6. Suppose that the joint pdf of the two dimensional r.v. (X, Y) is given by

$$f(x,y) = \begin{cases} x^2 + \frac{xy}{3}, & 0 < x < 1, \quad 0 < y < 2 \\ 0, & \text{elsewhere} \end{cases}$$

Compute i) $P(X > \frac{1}{2})$ ii) $P(Y < X)$ iii) $P(X < \frac{1}{2}, Y < \frac{1}{2})$
iv) $P(X+Y \geq 1)$ v) marginal pdf of X & Y
vi) Check for independence vii) $E(XY)$, $E(X)$, $E(Y)$

i). $P(X > \frac{1}{2}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy$

$$= \int_{y=0}^{2} \int_{x=1/2}^{1} \left(x^2 + \frac{xy}{3}\right) dx\, dy$$

$$= \int_0^2 \frac{x^3}{3} + \frac{y}{3}\left(\frac{x^2}{2}\right) \Big|_{1/2}^{1} dy$$

$$= \int_0^2 \left(\frac{1}{3} + \frac{y}{6}\right) - \left(\frac{1}{24} + \frac{y}{24}\right) dy$$

$$= \int_0^2 \left(\frac{y}{6} - \frac{y}{24} + \frac{7}{24}\right) dy$$

$$= \int_0^2 \left(\frac{3y}{24} + \frac{7}{24}\right) dy$$

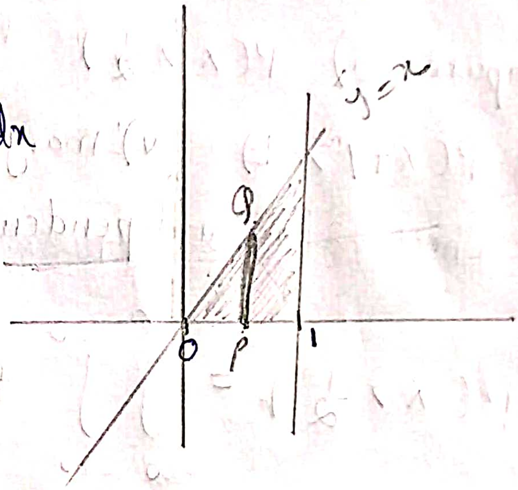
$$= \left[\frac{3}{24}\frac{y^2}{2} + \frac{7}{24}y\right]_0^2$$

$$P\left(X > \tfrac{1}{2}\right) = \frac{3}{48} \times 4 + \frac{7}{24} \times 2$$

$$= \frac{3}{12} + \frac{7}{12}$$

$$= \frac{10}{12} = \underline{\underline{\frac{5}{6}}}$$

ii) $$P(Y < X) = \int_{x=0}^{1} \int_{y=0}^{x} \left(x^2 + \frac{xy}{3}\right) dy\, dx$$



$$= \int_0^1 x^2 y + \frac{x}{3}\left(\frac{y^2}{2}\right)\Big|_0^x \, dx$$

$$= \int_0^1 x^3 + \frac{x^3}{6} \, \cancel{dy}\, dx$$

$$= \frac{x^4}{4} + \frac{x^4}{24}\Big|_0^1$$

$$= \frac{1}{4} + \frac{1}{24} = \underline{\underline{\frac{7}{24}}}$$

iii) $$P\left(X < \tfrac{1}{2}, Y < \tfrac{1}{2}\right) = \int_0^{1/2}\int_0^{1/2} \left(x^2 + \frac{xy}{3}\right) dx\, dy$$

$$= \int_0^{1/2} \frac{x^3}{3} + \frac{y}{3}\left(\frac{x^2}{2}\right)\Big|_0^{1/2} dy$$

$$= \int_0^{1/2} \left[\frac{1}{24} + \frac{y}{6}\left(\frac{1}{4}\right)\right] dy$$

$$P\left(X<\tfrac{1}{2}, Y<\tfrac{1}{2}\right) = \int_0^{1/2}\left(\frac{1}{24}+\frac{y}{24}\right)dy$$

$$= \frac{y}{24}+\frac{y^2}{48}\Bigg|_0^{1/2}$$

$$= \frac{1}{48}+\frac{1}{192}$$

$$= \frac{5}{192}$$

iv) $P(X+Y \geq 1) = 1 - P(X+Y<1)$

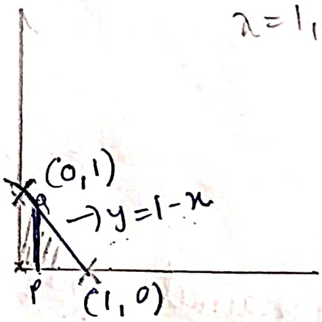
$x+y=1$

$y=1-x$

when $x=0, y=1$

$x=1, y=0$

$$= 1 - \int_{x=0}^{1}\int_{y=0}^{1-x}\left(x^2+\frac{xy}{3}\right)dy\,dx$$



$$= 1 - \int_0^1 \left[x^2y+\frac{x}{3}\left(\frac{y^2}{2}\right)\right]_0^{1-x} dx$$

