Explaining the decay of the distance of two kinks with linear dynamics and a sum of Gaussians

#twokinks #1D #linear_regime

Motivation

If $C(t) < 0$ for a long time (as it happens in the cases above) eventually $u(x) \ll C_0$ and we expect the non-linearity to play a negligible role in the dynamics. So it is **natural** to expect that **the LINEAR dynamics is SUFFICIENT** to describe the **steps** that we see in the decay of the distance.
Even if the initial state is, in principle, relevant for the decay $d(t)$, we build the initial state **by summing Gaussian** functions, for simplicity in the calculations.

Estimating the duration of a step

Here I present a way for estimating the duration of a step, that is **independent on the model** adopted to predict the decay $d(t)$ along a step.

At what time t_0 the step originates?

The **beginning of the step** corresponds **empirically** with the moment t_0 **when** $C(t_0) = 0; \dot{C}(t_0) < 0$ (when $C(t)$ becomes negative). For a quench experiment, then $t_0 = 0$.

Duration of a step

Assuming we can neglect the non-linearity **from** the moment **when** $C(t)$ **becomes negative** (this is supported by the fact that we see a good fit if we start to compare model and simulation from $t_0 : C(t_0) = 0$ and $\dot{C}(t_0) < 0$). Then

$$u_{q=0}(t) = u_{q=0}(t_0)e^{B(t)} \quad \text{if } t > t_0$$

$$B(t) = \int_{t_0}^{t} dt' C(t')$$

so $u_{q=0}(t)$ initially decreases, but then it increases again, becoming bigger than the initial value $u_{q=0}(t_0)$ and then the non-linearity in **no more negligible**.
As a consequence, we estimate the time when the decay finishes t_f as the time when $u_{q=0}(t) = u_{q=0}(t_0)$ **again** so

$$B(t_f) = B(t_0) = 0$$

$$\int_{t_0}^{t_f} dt' C(t') = 0$$

$$C(t') = \bar{C}[1 + \frac{A}{\bar{C}}\sin\left(\frac{2\pi t'}{T}\right)]$$

changing variable $\tau = \frac{t}{T}$ and integrating, we find

$$2\pi(\tau_f - \tau_0) = \frac{A}{\bar{C}}[\cos(2\pi\tau_f) - \cos(2\pi\tau_0)]$$

$$t_f = \tau_f T \quad t_0 = \tau_0 T \qquad \tau_0 = \frac{1}{2}\left[1 - \frac{1}{\pi}\arcsin\left(-\frac{\bar{C}}{A}\right)\right]$$

this means that τ_f, τ_0 do not depend on T and so the **duration of the decay** (step) is

$$t_f - t_0 \propto T$$

in general

$$t_f - t_0 = f\left(\frac{\bar{C}}{A}\right)T$$

so it **does NOT depend on the initial distance**, as we see in simulations. The duration of the last step will follow a different rule, as the collapse time t_c predicted with the model of the two gaussians will be smaller than $t_f - t_0$.
Notice: This estimate of the duration of the step does not depend on the model adopted to compute the decay $d(t)$.

Expansion of the step's duration for large amplitude

We define

$$2\pi\frac{\bar{C}}{A} = \epsilon \ll 1$$

Then the equation for τ_f is

$$\epsilon(\tau_f - \tau_0) = \cos(2\pi\tau_f) - \cos(2\pi\tau_0)$$

$$\tau_0 = \frac{1}{2}\left[1 - \frac{1}{\pi} - \arcsin\left(-\frac{\epsilon}{2\pi}\right)\right] \simeq \frac{1}{2} + \frac{\epsilon}{(2\pi)^2} + O(\epsilon^2)$$

As $\tau_f \to \frac{3}{2}$ in the limit $\epsilon \to 0$ (think that this limit is achieved by $\bar{C} \to 0$ with A finite. In this limit $\tau_0 = \frac{1}{2}$ and $\tau_f = \tau_0 + 1$) we estimate τ_f by expanding $\cos(2\pi\tau_f)$ close to $\tau_f \simeq \frac{3}{2}$

$$\cos(2\pi\tau_f) \simeq -1 + \frac{(2\pi)^2}{2}\left(\tau_f - \frac{3}{2}\right)^2 + \dots$$

using this in the first expression, and using

$$\sin(2\pi\tau_0) = -\frac{\epsilon}{2\pi} \implies \cos(2\pi\tau_0) = -\sqrt{1-\sin^2} \simeq -\left(1 - \frac{\epsilon^2}{2(2\pi)^2} + O(\epsilon^4)\right)$$

along with the estimate of τ_0 written above, we find (neglecting $O(\epsilon^2)$)

$$\epsilon\left(\tau_f - \frac{1}{2}\right) \simeq \frac{(2\pi)^2}{2}\left(\tau_f - \frac{3}{2}\right)^2 + O(\epsilon^2)$$

considering the root $< \frac{3}{2}$ (as we expect τ_f to decrease if A increases with fixed $\bar{C}$)

$$\tau_f = \frac{3}{2} - \frac{\sqrt{2}}{2\pi}\epsilon^{1/2} + \frac{\epsilon}{(2\pi)^2} + O(\epsilon^2)$$

So, the estimated duration of the step is

$$\tau_f - \tau_0 \simeq 1 - \frac{\sqrt{2}}{2\pi}\epsilon^{1/2} + O(\epsilon^2)$$

remembering $\epsilon = \frac{2\pi\bar{C}}{A}$, then

$$t_f - t_0 = f\left(\frac{\bar{C}}{A}\right)T$$

$$f\left(\frac{\bar{C}}{A}\right) \simeq 1 - \sqrt{1/\pi}(\frac{\bar{C}}{A})^{1/2}$$

We can use this expression to estimate A ($\bar{C} = 1$) from the measures of duration of the step Δt_{step} by inverting

$$\Delta t_{step} = t_f - t_0 = T(1 - 1/\sqrt{\pi}A^{-1/2})$$

here we see the ratio of the estimated amplitude respect to the true value.

- It is expected that it is always underestimated as I expect the non-linearity do become negligible a little bit after C(t) crosses zero and not when it crosses it.
- It is expected that the last datapoint is not 1, because the last steps lasts a time $t_c < \Delta t_{step}$.

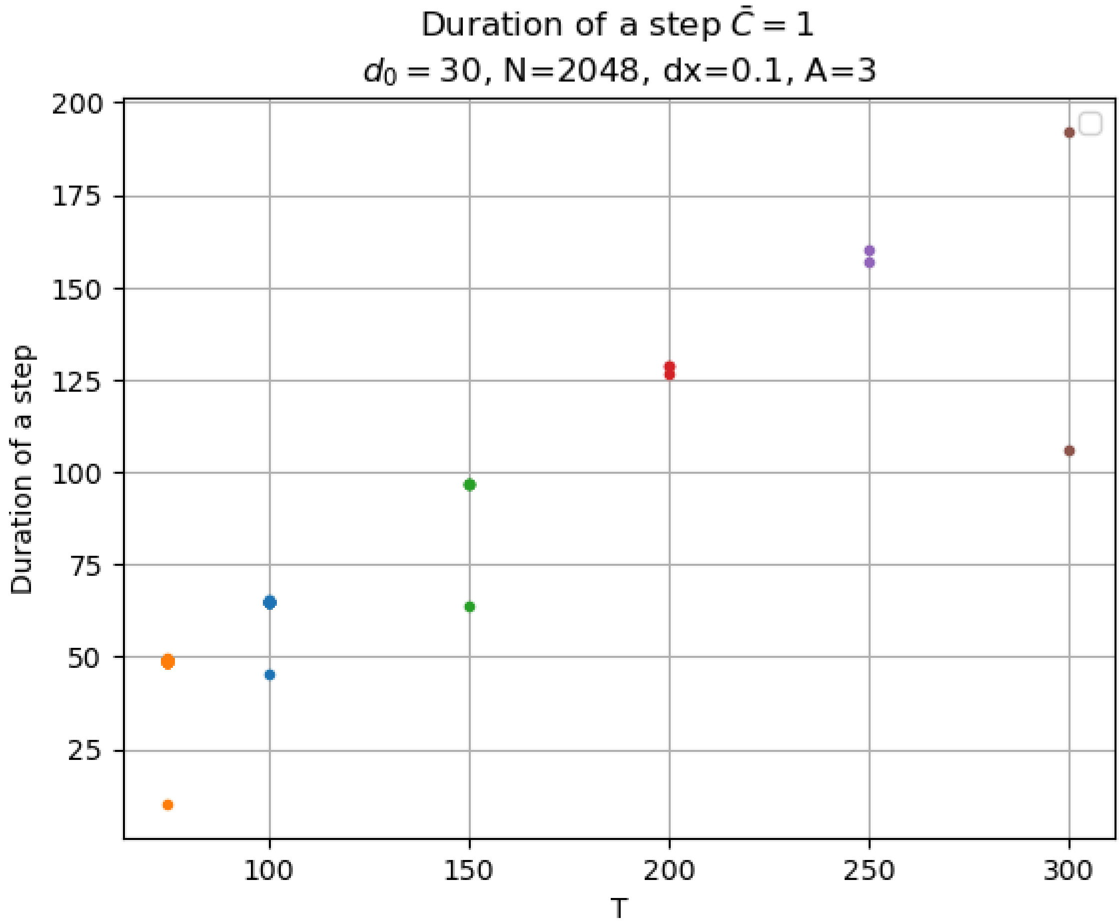
Measuring the duration of the steps

I measure the duration of the steps $t_1 - t_0$ by considering

- t_0 as the instant when $C(t)$ becomes negative
- t_1 is estimated as the first time $t_1 > t_0$ where the derivative $\partial_t d$ becomes smaller than a tollerance 10^{-5}

  Linear dependence on T

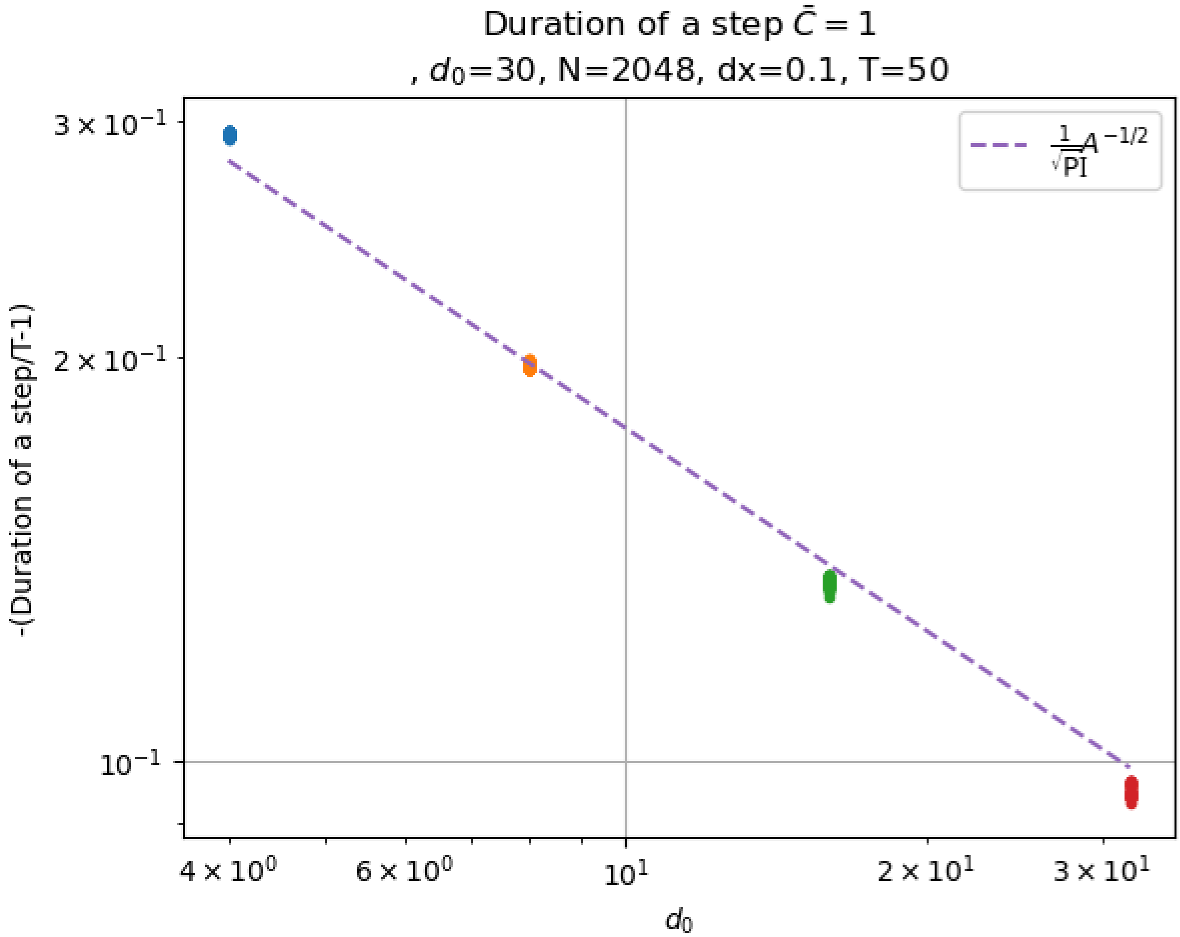
The points far away represent the duration of the last step that is not expected to be linear in T.



