PROBABILITY THEORY AND NUMERICAL METHODS

90 - 100 → A
80 - 90 → A

UNIT 3: PROBABILITY THEORY

Finite sample space, conditional probability and independence, Bayes' theorem (overview).

IMP

One dimensional random variable:

60 students → 35

discrete and continuous random variable,

100 students

probability functions,

cumulative distribution function, expectation and variance.

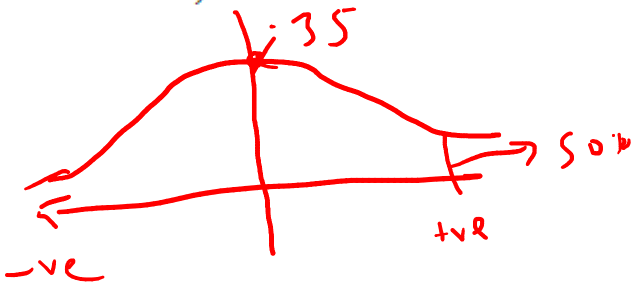
Maths: 2013 → 50 Marks

Two dimensional random variable, covariance and correlation coefficient.

A

Distributions:

Binomial,
Poisson,
Normal
exponential



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By,
Dr. Vasanth K R,
Assistant Professor Gd-III
Dept. Of Mathematics
NMAM Institute of Technology

Prerequisites :

coin → toss = (H or T)

Exhaustive event: an event consisting of all the various possibilities is called an exhaustive event

Mutually exclusive event: two or more events are said to be mutually exclusive if the happening of one event prevent the simultaneous happening of the other.

Examples;

In tossing a coin, getting head and tail are mutually exclusive inview of the fact that if head is the turn out getting tail is not possible.

Independent events: two or more events are said to be independent if the happening or non happening of one event does not prevent the happening or non happening of others.

Example: when two coins are tossed the event of getting head is an independent event as both the coins can turn out head.

Mathematical definition of probability

If the outcome of a trial consists n exhaustive, mutually exclusive, equally possible cases, of which m of them are favourable cases to an event E, then the probability of the happening of the event E, usually denoted by P(E) or simply p is defined to be equal to m/n

i.e, $P(E) = \dfrac{\textit{number of favourable cases}}{\textit{number of possible cases}} = \dfrac{m}{n}$

$P(E) = \dfrac{m}{n}$ → fav →

The probability of the non happening of the event usually denoted by q is given by

$q = 1 - p$ or $p + q = 1$

$\boxed{p + q = 1}$

p is also referred as the probability of success, and q as the probability of failure. Their sum is always equal to one.

If P (E) = 1, E is called the sure event and if P (E) = 0, E is called an impossible event.

$\boxed{P(E) \leq 1}$

VVIP

Coin Tossed
Prob of getting H

$P(H) = \frac{1}{2}$

$P(T) = \frac{1}{2}$ H or T

Example: Two fair coins are tossed. What is the probability of getting one heads and one tails?

Answer: For a fair or unbiased coin, for each toss of each coin

P[heads] =P[tails] = ½

Addition theorem of probability

The probability of the happening of the one or the other mutually exclusive events is equal to the sum of the probabilities of the two events

That is, if A,B are two mutually exclusive events then

$\boxed{P\ (A \text{ or } B) = P\ (A) + P(B)}$

Product theorem of probability

If a compound event is made up of independent events, the probability of the happening of the compound event is equal to the product of the probabilities of the independent events.

ie, if A and B are independent events then $\boxed{P\ (A \text{ and } B) = P\ (A)\ .\ P\ (B)}$

Conditional Probability: (Required for Baye's Theorem)

We write condition probability as $P(A|B)$ Probability of occurrence of event A when event B has already occurred

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Already occured ✓

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$P(A \cap B) = P(A).P(A|B)$ were A and B are dependent event,

Be careful.

Also we can write like

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$P(A \cap B) = P(B).P(B|A)$ were A and B are dependent event,

[Also, $P(A \cap B) = P(A).P(B)$ If A and B are independent events]

And

* $P(A|B) = \frac{P(A \cap B)}{P(B)}$

* $P(A \cap B) = P(A).P(B)$ Provided A & B are independent

* $P(A \cap B) = P(A).P(A|B)$

or $= P(B) P(B|A)$ Provided A & B are dependent

1. In a certain college 25% failed in maths, 15% failed in chemistry, 10% failed in both. The student is selected at random.

   a. If a student failed in chemistry then what is the probability that he has failed in maths.
   b. If a student failed in maths then what is the probability that he has failed in chemistry.
   c. What is the probability that a student failed in maths or chemistry
   d. What is the probability that a student failed in neither chemistry nor maths.

Given :-

$P(M) = 25\% = 0.25$

$P(C) = 15\% = 0.15$

$P(M \cap C) = 10\% = 0.10$

⇒ Let $M \rightarrow$ No. of Students failed in Maths

$C \rightarrow$ No. of Students faild in chem

(i) $P(M/C) = \dfrac{P(M \cap C)}{P(C)} = \dfrac{0.10}{0.15} = 0.67$

(ii) $P(C/M) = \dfrac{P(M \cap C)}{P(M)} = \dfrac{0.1}{0.25} = 0.4$

(iii) $P(M \cup C) = P(M) + P(C) - P(M \cap C)$

$= 0.5 + 0.15 - 0.1$

$= 0.30$

(iv) $P(\overline{M \cup C}) = 1 - P(M \cup C)$

$= 1 - 0.3$

$= 0.7$

$$P(A \cap B) = P(B) \cdot P(A/B)$$
$$= P(A) \cdot P(B/A)$$
$\rightarrow$ events are dependent

$$P(A \cap B) = P(A) \cdot P(B)$$
$\rightarrow$ events are independent

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$\rightarrow$ Conditional Probability

