1D Constant (Analytical)

The 1D TDGL equation for a constant C is

$$\partial_t u = \partial_{xx} u + Cu - u^3$$

Parameter-less equation

If C is constant, it is possible to rescale x, t and u to get a parameter-free equation

- $x \to C^{-1/2}$
- $t \to C^{-1} t$
- $u \to C^{1/2} u$
  then you simplify $C^{3/2}$ at each term you get

  $$\partial_t u = \partial_{xx} u + u - u^3$$

  This is not possible to to if $C(t)$ as the time-derivative will transform differently with the re-scaling.

Linear dynamics

If $u \simeq 0$ everywhere, then you can neglect the non-linearity and

$$\partial_t u \simeq \partial_{xx} u + Cu$$

taking the Fourier transform

$$\sigma = i\omega = -q^2 + C$$

As $\lambda = \frac{2\pi}{q}$:

- If $C > 0$ **large wavelength** modes ($q < C^{1/2}$, $\lambda > 2\pi C^{-1/2}$) are **unstable**
- If $C < 0$ all modes decay to zero exponentially fast.
  The fastest unstable mode is $q = 0$ and its characteristic time is $\tau = C^{-1}$, representing a **characteristic timescale** of the system.

Stationary states

The equation for stationary states $\partial_t u = 0$ is

$$\partial_{xx} u = -Cu + u^3$$

That can be written as the equation of motion of a system that conserves energy

$$\partial_{xx}u = -\frac{dV}{du} \quad V(u) = \frac{C}{2}u^2 - \frac{1}{4}u^4 = \frac{1}{4}(C-u^2)^2$$

so the stationary states of the TDGL are the trajectories of a particle experiencing no friction in the potential $V(u)$ (u: position of the particle, x: time).

- If $C \leq 0$ there is only one possible trajectory: $u = 0$
- if $C > 0$ there are two trivial trajectories $u = \pm\sqrt{C}$ and then the non-trivial trajectory with higher energy is

$$u(x) = \sqrt{C}u_k(\chi) \quad \chi = C^{1/2}x$$
$$u_k(\chi) = \tanh\left(2^{-1/2}(\chi - \chi_0)\right)$$

  where x_0 is the kink position ($\chi_0 = C^{1/2}x_0$) and can be any value. Then you have an infinite (that becomes finite if you apply PBC) number of lower energy trajectories that are **periodic trajectories** $u_p(\chi)$.

Kink dynamics

> Kink dynamics (Constant C)
>
> Due to the non-linearity in the TDGL equation, we cannot solve it analytically.
> We deal with this by looking for laws that **approximately** describe the dynamics of the system.
> Here we **prove** a law that describe how the distance $d(t)$ between **two isolated neighboring kinks** decays with time: so we state that this law describe an **effective interaction** between kinks.
>
> $$\dot{d}(t) \simeq -24\sqrt{2}C^{\frac{1}{2}}(t)e^{-2^{\frac{1}{2}}C(t)^{\frac{1}{2}}d}$$

Proof

Main idea

Close to a kink, we **assume** that the kink's shape **does not change**, but the kink just propagates:

$$u(x,t) = u_n(x - \dot{x_n}t) \tag{1}$$

where

- $u_n(x)$ is the shape of the n-th kink;

- x_n is the position of the n-th kink.
  Notice: This assumption, that the profile $u(x)$ does not change but just propagates, is very reasonable close to the kink's position ($x \simeq x_n$). However, later, we will assume this for a large interval $[x_{n-\frac{1}{2}}, x_{n+\frac{1}{2}}]$.

An equation for $\dot{x_n}$

Here we find an (approximated) formula for $\dot{x_n}$, the velocity of the n-th kink as a function of the distance from the neighboring kinks.