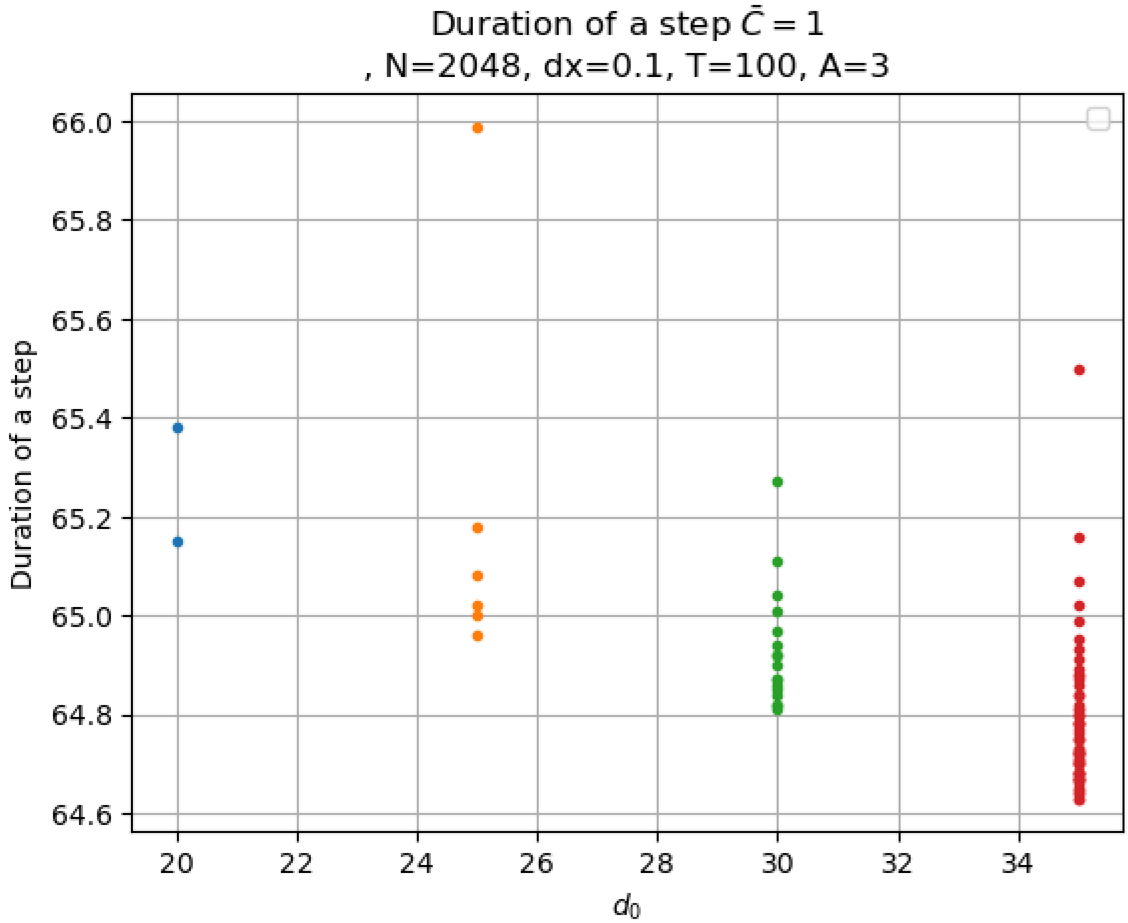
Dependence on A

$$\Delta t_{step} \simeq T(1 - 1/\sqrt{\pi} A^{-1/2}) \implies -\left[\frac{\Delta t_{step}}{T} - 1\right] \simeq \frac{1}{\sqrt{\pi}} A^{-1/2}$$

Here we don't see any point far away, because the simulation ends before the collapse (notice here T is smaller and I made this choice to lunch simulations with higher A having at least a couple of steps before collapse).
Maybe considering higher order terms in the asymptotic expansion, we could get a better match



Dependence on initial distance



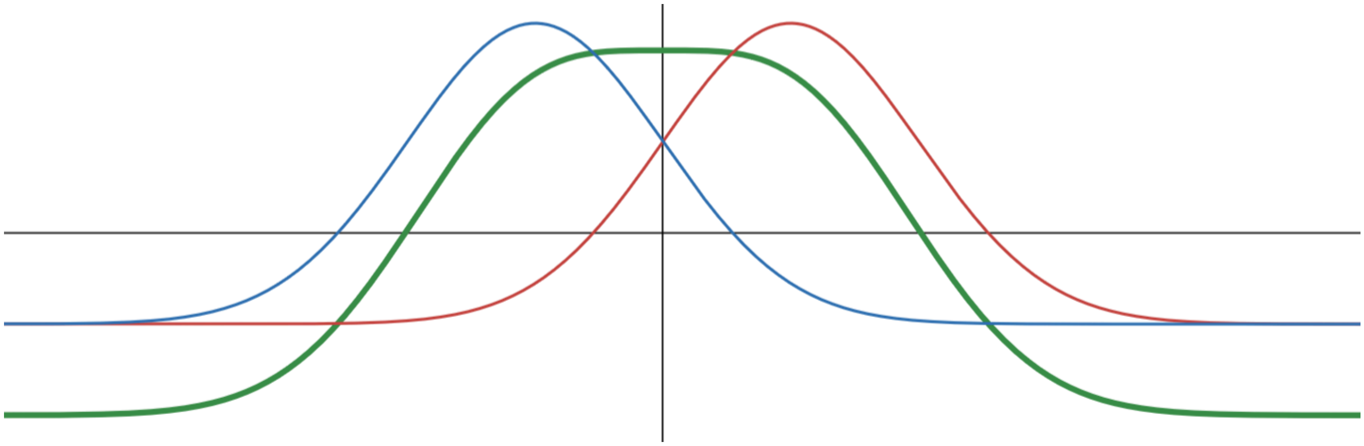
Comment: It seems there is a dependence of the duration of the step on the index of the step and on the initial distance. It could be related to **how I measure the step duration**. As I estimate the end of the decay by checking when $\partial_t d$ goes below a tollerance and not $\frac{\partial_t d}{d}$

Model 1: Sum of TWO Gaussian with the same σ

We compute analytically the decay of the distance of this initial state, according to linear dynamics.

$$u(x, t_0) = \mathcal{N}\left[g_+(x) + g_-(x) - \frac{e^{-1/2}}{\sqrt{2\pi}\sigma}\right]$$

where $\mathcal{N}$ is a normalization, that sets the value of the green curve at $x = 0$ (and $x = \pm\infty$).



$$g_\pm(x) = \frac{e^{-(x-x_\pm)^2/2\sigma^2}}{\sqrt{2\pi}\sigma}$$

As $\partial_{xx} g_\pm(x_\pm) = \frac{3e^{-2}}{\sigma^2}$, then we choose $L = 2\sigma$ (where $L = x_+ - x_-$) such that the second derivative is zero in the midpoint (**plateau**).

Parameters of the model

There is **only one parameter** to be set in order to compute the prediction of $d(t)$, that is **the initial width** σ of the kinks. That's because $x_+ - x_- = 2\sigma$ is the condition required to have a plateau between the gaussians (a property that is kept with time as the width of the two gaussian is the same).
The initial distance between kinks d_0, is related to the distance L between the centers of the Gaussians ($x_\pm$) and their initial width σ as

$$d_0 \simeq L + 2\sigma$$

then

$$\sigma \simeq \frac{d_0}{4}$$

When **approximating** the shape of $u(x)$ to a sum of two Gaussians, this is the **natural** way of determining the (only) parameter σ, by measuring the distance between the kinks.

Decay of the distance

In Linear dynamics twokinks with Gaussian profile is calculated analytically the evolution of the above profile, under the linear dynamics

$$u(x,t) = e^{B(t)} \mathcal{N} \left[\frac{e^{-(x-x_+)^2/2\sigma(t)^2}}{\sqrt{2\pi}\sigma(t)} + \frac{e^{-(x-x_-)^2/2\sigma(t)^2}}{\sqrt{2\pi}\sigma(t)} - \frac{e^{-1/2}}{\sqrt{2\pi}\sigma} \right]$$

$$\sigma(t)^2 = \sigma_0^2 + 2(t - t_0)$$

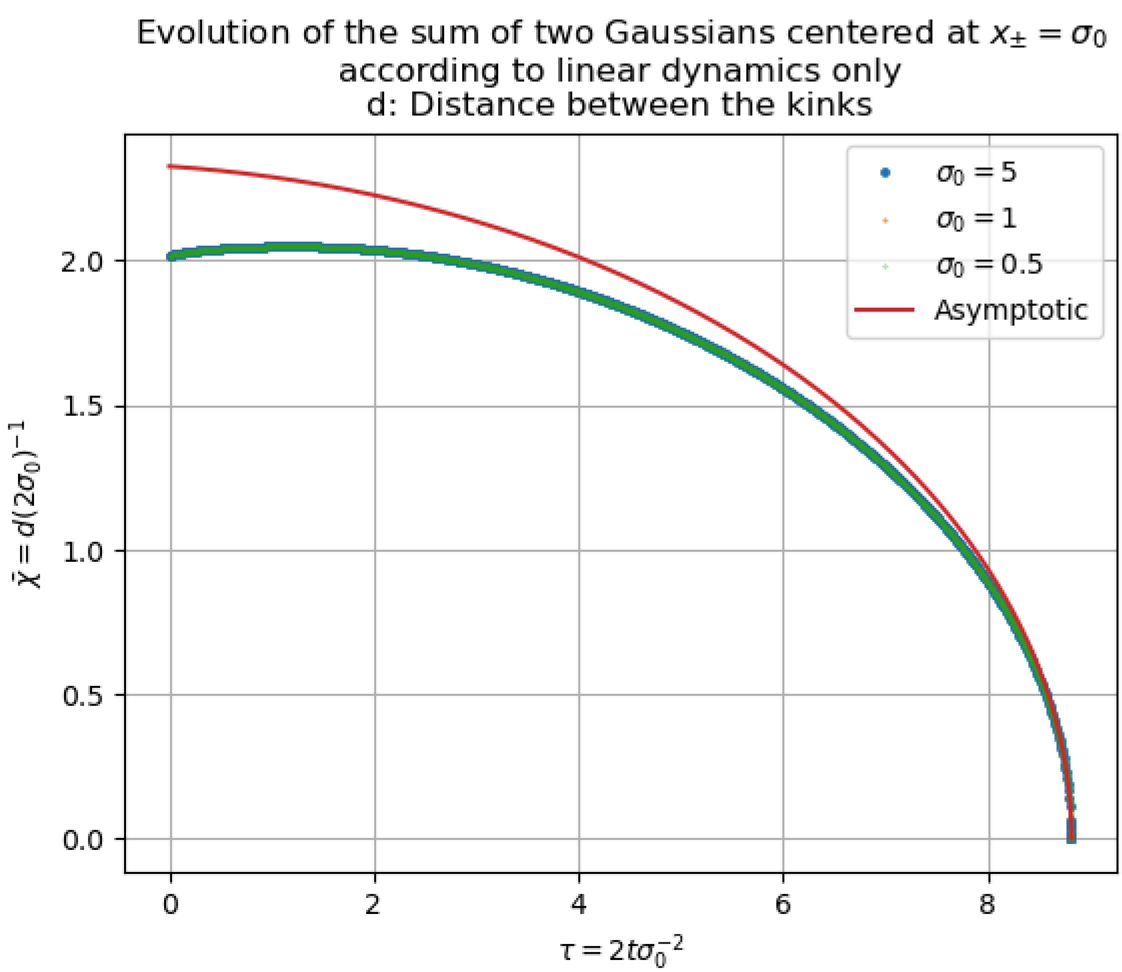
Notice: the positions of the zeros (kinks), and so their distance, does NOT depend on $C(t)$ and on the amplitude $\mathcal{N}$ of the initial profile.
Properly **rescaling** the axis (Linear dynamics twokinks with Gaussian profile), I find a profile whose shape is not dependent on the only parameter σ.

