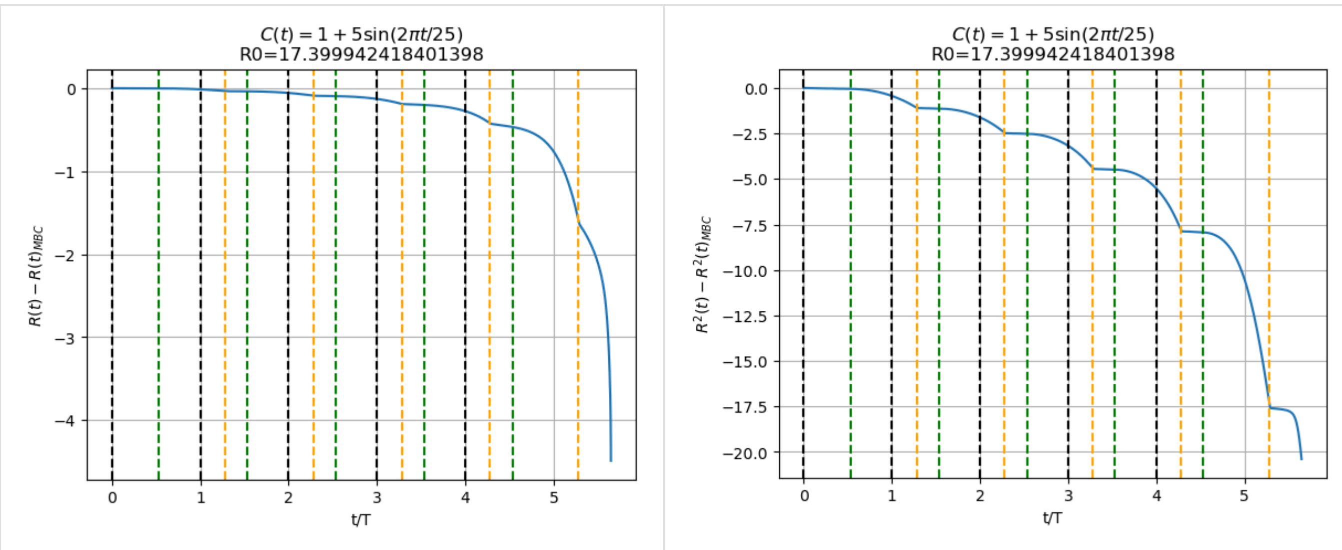
2D Slow oscillations (A>>C0) (Analytical)

Evidence of steps

Subtracting a simulation with $C = \bar{C}$ from a simulation with $C(t)$ shows **steps**.
In analogy to 1D, we try to explain steps by

- Assuming the non-linearity is negligible in intervals $[t_0, t_f]$ where t_0 is the time when $C(t)$ becomes negative and t_f the time when the non-linearity becomes relevant again;
- Considering an erf-shaped interface, to calculate **analytically** $R(t)$ during the linear dynamics.



Build a circular island with Erf-shaped interface

We sum an **infinite amount of gaussians**, by considering the points $(i * dx, j * dx)$ of a square lattice that are contained in a circle of radius R (initial radius of the domain) and then we sum Gaussians centered at these points and with the same width. Finally we take the limit $dx \to 0$:

$$G(\mathbf{r_0}) = \frac{\pi R^2}{N} \sum_{i,j \in \mathcal{Z}}^{i^2+j^2 \leq (R/dx)^2} g(\mathbf{r_{ij}}, \mathbf{r_0}, \sigma) \quad \mathbf{r_{ij}} = (i * dx, j * dx) \quad N = \frac{\pi R^2}{dx^2}$$

where N is the number of points of the square lattice inside the circle, so the numbers of elements of the sum. Taking the limit $dx \to 0$ (and subtracting $\frac{1}{2}$):

$$G(r_0) = \int_0^R r dr \int_0^{2\pi} d\theta g(\mathbf{r}, \mathbf{r_0}, \sigma) - \frac{1}{2}$$

$$g(\mathbf{r}, \mathbf{r_0}, \sigma) = \frac{1}{2\pi\sigma^2} e^{-(\mathbf{r}-\mathbf{r_0})^2/2\sigma^2}$$

