Deviation from MBC under slow oscillations

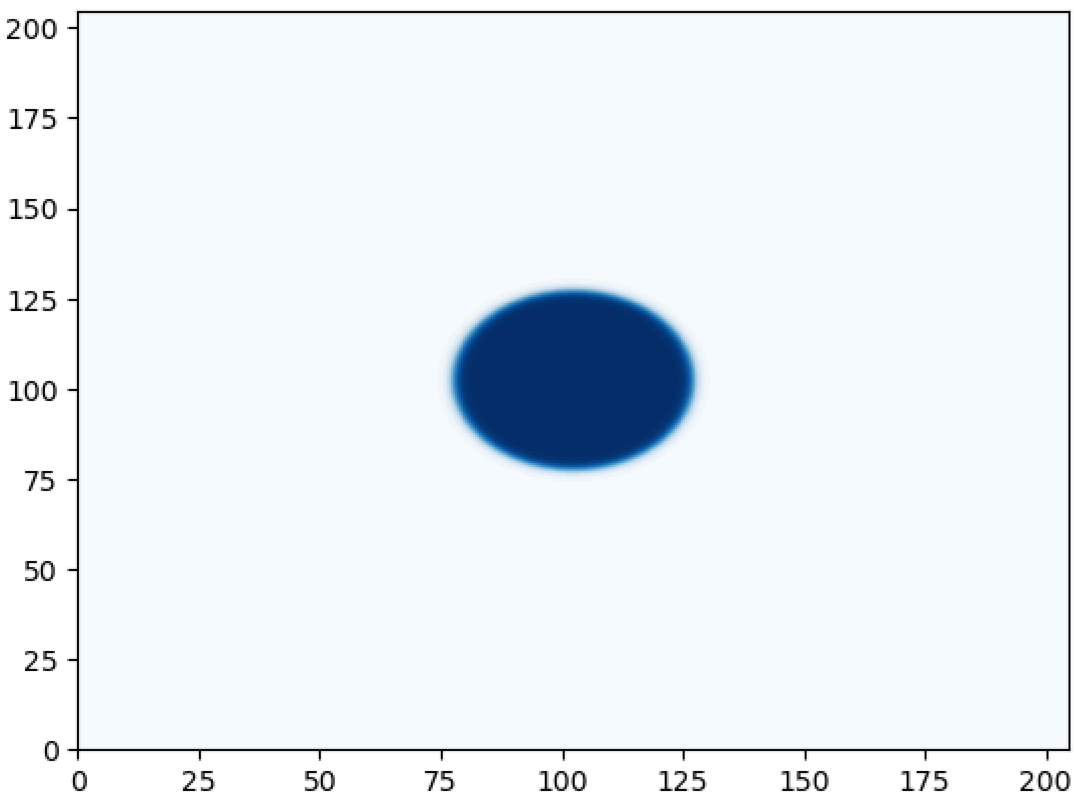
#2D #circular_island #mbc

Here we ran simulations of the 2D TDGL equation considering the initial state to be a circular island of radius R_0

$$u(x, y, t = 0) = \tanh\left((R - R_0)\sqrt{\frac{1}{2}}\right)$$

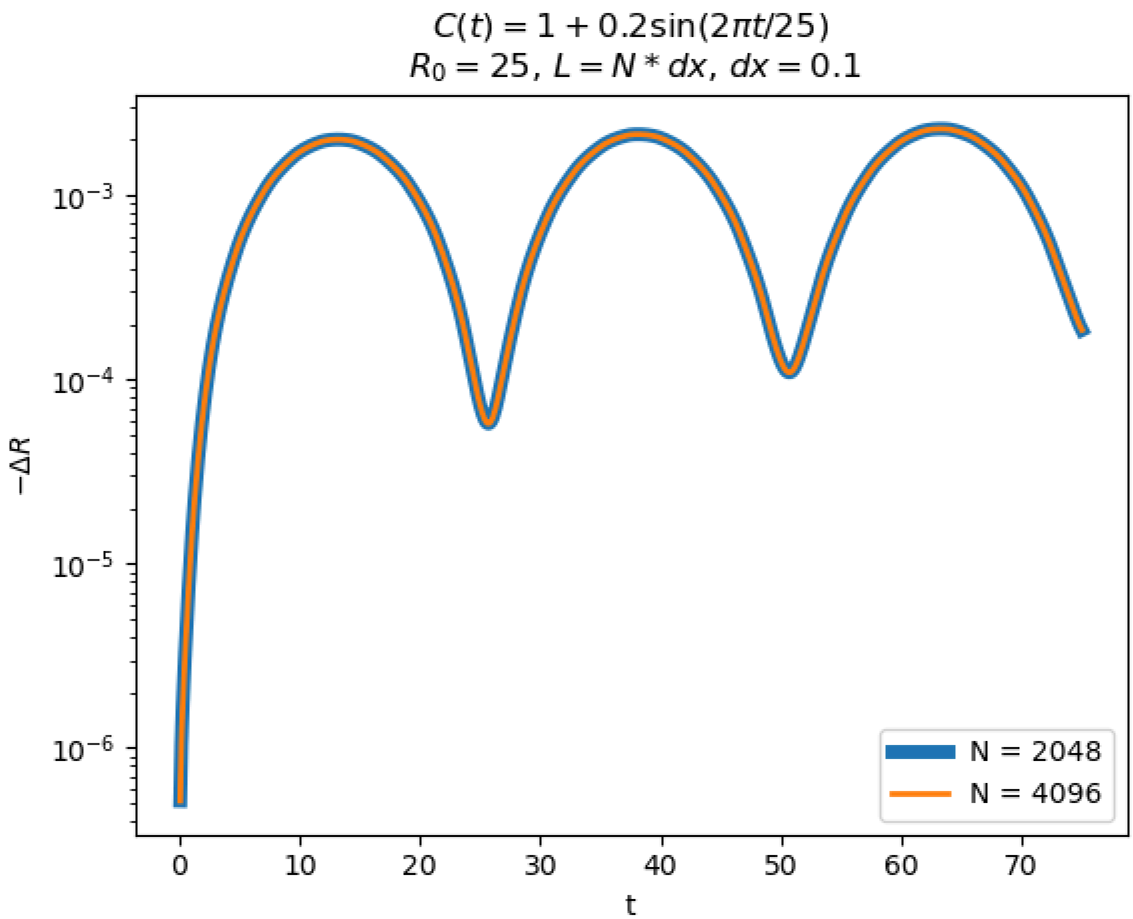
we considered the simulation box lenght

$$L = 204.8 \quad R_0 = 25$$



such that there is no interaction between opposite kinks due to the PBC conditions.
The size of the **simulation box is big enough** for this radius as you can see from here:

$$\Delta R = R_{C(t)} - R_{C=1}$$



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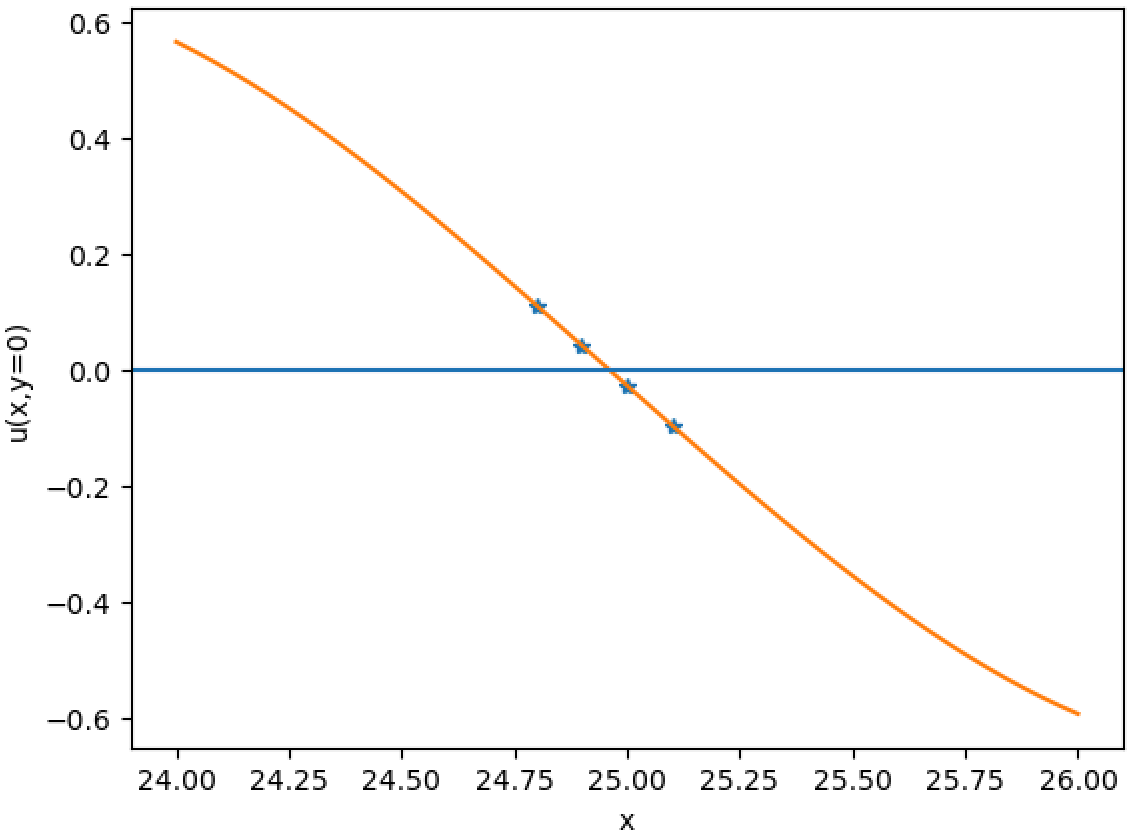
How we measure things

