Falkner-Skan flow of the nanofluid past over a moving wedge

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Here we consider a wedge immersed in a nanofluid subjected to suction or injection type of boundary conditions and having a constant free-stream flow velocity $u_e(x)$. We introduce a co-ordinate system by taking the horizontal x-axis to be along the wedge surface and the vertical y-axis to be normal to the wedge surface. Let T_∞ and C_∞ denotes the ambient temperature and nanoparticles volume fraction, respectively. Also, let Tw and Cw denotes the constant temperature and concentration at the wall of the wedge, respectively. The entire physical flow configuration is depicted in the Figure 1).

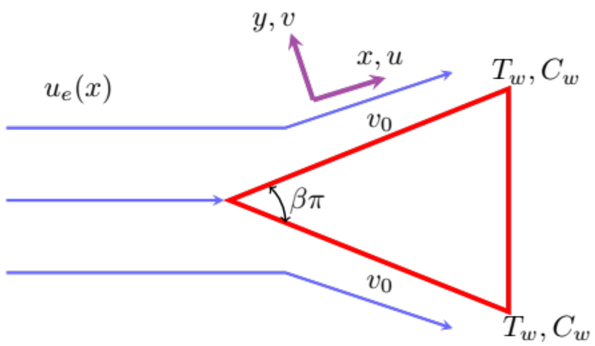


Figure 1: The physical flow configuration of the problem.

The fundamental assumptions to obtain the simplified mathematical equations are: the flow is steady; the ambient fluid is viscous and imcompressible; and the flow considered is two dimensional. The mathematical model which includes the fundamental conservation of mass, momentum, thermal energy incorporating the thermophysical properties of the nanofluids written in the cartesian coordinate system as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + u_e\frac{\partial u_e}{\partial x} \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \tau[D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial y} + \frac{D_T}{T_\infty}\frac{\partial T}{\partial y}^2] + Q(T - T_\infty) \tag{3}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_B\frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2} \tag{4}$$

where u and v are the velocity components in x and y directions, ν is the kinematic viscosity of nanofluid, T is the nanofluid temperature and C is the volume fraction, α is nanofluid thermal diffusivity, τ is the ratio of heat capacity of nanoparticle and the heat capacity of the fluid. The term D_B is the Brownian diffusion coefficient which defines the Brownian motion. The parameter Q is the specific heat of nanoparticle and heat absorption/generation coefficient. The constant D_T is the thermal diffusion. The boundary conditions for the flow problem are:

$$y = 0 : u = 0; v = v_0; T = T_w; C = C_w; \tag{5}$$

$$y \longrightarrow \infty : u = ax^m; v = 0; T = T_\infty; C = C_\infty \tag{6}$$

Our goal in this work is to develop a framework for the numerical solution for the problem described above. To reduce the partial differential equations model (1)-(4) to the corresponding nonlinear ordinary differential equations model, we first introduce scalar stream function $\psi(x, y)$ defined as:

$$u = \frac{\partial \psi}{\partial y}; v = -\frac{\partial \psi}{\partial x} \tag{7}$$

which satisfies the continuity equation (1) automatically. Then, to study the problem under meaningful boundary-layer assumptions, we introduce the following dimensionless variables:

$$\psi = \sqrt{\frac{2u_e \nu x}{m+1}} f(\eta) \tag{8a}$$

$$T = T_\infty + (T_w - T_\infty)\theta(\eta) \tag{8b}$$

$$C = C_\infty + (C_w - C_\infty)\phi(\eta) \tag{8c}$$

$$\eta = y\sqrt{\frac{(m+1)u_e}{2\nu x}}; \tag{9}$$

$$\beta = \frac{2m}{m+1} \text{ and } u_e(x) = ax^m \tag{10}$$

Now ψ and η can be simplified as -

$$\eta = y\sqrt{\frac{(m+1)u_e(x)}{2\nu x}} = y\sqrt{\frac{max^{m-1}}{\beta\nu}} = Pyx^{\frac{m-1}{2}}; \text{taking,} P = \sqrt{\frac{ma}{\beta\nu}}$$

$$\psi = \sqrt{\frac{\beta a x^{m+1}\nu}{m}} f(\eta) = \frac{a}{p} x^{\frac{m+1}{2}} f(\eta)$$

