Unit-II Linear Transformations

Def: Let V & W be vector spaces over the field F. A linear transformation from V into W is a function T from V into W such that

$T(cv_1+v_2) = cTv_1 + Tv_2$ for all $v_1, v_2 \in V$ & $\forall c \in F$.

1] Suppose the mapping $T: \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T(x,y) = (x+y, x)$. S.T. T is linear.

Soln: Let $v_1 = (x_1, y_1)$ $v_2 = (x_2, y_2)$, $c \in F$

$$T(cv_1+v_2) = T[c(x_1,y_1) + (x_2,y_2)]$$
$$= T[(cx_1, cy_1) + (x_2, y_2)]$$
$$= T(cx_1 + x_2, cy_1 + y_2)$$
$$= (cx_1 + x_2 + cy_1 + y_2, cx_1 + x_2)$$
$$= (c(x_1+y_1) + x_2 + y_2, cx_1 + x_2)$$
$$= \cancel{c(x_1+y_1) + cx}$$
$$= (c(x_1+y_1), cx_1) + (x_2+y_2, x_2)$$
$$= c(x_1+y_1, x_1) + (x_2+y_2, x_2)$$
$$= cT(x_1,y_1) + T(x_2,y_2)$$
$$= cTv_1 + Tv_2$$

$\therefore$ T is linear.

2] Let $T: V_3(\mathbb{R}) \to V_2(\mathbb{R})$ defined by $T(x,y,z) = (x+y,\ y+z)$. S.T. T is a linear transformation from $V_3(\mathbb{R})$ to $V_2(\mathbb{R})$.

Solns Let $v_1 = (x_1, y_1, z_1)$, $v_2 = (x_2, y_2, z_2) \in V_3(\mathbb{R})$

$c \in F$.

$$T(cv_1 + v_2) = T(c(x_1, y_1, z_1) + (x_2, y_2, z_2))$$

$$= T((cx_1, cy_1, cz_1) + (x_2, y_2, z_2))$$

$$= T((cx_1 + x_2,\ cy_1 + y_2,\ cz_1 + z_2))$$

$$= (cx_1 + x_2 + cy_1 + y_2,\ cy_1 + y_2 + cz_1 + z_2)$$

$$= (c(x_1 + y_1) + x_2 + y_2,\ c(y_1 + z_1) + y_2 + z_2)$$

$$= (c(x_1 + y_1),\ c(y_1 + z_1)) + (x_2 + y_2,\ y_2 + z_2)$$

$$= c(x_1 + y_1,\ y_1 + z_1) + (x_2 + y_2,\ y_2 + z_2)$$

$$= c\,T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$$

$$= c\,Tv_1 + Tv_2$$

$\therefore$ T is linear.

H.W

3] Let $T: V_3 \to V_2$ be defined by $T(x_1, x_2, x_3) = (x_1 - x_2,\ x_1 + x_3)$. S.T. T is a linear map from V_3 to V_2.

4] S.T. the transformation T defined by
$T(x_1, x_2, x_3) = (2x_1 - 3x_2,\ x_1 + 4,\ 5x_3)$ is not linear.

Soln: Let $v_1 = (x_1, x_2, x_3)$

$v_2 = (y_1, y_2, y_3)$

For any scalar c

$$T(cv_1 + v_2) = T[(cx_1, cx_2, cx_3) + (y_1, y_2, y_3)]$$

$$= T(cx_1 + y_1,\ cx_2 + y_2,\ cx_3 + y_3)$$

$$= (2(cx_1 + y_1) - 3(cx_2 + y_2),\ cx_1 + y_1 + 4,\ 5(cx_3 + y_3))$$

$$= (2(cx_1) - 3cx_2,\ cx_1,\ 5cx_3) + (2y_1 - 3y_2,\ y_1 + 4,\ 5y_3)$$

$$= c(2x_1 - 3x_2,\ x_1,\ 5x_3) + (2y_1 - 3y_2,\ y_1 + 4,\ 5y_3)$$

$$\neq cTv_1 + Tv_2$$

$\therefore$ T is not a linear transformation.

The Algebra of Linear Transformations:

Theorem: Let V & W be vector spaces over the field F. Let T & U be linear transformations from V into W. The function $T+U$ defined by $(T+U)\alpha = T\alpha + U\alpha$ is a linear transformation from V into W. If c is any element of F, the function cT defined by $(cT)\alpha = c(T\alpha)$ is a L.T. from V into W.

Proof: We will prove that $T+U$ is a linear transformation from V into W.