It follows

$$G(r_0) = \frac{1}{\sigma} \int_0^{R/\sqrt{2}\sigma} r e^{-(r^2+r_0^2)/2\sigma^2} B_I \left(0, \frac{2rr_0}{2\sigma^2}\right) dr - \frac{1}{2}$$

and in the limit where $r_0 \gg \sqrt{2}\sigma$, then we can asymptotically expand the Bessel function. And using that $re^{-(r-r_0)^2}$ is significatively different from zero only when $r \simeq r_0$ then

$$G(r_0) \simeq \frac{1}{2} \left[erf\left(\frac{r_0}{\sqrt{2}\sigma}\right) - erf\left(\frac{r_0 - R}{\sqrt{2}\sigma}\right) - 1 \right] \quad \text{if } r_0 \gg \sqrt{2}\sigma$$

$r_0 \gg \sqrt{2}\sigma$, because this is the width of the left kink, centered at $r_0 = 0$, that is a feature of the approximation and **not** of $G(r_0)$.

Estimating the depth of a step

Considering an erf-shaped interface and linear dynamics in the interval $[t_0, t_f]$, we can calculate **analytically** the variation of $R(t)$ in this interval, that we expect to estimate the depth of the step.

$$\Delta R_{step} = -\frac{8}{\sqrt{\pi}} TkR^{-1} e^{-R^2/4kT}$$

where $k = \left(1 - \frac{1}{\sqrt{\pi}} \left(\frac{\bar{C}}{A}\right)^{1/2}\right)$ if $\bar{C} \ll \pi A$ and R is the radius at the beginning of the step.

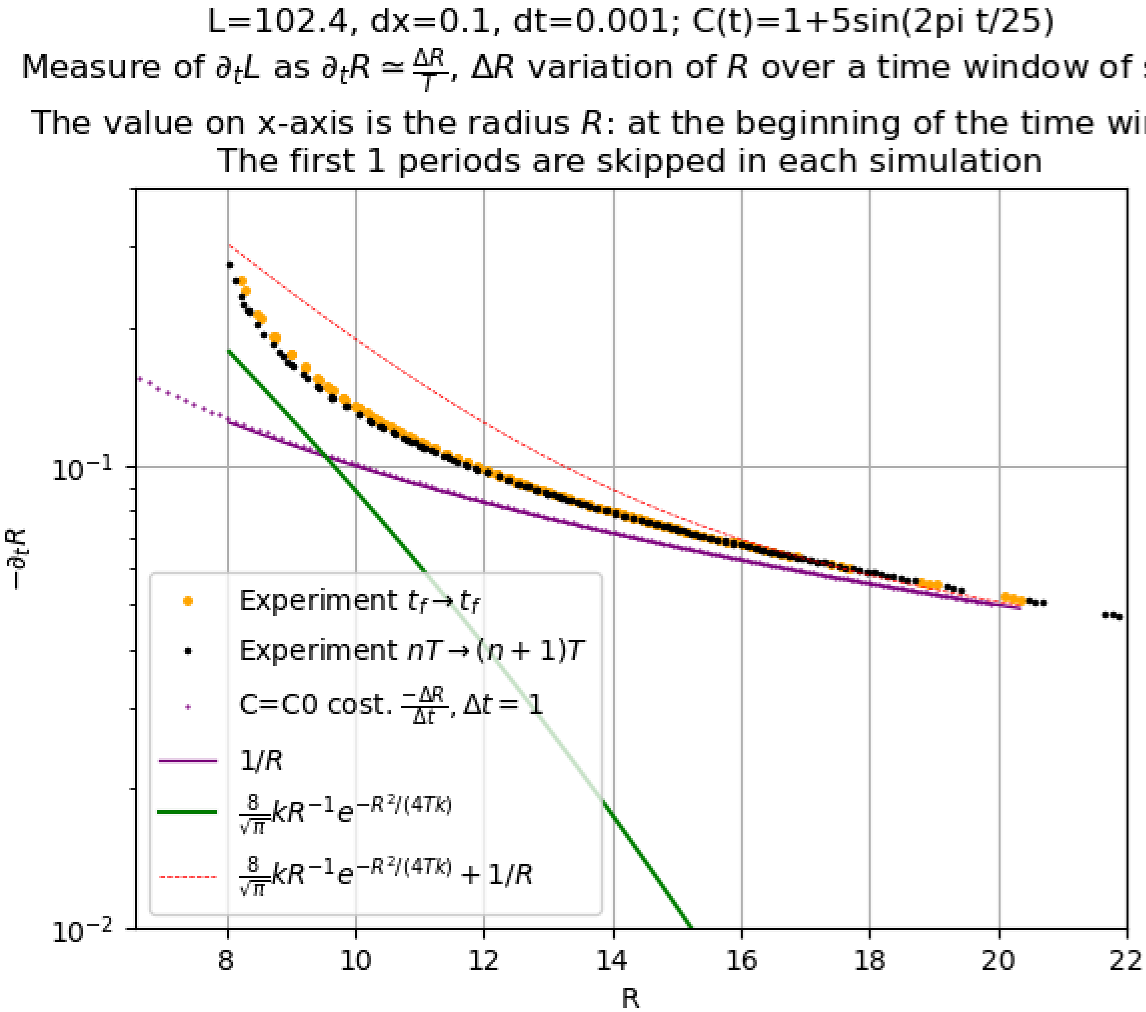
Notice (other contribution): We calculated the variation of R in the interval $[t_0, t_f]$, but this interval has lenght $\Delta t_{step} < T$, so the variation of R over an interval of one period should contain **another contribution**. However $\Delta t_{step} \to T$ very fast as $\frac{A}{C_0}$ grows, so likely the other contribution is negligible.

