UNIT II
DISTRIBUTIONS

Binomial distribution:

Consider a random experiment having only two outcomes, say success (**S**) and failure (**F**). Suppose that trial is conducted, say, n number of times. One might be interested in knowing what is the probability that getting success in k times.

Let **p** denotes the probability of obtaining a success in a single trial and **q** stands for the chance of getting a failure in one attempt implying that **p + q = 1.**

If the experiment has the following characteristics;

- the probability of obtaining failure or success is same for each and every trial
- trials are independent of one another
- probability of having a success is a finite number, then

We say that the problem is based on the binomial distribution. In this problem, we define **X** as the random variable equals the number of successes obtained in n trials. Then **X** takes the values 0, 1, 2, 3 . . . up to **n**. Therefore, one can view **X** as a discrete random variable, called binomial random variable. Since number of ways of obtaining **k** successes in **n** trials may be achieved in $\binom{n}{k} = \frac{n!}{(n-k)!\ k!}$ ways, therefore, probability function (is called binomial probability distribution) may be formulated as $b(n,p,k) = P(X=k) = \binom{n}{k}p^k q^{n-k}, k = 0,1,2,\ldots,n.$

Clearly, $\sum_{k=0}^{n} P(X=k) = \sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k} = (p+q)^n = 1.$

Expectation and Variance of Binomial distribution function:

Let X be a discrete random variable with following a binomial distribution function $b(n,p,k) = \binom{n}{k}p^k q^{n-k}, k = 0,1,2,\ldots,n.$

Consider the expectation of X, namely,

$$E(X) = \sum_{k=0}^{n} k\ P(X=k)$$

$$= \sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=0}^{n} k \frac{n!}{(n-k)!\ k!} p^k q^{n-k}$$

$$= \sum_{k=0}^{n} k \frac{n.\ (n-1)!}{k.\ (k-1)!\ (n-k+1-1)!} p.p^{k-1}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\ (n-1-(k-1))!} p^{k-1} q^{n-1-(k-1)}$$

$$= np\,(p+q)^{n-1}$$

$$\boldsymbol{E(X) = np} \qquad \text{(because p + q = 1)}$$

Variance of X is, $V(X) = E(X^2) - \big(E(X)\big)^2$

$$E(X^2) = \sum_{k=0}^{n} k^2\ P(X=k)$$

$$= \sum_{k=0}^{n} k^2 \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=0}^{n} (k^2 - k + k) \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=0}^{n} (k^2 - k) \binom{n}{k} p^k q^{n-k} + \sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=0}^{n} k(k-1) \frac{n!}{(n-k)!\ k!} p^k q^{n-k} \quad + \quad \sum_{k=0}^{n} k \frac{n!}{(n-k)!\ k!} p^k q^{n-k}$$

$$= \sum_{k=0}^{n} k(k-1) \frac{n.(n-1).(n-2)!}{k.(k-1).(k-2)!\ (n-k)!}\ p^k q^{n-k} \quad + \quad np$$

$$= n(n-1)p^2 \sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!\ \big(n-2-(k-2)\big)!}\ p^{k-2} q^{n-2-(k-2)} \quad + \quad np$$

$$= n(n-1)p^2 (p+q)^{n-2} \quad + \quad np \quad = \quad n^2p^2 - np^2 + \ np.$$

Therefore, $V(X) = n^2p^2 - np^2 + \ np - n^2p^2 \quad = \quad np(1-p) \quad = npq$

And Standard deviation = $\sqrt{npq}$.

Problem :- 1) 6 coins are tossed, find the probability of getting i) exactly 3 H ii) almost 3 H iii) atleast 3 H iv) atleast 1 H

Sol?:- The probability of getting a head ~~for~~ in each coin is $p = \frac{1}{2}$, here $n = 6$.

∴ Probability of getting x heads when 6 coin are tossed is

$$b(n, p, x) = {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$b(6, \tfrac{1}{2}, x) = {}^{6}C_{x} \left(\tfrac{1}{2}\right)^{x} \left(1 - \tfrac{1}{2}\right)^{n-x} = {}^{6}C_{x} \frac{1}{2^{x}} \frac{1}{2^{6-x}}$$

$$= \frac{1}{2^{6}}\, {}^{6}C_{x}$$

i) The probability of getting exactly 3H,

$x = 3$, $b(6, \frac{1}{2}, 3) = \frac{1}{2^{6}}\, {}^{6}C_{3} = 0.3125$

ii) The probability of getting almost 3H $x \leq 3$

$$p(x) = b(6, \tfrac{1}{2}, x) = p(x=0) + p(x=1) + p(x=2) + p(x=3)$$

$$= \frac{1}{2^{6}}\, {}^{6}C_{0} + \frac{1}{2^{6}}\, {}^{6}C_{1} + \frac{1}{2^{6}}\, {}^{6}C_{2} + \frac{1}{2^{6}}\, {}^{6}C_{3}$$

$$= \frac{42}{64} = 0.65625$$

iii) at least 3H, $x \geq 3$

$$b(6, \tfrac{1}{2}, x) = b(6, \tfrac{1}{2}, 3) + b(6, \tfrac{1}{2}, 4) + b(6, \tfrac{1}{2}, 5) + b(6, \tfrac{1}{2}, 6)$$

$$= \frac{1}{2^{6}}\left({}^{6}C_{3} + {}^{6}C_{4} + {}^{6}C_{5} + {}^{6}C_{6}\right) = \frac{42}{64} = 0.65625$$

iv) probability of getting atleast 1 H.

