Unit - I

Matrix and Vector Space

Abelian group: A non-empty set G with a binary operation $*$ is called a abelian group, denoted by $(G, *)$, if it satisfies the following axioms.

i) Closure law: $\forall\ a, b \in G,\ a*b \in G$

ii) Associative law: $\forall\ a, b, c \in G,\ a*(b*c) = (a*b)*c$.

iii) Existence of Identity element :- There exists an element $e \in G$, such that

$a*e = e*a = a, \quad \forall a \in G.$

The element e is called the identity element of G under $*$.

iv) Existence of inverse element: For every $a \in G$, $\exists$ an element $a^{-1} \in G$ s.t

$a*a^{-1} = a^{-1}*a = e$.

a^{-1} is called the inverse of a in G under $*$.

v) Commutative law: $\forall a, b \in G\quad a*b = b*a$.

Example: 1) The set of all integer under $+$

2) The set of all real number under $+$.

3) The set of all non zero real number under multiplication.

Note:- The set of all natural number N is not a group under $+$. since identity element doesnot exists

Field :-

Let F be a non empty set on which two operations addition $(+)$ and multiplication $(\cdot)$ are so defined that

i) $(F, +)$ is an abelian group.

ii) $(F - \{0\}, \cdot)$ is an abelian group.

iii) Multiplication is distributive w.r to addition

(i.e $a \cdot (b+c) = a \cdot b + a \cdot c$
$(b+c) \cdot a = b \cdot a + c \cdot a$, $\forall a, b, c \in F$)

Then the set F satisfying the above condition is called a field and is denoted by $(F, +, \cdot)$

ex:- $\mathbb{R} - \{0\}$, $\mathbb{C} - \{0\}$.

1) The set of non zero real numbers with usual addition and multiplication is a field.

2) The set of non zero complex numbers with usual addition and multiplication is a field.

Vector Space

Let $(F, +, \cdot)$ be a field, a non empty set V together with two binary operations vector addition $(+)$ and scalar multiplication $(\cdot)$ is called a vector space over the field F if it satisfies following axioms.

i) $(V, +)$ is an abelian group

a) Closure law : $\forall\ u, v \in V,\ u+v \in V$.

b) Associative law : $\forall\ u, v, w \in V,\ u+(v+w) = (u+v)+w$.

c) There exists a unique element 0 called **zero vector** in V such that $u+0 = 0+u = u,\ \forall\ u \in V$.

d) For each $u \in V$, $\exists$ a unique vector $-u \in V$ such that $u+(-u) = (-u)+u = 0$.

e) Commutative law : $\forall\ u, v \in V,\ u+v = v+u$.

ii) The scalar multiple of u by c, denoted by cu is in V (i.e. $\forall c \in F, \forall u \in V,\ cu \in V$)

iii) $c(u+v) = cu+cv,\ \forall\ u, v \in V,\ \forall\ c \in F$

iv) $(c+d)\,u = cu+du,\ \forall u \in V,\ \forall\ c, d \in F$

v) $c(du) = (cd)\,u,\ \forall c, d \in F,\ u \in V$

vi) $1 \cdot u = u,\ \forall u \in V$.

Note: 1) The elements of the set V are called vectors and elements of the field F are called scalars.

2) When F = R, the field of real numbers then V(R) is called the real vector space.

Euclidean-n space: Let R be an arbitrary field. The notation R^n is used to denote the set of all n-tuples of elements in R.

i.e $R^n = \{ (x_1, x_2, \dots x_n) \mid x_i \in R \}$

R^n is a vector space over R using the following operations.

i) Vector addition: $(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots y_n)$
$= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$

ii) Scalar multiplication: $k(x_1, x_2, \dots, x_n)$
$= (kx_1, kx_2, \dots, kx_n)$.

The zero vector in R^n is the n-tuple of zeros $0 = (0, 0, \dots 0)$

The negative of a vector is defined by
$-(x_1, x_2, \dots, x_n) = (-x_1, -x_2, \dots, -x_n)$

Show that the two dimensional Euclidean space is a vector space.

Sol^n:- Let $R^2 = \{(x, y) \mid x, y \in R\}$

vector addition is defined as

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

and scalar multiplication is defined as

$a \cdot (x, y) = (ax, ay)$, $(x, y) \in R^2$ $\forall$ $a \in R$.

i) Closure law:

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in R^2$.

$\forall$ $(x_1, y_1), (x_2, y_2) \in R^2$.

ii) Associative law:

$[(x_1, y_1) + (x_2, y_2)] + (x_3, y_3) = (x_1 + x_2, y_1 + y_2) + (x_3, y_3)$

$= (x_1 + x_2 + x_3, y_1 + y_2 + y_3)$

$(x_1, y_1) + [(x_2, y_2) + (x_3, y_3)]$

$= (x_1, y_1) + (x_2 + x_3, y_2 + y_3)$

$= (x_1 + x_2 + x_3, y_1 + y_2 + y_3)$

$\therefore [(x_1, y_1) + (x_2, y_2)] + (x_3, y_3) = (x_1, y_1) + [(x_2, y_2) + (x_3, y_3)]$

$\forall$ (x_1, y_1) (x_2, y_2), $(x_3, y_3) \in R^2$.

iii) Existence of Identity: $\exists$ $0 = (0, 0) \in R^2$ s.t

$(x, y) + (0, 0) = (x, y) = (0, 0) + (x, y)$, $\forall (x, y) \in R^2$.

$\therefore (0,0) \in R^2$ is an identity element of R^2.

iv) Existence of Inverse :

$\forall (x, y) \in R^2 \ \exists (-x, -y) \in R^2$ such that

$(x, y) + (-x, -y) = (0, 0)$.

$\therefore (-x, -y)$ is inverse of (x, y).

v) Commutative law :

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$= (x_2 + x_1, y_2 + y_1)$

$= (x_2, y_2) + (x_1, y_1)$

$\therefore (R^2, +)$ is an abelian group.

vi) $a(x, y) = (ax, ay) \in R^2$, $\forall a \in R$, $(x,y) \in R^2$.

vii) $a[(x_1, y_1) + (x_2, y_2)] = a(x_1 + x_2, y_1 + y_2)$

$= (ax_1, ay_1) + (ax_2, ay_2)$

$= a(x_1, y_1) + a(x_2, y_2)$.

