NOT FOR USE ?

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Gradient:

Defn

Introduction to differentiable functions:

A matrix A which is necessarily unique and which is written as $DF(x)$ or $F'(x)$ and is called as derivative point of F at x ; where $x = (x_1, x_2, \ldots, x_n)$ denoted as $P \in \mathbb{R}^n$

If $V = e_i = (0, 0, \ldots, 1, 0, \ldots 0)$ we write $D_{e_i} F(x)$ as $\frac{\partial}{\partial x_i} F(x)$ and is called the First order Partial derivative of F

Dell or gradient of f

Let $\nabla f_i(x) = \left(\frac{\partial f_i(x)}{\partial x_1}, \frac{\partial f_i(x)}{\partial x_2}, \frac{\partial f_i(x)}{\partial x_3}, \ldots, \frac{\partial f_i(x)}{\partial x_n} \right)$

or Dell

$\forall i$, & $\nabla f_i(x)$ is called Gradient of f_i at x

Imp

Imp If F is differentiable at x then all directional and Partial derivatives of F exist and is given by $F'(x) = \begin{pmatrix} \nabla f_1(x) \\ \nabla f_2(x) \\ \vdots \\ \nabla f_m(x) \end{pmatrix}$

Imp

(Ex) If $V = (V_1, V_2, \ldots, V_n) \in \mathbb{R}^n$ then $\boxed{D_v F(x) = F'(x) \cdot V}$

(Q1) Let $f: \mathbb{R}^4 \to \mathbb{R}^3$ be defined by $F(x, y, z, w) = (x^2 y, xyz, x^2 + y^2 + zw^2)$. Find $D_v F(X)$ where $X = (1, 2, -1, -2)$ & $V = (0, 1, 2, -2)$

W.K.T

$D_v F(x) = F'(x).v$

$F'(x,y,z,w) = \begin{pmatrix} \nabla f_1 \\ \nabla f_2 \\ \nabla f_3 \end{pmatrix}$

where

$F = (f_1, f_2, f_3)$

where $f_1 = x^2 y$; $f_2 = xyz$; $f_3 = x^2 + y^2 + zw^2$

$\nabla f_1 = \left(\frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y}, \frac{\partial f_1}{\partial z}, \frac{\partial f_1}{\partial w} \right)$

$= (2xy, x^2, 0, 0)$

$\nabla f_2 = \left(\frac{\partial f_2}{\partial x}, \frac{\partial f_2}{\partial y}, \frac{\partial f_2}{\partial z}, \frac{\partial f_2}{\partial w} \right)$

$\nabla f_2 = (yz, xz, xy, 0)$

$\nabla f_3 = \left(\frac{\partial f_3}{\partial x}, \frac{\partial f_3}{y}, \frac{\partial f_3}{z}, \frac{\partial f_3}{w} \right)$

$\nabla f_3 = (2x, 2y, w^2, 2wz)$

$\therefore F'(x) = F'(x,y,z,w) =$

$F'(x) = \begin{pmatrix} 2xy, x^2, 0, 0 \\ yz, xz, xy, 0 \\ 2x, 2y, w^2, 2wz \end{pmatrix}$

$F'(x)/_{(1,2,-1,-2)}$
$\quad x, y, z, w$

$$= \begin{pmatrix} 2(1)(2) & 1^2 & 0 & 0 \\ 2(-2) & 1(-2) & 1(2) & 0 \\ 2(1) & 2(2) \cdot & 4 & 2(-2)(-1) \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 1 & 0 & 0 \\ -2 & -1 & 2 & 0 \\ 2 & 4 & 4 & 4 \end{pmatrix}$$

$\therefore D_v F(x) = F'(x) \cdot v$

$$= \begin{pmatrix} 4 & 1 & 0 & 0 \\ -2 & -1 & 2 & 0 \\ 2 & 4 & 4 & 4 \end{pmatrix} \cancel{(0, 1, 2, -2)}$$

$$= \begin{pmatrix} 4 & 1 & 0 & 0 \\ -2 & -1 & 2 & 0 \\ 2 & 4 & 4 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

(Q2) If $F(x, y, z, w) = (x^2 - y^2, 2xy, zx, z^2w^2x^2)$ and $v = (2, 1, -2, -1)$. Find $F'(1, 2, -1, -2)$ and $D_v F(1, 2, -1, -2)$

= W.K.T

$D_v F(x, y, z, w) = F'(x, y, z, w) \cdot v$

$$F'(x, y, z, w) = \begin{pmatrix} \nabla f_1 \\ \nabla f_2 \\ \nabla f_3 \\ \nabla f_4 \end{pmatrix}$$

where $F = (f_1\ f_2\ f_3\ f_4)$

where $f_1 = x^2 - y^2$; $f_2 = 2xy$; $f_3 = zx$; $f_4 = x^2 z^2 w^2$

$$\therefore \nabla f_1 = \left(\frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y}, \frac{\partial f_1}{\partial z}, \frac{\partial f_1}{\partial w}\right)$$

$$= (2x, -2y, 0, 0)$$

$$\nabla f_2 = \left(\frac{\partial f_2}{\partial x}, \frac{\partial f_2}{\partial y}, \frac{\partial f_2}{\partial z}, \frac{\partial f_2}{\partial w}\right)$$

$$= (2y, 2x, 0, 0)$$

$$\nabla f_3 = \left(\frac{\partial f_3}{\partial x}, \frac{\partial f_3}{\partial y}, \frac{\partial f_3}{\partial z}, \frac{\partial f_3}{\partial w}\right)$$

$$= (z, 0, x, 0)$$

$$\nabla f_4 = \left(\frac{\partial f_4}{\partial x}, \frac{\partial f_4}{\partial y}, \frac{\partial f_4}{\partial z}, \frac{\partial f_4}{\partial w}\right)$$

$$= (2xz^2w^2, 0, 2x^2zw^2, 2x^2z^2w)$$

$$\therefore F'(x) = F'(x,y,z,w) = \begin{pmatrix} 2x & -2y & 0 & 0 \\ 2y & 2x & 0 & 0 \\ z & 0 & x & 0 \\ 2xz^2w^2 & 0 & 2x^2zw^2 & 2x^2z^2w \end{pmatrix}$$

$F'(x)/_{(1,2,-1,-2)}$

$$= \begin{pmatrix} 2(1) & -2(2) & 0 & 0 \\ 2(2) & 2(1) & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 2(1)(1)(4) & 0 & 2(1)(-1)(4) & 2(1)(1)(-2) \end{pmatrix}$$

$$F'(x)/_{(1,2,-1,-2)} = \begin{pmatrix} 2 & -4 & 0 & 0 \\ 4 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 8 & 0 & -8 & -4 \end{pmatrix}$$

$\therefore D_v F(x,y,z,w) = F'(x,y,z,w) \cdot v$

$$= \begin{pmatrix} 2 & -4 & 0 & 0 \\ 4 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 8 & 0 & -8 & -4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 10 \\ -4 \\ 36 \end{pmatrix}$$

(03) Let $f(x,y,z) = x^2 - xy + yz^3 - 6z$; Find all points (x,y,z) such that $\nabla f(x,y,z) = (0,0,0)$