How Radius is measured

The radius, as a function of time $R(t)$, has been measured by looking at the sections of $u(x, y)$ along the line $y = 0$. Then we focus on the $x > 0$ semi-axis, where there is a kink.
The position of the kink is **estimated** by finding the discrete x_n where u changes sign. Then **4 points** around x_n are considered and a **spline cubic interpolation** is made.
CubicSpline: "piecewise cubic polynomial which is twice continuously differentiable"
The position of the kink is represented by the central root of the polynomial and this is our estimate for the radius R.



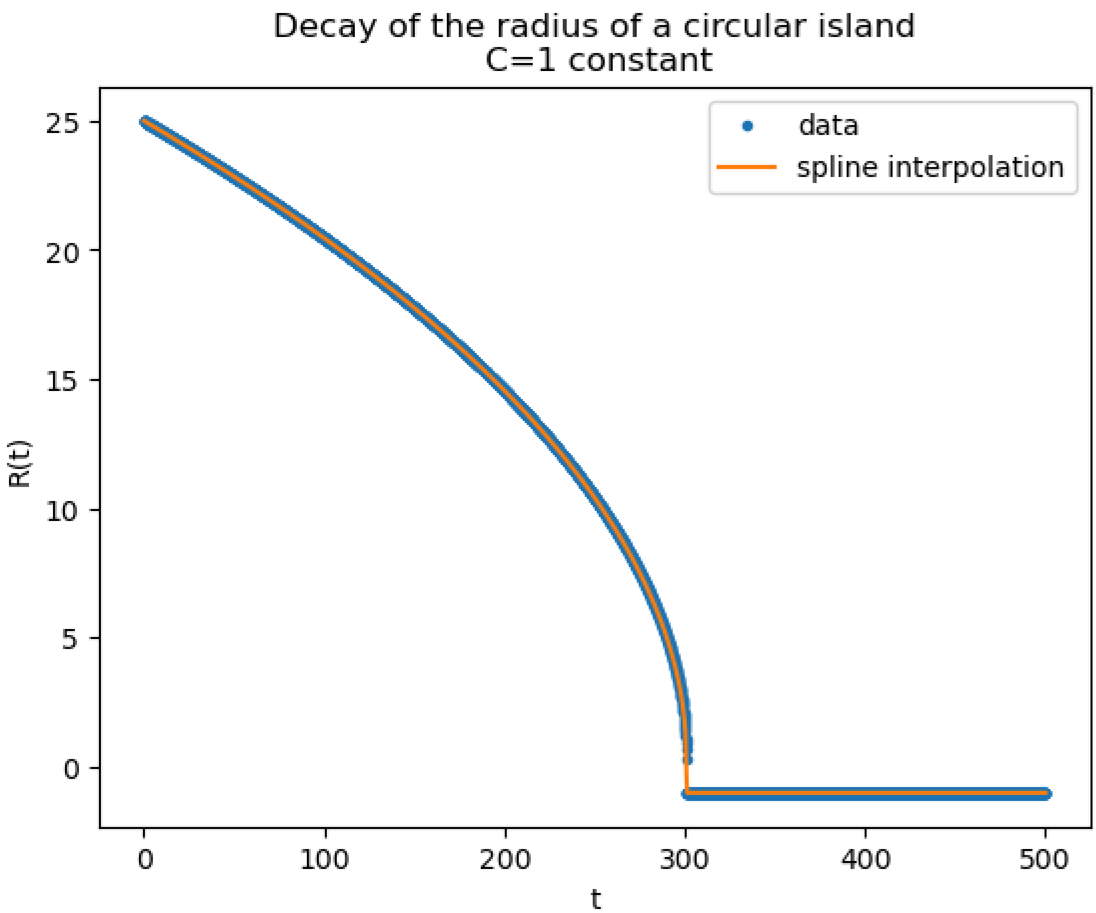
How constant C and C(t) data is compared

As time passes, the curve $R(t)$ deviates more and more from the curve $R_{C=1}(t)$ curve.
If we want to compare $\partial_t R_{C=1}$ and $\partial_t R$ at the **same value of R**, this is a problem, because this **does NOT mean** to compare the two derivatives at the **same time t**.

Solution

We interpolate with a **spline** of degree $k = 4$ (**UnivariateSpline**) the data $R_{C=1}(t)$. Obtaining a **continuos** function

$$f : t \to R_{C=1}(t)$$



Then we compute the **analytical derivative**

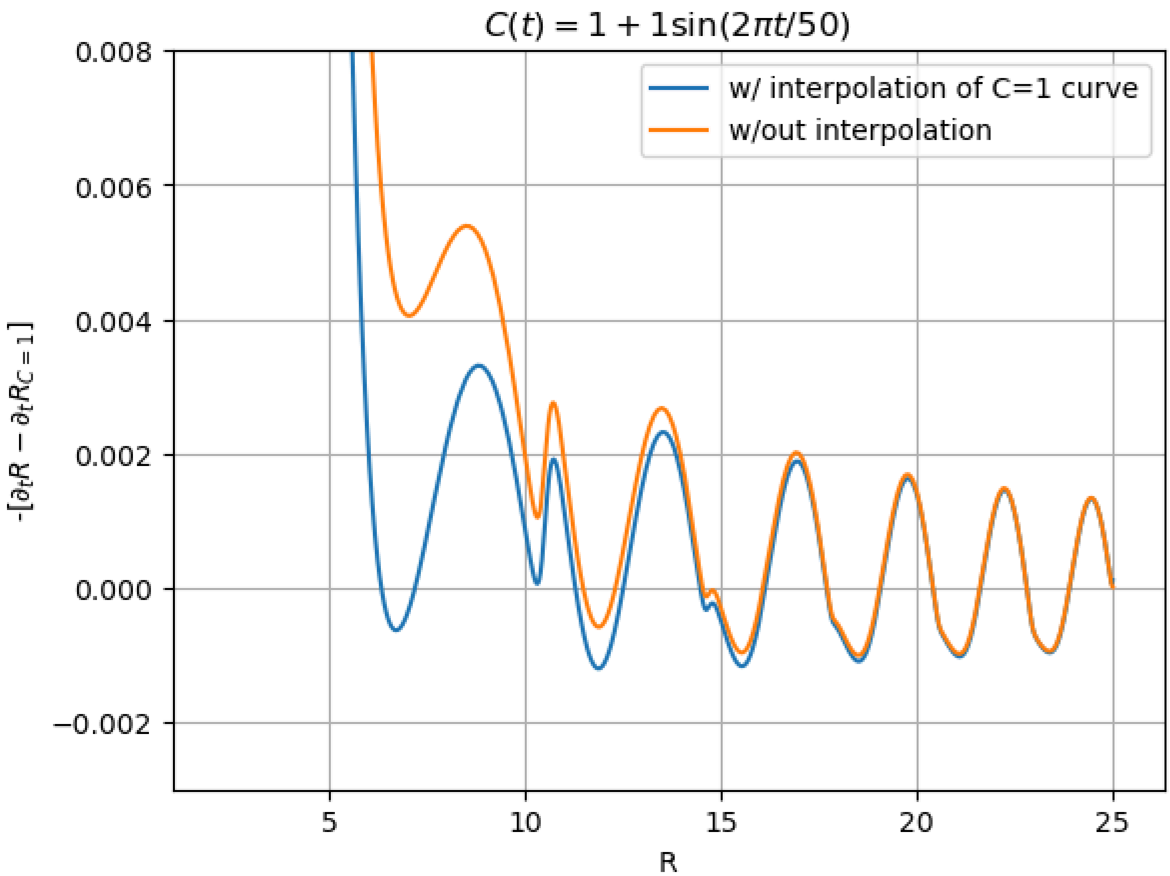
$$g : t \to \partial_t R_{C=1}(t)$$

Then we can find the inverse respect to f of an image R by looking at the root of $f(t) - R$.
And we can compose those functions to obtain

