Lecture Notes: Quantitative Reasoning and Mathematical Thinking[1]

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August 28, 2025

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Chapter 1

Introduction

Courant and Robbins, in What is Mathematics? (1941), present mathematics not as a dry collection of formulas and tools, but as a living, creative discipline rooted in human thought and curiosity. For them, mathematics is both a pathway for understanding the natural world and an autonomous intellectual pursuit that reveals structures of order, beauty, and generality. They stress that its essence lies in the interplay between abstraction and concrete problem-solving: starting from simple, practical problems, mathematics ascends to general concepts and theories that then illuminate new domains.

They emphasize accessibility and unity: mathematics belongs to everyone who is willing to think rigorously, and its spirit combines logic with imagination. Rather than reducing it to calculation or technical skill, Courant and Robbins describe mathematics as “an expression of the human mind” where precision, creativity, and aesthetic appreciation converge. Their central idea is that mathematics is at once useful, philosophical, and artistic—simultaneously a language of science, a training ground for reasoning, and a source of intellectual delight.

Early mathematics was computational when the emphasis was on finding methods to obtain solutions. However, over the years, the disciplines of mathematics and computer science – the subject of designing algorithms for problem solving – have diverged. In mathematics abstraction is symbolic and logical. It seeks general structures, patterns, and proofs independent of implementation. It often endeavours to seek and capture common structures across different abstractions. The primary aim is truth and understanding – developing rigorous proofs, ensuring logical consistency, and uncovering general laws. Utility often follows from this pursuit but is not always the main driver. In contrast, the role of abstraction in computational thinking is more operational and algorithmic. It emphasizes creating computational process models for natural, social and even abstract phenomena for operational analysis. The primary aim is effective procedure – designing algorithms that solve problems efficiently, often under constraints of time, memory, and real-world complexity. The power lies in execution and exploration—running a program can reveal insights about systems too complex to solve analytically. Both have become fundamental strands of epistemology that are essential for critical scientific thinking.

Data-driven inference represents a third way of knowing, distinct from the deductive rigour of mathematics and the constructive procedures of computational thinking. As practiced in modern data science and machine learning, it seeks knowledge not by proving theorems or designing explicit algorithms, but by discovering patterns and regularities directly from empirical data. Its epistemic core is induction at scale: hypotheses, models, or predictors are justified by their ability to capture hidden correlations and to generalize to new observations. Unlike mathematics, correctness is not absolute, and unlike computational thinking, procedures are not always fully transparent. Instead, credibility arises from empirical adequacy—the degree to which models explain, predict, or align with observed phenomena. This mode of inference expands our epistemic toolkit for a world where complexity and abundance of data overwhelm deductive or constructive methods, but it also brings new philosophical challenges: uncertainty about correctness, bias, and the gap between correlation and causation.

In this course we will try to cover some fundamentals of all of the above.

Chapter 2

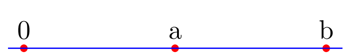
God gave us numbers, and human thought created algorithms

2.1 Numbers

Our discussion on mathematics and computing must start with numbers. What are numbers after all? The same number may be represented with symbols such as 3, III, or even as a line of a fixed length. But what is the underlying concept behind the different representations?

Bertrand Russell defined numbers as sizes (or *cardinality*) of *collections*. Some examples of *equinumerous* collections are $\{Red, Blue, Green\}$, $\{Amir, Salman, Shahrukh\}$, $\{Godavari, Kaveri, Krishna\}$. *Collections* are also called *Sets* or *Classes* in Mathematics. All three Sets above are of cardinality 3. Russel defined a number as – *the number of a class is the class of all those classes that are similar (equinumerous) to it*. So, according to Russel, a number is a class of classes.

Note that *cardinality* of sets is not the only way to describe the concept of a number. A number may also be a measure of a length. For example, in the straight line below, if we define the segment $\overline{0a}$ to be the *unit length* representing the number 1, then the line segment $\overline{0b}$ which is twice the length of $\overline{0a}$ may represent the number 2.



Without belabouring the point, it will suffice to say for our purpose that all of us intuitively understand what numbers mean.

2.1.1 Numbers may be represented in multiple ways

However, we need to do useful stuff with numbers – we need to add, subtract, multiply and divide them for obvious practical reasons. Indeed, the history of numbers date back to the Mesolithic stone age. The early humans had to figure out – due to a variety of practical considerations – that if they put two similar collections of size two and size three together, the larger collection becomes of size five.