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (0,0,0)$$

$$\Rightarrow \frac{\partial f}{\partial x} = 0 \ ; \ \frac{\partial f}{\partial y} = 0 \ ; \ \frac{\partial f}{\partial z} = 0 \ ;$$

$$2x - y = 0 \ ; \ -x + z^3 = 0 \ ; \ 3yz^2 - 6$$

$$\nabla f(x,y,z) = 0 \Leftrightarrow 2x - y = 0 ; -x + z^3 = 0 ; 3yz^2 - 6 = 0$$

$$\Leftrightarrow 2x = y \ ; \ x = z^3 \ ; \ yz^2 = 2$$

$$\Leftrightarrow 2x = y \ ; \ x = z^3 \ ; \ 2xz^2 = 2 \Rightarrow xz^2 = 1$$

$$\Leftrightarrow 2x = y \ ; \ x = z^3 \ ; \ z^3 z^2 = 1$$

$$2x = y \ ; \ x = z^3 \ ; \ z^5 = 1$$

$$\Rightarrow \boxed{z = 1}$$

$$\therefore x = z^3 = \boxed{1}$$

$$\therefore y = 2$$

$$\therefore x = 1, \ y = 2, \ z = 1$$

(04) If $f(x,y,z) = x^2 e^y$ and $g(x,y,z) = y^2 e^{xz}$. Then find ∇f, ∇g & $\nabla(fg)$ w.r.t. Verify ~~$= f\nabla g + g\nabla f$~~.

$\nabla(fg) = f\nabla g + g\nabla f$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) ; \ \nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right)$$

$$fg = x^2 e^{y + xz} \cdot y^2$$

$$\nabla(fg) = \left(\frac{\partial(fg)}{\partial x}, \frac{\partial(fg)}{\partial y}, \frac{\partial(fg)}{\partial z} \right)$$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$\nabla f = \left(2xe^{y}, x^2 e^{y}, 0 \right)$$

$$\nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right)$$

$$= \cancel{\left(y^2 e^{xz}, \right.}$$

$$\nabla g = \left(y^2 e^{xz} \cdot z, 2y \cdot e^{xz}, y^2 e^{xz} \cdot x \right)$$

$$\nabla (fg) = \left(\frac{\partial (fg)}{\partial x}, \frac{\partial (fg)}{\partial y}, \frac{\partial (fg)}{\partial z} \right)$$

$$\nabla (f,g) = \left(2x^2y^2e^{y+zx} + x^2y^2e^{y+zx} \cdot z, 2x^2ye^{y+xz} + x^2y^2e^{y+xz}, x^3y^2e^{y+xz} \right)$$

Verification:

$$f \Delta g = \left(x^2y^2e^{y+xz} \cdot z, 2x^2ye^{y+xz}, x^3y^2e^{y+xz} \right)$$

$$g \Delta f = \left(2xy^2e^{y+xz}, x^2y^2e^{y+xz}, 0 \right)$$

$$f\nabla g + g\nabla f = \left(2xy^2e^{y+xz} + x^2y^2e^{y+xz} \cdot z, 2x^2ye^{y+xz} + x^2y^2e^{y+xz}, x^3y^2e^{y+zx} \right)$$

$$= \nabla (fg)$$

$\therefore \nabla (fg) = f\nabla g + g\nabla f$ is verified.

Lecture

Hessian: Let $f: \mathbb{R}^n \to \mathbb{R}$ be a real valued fn. First and second order partial derivatives exists

Then the Hessian of f is denoted by $H_{f(p)}$ & denoted as the $n \times n$ matrix $\left(\frac{\partial^2 f(x)}{\partial x_i \partial x_j}\right)$ $1 \le i, j \le n$

Imp In terms of matrix

$$H_{(p)} = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2 x_2} (\partial x_2^2) & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & & \ddots & & \\ \frac{\partial^2 f}{\partial x_n x_1} & \cdots & \cdots & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

5m Imp

(Q1) Find gradient & Hessian of $f(x) = x^2 + y^2 + z^2 + w^2$

= (a) Gradient of $f(x)$ is

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial w}\right)$$

$$\nabla f = (2x, 2y, 2z, 2w)$$

Hessian of $f(x)$ is

$$H_{f(x)} = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial x \partial w} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial y \partial w} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} & \frac{\partial^2 f}{\partial z \partial w} \end{pmatrix}$$

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$$\therefore \begin{pmatrix} \frac{\partial^2 f}{\partial w \partial x} & \frac{\partial^2 f}{\partial w \partial y} & \frac{\partial^2 f}{\partial z \partial w} & \frac{\partial^2 f}{\partial w^2} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

(Q2) Find gradient & Hessian of $f(x) = 13x_1 + x_2 + 4x_3 + 5x_4$

= Gradient of $f(x)$ is

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_4} \right)$$

$$\nabla f = \left(13, 1 \quad 4 \quad 5 \right)$$

Here $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

Hessian of $f(x)$ is

$$H_{f(x)} = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} & \frac{\partial^2 f}{\partial x_1 \partial x_4} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \frac{\partial^2 f}{\partial x_2 \partial x_4} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} & \frac{\partial^2 f}{\partial x_3 \partial x_4} \\ \frac{\partial^2 f}{\partial x_4 \partial x_1} & \frac{\partial^2 f}{\partial x_4 \partial x_2} & \frac{\partial^2 f}{\partial x_4 \partial x_3} & \frac{\partial^2 f}{\partial x_4^2} \end{pmatrix}$$

$$\therefore H_1(X) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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UNIT-02: {Linear Transformations}

[CO1]

- → Defination
- → Algebra of linear transformation
- → Representation of transformation in matrix form
- → Isomorphism
- → Rank - Nullity theorem
- → Inner product
- → Orthogonal
- → Gram - Schmidt ortho process
- → Least Square

CEF
proof + diagram (7m)

Defination:

Let V and W be the vector space over a field F then a linear Transformation from V into W is a function of E such that $E(c\alpha+\beta) = cE(\alpha)+E(\beta)$ where $\forall c \in F$ and $\alpha, \beta \in V$

(Q1) Suppose the mapping transformation $T: R^2 \to R^2$ is defined by $T(x,y) = (x+y, x)$. Show that T is a linear transformation.

= Firstly, we need to show that

$$T(c\alpha+\beta) = c(T(\alpha)) + T(\beta) \quad \text{—} \ (1)$$

Let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$

$\therefore T(c\alpha+\beta) = T(c(\alpha_1,\alpha_2) + (\beta_1,\beta_2))$

$= T(c\alpha_1+\beta_1, c\alpha_2+\beta_2)$ (From given)

$= (c\alpha_1+\beta_1+c\alpha_2+\beta_2, c\alpha_1+\beta_1)$

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UNIT-02: {Linear Transformation}

[01]

- → Defination
- → Algebra of linear transformation
- → Representation of transformation in matrix form
- → Isomorphism
- → Rank - Nullity theorem
- → Inner product
- → Orthogonal
- → Gram - Schmidt ortho process
- → Least Square

CSE PTT (9m) — **Definition:**

Let V and W be the vector space over a field F then a linear Transformation from V into W is a function of T such that $T(c\alpha+\beta) = cT(\alpha)+T(\beta)$ where $\forall c \in F$ and $\alpha, \beta \in V$

(01) Suppose the mapping transformation $T: R^2 \to R^2$ is defined by $T(x,y) = (x+y, x)$. Show that T is a linear transformation.