$x \geq 1$, ~~$b(6, \frac{1}{2}, x) =$~~

$P(x \geq 1) = 1 - P(x < 1)$

$= 1 - P(x = 0) = 1 - \frac{1}{2^6}\, {}^6C_0 = 0.9844$

2) Find the probability that in a family of 4 children there will be atleast one boy (Assume that probability of a male birth is ½).

Sol:- Let x be the no. of boys in a family of 4 children

$P(x) = b(n, p, x) = b(4, \frac{1}{2}, x) = {}^4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x}$

$= \frac{1}{2^4}\, {}^4C_x$

∴ probability of atleast one boy.

$P(x \geq 1) = 1 - P(x < 1)$

$= 1 - P(x = 0)$

$= 1 - \frac{1}{2^4}\, {}^4C_0 = \frac{15}{16} = 0.93.$

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(10) The probability that a pen manufactured by a company will be defective is $1/10$. If 12 such pens are manufactured, find the probability that

a) exactly 2 will be defective

b) atmost 2 will be defective

c) none will be defective.

$p(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$.

Soln: p = prob(pen will be def) $= 1/10 = 0.1$

$\therefore q$ = prob(pen will be non defective) $= 1 - 0.1 = 0.9$

a) P(exactly 2 will be defective) $= P(x=2) = p(2) = {}^{12}C_{2}\,(0.1)^{2}(0.9)^{10}$ $= 0.2301$

b) P(atleast 2 will be defective) $= P(x=2,3,\ldots 12)$

$= 1 - P(x=0,1)$

$= 1 - \left[{}^{12}C_{0}\,(0.1)^{0}(0.9)^{12} + {}^{12}C_{1}\,(0.1)^{1}(0.9)^{11}\right]$

$= 0.3412$

c) P[none will be defective] $= P[x=0] = {}^{12}C_{0}\,(0.1)^{0}(0.9)^{12}$

$= 0.2833$

3) If the mean and variance of a Binomial distribution are 1.2 and 0.84 resp'ly, find n & p.

mean $= np = 1.2$

Variance $= npq = 0.84$

i.e $1.2q = 0.84 \Rightarrow q = \frac{0.84}{1.2} =$

$\therefore p =$ and $n =$

4) In playing with an opponent of equal ability which is more probable?

i) winning 3 games out of 4 or 5 out of 8.

ii) winning at least 3 games out of 4 or at least 5 out of 8

Soln: $p =$ prob-of winning a game $= \frac{1}{2}$. Now $p(x) = {}^nC_x p^x q^{n-x}$

i) P[winning 3 games out of 4] $= p(x=3) = {}^4C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^1$

$= 0.25$

P(winning 5 games out of 8] $= {}^8C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^3 = 0.21875$

$\therefore$ winning 3 games out of 4 is more probable.

ii) P(winning at least 3 games out of 4) $= P[x = 3, 4]$

$$= \sum_{x=3}^{4} {}^4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x} = \underline{0.3125}$$

P(winning at least 5 games out of 8) $= P[x = 5, 6, 7, 8]$

$$= \sum_{x=5}^{8} {}^8C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{8-x}$$

$$= \underline{0.37109}.$$

Thus winning at least 5 games out 8 is more probable.

Poisson distribution:

Let X be a random variable assuming the possible values 0, 1, 2, 3, …..

If $P(X = k) = \frac{e^{-\alpha} \alpha^{k}}{k!}, \ k = 0,1,2,3, \ldots.$

We say that X has poisson distribution with parameter α >0.

Property:
Poisson distribution is regarded as limiting form of binomial distribution.

Proof : We know that binomial distribution of X at k is $b(n,p,k) = P(X=k) = \binom{n}{k} p^k q^{n-k}$

$$\text{that is,} \qquad P(X=k) = \frac{n!}{k!\,(n-k)!} p^k q^{n-k}$$

$$= \frac{n(n-1)(n-2)\ldots\ldots(n-(k-1))}{k!} p^k q^{n-k}$$

$$= \frac{n^k(1-1/n)(1-2/n)\ldots\ldots(1-(k-1)/n)}{k!} p^k q^{n-k}$$

$$= (1-1/n)(1-2/n)\ldots\ldots\left(1-(k-1)/n\right) \frac{(np)^k}{k!} q^{n-k}$$

Put $np = \propto$ $\;$ then $q = 1 - \frac{\propto}{n}$,

therefore , $P(X=k) = (1-1/n)(1-2/n)\ldots\ldots\left(1-(k-1)/n\right) \frac{\propto^k}{k!}\left(1-\frac{\propto}{n}\right)^{n-k}$

take limit as $n \to \infty$, we get,

$$P(X) = \lim_{n\to\infty}(1-1/n)(1-2/n)\ldots\ldots\left(1-(k-1)/n\right) \frac{\propto^k}{k!}\left(1-\frac{\propto}{n}\right)^{n-k}$$