Macroscopic derivative

As in 1D, we can define a macroscopic derivative as

$$\partial_t R \equiv \frac{\Delta R}{T}$$

where ΔR is the variation of the radius over a window of time of size T. This is a function of R, the initial radius at the beginning of the variation ΔR.



Asymptotics

Asymptotically, the $C(t)$ data matches the $C = \bar{C}$ data. So MBC holds.

Low R dynamics

When the radius is comparable to the crossing point R_2^*, the data do not match neither one of the two behaviour, neither the sum (as it happens in 1D).

Why linear dynamics with erf-shaped interface does NOT predict MBC?

Asymptotically the green line does not resemble MBC, due to the decayng exponential term.
While in 1D we expected to not predict kink dynamics as the shape of the tail is responsible for that and we assume a wrong one, here we expected to predict MBC.
In fact, for a circular domain

$$\partial_t u(r,t) = \partial_{rr}u + \frac{1}{r}\partial_r u + Cu - u^3$$

evaluating at the interface $r = R$ and assuming the interface propagates without changing shape, at least for $r \simeq R$:

- $u(R) = 0 \implies Cu - u^3 = 0$ (so it does not matter if we consider linear dyn only or not)
- $u(r,t) = u(r - \dot{R}t)$

$$-\dot{R}\partial_r u|_{r=R} = \partial_{rr}u|_{r=R} + \frac{1}{R}\partial_r u|_{r=R}$$

  if the **shape of the interface** is well approximated by an **error function**, then $\partial_{rr}u|_{r=R} = 0$ and it follows MBC.
  So we prooved that for a circular island with erf-shaped interface, evolving under linear dynamics, MBC is expected. But we are not catching it in our calculations!.

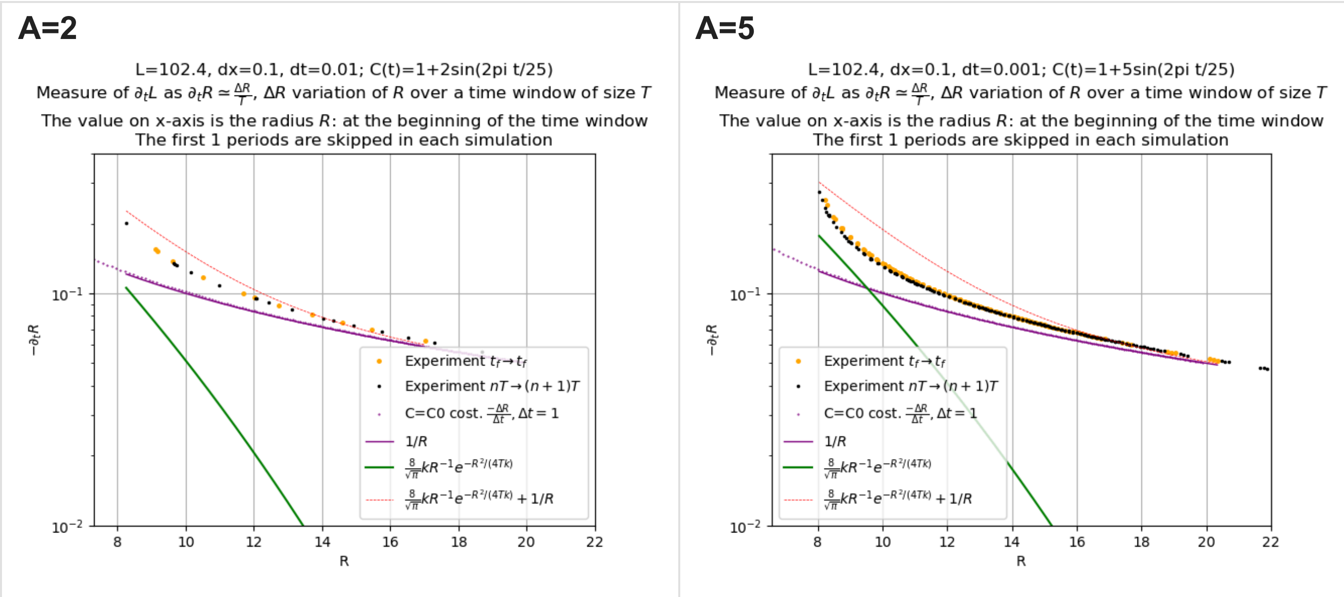
Is the macroscopic derivative $\partial_t R$ WELL defined in 2D?