- $d \to \frac{d}{\sigma_0}$ and $\bar{\chi} = \frac{d}{2}$ is the position of the positive kink.
- $t \to \tau = 2\frac{t-t_0}{\sigma_0^2}$
  I cannot write a formula for $d(t)$, but I can find it numerically with the **Newton's algorithm**.
  And I can find an asymptotic expansion, close to the collapse time t_c

$$\bar{\chi} \simeq \frac{1+\tau_c}{\tau_c^{1/2}}\left[\frac{\tau_c-\tau}{1+\tau_c} + \frac{(\tau_c-\tau)^2}{2(1+\tau_c)^2}\right]^{1/2} \quad \text{if } \frac{\tau_c-\tau}{\tau_c} \ll 1$$

where $\tau_c \simeq 8.82$ is determined by

$$2\frac{e^{-1/2(1+\tau_c)}}{(1+\tau_c)^{1/2}} = e^{-1/2}$$

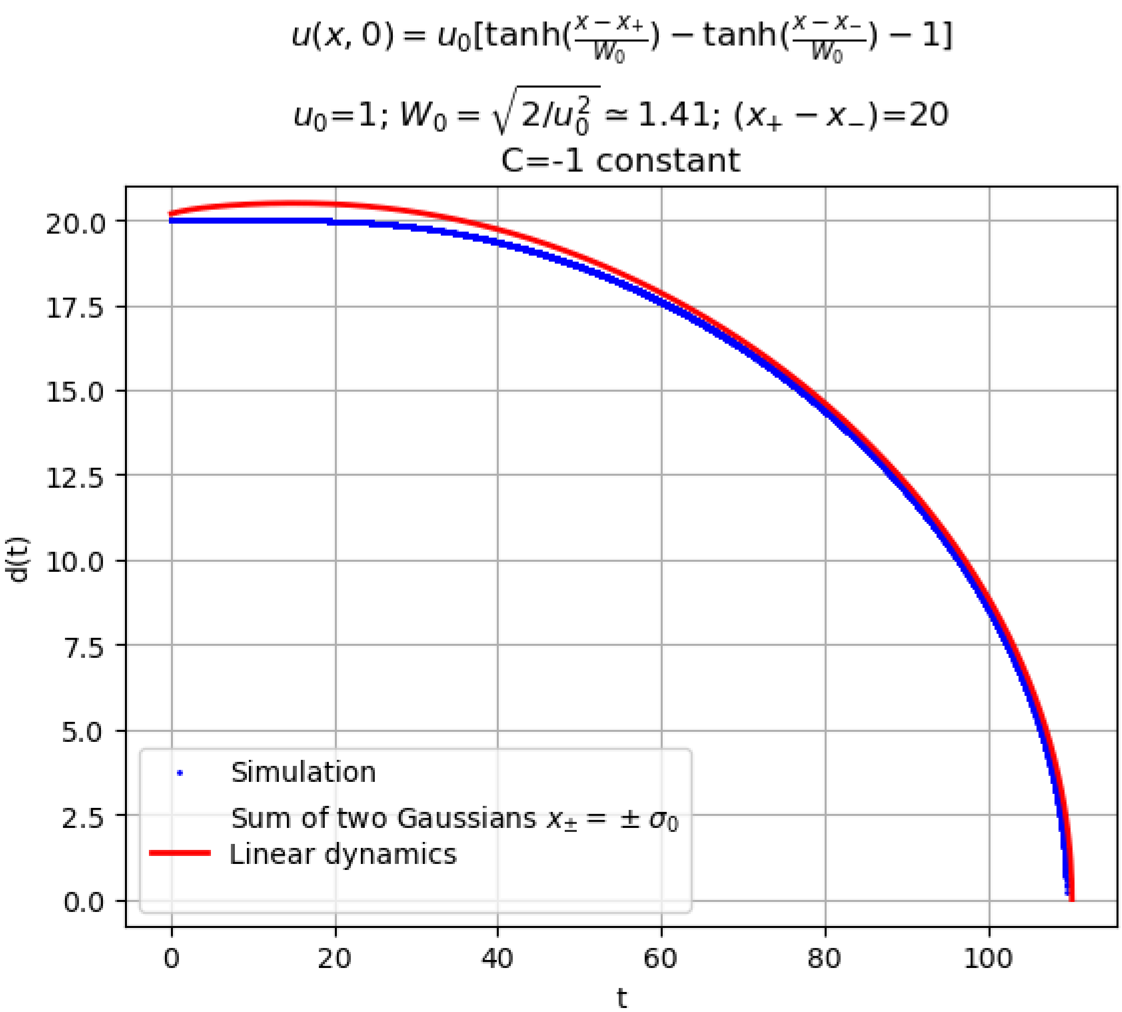


Experiments

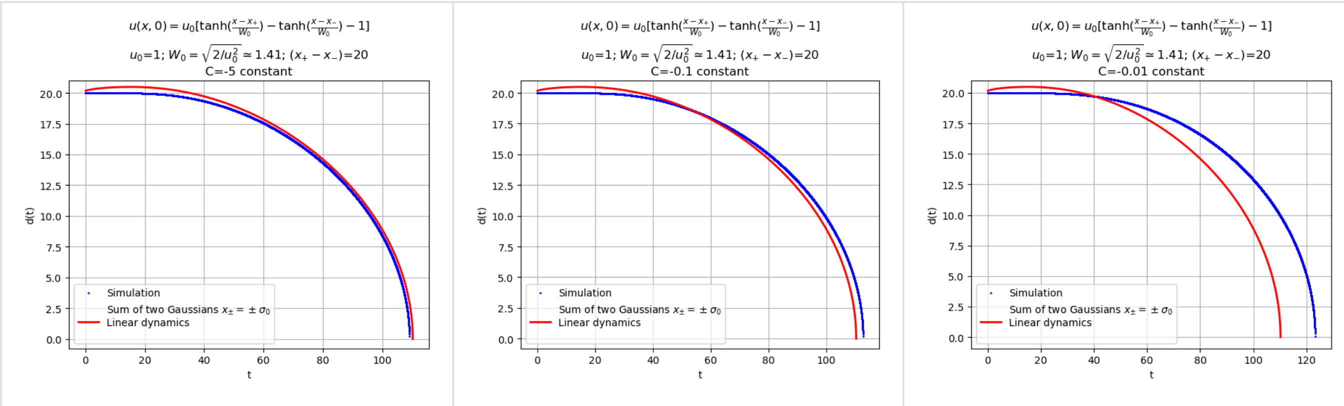
1) Quench to $C < 0$

Paramters (for the SIMULATION):

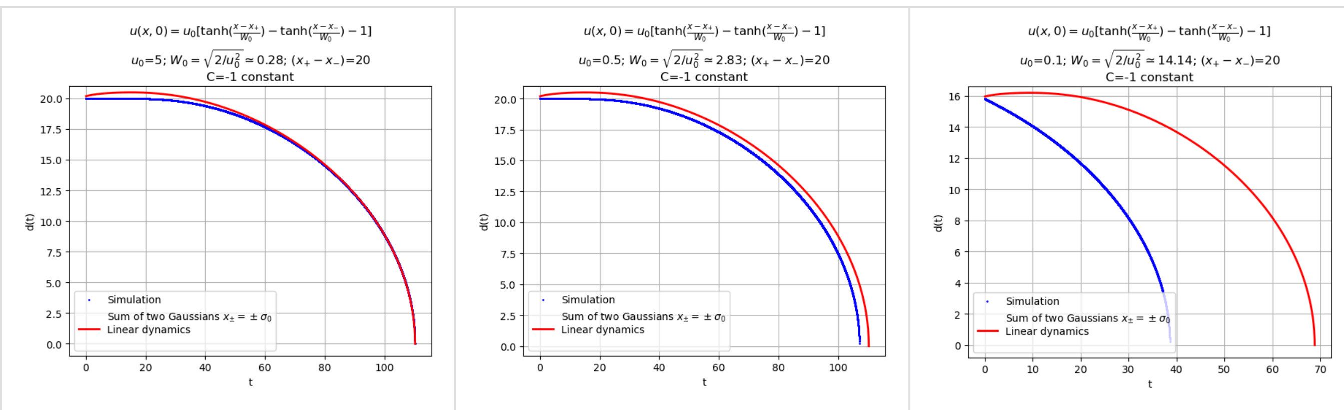
- u_0: amplitude of the initial state **and** $w_0 = u_0^{-1/2}$ initial width of the kinks
- d_0: initial distance beetween kinks
- $C < 0$: constant value of C



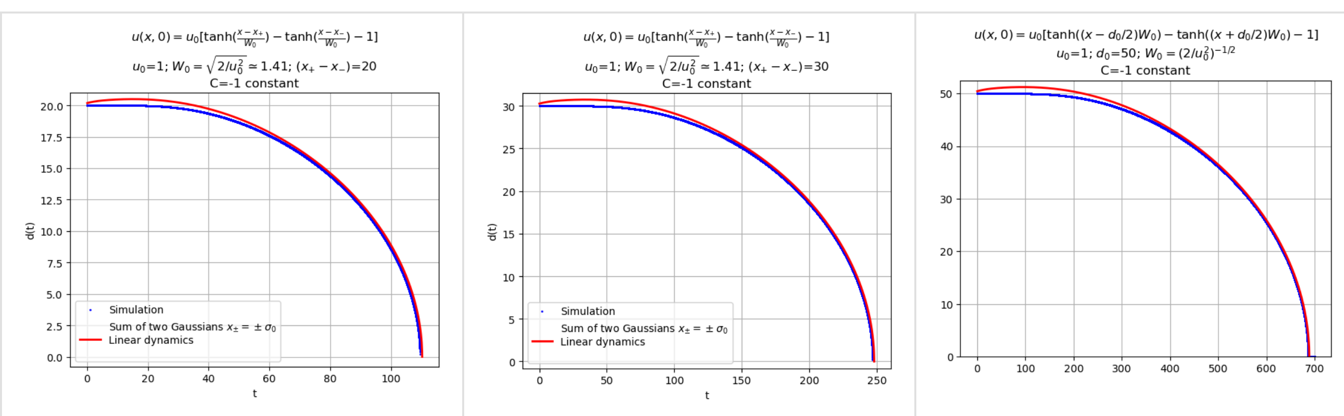
When $|C| \sim 0.1$ the fit is not good, not even at late times. I guess this is due to the relevance of the non linearity up to longer times, as u **decays slower** to zero in this case.



- **Surprisingly** if the initial amplitude is small $u_0 \sim 0.1$, the fit is bad also at late times. I would expect it to be better, as the non-linearity is less important!



While the goodness of the fit does not depend on the initial distance d_0.



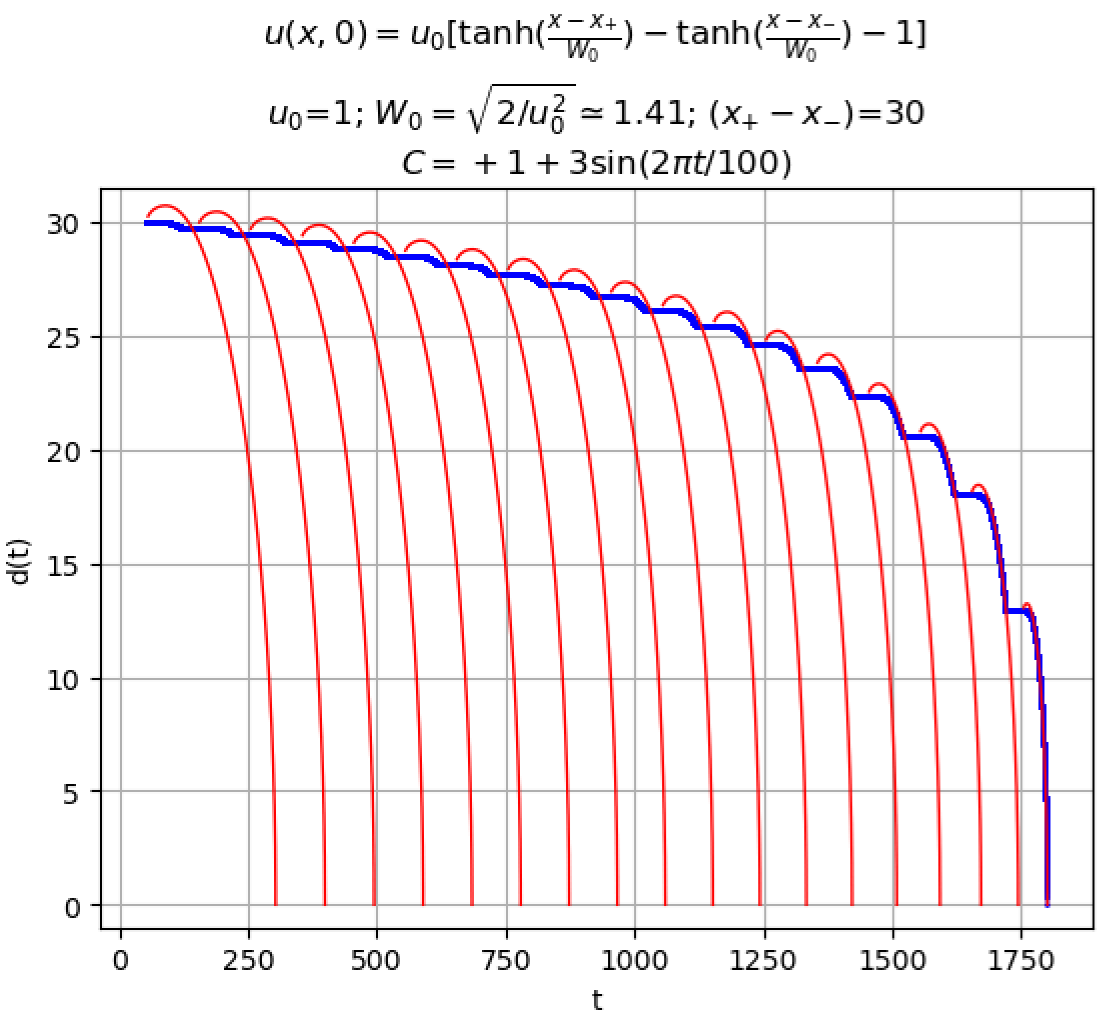
2) Slow oscillations $A \gg C_0$

$$C(t) = C_0 + A\sin\left(\frac{2\pi t}{T}\right); \quad C_0 = 1$$

At what time t_0 the step originates?

