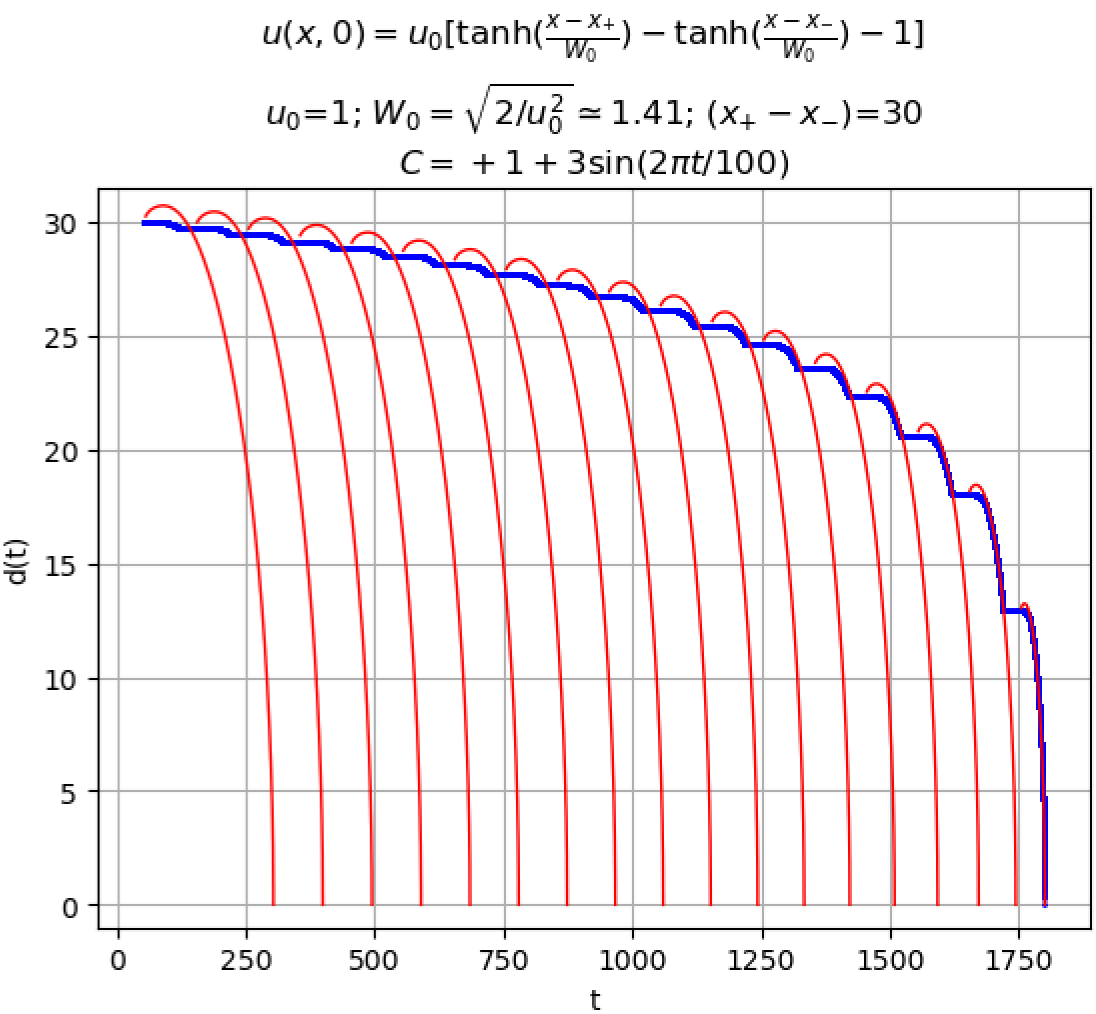

## 2) Slow oscillations $A \gg C_0$

$$C(t) = C_0 + A\sin\left(\frac{2\pi t}{T}\right); \quad C_0 = 1$$

### At what time $t_0$ the step originates?

Here we assume the beginning of the decay $t_0$ as the moment, within a period, **when $C(t)$ starts to take negative values:** $C(t_0) = 0; \dot{C}(t_0) < 0$.
This choice leads to a good fit, when the **depth of the step is large.**

- At the beginning of the step, the red curve is increasing, while the simulation is strictly decreasing. I expect this to be an effect of the **non-linearity, that still plays a role in the first istants** of negative C.
- After the beginning of the step, the fit is good, **without any shift**.




