8. A random variable X has the density function (19)

$$f(x) = \begin{cases} kx^2, & -3 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

a) Find k b) Find $P(1 \le X \le 2)$ c) $P(X \le 2)$ d) $P(X > 1)$

e) $E(X)$ f) $V(X)$

Soln: a) In order that $f(x)$ may be a probability density function, the two conditions to be satisfied are

$f(x) \geq 0$ & $\int_{-\infty}^{\infty} f(x)\,dx = 1$. The given function satisfies the first condition if $k \geq 0$. The second condition is satisfied if $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

i.e. $\int_{-\infty}^{-3} f(x)\,dx + \int_{-3}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_{-3}^{3} kx^2\,dx = 1$$

$$\Rightarrow k\left.\frac{x^3}{3}\right|_{-3}^{3} = 1 \quad \Rightarrow \frac{k}{3}(27+27) = 1$$

$$\Rightarrow k(18) = 1$$

$$\Rightarrow k = \frac{1}{18}$$

b) $P(1 \le X \le 2) = \int_{1}^{2} f(x)\,dx$

$$= \int_{1}^{2} \frac{1}{18} x^2\,dx$$

$$= \frac{1}{18}\left(\frac{x^3}{3}\right)\Big|_{1}^{2} = \frac{1}{54}(7) = 0.1296$$

c) $P(X \leq 2) = \int_{-\infty}^{2} f(x)\, dx$

$= \int_{-3}^{2} f(x)\, dx$

$= \int_{-3}^{2} \frac{1}{18} x^2\, dx$

$= \frac{1}{18}\left(\frac{x^3}{3}\right)\Big|_{-3}^{2}$

$= \frac{1}{54}(8+27) = \frac{35}{54} = \underline{\underline{0.6481}}$

d) $P(X>1) = 1 - P(X \leq 1)$

$= 1 - \int_{-\infty}^{1} f(x)\, dx$

$= 1 - \int_{-3}^{1} f(x)\, dx$

$= 1 - \int_{-3}^{1} \frac{1}{18} x^2\, dx$

$= 1 - \frac{1}{18}\left(\frac{x^3}{3}\right)\Big|_{-3}^{1}$

$= \frac{26}{54} = 0.48148$

e) $E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$

$= \int_{-3}^{3} x \frac{1}{18} x^2\, dx$

$= \frac{1}{18}\left[\frac{x^4}{4}\right]_{-3}^{3} = \frac{1}{54}\left[81 - 81\right] = 0$

f) $V(X) = E(X^2) - [E(X)]^2$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_{-3}^{3} x^2 \frac{1}{18} x^2\, dx$$

$$= \frac{1}{18} \int_{-3}^{3} x^4\, dx$$

$$= \frac{1}{18} \left[\frac{x^5}{5} \right]_{-3}^{3}$$

$$= \frac{1}{90} \left[243 - (-243) \right] = \underline{\underline{5.4}}$$

$\therefore V(X) = \underline{\underline{5.4}}$

cdf problems:

1] Let X be a continuous random variable with pdf given by
$$f(x)=\begin{cases} ax, & 0\le x<1 \\ a, & 1\le x<2 \\ -ax+3a, & 2\le x<3 \\ 0, & \text{elsewhere}\end{cases}$$

1] Find a ? 2] Also determine the cdf $F(x)$.

Soln:
$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{0}^{1} ax\,dx + \int_{1}^{2} a\,dx + \int_{2}^{3} (-ax+3a)\,dx = 1$$

$$\left.\frac{ax^2}{2}\right|_0^1 + \left.ax\right|_1^2 + \left.\left(-\frac{ax^2}{2}+3ax\right)\right|_2^3 = 1$$

