Isomorphism:

If V & W are vector spaces over the field F, any one-one linear transformation T of V onto W is called an isomorphism of V onto W.

If there exists an isomorphism of V onto W, we say that V is isomorphic to W & it is denoted by $V \cong W$.

Thm: Every finite dimensional vector space V is isomorphic to F^n (where F is $\mathbb{R}$ or $\mathbb{C}$)

Pf Let V be an n-dimensional vector space over the field F & $B = \{v_1, v_2, \ldots, v_n\}$ be a basis for V.

~~Define α~~

Then every vector $v \in V$ can be expressed as the linear combination of the elements of B.

i.e $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

Consider a mapping

$T: V \longrightarrow F^n$ such that $Tv = (\alpha_1, \alpha_2, \ldots \alpha_n)$ $\forall v \in V$

[i.e. $(\alpha_1, \alpha_2 \ldots \alpha_n) = [v]_B$ coordinates of v relative to the basis B]

T is linear:

Let v ~~$\exists$~~ $u \in V$ ~~such that~~

Then $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

$u = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$

Then $\forall c \in F$ we have

$T(cv+u) = T(c\alpha_1 v_1 + c\alpha_2 v_2 + \cdots + c\alpha_n v_n + \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n)$

$= T[(c\alpha_1+\beta_1)v_1 + (c\alpha_2+\beta_2)v_2 + \cdots + (c\alpha_n+\beta_n)v_n]$

$= (c\alpha_1+\beta_1, c\alpha_2+\beta_2, \ldots, c\alpha_n+\beta_n)$

$= (c\alpha_1, c\alpha_2, \ldots c\alpha_n) + (\beta_1, \beta_2, \ldots \beta_n)$

$= c(\alpha_1, \alpha_2, \ldots \alpha_n) + (\beta_1, \beta_2, \ldots \beta_n)$

$= cTv + Tu$

$\therefore$ T is linear

II

T is one-one:

Suppose $Tv = Tu$

$\Rightarrow (\alpha_1, \alpha_2 \ldots \alpha_n) = (\beta_1, \beta_2, \ldots \beta_n)$

$\Rightarrow \alpha_i = \beta_i \quad \forall i$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n$

$\Rightarrow v = u$

$\therefore$ T is one-one

T is onto:

Corresponding to each $(\alpha_1, \alpha_2, \ldots \alpha_n) \in F^n$ there exists a vector say $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$ in V

Such that $Tv = \cancel{(\alpha_1, \alpha_2, \ldots \alpha_n)}\, T(\alpha_1 v_1 + \ldots + \alpha_n v_n)$

$= (\alpha_1, \alpha_2, \ldots \alpha_n)$

$\therefore$ T is ~~onto~~ Onto.

$\therefore \quad V \cong F^n$

Inner product: Consider the vectors in $\mathbb{R}^n$ as $n \times 1$ matrices. For $u, v \in \mathbb{R}^n$, the matrix product $u^T v$ is a 1×1 matrix, called the inner product of u & v. It is denoted by $u \cdot v = u^T v$

Let $u, v, w \in \mathbb{R}^n$ & let c be a scalar. Then

1) $u \cdot v = v \cdot u$
2) $(u+v) \cdot w = u \cdot w + v \cdot w$
3) $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$
4) $u \cdot u \geq 0$ & $u \cdot u = 0 \iff u = 0$.

The length or norm of v defined by

$$\|v\| = \sqrt{v \cdot v}$$

A vector whose length is 1 is called a unit vector. If $u = \frac{1}{\|v\|} v$, then it is a unit vector. The process of creating u from v is called normalizing v & ~~v~~ is we say that u is in the same direction as v.

Example: Let $v = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}$. Find a unit vector u in the same direction as v.

Soln: $u = \frac{1}{\|v\|} v$

$$\|v\| = \sqrt{v \cdot v} = \sqrt{v^T v} =$$

$$v^T v = \begin{bmatrix} 1 & -2 & 2 & 0 \end{bmatrix}_{1 \times 4} \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}_{4 \times 1} = 1 + 4 + 4 + 0 = 9$$

$$\therefore \|v\| = \sqrt{9} = 3$$

$$\therefore u = \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix}$$

To check $\|u\| = 1$:

$\|u\| = \sqrt{u \cdot u}$

$$u \cdot u = u^T u = \begin{bmatrix} 1/3 & -2/3 & 2/3 & 0 \end{bmatrix} \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix} = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = 1$$

$\therefore \|u\| = \sqrt{1} = 1$

Def: Two vectors u & v in $\mathbb{R}^n$ are orthogonal (to each other) if $u \cdot v = 0$

A set of vectors $\{u_1 \dots u_p\}$ in $\mathbb{R}^n$ is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal, i.e. $u_i \cdot u_j = 0$ where $i \neq j$.