$$\gamma = g(f^{-1}(R)) : R \to \partial_t R_{C=1}$$

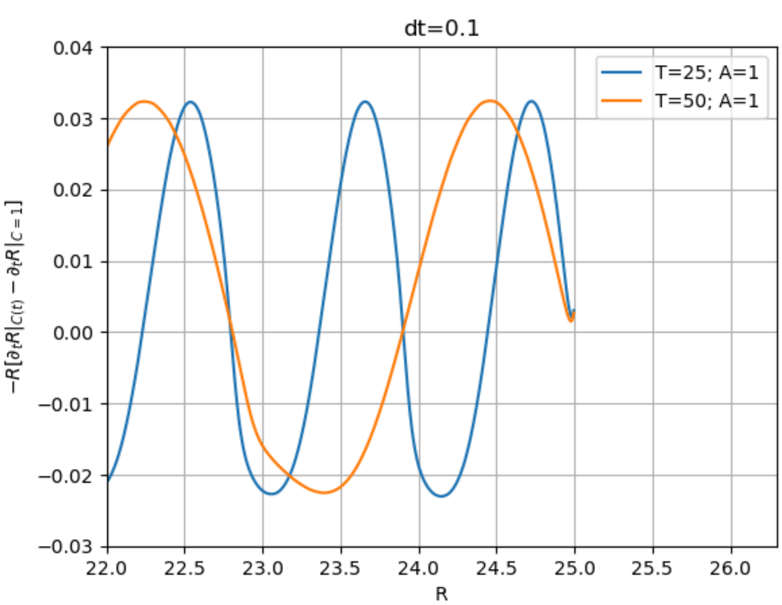
Now this function is defined **for any value of** R, so we can compare it with the **numerical derivative** of $R(t)$ such that R is the same for each datapoint compared.
In the plot below, the blue line represent the difference obtained using the interpolation, while the orange curve represents the difference obtained subtracting data points with the same time

t (and on x-axis is plotted the radius of the simulation with $C(t)$).



Model-free analysis

Here we seek for the effect of oscillations on the dynamics of a circular domain, by **subtracting data** collected with $C = 1$ constant to data collected with $C(t) = 1 + A\sin\left(\frac{2\pi t}{T}\right)$.
We see an oscillation of this quantity. Is it a **numerical effect**?

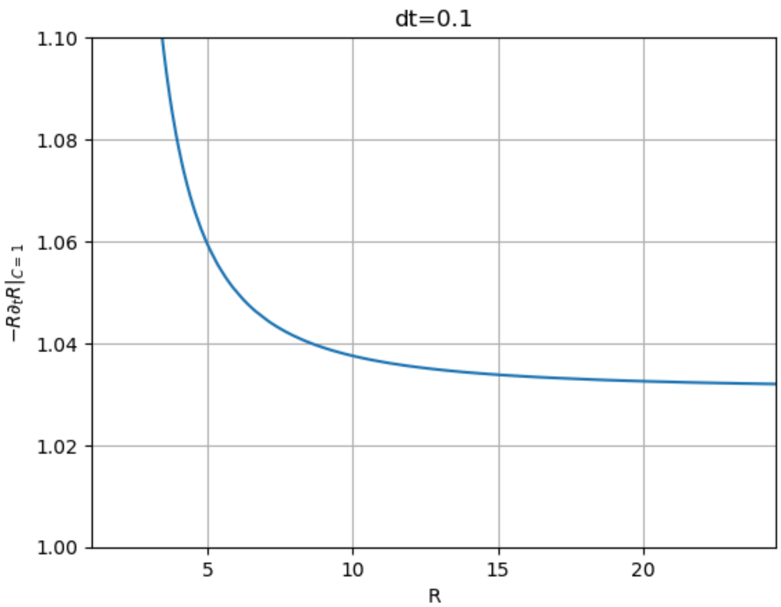


Numerical effect in the C constant case

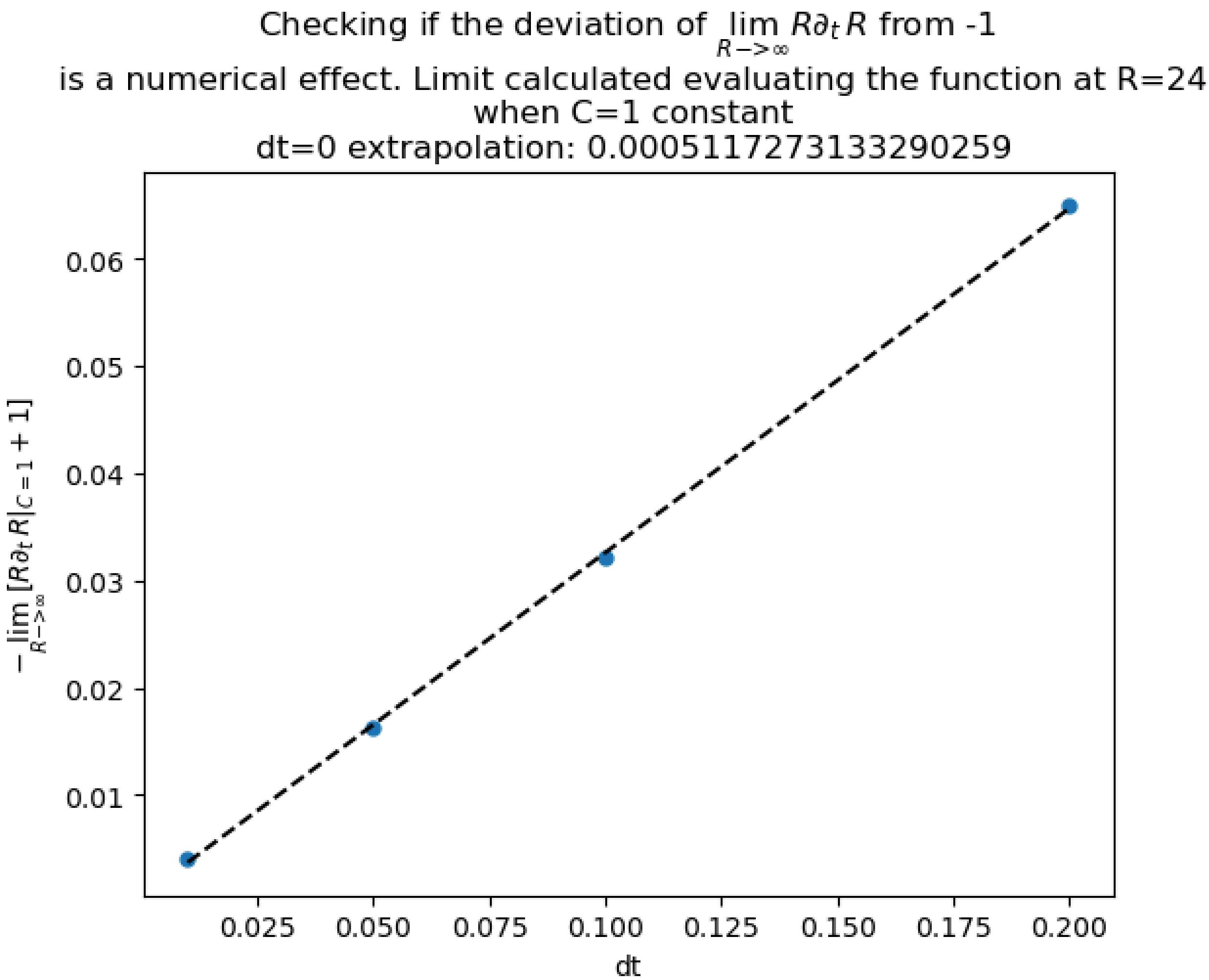
We expect that, in the large R limit, if C is constant

$$\partial_t R = -\frac{1}{R}$$

but we see that, at large R, $R\partial_t R$ saturates to a value different from -1.
Is it due to the time discretization?



