Unit-III

Probability Theory

Def: We say that the events $B_1, B_2 \cdots B_K$ represent a partition of the sample space S if

i) $B_i \cap B_j = \phi \quad \forall i \neq j$

ii) $\bigcup_{i=1}^{k} B_i = S$

iii) $P(B_i) > 0 \quad \forall i$

Bayes' Theorem:

If $B_1, B_2, \cdots, B_K$ are non-empty events which constitute a partition of the sample space S [i.e. $B_1, B_2 \cdots B_K$ are pairwise disjoint, $\bigcup_{i=1}^{k} B_i = S$ & $P(B_i) > 0 \ \forall i$] & A is any event of nonzero probability then

$$P(B_i | A) = \frac{P(A|B_i)\,P(B_i)}{\sum_{j=1}^{k} P(A|B_j)\,P(B_j)} \quad \text{for any } i = 1, \cdots k$$

Proof: For any event A,

$A = A \cap S$

$A = A \cap [B_1 \cup B_2 \cup \cdots \cup B_K]$

$A = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_K)$

$P(A) = P[(A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_K)]$ — (1)

$A \cap B_i$ & $A \cap B_j$ are respectively subsets of B_i & B_j.

W.K.T. B_i & B_j are disjoint for $i \neq j$.

$\therefore A \cap B_i$ & $A \cap B_j$ are also disjoint $\forall i \neq j,\ i, j = 1, 2 \dots k$

Using Kolmogorov's axiom in eq (1) we have

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \dots + P(A \cap B_k)$$

$$= P(A|B_1)\,P(B_1) + P(A|B_2)\,P(B_2) + \dots + P(A|B_k)\,P(B_k)$$

[$\because$ using multiplication theorem]

$$P(A) = \sum_{j=1}^{k} P(A|B_j)\,P(B_j) \quad \text{--- (2)}$$

↳ Theorem of total probability

Now $P(B_i|A) = \dfrac{P(A \cap B_i)}{P(A)}$ (using conditional probability)

$$= \frac{P(A|B_i)\,P(B_i)}{P(A)} \quad \text{(using multiplication theorem)}$$

$$= \frac{P(A|B_i)\,P(B_i)}{\sum_{j=1}^{k} P(A|B_j)\,P(B_j)} \quad \text{(using eq. (2))}$$

Problems:

1] Bag I contains 3 red & 4 black balls, while another bag II contains 5 red & 6 black balls. One ball is drawn at random from one of the bags & it is found to be red. Find the probability that it was drawn from bag II?

Soln: Let B_1 = {selecting ~~balls~~ ~~from~~ bag I}

B_2 = { " " " II }

A = { selecting red ball }

$P(B_1) = \frac{1}{2} = P(B_2)$ $\quad$ $P(B_2|A) = ?$

$P(A|B_1) = \frac{3}{7}$ $\quad$ $P(A|B_2) = \frac{5}{11}$

The probability of selecting a ball from bag II, given that it is red $= P(B_2|A)$

$$= \frac{P(A|B_2)\,P(B_2)}{P(A|B_1)\,P(B_1) + P(A|B_2)\,P(B_2)}$$

$$= \frac{\frac{5}{11} \times \frac{1}{2}}{\left(\frac{3}{7} \times \frac{1}{2}\right) + \left(\frac{5}{11} \times \frac{1}{2}\right)}$$

$$= \frac{35}{68} = 0.5147$$

2] Three machines A, B & C produce respectively 50%, 30%, 20% of the total number of items of a factory. The percentages of defective output of these machines are 3%, 4% & 5%. If an item is selected at random, find the probability that the item is defective.

$B_1 = \{$ items produced from machine A $\}$

$B_2 = \{$ " " " " B $\}$

$B_3 = \{$ " " " " C $\}$

$D = \{$ defective output $\}$

$P(D) = ?$ $\quad$ $P(B_1) = \frac{50}{100} = 0.5$ $\quad$ $P(B_2) = 0.3$ $\quad$ $P(B_3) = 0.2$

$P(D|B_1) = 0.03$ $\quad$ $P(D|B_2) = 0.04$ $\quad$ $P(D|B_3) = 0.05$

From theorem of total probability

$$P(D) = P(D|B_1)P(B_1) + P(D|B_2)P(B_2) + P(D|B_3)P(B_3)$$

$$= (0.5)(0.03) + (0.3)(0.04) + (0.2)0.05$$

$$= 0.037$$

3] Among the three men, the chances that a politician, a business man & an academician will be appointed as VC of VTU are 0.5, 0.3 & 0.2 respectively. Probabilities that research is promoted by these people if they are appointed as VC are 0.3, 0.7 & 0.8 respectively. If research is promoted in VTU, what is the probability that the VC is an academician?