Here we assume the beginning of the decay t_0 as the moment, within a period, **when $C(t)$ starts to take negative values:** $C(t_0) = 0; \dot{C}(t_0) < 0.$
This choice leads to a good fit, when the **depth of the step is large.**

- At the beginning of the step, the red curve is increasing, while the simulation is strictly decreasing. I expect this to be an effect of the **non-linearity, that still plays a role in the first istants** of negative C.
- After the beginning of the step, the fit is good, **without any shift**.



Predicting the depth of the step

If the duration of the step Δt_{step} is close to the decay time t_c computed in the model of two Gaussians, then, in the last istants of the step, we are in the regime of validity of

$$\bar{\chi} \simeq \frac{1+\tau_c}{\tau_c^{1/2}} \left[\frac{\tau_c - \tau}{1+\tau_c} + \frac{(\tau_c - \tau)^2}{2(1+\tau_c)^2} \right]^{1/2} \quad \text{if } \frac{\tau_c - \tau}{\tau_c} \ll 1$$

where $\tau = \frac{2(t-t_0)}{\sigma_0^2}$, $d(t) = \frac{d_0}{2}\bar{\chi}(\tau)$.
It holds in the last moments of the step, so also at its end $t - t_0 = \Delta t_{step}$

$$\Delta t_{step} \simeq T[1 - \sqrt{\frac{1}{\pi}}\left(\frac{\bar{C}}{A}\right)^{1/2}]$$

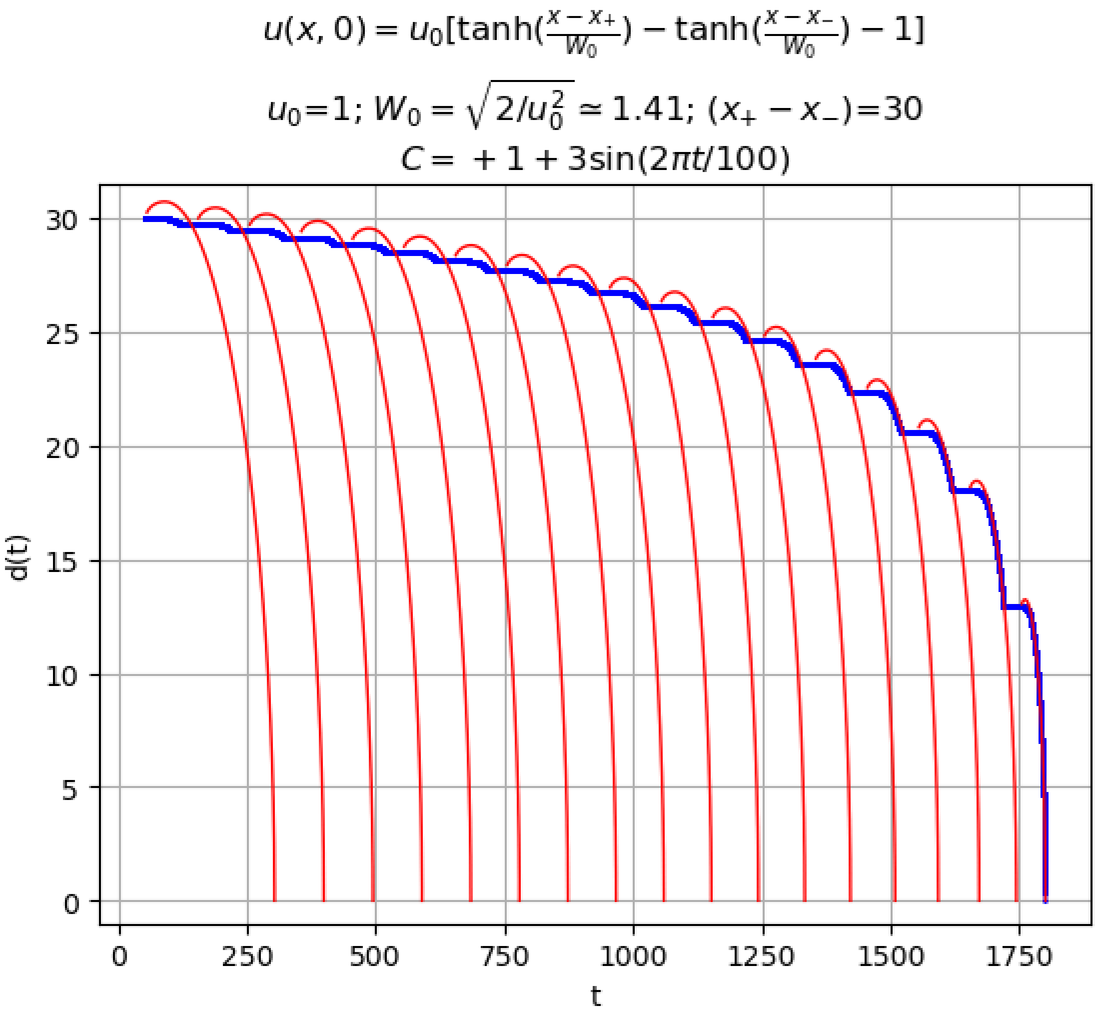
$$\Delta\tau_{step} = \frac{2\Delta t_{step}}{\sigma_0^2} \sim d_0^{-2}T[1 - \sqrt{\frac{1}{\pi}}\left(\frac{\bar{C}}{A}\right)^{1/2}]$$

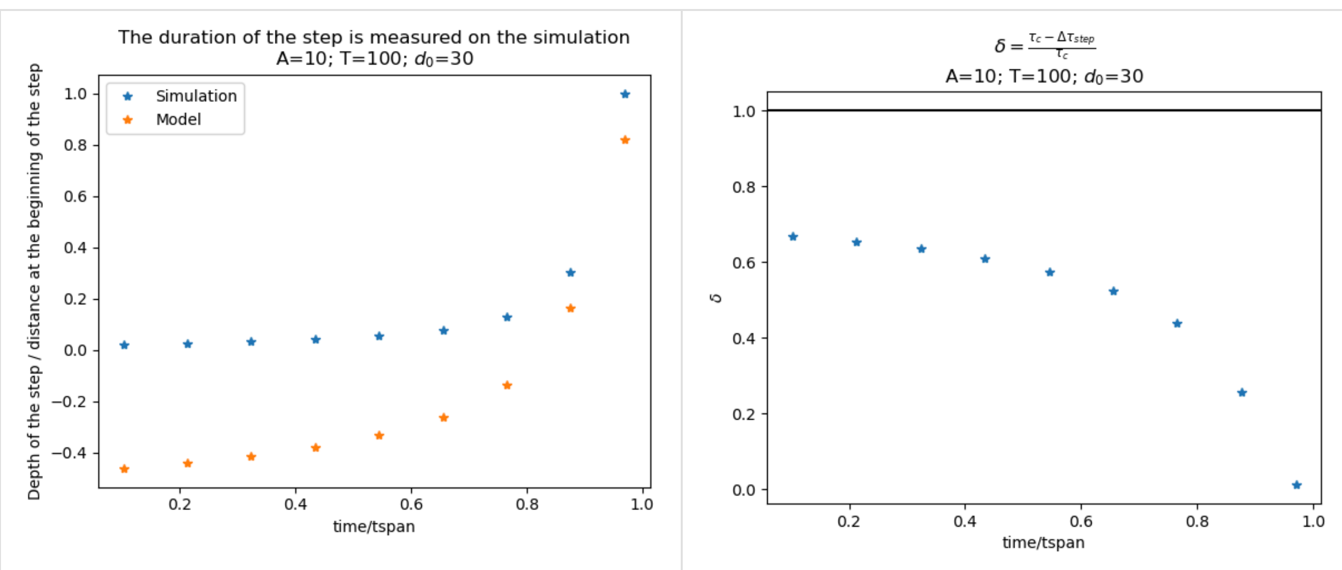
To estimate the depth of the step, we should insert $\Delta\tau_{step}$ as τ in the expression for $\bar{\chi}(\tau)$, where $\tau_c \simeq 8.82$. If, instead, $\Delta t_{step} > t_c$, the depth of the step is d_0 and its duration is t_c.

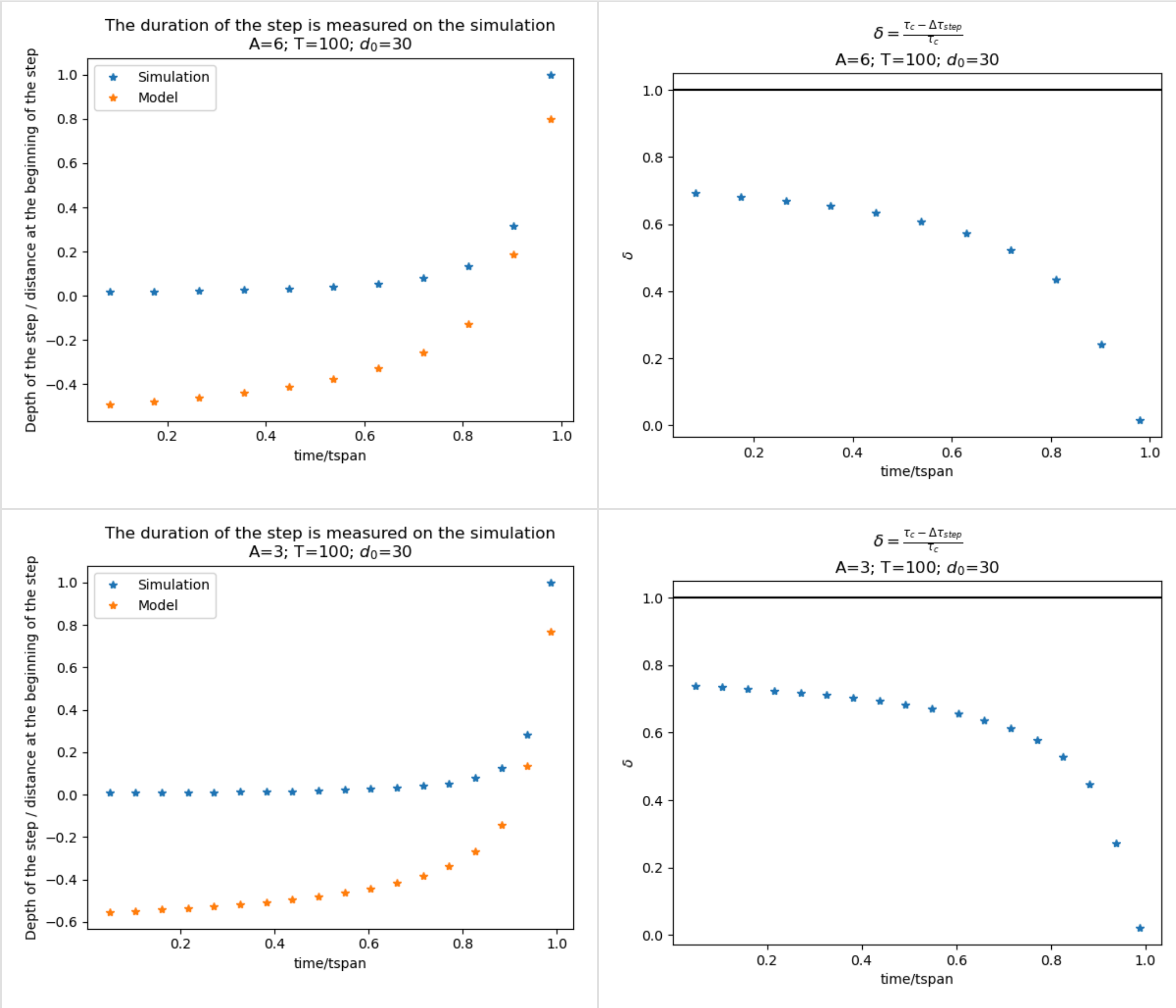
Predicting Δd by MEASURING Δt_{step}

We can evaluate the expression for $\bar{\chi}(\tau)$ at $\tau = \Delta\tau_{step}$ where the duration of the step is **measured**. In the following plots, it is reported the value of the small parameter $(\tau_c - \tau)/\tau_c$ for $\tau = \Delta\tau_{step}$.
The asymptotic expansion for $\bar{\chi}(\tau)$ is true when that parameter is very small, but the plots below show that it is not so small, except for the last period. This is coherent with the fact that the estimate is **bad** even when there is a good fit.
Notice: The prediction $\frac{\Delta d}{d} < 0$ when $\Delta\tau_{span} < 4$, because the asymptotic expansion, far from τ_c,

behaves like that.