= Firstly, we need to show that

$$T(c\alpha+\beta) = c(T(\alpha)) + T(\beta) \quad —① $$

Let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$

$\therefore T(c\alpha+\beta) = T(c(\alpha_1, \alpha_2) + (\beta_1, \beta_2))$

$= T(c\alpha_1+\beta_1, c\alpha_2+\beta_2)$ (From given)

$= (c\alpha_1+\beta_1+c\alpha_2+\beta_2, c\alpha_1+\beta_1)$

$= (c(\alpha_1+\alpha_2, \alpha_1) + (\beta_1+\beta_2, \beta_1))$

$= c(T(\alpha_1, \alpha_2)) + T(\beta_1, \beta_2)$

$= \underline{\underline{cT(\alpha) + T(\beta)}}$

Hence T is a Linear Transformation

(Q2) Let $T: V_3(IR) \xrightarrow{(R^3) \to R^2} V_2(IR)$ defined by $T(x, y, z) = (x+y, y+z)$. P.T T is LT

We need to P.T $T(c\alpha+\beta) = c(T(\alpha)) + T(\beta)$ — ①

= Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ & $\beta = (\beta_1, \beta_2, \beta_3)$

$\therefore T(c\alpha+\beta) = T(c(\alpha_1, \alpha_2, \alpha_3) + (\beta_1, \beta_2, \beta_3))$

$= T(\underset{x}{c\alpha_1+\beta_1}, \underset{y}{c\alpha_2+\beta_2}, \underset{z}{c\alpha_3+\beta_3})$ → Given $\because T(x, y, z) = (x+y, y+z)$

$= (c\alpha_1+\beta_1+c\alpha_2+\beta_2, c\alpha_2+\beta_2+c\alpha_3+\beta_3)$

$= (c(\alpha_1+\alpha_2, \alpha_2+\alpha_3) + (\beta_1+\beta_2, \beta_2+\beta_3))$ → Given

$= (c(T(\alpha_1, \alpha_2, \alpha_3)) + T(\beta_1, \beta_2, \beta_3))$

$= \underline{\underline{c(T(\alpha)) + T(\beta)}}$

Hence T is a Linear Transformation

(Q3) S.T Transformation defined by $T(x_1, x_2, x_3) = (2x_1 - 3x_2, x_1+4, 5x_3)$ is not Linear

= Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$

$\therefore T(c\alpha+\beta) = T(c(\alpha_1, \alpha_2, \alpha_3) + (\beta_1, \beta_2, \beta_3))$

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$= T(\underbrace{c\alpha_1+\beta_1}_{x_1}, \underbrace{c\alpha_2+\beta_2}_{y\ x_2}, \underbrace{c\alpha_3+\beta_3}_{z\ x_3})$

$= (2(c\alpha_1+\beta_1) - 3(c\alpha_2+\beta_2), c\alpha_1+\beta_1+4, 5(c\alpha_3+\beta_3))$

$= \cancel{X\alpha\alpha}$

$= \cancel{c(2\alpha_1 - 3\alpha_2}$

$= c(2\alpha_1 - 3\alpha_2, \alpha_1, 5\alpha_3) + \cancel{\beta}(2\beta_1 - 3\beta_2, \beta_1+4, 5\beta_3)$

$= \cancel{c(T(}$ No transformat~~ef~~ $T(\beta_1, \beta_2, \beta_3)$ defined $\neq \cancel{c(T(\alpha_1,\alpha_2,\alpha_3)) + T(\beta_1,\beta_2,\beta_3)}$

$\therefore$ Here $T(c\alpha+\beta) \neq c(T(\alpha)) + T(\beta)$

Since in first term the transformation doesn't satisfies the above ~~condit~~ transformation defined

$\therefore$ T is not a Linear Transformation.

(Q4) $T: V_3 \to V_3$ defined by $T(x_1, x_2, x_3) = (x_1 - x_2, x_1 + x_2)$

P.T T is a Linear.

= Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and ~~β be and~~ $\beta = (\beta_1, \beta_2, \beta_3)$

$\therefore T(c\alpha+\beta) = T(c(\alpha_1, \alpha_2, \alpha_3) + (\beta_1, \beta_2, \beta_3))$

$= T(\underbrace{c\alpha_1+\beta_1}_{x_1}, \underbrace{c\alpha_2+\beta_2}_{x_2}, \underbrace{c\alpha_3+\beta_3}_{x_3})$

$= ((c\alpha_1+\beta_1) - (c\alpha_2+\beta_2), (c\alpha_1+\beta_1) + (c\alpha_2+\beta_2))$

$= (c(\alpha_1-\alpha_2) + (\beta_1-\beta_2), c(\alpha_1+\alpha_2) + (\beta_1+\beta_2))$

$= c(\alpha_1-\alpha_2, \alpha_1+\alpha_2) + (\beta_1-\beta_2, \beta_1+\beta_2)$

$= c(T(\alpha_1, \alpha_2, \alpha_3)) + T(\beta_1, \beta_2, \beta_3)$

$= c(T(\alpha)) + T(\beta)$

$\therefore T$ is linear.

Matrix of Linear Transformation:

Let $T: U \to V$ be L.T from vector space V to V. Let $B_1 = \{u_1, u_2, \ldots, u_n\}$ and $B_2 = \{v_1, v_2, \ldots, v_n\}$ be the basis of u and v respectively. Such that transformation of $T(u_1) = a_{11}v_1 + a_{21}v_2 + \cdots + a_{n1}v_n$

$T(u_2) = a_{12}v_1 + a_{22}v_2 + \cdots + a_{n2}v_n$

$T(u_3) = a_{13}v_1 + a_{23}v_2 + a_{33}v_3 + \cdots + a_{n3}v_n$

Then $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$ is defined $m \times n$ as matrix of linear transformation

Eg: Q1 Find matrix of linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x, -y)$ w.r.t basis $B_1 = \{\underset{u_1}{(1,1)}, \underset{u_2}{(1,0)}\}$ $B_2 = \{\underset{v_1}{(2,3)}, \underset{v_2}{(4,5)}\}$

= Given $T(x, y) = (x, -y)$

$T(1,1) = (1,-1) = a_{11}v_1 = a_{11}(2,3) + a_{21}(4,5)$ — ①

$T(1,0) = (1,0) = a_{12}v_1 = a_{12}(2,3) + a_{22}(4,5)$ — ②

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From ①

$(1,-1) = (2a_{11}+4a_{21},\ 3a_{11}+5a_{21})$

$3\times\ 2a_{11}+4a_{21} = 1$
$2\times\ 3a_{11}+5a_{21} = -1$

$2(-1)$

$6a_{11}+12a_{21} = 3$
$\underline{-6a_{11}\overset{-}{+}10a_{21} = \overset{+}{-}2}$

$2(-9)+\cancel{4}^{2}\times 5/\cancel{2} =$
$-18+10 =$

$2a_{21} = 5$

$a_{21} = \underline{\underline{5/2}} = \underline{\underline{2.5}}$

$\therefore\ 2a_{11} = 1-\cancel{4}^{2}\times 5/\cancel{2} = \frac{1-10}{2} = -9/\cancel{2} = \underline{\underline{-4.5}}$

From ②

$1 = 2a_{12}+4a_{22}$
$0 = 3a_{12}+5a_{22}$

$3 = 6a_{12}+12a_{22}$
$\underline{\underset{-}{0} = \underset{-}{6}a_{12}\underset{-}{+}10a_{22}}$
$3 = 2a_{22}$

$a_{22} = 3/2 = \underline{\underline{1.5}}$

②.
$3a_{12}+5(1.5) = 0$
$3a_{12} = -7.5$
$a_{12} = -\frac{\cancel{7.5}^{2.5}}{\cancel{3}}$

$a_{12} = \underline{\underline{-2.5}}$

$\therefore$ Matirx of L.T is $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} -4.5 & -2.5 \\ 2.5 & 1.5 \end{bmatrix}$