Let $v_1, v_2 \in V$, $c \in F$

$$(T+U)(cv_1+v_2) = T(cv_1+v_2) + U(cv_1+v_2)$$

$$= cTv_1 + Tv_2 + cUv_1 + Uv_2 \quad (\because T, U \text{ are L.T.})$$

$$= c[Tv_1 + Uv_1] + Tv_2 + Uv_2$$

$$= c[(T+U)v_1] + (T+U)v_2$$

$$= c(T+U)v_1 + (T+U)v_2$$

$\therefore T+U$ is a linear transformation.

~~St~~ We will prove that cT is a l.T. from V into W. Let $c' \in F$

$$(cT)(c'v_1+v_2) = c[T(c'v_1+v_2)]$$

$$= c[c'Tv_1 + Tv_2] \quad (\because T \text{ is linear})$$

$$= cc'(Tv_1) + c(Tv_2)$$

$$= c'[c(Tv_1)] + c(Tv_2)$$

$$= c'[(cT)v_1] + (cT)v_2$$

$\therefore cT$ is a linear transformation.

Theorem: Let V, W and Z be vector spaces over the field F. Let T be a linear transformation from V into W & U be a linear transformation from W into Z. Then the composed function $U \circ T$ defined by $(U \circ T)(\alpha) = U(T(\alpha))$ is a linear transformation from V into Z.

Proof: Let $v_1, v_2 \in V$, $c \in F$

$$(U \circ T)(cv_1 + v_2) = U(T(cv_1 + v_2))$$
$$= U[c(Tv_1) + Tv_2] \quad (\because T \text{ is a L.T})$$
$$= c[U(Tv_1)] + U(Tv_2)$$
$$= c[(U \circ T)v_1] + (U \circ T)v_2$$

$\therefore U \circ T$ is a L.T.

Defi: Let U & V be vector spaces over the field F & Let T be a L.T. from U into V.

The null space of T is defined by

$$N(T) = \{u \in U \mid Tu = 0\}$$

The range space of T is defined by

$$R(T) = \{v \in V \mid Tu = v \text{ for some } u \in U\}$$

S.T. $N(T)$ is a subspace of U where $T: U \longrightarrow V$ is linear.

Proof: Clearly $N(T) \neq \phi$ since $0 \in N(T)$.

Let $u_1, u_2 \in N(T)$. $Tu_1 = 0$ $Tu_2 = 0$

For any scalar c, $T(cu_1 + u_2) = c(Tu_1) + Tu_2$ $(\because T$ is linear

$= c.0 + 0 = 0$

$T(cu_1 + u_2) = 0$

$\Rightarrow cu_1 + u_2 \in N(T)$

$\therefore N(T)$ is a subspace of U.

S.T. $R(T)$ is a subspace of V where $T: U \to V$ is a L.T..

Proof: Clearly $R(T) \neq \phi$ as $0 \in R(T)$.

Let $v_1, v_2 \in R(T)$.

Then $\exists\, u_1, u_2 \in U$ such that $Tu_1 = v_1$ & $Tu_2 = v_2$

For any scalar c,

$T(cu_1 + u_2) = c(Tu_1) + Tu_2$ ($\because$ T is linear)

$= cv_1 + v_2$

$\Rightarrow cv_1 + v_2 \in R(T)$.

$\therefore R(T)$ is a subspace of V.

Rank - Nullity Theorem:

Let U & V be two finite dimensional vector spaces over the field F. Let $T: U \to V$ be a L.T.. Then

$\dim(R(T)) + \dim(N(T)) = \dim U$

or $\text{Rank } T + \text{Nullity } T = \dim U$

Proof: Let $\dim U = n$ & $\dim(N(T)) = r$

Since $N(T)$ is a subspace of U, $r \leq n$.

Suppose $\{u_1, u_2, \ldots u_r\}$ is a basis of $N(T)$.

Since $\{u_1, u_2, \ldots u_r\}$ is linearly independent in $N(T)$,

$\{u_1, u_2, \ldots u_r\}$ is linearly independent in U.

So we can extend it to form a basis of U.
Now there exists vectors $\{u_{r+1}, u_{r+2}, \ldots u_n\}$ such that $\{u_1, u_2, \ldots u_r, u_{r+1}, u_{r+2}, \ldots u_n\}$ is a basis of U.
We claim that $\{T(u_{r+1}), T(u_{r+2}), \ldots T(u_n)\}$ is a basis of $R(T)$.
First we prove $\{T(u_{r+1}), T(u_{r+2}), \ldots T(u_n)\}$ spans $R(T)$.
For, let $v \in R(T)$. Then $\exists$ a vector $u \in U$ such that $v = T(u)$. Since $\{u_1, u_2, \ldots u_n\}$ is a basis of U, we can express $u = c_1u_1 + c_2u_2 + \ldots + c_nu_n$, where $c_1, c_2 \ldots c_n \in F$