$\forall a \in R$, $(x_1, y_1)(x_2, y_2) \in R^2$.

viii) $(a+b)(x, y) = ((a+b)x, (a+b)y)$

$= (ax, ay) + (bx, by)$

$= a(x, y) + b(x, y)$

$\forall a \in R \ \forall (x, y) \in R^2$.

ix) $(ab)(x, y) = ((ab)x, (ab)y)$

$= (a(bx), a(by))$

$= a(b(x,y))$ $\forall\ a, b \in R, (x,y) \in R^2$

x) $1 \cdot (x,y) = (1 \cdot x, 1 \cdot y) = (x, y)$ $\forall\ (x,y) \in R^2$

$\forall\ 1 \in R$.

$\therefore$ R^2 is a vector space over R.

How
2) Check whether $V = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \middle| x, y \in R \right\}$

is a vector space under usual matrix addition and scalar multiplication.

3) Check whether $V = \{ (x, y) \mid x, y \in R \}$ with vector addition defined by

$(x_1, y_1) + (x_2, y_2) = (x_1 y_2, x_2 y_1)$

and scalar multiplication defined by

$c(x, y) = (cx, cy)$ is a vector space.

Soln:- 1) closure law.

$(x_1, y_1) + (x_2, y_2) = (x_1 y_2, x_2 y_1) \in V$

$\forall\ (x_1, y_1), (x_2, y_2) \in V$

2) Associative law:

$((x_1, y_1) + (x_2, y_2)) + (x_3, y_3) = (x_1 y_2, x_2 y_1) + (x_3, y_3)$

$$= (x_1 y_2 y_3, \quad x_2 y_1 x_3).$$

$$(x_1, y_1) + ((x_2, y_2) + (x_3, y_3)) = (x_1, y_1) + (x_2 y_3, x_3 y_2)$$

$$= (x_1 x_3 y_2, \; x_2 y_2 y_1)$$

$$((x_1, y_1) + (x_2, y_2)) + (x_3, y_3) \neq (x_1, y_1) + ((x_2, y_2) + (x_3, y_3)).$$

$\therefore V$ is not a vector space

Subspaces

Defn:- Let V be a vector space over the field F. A non empty subset W is said to be a subspace of V if

i) $0 \in W$

ii) For each $u, v \in W \Rightarrow u+v \in W$

iii) For each $u \in W$ and each $c \in F \Rightarrow cu \in W$

If W_1 and W_2 are subspaces of a vectorspace V then $W_1 \cap W_2$ is also a subspace of V.

Proof:- i) Given W_1 and W_2 are subspaces of V

$\Rightarrow 0 \in W_1$ and $0 \in W_2$

$\Rightarrow 0 \in W_1 \cap W_2$.

ii) Let $x, y \in W_1 \cap W_2$

$\Rightarrow x, y \in W_1$ and $x, y \in W_2$

Since W_1 and W_2 are subspaces of V

$\Rightarrow x+y \in W_1$ and $x+y \in W_2$.

$\Rightarrow x+y \in W_1 \cap W_2$. , $\forall x, y \in W_1 \cap W_2$.

iii) Let $\alpha \in F$, $x \in W_1 \cap W_2$

$\Rightarrow x \in W_1$ and $x \in W_2$.

Since W_1 and W_2 are subspace.

$\alpha x \in W_1$ and $\alpha x \in W_2$

$\therefore \alpha x \in W_1 \cap W_2$, $\forall \alpha \in F$ $\forall x \in W_1 \cap W_2$

Hence $W_1 \cap W_2$ is a subspace of V.

2) If W_1 and W_2 are subspaces of a vectorspace V then $W_1 \cup W_2$ need not be a subspace of V.

<u>Proof</u>:- Given W_1 and W_2 are subspaces of a vector space V.

Let $W_1 = \{(a_1, a_2, 0) \mid a_1, a_2 \in R\}$

$W_2 = \{(a_1, 0, a_3) \mid a_1, a_3 \in \mathbb{R}\}$.

W_1 and W_2 are subspaces of $\mathbb{R}^3(\mathbb{R})$

~~Let~~ Now consider $\alpha = (1, 2, 0)$ $\beta = (3, 0, 5)$ $\in W_1 \cup W_2$

$\alpha + \beta = (4, 2, 5) \notin W_1 \cup W_2)$

$\therefore W_1 \cup W_2$ is not a subspace of $R^3(\mathbb{R})$.

Theorem:- A nonempty subset W of a vectorspace V over the field F is a subspace of V if and only if for any $\alpha, \beta \in W$ and $c \in F$, $c\alpha + \beta \in W$.

Proof:- ($\Rightarrow$) Given W is a subspace of vectorspace V over the field F.

Let $\alpha, \beta \in W$ and $c \in F$

$\Rightarrow c\alpha \in W$ ($\because$ W is a subspace of V)

$c\alpha \in W$ and $\beta \in W \Rightarrow c\alpha + \beta \in W$

($\because$ W is a subspace of V).

Conversely, Assume that in a non empty subset W of V for any $\alpha, \beta \in W$ and $c \in F$, $c\alpha + \beta \in W$ holds, we need to prove W is a subspace of V.

i) Since $W \neq \phi$, Let $\gamma \in W$, $-1 \in F$

By assumption $(-1)\gamma + \gamma \in W$

$\Rightarrow 0 \in W$.

ii) Let $c \in F$, $0, \alpha \in W \Rightarrow c \cdot \alpha + 0 \in W$

$\Rightarrow c\alpha \in W$, $\forall c \in F$, $\alpha \in W$

iii) Let $1 \in F$, $\alpha, \beta \in W$

$\Rightarrow 1 \cdot \alpha + \beta \in W$

$\Rightarrow \alpha + \beta \in W$, $\forall \alpha, \beta \in W$.

$\therefore$ W is a subspace of V.

If W_1 and W_2 are subspaces of V over F then P.T $W_1 + W_2 = \{(x+y) \mid x \in W_1, y \in W_2\}$ is a subspace of V.

<u>Proof:-</u> Given W_1 and W_2 are subspaces of V over F.

and $0 \in W_2$

$\Rightarrow 0 \in W_1$

$\Rightarrow 0 = 0 + 0 \in W_1 + W_2$

$\therefore 0 \in W_1 + W_2$.