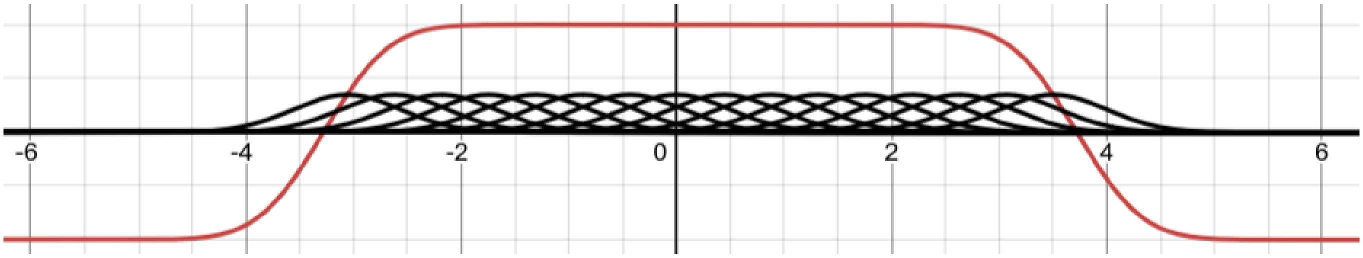


Model 2: Sum of INFINITE amount of Gaussians

Here we exploit the following results, about a sum of an infinite amount of Gaussian functions with the same σ

$$f(x) = \lim_{N\to\infty} \frac{2L}{N}\left(\sum_{n=1}^{N} g_n(x) - \frac{1}{2}\right) = \frac{1}{2}\left(\text{erf}\left(\frac{x}{\sqrt{2}\sigma}\right) - \text{erf}\left(\frac{x-L}{\sqrt{2}\sigma}\right) - 1\right)$$

$$g_n(x) = \frac{e^{-(x-nL/N)^2/2\sigma^2}}{\sqrt{2\pi}\sigma}$$



Proof

$$\frac{1}{\sigma\sqrt{2\pi}} \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \exp\left(-\frac{(x-\frac{i}{n})^2}{2\sigma^2}\right) = \frac{1}{\sigma\sqrt{2\pi}} \int_0^1 \exp\left(-\frac{(x-y)^2}{2\sigma^2}\right) dy =$$

$$= \frac{1}{2}\left(\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma}\right) - \operatorname{erf}\left(\frac{x-1}{\sqrt{2}\sigma}\right)\right)$$

Parameters of the model

The advantage of this profile is that the two parameters L, σ are **independent**!

- If $L \gg \sigma$, then $d \simeq L$ is the (initial) **distance** between kinks;
- while σ describes the **width** of the kinks $W = \sqrt{2}\sigma$.

Decay of the distance

Under **linear dynamics only**, this profile has a trivial evolution at time $t > 0$:

$$\sigma_0^2 \to \sigma(t)^2 = \sigma_0^2 + 2t$$

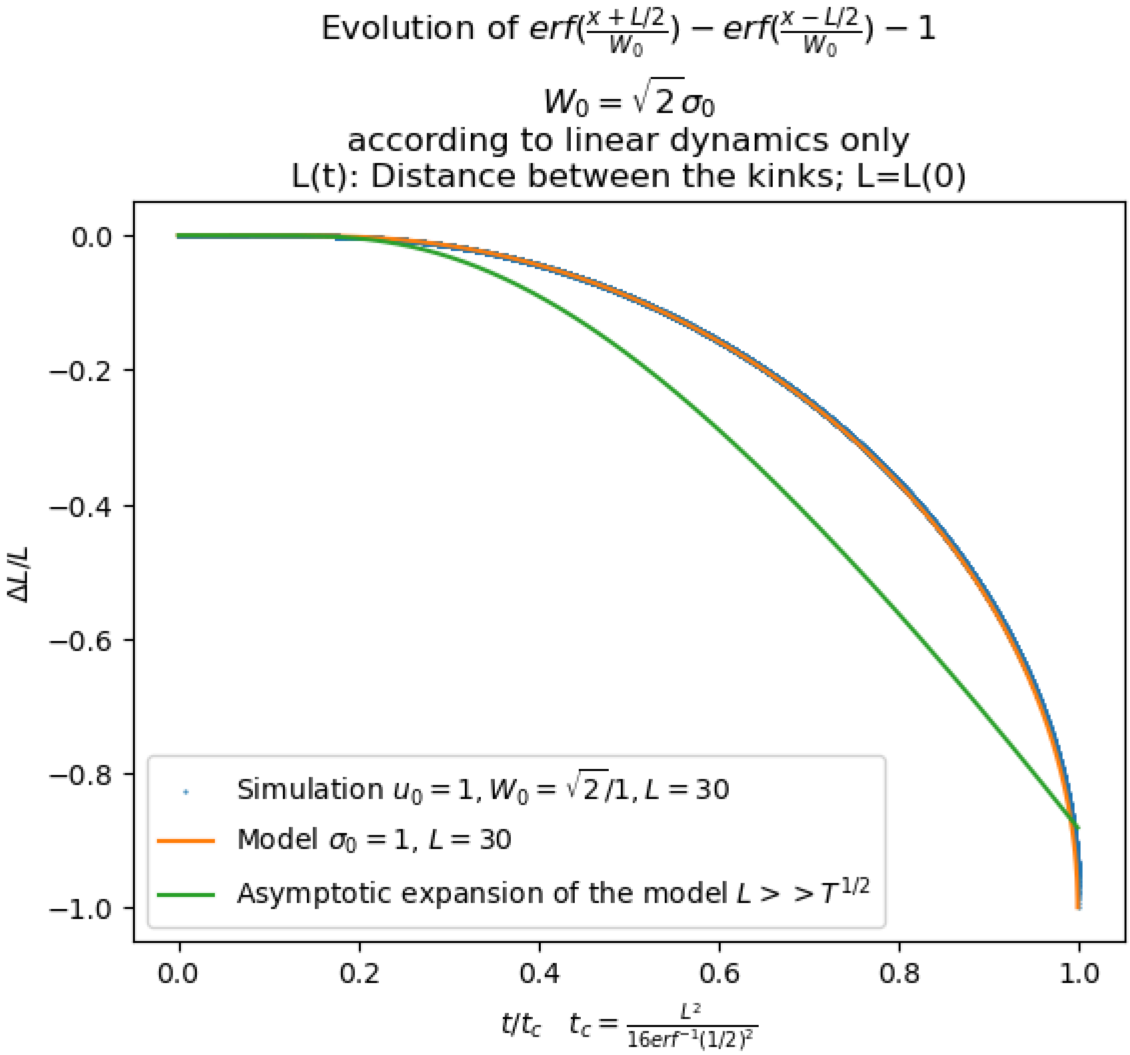
$$f(x) = \frac{1}{2}\left(\operatorname{erf}\left(\frac{x}{\sqrt{2}\sigma(t)}\right) - \operatorname{erf}\left(\frac{x-L}{\sqrt{2}\sigma(t)}\right) - 1\right)$$

To find the distance as a function of time, we look for the zeros of $f(x)$: $x^*_\pm$

$$d(t) = x^*_+(t) - x^*_-(t) = 2x^*$$

$$\operatorname{erf}\left(\frac{x^* + \frac{L}{2}}{\sqrt{2}\sigma(t)}\right) - \operatorname{erf}\left(\frac{x^* - \frac{L}{2}}{\sqrt{2}\sigma(t)}\right) = 1$$

Computing this with the **Newton's method** .



Notice: The prediction of $d(t)$ is always decreasing, also at the beginning: as in simulations.

Inital distance d_0 and collapse time t_c

If in the **initial state** the kinks are overlapping ($2\sigma \sim L$), then L does not represent anymore the initial distance.
It is possible to calculate the deviation of d_0 from L by expanding the equation for $d(t)$ in powers of $\alpha = \frac{L}{\sigma_0} \gg 1$ and for $t = 0$:

$$d(t=0) = 2x^*(t=0) = \simeq L - \frac{4\sigma_0}{L} e^{-(\sigma_0/L)^2/2}$$

While the collapse time can be found by requiring $x^*(t_c) = 0$

$$t_c = \frac{L^2}{16\left(erf^{-1}\left(\frac{1}{2}\right)\right)^2} - \frac{\sigma_0^2}{2}$$

Rescaling

If there is no overlap ($L \gg 2\sigma_0$) in the initial state, the previous formulas simplify to

- $d_0 \simeq L$
- $t_c \simeq \frac{L^2}{16\left(erf^{-1}\left(\frac{1}{2}\right)\right)^2}$

  And rescaling $\Delta d(t) = d(t) - d_0$ and t as
- $\frac{\Delta d(t)}{L}$
- $\frac{t}{t_c}$

  we see that both simulations and numerical solutions $x^*(t)$ of

$$\mathrm{erf}\left(\frac{x^* + \frac{L}{2}}{\sqrt{2}\sigma(t)}\right) - \mathrm{erf}\left(\frac{x^* - \frac{L}{2}}{\sqrt{2}\sigma(t)}\right) = 1$$

collapse on the same curve, independently on the choice of σ_0, L (if $L \gg 2\sigma_0$).
It means that, if $L \gg 2\sigma_0$

$$\frac{\Delta d(t)}{L} = 1 - \frac{d(t)}{L} = f\left(\frac{t}{t_c(L)}\right)$$

so we can **predict** the **depth** of the step, knowing the duration of the step

$$\Delta d_{step} = Lf\left(\frac{\Delta t_{step}}{t_c(L)}\right)$$

where L is the distance at the beginning of the step.