We estimate $\lim_{R\to\infty} R\partial R$ by evaluating the function (plotted above) at a large R. We chose $R = 24$ ($R_0 = 25$ is the initial radius). Then we repeat many simulation with $C = 1$ and different values of dt.



The value of the estrapolation to $dt = 0$ represent the order of magnitude of a real (non-numerical) effect. If we compare it to 1, that is the expected value of the limit.

$$5 * 10^{-4} \ll 1$$

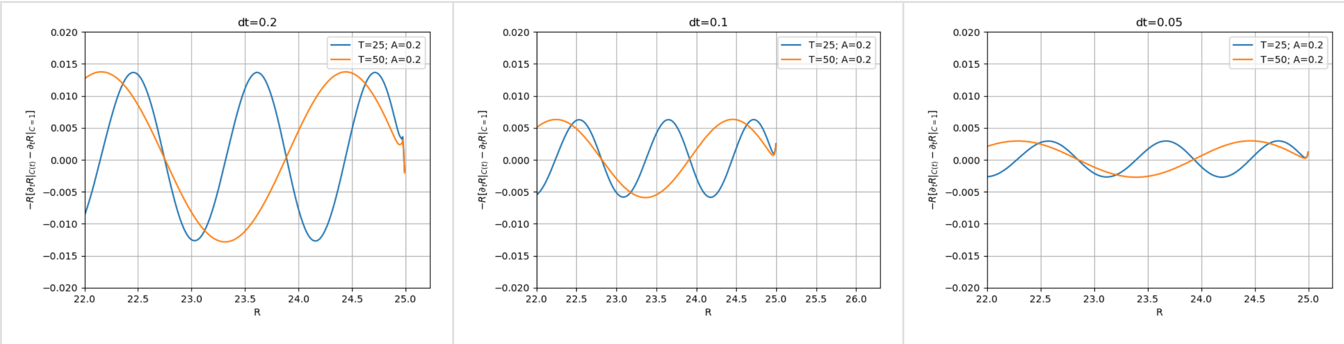
so we conclude **it is just a numerical effect.**

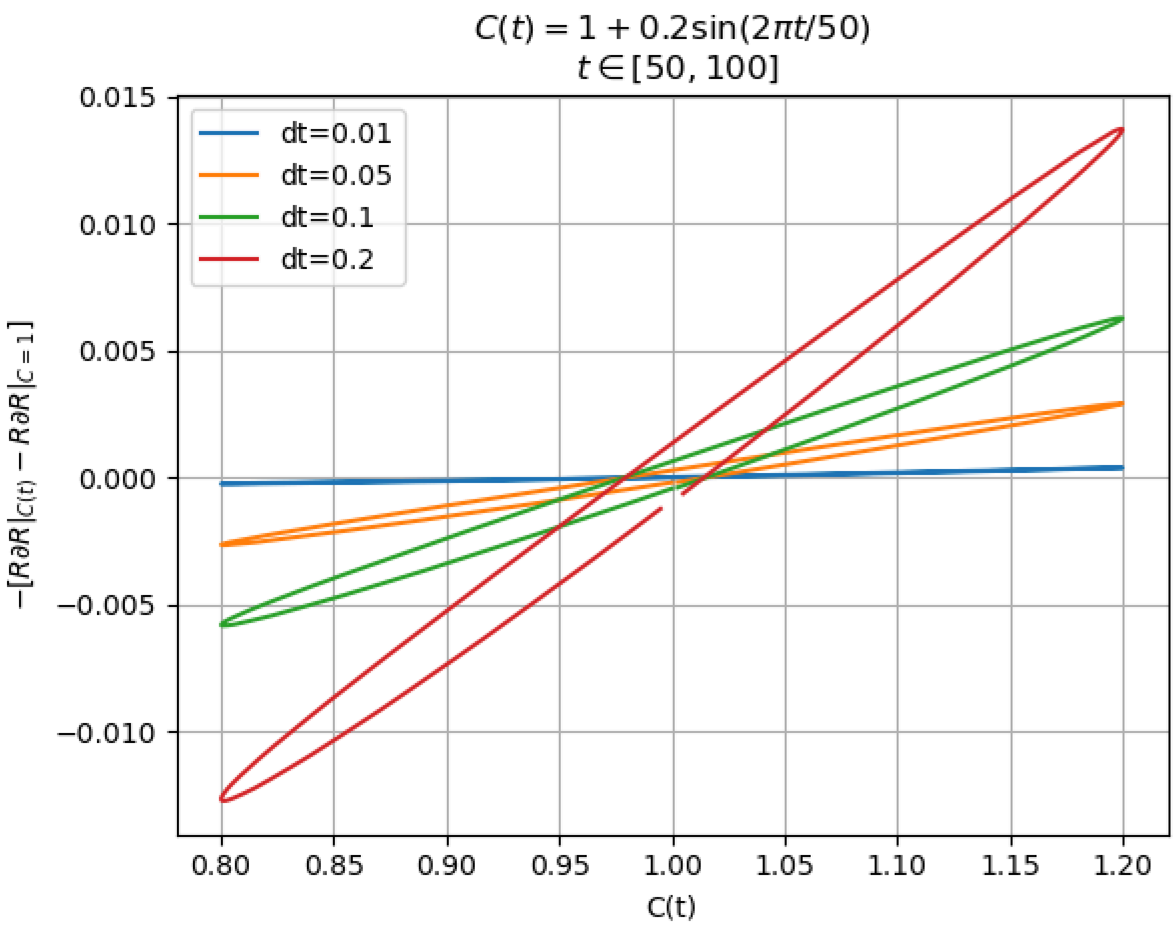
Numerical effect in the C(t) case

Oscillations in C introduce an oscillation in the deviation from the constant C curve.
We look at the difference between the curves:

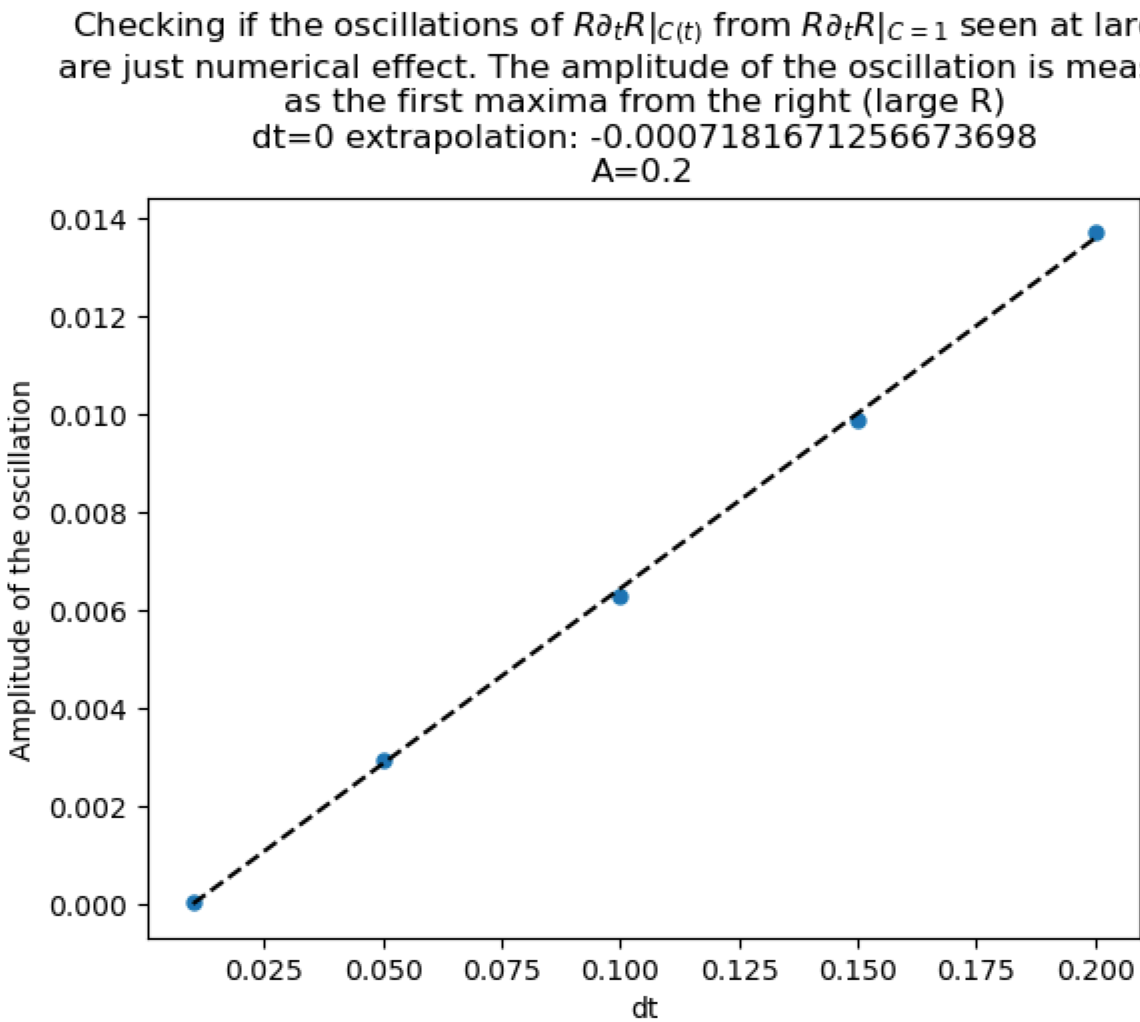
$$-[R\partial R|_{C(t)} - R\partial R|_{C=1}]$$