$$= \frac{\propto^k}{k!} \lim_{n\to\infty}(1-1/n)(1-2/n)\ldots\ldots\left(1-(k-1)/n\right) \left(1-\frac{\propto}{n}\right)^{n-k}$$

$$= \frac{\propto^k}{k!} \lim_{n\to\infty} \left(1-\frac{\propto}{n}\right)^{n} \times \left(1-\frac{\propto}{n}\right)^{-k}$$

$$= \frac{\propto^k}{k!} e^{-\propto} \qquad \left(\lim_{n\to\infty} \left(1-\frac{\propto}{n}\right)^{n} = e^{-\propto} \text{ and } \lim_{n\to\infty} \left(1-\frac{\propto}{n}\right)^{-k} = 1\right)$$

therefore P(X) is a poisson distribution.

Note: The distribution of probabilities of X = 0, 1, 2,3, …. is as follows:

X	0	1	2	3	4	…
P(X)	$e^{-\alpha}$	$\alpha\, e^{-\propto}$	$\frac{\propto^2}{2} e^{-\propto}$	$\frac{\propto^3}{6} e^{-\propto}$	$\frac{\propto^4}{24} e^{-\propto}$	…

Clearly, P(X) $\geq$ 0 for $\alpha > 0$ *and* $\sum P(X) = \sum_{k=0}^{\infty} \frac{\propto^k}{k!} e^{-\propto} = e^{-\propto} \sum_{k=0}^{\infty} \frac{\propto^k}{k!} = e^{-\propto} e^{\propto} = 1.$

Expectation and Variance of Poisson distribution:

Let X be a random variable with poisson distribution,

$P(X=k)=\frac{e^{-\propto}\propto^k}{k!},\ k=0,1,2,3,\ldots.$

Then,

$$Expectation = E(X)=\sum_{k=0}^{\infty}k\frac{e^{-\propto}\ \propto^k}{k!}=e^{-\propto}\sum_{k=0}^{\infty}k\frac{\propto.\propto^{k-1}}{k.(k-1)!}$$

$$=\propto.e^{-\propto}\sum_{k=1}^{\infty}\frac{\propto^{k-1}}{(k-1)!}=\propto e^{-\propto}e^{\propto}=\propto.$$

Variance= V(X) = $E(X^2)-(E(X))^2$

Where,

$$E(X^2)=\sum_{k=0}^{\infty}k^2\frac{e^{-\propto}\ \propto^k}{k!}=\sum_{k=0}^{\infty}(k^2-k+k)\frac{e^{-\propto}\ \propto^k}{k!}$$

$$=\sum_{k=0}^{\infty}(k^2-k)\frac{e^{-\propto}\ \propto^k}{k!}+\sum_{k=0}^{\infty}k\frac{e^{-\propto}\ \propto^k}{k!}$$

$$=e^{-\propto}\sum_{k=0}^{\infty}k(k-1)\frac{\propto^2\propto^{k-2}}{k.(k-1).(k-2)!}+E(X)$$

$$=\propto^2 e^{-\propto}\sum_{k=2}^{\infty}\frac{\propto^{k-2}}{(k-2)!}+E(X)$$

$$=\propto^2 e^{-\propto}e^{\propto}+E(X)=\propto^2+\propto$$

therefore,

$$V(X)=\propto^2+\propto-\propto^2=\propto$$

Note: Expectation of poisson variate is equal to its variance.

Problems:

1. 5 coins are tossed 6400 times. Using the poisson distribution. What is the approximate probability getting 5 heads 100 times?

$p = \frac{\lambda e^{-\lambda}}{0!}$

Soln: probability of getting one head with one coin $= \frac{1}{2}$

$\therefore$ The probability of getting 5 heads with 5 coins $= \left(\frac{1}{2}\right)^5 = \frac{1}{32} = p$

Here $n = 6400$, $p = \frac{1}{32}$

$\therefore \lambda = np = 6400 \times \frac{1}{32} = 200$

$\therefore p(X=100) = \frac{\lambda^{100} e^{-200}}{100!} = \frac{(200)^{100} e^{-200}}{100!}$

2. Suppose that the probability that an item produced by a particular machine is defective equals 0.2. If 10 items produced from this machine are selected at random, what is the probability that not more than one defective is found? Use the binomial and poisson distribution & compare the answer?

0.4295

Given n = No. of items produced = 10.

$p = 0.2$ (= probability that item is defective)

(i) Binomial distribution:

Here $q = 1 - p = 1 - 0.2 = 0.8$.

$P(X \le 1) = P(0) + P(1) = ({}^{10}C_0\, p^0 q^{10}) + ({}^{10}C_1\, p^1 q^9)$

$= [{}^{10}C_0 \times (0.2)^0 \times (0.8)^{10}] + [{}^{10}C_1\, (0.2)^1 \times (0.8)^9]$

$= 0.3758$

(ii) Poisson's distribution:

Here $\lambda = np = 10 \times 0.2 = 2$

$P(X \le 1) = P(0) + P(1) = \frac{e^{-\lambda} \lambda^0}{0!} + \frac{e^{-\lambda} \cdot \lambda^1}{1!}$

$= e^{-2} \cdot \frac{1}{1} + e^{-2} \cdot 2 = 3e^{-2}$

$= 0.4060$

(3) An insurance company has discovered that only about 0.1% of the population is involved in a certain type of accident each year. If its 10,000 policy holders were randomly selected from the population, what is the probably that not more than 5 of its clients are involved in such an accident next year?