Let $\alpha = x_1 + y_1$, $\beta = x_2 + y_2 \in W_1 + W_2$, $c \in F$

$c\alpha + \beta = c(x_1 + y_1) + (x_2 + y_2)$

$= (cx_1 + x_2) + (cy_1 + y_2) \in W_1 + W_2$

$\left(\because W_1 \text{ and } W_2 \text{ are subspace of } V \Rightarrow cx_1 + x_2 \in W_1 \text{ \& } cy_1 + y_2 \in W_2\right)$

$\Rightarrow c\alpha + \beta \in W_1 + W_2$

Hence $W_1 + W_2$ is a subspace of V.

Let V be a vectorspace of all 2×2 matrices then S.T $W = \left\{ \begin{bmatrix} x & -x \\ y & z \end{bmatrix} \middle| -x, x, y, z \in F \right\}$ is a subspace of V

Sol^n:- i) Clearly $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$.

ii) Let $\alpha = \begin{bmatrix} x_1 & -x_1 \\ y_1 & z_1 \end{bmatrix}$, $\beta = \begin{bmatrix} x_2 & -x_2 \\ y_2 & z_2 \end{bmatrix} \in W$.

and $c \in F$

$$c\alpha + \beta = c\begin{bmatrix} x_1 & -x_1 \\ y_1 & z_1 \end{bmatrix} + \begin{bmatrix} x_2 & -x_2 \\ y_2 & z_2 \end{bmatrix}$$

$$= \begin{bmatrix} cx_1 + x_2 & -cx_1 - x_2 \\ cy_1 + y_2 & cz_1 + z_2 \end{bmatrix} \in W$$

$\left(\because cx_1 + x_2,\ cy_1 + y_2,\ cz_1 + z_2 \in F \right)$.

Thus $c\alpha + \beta \in W$.

$\therefore$ W is a subspace of V.

Let $\vec{x} = (1, 2, -3)$, $\vec{y} = (-2, 3, 0)$ be 2 vectors in R^3. and W be the set of vectors of the form $a\vec{x} + b\vec{y}$ where a and b are real numbers. Show that W is a subspace of R^3

Sol^n:-

Given $W = \{a\vec{x} + b\vec{y} \mid a, b \in R\}$.

Clearly $0 = 0 \cdot \vec{x} + 0 \cdot \vec{y} \in W$.

Let $c \in F$, $\alpha = a_1\vec{x} + b_1\vec{y}$, $\beta = a_2\vec{x} + b_2\vec{y}$.

where $a_1, b_1, a_2, b_2 \in R$

$$c\alpha + \beta = c(a_1\vec{x} + b_1\vec{y}) + (a_2\vec{x} + b_2\vec{y})$$

$$= (ca_1 + a_2)\vec{x} + (cb_1 + b_2)\vec{y}$$

$\in W$ $\quad (\because ca_1 + a_2, cb_1 + b_2 \in R)$

$\therefore c\alpha + \beta \in W$.

Thus W is a subspace of R^3.

Let A be any $n \times n$ matrix with entries in a field. $V = F^n$ = the set of all n-dimensional column vectors with entries in F. Then prove that $W = \{X \in F^n \mid AX = 0\}$ (where 0 is the zero element of F^n & $A = (a_{ij})$) is a subspace of V.

$W \neq \phi$

Sol^n:- $W \neq \phi$ since 0 of F^n is in W.

Let $X_1, X_2 \in W$ and $c \in F$.

$\Rightarrow AX_1 = 0$ and $AX_2 = 0$

Consider, $A(cX_1 + X_2) = cAX_1 + AX_2$

$= 0$

$\Rightarrow cX_1 + X_2 \in W$.

$\therefore W$ is a subspace of V.

Linear Span

Suppose $u_1, u_2, \cdots, u_m$ are any vectors in a vector space V. Any vector of the form $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m$, where the α_i are scalars, is called a linear combination of $u_1, u_2, \cdots, u_m$. The collection of all such linear combinations, denoted by span $(u_1, u_2, \cdots, u_n)$ is called the linear span of $u_1, u_2, \cdots, u_m$.

Linear dependence and independence

Let V be a vector space over a field F. The vectors $\{v_1, v_2, \cdots, v_m\}$ in V are said to be linearly dependent if there exists scalar $\alpha_1, \alpha_2, \cdots, \alpha_m$ not all of them zero, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0.$$

Let V be a vector space over a field F. The vectors $\{v_1, v_2, \cdots, v_m\}$ in V are said to be linearly independent if for every expression of the type $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0$

$\Rightarrow \alpha_i = 0, \ \forall i$

Theorem: A set of vectors $\{u_1, u_2, \ldots, u_n\}$ is linearly dependent if and only if atleast one of them is a linear combination of the others

Proof: ($\Rightarrow$) Let $\{u_1, u_2, \ldots, u_n\}$ is linearly dependent then there exists a scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ not all are zero such that $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = 0$

Assume $\alpha_i \neq 0$

$$\Rightarrow \alpha_i u_i = -\alpha_1 u_1 - \alpha_2 u_2 - \cdots - \alpha_{i-1} u_{i-1} - \alpha_{i+1} u_{i+1} - \cdots - \alpha_n u_n$$

$$u_i = -\frac{\alpha_1}{\alpha_i} u_1 + \left(\frac{-\alpha_2}{\alpha_i}\right) u_2 + \cdots + \left(\frac{-\alpha_{i-1}}{\alpha_i}\right) u_{i-1} + \left(\frac{\alpha_{i+1}}{\alpha_i}\right) u_{i+1} + \cdots + \left(-\frac{\alpha_n}{\alpha_i}\right) u_n$$

Thus $u_i \in V$ is expressed as a linear combination of $u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n$.

Conversely, Let $u_i \in V$ is expressed as a linear combination of others

i.e $u_i = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_{i-1} u_{i-1} + \beta_{i+1} u_{i+1} + \cdots + \beta_n u_n$

where $\beta_1, \beta_2, \ldots, \beta_n \in F$

$\Rightarrow \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_{i-1} u_{i-1} + (-1) u_i + \beta_{i+1} u_{i+1} + \cdots + \beta_n u_n = 0$

where the co-efficient of u_i is $-1 \neq 0$.

Hence $\{u_1, u_2, \ldots, u_n\}$ are linearly dependent.