Bayes' Theorem

VVVIP *

$i = 1, 2, 3, 4 \dots n$

Statement: Let $A_1, A_2, A_3 \dots A_n$ be a set of exhaustive and mutually exclusive event $P(A_i) \neq 0$ for each i. Let A be any event associated with $A_i's$ such that $P(A) \neq 0$ then A is contained in the union of $A_i's$ $(A \subseteq \cup A_i's)$ then the probability of event A_i is given by,

$A_i \to$ set of

A

$$P(A_i|A) = \frac{P(A_i).P(A|A_i)}{\sum_{i=1}^{n} P(A_i)P(A|A_i)} \text{ where } i = 1,2,3, \dots n$$

$P(A_i|A) = \frac{P(A_i \cap A)}{P(A)}$

$\subseteq \to$ contained

$P(A) \neq 0$

Proof:

We have $S = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n$ Since $(A \subseteq \cup A_i's)$

$A \subseteq \cup (A_i)$

$A = S \cap A$

$A = (A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n) \cap A$

$A = (A_1 \cap A) \cup (A_2 \cap A) \cup \dots \cup (A_n \cap A)$

$P(A) = P(A_1 \cap A) + P(A_2 \cap A) + \dots + P(A_n \cap A)$ [i.e. $A_i \cap A$ are all mutually exclusive event for every i]

$$P(A) = P(A_1).P(A|A_1) + P(A_2).P(A|A_2) + \cdots + P(A_n).P(A|A_n)$$

$S = A_1 \cup A_2 \cup A_3 \ldots \cup A_n$ $[A \subseteq \cup A_i]$

For any i, the probability of A_i given A is written as,

$$P(A_i|A) = \frac{P(A_i \cap A)}{P(A)} \text{ (Using conditional probability)}$$

$A = S \cap A$

$A = [A_1 \cup A_2 \cup A_3 \ldots \cup A_n] \cap A$

$$P(A_i|A) = \frac{P(A_i).P(A|A_i)}{P(A_1).P(A|A_1) + P(A_2).P(A|A_2) + \cdots + P(A_n).P(A|A_n)}$$

$A = (A_1 \cap A) \cup (A_2 \cap A) \cup \ldots (A_n \cap A)$

$P(A) = P(A_1 \cap A) + P(A_2 \cap A) + \ldots + P(A_n \cap A)$

$$P(A_i|A) = \frac{P(A_i).P(A|A_i)}{\sum_{i=1}^{n} P(A_i)P(A|A_i)}$$

$P(A) = P(A_1).P(A|A_1) + P(A_2).P(A|A_2) + \ldots + P(A_n).P(A|A_n)$

$= \sum_{i=1}^{n} P(A_i).P(A|A_i)$

1. An office has 4 secretaries handling respectively 20%, 60%, 15%, 5% of the file of all government reports. The probability that they mis-files such reports are respectively 0.05, 0.10, 0.10, and 0.05. Find the probability that a misfiled report can be ~~blamed~~ blamed on the first secretary.

⇒ Let A : {misfiled Reports}

Let A_1, A_2, A_3 & A_4 be the 4 Secretaries Respectively

$P(A_1) = 20\% = 0.2$

$P(A_2) = 60\% = 0.6$

$P(A_3) = 15\% = 0.15$

$P(A_4) = 5\% = 0.05$

$P(A/A_1) = 0.05$

$P(A|A_2) = 0.1$

$P(A|A_3) = 0.1$

$P(A|A_4) = 0.05$

To find Probabili

To find Probability that the mis-filed report can be blamed on 1st Secretary (ie A_1)

$P(A_1/A) = ?$

We have Baye's theorem

$$P(A_i|A) = \frac{P(A_i).P(A|A_i)}{\sum_{i=1}^{n} P(A_i).P(A|A_i)}$$

$$P(A_1|A) = \frac{P(A_1).P(A|A_1)}{P(A_1).P(A|A_1) + P(A_2).P(A|A_2) + P(A_3).P(A|A_3) + P}$$

2. Of the three men, the chances that a politician, a business man and an academician will be appointed as VC of VTU are 0.50, 0.30 & 0.20 respectively. Probabilities that research is promoted by these people if they are appointed as VC are 0.3, 0.7, and 0.8 respectively. If research is promoted in the VTU, what is the probability that the VC is an academician?

⇒ $A = \{$ Promoting of Research $\}$

Let A_1, A_2, A_3 be chances that a Politician, A Businessman & an academician be appointed as VC respectively.

$P(A_1) = 0.5$ $\qquad P(A|A_1) = 0.3$

$P(A_2) = 0.3$ $\qquad P(A|A_2) = 0.7$

$P(A_3) = 0.2$ $\qquad P(A|A_3) = 0.8$

To find the Probability that the VC is an Academician

i.e $P(A_3|A) = \dfrac{P(A_3) \cdot P(A|A_3)}{P(A_1) \cdot P(A|A_1) + P(A_2) P(A|A_2) + P(A_3) P(A|A_3)}$

$= 0.3076$

3. In a certain college 25% of boys and 10% of girls are studying mathematics. The girls constitute 60% of the student boys. If a student is selected at random and is found to be studying mathematics, find the probability that the student is a girl? a boy?