(Q2) Find matrix of transformation $T: V_2(R) \to V_3(R)$ w.r.t Basis, defined by $T(x,y) = (x+y, x, 3x-y)$: such that $B_1 = \{(1,1),(3,1)\}$ $B_2 = \{(1,1,1),(1,1,0),(1,0,0)\}$

$3(1)-1=$

: Given: $T(x,y) = (x+y, x, 3x-y)$

$T(1,1) = (2,1,2) = a_{11}(1,1,1) + a_{21}(1,1,0) + a_{31}(1,0,0)$ — ①

Trick: order of matrix: $V \times U$

$T(3,1) = (4,3,8) =$

$a_{12}(1,1,1) + a_{22}(1,1,0) + a_{32}(1,0,0)$ — ②

$3(3)-1 = 8$

$\Rightarrow$ From ①

$2 = a_{11} + a_{21} + a_{31}$

$1 = a_{11} + a_{21}$

$\boxed{2} = a_{11}$

$\therefore a_{21} = 1 - 2 = \boxed{-1}$

$a_{31} = 2 - 2 - (-1) = \boxed{1}$

$a_{11} = 2, a_{21} = -1, a_{31} = 1$

From ②

$4 = a_{12} + a_{22} + a_{32}$

$3 = a_{12} + a_{22}$

$8 = a_{12}$

$a_{22} = 3 - 8 = \boxed{-5}$

$a_{32} = 4 - 8 + 5 = -4 + 5 = \boxed{1}$

$\therefore a_{12} = 8 \quad a_{22} = -5 \quad a_{32} = 1$

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$\therefore$ MOLT is say $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times 2}$

$= \begin{bmatrix} 2 & 8 \\ -1 & -5 \\ 1 & 1 \end{bmatrix}_{3\times 2}$

(03) $T : V_3(R) \to V_2(R)$ defined by $T(x,y,z) = (x-y+2z, 3x+y)$

w.r.t Basis such that

$B_1 = \{(1,1,1), (1,2,3), (1,0,0)\}$ $B_2 = \{(1,1), (1,-1)\}$

Given: $T(x,y,z) = (x-y+2z, 3x+y)$

$T(1,1,1) = (2,4) = a_{11}(1,1,1) + a_{21}(1,-1)$ — (1)

$T(1,2,3) = (5,5) = a_{12}(1,1) + a_{22}(1,-1)$ — (2)

$T(1,0,0) = (1,3) = a_{13}(1,1) + a_{23}(1,-1)$ — (3)

From (1)

$2 = a_{11} + a_{21}$

$4 = a_{11} + a_{21}(-1)$

$6 = 2a_{11}$

$\boxed{a_{11} = 3}$

$a_{21} = 2 - 3 = \boxed{-1}$

From (2)

$5 = a_{12} + a_{22}$

$5 = a_{12} + a_{22}(-1)$

$10 = 2a_{12}$

$a_{12} = 5$

$a_{22} = 0$

From (3)

$1 = a_{13} + a_{23}$

$3 = a_{13} - a_{23}$

$4 = 2a_{13}$

$a_{13} = 2$

$a_{23} = -1$

$\therefore$ MOLT is say $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} 3 & 5 & 2 \\ -1 & 0 & -1 \end{bmatrix}_{2\times 3}$

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(Q3) Given: Matrix $A = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$; Determine the linear transformation $T: V_3(R) \to V_2(R)$ relative to the Basis $B_1 = \{(1,2,0), (0,-1,0), (1,-1,1)\}$ and

$B_2 = \{(1,0), (2,-1)\}$

= Transformation $T(1,2,0) = a_{11}(1,0) + a_{21}(2,-1)$

$T(1,2,0) = -1(1,0) + 1(2,-1)$
$= (-1,0) + (2,-1)$
$T(1,2,0) = \underline{\underline{(1,-1)}}$ — (1)

|||ly,

$T(0,-1,0) = a_{12}(1,0) + a_{22}(2,-1)$

$= 2(1,0) + 0(2,-1)$
$= (2,0) + (0,0)$
$T(0,-1,0) = \underline{\underline{(2,0)}}$ — (2)

|||ly

$T(1,-1,1) = a_{13}(1,0) + a_{23}(2,-1)$
$= 1(1,0) + 3(2,-1)$
$= (1,0) + (6,-3)$
$T(1,-1,1) = \underline{\underline{(7,-3)}}$ — (3)

Let $(x,y,z) \in V_3(R)$

$(x,y,z) = \alpha_1(1,2,0) + \alpha_2(0,-1,0) + \alpha_3(1,-1,1)$

$(x,y,z) = (\alpha_1 + \alpha_3, 2\alpha_1 - \alpha_2 - \alpha_3, \alpha_3)$
$x = \alpha_1 + \alpha_3$ $\quad z = \alpha_3$
$y = 2\alpha_1 - \alpha_2 - \alpha_3$

$x = \alpha_1 + z$

$\alpha_1 = x - z$

$y = 2(x-z) - \alpha_2 - z$

$\alpha_2 = 2x - 2z - y - z$

$\alpha_2 = 2x - 3z - y$

$\alpha_2 = 2x - y - 3z$

~~(x, y, z) = Z~~

From ① ... F ②

$\therefore T(x,y,z) = (x-z)T(1,2,0) + (2x-y-3z)T(0,-1,0) + zT(1,-1,1)$

F ③

$= (x-z)(1,-1) + (2x-y-3z)(2,0) + z(7,-3)$

$= ((x-z), -(x-z)) + (2(2x-y-3z), 0) + (7z, -3z)$

$T(x,y,z) = ((x-z) + 2(2x-y-3z) + 7z, -(x-z) - 3z)$

$= (x - z + 4x - 2y - 6z + 7z, -x + z - 3z)$

$T(x,y,z) = (5x - 2y, -x - 2z)$

(a) Given: $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}$: Determine the L.T

$T: V_3 \to V_2$ relative to the Basis $B_1 = \{\underset{u_1}{(1,1,1)}, \underset{u_2}{(1,2,3)}, \underset{u_3}{(1,0,0)}\}$ $B_2 = \{\underset{v_1}{(1,1)}, \underset{v_2}{(1,-1)}\}$

$\therefore$ Transformation: $T(1,1,1) = a_{11}(1,1) + a_{21}(1,-1)$

$T(1,1,1) = 1(1,1) + 3(1,-1)$
$= (1,1) + (3,-3)$
$T(1,1,1) = (4,-2)$ — ①

$T(1,2,3) = a_{12}(1,1) + a_{22}(1,-1)$

$= -1(1,1) + 1(1,-1)$
$= (-1,-1) + (1,-1)$
$T(1,2,3) = (0,-2)$ — ②

$T(1,0,0) = a_{13}(1,1) + a_{23}(1,-1)$

$= 2(1,1) + 0(1,-1)$
$T(1,0,0) = (2,2)$ — ③

Let $(x,y,z) \in V_3(R)$

$(x,y,z) = \alpha_1(1,1,1) + \alpha_2(1,2,3) + \alpha_3(1,0,0)$

$x = \alpha_1 + \alpha_2 + \alpha_3$
$y = \alpha_1 + 2\alpha_2$
$z = \alpha_1 + 3\alpha_2$
$\therefore \cancel{y = z + 2\alpha_2}$
$\cancel{\frac{y-z}{2} = \alpha_2}$