Problem:- Write the vector $v = (1, -2, 5)$ in $\mathbb{R}^3$ as a linear combination of the vectors $u_1 = (1, 1, 1)$, $u_2 = (1, 2, 3)$ and $u_3 = (2, -1, 1)$

Sol^n:-

Let $v = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3$ where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$.

i.e $(1, -2, 5) = \alpha_1 (1, 1, 1) + \alpha_2 (1, 2, 3)$
$+ \alpha_3 (2, -1, 1)$

$\Rightarrow (1, -2, 5) = (\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 - \alpha_3, \alpha_1 + 3\alpha_2 + \alpha_3)$

$\Rightarrow \alpha_1 + \alpha_2 + 2\alpha_3 = 1 \quad - (1)$

$\alpha_1 + 2\alpha_2 - \alpha_3 = -2 \quad - (2)$

$\alpha_1 + 3\alpha_2 + \alpha_3 = 5 \quad - (3)$

$(1) - (2) \Rightarrow -\alpha_2 + 3\alpha_3 = 3 \quad - (4)$

$(2) - (3) \Rightarrow -\alpha_2 - 2\alpha_3 = -7 \quad - (5)$

$(4) - (5) \Rightarrow 5\alpha_3 = 10$

$\underline{\underline{\alpha_3 = 2}}$

Substitute in (4) $\quad -\alpha_2 + 3\alpha_3 = 3$

$-\alpha_2 + 3 \times 2 = 3$

$-\alpha_2 = -3$

$\underline{\underline{\alpha_2 = 3}}$

Sub in (1) $\Rightarrow \alpha_1 + 3 + 2 \times 2 = 1$

$\underline{\underline{\alpha_1 = -6}}$

Thus $v = -6u_1 + 3u_2 + 2u_3$.

2) Can $v = (2, -5, 3) \in \mathbb{R}^3$ be represented as a linear combination of the vectors $u_1 = (1, -3, 2)$, $u_2 = (2, -4, -1)$, $u_3 = (1, -5, 7)$.

Sol^n:- Suppose v is a linear combination of u_1, u_2, u_3. then

$$v = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3.$$

$$(2, -5, 3) = \alpha_1 (1, -3, 2) + \alpha_2 (2, -4, -1) + \alpha_3 (1, -5, 7)$$

$$\Rightarrow (2, -5, 3) = (\alpha_1 + 2\alpha_2 + \alpha_3, \; -3\alpha_1 - 4\alpha_2 - 5\alpha_3, \; 2\alpha_1 - \alpha_2 + 7\alpha_3)$$

$$\Rightarrow \alpha_1 + 2\alpha_2 + \alpha_3 = 2$$

$$-3\alpha_1 - 4\alpha_2 - 5\alpha_3 = -5$$

$$2\alpha_1 - \alpha_2 + 7\alpha_3 = 3$$

$$[A : B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ -3 & -4 & -5 & -5 \\ 2 & -1 & 7 & 3 \end{array}\right]$$

$R_2 \to R_2 + 3R_1$ $\quad$ $R_3 \to R_3 - 2R_1$

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$$[A:B] \sim \begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 2 & -2 & : & 1 \\ 0 & -5 & 5 & : & -1 \end{bmatrix}$$

$R_3 \longrightarrow 2R_3 + 5R_2$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 1 & : & 2 \\ 0 & 2 & -2 & : & 1 \\ 0 & 0 & 0 & : & 3 \end{bmatrix}$$

$\therefore \rho(A:B) = 3 \quad \because \rho(A) = 2$

$\therefore \rho(A) \neq \rho(A:B)$.

Thus the system is inconsistent.

and so has no solution.

Thus v cannot be written as a linear combination of u_1, u_2, u_3.

3) For which values of k, the vector $u = (1, -2, k)$ in $\mathbb{R}^3$ be a linear combination of the vectors $v = (3, 0, -2)$ and $w = (2, -1, -5)$.

Soln:- Let $u = \alpha_1 v + \alpha_2 w$

$(1, -2, k) = \alpha_1(3, 0, -2) + \alpha_2(-2, -1, -5)$

$(1, -2, k) = (3\alpha_1 - 2\alpha_2, \quad -\alpha_2, -2\alpha_1 - 5\alpha_2)$

$$3\alpha_1 + 2\alpha_2 = 1 \quad \text{--- (1)}$$

$$-\alpha_2 = -2 \quad \text{--- (2)}$$

$$-2\alpha_1 - 5\alpha_2 = k \quad \text{--- (3)}$$

(2) $\Rightarrow \alpha_2 = 2$

Sub in (1) $\Rightarrow 3\alpha_1 + 4 = 1$

$$3\alpha_1 = -3$$

$$\alpha_1 = -1$$

Sub in (3) $\Rightarrow -2\times -1 - 5\times 2 = k$

$$\Rightarrow k = -8$$

$\therefore k = -8$

4) Write the matrix $E = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$ as the linear combination of the matrices $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}$

Sol^n:- Let $E = \alpha_1 A + \alpha_2 B + \alpha_3 C$

where $\alpha_1, \alpha_2, \alpha_3 \in F$.

$$\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_1 + 2\alpha_3 \\ \alpha_1 + \alpha_2 & \alpha_2 - \alpha_3 \end{bmatrix}$$

$$\Rightarrow \quad \alpha_1 = 3$$

$$\alpha_1 + 2\alpha_3 = 1$$

$$\alpha_1 + \alpha_2 = 1$$

$$\alpha_2 - \alpha_3 = -1$$

$$\Rightarrow \alpha_1 + 2\alpha_3 = 1$$

$$\Rightarrow 2\alpha_3 = 1 - \alpha_1$$

$$2\alpha_3 = 1 - 3$$

$$\Rightarrow \alpha_3 = \frac{-2}{2}$$

$$\Rightarrow \underline{\alpha_3 = -1}$$

$$\alpha_1 + \alpha_2 = 1$$

$$\alpha_2 = 1 - \alpha_1$$

$$\alpha_2 = 1 - 3$$

$$\underline{\alpha_2 = -2}$$

$$\alpha_2 - \alpha_3 = -1$$

$$\alpha_2 = -1 + \alpha_3$$

$$\alpha_2 = -1 + 1$$

$$\therefore \quad E = 3A - 2B - C.$$

Note: Form the matrix whose columns are the given vectors. Reduce the matrix to echelon form. If all columns are pivot column (column with pivot element) then given vectors are linearly independent otherwise linearly dependent.

1) Check whether the following vectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$ are

linearly independent or not.