Small oscillations A=0.2





As the amplitude of the oscillation of this deviation increases with dt, we suspect it to be a numerical effect. We verify this:

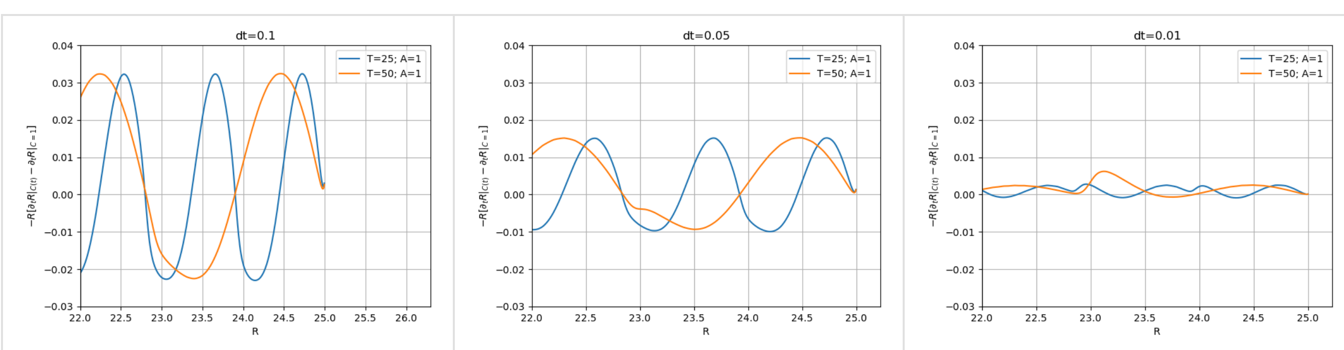


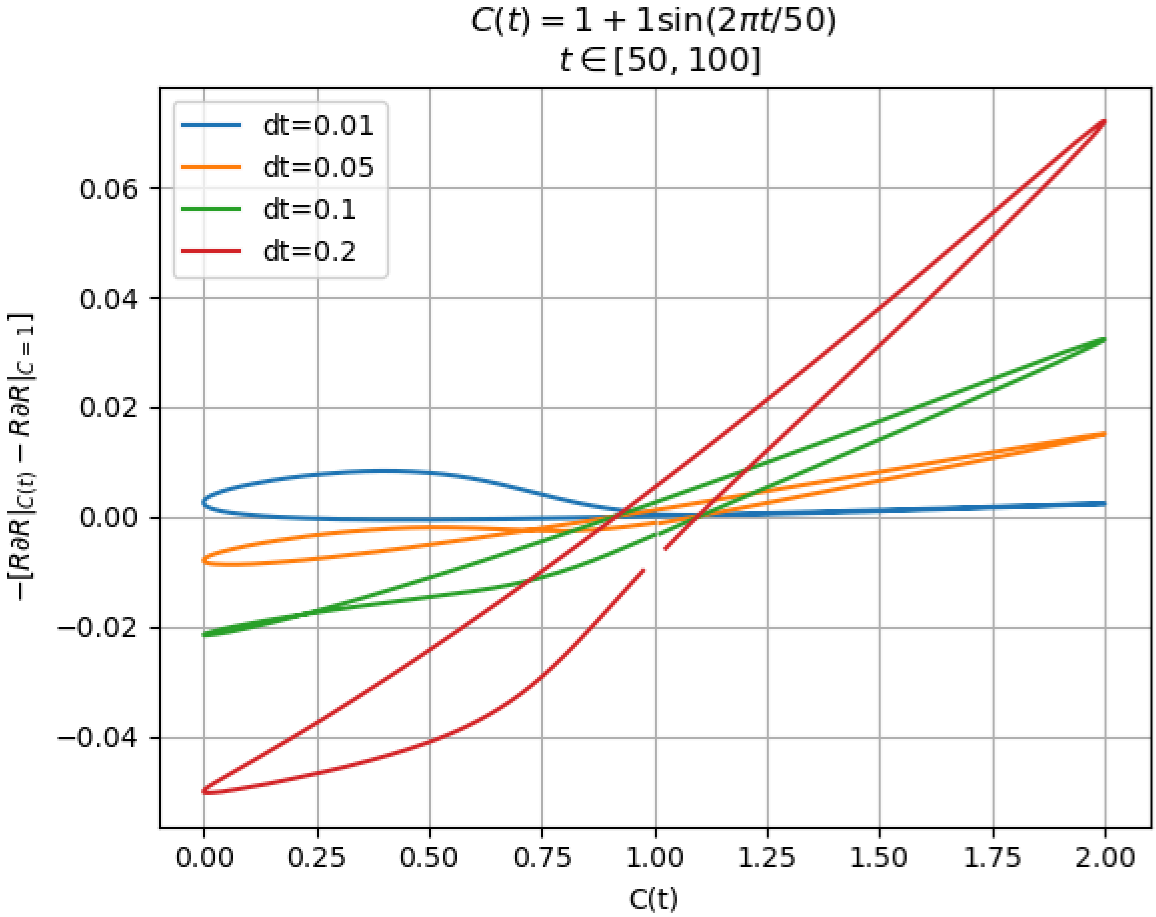
The extrapolated value at $dt = 0$ is the order of magnitude of the real (non-numerical) effect. Compared to the value of $R\partial R|_{C=1}$ at large R (that is -1)