Now to reduce the partial differential equations model (1)-(4) to the corresponding nonlinear ordinary differential equations model we are going to use following derivatives-

$$u = \frac{\partial \psi}{\partial y} = \frac{a}{P} x^{\frac{m+1}{2}} f'(\eta) \frac{\partial \eta}{\partial y} = ax^m f'(\eta)$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial(\frac{a}{p} x^{\frac{m+1}{2}} f(\eta))}{\partial x} = -[\frac{am}{\beta P} x^{\frac{m-1}{2}} f(\eta) + \frac{a(m-1)}{2} yx^{m-1} f'(\eta)]$$

$$\frac{\partial^2 \psi}{\partial y^2} = ax^m.f''(\eta).x^{\frac{m-1}{2}} = aPx^{\frac{3m-1}{2}} f''(\eta)$$

$$\frac{\partial^3 \psi}{\partial y^3} = aPx^{\frac{3m-1}{2}} f'''(\eta).Px^{\frac{m-1}{2}} = aP^2 x^{2m-1} f'''(\eta)$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = amx^{m-1} f'(\eta) + ax^m f''(\eta) \frac{\partial \eta}{\partial x} = amx^{m-1} f'(\eta) + \frac{aP(m-1)}{2} yx^{\frac{3m-3}{2}} f''(\eta)$$

$$\frac{\partial T}{\partial x} = (T_W - T_\infty)\theta'(\eta) Py \frac{m-1}{2} x^{\frac{m-3}{2}} = \frac{P(T_w - T_\infty)(m-1)y}{2} x^{\frac{m-3}{2}} \theta'(\eta)$$

$$\frac{\partial C}{\partial x} = (C_w - C_\infty)\phi'(\eta) Py(\frac{m-1}{2}) x^{\frac{m-3}{2}} = \frac{P(C_w - C_\infty)(m-1)y}{2} x^{\frac{m-3}{2}} \phi'(\eta)$$

$$\frac{\partial T}{\partial y} = (T_w - T_\infty)Px^{\frac{m-1}{2}} \theta'(\eta)$$

$$\frac{\partial C}{\partial y} = (C_w - C_\infty)Px^{\frac{x-1}{2}} \phi'(\eta)$$

$$\frac{\partial^2 C}{\partial y^2} = (C_w - C_\infty)P^2 x^{m-1} \phi''(\eta)$$

$$\frac{\partial^2 T}{\partial y^2} = (T_w - T_\infty)P^2 x^{m-1} \theta''(\eta)$$

Since, u and v are both function of ψ equation (1) can be satisfied by equation (7) as-

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial^2\psi}{\partial x\partial y}=0$$

Replacing the derivative values in (2),

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}+u_e\frac{\partial u_e}{\partial x}$$

$$\implies \frac{\partial\psi}{\partial y}\frac{\partial^2}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=\nu\frac{\partial^3\psi}{\partial y^3}+a^2mx^{2m-1}$$

$$\implies ax^m f'(\eta)[amx^{x-1}f'(\eta)+\frac{aP(m-1)y}{2}x^{\frac{3m-3}{2}f''(\eta)}]-aPx^{\frac{3m-1}{2}}f''(\eta)[\frac{am}{\beta P}x^{\frac{m-1}{2}}f(\eta)+\frac{a(m-1)y}{2}x^{m-1}f'(\eta)]$$
$$=\nu aP^2x^{2m-1}f'''(\eta)+a^2mx^{2m-1}$$