Civilisations have found many ways to represent numbers through the ages. Some examples are as tally marks in the prehistoric to early civilisations – as straight marks on bones, sticks, or stones – as can be observed in the archaeological evidence of the Ishango bones from around 20000 BCE; as Egyptian – a stroke for 1, heel bone for 10, coil of rope for 100, etc. – or Roman – I, V, X, L, C, D, M – numerals; as Base-60 (sexagesimal) numbers written as combinations of "1" and

"10" wedges by the Babylonians around 2000 BCE; as used rods arranged on counting boards in base-10 with positional notation in Chinese rod systems; as positional decimal systems in Indian numerals in the Gupta period around 5^{th} century CE; as beads or stones moved on rods or grooves to represent numbers in Abacus systems in China, Rome, Mesopotamia and Jerusalem; with Indo-Arabic numerals in the medieval period; with various mechanical calculators such as Napier's bones, Slide rules, Pascal's calculator, and Leibniz's stepped reckoner in the 17^{th} century; as gears and levers in Charles Babbage's first programmable computer – the Analytic Engine; and as bits and bytes in modern digital computers. However, note that the methods of carrying out these operations – the algorithms – will necessarily depend on the representation we choose for numbers.

2.2 Sets

We will use *Sets* quite a bit in this course. We may describe a *Set* or a *Collection* by explicitly listing out its elements without duplicates, such as in the examples above. We may sometimes also describe a Set with a property like "all students enrolled in the QRMT section FC-0306-3". We write this formally using a variable x as $\{x \mid x \text{ is a student in the QRMT section FC-0306-3}\}$. The symbol $\mid$ is read as "such that".

If an element x belongs to a set A, we usually write this as $x \in A$.

Here are some more examples of Sets:

1. $A = \{x \mid x \text{ is a student pursuing a degree in India}\}$
2. $B = \{x \mid x \text{ is a CS Major student at Ashoka University}\}$
3. $C = \{x \mid x \text{ is a CS Major student at Ashoka University } \textbf{and } x \text{ is female}\}$

Clearly, all members of C are also members of B, and all members of B are members of A. We then say that C is a *subset* of B ($C \subseteq B$), and B is a *subset* of A ($B \subseteq A$). Formally, a set B is a *subset* of another set A, denoted as $B \subseteq A$, if $x \in A$ whenever $x \in B$.The empty set is denoted by ϕ, its size is zero (0), and it is a subset of all sets.

Given two sets A and B, the *union* $A \cup B$ is the set of all elements that are in A, or in B, or in both. Formally, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$. For example, if $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$.

Given two sets A and B, the *intersection* $A \cap B$ is the set of all elements that are in both A and B. Formally, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$. For example, if $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \cap B = \{3\}$.

Clearly, for any set A, $A \cup \phi = A$ and $A \cap \phi = \phi$,

Exercise 2.1 *Suppose $B \subseteq A$. Argue that*

1. $A \cup B = A$
2. $A \cap B = B$

2.3 The set of Natural numbers

Some sets can also be unbounded or infinite. We define the set of *Natural numbers* as $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$[1].

While we all intuitively understand this set, note that the elements of the set are as yet uninterpreted and undefined. We can overcome this lacunae by assuming a *God-gifted* ability to count. Given a number n as the size of a Set or a length, let us assume that we can interpret and construct the successor of n as $S(n) = n + 1$. Then, we can formally define the set of Natural numbers $\mathbb{N}$ as

1. $0 \in \mathbb{N}$, where 0 is the symbol that denotes the size of the empty set, and

[1] 0 is usually not included in the Set of Natural numbers in Mathematics. We will however include 0 in the set of Natural numbers in this course. After all, it is quite natural to score a 0 in an examination

2. if $n \in \mathbb{N}$, then $S(n) = n + 1 \in \mathbb{N}$

We can then adopt a suitable representation for successive elements in the set $\mathbb{N}$. Note that the set $\mathbb{N}$ is unbounded, because every number – no matter how large – has a successor.

2.3.1 Addition

We observed that the underlying concept of a number is independent of specific representations. Ideally, so should be the concepts of carrying out various operations with numbers. We may think of addition – the sum $a + b$ of two numbers a and b – as just combining two similar sets of sizes a and b. However, the procedure for "combining" is not representation independent. While simple "putting together" may work if we represent the numbers as collections of stones or marbles, it is not well defined for adding two numbers in the place-value representation that we are familiar with from junior school. Hence "combining" is a somewhat unsatisfactory way of defining addition.