$$\frac{a}{2} + a + \left(-\frac{9a}{2} + 9a + 2a - 6a\right) = 1$$

$$\frac{3a}{2} + \cancel{\frac{11a}{2}} - \frac{9a}{2} + 5a = 1$$

$$\frac{3a - 9a + 10a}{2} = 1$$

$$\Rightarrow a = \frac{1}{2}$$

cdf:
$$F(x) = \int_{-\infty}^{x} f(s)\,ds$$

For $x < 0$,

$$F(x) = \int_{-\infty}^{x} 0 \, ds = 0$$

For $0 \le x < 1$, $F(x) = \int_{-\infty}^{x} f(s)\, ds$

$$= \int_{-\infty}^{0} 0 \, ds + \int_{0}^{x} as \, ds$$

$$= a \frac{s^2}{2} \Big|_0^x = \left(\frac{1}{2}\right)\left(\frac{x^2}{2}\right) = \frac{x^2}{4}$$

For $1 \le x < 2$, $F(x) = \int_{-\infty}^{x} f(s)\, ds$

$$= \int_{-\infty}^{0} 0 \, ds + \int_{0}^{1} \frac{1}{2} s \, ds + \int_{1}^{x} \frac{1}{2} \, ds$$

$$= \frac{1}{2} \frac{s^2}{2} \Big|_0^1 + \frac{1}{2} s \Big|_1^x$$

$$= \frac{1}{4} + \frac{1}{2}(x-1) = \frac{1+2x-2}{4} = \frac{2x-1}{4}$$

For $2 \le x < 3$, $F(x) = \int_{-\infty}^{x} f(s)\, ds$

$$= \int_{-\infty}^{0} 0 \, ds + \int_{0}^{1} \frac{s}{2} \, ds + \int_{1}^{2} \frac{1}{2} \, ds + \int_{2}^{x} -\frac{s}{2} + \frac{3}{2} \, ds$$

$$= \frac{s^2}{4} \Big|_0^1 + \frac{1}{2} s \Big|_1^2 + \left(-\frac{s^2}{4} + \frac{3}{2} s\right) \Big|_2^x$$

$$= \frac{1}{4} + \frac{1}{2} - \frac{x^2}{4} + \frac{3x}{2} + 1 - 3 = -\frac{5}{4} - \frac{x^2}{4} + \frac{3x}{2}$$

$$F(x) = \sum_j p(x_j) \quad \forall x_j \leq x$$

$$F(0) = \frac{1}{3}$$

$$F(1) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

$$F(2) = \frac{1}{3} + \frac{1}{6} + \frac{1}{2} = 1$$

3] The diameter of an electric cable say X is assumed to be a continuous random variable with pdf $f(x) = 6x(1-x)$, $0 < x < 1$. Check whether $f(x)$ is a valid pdf or not. Also find cdf.

Soln: Observe that $f(x) > 0$, $0 < x < 1$.

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\int_0^1 6x(1-x)\, dx = 1$$

$$\Rightarrow \int_0^1 6x - 6x^2\, dx = 1$$

$$\left. 6\frac{x^2}{2} - 6\frac{x^3}{3} \right|_0^1 = 1$$

$$3 - 2 = 1$$

$\therefore$ It is a valid pdf.

cdf: For $x \leq 0$, $F(x) = 0$

~~For $0 < x < 1$~~, ~~$F(x) = \int_0^x 6x(1-x)\, d$~~

For $x \geq 3$,

$$F(x) = \int_{-\infty}^{x} f(s)\,ds$$

$$= \int_{-\infty}^{0} 0\,ds + \int_{0}^{1} \frac{s}{2}\,ds + \int_{1}^{2} \frac{1}{2}\,ds + \int_{2}^{3} -\frac{s}{2} + \frac{3}{2}\,ds + \int_{3}^{x} 0\,ds.$$

$$= \frac{s^2}{4}\Big|_0^1 + \frac{1}{2}s\Big|_1^2 + \left(-\frac{s^2}{4} + \frac{3}{2}s\right)\Big|_2^3$$

$$= \frac{1}{4} + 1 - \frac{1}{2} - \frac{9}{4} + \frac{9}{2} + 1 - 3$$

$$= \underline{\underline{1}}$$

$$\therefore F(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{4}, & 0 \leq x < 1 \\ \frac{2x-1}{4}, & 1 \leq x < 2 \\ -\frac{5}{4} - \frac{x^2}{4} + \frac{3x}{2} & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

Q] Suppose the random variable X assumes three values 0, 1, 2 with probability $\frac{1}{3}, \frac{1}{6}, \frac{1}{2}$ respectively. Obtain the cdf of X.