Let $A = \{\text{Student Studying Maths}\}$

Let A_1 & A_2 be resp boys & girls respectively

then $P(A_1) = \frac{40}{100} = 0.4$; $P(A_2) = \frac{60}{100} = 0.6$

$P(A|A_2) = 0.10$

$P(A|A_1) = 0.25$

To find: 1) $\underbrace{P(A_1/A)}_{\text{Boy}}$ 2) $\underbrace{P(A_2|A)}_{\text{girl}}$

1) $P(A_1|A) = \frac{P(A_1) \cdot P(A|A_1)}{P(A_1) \cdot P(A|A_1) + P(A_2) \cdot P(A|A_2)} = \frac{(0.4)(0.25)}{(0.4)(0.25) + (0.6)(0.1)} = 0.625$

$\boxed{2)\ P(A_2|A) = 0.375}$

4. Three bags X, Y, Z contain respectively 6 green, 4red ; 2 green, 6 red ; 1green, 8 red balls. Two balls are drawn from one of the bag. If both the balls are found to be red. What is the probability that it came from bag Y.

Let A_1, A_2, A_3 be bags X, Y, Z respectively ; $A = \{\text{Selecting 2 Red Balls}\}$

$A_1 = X$

6 - G

4 - R

$P(A_1) = \frac{1}{3}$; $P(A_2) = \frac{1}{3}$; $P(A_3) = \frac{1}{3}$

4 → Total Red

2 → balls to select

$$P(A/A_1) = \frac{{}^{4}C_2}{{}^{10}C_2} = \frac{6}{45} = 0.133$$

10 ← Total balls

2 ← Balls to select

$$P(A/A_2) = \frac{{}^{6}C_2}{{}^{8}C_2} = 0.5357 = \frac{15}{28}$$

IIIly $P(A|A_3) = \frac{{}^{8}C_2}{{}^{9}C_2} = 0.777$

we have to find $P(A_2|A) = ??$

$$P(A_2|A) = \frac{P(A_2) \cdot P(A|A_2)}{P(A_1) \cdot P(A|A_1) + P(A_2)P(A|A_2) + P(A_3) \cdot P(A|A_3)}$$

$$\simeq 0.3702$$

5. In a factory there are 4 machines A, B, C and D manufacturing respectively 20%, 15%, 25%, 40% of the total output. Among those output 5%, 4%, 3% and 2% are defective outputs. An item is chosen at random and it's found to be defective. What is the probability that this item was manufactured by machine A or machine D.

Let $A = \{$ The defective Product $\}$

Let A_1, A_2, A_3, A_4 be ~~produc~~ respectively A, B, C & D

$P(A_1) = 20\% = 0.2$; $P(A_2) = 0.15$; $P(A_3) = 0.25$; $P(A_4) = 0.4$

$P(A|A_1) = 0.05$; $P(A|A_2) = 0.04$; $P(A|A_3) = 0.03$; $P(A|A_4) = 0.02$

To find : $P(A_1 \cup A_4 / A) = P(A_1|A) + P(A_4|A)$

$$= \frac{P(A_1)P(A|A_1) + P(A_4).P(A|A_4)}{P(A_1)P(A|A_1) + P(A_2)P(A|A_2) + P(A_3)P(A|A_3) + P(A_4)P(A|A_4)}$$

$= 0.5173$

Random variables – introduction

A random variable is a function that assigned a real number to every sample point in the sample space of a random experiment.

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Real No.

In other words it is a function from the sample space S to the set of all real numbers; denoted as $\boxed{X : \boldsymbol{S} \to \boldsymbol{R}}$.

IMP

Space

Random

For example, consider the random experiment of tossing three fair coins up.

Then S = {HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}.

Define X as the number of heads that appear.

Hence, $X(HHH) = \mathbf{3}$, $X(HHT) = \mathbf{2}$, $X(HTH) = \mathbf{2}$, $X(THH) = \mathbf{2}$, $X(HTT) = \mathbf{1}$,

$X(THT) = \mathbf{1}$, $X(TTH) = \mathbf{1}$ and $X(TTT) = \mathbf{0}$.

$X = \{3, 2, 2, 2, 1, 1, 0\}$

Random Variable

Discrete and continuous random variables:

If a random variable takes finite ~~or countably infinite~~ number of values then it is called a discrete random variable.

Example 1

a) Tossing a coin and observing the outcome.

b) Tossing coins and observing the number of heads turning up.

If a random variable takes ~~non countable~~ infinite number of values then it is called a non discrete or continuous random variable.

Example 2

a) Weight of articles.

b) Length of nails produced by a machine.

c) No. of tallest Trees in the world.

d) No. of stars in sky

Probability function and discrete probability distribution

$X = \{1, 2, 3 \dots n\}$

x_i

If for each value x_i of a discrete random variable X, we assign a real number p (x_i) such that

$$\boxed{(i)\ P(x_i) \geq 0 \quad (ii) \sum_i p(x_i) = 1}$$

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Then the function p(x) is called a probability function.

X	0	1
P(x)	1/2	1/2

x_1 x_2

$P(x_i\text{'s}) \geq 0$ $\quad \sum P(x_i) = 1$

If the probability that X take the values x_i is p_i, then P (X = x_i) = p_i or p (x_i)

The set of values [x_i, p(x_i)] is called a discrete (finite) probability distribution of the discrete random variable X. the function p(X) is called the probability density function (p.d.f) or the probability mass function (p.m.f)

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The distribution function f(x) defined by

f(x) = P(X ≤ x) = $\sum_{i=1}^{x} p(x_i)$, *x being an integer is called the* cumulative distribution function (c.d.f)

$f(x) = P(X \leq x) = \sum_{i=1} P(x_i)$, where x is a integer is called as c.d.f

Mean or expectation of a random variable

If X is a discrete random variable with possible values $x_1, x_2, x_3 \dots x_n$ and $P(x_i)$ as its probability, then Mean or expectation of X is

$$Mean, \xleftarrow{VVIP} \mu = E(X) = \sum_{i=1}^{n} \boxed{x_i P(x_i)}$$

X	0	1	2
$P(x)$	$1/3$	$1/3$	0

Find mean

$$\mu = \sum_{i=1}^{n} x_i P(x_i)$$

$$= 0 \times 1/3 + 1 \times 1/3 + 2 \times 0$$

$$= \boxed{\mu = 2/3}$$

If X is a random variable then $\boxed{Variance\ \ of\ X}$ is $\boxed{V(X) = E\big(X - E(X)\big)^2}$

OR

Remember $\boxed{V(x) = E(x^2) - [E(x)]^2}$

SD $\boxed{\sigma = \sqrt{V}}$

Prove That $V(X) = E(X^2) - [E(X)]^2$.

