Matrix of linear transformation:

Let $T : U \to V$ be a linear transformation from a vector space U into a vector space V. U, V be of finite dimensions n & m respectively. Let $B_1 = \{u_1, u_2, \ldots, u_n\}$ & $B_2 = \{v_1, v_2, \ldots, v_m\}$ be bases of U & V respectively such that

$$T(u_1) = a_{11} v_1 + a_{21} v_2 + \cdots + a_{m1} v_m$$

$$T(u_2) = a_{12} v_1 + a_{22} v_2 + \cdots + a_{m2} v_m$$

$$\vdots$$

$$T(u_n) = a_{1n} v_1 + a_{2n} v_2 + \cdots + a_{mn} v_m$$

Then
$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is defined as matrix of linear transformation T relative to the basis B_1 & B_2.

1] Find the matrix of linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x, -y)$ w.r.t. the bases $B_1 = \{(1,1), (1,0)\}$ $B_2 = \{(2,3), (4,5)\}$

$T(1,1) = (1,-1) = a_{11}(2,3) + a_{21}(4,5)$

$T(1,0) = (1,0) = a_{12}(2,3) + a_{22}(4,5)$

$1 = 2a_{11} + 4a_{21}$
$-1 = 3a_{11} + 5a_{21}$ $\Big\} \to a_{11} = -4.5, \; a_{21} = 2.5$

$1 = 2a_{12} + 4a_{22}$
$0 = 3a_{12} + 5a_{22}$ $\Big\} \to a_{12} = -2.5, \; a_{22} = 1.5$

$\therefore$ Matrix of transformation is
$$\begin{bmatrix} -4.5 & -2.5 \\ 2.5 & 1.5 \end{bmatrix}$$

3] Find the matrix of the L.T. $T: V_2(\mathbb{R}) \to V_3(\mathbb{R})$ (15)
defined by $T(x,y) = (x+y, x, 3x-y)$ w.r.t.
bases $B_1 = \{(1,1), (3,1)\}$ & $B_2 = \{(1,1,1), (1,1,0), (1,0,0)\}$

Soln:
$T(1,1) = (2,1,2) = a_{11}(1,1,1) + a_{21}(1,1,0) + a_{31}(1,0,0)$

$T(3,1) = (4,3,8) = a_{12}(1,1,1) + a_{22}(1,1,0) + a_{32}(1,0,0)$

$\Rightarrow 2 = a_{11} + a_{21} + a_{31}$

$1 = a_{11} + a_{21}$

$2 = a_{11}$

$\Rightarrow a_{21} = -1$

$a_{31} = 1$

$4 = a_{12} + a_{22} + a_{32}$

$3 = a_{12} + a_{22}$

$8 = a_{12}$

$\Rightarrow -5 = a_{22}$

$a_{32} = 1$

$\therefore$ Matrix of linear transformation

$$= \begin{bmatrix} 2 & 8 \\ -1 & -5 \\ 1 & 1 \end{bmatrix}$$

2) A linear transformation defined by $T(x, y, z) = (x - y + 2z, 3x + y)$. Find the matrix of transformation relative to the bases

$B_1 = \{(1,1,1), (1,2,3), (1,0,0)\}$

$B_2 = \{(1,1), (1,-1)\}$

Soln: $T(1,1,1) = (2, 4) = a_{11}(1,1) + a_{21}(1,-1)$

$T(1,2,3) = (5, 5) = a_{12}(1,1) + a_{22}(1,-1)$

$T(1,0,0) = (1, 3) = a_{13}(1,1) + a_{23}(1,-1)$

$$\left.\begin{aligned} 2 &= a_{11} + a_{21} \\ 4 &= a_{11} - a_{21} \end{aligned}\right\} \rightarrow \quad a_{11} = 3, \quad a_{21} = -1$$

$$\left.\begin{aligned} 5 &= a_{12} + a_{22} \\ 5 &= a_{12} - a_{22} \end{aligned}\right\} \rightarrow \quad a_{12} = 5, \quad a_{22} = 0$$

$$\left.\begin{aligned} 1 &= a_{13} + a_{23} \\ 3 &= a_{13} - a_{23} \end{aligned}\right\} \rightarrow \quad a_{13} = 2, \quad a_{23} = -1$$

$\therefore$ Matrix of transformation $= \begin{bmatrix} 3 & 5 & 2 \\ -1 & 0 & -1 \end{bmatrix}$

4] Given $A = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$, determine the L.T.