Soln:

X	0	1	2
$P(x)$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$

For $0 < x < 1$, $F(x) = \int_{-\infty}^{x} f(s)\,ds$

$$= \int_{-\infty}^{0} 0 + \int_{0}^{x} 6s(1-s)\,ds$$

$$= \int_{0}^{x} 6s - 6s^2\,ds$$

$$= 6\frac{s^2}{2} - 6\frac{s^3}{3}\Big|_{0}^{x}$$

$$= 3x^2 - 2x^3$$

For $x \geq 1$, $F(x) = \int_{-\infty}^{x} f(s)\,ds$

$$= \int_{-\infty}^{0} 0\,ds + \int_{0}^{1} 6s - 6s^2\,ds + \int_{1}^{x} 0\,ds$$

$$= 6\frac{s^2}{2} - \frac{6s^3}{3}\Big|_{0}^{1} = 3 - 2 = \underline{1}$$

$$\therefore F(x) = \begin{cases} 0, & x \leq 0 \\ 3x^2 - 2x^3, & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

Two & higher dimensional random variables:

Def: Let E be an experiment & S is a sample space associated with E. Let $X = X(s)$, & $Y = Y(s)$ be two functions each assigning a real number to each outcome $s \in S$. We call (X, Y) a two dimensional r.v.

If $X = X_1(s)$, $X_2 = X_2(s)$, ---- $X_n = X_n(s)$ over n functions each assigning a real number to every outcome $s \in S$, we call $(X_1, X_2, \ldots X_n)$ an n-dimensional r.v.

Range space of (X, Y) is denoted by R_{XY} which is the set of all possible values of (X, Y).

Def: (X, Y) is a two-dimensional discrete r.v., if the possible values of (X, Y) are finite or countably infinite. i.e., the possible values of (X, Y) may be represented as (x_i, y_j) $i = 1, 2, \ldots n, \ldots$ $j = 1, 2, \ldots m, \ldots$

Def: Let (X, Y) be a two dimensional discrete r.v. With each possible outcome (x_i, y_j) we associate a number $p(x_i, y_j)$ representing $P(X = x_i, Y = y_j)$ & satisfying the following conditions:

1] $p(x_i, y_j) \geq 0 \quad \forall (x_i, y_j)$

2] $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} p(x_i, y_j) = 1$

The function p defined for all (x_i, y_j) in the range space of (X, Y) is called the joint probability function of (X, Y). The set of triples $(x_i, y_j, p(x_i, y_j))$ $i, j = 1, 2, \ldots$ is called

The probability distribution of (X, Y).

The cumulative distribution function (cdf) F of the two dimensional r.v. is defined by

$$F(x, y) = P(X \le x, Y \le y).$$

Def: Let (X, Y) be a two dimensional discrete r.v. & pdf of (X, Y) is known. Suppose that $p(x_i) = P(X = x_i) = \sum_{j=1}^{\infty} p(x_i, y_j)$. The function p defined for $x_1, x_2 - -$ represents the marginal probability distribution of X (or marginal pdf of X).

Analogously, we define $q(y_j) = P(Y = y_j) = \sum_{i=1}^{\infty} p(x_i, y_j)$ as marginal pdf of Y.

Def: Let (X, Y) be a two dimensional discrete r.v. We say that X & Y are independent r.v. if and only if $p(x_i, y_j) = p(x_i)\, q(y_j)$ $\forall i, j$.

i.e. $P(X = x_i, Y = y_j) = P(X = x_i)\, P(Y = y_j)$ $\forall i, j$.

Def: Let (X, Y) be a discrete r.v. Let $p(x_i), q(y_j)$ be the marginal pdf's of X & Y respectively.