A better way of defining $a + b$ is by using the successor operation $S(a) = a + 1$, b times. As long as we have a primitive method for computing $a + 1$ in any representation for an arbitrary a, this definition of $a + b$ becomes representation independent. We may define the basic property of addition using counting as:

For all $m, n \in \mathbb{N}$:

1. $m + 0 = m$
2. $m + S(n) = S(m + n)$

In the above definition we have used the same trick as in definition of the set $\mathbb{N}$ above, of defining a larger concept as a successor of a smaller concept. The process repeats, and the actual additions happen in the return path. For example,

$$
\begin{aligned}
&7 + 5 \\
&\quad = (7 + 4) + 1 \\
&\quad = ((7 + 3) + 1) + 1 \\
&\quad = (((7 + 2) + 1) + 1) + 1 \\
&\quad = ((((7 + 1) + 1) + 1) + 1) + 1 \\
&\quad = (((((7 + 0) + 1) + 1) + 1) + 1) + 1 \\
&\quad = ((((7 + 1) + 1) + 1) + 1) + 1 \\
&\quad = (((8 + 1) + 1) + 1) + 1 \\
&\quad = ((9 + 1) + 1) + 1 \\
&\quad = (10 + 1) + 1 \\
&\quad = 11 + 1 \\
&\quad = 12
\end{aligned}
$$

Note that the repeated substitution of a larger problem with a smaller problem is bounded, because the first condition of the definition works as a sentinel that we are bound to encounter as we keep reducing n.

We can then describe a procedure for computing $a+b$ (Algorithm 1) based on the above principle, but avoiding the deferred computations. The procedure takes a and b as input and returns *sum* as the output. $sum \leftarrow sum + 1$ denotes the operation "*sum* is *assigned* $sum + 1$" indicating that *sum* is incremented by 1.

Exercise 2.2 *1. Assuming that the operation $a + 1$ is available as a primitive, convince yourself that the above procedure for adding two numbers are correct.*

2. Argue that if the operation $a + 1$ is available as a primitive, then the above algorithm for addition is representation independent.

3. Describe how the algorithm may be implemented using pebbles or marbles to represent numbers.

Algorithm 1 An algorithm for $a + b$ by $+1$ b-times.

```
procedure ADD(a, b)
    counter ← 0
    sum ← a
    while counter < b do
        sum ← sum + 1
        counter ← counter + 1
    return sum
```

2.3.2 Multiplication

We can now define multiplication as repeated additions:

1. $n \times 0 = 0$, for all $n \in \mathbb{N}$
2. $n \times S(m) = n \times m + n$, for all $n, m \in \mathbb{N}$

Note that here again we have defined $n \times S(m)$, in terms of a smaller problem $m \times n$ of the same type.

Exercise 2.3 1. *Convince yourself that according to the above definition* $n \times m = \underbrace{n + n + n + \ldots + n}_{m \text{ times}}$.

2. *Provide a representation independent algorithm, using only the successor function and addition, for multiplication of two numbers.*

3. *Describe how the algorithm may be implemented using pebbles or marbles to represent numbers.*

2.3.3 Subtraction

To define the subtraction operation $m - n$, we may first define a predecessor operation $P(n)$ – analogous to $S(n)$ – as

1. $P(0)$ is undefined
2. $P(n) = n - 1$ for all $n > 0$.

We assume, as before, that we have a primitive counting based procedure for computing $P(n) = n - 1$ in any representation. We can define the subtraction operation $m - n$ similarly to addition:

For all $m, n \in \mathbb{N}, m \geq n$

1. $m - m = 0$
2. $m - n = S(P(m) - n)$

As before, note that $P(m) - n$ is a smaller problem that $m - n$.

The subtraction algorithm may then be given as:

Algorithm 2 An algorithm for $a - b$, $a \geq b$ by -1 b-times.

```
procedure SUBTRACT(a, b)
    counter ← 0
    while counter < b do
        a ← a − 1
        counter ← counter + 1
    return a
```

Exercise 2.4 *Provide alternative versions of Algorithms 1 and 2 without using the counter. Instead decrement b using $b \leftarrow b - 1$ repeatedly till $b = 0$.*

2.3.4 Division

Division is a natural requirement in civilised societies, mainly for sharing. However, it may not always be possible to divide natural numbers in equal proportions. For example, a collection of size 3 cannot be divided in two proportions of equal sizes without breaking up at least one member element. We have the *division theorem*:

Theorem 2.1 *Given two numbers $a, b \in \mathbb{N}$, there exist unique $q, r \in \mathbb{N}$ (quotient and remainder, respectively) such that $a = bq + r$ and $0 \leq r < b$.*

Proof: Let us first argue that such q and r exist. Repeatedly compute $a-b, a-2b, a-3b, \ldots, a-kb$, $k \geq 0$, till $a - kb < b$ and subtraction is possible no more. Set $q = k$ and $r = a - kb$. Clearly, q is the total number of times b can be subtracted from a, and $0 \leq r < b$. If $r = 0$ then b divides a exactly.