$$\implies a^2mx^{2m-1}f'(\eta)^2+\frac{a^2P(m-1)y}{2}x^{\frac{5m-3}{2}}f(\eta)f''(\eta)-\frac{a^2m}{\beta}x^{2m-1}f(\eta)f'(\eta)-\frac{a^2Py(m-1)}{2}x^{\frac{5m-3}{2}}f'(\eta)f''(eta)$$
$$=\nu a(\frac{ma}{\beta\nu})x^{2m-1}f'''(\eta)+ma^2x^{2m-1}$$

$$\implies a^2mx^{2m-1}[f'(\eta)^2-\frac{f(\eta)f''(\eta)}{\beta}]=a^2mx^{2m-1}[\frac{f'''(\eta)}{\beta}+1]$$

Divide both sides by a^2mx^{2m-1} we get,

$$\implies f'(\eta)^2-\frac{f(\eta)f''(\eta)}{\beta}=\frac{f'''(\eta)}{\beta}+1$$

$$\implies \beta f'(\eta)^2-f(\eta)f''(\eta)-f'''(\eta)-\beta=0$$

$$\text{Hence, } f'''(\eta)+f(\eta)f''(\eta)+\beta(1-f'(\eta)^2)=0 \tag{11a}$$

From equation (3) replacing the derivative values,

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}+\tau[D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial y}+\frac{D_T}{T_\infty}\frac{\partial T}{\partial y}^2]+Q(T-T_\infty)$$
$$\implies [ax^mf'(\eta)][\frac{P(T_w-T_\infty)(m-1)y}{2}x^{\frac{m-3}{2}}\theta'(\eta)]-(T_w-T_\infty)Px^{\frac{m-1}{2}}\theta'(\eta)[\frac{am}{\beta P}x^{\frac{m-1}{2}}f(\eta)+\frac{a(m-1)y}{2}x^{m-1}f'(\eta)]$$
$$=\alpha(T_w-T_\infty)P^2x^{m-1}\theta''(\eta)+\tau[D_B(C_w-C_\infty)(T_w-T_\infty)P^2x^{m-1}\theta'\phi'+\frac{D_T}{T_\infty}(T_w-T_\infty)^2P^2x^{m-1}(\theta'')^2]+Q(T-T_\infty)$$
$$\implies \frac{aP(T_w-T_\infty)(m-1)y}{2}x^{\frac{3m-3}{2}}f'\theta'-\frac{am(T_w-T_\infty)}{\beta}x^{m-1}f\theta'-\frac{aP(T_w-T_\infty)(m-1)y}{2}x^{\frac{3m-3}{2}}f'\theta'$$
$$=(T_w-T_\infty)x^{m-1}[\alpha P^2\theta''+\tau D_BP^2(C_w-C_\infty)\phi'\theta'+\tau\frac{D_T}{T_\infty}P^2(T_w-T_\infty)(\theta')^2]+Q(T_w-T_\infty)\theta$$

Divide both sides by $(T_w-T_\infty)x^{m-1}$,

$$\implies -\frac{am}{\beta}f\theta'=\alpha P^2\theta''+\tau P^2[D_B(C_w-C_\infty)\phi'\theta'+\frac{D_T}{T_\infty}(T_w-T_\infty)(\theta')^2]+\frac{Q\theta(\eta)}{x^{m-1}}$$

$$\implies f\theta'+\frac{\beta\alpha P^2}{am}\theta''+\frac{\tau\beta P^2}{am}[D_B(C_w-C_\infty)\phi'\theta'+\frac{D_T}{T_\infty}(T_w-T_\infty)(\theta')^2]+\frac{\beta Q\theta x}{amx^m}$$

$$\implies f\theta'+\frac{\alpha}{\nu}\theta''+\frac{\tau}{\nu}[D_B(C_w-C_\infty)\phi'\theta'+\frac{D_T}{T_\infty}(T_w-T_\infty)(\theta')^2]+\frac{\beta\delta\theta}{m}=0$$

$$\implies \frac{1}{Pr}\theta'' + f\theta' + N_B\phi'\theta' + N_t\theta'^2 + \frac{\beta\delta\theta}{m} = 0 \tag{11b}$$