$$= 1 - \int_0^1 x^2(1-x)+\frac{x}{6}(1-x)^2\,dx$$

$$= 1 - \int_0^1 x^2-x^3+\frac{1}{6}\left(x+x^3-2x^2\right)dx$$

$$= 1 - \left[\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^2}{12}+\frac{x^4}{24}-\frac{2x^3}{18}\right]_0^1$$

$$= 1 - \left[\frac{1}{3}-\frac{1}{4}+\frac{1}{12}+\frac{1}{24}-\frac{1}{9}\right]$$

$$P(X+Y\geq 1) = 1-\left[\frac{24-18+6+3-8}{72}\right] = 1-\frac{7}{72} = \frac{65}{72}$$

v) Marginal pdf of $X = g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$$= \int_0^2 \left(x^2+\frac{xy}{3}\right)dy$$

$$= x^2y+\frac{x}{3}\left(\frac{y^2}{2}\right)\Big|_0^2$$

$$= x^2 2+\frac{x}{6}(4)$$

$$= 2x^2+\frac{2x}{3}$$

Marginal pdf of $Y = h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

$$= \int_0^1 \left(x^2+\frac{xy}{3}\right)dx$$

$$= \frac{x^3}{3}+\frac{y}{3}\left(\frac{x^2}{2}\right)\Big|_0^1$$

$$= \frac{1}{3}+\frac{y}{6}$$

vi) X & Y are independent $\Leftrightarrow f(x,y) = g(x)h(y)$

$$x^2+\frac{xy}{3} \neq \left(2x^2+\frac{2x}{3}\right)\left(\frac{1}{3}+\frac{y}{6}\right)$$

$\therefore$ X & Y are not independent.

vi) $$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x, y)\, dx\, dy$$

$$= \int_0^2 \int_0^1 xy\left(x^2 + \frac{xy}{3}\right) dx\, dy$$

$$= \int_0^2 \left[\int_0^1 \left(x^3 y + \frac{x^2 y^2}{3}\right) dx\right] dy$$

$$= \int_0^2 y\, \cancel{x^3}\, \frac{x^4}{4} + \frac{y^2}{3}\left(\frac{x^3}{3}\right)\Big|_0^1 dy$$

$$= \int_0^2 \frac{y}{4} + \frac{y^2}{9}\, dy$$

$$= \frac{y^2}{8} + \frac{y^3}{27}\Big|_0^2$$

$$= \frac{4}{8} + \frac{8}{27} = \frac{27+16}{54} = \underline{\underline{\frac{43}{54}}}$$

$$E(X) = \int_{-\infty}^{\infty} x\, g(x)\, dx$$

$$= \int_0^1 x\left(2x^2 + \frac{2x}{3}\right) dx$$

$$= \int_0^1 2x^3 + \frac{2x^2}{3}\, dx$$

$$E(X) = \frac{x^4}{2} + \frac{2x^3}{9} \Big|_0^1$$

$$= \frac{1}{2} + \frac{2}{9} = \frac{9+\cancel{2}4}{18} = \underline{\underline{\frac{13}{18}}}$$

$$E(Y) = \int_{-\infty}^{\infty} y\, h(y)\, dy$$

$$= \int_0^2 y\left(\frac{1}{3} + \frac{y}{6}\right) dy$$

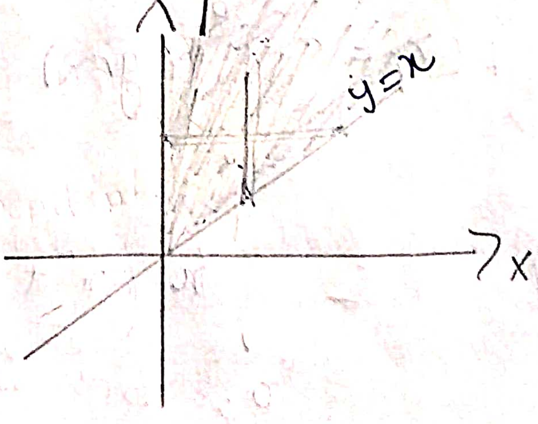
$$= \frac{y^2}{6} + \frac{y^3}{18} \Big|_0^2$$

$$= \frac{4}{6} + \frac{8}{18} = \frac{12+8}{18} = \underline{\underline{\frac{20}{18}}}$$