$\Rightarrow$ WKT $V(x) = E[X - E(x)]^2$

$$= E[X^2 - 2X E(x) + (E(x))^2]$$

$$= E(x^2) - 2E(x) \cdot E(x) + \underbrace{E[E(x)]^2}$$

$$= E(x^2) - 2[E(x)]^2 + (E(x))^2$$

$$\boxed{V(x) = E(x^2) - [E(x)]^2}$$

Prove That $E(aX+b)=aE(X)+b$

$$E(x)=\sum_{i=1}^{n} x_i\,P(x_i)$$

$\Rightarrow$ WKT $E(x)=\sum_{i=1}^{n} x_i\,P(x_i)$

$$E(ax+b)=\sum_{i=1}^{n}(ax_i+b)\,P(x_i)$$

$$P(x=x_i)=P(x_i)$$

$$=\sum_{i=1}^{n} a x_i P(x_i)+\sum_{i=1}^{n} b P(x_i)$$

$$=a\underbrace{\sum_{i=1}^{n} x_i P(x_i)}_{E(x)}+b\underbrace{\sum_{i=1}^{n} P(x_i)}_{1}$$

$\left\{ \because \sum_{i=1}^{n} P(x_i)=1 \text{ \& } P(x_i)\geq 0 \right\} \rightarrow$ def^n of p.d.f

$$=a.E(x)+b(1)$$

$$\therefore \boxed{E(ax+b)=aE(x)+b}$$

$$V(aX+b)=a^2V(X)$$

$\Rightarrow$ W.K.T $V(X)=E[X-E(X)]^2$

$V(aX+b)=E[aX+b-E(aX+b)]^2$

$=E[aX+b-(aE(X)+b)]^2$

$=E[aX+\cancel{b}-aE(X)-\cancel{b}]^2$

$=E[aX-aE(X)]^2$

$=E[a(X-E(X))]^2$

$=a^2.E[X-E(X)]^2=a^2V(X)$

$p(x_i)\geq 0$
$\sum p(x_i)=1$ $\Big\}\rightarrow$ pdf

$\mu=\sum_{i=1}^{n}x_i\,p(x_i)$

$V(X)=E(X-E(X))^2$

$V(X)=E(X^2)-[E(X)]^2$

$\hookrightarrow$ PT

* Random variable
* Discrete
* Continuous

1) The probability distribution of a random variable is given by following table.

X	-2	-1	0	1	2	3
P(X)	0.1	k	0.2	2k	0.3	k

(i) Find the value of k. Also find the mean and variance. (ii) Find $P(X > -1)$ (iii) P(X<0) (iv) $P(-2 \le X \le 2)$

$p(x_i) \ge 0$ & $\sum p(x_i) = 1$ } condn for prob dist fun (PDF)

⇒ WKT pdf is given by condns

$p(x_i) \ge 0$ & $\sum p(x_i) = 1$

(i) ∴ $0.1 + k + 0.2 + 2k + 0.3 + k = 1$

$4k + 0.6 = 1$

$4k = 1 - 0.6$

$k = \frac{0.4}{4} = 0.1$

$\boxed{k = 0.1}$

Mean $\mu = \boxed{\sum x_i p(x_i) = E(x)}$

$V = E(x^2) - [E(x)]^2$

How to calculate ??

$E(x) = \sum x_i p(x_i)$

$E(x^2) = \sum x_i^2 p(x_i)$

mean $= \mu = E(x) = \sum x_i p(x_i)$

x	-2	-1	0	1	2	3
$P(x)$	0.1	0.1	0.2	0.2	0.3	0.1
$xP(x)$	-0.2	-0.1	0	0.2	0.6	0.3
$x^2P(x)$	0.4	+0.1	0	0.2	1.2	0.9

$\Rightarrow \sum x_i P(x_i) = 0.8$

$\Rightarrow \sum x^2 P(x) = 2.8$

$(-2)^2 \cdot (0.1)$

Mean $\mu = E(x) = \sum x_i P(x_i)$

$\boxed{\therefore E(x) = \mu = 0.8}$

$V(x) = E(x^2) - [E(x)]^2$??

$\hookrightarrow$ where

$E(x^2) = \sum x_i^2 P(x_i)$

$\therefore V(x) = 2.8 - (0.8)^2$

$\boxed{V(x) = 2.16}$

$SD = \sqrt{V}$

$= \sqrt{2.16}$

2) $P(x > -1) = P(0) + P(1) + P(2) + P(3)$

$= 0.2 + 0.2 + 0.3 + 0.1$

$= 0.8$

OR

$P(x > -1) = 1 - P(x \le -1)$??