Where $Pr = \frac{\tau}{\alpha}$ is the Prandtl number, $\delta = \frac{Qx}{ax^m}$ is the heat generation/absorption parameter, $N_B = \frac{\tau D_B(C_w - C_\infty)}{\nu}$ is the Brownian motion parameter and $N_t = \frac{\tau D_T}{\nu T_\infty}(T_w - T_\infty)$ is the thermophoresis parameter.
Now from equation (4) we get,

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_B\frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2}$$

$$\implies amf'(\eta)[\frac{P(m-1)(C_w - C_\infty}{2}yx^{\frac{m-3}{2}}\phi'(\eta)] - P(C_w - C_\infty)x^{\frac{m-1}{2}}\phi'(\eta)[\frac{a(m+1)}{2P}f(\eta) + \frac{a(m-1)y}{2}x^{m-1}f'(\eta)]$$
$$= D_B P^2(C_w - C_\infty)x^{m-1}\phi''(\eta) + \frac{D_T}{T_\infty}P^2(T_w - T_\infty)x^{m-1}\phi''(\eta)$$

$$\implies \frac{aP(m-1)(C_w - C_\infty)y}{2}x^{\frac{3m-3}{2}}f'(\eta)\phi'(\eta) - \frac{a(m-1)(C_w - C_\infty)}{2}x^{m-1}\phi'(\eta)f(\eta) - \frac{aP(m-1)(C_w - C_\infty)y}{2}$$
$$x^{\frac{3m-3}{2}}\phi'(\eta)f'(\eta) = P^2x^{m-1}[D_B(C_w - C_\infty)\phi'(\eta) + \frac{D_T}{T_\infty}(T_w - T_\infty)\theta''(\eta)]$$

$$\implies \frac{-am}{\beta}(C_w - C_\infty)\phi'(\eta)f(\eta) = \frac{ma}{\beta\nu}(C_w - C_\infty)[D_B\phi''(\eta) + \frac{D_T(T_w - T_\infty)}{T_\infty(C_w - C_\infty)}\theta''(\eta)]$$

$$\implies f(\eta)\phi'(\eta) + \frac{D_B}{\nu}\phi''(\eta) + \frac{1}{\nu}\frac{D_T}{T_\infty}\frac{(T_w - T_\infty)}{(C_w - C_\infty)}\theta''(\eta) = 0$$

$$\implies \phi''(\eta) + \frac{\nu}{D_B}\phi'(\eta)f(\eta) + \frac{D_T(T_w - T_\infty)}{D_B T_\infty(C_w - C_\infty)}\theta''(\eta) = 0$$

$$\implies \phi''(\eta) + Lef(\eta)\phi'(\eta) + \frac{N_t}{N_B}\theta''(\eta) = 0 \tag{11c}$$

Where, $Le = \frac{\nu}{D_B}$ is the Lewis number.
Now from given boundary conditions,

$$\text{When } y \longrightarrow 0 : \eta = y\sqrt{\frac{(m+1)u_e(x)}{2\nu x}} = 0$$

$$u = \frac{\partial\psi}{\partial y} = ax^m f'(\eta) = ax^m f'(0) = 0$$

$$\text{Therefore, } f'(0) = 0$$

$$v = v_0 \implies -\frac{\partial\psi}{\partial x} = v_0 \implies -\frac{a(m+1)}{2P}x^{\frac{m-1}{2}}f(0) - 0 = v_0$$

$$\frac{a(m+1)}{2}\sqrt{\frac{\beta\nu}{ma}}x^{\frac{m-1}{2}}f(0) = -v_0$$

$$\frac{m+1}{2}\sqrt{\frac{2ma\nu x^{m-1}}{m(m+1)}}f(0) = -v_0$$

$$\frac{m+1}{2}\sqrt{\frac{2\nu u_e(x)}{x(m+1)}}f(0) = -v_0$$

$$f(0) = -\frac{2v_0}{m+1}\sqrt{\frac{x(m+1)}{2\nu u_e(x)}}$$

$$\text{Hence, } f(0) = \frac{2}{m+1}s; \text{ where, } s = -v_0\sqrt{\frac{x(m+1)}{2\nu u_e(x)}}$$