Sol^n:-

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$A \sim \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & -6 & -6 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_2$

$$A \sim \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

In echelon form of matrix A, has only first 2 columns are pivotal coulumn.

$\therefore$ Given set of vectors are linearly dependent.

Q) Check whether the following set of vectors $\left\{ \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \right.$ are linearly independent or not.

Soln:-

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 4 & -3 \\ 5 & 8 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 4R_1$, $R_3 \rightarrow R_3 - 5R_1$

$$A \sim \begin{bmatrix} 1 & 4 & 3 \\ 0 & -12 & -15 \\ 0 & -12 & -15 \end{bmatrix}$$

$R_3 \longrightarrow R_3 - R_2$

$$A \sim \begin{bmatrix} 1 & 4 & 3 \\ 0 & -12 & -15 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus in echelon form of A, only first 2 columns are pivot columns.

$\therefore$ Given set of vectors are linearly dependent.

3) Check whether the following set of vectors

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ -2 \\ 3 \end{bmatrix}$$ are linearly independent

or not.

Solⁿ:-

$$A = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 4 \\ -1 & 0 & -2 \\ 2 & -1 & 3 \end{bmatrix}$$

$R_3 \longrightarrow R_3 + R_1, \quad R_4 \longrightarrow R_4 - 2R_1$

$$A \sim \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 4 \\ 0 & 4 & 4 \\ 0 & -9 & 9 \end{bmatrix}$$

$R_4 \to 4R_4 + 9R_3$

$$A \sim \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 6 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$A \sim \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 6 \\ 0 & 0 & -8 \\ 0 & 0 & 0 \end{bmatrix}$$

All the columns are pivotal columns

$\therefore$ Given set of vectors are linearly independent.

4) Check whether the following set of vectors are linearly independent or not. (15)

4 1) $\begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}1\\-2\\1\end{bmatrix}, \begin{bmatrix}-3\\2\\-1\end{bmatrix}$ ✗

2) $\begin{bmatrix}1\\2\\4\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ ✓

Basis and Dimension

Defn:- A set $S = \{u_1, u_2, \ldots, u_n\}$ of vectors is a basis of vectorspace V over a field F if it has the following two properties

i) S is linearly independent

ii) S spans V.

Note: A set $S = \{u_1, u_2, \ldots, u_n\}$ of vectors is a basis of V if every $v \in V$ can be written uniquely as a linear combination of the basis vectors.

Def^n:- The dimension of a vectorspace V over a field F is the number of vectors in any basis set of V and denoted by $\dim V$.

Problem:

Check whether the set $\{(1,3,1), (0,2,0), (0,0,7)\}$ is a basis for $\mathbb{R}^3$. If it is, find the dimension of $\mathbb{R}^3(\mathbb{R})$.

Sol^n:- i) First we have To check the given set of vectors linearly independent

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 0 & 7 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1, \quad R_3 \to R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

In the echelon form of A, all the columns are pivot columns.

$\therefore$ The given set of vectors are linearly independent. (16)

ii) To check the given set of vectors spans $\mathbb{R}^3$.

Let $(a, b, c) \in \mathbb{R}^3$

Suppose the given

$(a, b, c) = \alpha_1 (1, 3, 1) + \alpha_2 (0, 2, 0) + \alpha_3 (0, 0, 7)$

$(a, b, c) = (\alpha_1, 3\alpha_1 + 2\alpha_2, \alpha_1 + 7\alpha_3)$

$\alpha_1 = a$ — (1)

$3\alpha_1 + 2\alpha_2 = b$ — (2)

$\alpha_1 + 7\alpha_3 = c.$ — (3)

(2) $\Rightarrow$ $\alpha_2 = \dfrac{b - 3\alpha_1}{2}$

$\alpha_2 = \dfrac{b - 3a}{2}.$

(3) $\Rightarrow$ $\alpha_3 = \dfrac{c - \alpha_1}{7}$

$\alpha_3 = \dfrac{c - a}{7}$, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$.

Thus system of equation has solution.

Hence the given set of vectors spans $\mathbb{R}^3$.

$\therefore$ The given set of vectors is a basis for $\mathbb{R}^3$.

2) Check whether the set $\{(2,3),(0,4)\}$ is a basis for $\mathbb{R}^2$.

Sol^n:- i) To check the set $\{(2,3),(0,4)\}$ is linearly independent.

$$A = \begin{bmatrix} 2 & 0 \\ 3 & 4 \end{bmatrix}$$

$R_2 \rightarrow 2R_2 - 3R_1$

$$A \sim \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}$$

In echelon form of A, all the columns are pivot columns.

$\therefore$ The set $\{(2,3),(0,4)\}$ is linearly independent vectors.

ii) To check the set $\{(2,3),(0,4)\}$ is spans $\mathbb{R}^2$.

(17)

Let $(a, b) \in R^2$

$$(a, b) = \alpha_1 (2, 3) + \alpha_2 (0, 4)$$

$$(a, b) = (2\alpha_1, 3\alpha_1 + 4\alpha_2)$$

$$2\alpha_1 = a \quad \text{--- (1)}$$

$$3\alpha_1 + 4\alpha_2 = b \quad \text{--- (2)}$$

$$(1) \Rightarrow \alpha_1 = \frac{a}{2}$$

$$(2) \Rightarrow \alpha_2 = \frac{b - 3\alpha_1}{4}$$

$$\alpha_2 = \frac{b - \frac{3a}{2}}{4}$$

$$\alpha_2 = \frac{2b - 3a}{8}, \quad \alpha_1, \alpha_2 \in \mathbb{R}.$$

Thus the system of equation has solution.

~~i.e~~ ∴ The given set $\{(2,3), (0,4)\}$ spans R^2.

~~∴~~ Thus the given set $\{(2,3), (0,4)\}$ is a bases for $\mathbb{R}^2$.

3) Show that the set $S = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ is a basis of $\mathbb{R}^3(\mathbb{R})$

1. ✓

2. $\alpha_3 = c$

Note:- standard bases of $\mathbb{R}^3(\mathbb{R})$ is

$\alpha_2 = b - a$

$\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

$\alpha_1 = b - c$

Standard bases of $\mathbb{R}^2(\mathbb{R})$ is $\{(1, 0), (0, 1)\}$

To find the basis for subspace W of vectorspace V:-

Let $u_1, u_2, \ldots, u_n$ are the set of vectors which spans W. Form the matrix A whose column are the given vectors and reduce it to an echelon form. Identify the pivot columns, corresponding original column vector forms a basis for W. and hence dimension of W can be calculated.