Soln: $B_1 = \{\text{politicians}\}$ $B_2 = \{\text{business man}\}$ $B_3 = \{\text{academician}\}$

$P(B_1) = 0.5$ $P(B_2) = 0.3$ $P(B_3) = 0.2$

$R = \{\text{research activity}\}$

$P(R|B_1) = 0.3$ $P(R|B_2) = 0.7$ $P(R|B_3) = 0.8$

$$P(B_3|R) = \frac{P(R|B_3)\,P(B_3)}{P(R|B_1)P(B_1) + P(R|B_2)P(B_2) + P(R|B_3)P(B_3)}$$

$$= \frac{(0.8)(0.2)}{(0.3)(0.5) + (0.7)(0.3) + (0.8)(0.2)}$$

$$= 0.3077$$

4] In a certain college 25% of boys & 10% of girls are studying mathematics. The girls constitute 60% of the student. If a student is selected at random & is found to be studying mathematics, find the prob. that the student is a girl?

Soln: $B_1 = \{ \text{girls} \}$ $B_2 = \{ \text{boys} \}$ $M = \{ \text{student studying mathematics} \}$

$P(B_1) = 0.6$

$P(B_2) = 0.4$

$P(M|B_2) = 0.25$ $P(M|B_1) = 0.1$

$$P(B_1|M) = \frac{P(M|B_1)\,P(B_1)}{P(M|B_1)\,P(B_1) + P(M|B_2)\,P(B_2)}$$

$$= \frac{(0.1)(0.6)}{(0.1)(0.6) + (0.25)(0.4)}$$

$$= 0.375$$

5] In a bolt factory, machines A, B & C manufacture 25%, 35% & 40% of the total output. Out of which 5%, 4%, 2% are defective bolts. A bolt is selected at random & found to be defective. What is the prob. that the bolt came from machine A?

Soln: $B_1 = \{ \text{bolts from machine A} \}$

$B_2 = \{ \text{" " " B} \}$

$B_3 = \{ \text{" " " C} \}$

$D = \{ \text{defective bolts} \}$

$P(B_1) = 0.25 \quad P(B_2) = 0.35 \quad P(B_3) = 0.4$

$P(D|B_1) = 0.05 \quad P(D|B_2) = 0.04 \quad P(D|B_3) = 0.02$

$$P(B_1|D) = \frac{P(D|B_1)\,P(B_1)}{P(D|B_1)\,P(B_1) + P(D|B_2)\,P(B_2) + P(D|B_3)\,P(B_3)}$$

$$= \frac{(0.05)(0.25)}{(0.05)(0.25) + (0.04)(0.35) + (0.02)(0.4)}$$

$$= \underline{\underline{0.3623}}$$

6] Mr. Sharma who lives in the outskirts of a city wants to catch an early morning train. The probability that he gets an auto at that time is 0.23. If he gets an auto, the probability of catching the train is 0.85. If he fails to get an auto, he has to take a city bus & the prob. that he gets the train is 0.43. Find the prob. that he catches the train.

<u>Soln:</u> $P(A) = 0.23 \quad P(T|A) = 0.85$

$P(T|\bar{A}) = 0.43$

$P(\bar{A})$ = Prob. that Mr. Sharma has to take a city bus $= 1 - P(A)$

$= 1 - 0.23$

$= 0.77$

$\therefore$ The prob. that Mr. Sharma catches the train

$= P(T) = P(A)P(T|A) + P(\bar{A})P(T|\bar{A})$

$= (0.23)(0.85) + (0.77)(0.43)$

$= \underline{\underline{0.5266}}$

H.W

7] An office has 4 secretaries handling 20%, 60%, 15%, 5% of the file of all govt. reports. The prob. that they mis-files such reports are 0.05, 0.1, 0.1, 0.05. Find the prob. that a misfiled report can be blamed on the first secretary?

Ans: 0.11428