## Predicting the depth of the step

If the duration of the step $\Delta t_{step}$ is close to the decay time $t_c$ computed in the model of two Gaussians, then, in the last istants of the step, we are in the regime of validity of

$$\bar{\chi} \simeq \frac{1+\tau_c}{\tau_c^{1/2}} \left[ \frac{\tau_c - \tau}{1+\tau_c} + \frac{(\tau_c-\tau)^2}{2(1+\tau_c)^2} \right]^{1/2} \quad \text{if } \frac{\tau_c - \tau}{\tau_c} \ll 1$$

where $\tau = \frac{2(t-t_0)}{\sigma_0^2}$, $d(t) = \frac{d_0}{2}\bar{\chi}(\tau)$.
It holds in the last moments of the step, so also at its end $t - t_0 = \Delta t_{step}$

$$\Delta t_{step} \simeq T[1 - \sqrt{\frac{1}{\pi}\left(\frac{\bar{C}}{A}\right)^{1/2}}]$$

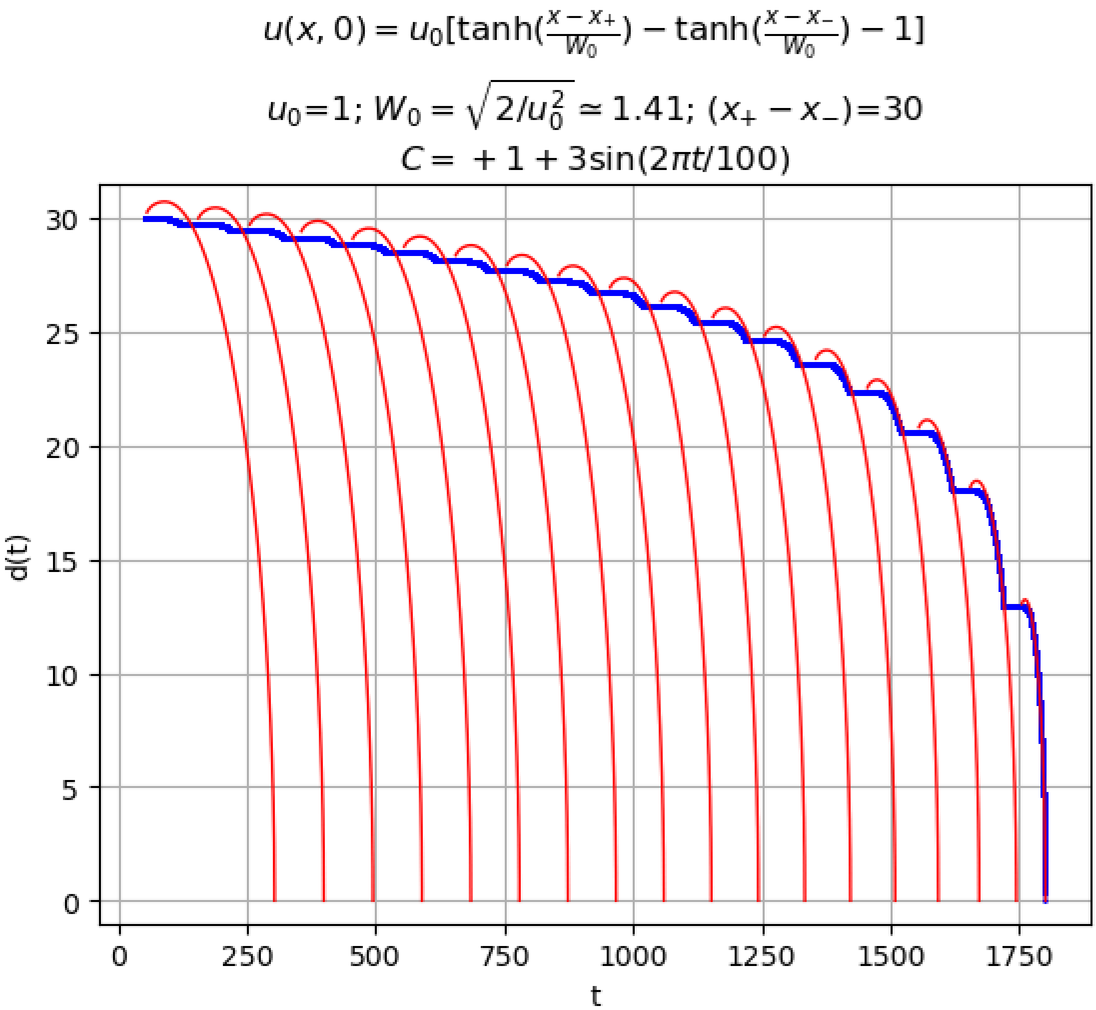
$$\Delta \tau_{step} = \frac{2\Delta t_{step}}{\sigma_0^2} \sim d_0^{-2} T[1 - \sqrt{\frac{1}{\pi}\left(\frac{\bar{C}}{A}\right)^{1/2}}]$$