$$\text{When, } T = T_w \longrightarrow T_\infty + (T_w - T_\infty)\theta(0) = T_w$$

$$\implies (T_w - T_\infty)\theta(0) = (T_w - T_\infty)$$

$$\text{Therefore, } \theta(0) = 1$$

$$C = C_w \longrightarrow C_\infty + (C_w - C_\infty)\phi(0) = C_w$$

$$\implies (C_w - C_\infty)\phi(0) = C_w - C_\infty$$

$$\text{Therefore, } \phi(0) = 1$$

Again when y $\longrightarrow \infty$, then $\eta \longrightarrow \infty$. According to the boundary condition,

$$u = \frac{\partial \psi}{\partial y} = ax^m f'(\infty) = ax^m$$

$$\implies f'(\infty) = 1$$

$$T = T_\infty \implies T_\infty + (T_w - T_\infty)\theta(\infty) = T_\infty$$

$$\implies (T_w - T_\infty)\theta(\infty) = 0 \implies \theta(\infty) = 0$$

$$C = C_\infty \implies C_\infty + (C_w - C_\infty)\phi(\infty) = C_\infty$$

$$\implies (C_w - C_\infty)\theta(\infty) = 0 \implies \phi(\infty) = 0$$

Thus the boundary conditions (5) and (6) reduces to,

$$f'(0) = 0, f(0) = \frac{2s}{m+1}, \theta(0) = 1, \phi(0) = 1 \tag{12a}$$

$$f'(\infty) = 1, \theta(\infty) = 0. \phi(\infty) = 0 \tag{12b}$$

From equations (11a), (11b), and (11c), we can form a system of first order differential equations by letting

$$u_1 = f, u_2 = f', u_3 = f'', u_4 = \theta, u_5 = \theta', u_6 = \phi, u_7 = \phi'$$

From (11a) and u_1, u_2, and u_3, we get

$$\frac{du_1}{d\eta} = u_2, \frac{du_2}{d\eta} = u_3, \frac{du_3}{d\eta} = -u_1u_3 - \beta(1 - u_2^2)$$

Equations (11b) and (11c) can be used to get

$$\theta'' = -Pr(f\theta + N_b\phi'\theta' + \frac{\beta\delta\theta}{m} + N_t\theta^2) \tag{13}$$

$$\frac{N_t}{N_b}\theta'' = -\phi'' - Lef\phi' \tag{14}$$

By multiplying (13) with $\frac{-N_t}{N_b}$ and adding to (14), we get

$$\phi'' = \frac{PrN_t}{N_b}(f\theta' + N_b\phi'\theta' + \frac{\beta\delta\theta}{m} + N_t\theta'^2) - Lef\phi' \tag{15}$$

Using equations (13) and (15), we get the rest of the system of first order differential equations,

$$\frac{du_4}{d\eta} = u_5, \frac{du_5}{d\eta} = -Pr(u_1u_5 + N_bu_5u_7 + \frac{\beta\delta u_4}{m} + N_tu_5^2)$$