1) Find the dimension of the subspace W of $\mathbb{R}^3$ spanned by the vectors $\{(3, 1, 0), (2, 1, 0), (1, 1, -2)\}$.

Soln:- Let
$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$R_2 \to 3R_2 - R_1$

$$A \sim \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

which is the echlon form of A.

In echelon form of a matrix all the ~~[illegible]~~ columns are pivot columns

$\therefore$ Bases of W is $\{(3,1,0), (2,1,0), (1,1,-2)\}$

$\therefore$ dim W = 3.

2) Find the dimension and basis of the subspace spanned by the vectors $u_1 = (2,4,2)$ $u_2 = (1,-1,0)$, $u_3 = (1,2,1)$ $u_4 = (0,3,1)$ in $V_3(\mathbb{R})$.

Sol^n:- Let $A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & -1 & 2 & 3 \\ 2 & 0 & 1 & 1 \end{bmatrix}$

$R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - R_1$

$A \sim \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -3 & 0 & 3 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

$R_3 \rightarrow 3R_3 - R_2$

$A \sim \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -3 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

In the echelon form of matrix A, only first two columns are pivot columns.

$\therefore$ Bases of subspace spanned by given vector is $\{(2, 4, 2), (1, -1, 0)\}$

$\therefore$ dimension of subspace is 2.

3) Let W be the subspace of $\mathbb{R}^4$, spanned by the vectors $u_1 = (1, -2, 5, -3)$, $u_2 = (2, 3, 1, -4)$, $u_3 = (3, 8, -3, -5)$.

i) Find the bases and dimension of W

ii) Extend the bases of W to a basis of $\mathbb{R}^4$.

Sol^n:- Form the matrix whose columns are the given vectors and reduce to echelon form

$$A = \begin{bmatrix} 1 & 2 & +3 \\ -2 & 3 & 8 \\ 5 & 1 & -3 \\ -3 & -4 & -5 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - 5R_1$, $R_4 \to R_4 + 3R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 7 & 14 \\ 0 & -9 & -18 \\ 0 & 2 & 4 \end{bmatrix}$$

$R_4 \to 7R_4 - 2R_2$, $R_3 \to 7R_3 + 9R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 7 & 14 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In the echelon form of matrix A only first two columns are pivot columns.

$\therefore$ Basis of subspace W is

$\{(1, -2, 5, -3), (2, 3, 1, -4)\}$

and $\dim W = 2$

ii) Let $w_1 = (0, 0, 1, 0)$, $w_2 = (0, 0, 0, 1)$

Then u_1, u_2, w_1, w_2 form a matrix in echelon form. Thus they are linearly independent and they form a basis of $\mathbb{R}^4$.

Hence Basis of $\mathbb{R}^4$ is $\{(1, -2, 5, -3)$, $(2, 3, 1, -4), (0, 0, 1, 0), (0, 0, 0, 1)\}$

4) Determine whether $(1,1,1,1)$, $(1,2,3,2)$, $(2,5,6,4)$, $(2,6,8,5)$ form a basis of $\mathbb{R}^4$. If not find the dimension of the subspace they span.

Sol^n:- Let

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 5 & 6 \\ 1 & 3 & 6 & 8 \\ 1 & 2 & 4 & 5 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$

$$A \sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 4 & 6 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$, $R_4 \to R_4 - R_2$

$$A \sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

$R_4 \to 2R_4 - R_3$

$$A \sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In echelon form of a matrix only first three columns are pivot columns

$\therefore$ Given set of vectors doesnot form a basis of $\mathbb{R}^4$.

Basis of subspace W spanned by given set vectors is $\{(1,1,1,1)\ (1,2,3,2)\ (2,5,6,4)\}$

$\therefore$ $\boxed{\dim W = 3.}$

5) Determine the dimension of a subspace H of $\mathbb{R}^3$ spanned by the vectors $v_1 = (2,-8,6)$ $v_2 = (3,-7,-1)$ $v_3 = (-1,6,-7)$.

Theorem:- If $\{u_1, u_2, \cdots, u_n\}$ is a basis for a vector space V, then prove that any vector in V can be uniquely expressed as a linear combination of vectors in the basis.

Proof:- Let $\{u_1, u_2, \cdots, u_n\}$ is a basis for a vectorspace V.

Let $v \in V$, then there exists a scalars $\alpha_1, \alpha_2, \cdots, \alpha_n \in F$ such that

$v = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$

($\because \{u_1, u_2, \ldots, u_n\}$ is a basis for V)

To prove above representation is unique:-

Suppose that $v \in V$ has one more representation,

i.e $v = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n$, (where $\beta_1, \beta_2, \ldots \ldots \beta_n \in F$)

$\Rightarrow \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n$

$\Rightarrow (\alpha_1 - \beta_1) u_1 + (\alpha_2 - \beta_2) u_2 + \cdots + (\alpha_n - \beta_n) u_n = 0.$

Since $\{u_1, u_2, \ldots, u_n\}$ is linearly independent.

$\therefore \alpha_i - \beta_i = 0, \quad \forall i, \ 1 \leq i \leq n$

$\Rightarrow \alpha_i = \beta_i, \quad \forall i, \ 1 \leq i \leq n.$

Hence any vector in V can be uniquely expressed as a linear combination of basis vectors $\{u_1, u_2, \ldots, u_n\}$.

Theorem:- Let v_1, and v_2 are vectors in a vectors space V, let $H = \text{span}\{v_1, v_2\}$. Show that H is a subspace of V.

Proof:- $0 = 0 \cdot v_1 + 0 \cdot v_2$

$\Rightarrow 0 \in H$

Let $\alpha = a_1 \cdot v_1 + b_1 v_2$, $\beta = a_2 v_1 + b_2 v_2 \in H$

and $c \in F$

$c\alpha + \beta = c(a_1 v_1 + b_1 v_2) + (a_2 v_1 + b_2 v_2)$

$= (ca_1 + a_2) v_1 + (cb_1 + b_2) v_2$

$\in H$

$\therefore c\alpha + \beta \in H$

Thus H is a subspace of V.