$= 1 - [P(-2) + P(-1)]$

$= 1 - [0.1 + 0.1] = 0.8$

3) $P(X<0) = P(X=-2) + P(X=-1)$

$= 0.1 + 0.1$

$= 0.2$ //

4) $P(-2 \leq X \leq 2) = P(-2) + P(-1) + P(0) + P(1) + P(2)$

$= 0.1 + 0.1 + 0.2 + 0.2 + 0.3$

$= 0.9$ //

X	0	1	2	3	4	5	6
P(X)	k	3k	5k	7k	9k	11k	13k
OR	1/49	3/49	5/49	7/49	9/49	11/49	13/49

For what value of k, does this represent a valid probability distribution? Also find $P(X<4)$, $P(X\geq 5)$ and $P(3<X\leq 6)$

$\sum P(x_i) = 1$

$\Rightarrow \boxed{K = 1/49}$

$P(x<4) = P(0)+P(1)+P(2)+P(3)$

$\boxed{= \frac{16}{49}}$

$P(x\geq 5) = P(5)+P(6)$

$\boxed{= 24/49}$

$P(3<X\leq 6) =$

$P(4)+P(5)+P(6)$

$= \frac{33}{49}$

3) A fair coin is tossed 3 times. Let X denote the no. of heads obtained. Find the distribution of X. Also find the mean and Variance.

⇒ A fair coin is tossed 3 times, ∴ possible outcomes are

$S = \{HHH, TTT, HTT, THH, HHT, TTH, HTH, THT\}$

Let X denote the no. of heads obtained

$X = \{0, 1, 2, 3\}$

X	0	1	2	3
$P(x)$	$1/8$	$3/8$	$3/8$	$1/8$

$\frac{\text{Possib}}{\text{Total}} \to \frac{1}{8}$

Mean $\mu = E(x) = \sum x_i P(x_i)$

$= \left(0 \times \frac{1}{8}\right) + \left(1 \times 3/8\right) + \left(2 \times 3/8\right) + \left(3 \times \frac{1}{8}\right)$

$= \frac{12}{8} = 1.5$

$V(x) = E(x^2) - [E(x)]^2$

$= \left[\left(0^2 \times 1/8\right) + \left(1^2 \times 3/8\right) + \left(2^2 \times 3/8\right) + \left(3^2 \times 1/8\right)\right] - (1.5)^2$

$= 0.75$

4) A box contains 12 items of which 4 are defective. A sample of 3 items are selected from the box.

a. Find the p.d.f of X (3 Marks)   b. Find mean and SD

⇒ Let X = {No. of defective items selected}

∴ X can take the values = {0, 1, 2, 3}

X	0	1	2	3
P(X)	0.2545	0.5090	0.2181	0.0181

$$P(X=0) = \frac{^{8}C_3}{^{12}C_3} = 0.2545$$

favorable → $^{8}C_3$ → 12 − 4

no. of possible → $^{12}C_3$

$$P(X=1) = \frac{^{8}C_2 \times {^{4}C_1}}{^{12}C_3} = 0.5090$$

$$P(X=2) = \frac{^{8}C_1 \times {^{4}C_2}}{^{12}C_3} = 0.2181$$

$$P(X=3) = \frac{^{4}C_3}{^{12}C_3} = 0.0181$$

$$\text{mean} = \mu = E(x) = \sum x_i \, P(x_i)$$

$$= \left(x_0 \cdot P(x_0)\right) + \left(x_1 \cdot P(x_1)\right) + \left(x_2 \cdot P(x_2)\right) + \left(x_3 \cdot P(x_3)\right)$$

$$= (0 \times 0.2545) + (1 \times 0.5090) + (2 \times 0.2181) + (3 \times 0.0181)$$

$$E(x) = 0.999$$

$$\text{Variance } V(x) = E(x^2) - [E(x)]^2$$

$$= \left\{\left(x_0^2 \, P(x_0)\right) + \left(x_1^2 \cdot P(x_1)\right) + \left(x_2^2 \, P(x_2)\right) + \left(x_3^2 \, P(x_3)\right)\right\} - [E(x)]^2$$

$$= \left[(0^2 \times 0.2545) + (1^2 \times 0.5090) + (2^2 \times 0.2181) + (3^2 \times 0.0181)\right] - (0.999)^2$$

$$V(x) = 0.5452$$

$$SD = \sqrt{V}$$

$$= \sqrt{0.5452} =$$

5) Suppose that a random variable X takes values 1, 2, 3, ... and $P(X=j)=\frac{1}{2^j}, j=1, 2, 3, \cdots$

b) ~~a)~~ Compute P(X is even) c) ~~b)~~ Compute $P(X \geq 5)$ d) ~~c)~~ Compute P(X is divisible by 3)

a) check whether it is a valid pdf

$\Rightarrow$ a) a valid pdf means $P(x=j) \geq 0$ & $\sum P(x=j)=1$

Given $p(x=j)=\frac{1}{2^j}$; $j=1,2,3\ldots$ (+ve values) $\therefore P(X=j) \geq 0$

$$\left[S_n = \frac{a}{1-r} \right]$$

$$\sum p(x_j s) = \frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots$$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

$$r = \frac{\frac{1}{2^2}}{\frac{1}{2}} = \frac{1}{2^2} \times 2 = \frac{1}{2}$$

$\therefore$ It is a valid pdf

b) $P(x \text{ is even}) = P(x=2) + P(x=4) + P(x=6) + \cdots$

$$= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots$$

$$= \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}$$

$$S_n = \frac{a}{1-r}$$

c) $P(x \geq 5) = P(x=5) + P(x=6) + P(x=7) + \cdots$

$$= \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} + \cdots$$

$$= \frac{\frac{1}{2^5}}{1-\frac{1}{2}} = \frac{1}{16}$$

d) $P(x \text{ divisible by } 3)$

$$= P(x=3) + P(x=6) + P(x=9) + \cdots$$

$$= \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \frac{1}{2^{12}} + \cdots$$

$$= \frac{\frac{1}{2^3}}{1-\frac{1}{2^3}} = \frac{1}{7}$$

6) Suppose that a Random variable X assumes values 0,1,2 with the probabilities 1/3, 1/6, ½ respectively. Obtain the C.D.F of X