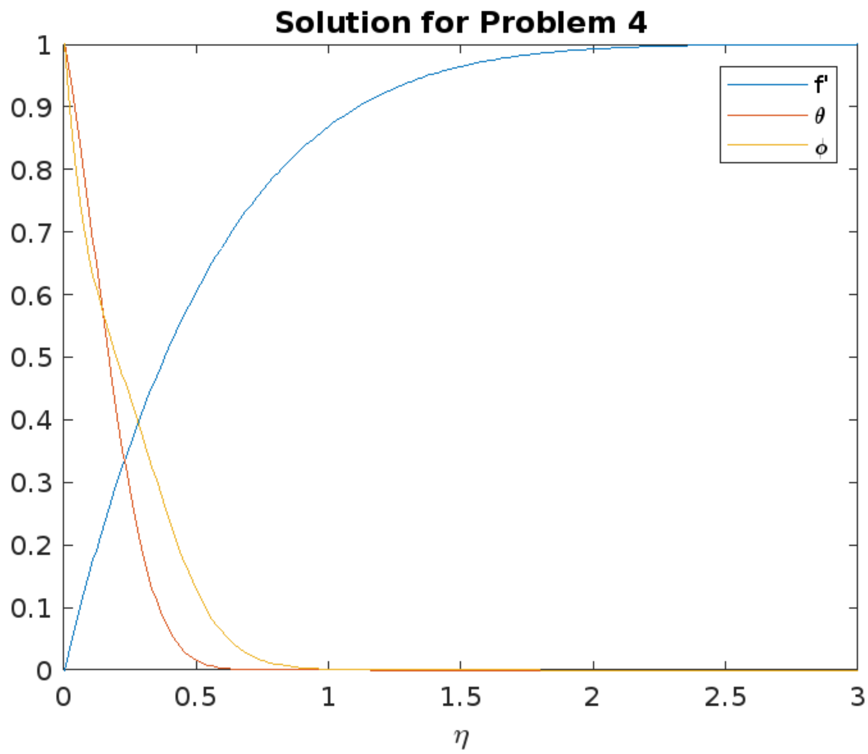
$$\frac{du_6}{d\eta} = u_7, \frac{du_7}{d\eta} = \frac{PrN_t}{N_b}(u_1u_5 + N_bu_5u_7 + \frac{\beta\delta\theta}{m} + N_tu_5^2) - Leu_1u_7$$

The following graph shows the plot of f', θ, and ϕ from equations (11a), (11b), and (11c) with the following boundary conditions.

$$f(0) = \frac{2s}{m+1}, f'(0) = 0, f'(3) = 1, \theta(0) = 1, \theta(3) = 0, \phi(0) = 1, \phi(3) = 0$$

And given values

$$N_t = N_B = .3, Pr = 10, Le = 5, m = 0, \beta = 0, s = .75$$



The following code was used to create the previous plot of f', θ, and ϕ. The Matlab code uses the bvp4c solver for a system of first order differential equations with given boundary conditions.