$$\therefore v = T(u) = T(c_1u_1 + c_2u_2 + \ldots + c_ru_r + c_{r+1}u_{r+1} + \ldots + c_nu_n)$$

$$= c_1T(u_1) + c_2T(u_2) + \ldots + c_rT(u_r) + c_{r+1}T(u_{r+1}) + \ldots + c_nT(u_n)$$

$$= c_1 \cdot 0 + c_2 \cdot 0 + \ldots + c_r \cdot 0 + c_{r+1}T(u_{r+1}) + c_{r+2}T(u_{r+2}) + \ldots + c_nT(u_n)$$

$$(\because \{u_1, u_2 \ldots u_r\} \in N(T))$$

$$v = T(u) = c_{r+1}T(u_{r+1}) + c_{r+2}T(u_{r+2}) + \ldots + c_nT(u_n)$$

$$\therefore R(T) = \operatorname{span}\{T(u_{r+1}), \ldots T(u_n)\}$$

Next we shall p.T. $\{T(u_{r+1}), \ldots T(u_n)\}$ is linearly independent.
For, consider

$$c_{r+1}T(u_{r+1}) + c_{r+2}T(u_{r+2}) + \ldots + c_nT(u_n) = 0$$

$$T(c_{r+1}u_{r+1}) + T(c_{r+2}u_{r+2}) + \ldots + T(c_nu_n) = 0$$

$T(c_{r+1}u_{r+1} + c_{r+2}u_{r+2} + \cdots + c_n u_n) = 0$ [$\because T$ is linear]

$\Rightarrow c_{r+1}u_{r+1} + c_{r+2}u_{r+2} + \cdots + c_n u_n \in N(T)$.

$\therefore \exists$ scalars $d_1, d_2 \ldots d_r$ such that

$$c_{r+1}u_{r+1} + c_{r+2}u_{r+2} + \cdots + c_n u_n = d_1u_1 + d_2u_2 + \cdots + d_r u_r$$

$$-d_1u_1 - d_2u_2 - \cdots - d_r u_r + c_{r+1}u_{r+1} + c_{r+2}u_{r+2} + \cdots + c_n u_n = 0$$

Since the set $\{u_1, u_2 \ldots u_n\}$ is linearly independent,

we have $d_1 = d_2 = \cdots = d_r = c_{r+1} = \cdots = c_n = 0$

$\therefore \{T(u_{r+1}), T(u_{r+2}) \ldots T(u_n)\}$ is linearly independent.

Hence $\{T(u_{r+1}), T(u_{r+2}), \ldots T(u_n)\}$ is a basis for $R(T)$.

$\therefore \dim(R(T)) = \text{Rank } T = n - r$

$\therefore (n-r) + r = n$

Rank T + Nullity T = $\dim U$

1] Let a linear map $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x,y,z) = (y-x, y-z)$. Find the null space & range space of T. Find the nullity of T & Rank(T).

Soln: $R(T) = \{T(x,y,z) / (x,y,z) \in \mathbb{R}^3\}$

$= \{(y-x, y-z) / (x,y,z) \in \mathbb{R}^3\}$

$N(T) = \{(x,y,z) \in \mathbb{R}^3 / T(x,y,z) = (0,0)\}$

$= \{(x,y,z) \in \mathbb{R}^3 / (y-x, y-z) = (0,0)\}$

$\Downarrow$ $y-x=0$ $\quad y-z=0$

$\Rightarrow$ $y=x,$ $\quad y=z$

$\therefore N(T) = \{(x,y,z) / x=y=z\}$

Let $x=y=z=k$, $k \neq 0$.

Then $\{(k,k,k)\}$ linearly independent & spans.

$\therefore$ This forms a basis for $N(T)$. Hence $\dim N(T) =$ Nullity$(T) = 1$. Using Rank-Nullity theorem,

$\text{Rank}(T) + \text{Nullity}(T) = \dim \mathbb{R}^3$

$\text{Rank}(T) = \dim \mathbb{R}^3 - \text{Nullity}(T) = 3-1 = 2$

2] Let $T: V_3 \to V_3$ be a linear map defined by $T(x_1, x_2, x_3) = (x_1, x_2, 0)$. Find the nullspace & range space of T. Also find nullity(T) & Rank(T).

Soln: $N(T) = \{(x_1, x_2, x_3) \in V_3 \mid T(x_1, x_2, x_3) = (0,0,0)\}$

$= \{(x_1, x_2, x_3) \in V_3 / (x_1, x_2, 0) = (0,0,0)\}$

$\Rightarrow x_1 = x_2 = 0$.

$\therefore N(T) = \{(0,0,x_3)\}$

Basis for $N(T) = \{ \cancel{(0,0,0)}\ (0,0,x_3)\}$ where $x_3 \neq 0$.