Soln:- Here, $p = 0.1\% = 0.001$

$n = 10{,}000$, $np = \lambda = 10$

By using poisson's distribution

$P(X \le 5) = P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$

$= \frac{e^{-10} \cdot \lambda^0}{0!} + \frac{e^{-10} (10)^1}{1!} + \frac{e^{-10} \cdot (10)^2}{2!} + \frac{e^{-10} \cdot (10)^3}{3!} + \frac{e^{-10} (10)^4}{4!} + \frac{e^{-10} (10)^5}{5!}$

$= 0.0670$

(4). Suppose that X has a poission distribution. If $P(X=2) = \frac{2}{3} P(X=1)$. Evaluate $P(X=0)$ & $P(X=3)$.

Soln:- First we have to find λ from given data.

$$P(X=2) = \frac{2}{3} P(X=1). \quad \text{---} (*)$$

Since P is a poission distribution function,

$$P(X=2) = \frac{e^{-\lambda} \cdot \lambda^2}{2!} = e^{-\lambda} \frac{\lambda^2}{2}$$

& $$P(X=1) = e^{-\lambda} \cdot \frac{\lambda^1}{1!} = e^{-\lambda} \frac{\lambda}{1} = e^{-\lambda} \lambda$$

from (*) $\Rightarrow$ $$e^{-\lambda} \cdot \frac{\lambda^2}{2} = \frac{2}{3} e^{-\lambda} \cdot \lambda$$

$$\Rightarrow \boxed{\lambda = 4/3}$$

$$\therefore P(X=3) = e^{-\lambda} \frac{\lambda^3}{3!} = e^{-4/3} \frac{(4/3)^3}{3!} = \underline{0.1041}$$

$$P(X=0) = e^{-\lambda} \cdot \frac{\lambda^0}{0!} = e^{-4/3} \cdot \frac{1}{1} = e^{-4/3} = \underline{0.2636}$$

(5) In a certain factory turning out blades there is a small chance, $\frac{1}{500}$ for any blade to be defective. The blades are supplied in packets of 10. Use P.D. to calculate the approximate no. of packets containing (i) a no defective blades (ii) one defective (iii) 2 defective in a consignment of 1000 packets?

$p = 1/500$, $n = 10$. P

Ans:

(6) If the probability that an individual suffers a bad reaction from an injection is 0.001, determine the probability that out of 2000 individuals (i) exactly 3 (ii) more than 2 indivi uals suffer a bad reaction.

(Ans: 0.3233 & 0.18045)

(7) Suppose that a book of 585 pages contains 43 typographical errors. If these errors are randomly distributed through out the book, what is the probability that 10 pages, selected at random, will be

$\frac{43}{585}$

$\lambda = \frac{43}{585}$

Normal distribution

Let X be a continuous random variable with pdf

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty.$$

Then X is called a normal variate or we say that X has normal distribution. Here μ and σ^2 are the parameters and the distribution is denoted by $N(\mu, \sigma^2)$.

* Obtain the mean and variance of normal distribution

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} x\, e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2} dx.$$

put $\frac{x-\mu}{\sqrt{2}\sigma} = z$

i.e. $x = \sqrt{2}\sigma z + \mu$

$dx = \sqrt{2}\sigma\, dz$

$$= \frac{1}{\cancel{\sqrt{2}}\sqrt{\pi}\,\cancel{\sigma}} \int_{-\infty}^{\infty} (\sqrt{2}\sigma z + \mu)\, e^{-z^2} \cancel{\sqrt{2}}\cancel{\sigma}\, dz$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(\sqrt{2}\sigma z\, e^{-z^2} + \mu e^{-z^2}\right) dz$$

$$= \frac{\mu}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2} dz \quad \text{since} \quad \int_{-\infty}^{\infty} \sqrt{2}\sigma z e^{-z^2} dz = 0$$

as the function is odd.

$$\therefore E[X] = \frac{\mu}{\sqrt{\pi}} \times \sqrt{\pi} = \mu \qquad \left(\because \int_{-\infty}^{\infty} e^{-z^2} dz = 2\int_0^{\infty} e^{-z^2} dz = \sqrt{\pi} \right.$$

because put $z^2 = t$ i.e. $2z\,dz = dt$

$$\left. \therefore 2\int_0^{\infty} e^{-z^2} dz = \cancel{2}\int_0^{\infty} e^{-t}\, t^{-\frac{1}{2}} \frac{dt}{\cancel{2}} = \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \right)$$

$$V[x] = E\left[X - E[x]\right]^2 = E\left[X - \mu\right]^2$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Again put $\frac{x-\mu}{\sqrt{2}\,\sigma} = z$

$$= \int_{-\infty}^{\infty} 2\sigma^2 z^2 \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}} e^{-z^2} \cancel{\sqrt{2}\,\sigma}\, dz$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \underbrace{z^2 e^{-z^2}}_{\text{even}} dz$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \times 2 \int_{0}^{\infty} z^2 e^{-z^2} dz$$

put $z^2 = u$
$2z\,dz = du$

$$= \frac{\cancel{4}^{2}\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} u\, e^{-u} \frac{du}{\cancel{2}u^{1/2}}$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} u^{1/2} e^{-u} du$$