$\Rightarrow$

X	0	1	2
$P(x)$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$

c.d.f of X is given by

$$F(x) = P(X \leq x) = \sum_{x < x_i} p(x_i)$$

for $x < 0 \Rightarrow F(x) = 0$

for $x < 1 \Rightarrow F(x) = P(0) = \frac{1}{3}$

for $x < 2 \Rightarrow F(x) = \sum_{x<2} p(x) = P(0) + P(1) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$

for $x < \infty \Rightarrow F(x) = P(0) + P(1) + P(2) = \frac{1}{3} + \frac{1}{6} + \frac{1}{2} = 1$

The cumulative dist fn is

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{3} & \text{for } x < 1 \\ \frac{1}{2} & \text{for } x < 2 \\ 1 & \text{for } x < \infty \end{cases}$$

defn; $V(x) = E(x^2) - [E(x)]^2$

$E(ax+b) = aE(x)+b$

$V(ax+b) = a^2 V(x)$

Continuous probability Distribution :

Continous Random Variable :- Infinitely uncountable values $\int_{-\infty}^{\infty}$

Definition: If for every x belonging to the range of a continuous random variable X, we assign a real number f(x) satisfying the conditions

i) $f(x) \geq 0$ **(ii)** $\int_{-\infty}^{\infty} f(x)dx = 1$

In case of discrete
i) $p(x_i) \geq 0$ & ii) $\sum p(x_i) = 1$
$\hookrightarrow i = 1, 2, \ldots n$

Then f(x) is called a continuous probability function or probability density function (p.d.f)

If (a, b) is a subinterval of the range space of X then the probability that x lies in (a, b) is defined to be the integral of f(x) between a and b. That is ,

$$P(a \leq x \leq b) = \int_{a}^{b} f(x)\, dx$$

The Cumulative Distribution Function

Definition : The cumulative distribution function (CDF) of a random variable X is, $F_X(x) = \Pr(X \leq x)$.

Mean and Variance :

The mean and variance of the continuous probability distribution with pdf p(x), is defined as follows.

Continous

$$\textit{Mean,}\qquad \mu = E(X) = \int_{-\infty}^{\infty} x\,p(x)\,dx$$

In discrete :-

$\mu = \sum x_i\, P(x_i)$

Similarly

$$E(X^2) = \int_{-\infty}^{\infty} x^2\,p(x)\,dx$$

Variance,

$$(or)\ \sigma^2 = E(X^2) - (E(X))^2$$

$$\textit{Standard deviation } (\sigma) = \sqrt{V}$$

In discrete :-

$V(x) = E(x^2) - [E(x)]^2$

$E(x^2) = \sum x_i^2\, P(x_i)$

1) A random variable X has a density function $f(x) = \begin{cases} Kx^2 & ; -3 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$

Find

a) Value of K b) $P(1 \le x \le 2)$ c) $P(x \le 2)$

d) $P(X > 1)$ e) $E(X)$ f) $V(X)$

$\Rightarrow$ a) Find value of K :-

Since it's a Probability density function

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{-3} f(x)\,dx + \int_{-3}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$$

$$0 + \int_{-3}^{3} Kx^2\,dx + 0 = 1$$

$$K\left[\frac{x^3}{3}\right]_{-3}^{3} = 1$$

$$\frac{K}{3}\left[3^3 - (-3)^3\right] = 1$$

$$\frac{K}{3}\left[54\right] = 1$$

$$K = \frac{3}{54} \Rightarrow \boxed{K = \frac{1}{18}}$$ //

b) $P(1 \le x \le 2) = \int_{1}^{2} f(x)\, dx$

$= \int_{1}^{2} Kx^2\, dx$

$= \int_{1}^{2} \frac{1}{18} x^2\, dx$

$= \frac{1}{18} \left[\frac{x^3}{3} \right]_{1}^{2}$

$= \frac{1}{18} \left(\frac{2^3}{3} - \frac{1^3}{3} \right)$

$= 7/54$

c) $P(x \le 2) = \int_{-\infty}^{2} f(x)\, dx$

$= \int_{-\infty}^{-3} f(x)\, dx + \int_{-3}^{2} f(x)\, dx$

$= 0 + \int_{-3}^{2} Kx^2\, dx$

$= \frac{1}{18} \int_{-3}^{2} x^2\, dx$

$= \frac{35}{54}$

d) $P(X>1) = \int_1^{\infty} f(x)\,dx$

$= \int_1^3 f(x)\,dx + \int_3^{\infty} f(x)\,dx$

$= \int_1^3 kx^2\,dx + 0$

$= \frac{1}{18}\int_1^3 x^2\,dx$

$= \frac{26}{54}$ //

e) $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

$= \int_{-3}^{3} x \cdot kx^2\,dx$

$= \frac{1}{18}\int_{-3}^{3} x^3\,dx$

$= \frac{1}{18}\left[\frac{x^4}{4}\right]_{-3}^{3}$

$= \frac{1}{64}\left[3^4 - (-3)^4\right]$

$= \frac{1}{64}\left[3^4 - 3^4\right]$

$= 0$ //

$V(X) = E(X^2) - [E(X)]^2$

$E(X^2) = \int_{-3}^{3} x^2 f(x)\,dx$

$= \int_{-3}^{3} x^2 \cdot \cancel{k}x^2\,dx$

$= \frac{1}{18}\int_{-3}^{3} x^4\,dx$

$= 27/5$

$\therefore V(X) = \frac{27}{5} - 0^2$

$= \frac{27}{5}$ //

1) A random variable x has the density fn $f(x) = \frac{k}{1+x^2}$; $-\infty \le x \le \infty$