The conditional pdf of $X = x_i$ for given $Y = y_j$ is defined by $p(x_i | y_j) = \dfrac{p(x_i, y_j)}{q(y_j)}$

The conditional pdf of $Y = y_j$ for given $X = x_i$ is defined by ~~p(x_i)~~ $p(y_j | x_i) = \dfrac{p(x_i, y_j)}{p(x_i)}$

Let (X,Y) be a discrete r.v.

$$E(X) = \sum_j \sum_i x_i \, p(x_i, y_j)$$

$$= \sum_i x_i \, p(x_i)$$

$$E(Y) = \sum_j \sum_i y_j \, p(x_i, y_j)$$

$$= \cancel{\sum_j y_j \, p(x_i, y_j)}$$

$$= \sum_j y_j \, q(y_j)$$

$$E(XY) = \sum_j \sum_i x_i y_j \, p(x_i, y_j)$$ ~~where (X, Y)~~

<u>Def:</u> (X, Y) is a two dimensional continuous r.v. if (X, Y) can assume all values in some noncountable set of the Euclidean plane.

<u>Ex:</u> (X, Y) assume all values in the rectangle

$\{(x, y) : a \le x \le b, \; c \le y \le d\}$.

<u>Def:</u> Let (X, Y) be a continuous r.v. assuming all values in some region R of the Euclidean plane. The joint probability density function f (joint pdf) is a function satisfying the following conditions:

1] $f(x, y) \geq 0 \quad \forall (x, y) \in R$

2] $\iint_R f(x, y) \, dx \, dy = 1$

<u>Def:</u> Let f be the joint pdf of the continuous two dimensional r.v. (X, Y). We define g & h, the marginal probability density functions of X & Y

respectively as follows:

$$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

$$h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

Def; Let (X, Y) be a continuous r.v. with joint pdf f. Let g & h be the marginal pdf's of X & Y respectively. The conditional pdf of X for given $Y=y$ is defined by $g(x|y) = \dfrac{f(x,y)}{h(y)}$, $h(y) > 0$

The conditional pdf of Y for given $X = x$ is defined by $h(y|x) = \dfrac{f(x,y)}{g(x)}$, $g(x) > 0$.

Def: Let (X, Y) be a two dimensional continuous r.v.. We say that X & Y are independent if and only if $f(x,y) = g(x)\, h(y) \;\forall\; (x, y)$

Let (X, Y) be a continuous r.v.

$$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x,y)\, dx\, dy = \int_{-\infty}^{\infty} x\, g(x)\, dx$$

$$E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(x,y)\, dx\, dy = \int_{-\infty}^{\infty} y\, h(y)\, dy$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$$

1) A joint distribution of (X, Y) is given by the following table. Determine i) marginal distributions of X & Y. ii) Test for independence of X & Y.
iii) $E(X), E(Y), E(XY)$ iv) σ_X, σ_Y {i.e standard deviation of X & Y)

X \ Y	-4	2	7
1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
5	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$

Soln: Marginal distribution of X: $p(x_i) = \sum_{j=1}^{\infty} p(x_i, y_j)$

$$p(x_1) = p(1) = \sum_{j=1}^{3} p(x_1, y_j)$$
$$= p(x_1, y_1) + p(x_1, y_2) + p(x_1, y_3)$$
$$= \frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{1}{2}$$

$$p(x_2) = p(5) = \sum_{j=1}^{3} p(x_2, y_j)$$
$$= p(x_2, y_1) + p(x_2, y_2) + p(x_2, y_3)$$
$$= \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

Marginal distribution of X:

X	1	5
$p(x_i)$	$\frac{1}{2}$	$\frac{1}{2}$

Marginal distribution of Y: $q(y_j) = \sum_{i=1}^{\infty} p(x_i, y_j)$

$$q(y_1) = \sum_{i=1}^{2} p(x_i, y_1) = p(x_1, y_1) + p(x_2, y_1)$$
$$= \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$$

$$q(y_2) = \sum_{i=1}^{2} p(x_i, y_2) = p(x_1, y_2) + p(x_2, y_2)$$
$$= \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$$

$$q(y_3) = \sum_{i=1}^{2} p(x_i, y_3) = p(x_1, y_3) + p(x_2, y_3)$$
$$= \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