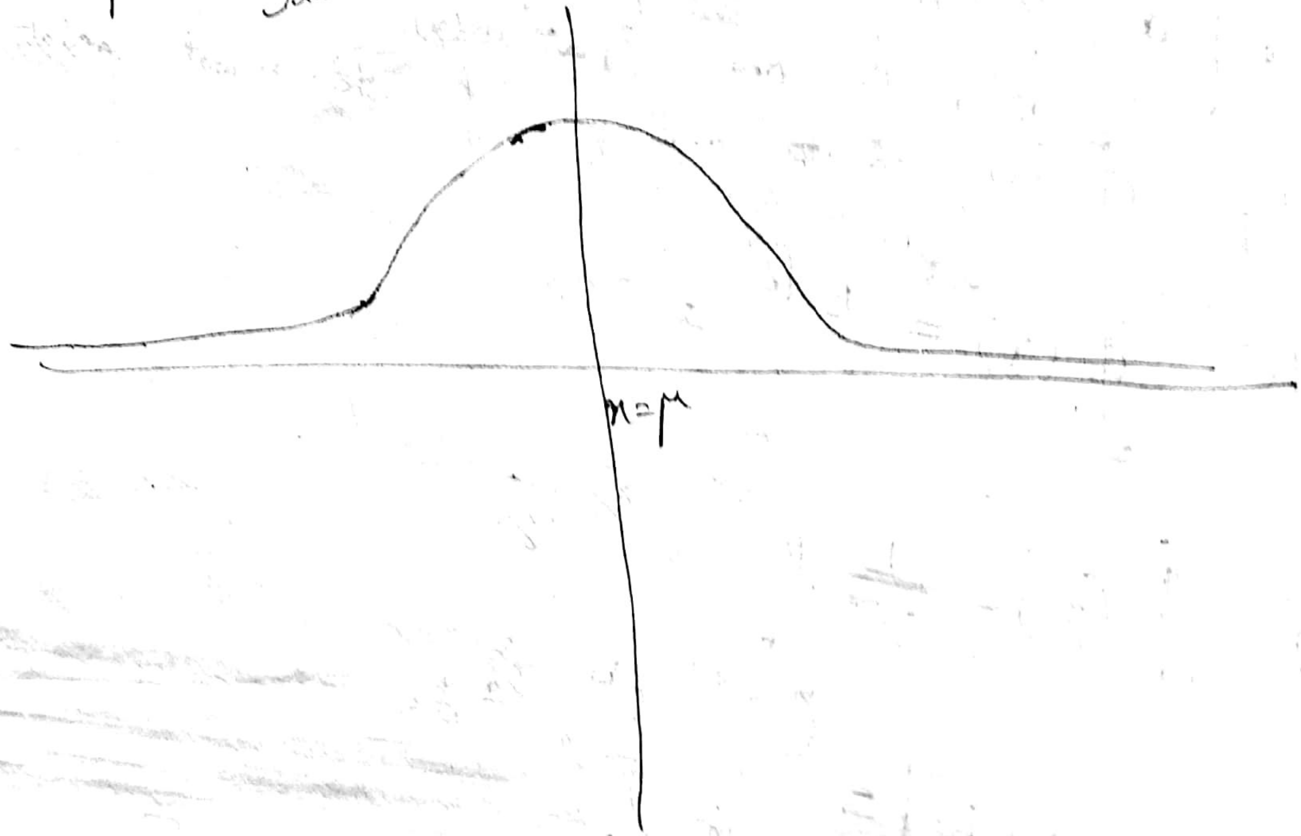
$$= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\tfrac{1}{2}+1\right)$$

$$= \frac{2\sigma^2}{\cancel{\sqrt{\pi}}} \times \frac{1}{2} \cancel{\sqrt{\pi}}$$

$$\cancel{2\sigma^2} \quad \sigma^2$$

If X has normal distribution with mean μ and variance σ^2, then we ~~say~~ write $X \sim N(\mu, \sigma^2)$. The curve

$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ is symmetric about $x=\mu$.



Note: 1) $f(x)$ has max. value at $x=\mu$ and that max. value is $\frac{1}{\sqrt{2\pi}\,\sigma}$

2) As $x \to \pm\infty$, $f(x) \to 0$.

It is a continuous bell-shaped curve symmetric about the line $x=\mu$.

<u>Standard normal distribution</u>: If $\mu = 0$ and $\sigma = 1$ in $N(\mu, \sigma^2)$, then the pdf $f(x)$ becomes

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$ and this is called the standard normal density function. This is symmetric about y-axis ($\because x = \mu = 0 \Rightarrow$ y axis is). Standard normal distribution is denoted by $N(0,1)$.