Determine k & hence find

a) $P(X \geq 0)$ b) $P(0 < x < 1)$

$\Rightarrow$ WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{\infty} \frac{k}{1+x^2}\,dx = 1$$

$$2\int_{0}^{\infty} \frac{k}{1+x^2}\,dx = 1$$

$$2k.\left[\tan^{-1}x\right]_0^{\infty} = 1$$

$$2k.\left[\tan^{-1}\infty - \tan^{-1}0\right] = 1$$

$$\cancel{2}k.\left[\frac{\pi}{\cancel{2}} - 0\right] = 1$$

$$\boxed{k = \frac{1}{\pi}}$$

//

a) $P(X \geq 0) = \int_{0}^{\infty} f(x)\,dx$

$$= \int_{0}^{\infty} \frac{k}{1+x^2}\,dx =$$

$$= \frac{1}{\pi}\left[\tan^{-1}x\right]_0^{\infty}$$

$$= \frac{1}{\pi}\left[\tan^{-1}\infty - \tan^{-1}0\right]$$

$$= \frac{1}{\cancel{\pi}}\left[\frac{\cancel{\pi}}{2}\right]$$

$$\boxed{= \frac{1}{2}}$$

b) $P(0 < x < 1)$

$$= \int_{0}^{1} f(x)\,dx$$

$$= \int_{0}^{1} \frac{k}{1+x^2}\,dx$$

$$= k.\left[\tan^{-1}x\right]_0^{1}$$

$$= \frac{1}{\pi}\left[\tan^{-1}1 - \tan^{-1}0\right]$$

$$= \frac{1}{\cancel{\pi}} \times \frac{\cancel{\pi}}{4}$$

$$\boxed{= \frac{1}{4}}$$

//

* The diameter of an electric cable $X \to$ Continuous Random Variable whose p.d.f is $f(x) = 6x(1-x)$; $0 \le x \le 1$. Check whether

a) It is a valid pdf

b) c.d.f of X

1) All $f(x) \ge 0$ ~~btw~~ $0 \le x \le 1$

$\Rightarrow$ Since $f(x) = 6x(1-x)$; $0 \le x \le 1$

2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

LHS: $\int_{-\infty}^{\infty} f(x)\,dx$

$= \int_{-\infty}^{\infty} 6(x)(1-x)\,dx$

$= \int_0^1 6[x - x^2]\,dx$

$= 6 \cdot \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$

$= 6\left[\frac{1}{2} - \frac{1}{3}\right]$

$= 6\left[\frac{3-2}{6}\right] = 1$

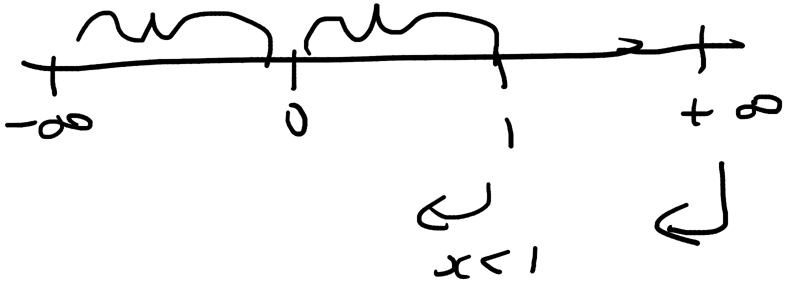
$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$

$\therefore$ The given function is a valid p.d.f

11) c.d.f of x is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$$

$$\left[f(x) = 6x(1-x) \;;\; 0 \leq x \leq 1 \right]$$



we have:

1) $x < 0 \Rightarrow F(x) = \int_{-\infty}^{0} f(x)\,dx = \boxed{0}$

2) $x < 1 \Rightarrow F(x) = \int_{-\infty}^{x} f(x)\,dx$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{x} 6x(1-x)\,dx$$

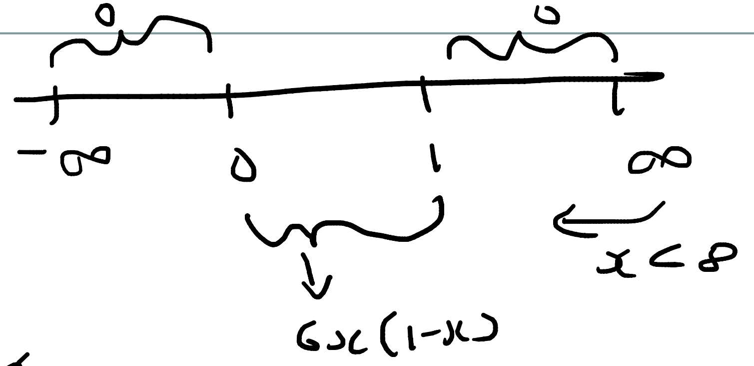
$$= \int_{0}^{x} (6x - 6x^2)\,dx$$

$$= \left[\frac{\cancel{6}^{3} x^2}{\cancel{2}} - \frac{\cancel{6}^{2} x^3}{\cancel{3}} \right]_0^x$$

$$\boxed{F(x) = 3x^2 - 2x^3} \quad \forall\, x < 1$$

3) for $x < \infty$

$$\Rightarrow F(x) = \int_{-\infty}^{x} f(x)\,dx$$



$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{1} 6x(1-x)\,dx + 0$$

$$= 1$$

$\therefore$ c.d.f

$$F(x) = \begin{cases} 0 & ; \quad x < 0 \\ 3x^2 - 2x^3 & ; \quad x < 1 \\ 1 & ; \quad x < \infty \end{cases}$$