7] Suppose that the joint pdf of (X, Y) is given by

$$f(x, y) = \begin{cases} e^{-y}, & x > 0,\ y > x \\ 0, & \text{elsewhere.} \end{cases}$$

Find marginal pdf of X & Y. Verify that X & Y are not independent.



$$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

$$= \int_{y=x}^{\infty} e^{-y}\, dy$$

$$g(x) = \frac{e^{-y}}{-1}\Big|_{x}^{\infty}$$

$$= \frac{0 - e^{-x}}{-1}$$

$$= \underline{\underline{e^{-x}}}$$

$$h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

$$= \int_{0}^{y} e^{-y}\, dx$$

$$= e^{-y} x \Big|_{0}^{y}$$

$$= e^{-y} y$$

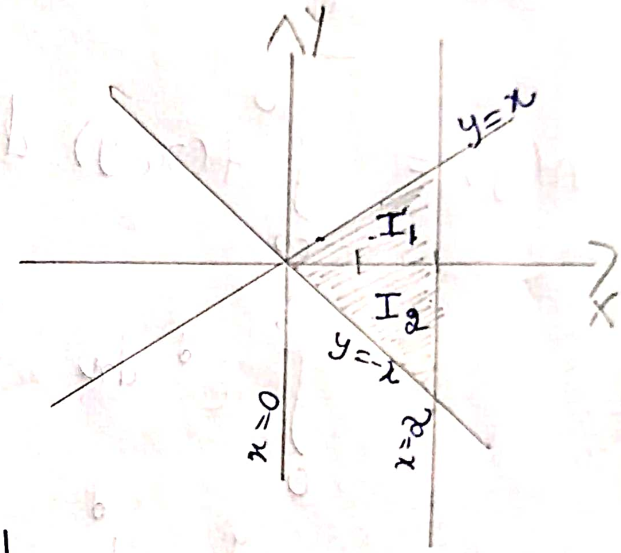
$$g(x)h(y) = e^{-x} y e^{-y} \neq e^{-y} = f(x, y)$$

$\therefore$ X & Y are not independent.

8. Suppose that the two dimensional r.v. (X, Y) has joint probability distribution function

$$f(x,y) = \begin{cases} kx(x-y), & 0 < x < 2, \quad -x < y < x \\ 0, & \text{elsewhere} \end{cases}$$

i) Find k ii) Find the marginal pdf of X & Y.



Soln:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$$

$$\int_{x=0}^{2}\int_{y=-x}^{x} kx(x-y)\,dy\,dx = 1$$

$$\int_{0}^{2}\left[\int_{-x}^{x}\left(kx^2 - kxy\right)dy\right]dx = 1$$

$$\int_{0}^{2} kx^2y - kx\frac{y^2}{2}\Big|_{-x}^{x}\,dx = 1$$

$$\int_{0}^{2} kx^2(x+x) - \frac{kx}{2}\left(x^2 - x^2\right)\,dx = 1$$

$$\int_{0}^{2} 2kx^3\,dx = 1$$

$$2k\frac{x^4}{4}\Big|_{0}^{2} = 1$$

$$k(8) = 1 \Rightarrow k = \frac{1}{8}$$

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{-x}^{x} \frac{1}{8} x(x-y)\,dy$$

$$= \frac{1}{8}\int_{-x}^{x} x^2 - xy\,dy$$

$$= \frac{1}{8}\left[x^2 y - x\frac{y^2}{2}\right]_{-x}^{x}$$

$$= \frac{1}{8}\left[x^2(2x) - \frac{x}{2}(0)\right] = \underline{\underline{\frac{x^3}{4}}}$$

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

In I_1: $$h(y) = \int_{y}^{2} \frac{1}{8}(x^2 - xy)\,dx$$

$$= \frac{1}{8}\left[\frac{x^3}{3} - \left(y\frac{x^2}{2}\right)\right]_{y}^{2}$$

$$= \frac{1}{8}\left[\frac{8}{3} - \frac{y^3}{3} - \frac{y}{2}(4 - y^2)\right]$$

$$= \frac{1}{8}\left[\frac{8}{3} - \frac{y^3}{3} - 2y + \frac{y^3}{2}\right]$$

$$= \frac{1}{3} - \frac{y^3}{24} - \frac{y}{4} + \frac{y^3}{16}$$

$$= \frac{1}{3} + y^3\left(-\frac{1}{24} + \frac{1}{16}\right) - \frac{y}{4} = \frac{1}{3} + y^3\left(\frac{-2+3}{48}\right) - \frac{y}{4}$$

$$= \frac{1}{3} + \frac{y^3}{48} - \frac{y}{4}$$

In I_2:

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

$$= \int_{-y}^{2} \frac{1}{8}(x^2 - xy)\, dx$$

$$= \frac{1}{8}\left[\frac{x^3}{3} - y\frac{x^2}{2}\right]_{-y}^{2}$$

$$= \frac{1}{8}\left[\frac{8-(-y)^3}{3} - \frac{y}{2}\left(4 - y^2\right)\right]$$

$$= \frac{1}{8}\left[\frac{8+y^3}{3} - 2y + \frac{y^3}{2}\right]$$

$$= \frac{1}{8}\left[\frac{8}{3} + \frac{y^3}{3} - 2y + \frac{y^3}{2}\right]$$

$$= \frac{1}{3} + \frac{y^3}{24} - \frac{y}{4} + \frac{y^3}{16}$$

$$= \frac{1}{3} - \frac{y}{4} + y^3\left(\frac{1}{24} + \frac{1}{16}\right)$$

$$= \frac{1}{3} - \frac{y}{4} + y^3\left(\frac{5}{48}\right)$$

9] (X, Y) have the following joint distribution.

$$f(x,y) = \begin{cases} 4xy, & 0 < x < 1, \quad 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

i) Find $P(0 \le X \le \frac{1}{2}, \frac{1}{8} \le Y \le \frac{1}{4})$ ii) $P(Y > X)$

i) $P(0 \le X \le \frac{1}{2},\ \frac{1}{8} \le Y \le \frac{1}{4})$

$$= \int_{1/8}^{1/4} \left[\int_{0}^{1/2} 4xy \, dx \right] dy$$

$$= \int_{1/8}^{1/4} 4y \left. \frac{x^2}{2} \right|_{0}^{1/2} dy$$

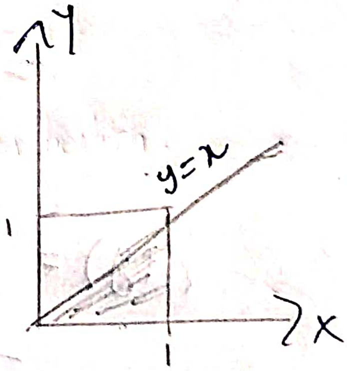
$$= \int_{1/8}^{1/4} 2y \left(\frac{1}{4} \right) dy$$

$$= \frac{1}{2} \left. \frac{y^2}{2} \right|_{1/8}^{1/4}$$

$$= \frac{\frac{1}{16} - \frac{1}{64}}{4}$$

$$= \frac{\frac{3}{64}}{4} = \underline{\underline{\frac{3}{256}}}$$

ii) $P(Y > X) = 1 - P(Y \le X)$



$$= 1 - \int_{x=0}^{1} \int_{y=0}^{x} 4xy \, dy \, dx$$

$$= 1 - \int_{0}^{1} 4x \left. \frac{y^2}{2} \right|_{0}^{x} dx$$

$$= 1 - \int_{0}^{1} 2x(x^2) \, dx = 1 - \int_{0}^{1} 2x^3 \, dx$$

$$P(Y > X) = 1 - 2\frac{x^4}{4}\Big|_0^1$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

10) $$f(x, y) = \begin{cases} 8xy, & 0 < x < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

i) Find the marginal pdf of X & Y.

ii) Examine for independence.

Soln: $$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

$$= \int_x^1 8xy\, dy$$

$$= 8x \frac{y^2}{2}\Big|_x^1$$

$$= 4x(1 - x^2)$$

$$h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

$$= \int_0^y 8xy\, dx$$

$$= 8y \frac{x^2}{2}\Big|_0^y = 4y^3$$

ii) $f(x, y) \neq g(x)\, h(y)$

$\therefore$ X, Y are not independent.

H.W

1] Two random variables X & Y have joint pdf

$$f(x, y) = \begin{cases} \frac{1}{3}(4-x-y), & 1 < x < 2, \quad 0 < y < 2 \\ 0, & \text{elsewhere} \end{cases}$$

Compute $E(X)$, $E(Y)$, $E(XY)$.

2] Suppose that the joint pdf of (X, Y) is given by

$$f(x, y) = \begin{cases} k e^{-y}, & 0 < x < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

i) Find k

ii) Find marginal pdf of X, Y.

3]

X \ Y	-2	-1	1	2
-1	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{16}$
0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
1	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{16}$

i) Find the marginal pdf of X & Y.

ii) Check for independence