Show that if $X \sim N(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma}$ is in $N(0,1)$.

<u>Soln:</u> $$E[Z] = E\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma} E[X] - E\left[\frac{\mu}{\sigma}\right]$$

$$= \frac{\mu}{\sigma} - \frac{\mu}{\sigma} \quad (\because E[X] = \mu)$$

$$= 0$$

$$V[Z] = V\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma^2} V[X] = \frac{1}{\sigma^2} \times \sigma^2 = 1$$
$$(\because V[X] = \sigma^2)$$

Thus Z has mean '0' and variance '1'

$\therefore Z = \frac{X-\mu}{\sigma} \sim N(0,1)$, standard normal distribution.

Note :- If X is in standard normal distribution, i.e. $X \sim N(0,1)$, then $P[a \le X \le b] = \Phi(b) - \Phi(a)$ where $\Phi(x)$ is the c.d.f of standard normal variate

For, the pdf of $X \sim N(0,1)$ is $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $-\infty < x < \infty$. If $\Phi(x)$ is the c.d.f of X,

then $\Phi(x) = P[X \le x] = \int_{-\infty}^{x} \phi(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2}\,dx$

$$\therefore P[a \le X \le b] = \int_a^b \phi(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{b} e^{-x^2/2}\,dx - \int_{-\infty}^{a} e^{-x^2/2}\,dx \right]$$

$$= \Phi(b) - \Phi(a)$$

Note :- $\Phi(-z) = 1 - \Phi(z)$.

Note :- $P[a \le X \le b] = P\left[\frac{a-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right]$

$= P\left[\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right]$ where $Z = \frac{X-\mu}{\sigma}$ is in $N(0,1)$

given $X \sim N(\mu, \sigma^2)$.

(For student's information while working out problem, follow the remark given below:)

Remark : Whenever X has normal distribution, to obtain the tabular values always convert X into standard normal variate and then refer standard normal distribution table.

If

Problems:

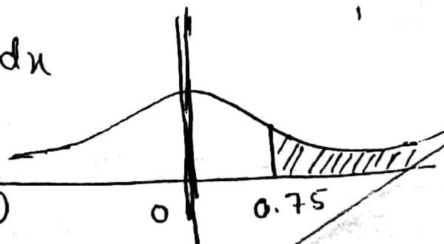
(1) Suppose that X has distribution $N(2, 0.16)$
evaluate $P(X \geq 2.3)$ & $P(1.8 \leq X \leq 2.1)$.

Soln:- Given Normal distribution $N(2, 0.16) = N(\mu, \sigma^2)$

$\mu = 2$, $\sigma^2 = 0.16 \Rightarrow \sigma = 0.4$, let $Y = \frac{X-\mu}{\sigma} = \frac{X-}{0.4}$

$$P(X \geq 2.3) = P\left(Y \geq \frac{2.3-2}{0.4}\right) = P(Y \geq 3/4)$$
$$= P(Y \geq 0.75)$$

$$P(0.75 \leq Y) = \int_{0.75}^{\infty} \varphi(x)\,dx$$



$$P(0.75 \leq Y) = \Phi(\infty) - \Phi(0.75)$$
$$= 1 - \Phi(0.75) = 1 - 0.7764 = \underline{0.2266}$$

(ii) $P(1.8 \le X \le 2.1) = P\left(\frac{1.8-2}{0.4} \le \frac{X-\mu}{\sigma} \le \frac{2.1-2}{0.4}\right)$

$= P\left(\frac{-0.2}{0.4} \le \frac{X-\mu}{\sigma} \le \frac{0.1}{0.4}\right)$

$= P\left(-0.5 \le \frac{X-\mu}{\sigma} \le 0.25\right)$.

$= \Phi(0.25) - \Phi(-0.5)$.

$= 0.5987 - 0.3085 = \underline{0.2902}$

2) The diameter of an electric cable is normally distributed with mean 0.8 & variance 0.0004

(i) what is the probability that the diameter will exceed 0.81 inch? (ii) Cable is considered defective if the diameter differs from the mean by more than 0.025 what is the probability of obtaining a defective cable?

Soln:- given $\mu = 0.8$, $\sigma^2 = 0.0004 \Rightarrow \sigma = 0.02$

obtaining a ...

Soln :- given $\mu = 0.8$, $\sigma^2 = 0.0004 \Rightarrow \sigma = 0.02$

(i) $P(X \geq 0.81) = P\left(\frac{X-\mu}{\sigma} \geq \frac{0.81-0.8}{0.02}\right) = P(Y \geq 0.5)$

$$\left(= \frac{1}{\sqrt{2\pi}} \int_{0.5}^{\infty} e^{-x^2/2}\, dx\right) = \Phi(\infty) - \Phi(0.5)$$
$$= 1 - 0.6915 = \underline{\underline{0.3085}}$$

(ii) Mean differs from 0.025 is X lies between

$(0.8 - 0.025) \leq X \leq (0.8 + 0.025)$

$0.775 \leq X \leq 0.825$

$$P(0.0775 \leq X \leq 0.825) = P\left(\frac{-0.025}{0.02} \leq Y \leq \frac{0.025}{0.02}\right)$$
$$= P(-1.25 \leq Y \leq 1.25)$$
$$= \Phi(1.25) - \Phi(-1.25)$$
$$= 0.8944 - 0.1056 = \underline{0.7888}$$

4) Suppose that the life length of two electronic devices, say D_1 and D_2, have distribution $N(40, 36)$ and $N(45, 9)$ respectively.