Coordinates

Let V be an n-dimensional vector space over F with basis $B = \{u_1, u_2, \dots, u_n\}$. Then each vector $v \in V$ can be expressed uniquely as a linear combination of the basis vectors in S, say

$$v = a_1 u_1 + a_2 u_2 + \dots + a_n u_n$$

These n scalars $a_1, a_2, \dots, a_n$ are called the coordinates of v relative to basis B, and they form a vector $[a_1\ a_2 \dots a_n]$ in F^n called the coordinate vector of v relative to B, denoted by $[v]_B$ or $[v]$.

1) Find the coordinate vector of $(a, b, c) \in \mathbb{R}^3$ with respect to the basis $\{(1,1,1), (1,1,0), (1,0,0)\}$

sol^n:- $(a, b, c) = \alpha_1(1,1,1) + \alpha_2(1,1,0) + \alpha_3(1,0,0)$

$(a, b, c) = (\alpha_1 + \alpha_2 + \alpha_3,\ \alpha_1 + \alpha_2,\ \alpha_1)$

$$\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 = a$$

$$\alpha_1 + \alpha_2 = b$$

$$\alpha_1 = c$$

$\Rightarrow \underline{\alpha_1 = c}$ $\underline{\alpha_2 = b - c}$

$\alpha_3 = a - \alpha_1 - \alpha_2$

$= a - c - b + c$

$\underline{\alpha_3 = a - b}$

Thus $[c, b-c, a-b]$ is the co-ordinate vector of (a, b, c).

2) Find the coordinate vector of $(2, 7, -4)$ in $\mathbb{R}^3$ w.r. to the basis

$\{(1, 2, 0), (1, 3, 2), (0, 1, 3)\}$

Sol^n:-

$(2, 7, -4) = \alpha_1(1, 2, 0) + \alpha_2(1, 3, 2) + \alpha_3(0, 1, 3)$

$(2, 7, -4) = (\alpha_1 + \alpha_2, 2\alpha_1 + 3\alpha_2 + \alpha_3, 2\alpha_2 + 3\alpha_3)$

$\alpha_1 + \alpha_2 = 2$

$2\alpha_1 + 3\alpha_2 + \alpha_3 = 7$ Calc

$2\alpha_2 + 3\alpha_3 = -4$

$\alpha_1 = -11, \quad \alpha_2 = 13, \quad \alpha_3 = -10$

Thus $[-11, 13, -10]$ is the coordinate vector of $(2, 7, -4)$ in $\mathbb{R}^3$ w.r. to the given basis.

3) Relative to the basis $B = \{(2,1), (-1,1)\}$ of $\mathbb{R}^2$, find the coordinate vector of $v = (4,5)$.

Solⁿ:- $v = \alpha_1 (2,1) + \alpha_2 (-1,1)$

$$(4,5) = (2\alpha_1 - \alpha_2, \alpha_1 + \alpha_2)$$

$$2\alpha_1 - \alpha_2 = 4$$

$$\alpha_1 + \alpha_2 = 5$$

$$\alpha_1 = 3, \quad \alpha_2 = 2$$

∴ The coordinate vector of v is $[3, 2]$.

4) Relative to the basis $B = \{(1,0), (1,2)\}$ in $\mathbb{R}^2$, x in $\mathbb{R}^2$ has a coordinate vector $[x]_B = [-2, 3]$. Find x.

Solⁿ:- $x = \alpha_1 (1,0) + \alpha_2 (1,2)$

$$x = -2(1,0) + 3(1,2)$$

$$x = (1, 6)$$

The Column Space of a Matrix

The column space of an $m \times n$ matrix A, written as $\text{Col} A$, is the set of all linear combinations of the columns of A. If $A = [a_1 \; a_2 \; \cdots \; a_n]$ then

$\text{Col} A = \text{Span}\{a_1, a_2, \ldots, a_n\}$.

Note: 1) The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

2) $\text{Col} A = \{ b : b = Ax \text{ for some } x \text{ in } \mathbb{R}^n \}$.

Problem: 1) Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$

Determine whether b is in column space of A.

Sol^n:- The vector b is a linear combination of the columns of A $\Longleftrightarrow$ b can be written as Ax for some x

i.e. $\Longleftrightarrow$ the equation $Ax = b$ has a solution.

$$[A:b] = \left[\begin{array}{ccc:c} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{array}\right]$$

$R_2 \to R_2 + 4R_1, \quad R_3 \to R_3 + 3R_1$

$$[A:b] \sim \left[\begin{array}{ccc:c} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{array}\right]$$

$R_3 \to 3R_3 - R_2$

$$[A:b] \sim \left[\begin{array}{ccc:c} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\overset{2}{\rho(A)} = \overset{2}{\rho(A:b)} = 2 <$ number of unknown

$\Rightarrow$ The equation $AX = b$ has a solution.

Thus b is in column space of A.

~~2)~~ Check whether $B = \begin{bmatrix} -7 \\ 3 \\ 2 \end{bmatrix}$ is in the column space of $A = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 0 & 7 \\ -3 & -5 & -3 \end{bmatrix}$

Solⁿ:- The vector B is a linear combination of the columns of A $\iff$ B can be written as Ax for some x

$\iff$ the equation $Ax = B$ has a solution.

$$[A:B] = \left[\begin{array}{ccc:c} 1 & -1 & 5 & -7 \\ 2 & 0 & 7 & 3 \\ -3 & -5 & -3 & 2 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + 3R_1$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & -1 & 5 & -7 \\ 0 & 2 & -3 & 17 \\ 0 & -8 & 12 & -19 \end{array}\right]$$

$R_3 \to R_3 + 4R_2$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & -1 & 5 & -7 \\ 0 & 2 & -3 & 17 \\ 0 & 0 & 0 & 49 \end{array}\right]$$

$$\overset{2}{\rho(A)} \neq \overset{3}{\rho(A:B)}$$

$\Rightarrow$ System has no solution

Thus B is not in Col A.

Basis of column space

The pivot columns of a matrix A form a basis for Col A

Find a basis for the column space of

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

Sol^n:-

$R_2 \to 2R_2 + R_1$, $R_3 \to 2R_3 + 3R_2$, $R_4 \to 2R_4 + R_1$

$$A \sim \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 7 & -14 & 14 & -49 \\ 0 & 9 & -18 & 10 & -23 \end{bmatrix}$$

$R_3 \to R_3 - 7R_2$, $R_4 \to R_4 - 9R_2$

$$A \sim \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -8 & 40 \end{bmatrix}$$

$R_3 \longleftrightarrow R_4$

$$A \sim \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -8 & 40 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1^{st}, 2^{nd} and 4^{th} column is pivot columns.