Example: S.T. $\{u_1, u_2, u_3\}$ is an orthogonal set where

$$u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \quad u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \quad u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$$

$u_1 \cdot u_2 = 3(-1) + (1)(2) + 1(1) = 0$

$u_1 \cdot u_3 = 3(-1/2) + (1)(-2) + (1)(\frac{7}{2}) = 0$

$u_2 \cdot u_3 = -1(-1/2) + 2(-2) + 1(7/2) = 0$

Thm: If $S = \{u_1, u_2 \dots u_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$ then S is linearly independent & hence is a basis for the subspace spanned by S.

Def: An orthogonal basis for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

An orthogonal projection:

Consider two non zero vectors $u, y \in$ in $\mathbb{R}^n$. The orthogonal projection of y onto u is given by $\frac{y \cdot u}{u \cdot u} u$

Ex: Let $y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Find the orthogonal projection of y onto u.

$y \cdot u = 40$

$u \cdot u = 20$

∴ The orthogonal projection of y onto u is

$$\frac{y \cdot u}{u \cdot u} u = \frac{40}{20} u = 2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

The Gram - Schmidt Process:

Given a basis $\{x_1, \ldots x_p\}$ for a subspace W of $\mathbb{R}^n$, define $v_1 = x_1$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$\vdots$$

$$v_p = x_p - \frac{x_p \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_p \cdot v_2}{v_2 \cdot v_2} v_2 - \cdots - \frac{x_p \cdot v_{p-1}}{v_{p-1} \cdot v_{p-1}} v_{p-1}$$

Then $\{v_1, \ldots v_p\}$ is an orthogonal basis for W.

The given set is a basis for a subspace W. Use the Gram - Schmidt process to produce an orthogonal basis for W.

1) $\{ [3, 0, -1]^T, \ [8, 5, -6]^T \}$

Let $x_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$ $\quad x_2 = \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix}$

$$v_1 = x_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$x_2 \cdot v_1 = x_2^T v_1 = \begin{bmatrix} 8 & 5 & -6 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = 24 + 6 = 30$$

$$v_1 \cdot v_1 = v_1^T v_1 = \begin{bmatrix} 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = 9 + 1 = 10$$

$$v_2 = \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix} - \frac{30}{10} \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix} - \begin{bmatrix} 9 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -\cancel{9}3 \end{bmatrix}$$

Orthogonal basis = $\left\{ \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} \right\}$

2) $\left\{ \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \right\}$

Let $x_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$ $\quad x_2 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$

$$v_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$x_2 \cdot v_1 = x_2^T v_1 = \begin{bmatrix} 4 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = 8 + 5 + 2 = 15$$

$$v_1 \cdot v_1 = v_1^T v_1 = \begin{bmatrix} 2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = 4 + 25 + 1 = 30$$

$$v_2 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} - \frac{15}{30} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 01 \\ -5/2 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix}$$

$\therefore$ Orthogonal Basis = $\left\{ \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3/2 \\ 3/2 \end{bmatrix} \right\}$

3] $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -4 \\ -2 \end{bmatrix} \right\}$

Let $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \end{bmatrix}$ $x_3 = \begin{bmatrix} 1 \\ -3 \\ -4 \\ -2 \end{bmatrix}$

$$v_1 = x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$x_2 \cdot v_1 = x_2^T v_1 = \begin{bmatrix} 1 & 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 1+2+4+5 = 12$$

$$v_1 \cdot v_1 = v_1^T v_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 4$$

$$\therefore v_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \end{bmatrix} - \frac{12}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix}$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$x_3 \cdot v_1 = x_3^T v_1$$

$$= \begin{bmatrix} 1 & -3 & -4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 1 - 3 - 4 - 2 = -8$$

$$x_3 \cdot v_2 = x_3^T v_2 = \begin{bmatrix} 1 & -3 & -4 & -2 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix} = -7$$

$$v_2 \cdot v_2 = v_2^T v_2 = \begin{bmatrix} -2 & -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix} = 10$$

$$v_3 = \begin{bmatrix} 1 \\ -3 \\ -4 \\ -2 \end{bmatrix} - \left(\frac{-8}{4}\right) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{-7}{10}\right) \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 16/10 \\ -17/10 \\ -13/10 \\ 14/10 \end{bmatrix}$$

$$= \begin{bmatrix} 8/5 \\ -17/10 \\ -13/10 \\ 7/5 \end{bmatrix}$$

$\therefore$ Orthogonal basis $= \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 8/5 \\ -17/10 \\ -13/10 \\ 7/5 \end{bmatrix} \right\}$