As in 1D, we can define a macroscopic derivative as

$$\partial_t R \equiv \frac{\Delta R}{T}$$

where ΔR is the variation of the radius over a window of time of size T. This is a function of R, the initial radius at the beginning of the variation ΔR.
Well defined: The macroscopic derivative is well defined if its value does not change significatively if the window of time of size T is shifted of a value $\ll T$.

If you look at the depth of the steps, this starts to change fast approaching the last steps. Intuitively, this leads to strong variations of $\partial_t R$ due to a small shift of the time window. But experiments do not show big differences. So the **macroscopic derivative is WELL defined**!



Explaination: Our simulations **never** show the region where the green curve dominates over the purple one: they are limited to the cross-over point. This means that the value of ΔR is mainly determined by the MBC contribution, so a difference in the height of consecutive steps will not significatively influence the value of the macroscopic derivative.

Can we push our measures of $\partial_t R$ to the region where steps are dominant over MBC?

This is the region where

$$\frac{\frac{8}{\sqrt{\pi}} kTe^{-R^2/4kT}}{R} \gg \frac{1}{R}$$

$$R \ll R_+^* \quad {R_+^*}^2 = 4kT \log\left(\frac{8kT}{\sqrt{\pi}}\right)$$

to sample $\partial_t R(R) = \frac{\Delta R}{T}$ for $R < R_+^*$, I need a simulation where there is a **full step** with initial radius R. Where 'full step' means that the **step ends before collapse**.
A necessary condition for this to happen is that the collapse time of R, that we call $t_c(R)$, must be smaller than the step's duration, that is kT.

$$t_c(R) = \int_0^R dR' \frac{1}{\partial_t R'_{steps+MBC}} \qquad t_c(R) > kT$$

$$\partial_t R_{steps+MBC} = -\frac{\frac{8}{\sqrt{\pi}} kTe^{-R^2/4kT}}{R} - \frac{1}{R}$$

the integral can be solved analytically

$$t_c(R) = 2kT \log\left(\frac{e^{R^2/4kT}\sqrt{\pi} + 8kT}{\sqrt{\pi} + 8kT}\right)$$

that is higher than kT when

$$R^2 \gg 4kT \log\left(\frac{8kT}{\sqrt{\pi}}(\sqrt{e} - 1) + \sqrt{e}\right)$$

the two conditions are **not compatible** as $\sqrt{e} > 1$.
So we cannot measure the macroscopic derivative in the region where the steps contribution uis