$$7 * 10^{-4} \ll 1$$

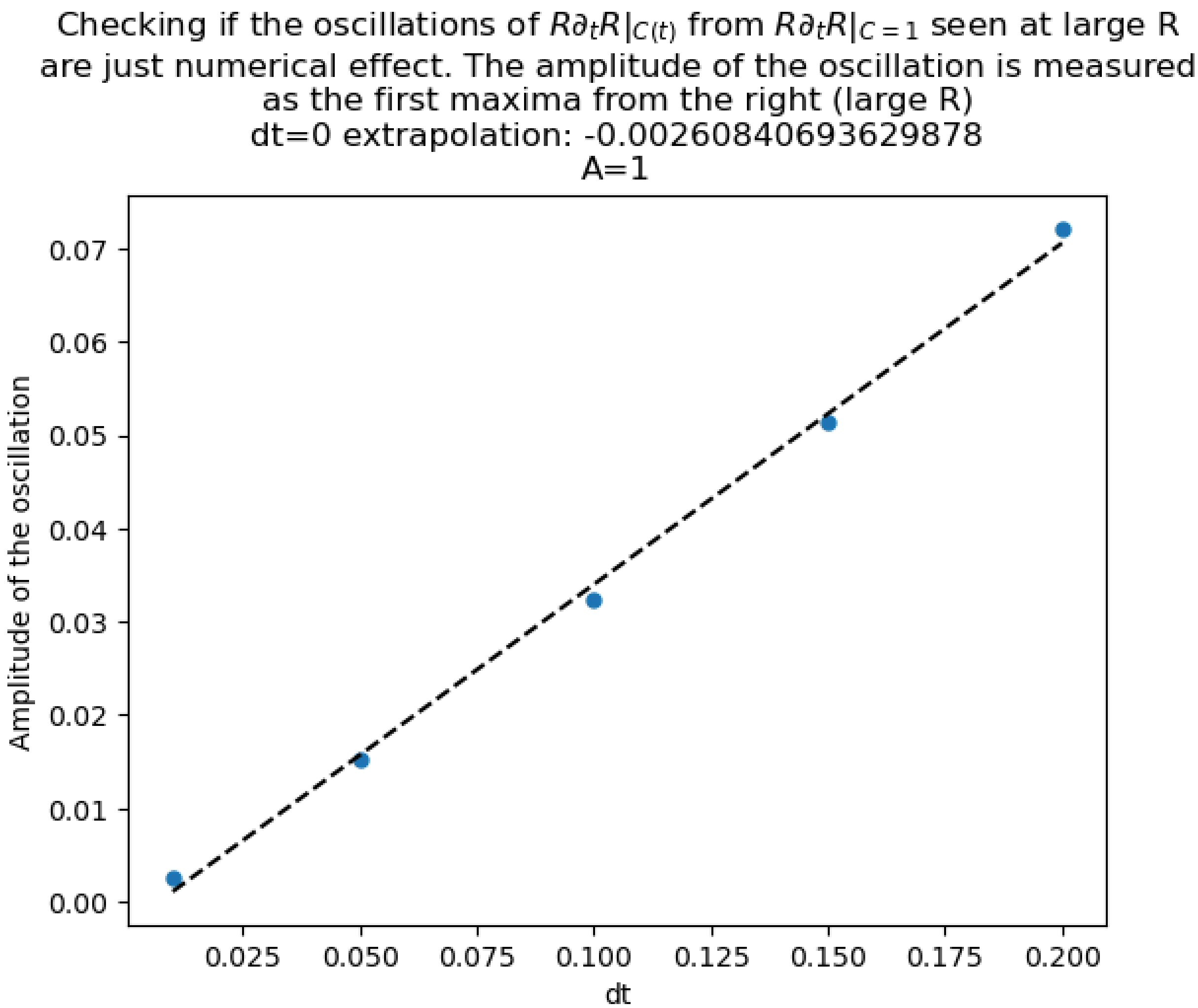
so we conclude this **is just a numerical effect** or, if there is an effect, it is 4 order of magnitude smaller than the limit value.

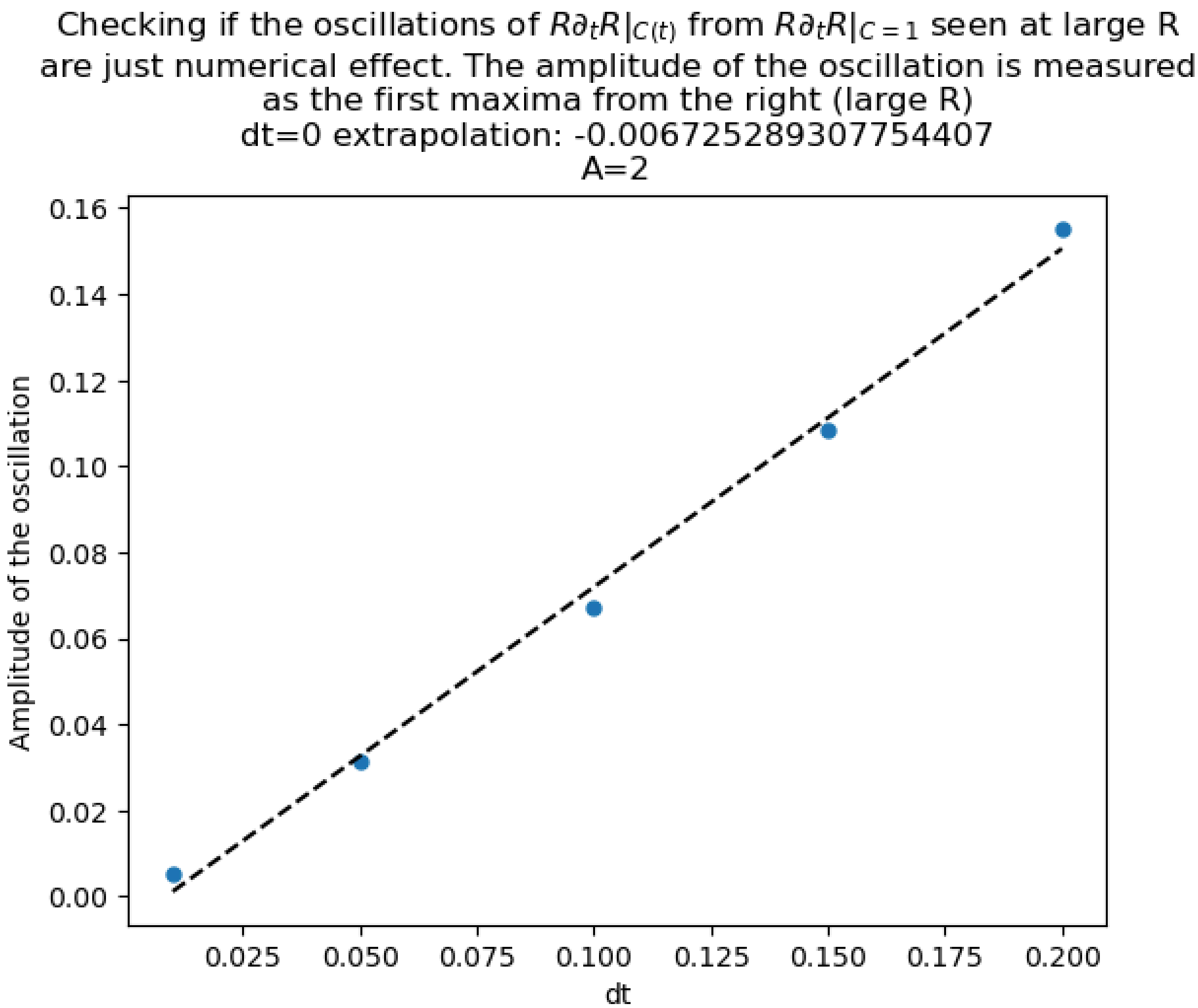
Large oscillations $A \geq 1$





$C(t) = 1 + 1\sin(2\pi t/50)$
$t \in [50, 100]$
dt=0.01
dt=0.05
dt=0.1
dt=0.2
$-[R\partial R|_{C(t)} - R\partial R|_{C=1}]$
C(t)
0.06
0.04
0.02
0.00
−0.02
−0.04
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00


Checking if the oscillations of $R\partial_t R|_{C(t)}$ from $R\partial_t R|_{C=1}$ seen at large R
are just numerical effect. The amplitude of the oscillation is measured
as the first maxima from the right (large R)
dt=0 extrapolation: -0.00260840693629878
A=1
Amplitude of the oscillation
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.025
0.050
0.075
0.100
0.125
0.150
0.175
0.200
dt



Now the extrapolated value at $dt = 0$ is one order of magnitude higher, but still

$$6 * 10^{-3} \ll 1$$

so it is a numerical effect or, if there is an effect, it is 3 orders of magnitude smaller than the real effect.

Model-dependent analysis

Here we investigate how MBC is affected by the oscillations of C.
We will in this order