$y - z = -\alpha_2$
$\alpha_2 = z - y$

$\cancel{x = z + z - y + \alpha_3}$

$x = z + \frac{y-z}{2} + \alpha_3$

$y = \alpha_1 + 2(z-y)$
$y - 2z + 2y = \alpha_1$
$\alpha_1 = 3y - 2z$

$\cancel{2\alpha_3 = 2x - z + y - z}$
$\cancel{2\alpha_3 = 2x + y - 2z}$
$\cancel{\alpha_3 = \frac{2x+y-2z}{2} = x + y/2 - z}$

$\therefore T(x,y,z) = \alpha_1 T(1,1,1) + \alpha_2(1,2,3) + \alpha_3(1,0,0)$ — (A)

$= \cancel{z\,T(1,1,1) + \frac{y-z}{2}T(1,2,3) + \frac{2x+y-2z}{2}T(1,0,0)}$

$= \cancel{z(4,-2) + \frac{y-z}{2}(0,2) + \frac{2x+y-2z}{2}(2,2)}$

$= \cancel{(4z,-2z) + \left(0, -\frac{2(y-z)}{2}\right)}$

$=$

$x = 3y - 2z + z - y + \alpha_3$

$x - 3y + 2z - z + y = \alpha_3$

$\underline{\underline{x - 2y + z = \alpha_3}}$

$\therefore$ From (A), (1), (2), (3)

$T(x,y,z) = (3y-2z)(4,-2) + (z-y)(0,-2) + (x-2y+z)(2,2)$

$= ((3y-2z)4, -2(3y-2z))$

$\quad + (0, -2(z-y)) + (2(x-2y+z), 2(x-2y+z))$

$= ((3y-2z)4 + 2(x-2y+z), -2(3y-2z) - 2(z-y) + 2(x-2y+z))$

$= (12y - 8z + 2x - 4y + 2z, -6y + 4z - 2z + 2y + 2x - 4y + 2z)$

$T(x,y,z) = \underline{\underline{(2x + 8y - 6z, 2x - 8y + 2z + 2z)}}$

Algebra of Linear Transformation

Let V and W be a vector space over a field F.

Let T_1 & T_2 be linear transformation from V to W then the function $(T_1 + T_2)$ defined by $(T_1 + T_2)\alpha = T_1(\alpha) + T_2(\alpha)$ is a linear ~~construction~~ transformation from V into W.

Also if C is any element in F ($C \in F$) then the function CT is defined by ~~CT(α)~~ $(CT)\alpha = C\,T(\alpha)$

(10m) (Q1) P.T above is the linear transformation from V into W

sometimes O(zero to W)

Proof: Suppose T_1 and T_2 are the linear transformation from V to W then $(T_1 + T_2)$ can be defined $(T_1 + T_2)(C\alpha + \beta)$

$$\cancel{(T_1+T_2)(C\alpha+\beta) = C\alpha T_1(\ldots)}$$

$$
\begin{aligned}
(T_1 + T_2)(C\alpha + \beta) &= T_1(C\alpha + \beta) + T_2(C\alpha + \beta) \\
&= T_1(C\alpha) + T_1(\beta) + T_2(C\alpha) + T_2(\beta) \\
&= C\,T_1(\alpha) + T_1(\beta) + C\,T_2(\alpha) + T_2(\beta) \\
&= C(T_1(\alpha) + T_2(\alpha)) + (T_1 + T_2)\beta
\end{aligned}
$$

$$(T_1 + T_2)(C\alpha + \beta) = C(T_1 + T_2)(\alpha) + (T_1 + T_2)\beta$$

- $(T_1 + T_2)(C\alpha + \beta)$ is a linear transformation from V ~~to~~ into W

(ii) We have p.T $(CT)\alpha = C\,T(\alpha)$

Let $a \in F$
$c \in \Omega$

$$cT(a\alpha + \beta) = cT(a\alpha) + cT(\beta)$$
$$= cTa(\alpha) + cT(\beta)$$
$$= a \cdot cT(\alpha) + c \cdot T(\beta)$$
$$= c(a \cdot T(\alpha) + T(\beta))$$

$= \cancel{c(T(a\alpha + \beta))}$
$= \cancel{cT}$

cT is a Linear Transformation

Range and Null Space

Let V and W be vector space over field F and T be the Linear Transformation from V to W then the Null space of T is a set of all vectors α in V such that $\boxed{T(\alpha) = 0}$ and it is denoted by

$N(T)$ or $Ker(T) = \{\alpha \in V \mid T(\alpha) = 0\}$

~~Kernal~~
Kernel

(Range Space)
The Range space of T is a set of all vectors β in W such that $T(\alpha) = \beta$ denoted by

$R(T)$ or $image(T) = \{\beta \in W \mid T(\alpha) = \beta\}$

Note: If T is a linear transformation then $T(0) = 0$;

(a) Show that Null space of T is a subspace of V

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= Proof: W.K.T $N(T) = \{\alpha \in V \mid T(\alpha) = 0\}$

Let $T: V \to W$ be a L.T then Null space of T
$N(T) = \{\alpha \in V \mid T(\alpha) = 0\}$

(i) W.K.T, $T(0) = 0 \Rightarrow 0 \in$ Null space of T i.e, $0 \in N(T)$
$\Rightarrow N(T) \in \phi$

(ii) Let α_1 and $\alpha_2 \in N(T)$ & $C \in F$

$\Rightarrow T(\alpha_1) = 0$ & $T(\alpha_2) = 0$
Then,
$$T(C\alpha_1 + \alpha_2) = C\,T(\alpha_1) + T(\alpha_2)$$
$$= 0 + 0$$
$$T(C\alpha_1 + \alpha_2) = 0$$
$$\Rightarrow T(C\alpha_1 + \alpha_2) \in N(T)$$

Hence Null space of T is a subspace of V

(Q2) Show that Range space of T is a subspace of W
Proof:
> Let $T: V \to W$ be a L.T then Range space of T
$R(T) = \{\beta \in W \mid T(\alpha) = \beta\}$

(i) W.K.T $T(0) = 0$
$\Rightarrow 0 \in R(T)$
$\Rightarrow R(T) \in \phi$

(ii) Let β_1 and $\beta_2 \in W$ and α_1 and $\alpha_2 \in V$ then
$T(\alpha_1) = \beta_1 \quad T(\alpha_2) = \beta_2$

$\therefore$ ~~$T(C\beta_1 + \beta_2) = C$~~
$T(C\alpha_1 + \alpha_2) = C\,T(\alpha_1) + T(\alpha_2) = C\beta_1 + \beta_2 \in R(T)$

$\therefore$ R(T) is a subspace of W

R.p.A.A. vimp **Rank Nullity theorem**

Statement: Let U and V be two finite dimensional vectorspace over a field F. Let $T: U \to V$ be a linear transformation then,

$$\dim(R(T)) + \dim(N(T)) = \dim(V)$$

OR

$$\text{Rank}(T) + \text{Nullity}(T) = \dim V$$

Imp **Proof:** (Later days) By me and Sirish.