The TDGL equation can be written in the form

$$\partial_t u = \partial_{xx} u - V'(u)$$

$$V(u) = \frac{1}{4}(C - u^2)^2$$

using the "propagating kink" assumption (1) :

$$-\dot{x_n}\partial_x u_n = \partial_{xx} u_n - V(u_n)$$

Multiplying by $\partial_x u_n$ and integrating $\int_{x_{n-\frac{1}{2}}}^{x_{n+1/2}} dx$

$$-\dot{x_n}\int (\partial_x u_n)^2 dx = \int \frac{1}{2}\partial_x[(\partial_x u_n)^2] - \int \frac{\frac{dV}{du}du}{dx}dx$$

$$-\dot{x_n}\int_{x_{n-\frac{1}{2}}}^{x_{n+1/2}} (\partial_x u_n)^2 dx = \frac{1}{2}[(\partial_x u_n)^2]_{x_{n-\frac{1}{2}}}^{x_{n+1/2}} - [V(u_n)]_{x_{n-\frac{1}{2}}}^{x_{n+1/2}}$$

The **reason** why we integrate from the mid-point between the (n-1)-th kink and the n-th kink ($x_{n-\frac{1}{2}}$) and the mid-point between the n-th and the (n+1)-th kink ($x_{n+\frac{1}{2}}$) is that, at the midpoint between two kinks, the derivative of $u(x)$ is zero! (You clearly see it from simulations)
So the first term in the RH side is zero and it remains

$$-\dot{x_n}\int_{x_{n-\frac{1}{2}}}^{x_{n+1/2}} (\partial_x u_n)^2 dx = -[V(u_n)]_{x_{n-\frac{1}{2}}}^{x_{n+1/2}}$$

Now we are left calculating the two objects

$$\mathcal{I}_1 = \int_{x_{n-\frac{1}{2}}}^{x_{n+1/2}} (\partial_x u_n)^2 dx$$

$$\mathcal{I}_2 = [V(u_n)]_{x_{n-\frac{1}{2}}}^{x_{n+1/2}}$$

$\mathcal{I}_1$

The **idea** we present for **approximating** this integral is the following:
Close to the kink's position ($x \simeq x_n$) the shape of the n-th kink reasonably resembles the shape of an isolated kink ($u_n(x) \simeq u_k(x)$ if $x \simeq x_n$) and the derivative is large close to x_n and decreasing exponentially fast as we get close to $x_{n\pm\frac{1}{2}}$. So we make the approximation

$$\mathcal{I}_1 \simeq \int_{x_{n-\frac{1}{2}}}^{x_{n+1/2}} (\partial_x u_k(x))^2 dx$$

then, as the derivative of the single kink profile decays exponentially fast, we can make a further approximation, by integrating along the whole real axis. And changing variable $\chi = C^{1/2}x$

$$\mathcal{I}_1 = C^{3/2} \int_{-\infty}^{+\infty} d\chi (\partial_\chi u_k(\chi))^2$$

where the single kink profile is

$$u_k(\chi) = \tanh(2^{-1/2}\chi)$$

in the Master Report.pdf that integral has been calculated

$$\mathcal{I}_1 = \frac{2\sqrt{2}}{3} C^{3/2}$$

$\mathcal{I}_2$

We need to evaluate $V(u)$ at $u_n\left(x_{n\pm\frac{1}{2}}\right)$.
The **idea** to **approximate** $u_n\left(x_{n\pm\frac{1}{2}}\right)$ is to write an **expansion** of $u_n(x)$ **valid close to the mid-points** between two kinks. Then we calculate the derivative ∂_x and then we evaluate the derivative in $x = x_{n\pm\frac{1}{2}}$.

We said that the profile $u(x)$ is well approximated by the single kink profile close to the kink's position. While it is not good close to the mid-point of two kinks, where the presence of the next kink will be relevant for the profile $u(x)$.

Approximation to the periodic kink stationary state $u_p(x)$

A **better approximation** is represented by the stationary solution of the TDGL equation with period $x_{n+1} - x_{n-1}$ that we will call $u_p(x)$.

This is still an approximation, but it is the **best we can do** up to now.