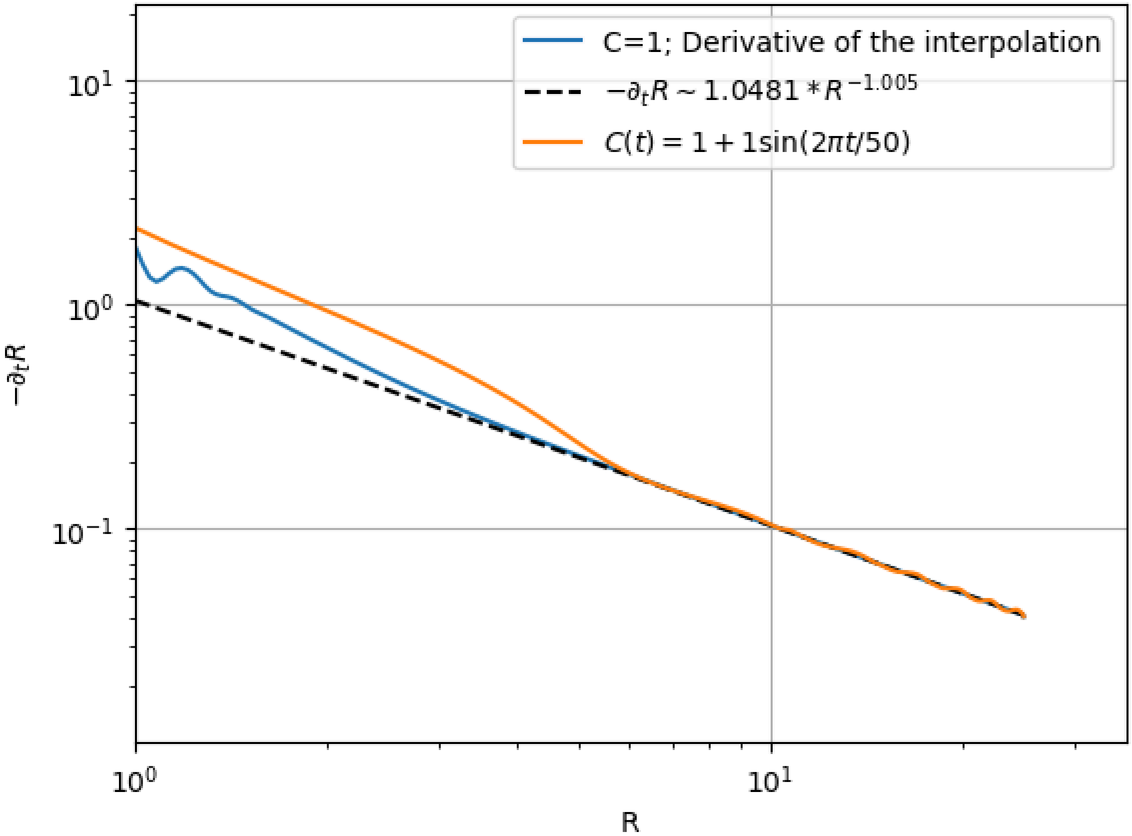
- Verify that, up to leading order, MBC is true
- Remove the leading order contribution and seek for the R and $C(t)$ dependence of the next leading effect.

Leading order effect (Motion by curvature)

The MBC law for a circular island

$$\dot{R} \simeq -\frac{1}{R} \quad R \gg 1$$

can be verified for a $C = 1$ simulation



Now the question is what happens if $C = C(t)$ and the oscillations are slow

$$\epsilon \sim \frac{1}{T}$$

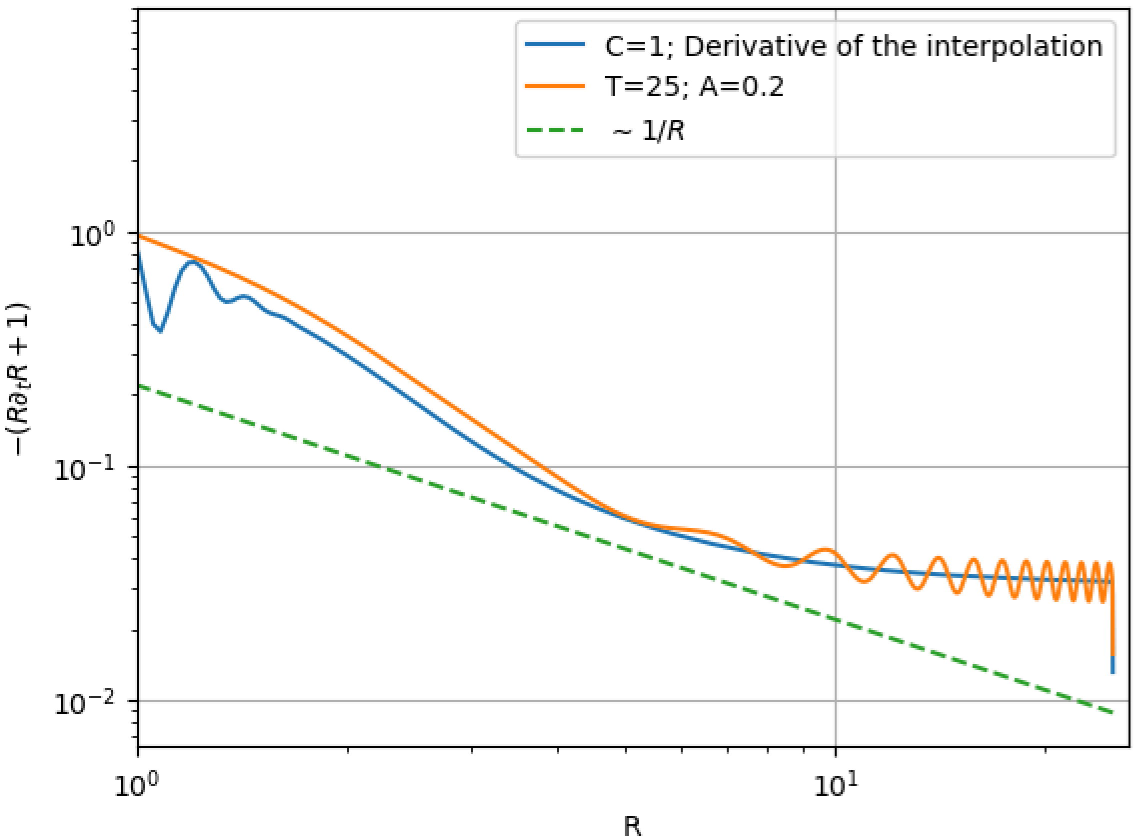
Next-to-leading order effect

Now we advance **two hypotesis** about the next order effect

- $\dot{R} = -\frac{1}{R} + \frac{f(C(t))}{R}$
- $\dot{R} = -\frac{1}{R} + \frac{f(C(t))}{R^2}$
  If the second model is true, it means that $(R\dot{R} + 1) \sim \frac{f(C(t))}{R}$.
  If, instead, the first is true $(R\dot{R} + 1) \sim f(C(t))$.

So we can **exclude** one model or the other by plotting, in **log-log** scale, $(R\dot{R} + 1)$ v.s. R.

Small amplitude A=0.2



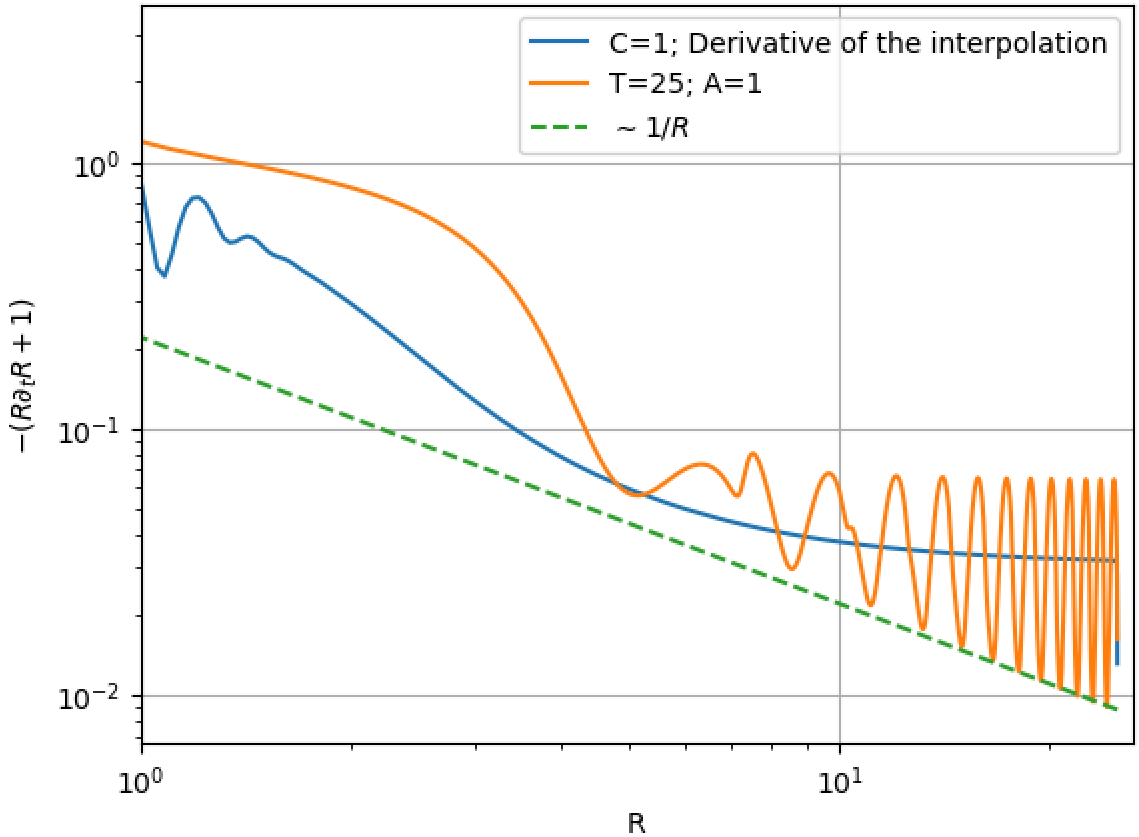
By adding oscillations, $-(R\dot{R}+1)$ decays slower that a power law $\sim \frac{1}{R}$, so the second model is false and the **first model** is more suitable.