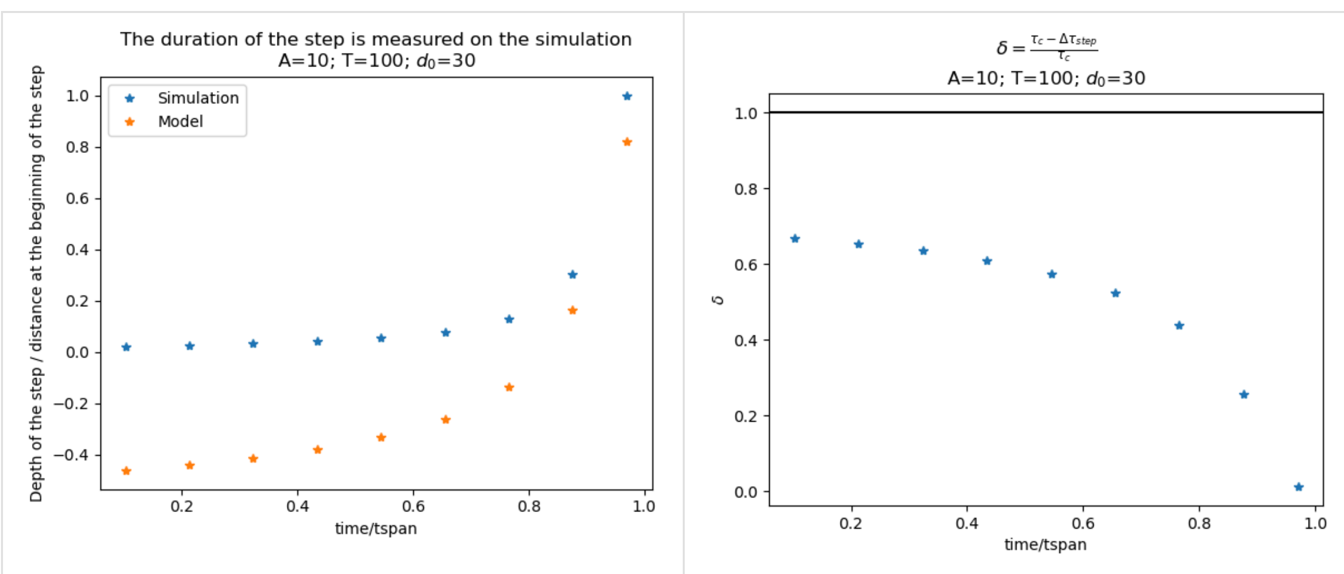
To estimate the depth of the step, we should insert $\Delta\tau_{step}$ as $\tau$ in the expression for $\bar{\chi}(\tau)$, where $\tau_c \simeq 8.82$. If, instead, $\Delta t_{step} > t_c$, the depth of the step is $d_0$ and its duration is $t_c$.