(i) If the electronic device is to be used for a 45-hour period, which device is to be preferred?

(ii) If it is to be used for 48-hours period, which device is preferred?

Soln:- (i) $N(40, 36) \Rightarrow \sigma^2 = 36 \Rightarrow \sigma = 6$ & $\mu = 40$.

For D_1,

$P(\quad X \leq 45) = P(\quad Y \leq 0.83)$

$= \Phi(0.83) - \Phi(-6.67) = 0.7967 - 0$

For D_2, $N(45, 9) \Rightarrow \sigma^2 = 9 \Rightarrow \sigma = 3$, & $\mu = 45$

$P(\quad X \leq 45) = P(-15 \leq X \leq 0)$

$= \Phi(0) - \Phi(-15) = 0.5 - 0 = 0.5$

$(\because \Phi(x \geq 3) = 1$ & $\Phi(x \leq -3) = 0)$.

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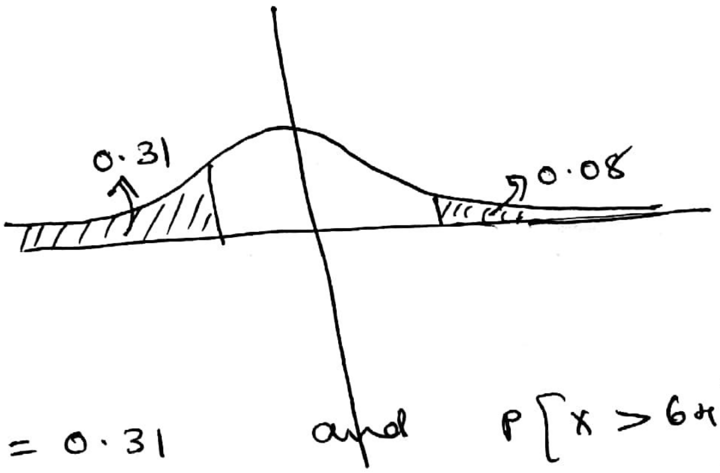
(2) In a normal distribution 31% of the items are under 45 and 8% are over 64. Find the mean and standard deviation.

<u>Soln:</u>



$P[X < 45] = 0.31$ and $P[X > 64) = 0.08$

i.e. $P\left[\frac{X-\mu}{\sigma} < \frac{45-\mu}{\sigma}\right] = 0.31$ and $P\left[\frac{X-\mu}{\sigma} > \frac{64-\mu}{\sigma}\right) = 0.08$

$P\left[Z < \frac{45-\mu}{\sigma}\right] = 0.31$ and $P\left[Z > \frac{64-\mu}{\sigma}\right] = 0.08$

$1 - P\left(Z < \frac{45-\mu}{\sigma}\right) = 1 - 0.31$ $\qquad$ $1 - P\left[Z < \frac{64-\mu}{\sigma}\right] = 0.08$

i.e. $1 - \Phi\left(\frac{45-\mu}{\sigma}\right) = 0.69$ $\qquad$ i.e. $1 - \Phi\left(\frac{64-\mu}{\sigma}\right) = 0.08$

i.e. $\Phi\left(\frac{\mu-45}{\sigma}\right) = 0.69$ $\qquad$ i.e. $0.92 = \Phi\left(\frac{64-\mu}{\sigma}\right)$

$\therefore \frac{\mu-45}{\sigma} = 0.5$ $\qquad$ $\therefore \frac{64-\mu}{\sigma} = 1.41$

i.e. $\mu - 0.5\sigma = 45 \rightarrow (1)$

$+\mu + 1.41\sigma = +64 \rightarrow (2)$

Solving (1) & (2), $\underline{\underline{\mu = 49.9738}}$, $\underline{\underline{\sigma = 9.9476}}$

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When the mean of marks was 50% and standard deviation 5% then 60% of the students failed in examination. Determine grace marks to be awarded in order to show that 70% of the students passed. Assume that the marks are normally distributed?

Sol^n:- Given $\mu = 50\% = 0.5$

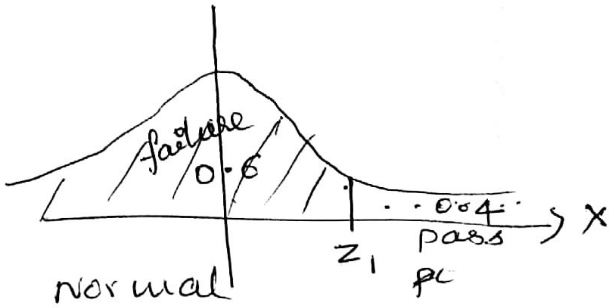
$S.D = \sigma = 5\% = 0.05$

Let X_1 be the marks obtained in an examination

X_2 be the marks awarded after the grace marks are given

$\because$ 60% failure corresponds to 0.6 area,

Determine z_1 such that z_1 must be positive. area to left is 0.6.