Asymptotics for $L \gg T^{1/2}$

We can evaluate numerically $f(\xi)$, but we lack of an analytical expression.
Although, it is possible to make an expansion of

$$\mathrm{erf}\left(\frac{x^*(t) + \frac{L}{2}}{\sqrt{2}\sigma(t)}\right) - \mathrm{erf}\left(\frac{x^*(t) - \frac{L}{2}}{\sqrt{2}\sigma(t)}\right) = 1$$

around $x^*(t) = \frac{L}{2}$, remembering that $d(t) = 2x(t)$

$$x^* = \frac{L}{2}(1 - \epsilon) \quad 0 < \epsilon \ll 1$$

about the time, we can write $\sigma(t) = \sigma_0(1 + \tau)^{1/2}$ where $\tau = \frac{2t}{\sigma_0^2}$ (where $\sigma_0 \sim W_0 \sim \bar{C}^{-1/2}$).
We will be interested, to estimate the depth of a step, in evaluating $L(t)$ at $t = \Delta t_{step} \sim T$, so

$$\tau \sim T\bar{C}$$

and in the **slow oscillation limit** $T \gg \bar{C}$ and so $\tau \gg 1$ and $(1 + \tau) \simeq \tau$.

$$erf(2\alpha) - erf(-\epsilon\alpha) = 1$$

where, evaluating $\tau = \Delta\tau_{step} = \frac{2\Delta t_{step}}{\sigma_0^2}$

$$\alpha = \frac{L}{4\Delta t_{step}^{1/2}} \gg 1 \quad \text{as } \Delta t_{step} \sim T \text{ and } L \gg T^{1/2}$$

we use the asymptotic expansion of erf for the first term and the taylor expansion at zero for the second.

$$\epsilon \simeq \frac{e^{-4\alpha^2}}{2\sqrt{\pi}\alpha^2}$$

To put back Δt_{step}, we define

$$\xi = \frac{\Delta t_{step}}{t_c(L)} \quad \gamma = (erf^{-1}\left(\frac{1}{2}\right))^2$$

and we recognize that

$$\alpha^2 = \frac{t_c\gamma}{\Delta t_{span}} = \frac{\gamma}{\xi}$$

then

$$\epsilon = \frac{\xi}{2\sqrt{\pi}\gamma} e^{-4\gamma/\xi} \equiv f(\xi)$$

and remembering $\frac{\Delta L(\Delta t_{span})}{L} = f\left(\frac{\Delta t_{step}}{t_c(L)}\right)$, we conclude that

$$\frac{\Delta L(\Delta t_{step})}{L} = f\left(\frac{\Delta t_{step}}{t_c(L)}\right) \simeq \frac{\frac{\Delta t_{step}}{t_c(L)}}{2\sqrt{\pi}\gamma} e^{-4\gamma/\frac{\Delta t_{step}}{t_c(L)}} \quad \text{where } \frac{\Delta t_{step}}{t_c(L)} \sim \frac{T}{L^2} \ll 1$$

Note: Here $t_c(L)$ is the collapse time of the Infinite Gaussian packet under linear dynamics, when the initial distance is L. In a simulation of the TDGL, the collapse time is indicated by T_c and is way larger!
Note: We used other approximations over $L \gg T^{1/2}$ to find this result: $L \gg \sigma_0, T \gg \sigma_0^2$.
Intuitively, as σ_0 is the order of the width of the kinks when $C(t)$ crosses zero, then we state $\sigma_0 \sim \bar{C}^{-1/2}$ and so the result holds if

$$L \gg max(4T^{1/2}, \bar{C}^{-1/2}, \bar{C}^{-1})$$

Attraction in the $L \gg T^{1/2}$ limit

We will call the collapse time $T_c(L, T, A)$ to distinguish it from the collapse time of the Infinite gaussian $\tau_c(L)$.
If $T_c \gg T$, then we can define a "macroscopic derivative"

$$\frac{\Delta L_{step}}{T} \simeq \partial_t L$$

If $L \gg T^{1/2}$ then

$$\partial_t L \simeq f(\xi)\frac{L}{T} = -\frac{8}{\sqrt{\pi}} L^{-1} e^{-L^2/4T}$$

intuitively we will say that this **attraction** is negligible respect to the kink dynamics (that we expected still to have, this is ad **additional** effect **we expect**)

$$\partial_t L|_{Ccost} = -(48C)^{-1} e^{-\sqrt{2C}L}$$

but let's be **quantitative**!

3 Regimes

When this effect **dominates on the other**? When the following inequality holds:

$$(48C)^{-1} e^{-\sqrt{2C}L} \ll 8L^{-1} e^{-L^2/4T}$$

if we apply the log at both sides, in the limit $L \gg max(192\bar{C}, 1)$ we can make an approximation

$$\frac{L^2}{4T} \ll (2\bar{C})^{1/2} L$$

$$L \ll 4\sqrt{2}\bar{C}^{1/2} T$$

So there are **three regimes**

$$L_1^* = 4T^{1/2} \quad L_2^* = 4\sqrt{2}\bar{C}^{1/2} T$$

- **Small distances**: If $L \ll L_1^*$ the steps are well described by the Infinite Gaussian model, but we do not know analytically $f(\xi)$ for not so small ξ. So we do not know how $L(t)$ scales, but we can compute it numerically (Netwton's method).
- **Intermediate regime**: If $L_1^* \ll L \ll L_2^*$ we can use the asymptotic expansion of $f(\xi)$ and we know how $L(t)$ scales, as

$$\partial_t L \simeq= -\frac{8}{\sqrt{\pi}} L^{-1} e^{-L^2/4T}$$

- **Asymptotic regime**: If $L \gg L_2^*$, then the effect due to the large oscillations disappears as it becomes negligible respect to the kinks dynamics. So we expect to see the well-known behaviour

$$\partial_t L = -(48\bar{C})^{-1} e^{-\sqrt{2\bar{C}}L}$$

Notice that we did many approximations, so L_1^* and L_2^* do not necessarily scale with T and $\bar{C}$ as stated, more generally:

- $L_1^* = max(4T^{1/2}, \bar{C}^{-1/2}, \bar{C}^{-1})$
- $L_2^* = max(4\sqrt{2}\bar{C}^{1/2})$ and bigger than the (higher) intersection between $(2\bar{C})^{1/2}L$ and $\log L$, if it exists.

Estimating the collapse time T_c using the asymptotic expansion

From

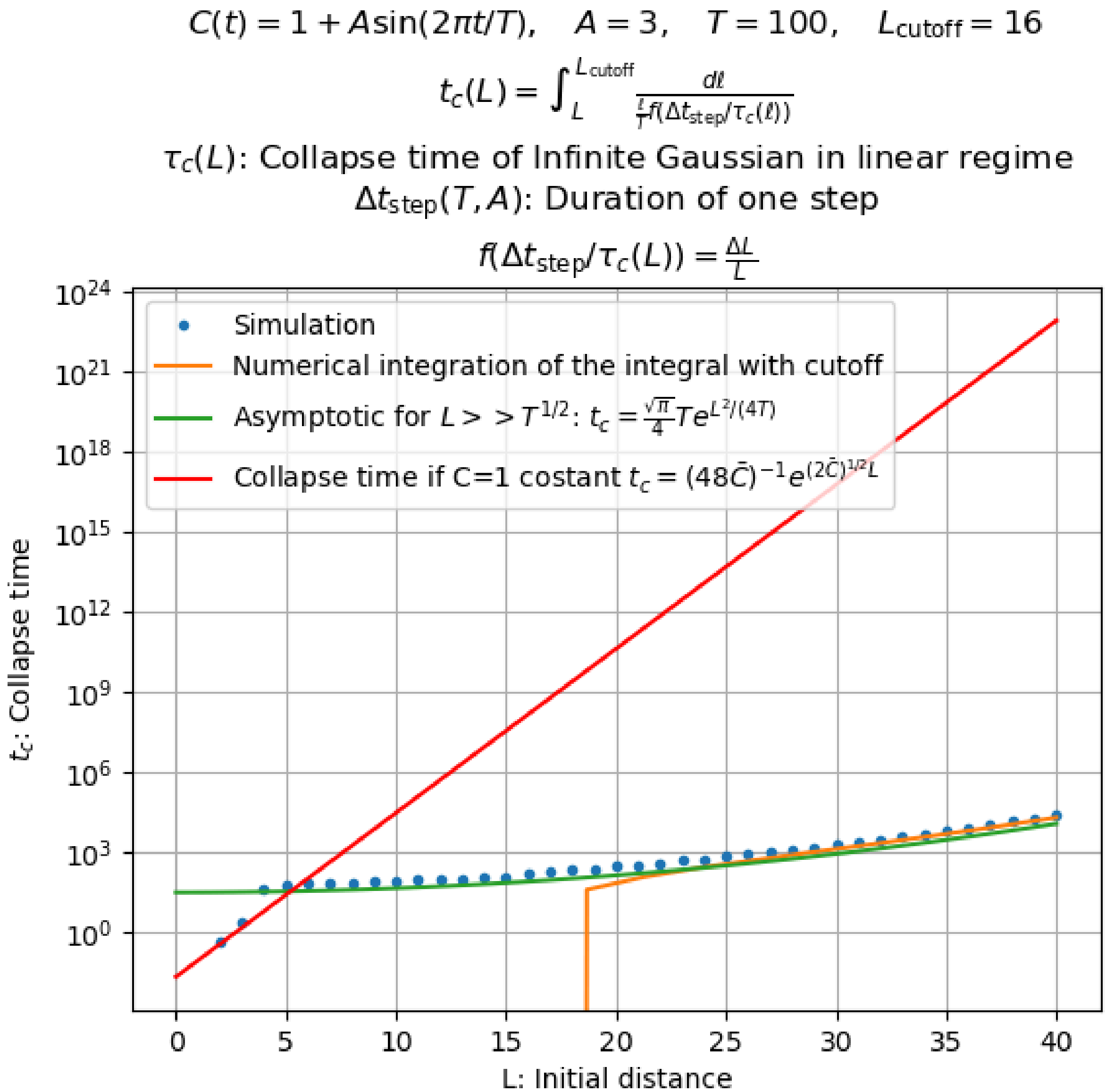
$$\partial_t L \simeq= -\frac{8}{\sqrt{\pi}} L^{-1} e^{-L^2/4T}$$

we can estimate the collapse time T_c if $L \gg L_1^*$ **BUT** still $L \ll L_2^*$

$$T_c = \int_L^0 \frac{dL'}{\partial_t L'}$$

we can put a cutoff $0 \to L_{cutoff}$ to the superior extreme of integration, to be able to use the asymptotic approximation, so $L_{cutoff} \gg T^{1/2}$. The cutoff will be irrelevant if $L \gg L_{cutoff}$. This leads to

$$T_c = \frac{\sqrt{\pi}}{4} T e^{L^2/4T}$$



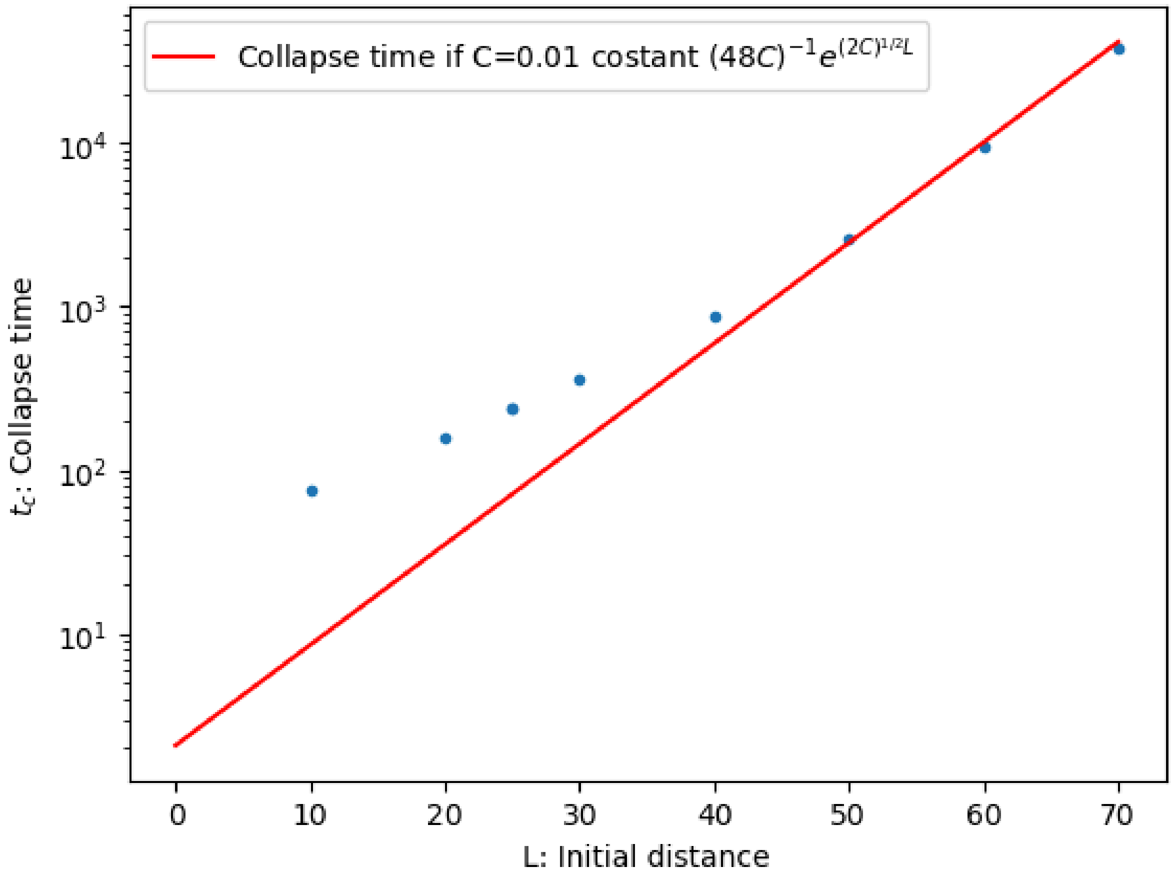
while, if $L \gg L_2^*$, the contribution from lower values of L' will be negligible and using

$$\partial_t L' = -(48\bar{C})^{-1} e^{-\sqrt{2\bar{C}}L'}$$

in the integral, and neglecting the cutoff, will lead to

$$T_c = \frac{1}{48\bar{C}} e^{\sqrt{2}\bar{C}^{1/2} L}$$

$\bar{C} = 10^{-2}, T = 100, A = 1$



How to see the asymptotic behaviour (L>>L2*) in simulations?