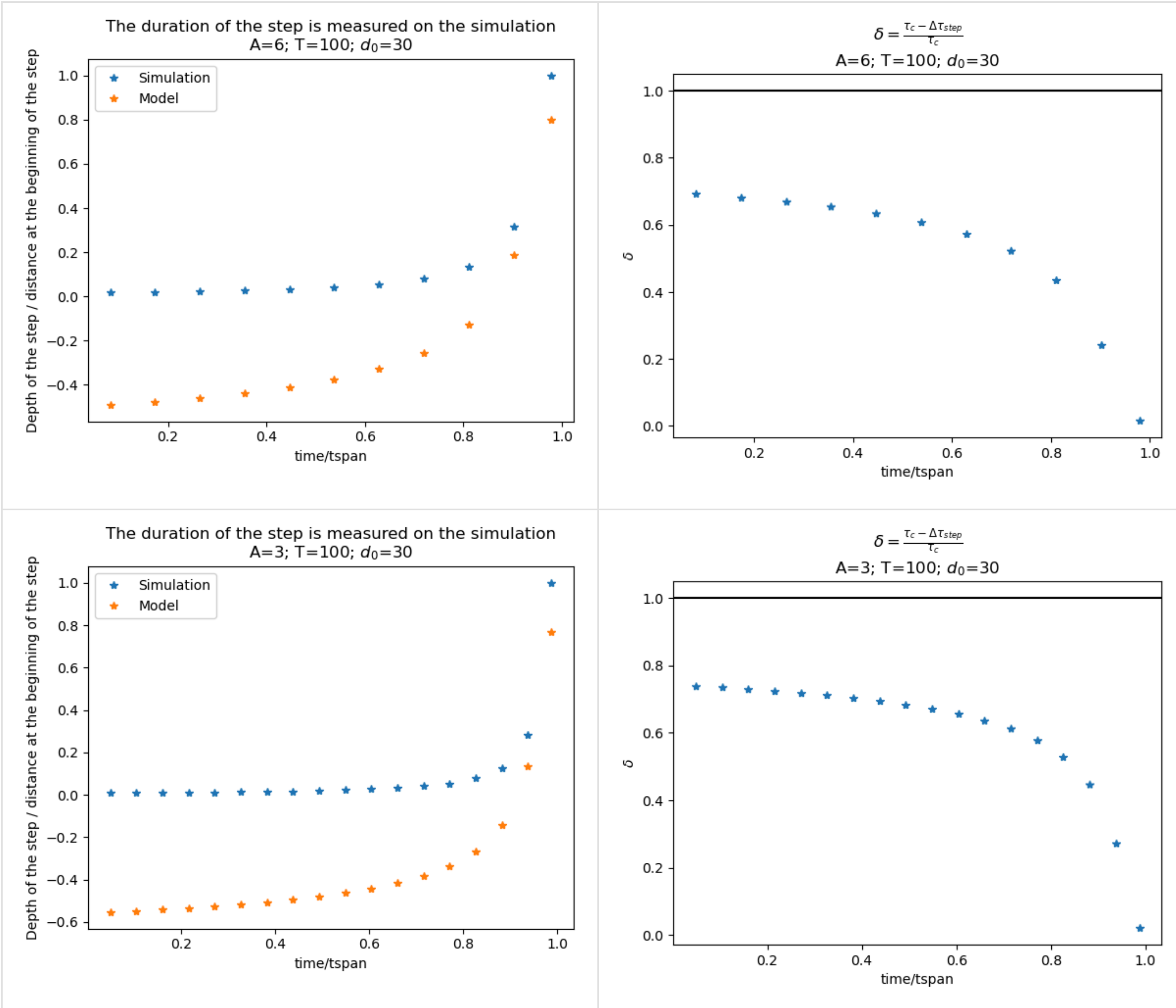
## Predicting $\Delta d$ by MEASURING $\Delta t_{step}$

We can evaluate the expression for $\bar{\chi}(\tau)$ at $\tau = \Delta\tau_{step}$ where the duration of the step is **measured**. In the following plots, it is reported the value of the small parameter $(\tau_c - \tau)/\tau_c$ for $\tau = \Delta\tau_{step}$.
The asymptotic expansion for $\bar{\chi}(\tau)$ is true when that parameter is very small, but the plots below show that it is not so small, except for the last period. This is coherent with the fact that the estimate is **bad** even when there is a good fit.
**Notice**: The prediction $\frac{\Delta d}{d} < 0$ when $\Delta\tau_{span} < 4$, because the asymptotic expansion, far from $\tau_c$,

behaves like that.



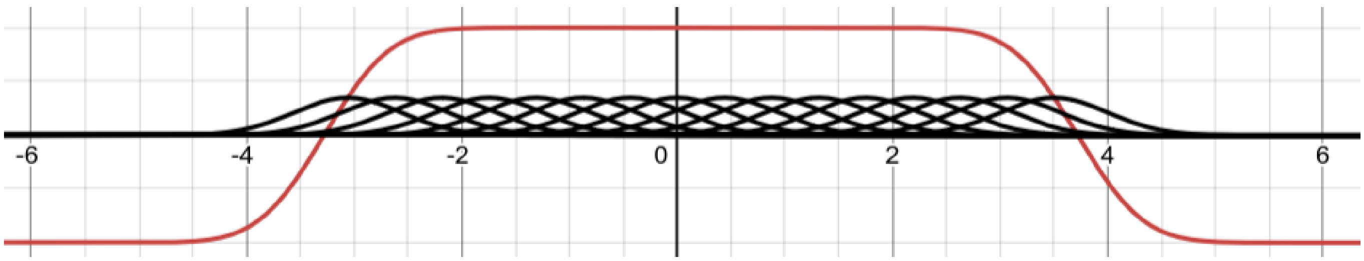



# Model 2: Sum of INFINITE amount of Gaussians

Here we exploit the following results, about a sum of an infinite amount of Gaussian functions with the same $\sigma$