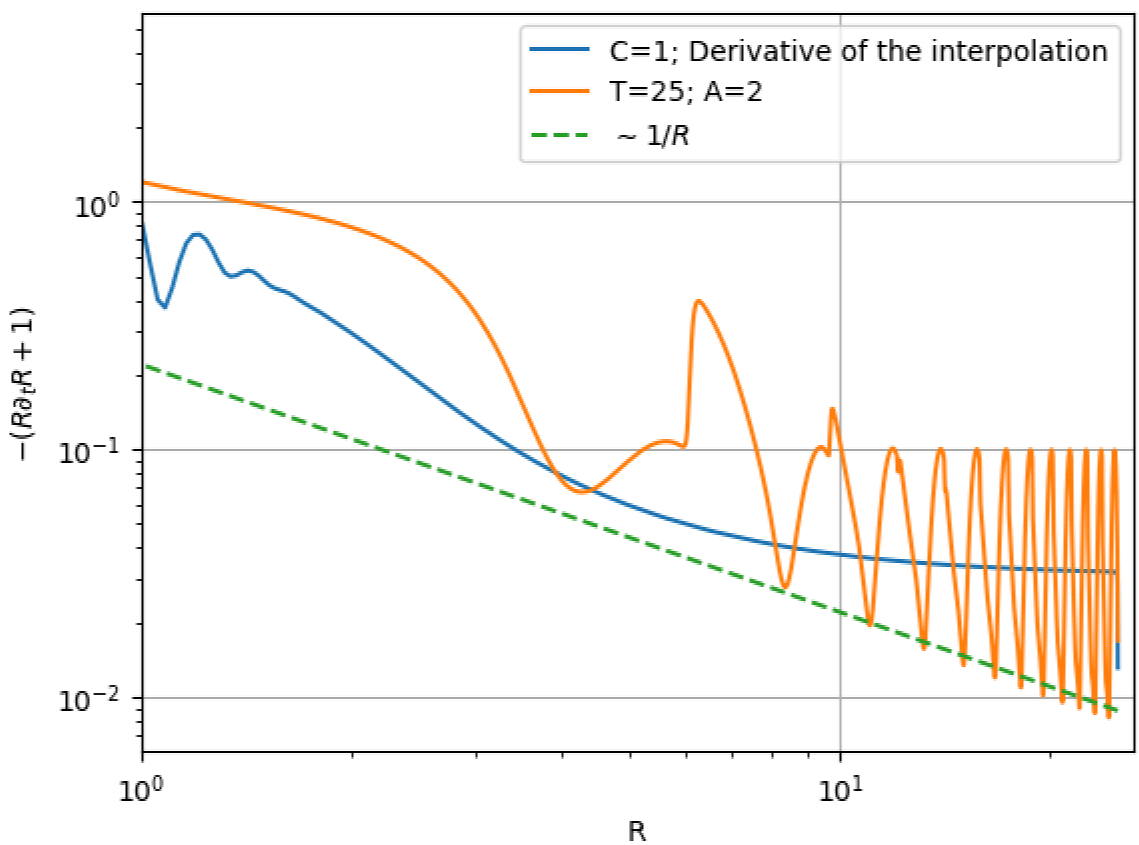
Large amplitude $A \geq 1$

By increasing the amplitude of $C(t)$, we see that

- The maximas of the oscillation do not depend on R

- The minimas decays as a power-law R^{-1}





How the correction scales with ϵ

For small amplitude oscillations, it seems that the model is

$$\dot{R} = -\frac{1}{R} + \frac{f(C(t))}{R}$$

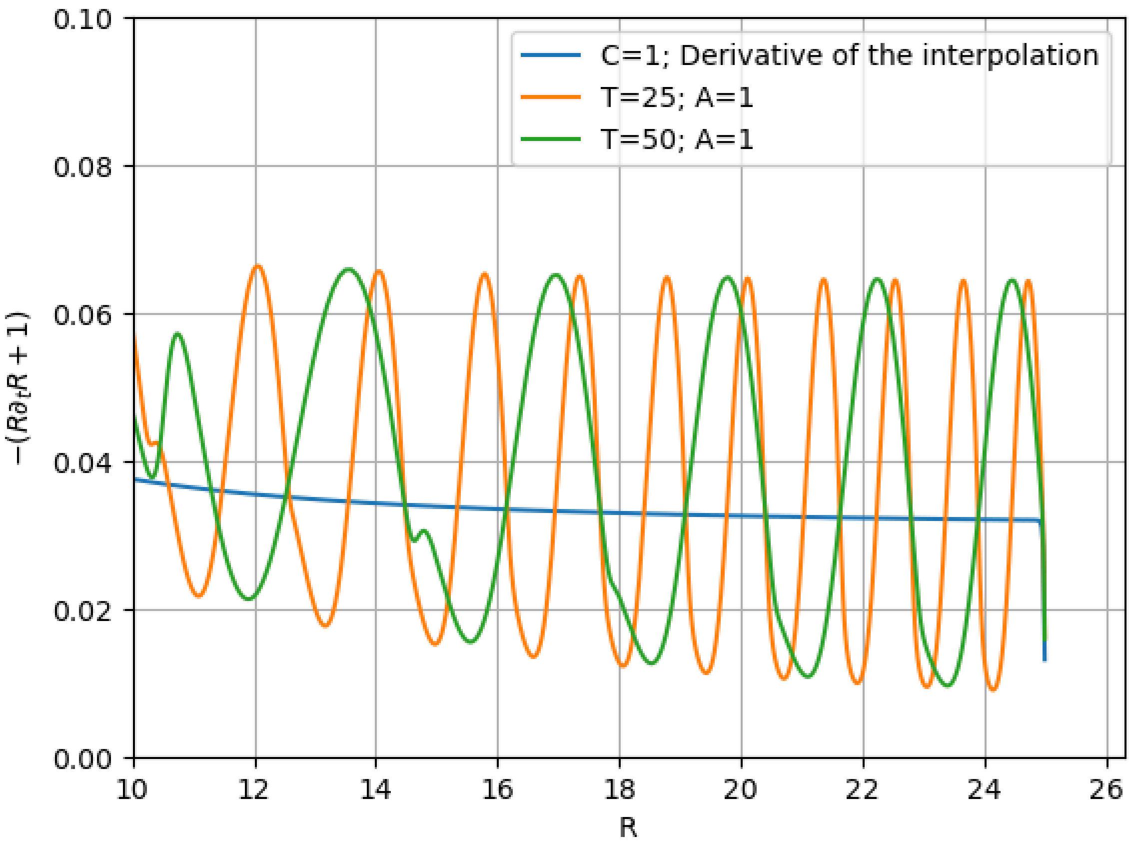
In order for this model to be coherent with our result on slow oscilliations

$$\dot{R} = -\frac{1}{R} + O(\epsilon^2)$$

as $\kappa = \frac{1}{R} \sim \epsilon$ (because $R_0 = 25 = T \sim \epsilon^{-1}$) we should see that

$$f(C(t)) = (R\dot{R} + 1) \sim \epsilon^n \quad n \geq 1$$

and NOT of order 1!



This plot shows experimental curves associated to two values of the period that are one the double of the other. So $\epsilon_{orange} = 2\epsilon_{green}$.
So the next-to-leading order contribution of the green line should be smallest than the half of the contribution of the orange line, but it is not so!

Dependence on C

We have plot how the leading and next-to leading contributions to $\dot{R}$ behave with R. But what about C?
Following the idea that the previous plots show a weak dependence on R of the next-to-leading order contribution (for sure slower than a power law R^{-1} for small amplitudes), then we can look

for the dependence on C comparing data **at different R** (assuming there is no dependance on R)