* Let x be a continous random variable with p.d.f

$$f(x) = \begin{cases} ax & ; \; 0 \le x \le 1 \\ a & ; \; 1 \le x \le 2 \\ -ax + 3a & ; \; 2 \le x \le 3 \\ 0 & ; \; \text{otherwise} \end{cases}$$

a) Find the value of a

b) Find c.d.f of x

2) $x < 1$

$\Rightarrow$ WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (-ax + 3a)\,dx + \int_3^{\infty} 0\,dx = 1$$

$$\left. \frac{ax^2}{2} \right|_0^1 + \left. ax \right|_1^2 + \left. \left(-\frac{ax^2}{2} + 3ax\right) \right|_2^3 = 1$$

$$\frac{a}{2} + a - \frac{a}{2}(9-4) + 3a(3-2) = 1$$

$$\frac{a}{2} + a - \frac{5a}{2} + 3a = 1$$

$$4a - \frac{4a}{2} = 1$$

$$2a = 1$$

$$\boxed{a = 1/2}$$

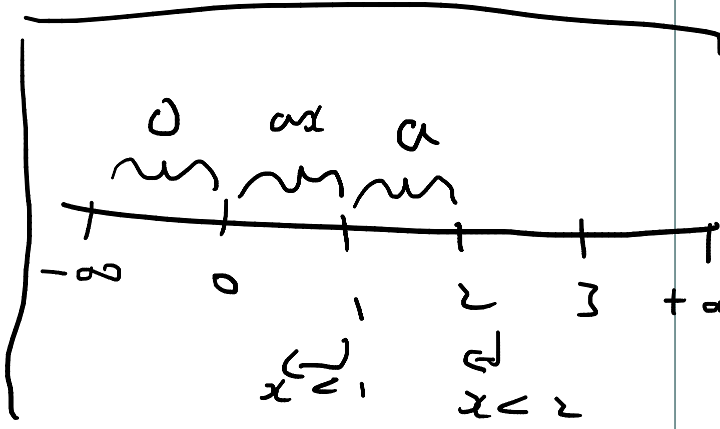
b) c.d.f of X = is $F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$

1) for $x < 0 \Rightarrow \boxed{F(x) = 0}$

2) for $x < 1 \Rightarrow F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$

$$= 0 + \int_{0}^{x} ax\,dx$$

$$\boxed{= \frac{ax^2}{2}} = \frac{x^2}{4}$$



3) for $x < 2$

$\Rightarrow F(x) = \int_{-\infty}^{x} f(x)\,dx$

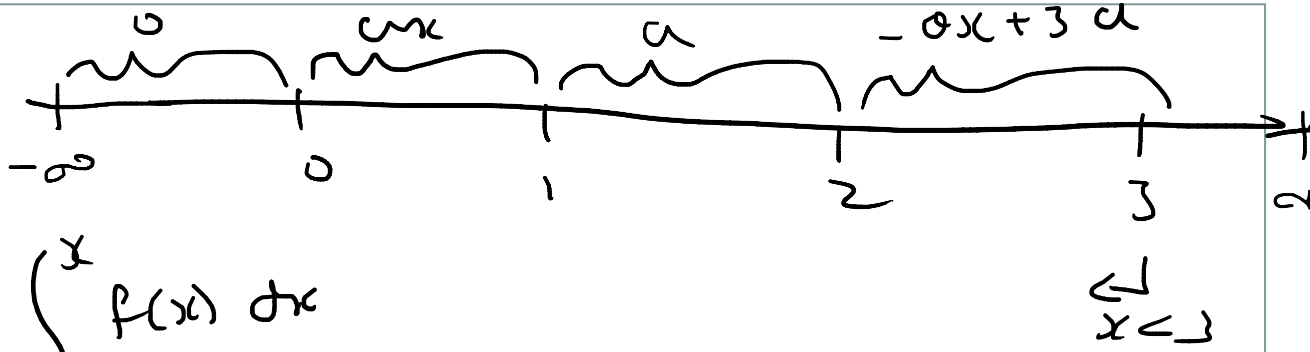
$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} ax\,dx + \int_{1}^{x} a\,dx$$

$$= 0 + \frac{ax^2}{2}\Big|_{0}^{1} + ax\Big|_{1}^{x}$$

$$\boxed{F(x) = \frac{1}{4} + \frac{1}{2}(x-1)} \text{ for } x < 2$$

4) $x < 3$



$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{1} ax\,dx + \int_{1}^{2} a\,dx + \int_{2}^{x} (-ax + 3a)\,dx$$

$$= \frac{1}{2}\frac{x^2}{2}\Big|_0^1 + \cancel{a}\,\frac{1}{2}x\Big|_1^2 + \left(-\frac{1}{2}\frac{x^2}{2} + \frac{3}{2}x\right]_2^x$$

$$= \frac{1}{4} + \frac{1}{2}(2-1) - \frac{1}{4}(x^2 - 2^2) + \frac{3}{2}(x-2)$$

$$= \frac{1}{4} + \frac{1}{2} - \frac{x^2}{4} + 1 + \frac{3x}{2} - 3$$

$$F(x) = \frac{3x}{2} - \frac{x^2}{4} - 5/4$$

for $x < 3$

3) For $x < \infty$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx$$

$$= 0 + \int_{0}^{1} ax\,dx + \int_{1}^{2} a\,dx + \int_{2}^{3} -ax+3a + 0$$

$$= \frac{1}{2}\frac{x^2}{2}\Big|_0^1 + \frac{1}{2}x\Big|_1^2 - \frac{1}{2}\frac{x^2}{2}\Big|_2^3 + 3\frac{1}{2}x\Big|_2^3$$

$$= \frac{1}{4} + \frac{1}{2} - \frac{1}{4}(3^2-2^2) + \frac{3}{2}$$

$$= 1$$

$\therefore$ c.d.f of X is

$$F(x) = \begin{cases} 0 \;; & x<0 \\ x^2/4 \;; & x<1 \\ \frac{1}{4}+\frac{1}{2}(x-1) \;; & x<2 \\ 3x/2 - x^2/4 - 5/4 \;; & x<3 \\ 1 \;; & x<\infty \end{cases}$$