Expansion of $u_p(x)$ close to the mid-point between kinks

Lets's change variable $\chi = C^{1/2}x$ and let's look for an **expansion** of $u_p(\chi)$ close to $x_{n\pm\frac{1}{2}}$ (notice that the kinks are located at x_{n-1}, x_n, x_{n+1}).
Close to $x_{n+\frac{1}{2}}$, if the kinks are sufficiently far from each other, $u_p(\chi) \simeq 1$, so

$$u_p(\chi) = 1 + \delta u_p(\chi)$$

inside the TDGL equation (and using that $u_p(\chi)$ is a stationary solution: $\partial_t u_p = 0$)

$$\partial_{xx}\delta u_p = -2\delta u_p + O(\delta u_p^2)$$

This equation, up to first order, admits exponential solutions. These solutions are infinite, but boundary conditions will *grant only one* solution.
As $u_p(\chi)$ decays/increases exponentially fast moving towards $x_{n\pm1}$, we look for a solution of this kind (if you look to the profile of $u(\chi)$ in simulations, it looks like the following ansatz)

$$\delta u_p(\chi) = a_- e^{-2^{1/2}(\chi-\chi_n)} + a_+ e^{-2^{1/2}(\chi_{n+1}-\chi)}$$

where $a_\pm$ are determined (if the ansatz is correct) by boundary conditions.

Asymptotic match

We require $u_p(\chi) = 1 + \delta u_p(\chi)$ to match $u_k(\chi)$ in a point that is **both**

- Far from the mid-point $x_{n+\frac{1}{2}}$
- Far from the n-th kink's position

  $$u_k(\chi) \simeq 1 + \delta u_p(\chi) \quad \chi_n \ll \chi \ll \chi_{n+\frac{1}{2}}$$

  as we are far from the n-th kink, we can expand $u_k(\chi)$ for $\chi \gg \chi_n$

  $$u_k(\chi) = \tanh(2^{-1/2}(\chi - \chi_n)) \simeq 1 - 2e^{-2^{1/2}(\chi-\chi_n)}$$

  imposing the boundary condition

  $$1 - 2e^{-2^{1/2}(\chi-\chi_n)} = 1 + a_- e^{-2^{1/2}(\chi-\chi_n)} + a_+ e^{-2^{1/2}(\chi_{n+1}-\chi)}$$

  as $\chi_n \ll \chi \ll \chi_{n+\frac{1}{2}}$ we can neglect the exponential with χ_{n+1}, then

  $$a_+ = -2$$

and, following a similar idea, also $a_- = -2$. So if $\chi \simeq \chi_{n+\frac{1}{2}}$

$$u_p(\chi) \simeq 1 - 2(e^{-2^{1/2}(\chi - \chi_n)} + e^{-2^{1/2}(\chi_{n+1} - \chi)})$$

Evaluating $V(u) = \frac{1}{4}(C - u^2)^2$ at $u = u_p\left(\chi_{n\pm\frac{1}{2}}\right)$, we find

$$\mathcal{I}_2 = [V(u)]_{\chi_{n-\frac{1}{2}}}^{\chi_{n+1/2}} = [C^2 \delta u_p^2 + O(\delta u_p^3)]]_{\chi_{n-\frac{1}{2}}}^{\chi_{n+1/2}}$$

Notice: Above we calculated $\delta u(\chi)$ close to $\chi_{n+\frac{1}{2}}$. When you evaluate it close to $\chi_{n-\frac{1}{2}}$ you should keep in mind that the expression for $\delta u(\chi)$ **is different!** This leads to the following result :
To leading order

$$\mathcal{I}_2 = 16C^2(e^{-2^{1/2} l_n C^{1/2}} - e^{-2^{1/2} l_{n-1} C^{1/2}})$$

where $l_n = x_{n+1} - x_n$ is the distance between the n-th kink and the (n+1)-th kink.

Putting our results together

$$\dot{x_n} = \frac{\mathcal{I}_2}{\mathcal{I}_1} = \frac{16}{\frac{2\sqrt{2}}{3}} C^{1/2}(e^{-2^{1/2} l_n C^{1/2}} - e^{-2^{1/2} l_{n-1} C^{1/2}})$$

Then, if in your system there are **only TWO kinks** at distance d (and you adopt **periodic boundary conditions**)

$$\dot{d}(t) \simeq -24\sqrt{2} C^{\frac{1}{2}}(t)(e^{-2^{\frac{1}{2}} C(t)^{\frac{1}{2}} d} - e^{-2^{\frac{1}{2}} C(t)^{\frac{1}{2}} (L-d)})$$

where L is the size of the simulation box. If $d \gg L$, the second exponential is negligible.