$$f(x) = \lim_{N\to\infty} \frac{2L}{N} \left( \sum_{n=1}^{N} g_n(x) - \frac{1}{2} \right) = \frac{1}{2} \left( \operatorname{erf}\left( \frac{x}{\sqrt{2}\sigma} \right) - \operatorname{erf}\left( \frac{x-L}{\sqrt{2}\sigma} \right) - 1 \right)$$

$$g_n(x) = \frac{e^{-(x-nL/N)^2/2\sigma^2}}{\sqrt{2\pi}\sigma}$$




**Proof**

$$\frac{1}{\sigma\sqrt{2\pi}} \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \exp\left(-\frac{(x-\frac{i}{n})^2}{2\sigma^2}\right) = \frac{1}{\sigma\sqrt{2\pi}} \int_0^1 \exp\left(-\frac{(x-y)^2}{2\sigma^2}\right) dy =$$

$$= \frac{1}{2}\left(\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right) - \operatorname{erf}\left(\frac{x-1}{\sqrt{2}\sigma}\right)\right)$$

# Parameters of the model

The advantage of this profile is that the two parameters $L, \sigma$ are **independent**!
If $L \gg \sigma$, then $d \simeq L$, while $\sigma$ describes the **width** of the kinks $W = \sqrt{2}\sigma$.

# Decay of the distance

We can exploit the fact that the initial profile is a sum of Gaussian functions, to conclude that the profile at time $t > 0$ due to the **linear dynamics only** is itself with $\sigma_0^2 \to \sigma_0^2 + 2t$

$$f(x) = \frac{1}{2}\left(\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma(t)}\right) - \operatorname{erf}\left(\frac{x-L}{\sqrt{2}\sigma(t)}\right) - 1\right)$$