To argue that that q and r obtained by the above procedure are unique, let us suppose they are not. Then, there exist q_1, r_1 and q_2, r_2 such that

$$a = bq_1 + r_1, 0 \leq r_1 < 0$$
$$a = bq_2 + r_2, 0 \leq r_2 < 0$$

Without loss of generality, let us assume that $q_1 \geq q2$. The above implies that $b(q_1 - q_2) = r_2 - r_1$. One of two cases arise:

1. $q_1 = q_2$. This implies that $r_1 = r_2$, and hence uniqueness.

2. $q_1 > q_2$. This implies that $q_1 - q_2 \geq 1 \in \mathbb{N}$. Hence $r_2 - r_1 \geq b$. But this is not possible because $0 \leq r_1, r_2 < b$. Hence q and r must be unique.

□

In the above proof, we used *explicit construction* as a proof technique for establishing existence of such q and r, and *contradiction* for establishing their uniqueness. We will revisit these techniques later in the course when we discuss proofs.

Exercise 2.5 *Describe an algorithm using repeated subtraction that computes q and r given a and b.*

2.4 The Sets of Integers

We defined the subtraction operation $m - n, m \geq n, m, n \in \mathbb{N}$ as the number of times the successor operation $S()$ needs to be applied to reach m from n. This definition requires the restriction that $m \geq n$. An obvious generalisation is to remove the restriction and measure the difference in terms of either the successor $S()$ or the predecessor $P()$ operator. Subtraction then becomes directional, and we require negative numbers to represent the direction. This leads us to the set of integers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, \ldots\}$$

Arithmetic in the set $\mathbb{Z}$ follows the same principles as in $\mathbb{N}$, except that they are now directional.

Exercise 2.6 *Rework the definitions and the algorithms for addition, multiplication, subtraction and division in $\mathbb{Z}$.*

2.5 The Sets of Rationals

The division theorem tells us that given $m, n \in \mathbb{N}$, there exist $q, r \in \mathbb{N}$, such that m can be divided in to q parts of size n, possibly leaving a remainder $0 \leq r < n$. Division is an obvious fundamental need for resource sharing. If each unit is indivisible – like live cattle, for example – then the division theorem is the best we can do. However, items measured in units such as weight, volume or length – such as meat from a hunted animal, or a pile of grains – are often divisible in smaller proportions like $1/3^{rd}$, $2/25^{th}$ etc. So, division inevitably leads us to fractions. We define the set of Rational numbers as

$$\mathbb{Q} = \{x | x = p/q, p \in \mathbb{Z}, q \in \mathbb{N}, q \neq 0\}$$

We often also write this as $\frac{p}{q}$. These are numbers of the type $\pm 1/1, \pm 1/2, \pm 1/3, \pm 1/4, \pm 2/5$ etc. We may also insist that p and q should have no common factors (i.e., $gcd(p, q) = 1$) to avoid multiple representations for the same Rational number. Clearly $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$.

Note, however, that we now have a situation where between any two rational numbers there are infinitely many other rational numbers.

Exercise 2.7 *Convince yourself of the above statement.*

This implies that there is no well defined successor function for a rational number, and we need to revisit out definition of addition for rationals. We start by noting that, for example,

$$\frac{1}{3} + \frac{2}{3} = \frac{1+2}{3} = 1$$

i.e., we can add the numerators as in $\mathbb{Z}$ if the denominators are the same. However, the addition

$$\frac{2}{3} + \frac{3}{4}$$

is not well defined unless the two fractions can be expressed in the same unit. But we can multiply the numerator and denominator of the first fraction by 4, and the second by three to convert to the same unit where the denominator of both is 12

$$\frac{2 \times 4}{3 \times 4} + \frac{3 \times 3}{4 \times 3} = \frac{8}{12} + \frac{9}{12} = \frac{8+9}{12} = \frac{17}{12}$$

Note that multiplying the numerator and the denominator of a fraction with the same number does not change the fraction. So, we can define the general rule for addition of two rational numbers as

$$\frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1 \times q_2 + p_2 \times q_1}{q_1 \times q_2}$$

where all the additions and multiplications are defined on the set $\mathbb{Z}$.

Exercise 2.8 *Extend the above idea to define subtraction, multiplication and division in the set $\mathbb{Q}$.*

2.6 Ruler and compass algorithms

TO DO.