Marginal distribution of Y:

Y	−4	2	7
$q(y_j)$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{4}$

ii) (X, Y) are independent $\Leftrightarrow p(x_i, y_j) = p(x_i)\, q(y_j) \ \forall i, j$

$p(x_1, y_1) = \frac{1}{8}$

$p(x_1)\, q(y_1) = \frac{1}{2} \times \frac{3}{8} = \frac{3}{16}$

$p(x_1, y_1) \neq p(x_1)\, q(y_1)$

X & Y are not independent

iii) ~~Cov(X,Y) = E(XY) − E(X)E(Y)~~

$$E(X) = \sum_{i=1}^{2} x_i p(x_i)$$
$$= (1)\left(\frac{1}{2}\right) + 5\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{5}{2} = 3$$

$$E(Y) = \sum_{j=1}^{3} y_j q(y_j)$$

$$= (-4)\left(\frac{3}{8}\right) + (2)\left(\frac{3}{8}\right) + 7\left(\frac{1}{4}\right)$$

$$= \frac{-12}{8} + \frac{6}{8} + \frac{7}{4} = \frac{-12+6+14}{8} = 1$$

$$E(XY) = \sum_j \sum_i x_i y_j p(x_i, y_j)$$

$$= (1)(-4)\left(\frac{1}{8}\right) + (1)(2)\left(\frac{1}{4}\right) + (1)(7)\left(\frac{1}{8}\right) +$$

$$(5)(-4)\left(\frac{1}{4}\right) + (5)(2)\left(\frac{1}{8}\right) + (5)(7)\left(\frac{1}{8}\right)$$

$$= -\frac{1}{2} + \frac{1}{2} + \frac{7}{8} - 5 + \frac{10}{8} + \frac{35}{8}$$

$$= \frac{3}{2}$$

$$\cancel{Cov(X,Y) = E(XY) - E(X)E(Y)}$$

$$\cancel{= \frac{3}{2} - 3 = -\frac{3}{2}}$$

iv) $V(X) = E(X^2) - [E(X)]^2$

$$E(X^2) = \sum_{i=1}^{2} x_i^2 p(x_i) = (1)^2\left(\frac{1}{2}\right) + (5)^2\left(\frac{1}{2}\right) = 13$$

$$\therefore V(X) = 13 - (3)^2 = 4$$

$$\sigma_X = \sqrt{4} = 2$$

$$V(Y) = E(Y^2) - [E(Y)]^2$$

$$E(Y^2) = \sum_{j=1}^{3} y_j^2 q(y_j) = (-4)^2\left(\frac{3}{8}\right) + (2)^2\left(\frac{3}{8}\right) + (7)^2\left(\frac{1}{4}\right)$$

$$= \frac{48}{8} + \frac{12}{8} + \frac{49}{4} = \frac{60+98}{8} = \frac{158}{8}$$

$$= 19.75$$

$V(Y) = 19.75 - 1 = 18.75$

$\sigma_Y = \sqrt{18.75} = 4.3301$

2] Let X & Y are independent random variables. X takes values $2, 5, 7$ with probability $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$ respectively. Y takes values $3, 4, 5$ with prob. $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$. Find the joint pdf of X & Y.

Soln:

X	2	5	7
$p(x_i)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

Y	3	4	5
$q(y_j)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

X & Y are independent r.v. $\Rightarrow$

$p(x_i, y_j) = p(x_i)\, q(y_j) \quad \forall i, j$

X \ Y	3	4	5
2	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
5	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$
7	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

3] Following table represents the joint pdf of two dimensional discrete r.v. (X, Y).

Y \ X	1	2	3
1	$\frac{1}{12}$	$\frac{1}{6}$	0
2	0	$\frac{1}{9}$	$\frac{1}{5}$
3	$\frac{1}{18}$	$\frac{1}{4}$	$\frac{2}{15}$

i) Obtain the marginal distribution functions of X & Y.
ii) Test for independence.
iii) Find $P(X=2 | Y=2)$.