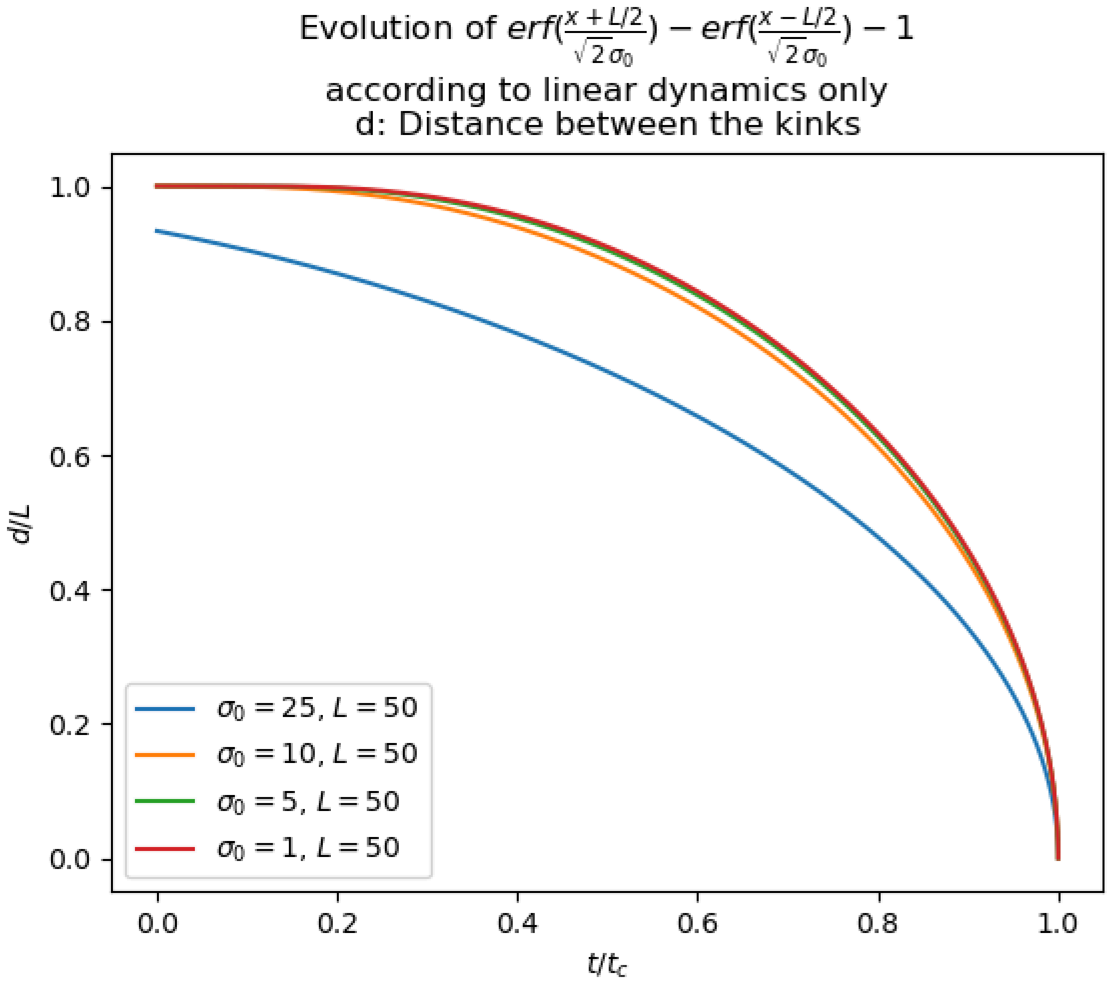
$$\sigma(t)^2 = \sigma_0^2 + 2t$$

To find the distance as a function of time, we look for the zeros of $f(x)$

$$\operatorname{erf}\left(\frac{x+\frac{L}{2}}{\sqrt{2}\sigma(t)}\right) - \operatorname{erf}\left(\frac{x-\frac{L}{2}}{\sqrt{2}\sigma(t)}\right) = 1$$

Computing this with the **Newton's method** .

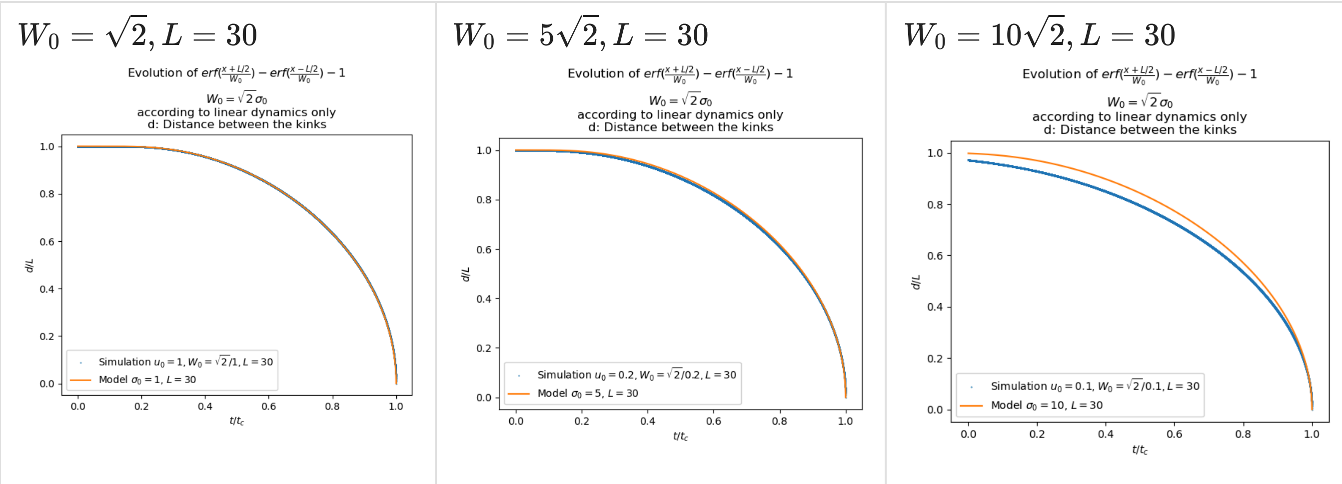
**Notice**: With this model, the prediction of $d(t)$ is always decreasing, **also at the beginning**!