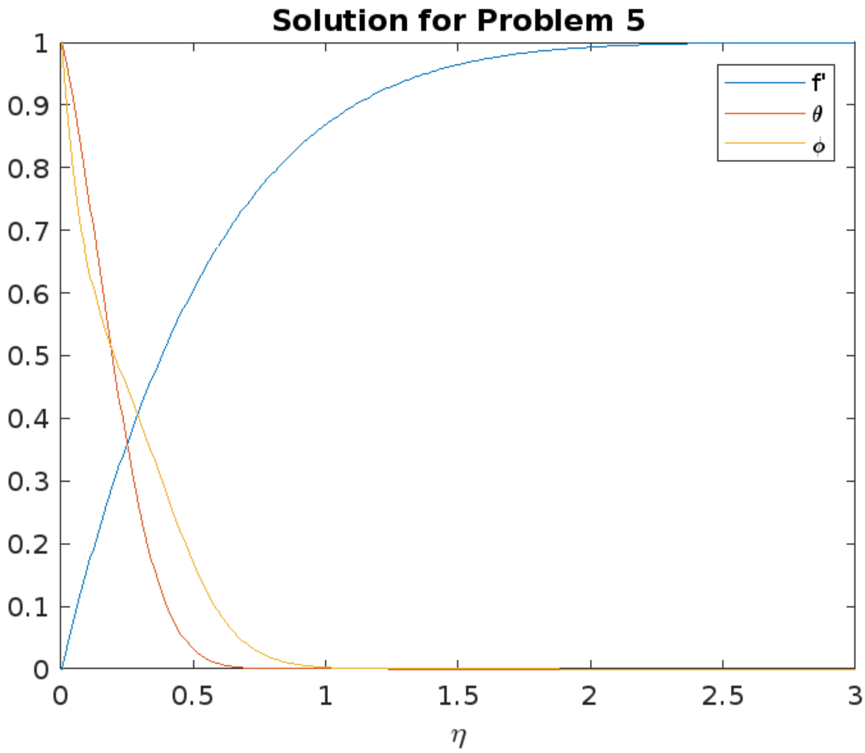
```
function sol = Project
solinit = bvpinit(linspace(0,3),@guess);
sol = bvp4c(@odes,@bcs,solinit);
plot(sol.x,sol.y([2,4,6],:));
ylim([0 1]);
title('Solution for Problem 4')
%ylabel("y'(x)")
xlabel('\eta')
legend("f'","\theta","\phi")
%================================================
function v = guess(x)
v = [sin(x); sin(x)+cos(x); sin(x)+cos(x)
   sin(x); sin(x)+cos(x)
   sin(x); sin(x)+cos(x)];

function dydx = odes(x,y)
nt = .3;
nb = .3;
pr = 10;
le = 5;
dydx = [y(2); y(3); -y(1)*y(3);
    y(5); -pr*(y(1)*y(5)+nb*y(7)*y(5)+nt*y(5).^2)
    y(7); (pr*nt/nb)*(y(1)*y(5)+nb*y(7)*y(5)+nt*y(5).^2)-le*y(1)*y(7)];

function res = bcs(ya,yb)
m = 0;
s = .75;
res = [ya(1)-((2*s)/(m+1)); ya(2); yb(2)-1
    ya(4)-1; yb(4)
    ya(6)-1; yb(6)];
```

The following graph is similar to Problem 4 with the exception of the value of m changing. The plots of f', θ, and ϕ are shown using Equations (11a), (11b), and (11c). The boundary conditions remain the same

$$f(0) = \frac{2s}{m+1}, f'(0) = 0, f'(3) = 1, \theta(0) = 1, \theta(3) = 0, \phi(0) = 1, \phi(3) = 0$$

And the given value for m changed

$$N_t = N_B = .3, Pr = 10, Le = 5, m = .0909, \beta = \frac{2m}{m+1}, s = .75, \delta = .2$$



The following Matlab code was used for Problem 5. Similar to Problem 4, bvp4c was used to solve a system of first order differential equations with the set of given boundary conditions.

```
function sol = Project1
solinit = bvpinit(linspace(0,3),@guess);
sol = bvp4c(@odes,@bcs,solinit);
plot(sol.x,sol.y([2,4,6],:));
ylim([0 1]);
title('Solution for Problem 5')
%ylabel("y'(x)")
xlabel('\eta')
legend("f'","\theta","\phi")
%=================================================
function v = guess(x)
v = [sin(x); sin(x)+cos(x); sin(x)+cos(x)
     sin(x); sin(x)+cos(x)
     sin(x); sin(x)+cos(x)];

function dydx = odes(x,y)
m = .0909;
nt = .3;
nb = .3;
delta = .2;
pr = 10;
le = 5;
b = 2*m/(m+1);
```

```
dydx = [y(2); y(3); -y(1)*y(3)-b*(1-y(2).^2)
     y(5); -pr*(y(1)*y(5)+nb*y(7)*y(5)+(b*delta*y(4)/m)+nt*y(5).^2)
     y(7); (pr*nt/nb)*(y(1)*y(5)+nb*y(7)*y(5)+(b*delta*y(4)/m)+nt*y(5).^2)-le*y
         (1)*y(7)];

function res = bcs(ya,yb)
m = .0909;
s = .75;
res = [ya(1)-((2*s)/(m+1)); ya(2); yb(2)-1
     ya(4)-1; yb(4)
     ya(6)-1; yb(6)];
```