The simulations we can do are limited by computational cost.
If the initial distance satisfies $L \gg L_2^*$, then the collapse time will be

$$T_c \simeq \frac{1}{48\bar{C}} e^{-\sqrt{2}\bar{C}^{1/2}L}$$

so it scales exponentially with the initial distance L!
If we set a **maximum simulation time** T_{max}, such that we require $T_c < T_{max}$, the maximum simulable L scales as

$$L_{max}(\bar{C}) \sim L_{max}(\bar{C} = 1)\bar{C}^{-1/2}$$

while

$$L_2^* = \bar{C}^{1/2}T$$

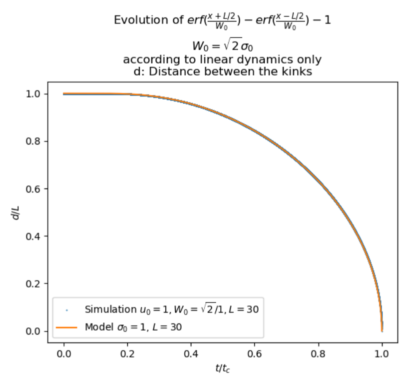
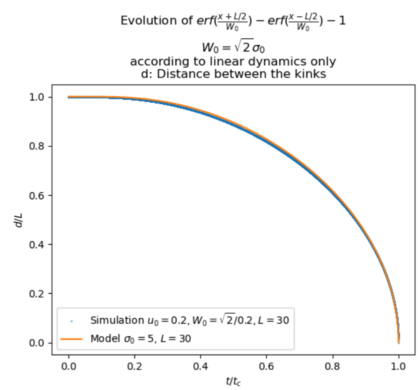
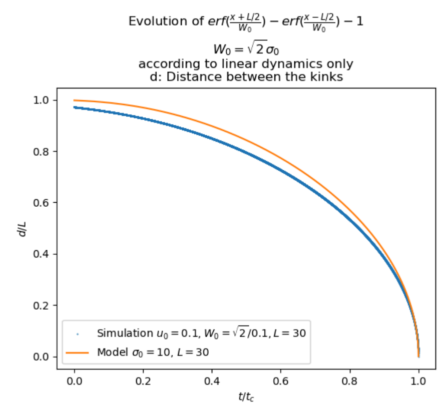
so we can find the values of $\bar{C}, T$ such that $L_{max}(\bar{C}) \gg L_2^*(\bar{C}, T)$.
Example: $\bar{C} = 10^{-2}, T = 100$ (the value of A is not relevant as soon as $A \gg \bar{C}$, so you can take $A = 1$ for instance).
Problem: This computational limit, brings to choices of $\bar{C}, T$ such that you cannot see the

intermediate behaviour, because they lead to $L_1^* > L_2^*$. I don't know if it exists a **sweet spot** such that, in the same simulations, you can see both the behaviours.

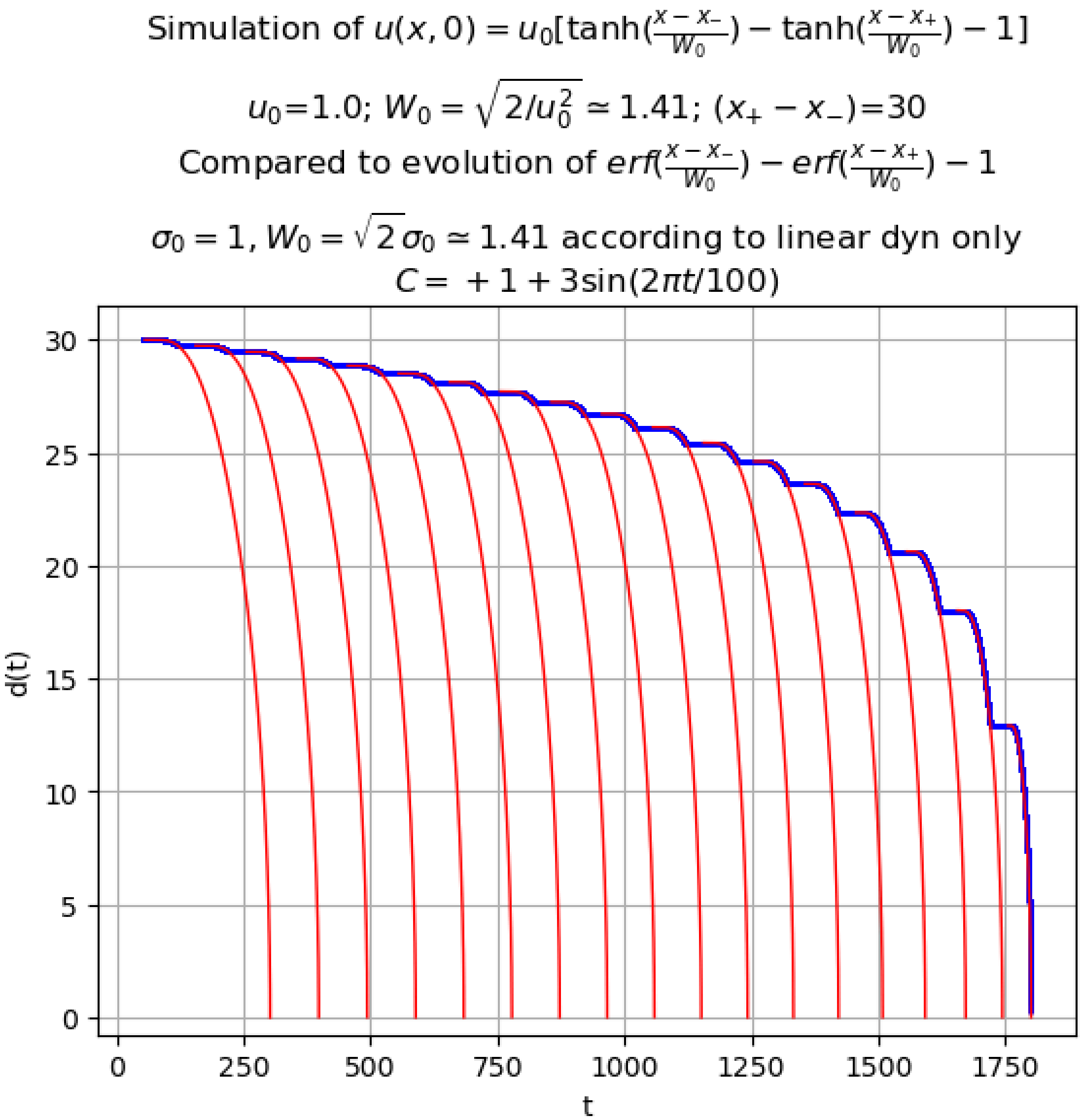
Experiments

1) Quench to C=-1

Comparing with a simulations, we can choose the initial width in the model as the initial width in the simulation. There are problems when there is overlap between the two kinks in the initial state ($\sigma \sim L$).

$W_0 = \sqrt{2}, L = 30$	$W_0 = 5\sqrt{2}, L = 30$	$W_0 = 10\sqrt{2}, L = 30$
		

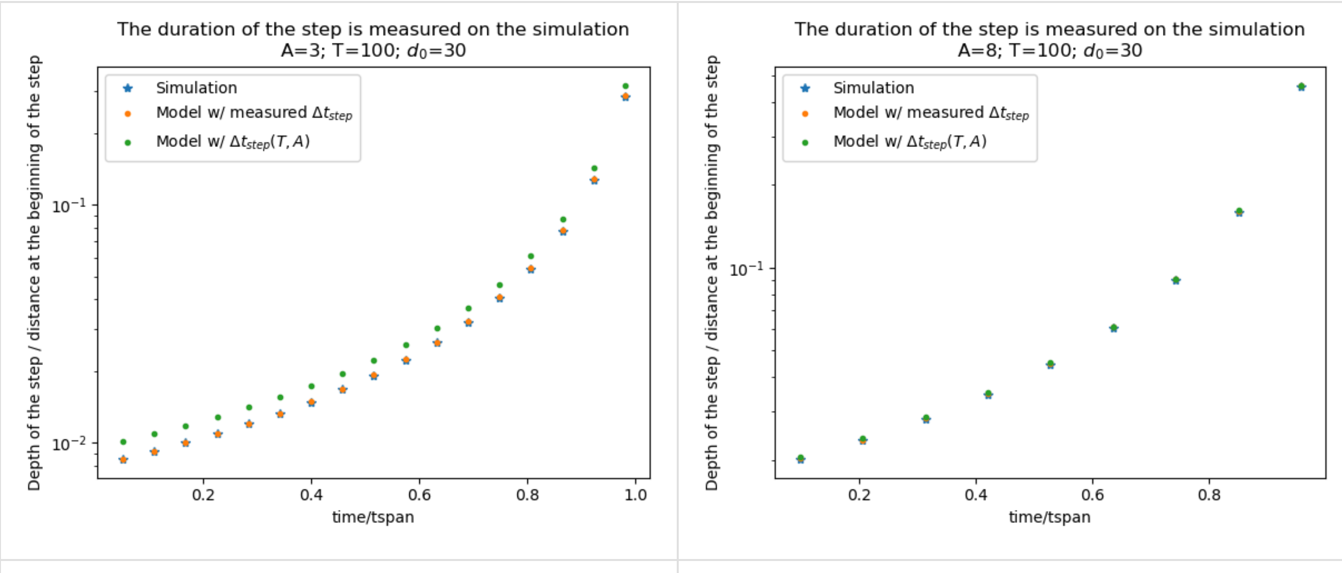
2) Oscillations C(t)



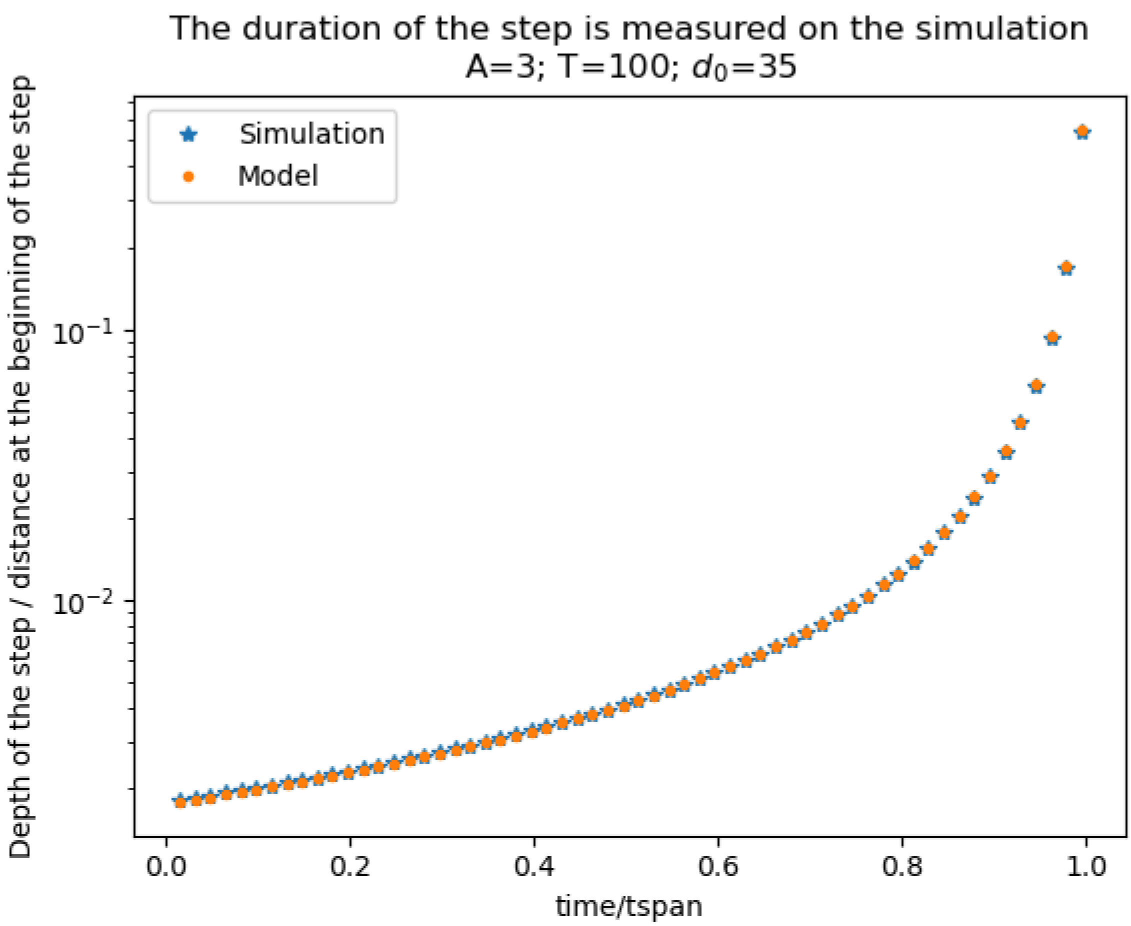
Predicting the depth of the step with the Infinite Gaussian model and the estimate of the step duration

The estimation of Δt_{step} is better when A is larger, as we are using an asymptotic expansion in $A \gg \bar{C}$.