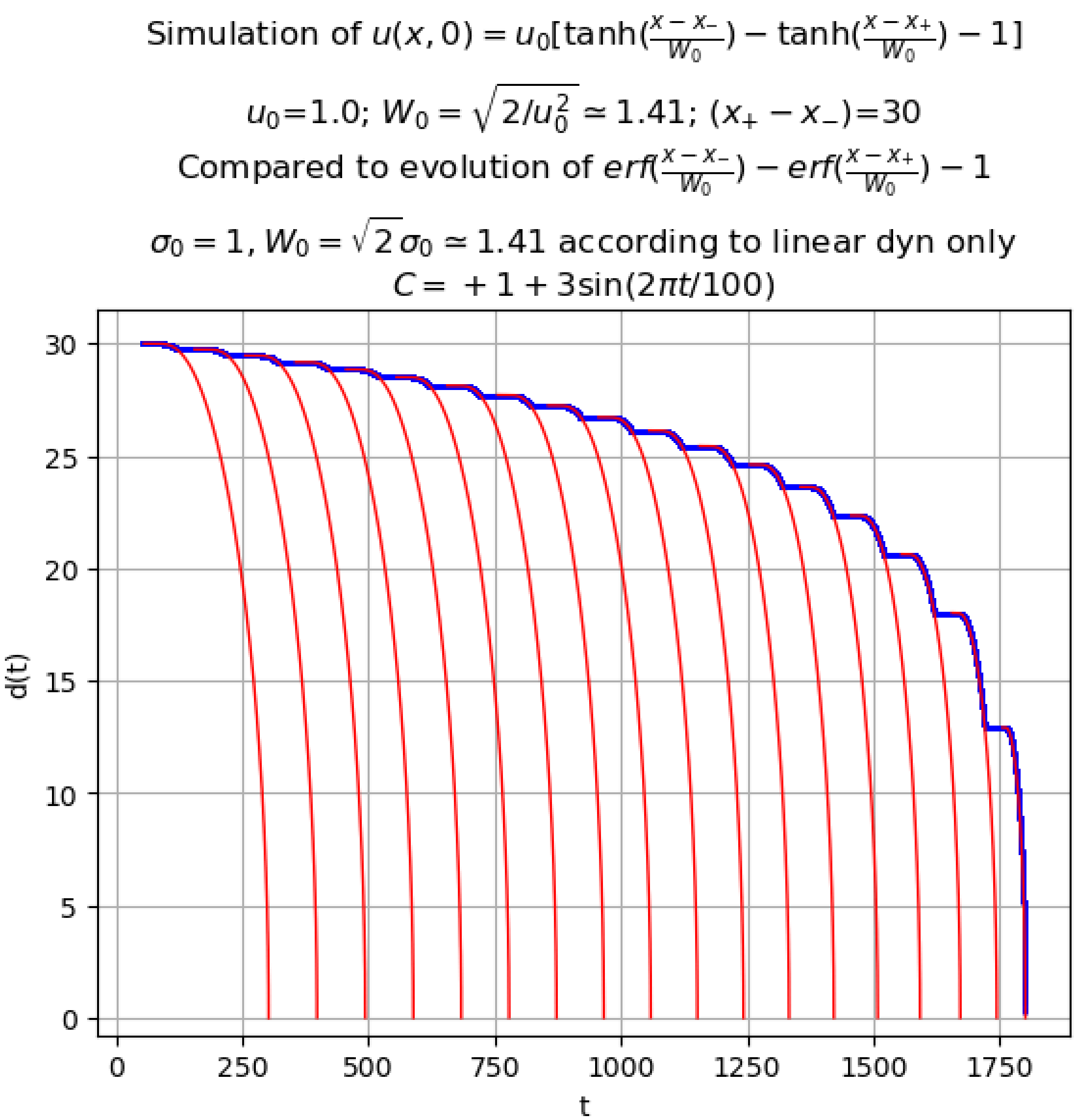

## Experiments

### 1) Quench to C=-1

**Comparing with a simulations**, we can choose the initial width in the model as the initial width in the simulation. There are problems when there is overlap between the two kinks in the initial state ($\sigma \sim L$).




## 2) Oscillations C(t)




## Problems with this model

- Lack of an analytical expansion, also asymptotical (I did not tried much)
- If the kinks overlap when $C(t)$ becomes negative, the model fails. But it is never the case in the simulations where $A >> \bar{C}$ and slow oscillations.