Problems:

(01) Let a linear map $T: R^3 \to R^2$ be defined by $T(x,y,z) = (y-x, y-z)$. Find the Null space & Range space of T and also find the dimension of Null space of T and Range space of T

Given: $T(x,y,z) = (y-x, y-z)$

$$\cancel{R(T) = \{T(x,y,z) \mid T(x,y,z) = (y-x, y-z)\}}$$

$$N(T) = \{(x,y,z) \in R^3 \mid T(x,y,z) = 0\}$$

$$N(T) = \{(x,y,z) \in R^3 \mid (y-x, y-z) = 0\}$$

$$R(T) = \{x,y,z \in R^3 \mid T(x,y,z) = (y-x, y-z)\}$$

OR

$$R(T) = \{y-x, y-z \mid T(x,y,z) = ((y-x), (y-z))\}$$

$\Rightarrow N(T) = \{ x, y, z \in \mathbb{R}^3 \mid y - x = 0 \text{ and } y - z = 0 \}$

$= \{ x, y, z \in \mathbb{R}^3 \mid y = x \text{ and } y = z \}$

$N(T) = \{ x, y, z \in \mathbb{R}^3 \mid x = y = z \}$

$\text{Basis}(N(T)) = \{ (x, x, x) \mid x \neq 0 \}$

$\therefore \text{Dim}(N(T)) = 1$ $\quad [\because (x, x, x)$ is a whole $x]$

$\therefore \text{Dim } U$ or $\text{Dim } \mathbb{R}^3 = 3$

By Rank Nullity theorem

$\dim N(T) + \dim R(T) = \dim U$

$\dim R(T) = \dim U - \dim N(T)$

$\therefore \dim R(T) = 3 - 1 = \boxed{2}$

(Q2) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be Linear Transformation such that $T(x, y, z) = (x + 2y - z, y + z, x + y - 2z\}$. Find $\text{Dim } R(T)$, $\text{Dim } N(T)$ and also $\text{Dim}(V)$

= Given: $T(x, y, z) = (x + 2y - z, y + z, x + y - 2z)$

$N(T) = \{ x, y, z \in \mathbb{R}^3 \mid T(x, y, z) = 0 \}$

$= \{ x, y, z \in \mathbb{R}^3 \mid (x + 2y - z, y + z, x + y - 2z \}$

$N(T) = \{ x, y, z \in \mathbb{R}^3 \mid x + 2y - z = 0, y + z = 0, x + y - 2z = 0 \}$

$\Rightarrow x + 2y - z = 0$

$y + z = 0$

$x + y - 2z = 0$

$$= \begin{bmatrix} 1 & 2 & -1 & : & 0 \\ 0 & 1 & 1 & : & 0 \\ 1 & 1 & -2 & : & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

$$= \begin{bmatrix} 1 & 2 & -1 & : & 0 \\ 0 & 1 & 1 & : & 0 \\ 0 & -1 & -1 & : & 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$= \begin{bmatrix} 1 & 2 & -1 & : & 0 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$\rho(A) = 2 \quad \rho(\overset{A:}{B}) = 2 \quad n = 3$

$\therefore$ The system is consistent and has infinite many solutions.

From the ~~rows~~ equation from corresponding matrix are

$x + 2y - z = 0$

$y + z = 0$

Let $\underline{\underline{z = t}}$

$\Rightarrow y + t = 0$

$\underline{\underline{y = -t}}$

$\therefore x + 2y - z = x + 2(-t) - t = 0$

$= x - 2t - t = 0$

$\underline{\underline{x = 3t}}$

All 3 are written in terms of constant 't'

$\therefore x = 3t,\ y = -t,\ z = t$

$\therefore N(T) = \{x, y, z \in R^3 \mid x = 3t, y = -t, z = t\}$

$\therefore$ Basis $\{N(T)\} = \{(3t, -t, t) \mid t \neq 0\} \Rightarrow N(T) = \{(3, -1, 1)\}$

$\therefore Dim(N(T)) = 1$

Since $Dim\, U = Dim\, R^3 = 3$

By Rank Nullity theorem

$\Rightarrow Dim\, R(T) = Dim\, U - Dim\, N(T)$

$= 3 - 1$

$= \boxed{2}$

Imp ☆☆☆

(Q3) Let $T: R^3 \to R^4$ be a linear map defined by $T(x,y,z) = (y+z, x+z, x+y, x+y+z)$. Find $Ker(T)$, $dim(Ker(T))$ and $Img(T)$ & $dim(Img\, T)$.

= Given: $T(x,y,z) = (y+z, x+z, x+y, x+y+z)$

$N(T) = \{x,y,z \in R^3 \mid T(x,y,z) = 0\}$

$= \{x,y,z \in R^3 \mid (y+z, x+z, x+y, x+y+z) = 0\}$

$\Rightarrow y+z=0 \Rightarrow y=-z$

$x+z=0 \Rightarrow x=-z$

$x+y=0 \Rightarrow x=-y$

$x+y+z=0$

Solution:

$\Rightarrow \boxed{x=y=z=-k}$

also $x=y=z=0$ (Since x,y,z has ∞ many solution)

$\therefore N(T) = \{x,y,z \in R^3 \mid x=y=z=-k\}$

$\therefore Basis(N(T)) = Basis(Ker(T)) = \{(-k,-k,-k) \mid -k \neq 0\}$

$\therefore Dim(Ker(T)) = Dim(N(T)) = \boxed{1}$

Since $Dim(V) = R^4 = \boxed{4}$

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$\Rightarrow$ ~~Dim(R(T)) + Dim(N(T)) = Dim(V)~~

~~= Dim~~

Since Basis(N(T)) $= \{(0,0,0) \mid x=y=z=0\}$

Since Basis(N(T)) contains ~~also~~ all zeros (Refer definition of Basis)
To get
Basis : ~~the~~ the given or obtained vectors should be (i) linearly independent $\to$ (No all columns)
(ii) One should ~~be~~ be written as linear combination of others.

Since in Basis(N(T)), x, y, z all contain zeros, indicates ~~the~~ that the vector is ~~not~~ said to Basis
$\Rightarrow$ Dimension doesn't exists ~~now~~ in Basis set

$\Rightarrow$ $\boxed{\text{Dim}(N(T)) = 0}$

)

$y + z = 0$ ④
$x + z = 0$ ③
$x + y = 0$ ②
$x + y + z = 0$ ①

consider $[A:O] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$

$R_2 \to R_2 - R_1 \quad R_3 \to R_3 - R_1$

$\therefore [A:O] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$

$R_3 \to R_3 + R_4$

$$\therefore [A:O] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$R_4 \to R_4 - R_3$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow$$ It is linearly Independent Hence, Basis doesn't exists $\Rightarrow \boxed{\dim(N(T)) = 0}$

$\Rightarrow \rho(A) = 3;\ \rho(A:B) = 3;\ \cancel{\ }\ n = 3$

$\Rightarrow r = r' = n = 3$

$\therefore$ The system is consistent and has ~~a~~ unique solution

$\therefore$ The equation from corresponding matrix are

$\boxed{z = 0}$

$\boxed{-y = 0} \Rightarrow y = 0$

$\Rightarrow x + y + z = 0$

$\Rightarrow \boxed{x = 0}$

$\Rightarrow (x, y, z) = (0, 0, 0)$

$\cancel{\text{...}} \Rightarrow \text{Dim}(N(T)) = 0$

~~Basis(N(T)) = {x, y, z ∈ R³ | (0,0,0)}~~

$\text{Basis}(N(T)) = \{(0,0,0) \mid x = y = z\}$

$\therefore$ By Rank Nullity theorem,

$Dim(R(T)) + Dim(N(T)) = Dim\,U$

$U = R^3$

$\therefore Dim(R(T)) = Dim\,U - Dim(N(T))$ $\therefore \underline{\underline{Dim\,U = 3}}$

$= 3 - 0 = \boxed{3}$

H.W.