Soln: i) Marginal pdf of X: $p(x_i) = \sum_{j=1}^{\infty} p(x_i, y_j)$

X	1	2	3
$p(x_i)$	$\frac{5}{36}$	$\frac{19}{36}$	$\frac{1}{3}$

Marginal pdf of Y: $q(y_j) = \sum_{i=1}^{\infty} p(x_i, y_j)$

Y	1	2	3
$q(y_j)$	$\frac{1}{4}$	$\frac{14}{45}$	$\frac{79}{180}$

ii) $p(x_1, y_1) = \frac{1}{12}$

$p(x_1)\, q(y_1) = \frac{5}{36} \times \frac{1}{4} = \frac{5}{144}$

$p(x_1, y_1) \neq p(x_1)\, q(y_1)$

X & Y are not independent.

iii) $P(X=2 | Y=2) = \frac{P(X=2, Y=2)}{q(2)} = \frac{\frac{1}{9}}{14/45} = \frac{5}{14}$

4) If X & Y are independent r.v. then S.T.

i) $E(XY) = E(X)\,E(Y)$

ii) $V(X+Y) = V(X) + V(Y)$

Soln: i) Case 1: X & Y are discrete r.v.

$$E(XY) = \sum_j \sum_i x_i y_j\, p(x_i, y_j)$$

$$= \sum_j \sum_i x_i y_j\, p(x_i)\, q(y_j) \quad (\because X \& Y \text{ are independent})$$

$$= \sum_i x_i\, p(x_i) \sum_j y_j\, q(y_j)$$

$$= E(X)\, E(Y)$$

Case 2: X & Y are continuous r.v.

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, g(x)\, h(y)\, dx\, dy \quad (\because X, Y \text{ are independent})$$

$$= \int_{-\infty}^{\infty} x\, g(x)\, dx \int_{-\infty}^{\infty} y\, h(y)\, dy$$

$$= E(X)\, E(Y)$$

ii) $V(X+Y) = E[(X+Y)^2] - [E(X+Y)]^2$

$$= E[X^2+Y^2+2XY] - [E(X)+E(Y)]^2$$

$$= E(X^2) + E(Y^2) + 2E(XY) - [(E(X))^2 + (E(Y))^2 + 2E(X)E(Y)]$$

$$= E(X^2) + E(Y^2) + 2E(X)E(Y) - (E(X))^2 - (E(Y))^2 - 2E(X)E(Y)$$

$$V(X+Y) = E(X^2) - (E(X))^2 + E(Y^2) - (E(Y))^2$$
$$= V(X) + V(Y)$$

5] A coin is tossed three times. Let X denote 0 or 1 according as a tail or a head occurs on the first toss. Let Y denote the total no. of tails which occur. Determine

i) the marginal distributions of X & Y.

ii) the joint distribution of X & Y

iii) E(X+Y).

Soln)

S	HHH	HHT	HTH	THH	HTT	TTH	THT	TTT
X	1	1	1	0	1	0	0	0
Y	0	1	1	1.	2	2	2	3

The marginal distributions of X

X	0	1
$p(x_i)$	$\frac{4}{8}$	$\frac{4}{8}$

The marginal distributions of Y

Y	0	1.	2	3
$q(y_j)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

The joint distribution of X & Y is given by

X \ Y	0	1	2	3
0	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0

$$E(X+Y) = \sum_j \sum_i (x_i + y_j)\, p(x_i, y_j)$$

$$= \sum_j \sum_i \left(x_i\, p(x_i, y_j) + y_j\, p(x_i, y_j) \right)$$

$$= \sum_j \sum_i x_i\, p(x_i, y_j) + \sum_j \sum_i y_j\, p(x_i, y_j)$$

$$= \sum_i x_i\, p(x_i) + \sum_j y_j\, q(y_j)$$

$$= \frac{4}{8} + \frac{3}{8} + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right)$$

$$= 2$$