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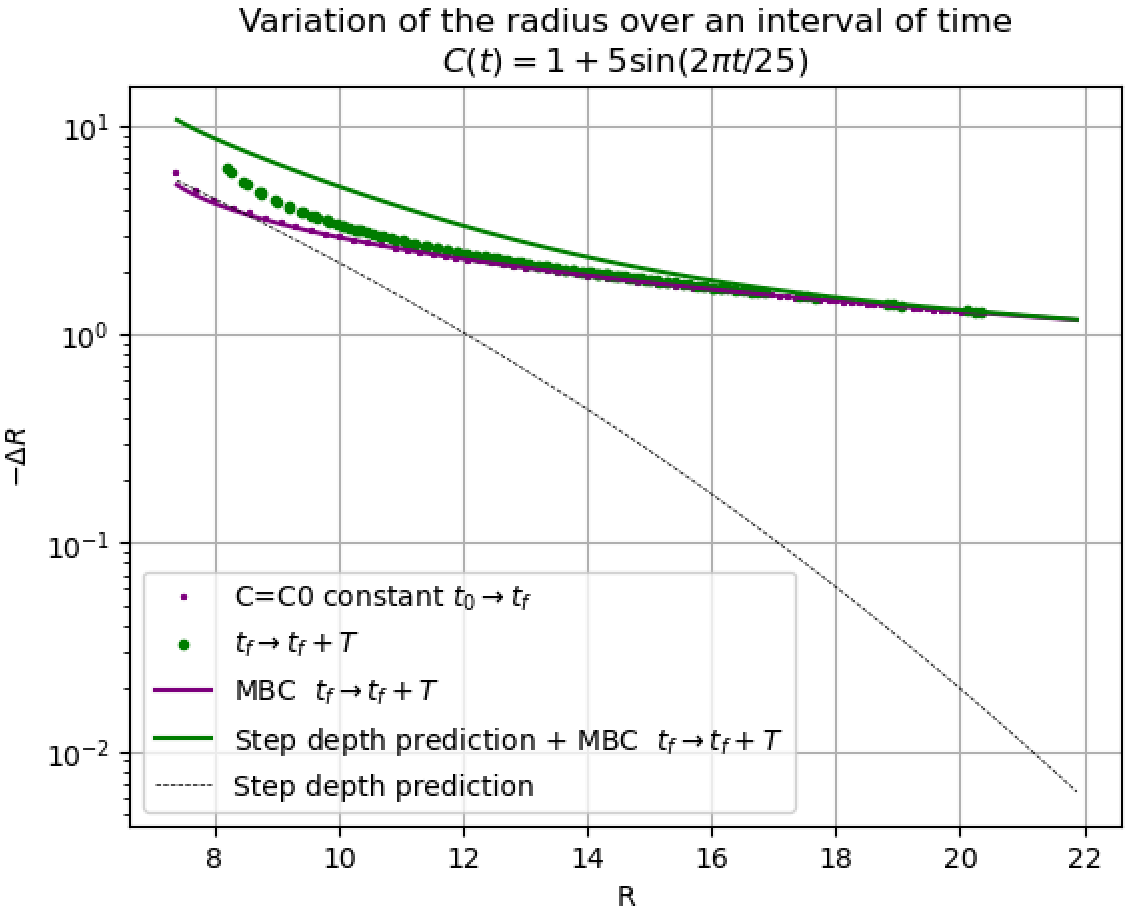
Can we SAMPLE the region where the steps dynamics dominates over MBC?

$\Delta R_{step}(R)$: the variation of the radius **along a step**

$$\Delta R_{step}(R(nT + t_f)) = R((n+1)T + t_f) - R((nT + t_f))$$

where $nT + t_f, n \in N$ represents the end of a step (**orange** lines in the steps figure).



When the steps contribution is dominant over MBC?

This value is expected to have a contribution from steps and a contribution from MBC

$$\Delta R^{steps}_{step} = -\frac{8}{\sqrt{\pi}} TkR^{-1} e^{-R^2/4kT} \qquad \Delta R^{MBC}_{step} = R\left(\sqrt{1 - \frac{2T}{R^2}} - 1\right)$$

so the steps contribution is dominant when

$$-R\left(\sqrt{1 - \frac{2T}{R^2}} - 1\right) \ll \frac{8}{\sqrt{\pi}} kTRe^{-R^2/4kT}$$

Can we sample this region?

There are some constraint that limit the region we can sample

- $\Delta R_{step}(R)$ makes sense only if the depth of the step is smaller than the initial value R (otherwise the step is not a full step)

$$-\Delta R_{step}(R) < R$$

  considering both the contributions

$$R^2 > T\left(1 + \frac{8}{\sqrt{\pi}} k e^{-R^2/4kT}\right)$$

- In 2D the erf profile is well approximated by a sum of gaussians if the width of the interface is much bigger than the radius. This means that the formula for the step's depth makes sense if

$$\sigma(t) = \sigma_0 \sqrt{1 + \frac{2t}{\sigma_0^2}} \ll \frac{R}{\sqrt{2}}$$

using $\sigma_0 \sim \bar{C}^{-1/2}$

$$R \gg 2T^{1/2} k^{1/2} \bar{C}^{-1/2} \qquad kT \gg \frac{\bar{C}^{-1}}{2}$$

We have three conditions, that can be satisfied simultaneously only for very large values of A, but then the state of the system will collapse into a flat state during the $C(t) < 0$ phase.
https://www.desmos.com/calculator/fmwjr1uzoo