(Q) $T: R^3 \to R^3$ defined by $T(x, y, z) = (x+y, x-y, x+2z)$
Verify ~~dim(R(T))~~ & Rank Nullity theorem

Inner Products : (Dot Product)

Scalar ~~prod~~ quantity

Inner product of two vectors u and v where $u = (u_1, u_2, u_3, \cdots u_n)$ and $v = (v_1, v_2, \cdots, v_n)$ is defined as $\boxed{u \cdot v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n}$

Note Dot Product is a $\boxed{\text{Scalar quantity}}$

Orthogonal Vector:

If u and v are 2 vectors ~~which~~ are said to be orthogonal to each other, if $\boxed{u \cdot v = 0}$

$|\theta_1 - \theta_2| = \pi/2$
$\tan\theta_1 \tan\theta_2 = -1$

Length (Norm):

The Length (Norm) of a vector say $v = (v_1, v_2 \cdots v_n)$ is denoted by "$\|v\|$" and defined as

$$\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

(01) Show that $\{u_1, u_2, u_3\}$ is an orthogonal set where

$$u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} ; \quad u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} ; \quad u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$$

= To prove, we need to show that

$u_1 \cdot u_2 = 0$; $u_2 \cdot u_3 = 0$; $u_3 \cdot u_1 = 0$

$\therefore u_1 \cdot u_2 = u_1^T \cdot u_2$

$$= \begin{bmatrix} 3 & 1 & 1 \end{bmatrix}_{1\times 3} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}_{3\times 1}$$

$$= \begin{bmatrix} 3(-1) + 1(2) + 1(1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \end{bmatrix}_{1\times 1} = 0 \text{ RHS}$$

IIIly, $u_2 \cdot u_3 = u_2^T \cdot u_3 = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$

$$= \begin{bmatrix} -1(-1/2) + 2(-2) + 1(7/2) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \end{bmatrix}_{1\times 1} = 0 = \text{RHS}$$

LHS $= u_3 \cdot u_1 = u_3^T \cdot u_1 = \begin{bmatrix} -1/2 & -2 & 7/2 \end{bmatrix}_{1\times 3} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}_{3\times 1}$

$$= \begin{bmatrix} -1/2(3) - 2(1) + 7/2(1) \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}_{1\times 1} = 0$$

(ii) B is linearly independent

Suppose,

$\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_r w_r + \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_s v_s = 0$ — ①

where α_i & $\beta_j \in F$ Then,

$F(\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_r w_r + \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_s v_s) = F(0) = 0$

By defn of L.T,

$\alpha_1 F(w_1) + \alpha_2 F(w_2) + \cdots + \alpha_r F(w_r) + \beta_1 F(v_1) + \beta_2 F(v_2) + \cdots$
$\cdots + \beta_s F(v_s) = 0$ — ②

But, w.r.t (nullity defn) (Range space defn)

$F(w_i) = 0$ since $w_i \in N(F)$ and

$F(v_j) = u_j$ then sub ② we get

$\beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_s v_s = 0$

Since u_j is linearly independent for each $\beta_i = 0$

Then, sub in ①

$\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_r w_r = 0$

$\Rightarrow$ w_i is a linearly independent for each $\alpha_i = 0$

Therefore, B is a linearly independent.

(ii) B spans V

Let $v \in V$.

Then $F(v) \in R(F)$ $\because u_i$ spans $R(F)$, $\exists$ scalars $a_1, a_2, \ldots, a_s$ such that

$F(\hat{v}) = 0$

$F(v) = a_1 u_1 + a_2 u_2 + \cdots + a_s u_s$.

And set $\hat{v} = a_1 v_1 + a_2 v_2 + \cdots + a_s v_s - v$

Then,

$F(\hat{v}) = F(a_1 v_1 + a_2 v_2 + \cdots + a_s v_s - v) =$

$= a_1 F(v_1) + a_2 F(v_2) + \cdots + a_s F(v_s) - F(v)$.

$F(\hat{v}) = a_1 u_1 + a_2 u_2 + \cdots + a_s u_s - F(v) = 0$ — ①

Thus, $\hat{v} \in N(F)$ $\because w_i$ span $N(F) \Rightarrow$
$\exists$ scalar $b_1, \ldots, b_r$ such that

$\hat{v} = b_1 w_1 + b_2 w_2 + \cdots + b_r w_r = 0$ — ②

$\hat{v} = a_1 u_1 + a_2 u_2 + \cdots + a_s u_s - v = b_1 w_1 + b_2 w_2 + \cdots + b_r w_r$

$v = a_1 u_1 + a_2 u_2 + \cdots + a_s u_s - b_1 w_1 - b_2 w_2 \cdots - b_r w_r$

Thus, B spans V

Orthogonal Projections

Consider a two non zero vectors $y, u \in \mathbb{R}^3$ in then the orthogonal projection of y on u is $\boxed{\left(\frac{y \cdot u}{u \cdot u}\right) u}$

Orthonormal Basis:

An orthonormal Basis is constructed from the orthogonal Basis by normalizing the each vector.

Gramschmidt's Orthogonalization process

Given a Basis $\{x_1, x_2, \ldots, x_n\}$ for the subspace W of $\mathbb{R}^n$ define $v_1 = x_1$; $v_2 = x_2 - \frac{x_2 \cdot x_1}{v_1 \cdot v_1} \cdot v_1$

and $v_3 = x_3 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} \cdot v_2 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} \cdot v_1$

– pending

$$\boxed{v_p = x_p - \frac{x_p v_1}{v_1 \cdot v_1} \cdot v_1 - \frac{x_p v_2}{v_2 \cdot v_2} \cdot v_2 - \cdots \frac{x_p v_{n+1}}{v_{n+1} \cdot v_{n+1}} \cdot v_{n+1}}$$

Then $\{v_1, v_2, \ldots, v_p\}$ are the orthogonal Basis

By normalizing each vector in orthogonal Basis we get orthonormal Basis $\{u_1, u_2, \ldots, u_p\}$ as

$u_1 = \frac{v_1}{\|v_1\|}$; $u_2 = \frac{v_2}{\|v_2\|}$

$u_p = \frac{v_p}{\|v_p\|}$

Imp:

$v_1, v_2, \ldots$ are orthogonal Basis

$u_1, u_2, \ldots u_p$ are orthonormal Basis

Q2) Let $W = \text{span}[x_1, x_2]$ where $x_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ $x_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Construct an orthogonal Basis of $\{v_1, v_2\}$ using Gram-Schmidt ortho process.

⇒ W.K.T

$$v_1 = x_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$

Given: $x_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} \cdot v_1$$

$$x_2 \cdot v_1 = x_2^T \cdot v_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1\times 3} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 3+12+0 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 15 \end{bmatrix}_{1\times 1}}}$$

$$v_1 \cdot v_1 = v_1^T \cdot v_1 = \begin{bmatrix} 3 & 6 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 9+36+0 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 45 \end{bmatrix}_{1\times 1}}}$$

$$\therefore v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} \cdot v_1$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{\cancel{15}^{1}}{\cancel{45}_{3}} \times \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$v_2 = \underline{\underline{\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}_{3\times 1}}}$$

$\therefore v_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ & $v_2 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$ are orthogonal Basis of $\{v_1, v_2\}$

(Q2) Given a Basis $\{x_1, x_2, x_3\}$ of a subspace W of $\mathbb{R}^3$

where $x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$; $x_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; $x_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. Find Orthonormal Basis of $\{v_1, v_2, v_3\}$ using Gram-Schmidt's Process