$\Phi(z_1) = 0.6$

$\Rightarrow z_1 = 0.25$ (from standard normal distribution table)

$z_1 = \frac{X_1 - \mu}{\sigma}$ ie $z_1 = \frac{X_1 - 50}{5}$

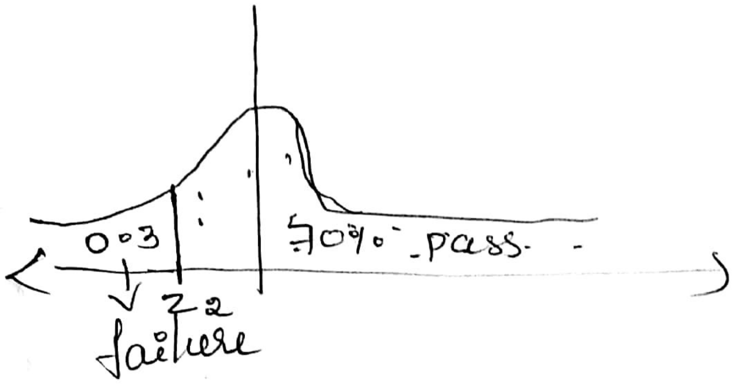
$\Rightarrow 0.25 = \frac{X_1 - 50}{5}$ or $X_1 = 51.25$

Thus the minimum pass mark is 51.25 before grace mark were given. After grace were awarded 70% pass the exam

~~Two~~

$\Phi(z_2) = 0.3$

$\therefore z_2 = -0.52$



Thus $z_2 = \dfrac{x_2 - 50}{5}$

$\Rightarrow x_2 = 50 - 0.52 \times 5$

$x_2 = 47.40$, i.e minimum pass mark after grace mark were given.

So grace mark awarded is $51.25 - 47.40$

$= 3.85$

Exponential distribution:

Definition: A continuous random variable X assuming all non-negative values is said to ~~be~~ have an exponential distribution with parameter $\alpha > 0$ if its pdf is given by,

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

clearly $\int_{-\infty}^{\infty} f(x)\,dx = 1.$

Mean & variance of X:

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} x \alpha e^{-\alpha x}\,dx = \alpha \int_0^{\infty} x e^{-\alpha x}\,dx$$

$$= \alpha \left[\frac{x e^{-\alpha x}}{-\alpha} - \frac{e^{-\alpha x}}{\alpha^2} \right]_0^{\infty} = \alpha \left[0 + \frac{1}{\alpha^2} \right] = \alpha/\alpha^2$$

$$\therefore E(X) = \alpha/\alpha^2 = 1/\alpha$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^{\infty} x^2 \alpha e^{-\alpha x}\,dx$$

$$= \alpha \int_0^{\infty} x^2 e^{-\alpha x}\,dx$$

$$= \alpha \left[x^2 \frac{e^{-\alpha x}}{-\alpha} - \frac{2x e^{-\alpha x}}{\alpha^2} - \frac{2 e^{-\alpha x}}{\alpha^3} \right]_0^{\infty}$$

$$E(X^2) = \alpha \left[0 + \frac{2}{\alpha^3} \right] = \frac{2}{\alpha^2}$$

$$V(X) = E(X^2) - (E(X))^2 = \frac{2}{\alpha^2} - \frac{1}{\alpha^2} = \frac{1}{\alpha^2}$$

(1) The length of a telephone conversation has an exponential distribution with a mean of 3 minutes. Find the probability that a call ends in less than 3 minutes?

Soln:- Since mean $\mu = \frac{1}{\alpha} = 3 \Rightarrow \boxed{\alpha = 1/3}$

The exponential distribution is $\frac{1}{3} e^{-1/3 x}$

$$P(X < 3) = 1 - P(X \geq 3)$$

$$= 1 - \int_3^\infty \frac{1}{3} e^{-1/3 x} \, dx = 1 - \frac{1}{3} \left\{ \frac{e^{-1/3 x}}{-1/3} \right\}_3^\infty$$

$$= 1 - \frac{1}{3} \left\{ 3 e^{-1} \right\} = 1 - \frac{1}{e} = 0.63212$$

(*) In a certain town the duration of a shower is exponentially distributed with mean 5 minutes. What is the probability that a shower will last for

(i) less than 10 minutes

ii) 10 minutes or more.

Soln: Let X denote the duration of the shower.

Mean of exponential distribution is $\frac{1}{\alpha} = 5$

$\therefore$ the p.d.f is $f(x) = \frac{1}{5} e^{-\frac{1}{5}x}$, $x > 0$.

i) P[shower will last less than 10 minutes] $= P[X < 10]$

$$= \int_0^{10} f(x)\,dx = \frac{1}{5}\int_0^{10} e^{-x/5}\,dx = \frac{1}{5}\left[\frac{e^{-x/5}}{-1/5}\right]_0^{10}$$

$$= -\left[e^{-2} - 1\right] = 1 - e^{-2} = 1 - 0.1353 = \underline{0.8647}$$

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