Least square problems:

Consider the system of equations $Ax=b$. In many applications inconsistent system arise. When a solution is demanded & none exists, the best one can do is to find x that makes Ax as close as possible to b.

Def: If A is an $m\times n$ matrix, $b\in\mathbb{R}^m$, a least-square solution of $Ax=b$ is an $\hat{x}\in\mathbb{R}^n$ such that

$$\|b-A\hat{x}\| \le \|b-Ax\| \quad \forall\, x\in\mathbb{R}^n.$$

Theorem: The set of least-square solutions of $Ax=b$ coincides with the nonempty set of solutions of the normal equations $A^TAx=A^Tb$.

Note: To find $\hat{x}$:

$$\hat{x} = (A^TA)^{-1}A^Tb$$

Find a least-square solution of the inconsistent system $Ax=b$ for

1) $A=\begin{bmatrix}4&0\\0&2\\1&1\end{bmatrix}$ $\quad b=\begin{bmatrix}2\\0\\11\end{bmatrix}$

$$A^TA=\underset{2\times3}{\begin{bmatrix}4&0&1\\0&2&1\end{bmatrix}}\underset{3\times2}{\begin{bmatrix}4&0\\0&2\\1&1\end{bmatrix}}=\begin{bmatrix}17&1\\1&5\end{bmatrix}$$

$$A^T b = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}_{2\times 3} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{84} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}$$

$$\hat{x} = (A^T A)^{-1} A^T b$$

$$= \frac{1}{84} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix} \begin{bmatrix} 19 \\ 11 \end{bmatrix} = \frac{1}{84} \begin{bmatrix} 84 \\ 168 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

2) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ $\quad b = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}_{3\times 4} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}_{4\times 3} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}_{3\times 4} \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}_{4\times 1} = \begin{bmatrix} 14 \\ 4 \\ 10 \end{bmatrix}$$

The augmented matrix for $A^T A x = A^T b$

$$\begin{bmatrix} 4 & 2 & 2 & : & 14 \\ 2 & 2 & 0 & : & 4 \\ 2 & 0 & 2 & : & 10 \end{bmatrix}$$

$R_2^* \rightarrow 2R_2 - R_1$
$R_3^* \rightarrow 2R_3 - R_1$

$$\sim \begin{bmatrix} 4 & 2 & 2 & : & 14 \\ 0 & 2 & -2 & : & -6 \\ 0 & -2 & 2 & : & 6 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_2$

$$\sim \begin{bmatrix} 4 & 2 & 2 & : & 14 \\ 0 & 2 & -2 & : & -6 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$R[A^TA] = 2$ $\quad R[A^TA : A^Tb] = 2$ < no. of unknowns.

$\therefore$ Infinite solutions.

$4x_1 + 2x_2 + 2x_3 = 14$

$2x_2 - 2x_3 = -6$

Let $x_3 = k$, $\quad k \rightarrow$ arbitrary

$2x_2 = -6 + 2x_3 = -6 + 2k$

$x_2 = -3 + k$

$4x_1 + 2(-3+k) + 2k = 14$

$4x_1 - 6 + 4k = 14$

$4x_1 = 20 - 4k \Rightarrow x_1 = 5 - k$

$x_1 = 5-K \quad x_2 = -3+k, \quad x_3 = K$

$\therefore$ The solution $\hat{x} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} + K \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

3) $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ $\quad b = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 4 \\ -4 \\ 2 \\ 6 \end{bmatrix}$$

$$\left[A^TA : A^Tb\right] = \left[\begin{array}{cccc|c} 6 & 2 & 2 & 2 & 4 \\ 2 & 2 & 0 & 0 & -4 \\ 2 & 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 2 & 6 \end{array}\right]$$

$R_2 \to 3R_2 - R_1 \quad R_3 \to 3R_3 - R_1 \quad R_4 \to 3R_4 - R_1$

$$\left[A^TA : A^Tb\right] \sim \left[\begin{array}{cccc|c} 6 & 2 & 2 & 2 & 4 \\ 0 & 4 & -2 & -2 & -16 \\ 0 & -2 & 4 & -2 & 2 \\ 0 & -2 & -2 & 4 & 14 \end{array}\right]$$

$R_3 \to 2R_3 + R_2 \quad R_4 \to 2R_4 + R_2$

$$\left[A^TA : A^Tb\right] \sim \left[\begin{array}{cccc|c} 6 & 2 & 2 & 2 & 4 \\ 0 & 4 & -2 & -2 & -16 \\ 0 & 0 & 6 & -6 & -12 \\ 0 & 0 & -6 & 6 & 12 \end{array}\right]$$

$R_4 \to R_4 + R_3$

$$\left[A^TA : A^Tb\right] \sim \left[\begin{array}{cccc|c} 6 & 2 & 2 & 2 & 4 \\ 0 & 4 & -2 & -2 & -16 \\ 0 & 0 & 6 & -6 & -12 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$R[A^TA] = R[A^TA : A^Tb] = 2 <$ no of unknowns.

$\therefore$ Infinite solutions.

$$6x_1 + 2x_2 + 2x_3 + 2x_4 = 4$$
$$4x_2 - 2x_3 - 2x_4 = -16$$
$$6x_3 - 6x_4 = -12$$

Let $x_4 = k$, $k \to$ arbitrary

$$6x_3 = -12 + 6k$$
$$x_3 = -2 + k$$
$$4x_2 = 2x_3 + 2x_4 - 16$$
$$= -4 + 2k + 2k - 16 = 4k - 20$$
$$x_2 = k - 5$$
$$6x_1 = 4 - 2x_2 - 2x_3 - 2x_4$$
$$= 4 - 2k + 10 + 4 - 2k - 2k$$
$$= -6k + 18$$
$$x_1 = -k + 3$$

$$\hat{x} = \begin{bmatrix} 3 \\ -5 \\ -2 \\ 0 \end{bmatrix} + k \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

4] $A = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}$ $b = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

$$A^T A = \underset{2\times 3}{\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}} = \begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix}$$

$$A^T b = \underset{2\times 3}{\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & 3 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}$$

$$A^T A x = A^T b$$

$$\begin{bmatrix} 6 & -11 \\ -11 & 22 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \end{bmatrix}$$

$$6x_1 - 11x_2 = -4$$
$$-11x_1 + 22x_2 = +11$$

$$\Rightarrow x_1 = 3 \qquad x_2 = 2$$

$$\therefore \hat{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$