$\therefore \dim N(T) = 1 = \text{Nullity}(T)$

$R(T) = \{T(x_1,x_2,x_3) \mid (x_1,x_2,x_3) \in V_3\}$

$= \{(x_1,x_2,0) \mid (x_1,x_2,x_3) \in V_3\}$

Using Rank-Nullity theorem,

$\text{Rank}(T) = \dim V_3 - \text{Nullity}(T)$

$= 3 - 1 = 2$

3. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear mapping defined by $T(x,y,z) = (x+2y-z,\ y+z,\ x+y-2z)$. Find the range space & null space of T. Also find the Rank(T) & nullity (T).

$R(T) = \{T(x,y,z) \mid (x,y,z) \in \mathbb{R}^3\} = \{(x+2y-z,\ y+z,\ x+y-2z) \mid (x,y,z) \in \mathbb{R}^3\}$

Soln: $N(T) = \{(x,y,z) \in \mathbb{R}^3 \mid T(x,y,z) = (0,0,0)\}$

$= \{(x,y,z) \in \mathbb{R}^3 \mid (x+2y-z,\ y+z,\ x+y-2z) = (0,0,0)\}$

$x+2y-z=0$

$y+z=0$

$x+y-2z=0$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{ccc:c} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$\cancel{R_3 \to R_3 -}$

$R_3 \to R_3 + R_2$ $\left[\begin{array}{ccc:c} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$R[A] = R[A:B] = 2 <$ number of unknown

Hence $x + 2y - z = 0$
$y + z = 0$ will have infinite solution.

Let $z = k$, k - arbitrary.

$\therefore y = -z \Rightarrow y = -k$.

$x = z - 2y = k + 2k = 3k$.

$\therefore N(T) = \{(x, y, z) \in \mathbb{R}^3 / x = 3k, y = -k, z = k\}$

Basis for $N(T) = \{(3k, -k, k)\}$, $k \neq 0$

$\therefore$ Nullity $(T) = 1$.

$\therefore$ Rank $(T) = \dim \mathbb{R}^3 -$ Nullity $(T) = \underline{\underline{3 - 1 = 2}}$

Theorem:
Let V be a finite dimensional vector space over the field F. Let $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ be an ordered basis for V. Let W be a vector space over the same field F. Let $\beta_1, \beta_2 \dots \beta_n$ be any vectors in W. Then there is precisely one L.T. T from V into W such that
$T\alpha_j = \beta_j \quad j = 1, 2 \dots n$.

Proof: $T : V \to W$
Let $\alpha \in V$. Since $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ forms a basis for V
$\alpha = c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n$ uniquely.
For this vector α we define
$T\alpha = c_1\beta_1 + c_2\beta_2 + \dots + c_n\beta_n$
Then T is a well-defined rule for associating with each vector $\alpha \in V$ a vector $T\alpha$ in W.
From the definition it is clear that $T\alpha_j = \beta_j$ for each j.
To show that T is linear:
Let $\beta = d_1\alpha_1 + d_2\alpha_2 + \dots + d_n\alpha_n \in V$ & $d_1, d_2 \dots d_n \in F$.
For any scalar c,

$$c\alpha + \beta = c[c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n] + [d_1\alpha_1 + d_2\alpha_2 + \dots + d_n\alpha_n]$$

$$= (cc_1 + d_1)\alpha_1 + (cc_2 + d_2)\alpha_2 + \dots + (cc_n + d_n)\alpha_n$$

$$T(c\alpha+\beta) = (cc_1+d_1)\beta_1 + (cc_2+d_2)\beta_2 + \cdots + (cc_n+d_n)\beta_n \quad \text{— (1)}$$

On the other hand

$$c(T\alpha) + T\beta = c\left(\sum_{i=1}^{n} c_i\beta_i\right) + \sum_{i=1}^{n} d_i\beta_i$$

$$= \sum_{i=1}^{n} (cc_i + d_i)\beta_i \quad \text{— (2)}$$

From eq. (1) & eq. (2)

$$T(c\alpha+\beta) = c(T\alpha) + T\beta$$

If U is a L.T. from V into W with

$U\alpha_j = \beta_j$, $j = 1, 2, \ldots, n$, then for the vector

$\alpha = \sum_{i=1}^{n} c_i\alpha_i$ we have

$$U\alpha = U\left(\sum_{i=1}^{n} c_i\alpha_i\right)$$

$$= \sum_{i=1}^{n} c_i(U\alpha_i)$$

$$= \sum_{i=1}^{n} c_i\beta_i$$

$$= T\alpha$$

This shows that the linear transformation T with

$T\alpha_j = \beta_j$ is unique.