$\therefore$ Basis for column space of A is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}.$$

2) Find a basis for the column space of

$$A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}$$

Sol^n:- $R_2 \longrightarrow R_2 + R_1$, $R_3 \longrightarrow R_3 - 5R_1$

$$A \sim \begin{bmatrix} 1 & -4 & 9 & -7 \\ 0 & -2 & 5 & -6 \\ 0 & 14 & -35 & 42 \end{bmatrix}$$

$R_3 \rightarrow R_3 + 7R_2$

$$A \sim \begin{bmatrix} 1 & -4 & 9 & -7 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

First and second columns are pivot columns.

$\therefore$ Basis for column space of A is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix} \right\}.$$

ule

The Null space of a matrix

The null space of an $m \times n$ matrix A, written as Nul A, is the set of all solutions to the homogeneous equation $Ax = 0$.

i.e Nul A $= \{ x : x$ is in $\mathbb{R}^n$ and $Ax = 0 \}$

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

Problem:- Let $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$, $u = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$

Determine whether u is in Nul A.

Soln:- To test if u satisfies $Au = 0$.

$$Au = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 - 9 + 4 \\ -25 + 27 - 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus u is in Nul A.

2) Check whether $\begin{bmatrix} -7 \\ 3 \\ 2 \end{bmatrix}$ is in the null space of $A = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 0 & 7 \\ -3 & -5 & -3 \end{bmatrix}$

Sol^n:-

Let $u = \begin{bmatrix} -7 \\ 3 \\ 2 \end{bmatrix}$

$$Au = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 0 & 7 \\ -3 & -5 & -3 \end{bmatrix} \begin{bmatrix} -7 \\ 3 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -7 - 3 + 10 \\ -14 + 0 + 21 \\ 21 - 15 - 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 7 \\ 0 \end{bmatrix}$$

$$\neq 0$$

$\therefore Au \neq 0$

$\Rightarrow u \notin \text{Nul } A.$

Row reduced echelon form:

A matrix A is said to be in row reduced echelon form if in the echelon form each pivotal entry is equal to 1 and in the pivotal column, pivotal entry is the only nonzero entry.

1) Find a basis for null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Sol^n: The First step is to find the general solution of $AX = 0$ interms of free variable.

Row reduce the augmented matrix $[A : 0]$ to reduced echelon form in order to write the basic variable in terms of the free variable.

$R_1 \leftrightarrow R_2$

$$[A:O] = \left[\begin{array}{ccccc:c} 1 & -2 & 2 & 3 & -1 & 0 \\ -3 & 6 & -1 & 1 & -7 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{array}\right]$$

$R_2 \to R_2 + 3R_1, \quad R_3 \to R_3 - 2R_1$

$$[A:O] \sim \left[\begin{array}{ccccc:c} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 5 & 10 & -10 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right]$$

$R_3 \leftrightarrow R_2$

$$[A:O] \sim \left[\begin{array}{ccccc:c} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 5 & 10 & -10 & 0 \end{array}\right]$$

$R_3 \to R_3 - 5R_2$

$$[A:O] \sim \left[\begin{array}{ccccc:c} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 \to R_1 - 2R_2$

$$[A:0] \sim \left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

which is the row reduced echelon form.

$AX = 0$

$$\Rightarrow \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$$

$$\Rightarrow x_1 - 2x_2 + \cancel{0} x_3 - x_4 + 3x_5 = 0$$

$$\cancel{0} \cdot x_1 + \cancel{0} \cdot x_2 + x_3 + 2x_4 - 2x_5 = 0.$$

The General solution is

$$x_1 = 2x_2 + x_4 - 3x_5$$ vale

$$x_3 = -2x_4 + 2x_5$$

where x_2, x_4, x_5 are free variable

and x_1, x_3 are basic variable.

Next decompose the vector giving the general solution into a linear combination of vectors, where the scalars are the free variables.

i.e

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} x_5$$

$$x = u x_2 + v x_4 + w x_5$$

every linear combination of u, v, w is an element of Nul A.

Thus $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul A.

The Row Space

Let A be an $m \times n$ matrix, each ~~R~~ row of A has n entries and that can be identified with a vector in $\mathbb{R}^n$.

The set of all linear combinations of the row vectors is called the row space of A and is denoted by $\text{Row}\, A$.

Each row has n entries, so $\text{Row}\, A$ is a subspace of $\mathbb{R}^n$.

Problem:

$\boxed{\text{Check}}$ whether $(5 \;\; 17 \;\; -20)$ is in the row space of

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & 1 \\ -2 & 4 & -3 \end{bmatrix}.$$

Sol^n:-

$$(5, 17, -2) = \alpha_1 (3, 1, -2) + \alpha_2 (4, 0, 1) + \alpha_3 (-2, 4, -3)$$

$$(5, 17, -2) = (3\alpha_1 + 4\alpha_2 - 2\alpha_3,\; \alpha_1 + \alpha_3,\; -2\alpha_1 + \alpha_2 - 3\alpha_3)$$

$3\alpha_1 + 4\alpha_2 - 2\alpha_3 = 5$

$\alpha_1 + 4\alpha_3 = 17$

$-2\alpha_1 + \alpha_2 - 3\alpha_3 = -20$

$\alpha_1 = 5, \quad \alpha_2 = -1, \quad \alpha_3 = 3$

$\therefore (5, 17, -20) = 5(3, 1, -2) - 1(4, 0, 1) + 3(-2, 4, -3).$

Thus $(5, 17, -20)$ is in the Row A.

<u>Note</u>:- If 2 matrices A and B are row equivalent then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

1) Find a basis for the rowspace of

$$A = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix}$$

Sol^n:-

$R_2 \to R_2 + 2R_1$, $R_3 \to R_3 + 3R_1$, $R_4 \to R_4 - 3R_1$

$$A \sim \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 6 & -9 & 24 \\ 0 & 0 & -8 & 12 & -27 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$, $R_4 \to R_4 + 4R_2$

$$A \sim \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$A \sim \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Ans

Non zero rows of echelon form of A forms a basis for row space of A.

∴ Basis for Row A is

$\{(1,-3,4,-1,9), (0,0,2,-3,8), (0,0,0,0,5)\}$

2) Find basis for the rowspace of

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$