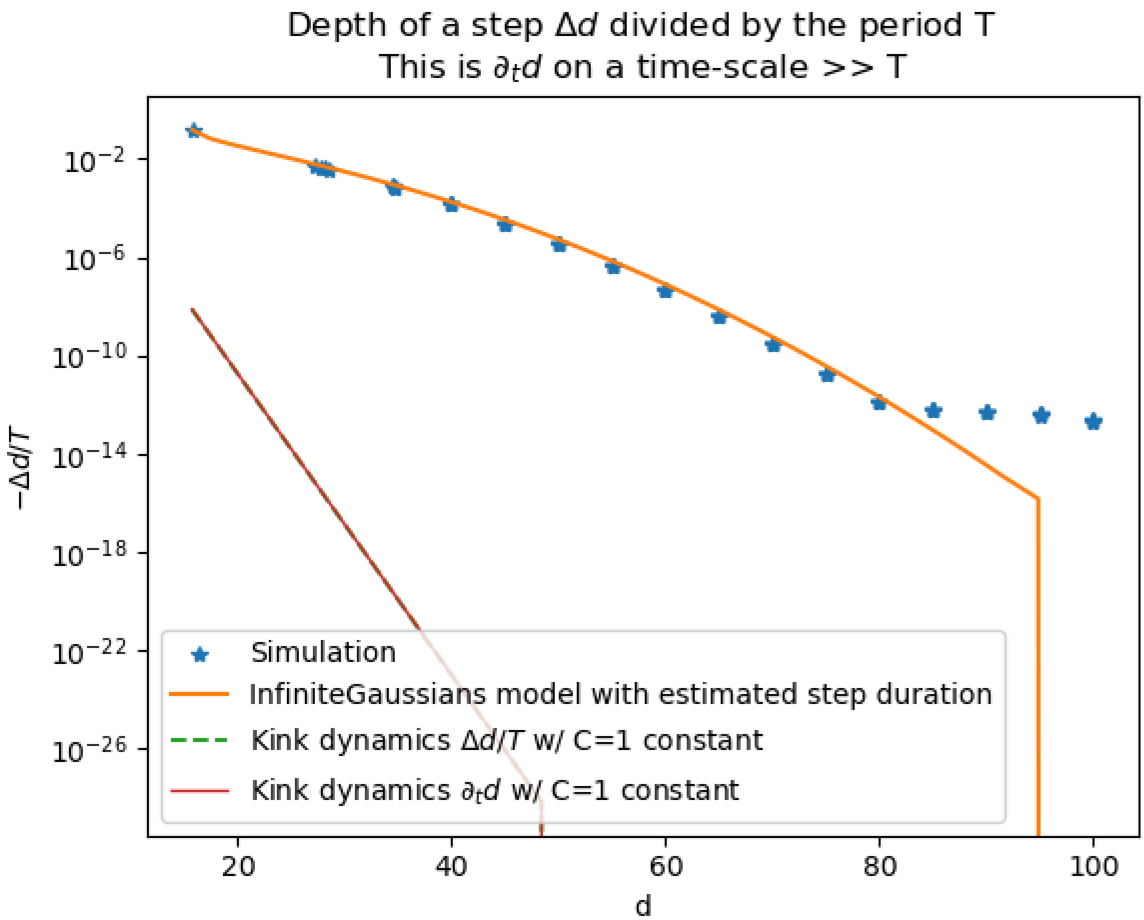
The prediction of Δt_{step} is better if A is larger



Also for the first steps it works really well.



It works really well, until the variation of d is very small and there I think there is numerical error!



Collapse time Tc

Here we present two scenarios.

- $L_1^* \ll L_{max} \ll L_2^*$ (intermediate)
- $L_{max} \gg L_2^*$ (asymptotic) and there is no intermediate regime, because $L_1^* > L_2^*$

Verifying the intermediate regime

Here we chose $\bar{C} = 1, A = 3, T = 100$.
Neglecting effects of order $\frac{A}{\bar{C}}$, we have

$$L_1^* = 4T^{1/2} = 40$$

but if we consider these effects, then

$$L_1^* = 4T^{1/2}\left[1 - \frac{1}{\sqrt{\pi}}\left(\frac{A}{\bar{C}}\right)^{-1/2}\right]^{1/2} \simeq 33$$

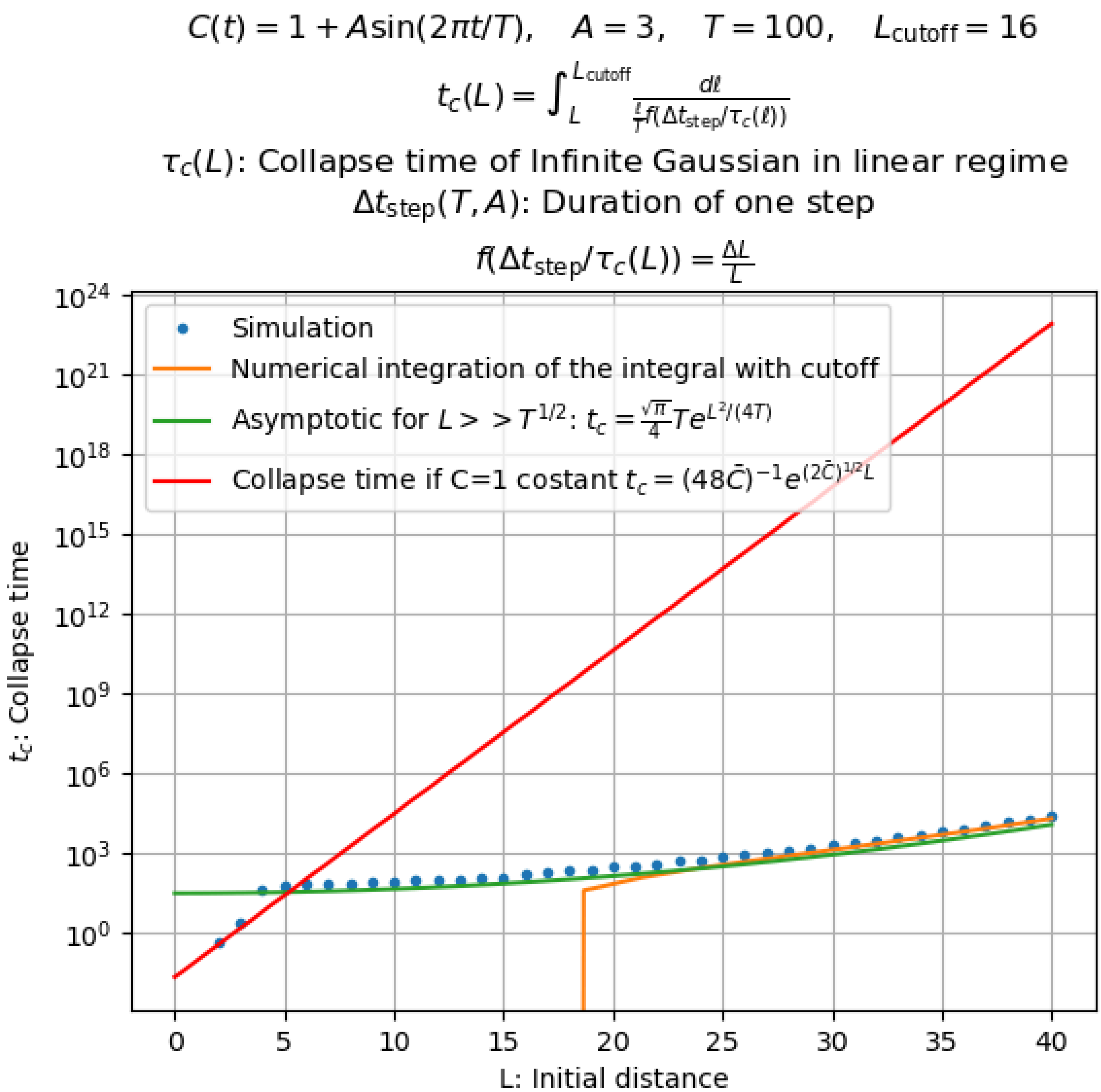
While

$$L_2^* = 4\sqrt{2}T\bar{C}^{1/2} \simeq 566$$

In the following simulation, we can see there is an agreement with the prediction when

$$L_1^* \ll L \ll L_2^*$$

and we cannot see the asymptotic behaviour, because $e^{\sqrt{2}\bar{C}^{1/2}L_2^*}$ is too big to be simulated.



The analytical prediction of the intermediate regime is

$$T_c = \frac{\sqrt{\pi}}{4} T e^{L^2/4T}$$

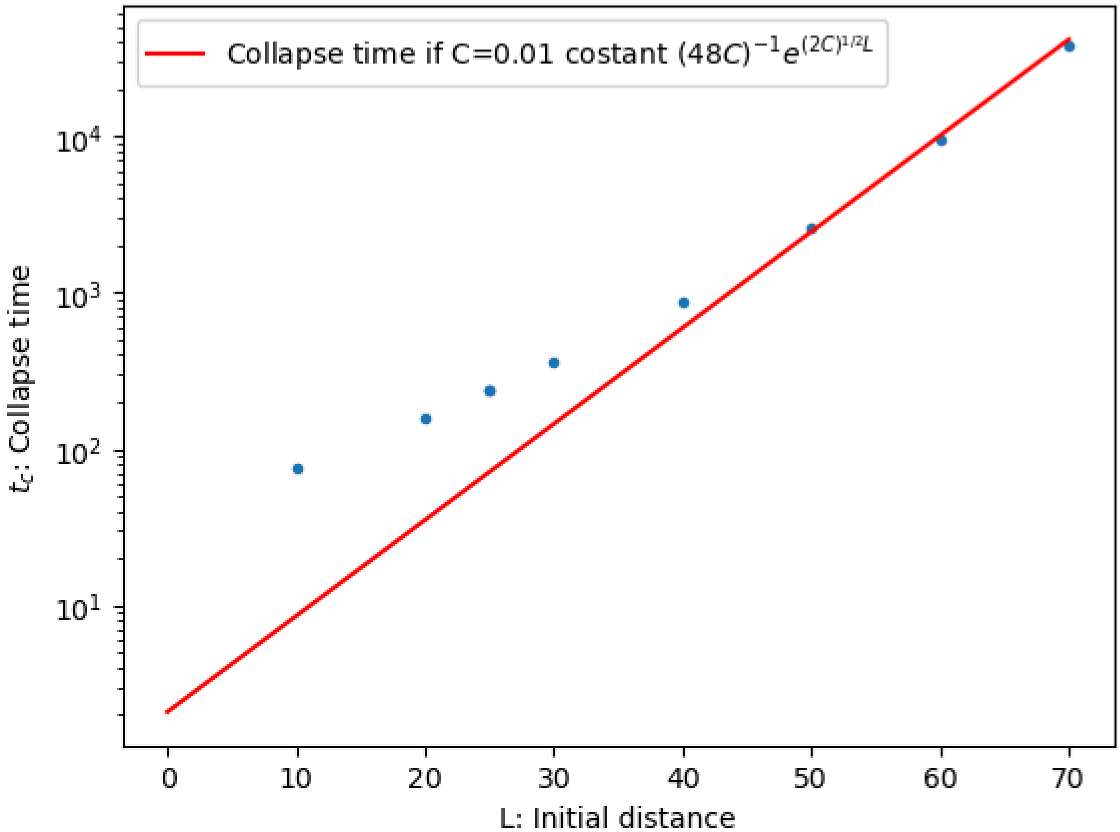
Verifiying the asymptotic behaviour

We find the parameters such that the maximum simulable initial distance is larger than L_2^* by requiring $L_2^* \ll L_{max}$ and we find, as an option, $\bar{C} = 10^{-2}, T = 100$ (and we take $A = 3$). Then

$$L_2^* = 4\sqrt{2}T\bar{C}^{1/2} \simeq 56$$

while we cannot see the intermediate behaviour, because

$$L_1^* \sim C^{-1} = 100 > L_2^*$$



There is a good agreement with the prediction when $L \gg L_2^*$

Problems with this model

- Cannot find a **sweet spot** $\bar{C}, T$ for which I can see both the intermediate and asymptotic behaviours in the same simulation.
- If the kinks overlap when $C(t)$ becomes negative, the model fails. But it is never the case in the simulations where $A >> \bar{C}$ and slow oscillations.