$T : V_3(R) \longrightarrow V_2(R)$ relative to the bases

$B_1 = \{(1,2,0),\ (0,-1,0),\ (1,-1,1)\}$

$B_2 = \{(1,0),\ (2,-1)\}$

Soln: $T(1,2,0) = -1(1,0) + 1(2,-1) = (1,-1)$

$T(0,-1,0) = 2(1,0) = (2,0)$

$T(1,-1,1) = 1(1,0) + 3(2,-1) = (7,-3)$

Let $(x,y,z) = \alpha_1(1,2,0) + \alpha_2(0,-1,0) + \alpha_3(1,-1,1)$

$x = \alpha_1 + \alpha_3$

$y = 2\alpha_1 - \alpha_2 - \alpha_3$

$z = \alpha_3$

$\therefore \cancel{T(x,y,z)} =$ $\quad \alpha_1 = x - \alpha_3$

$= x - z$

$\alpha_2 = 2\alpha_1 - y - \alpha_3$

$= 2(x-z) - y - z$

$= 2x - y - 3z$

$\therefore T(x,y,z) = \alpha_1 T(1,2,0) + \alpha_2 T(0,-1,0) + \alpha_3 T(1,-1,1)$

$= (x-z)(1,-1) + (2x-y-3z)(2,0) + z(7,-3)$

$= (x - z + 4x - 2y - 6z + 7z,\ z - x - 3z)$

$T(x,y,z) = (5x - 2y,\ -x - 2z)$

5) Given $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}$, determine the L.T.

$T: V_3(\mathbb{R}) \to V_2(\mathbb{R})$ relative to the basis

$B_1 = \{(1,1,1), (1,2,3), (1,0,0)\}$

$B_2 = \{(1,1), (1,-1)\}$

Soln: $T(1,1,1) = 1(1,1) + 3(1,-1) = (4,-2)$

$T(1,2,3) = -1(1,1) + 1(1,-1) = (0,-2)$

$T(1,0,0) = 2(1,1) + 0(1,-1) = (2,2)$

Let $(x,y,z) = \alpha_1(1,1,1) + \alpha_2(1,2,3) + \alpha_3(1,0,0)$

$x = \alpha_1 + \alpha_2 + \alpha_3$

$y = \alpha_1 + 2\alpha_2$

$z = \alpha_1 + 3\alpha_2$

$y - z = -\alpha_2$

$\boxed{-y + z = \alpha_2}$

$\therefore \alpha_1 = z - 3\alpha_2$

$= z - 3(-y+z)$

$= z + 3y - 3z$

$\boxed{\alpha_1 = 3y - 2z}$

$\therefore \alpha_3 = x - \alpha_1 - \alpha_2$

$= x - 3y + 2z + y - z$

$\boxed{\alpha_3 = x - 2y + z}$

20

$$\therefore T(x,y,z) = \alpha_1 T(1,1,1) + \alpha_2 T(1,2,3) + \alpha_3 T(1,0,0)$$

$$= (3y-2z)(4,-2) + (z-y)(0,-2) + (x-2y+z)(2,2)$$

$$= (12y-8z+2x-4y+2z,\ -6y+4z-2z+2y+2x-4y+2z)$$

$$= (8y+2x-6z,\ 2x-8y+4z)$$

Verify Rank-Nullity theorem for the linear transformation (21)

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by

$T(x,y,z) = (x+y,\ x-y,\ x+2z)$

Soln: $N(T) = \{(x,y,z) \mid T(x,y,z) = (0,0,0)\}$

$= \{(x,y,z) \mid (x+y,\ x-y,\ x+2z) = (0,0,0)\}$

$= \{(0,0,0)\}$

$N(T)$ contains only zero vector. $\therefore$ there are no basis vectors in $N(T)$.

$\therefore$ Nullity $(T) = 0$

$R(T) = \{T(x,y,z) \mid (x,y,z) \in \mathbb{R}^3\}$

$= \{(x+y,\ x-y,\ x+2z) \mid (x,y,z) \in \mathbb{R}^3\}$

Standard basis for $\mathbb{R}^3$. $\{(1,0,0),\ (0,1,0),\ (0,0,1)\}$

$T(1,0,0) = (1,1,1)$

$T(0,1,0) = (1,-1,0)$

$T(0,0,1) = (0,0,2)$

Now $\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & -1 & 2 \end{bmatrix}$

$\xrightarrow{R_3 \to 2R_3 - R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

All the columns are pivot columns. Hence $\{(1,1,1),$

$(1,-1,0)$, $(0,0,2)\}$ are linearly independent & they span.

$\therefore$ They form a basis.

$\therefore$ $\dim R(T) = \text{Rank}(T) = 3$

$\therefore$ $\text{Rank}(T) + \text{Nullity}(T) = 3 + 0 = \dim \mathbb{R}^3$.