<u>Orthonormal Bases:</u> An orthonormal basis is constructed from an orthogonal basis $\{v_1, \dots, v_p\}$ by normalizing all v_K.

Find an orthonormal basis of the subspace spanned by the vectors $\left\{ \underset{x_1}{\begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}}, \underset{x_2}{\begin{bmatrix} 7 \\ -7 \\ -4 \\ 1 \end{bmatrix}} \right\}$ using Gram-Schmidt process.

<u>Soln:</u> $v_1 = x_1 = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$v_1 \cdot v_1 = v_1^T v_1 = \begin{bmatrix} 1 & -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix} = 18$$

$$x_2 \cdot v_1 = x_2^T v_1 = \begin{bmatrix} 7 & -7 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix} = 36$$

$$\therefore v_2 = \begin{bmatrix} 7 \\ -7 \\ -4 \\ 1 \end{bmatrix} - \frac{36}{18} \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -7 \\ -4 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ -8 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -4 \\ -1 \end{bmatrix}$$

$\therefore$ Orthogonal basis $= \left\{ \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ -4 \\ -1 \end{bmatrix} \right\}$

Normalized form of $v_1 = \dfrac{1}{\sqrt{1+16+0+1}} \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix} = \dfrac{1}{\sqrt{18}} \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix} = \dfrac{1}{3\sqrt{2}} \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3\sqrt{2} \\ -4/3\sqrt{2} \\ 0 \\ 1/3\sqrt{2} \end{bmatrix}$

$\therefore$ Normalized form of $v_2 = \frac{1}{\sqrt{25+1+16+1}} \begin{bmatrix} 5 \\ 1 \\ -4 \\ -1 \end{bmatrix}$

$$= \frac{1}{\sqrt{43}} \begin{bmatrix} 5 \\ 1 \\ -4 \\ -1 \end{bmatrix} = \begin{bmatrix} 5/\sqrt{43} \\ 1/\sqrt{43} \\ -4/\sqrt{43} \\ -1/\sqrt{43} \end{bmatrix}$$

$\therefore$ Orthonormal basis $= \left\{ \begin{bmatrix} 1/3\sqrt{2} \\ -4/3\sqrt{2} \\ 0 \\ -1/3\sqrt{2} \end{bmatrix}, \begin{bmatrix} 5/\sqrt{43} \\ 1/\sqrt{43} \\ -4/\sqrt{43} \\ -1/\sqrt{43} \end{bmatrix} \right\}$

2] Find an orthonormal basis of the subspace spanned by the vectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

Soln: Let $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

$$v_1 = x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$x_2 \cdot v_1 = x_2^T v_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 3$$

$v_1 \cdot v_1 = v_1^T v_1 = 4$

$$\therefore v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$x_3 \cdot v_1 = x_3^T v_1 = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 2$$

$$x_3 \cdot v_2 = x_3^T v_2 = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \frac{1}{2}$$

$$v_2 \cdot v_2 = v_2^T v_2 = \begin{bmatrix} -\frac{3}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} -3/4 \\ 1/4 \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} = \frac{9}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}$$

$$= \frac{12}{16} = \frac{3}{4}$$

$$\therefore v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\frac{1}{2}}{\frac{3}{4}}\begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 1/6 \\ 1/6 \\ 1/6 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

Normalized form of $v_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$

Normalized form of $v_2 = \frac{1}{\sqrt{3}/2} \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$

$$= \begin{bmatrix} \frac{-3}{2\sqrt{2}} \\ \frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} \end{bmatrix}$$

Normalized form of $v_3 = \frac{1}{\sqrt{2/3}} \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$

$$= \begin{bmatrix} 0 \\ -\frac{2}{3}\sqrt{\frac{3}{2}} \\ \frac{1}{3}\sqrt{\frac{3}{2}} \\ \frac{1}{3}\sqrt{\frac{3}{2}} \end{bmatrix}$$