- To find orthonormal, we need to find the orthogonal Basis Hence,

$$v_1 = x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} \cdot v_1 \quad \text{--- (1)}$$

$$x_2 \cdot v_1 = x_2^T \cdot v_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}_{1\times3} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}_{3\times1} = \underline{\underline{\begin{bmatrix} 1 \end{bmatrix}_{1\times1}}} = \underline{\underline{1}}$$

$$v_1 \cdot v_1 = v_1^T \cdot v_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix}_{1\times1} = \underline{\underline{2}}$$

From (1)

$$\therefore v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2} \times \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 0/2 \\ 1/2 \end{bmatrix} = \underline{\begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}}$$

$$v_3 = x_3 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 \quad \text{———} \quad (2)$$

$$x_3 \cdot v_2 = x_3^T \cdot v_2 = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 0 + 0 \end{bmatrix} = [1]_{1\times1} = \underline{1}$$

$$v_2 \cdot v_2 = v_2^T \cdot v_2 = \begin{bmatrix} 1/2 & 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}$$

$$= \cancel{[2]_{1\times1} = 2} = \begin{bmatrix} 1/4 + 1/4 \end{bmatrix} = [1/2]_{1\times1}$$

$$x_3 \cdot v_1 = x_3^T \cdot v_1 = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 + 0 + 0 \end{bmatrix} = [2]_{1\times1} = \underline{2}$$

From (2)

$$v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{\cancel{2}} \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/\cancel{4} \\ 0 \\ -1/\cancel{4} \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\frac{8-1-4}{4} \qquad \frac{3}{4}$$

$$v_3 = \begin{bmatrix} 2 - 1/\cancel{4} - 1 \\ 1 - 0 - 0 \\ 0 + 1/\cancel{4} - 1 \end{bmatrix} = \begin{bmatrix} \cancel{3/4}\ 0 \\ 1 \\ \cancel{-3/4}\ 0 \end{bmatrix}_{3\times1}$$

$$\frac{1-4}{4} =$$

(Q4) Let $W = \{x_1, x_2, x_3\}$ where $x_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ $x_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}$ $x_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$

Construct an orthogonal normal Basis using GramSchmidt method.

= W.K.T $v_1 = x_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ —— ①

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$v_3 = x_3 - \frac{x_3 v_2}{v_2 \cdot v_2} v_2 - \frac{x_3 v_1}{v_1 \cdot v_1} v_1 \quad \text{—— (*)}$$

$$x_2 \cdot v_1 = x_2^T \cdot v_1 = \begin{bmatrix}0 & 1 & 1 & 1\end{bmatrix}_{1\times4} \begin{bmatrix}1\\1\\1\\1\end{bmatrix}_{4\times1} = [3]_{1\times1} = \underline{3}$$

$$v_1 \cdot v_1 = v_1^T \cdot v_1 = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix} \begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}4\end{bmatrix} = \underline{4}$$

$$\therefore v_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \frac{3}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$v_2 = \begin{bmatrix}-3/4\\1/4\\1/4\\1/4\end{bmatrix} \quad \text{—— ②}$$

$$x_3 \cdot v_2 = x_3^T v_2 = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \end{bmatrix}_{1\times 1} = \underline{\underline{1/2}}$$

$$v_2 \cdot v_2 = v_2^T \cdot v_2 = \begin{bmatrix} -3/4 & 1/4 & 1/4 & 1/4 \end{bmatrix} \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \underline{\underline{\frac{12}{16}}} = 3/4$$

$$x_3 \cdot v_1 = x_3^T \cdot v_1 = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = [2]_{1\times 1} = \underline{\underline{2}}$$

∴ From (*)

$$v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{\frac{1}{2}}{3/4} \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\frac{1}{6} \begin{bmatrix} -3/4 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$-\frac{1}{4} \times -\frac{3}{4} \cdot 2 = \frac{1}{8}$$

$$-\frac{1}{6} \times \frac{1}{4} =$$

$$\frac{1}{2} \times \frac{4}{3}$$

$$\cancel{v_3 = \begin{bmatrix} -3/8 \\ -13/24 \\ 11/24 \\ 11/24 \end{bmatrix}}$$

$$\therefore \; v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/2 \\ -1/6 \\ -1/6 \\ -1/6 \end{bmatrix} + \begin{bmatrix} -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}$$

$$\|v_1\| = \sqrt{1^2+1^2+1^2+1^2} = \sqrt{4} = 2$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{2}{\sqrt{3}}\begin{bmatrix}-3/4\\1/4\\1/4\\1/4\end{bmatrix}$$

$$\|v_2\| = \sqrt{(3/4)^2+(1/4)^2+(1/4)^2+(1/4)^2} = \frac{\sqrt{3}}{2}$$

$$u_3 = \frac{v_3}{\|v_3\|} = \sqrt{\frac{3}{2}}\begin{bmatrix}0\\-2/3\\1/3\\1/3\end{bmatrix}$$

$$\|v_3\| = \sqrt{2/3} = \sqrt{2/3}$$

(05) Let $W = \text{Span}\{x_1, x_2, x_3\}$

where $x_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$; $x_2 = \begin{bmatrix}1\\-1\\2\\2\end{bmatrix}$; $x_3 = \begin{bmatrix}1\\2\\-3\\-4\end{bmatrix}$

Construct Orthogonal Basis using GSM

= W.K.T

$$v_1 = x_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad \text{—} (1)$$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$x_2 \cdot v_1 = x_2^T \cdot v_1 = \begin{bmatrix}1 & -1 & 2 & 2\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \underline{4}$$

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$$v_1 \cdot v_1 = v_1^T \cdot v_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \underline{\underline{4}}$$

$$\therefore v_2 = \begin{bmatrix} 1 \\ -1 \\ +2 \\ 2 \end{bmatrix} - \frac{4}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

$$v_3 = x_3 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$x_3 \cdot v_2 = x_3^T \cdot v_2 = \begin{bmatrix} 1 & 2 & -3 & -4 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} = \underline{\underline{-11}}$$

$$v_2 \cdot v_2 = v_2^T \cdot v_2 = \begin{bmatrix} 0 & -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} = \underline{\underline{6}}$$

$$x_3 \cdot v_1 = x_3^T \cdot v_1 = \begin{bmatrix} 1 & 2 & -3 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \underline{\underline{-4}}$$

$$\therefore v_3 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ -4 \end{bmatrix} - \frac{(-11)}{6} v_2 - \frac{(-4)}{4} v_1$$

$$= x_3 + \frac{11}{6} v_2 + v_1$$

$$= \begin{bmatrix} 1 \\ 2 \\ -3 \\ -4 \end{bmatrix} + \begin{bmatrix} 0 \\ -22/6 \\ 11/6 \\ 11/6 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -2/3 \\ -1/6 \\ \cancel{-1/6}\ -7/6 \end{bmatrix}$$

$\therefore$ v_1, v_2 & v_3 are the required Orthogonal Basis

Is used to find the solution of inconsistent ~~some~~ system $AX = B$.
When a solution doesn't exists, the best one we can do is find an 'x' that makes AX as closely as possible to B
The General Least Square Method is to find 'x' that makes norm$(B - AX)$ or $\|B - AX\|$ as small as possible

Imp: $\boxed{\text{Defination}}$ If A is an $n \times n$ Matrix and B is in $\mathbb{R}^n$. A Least Square solution of $AX = B$ is $\hat{x}$ such that $\|B - A\hat{x}\| \leq \|B - AX\|$ $\forall \hat{x} \in \mathbb{R}^n$

(i) The Least Square Solution of the Equation: $AX = B$ satisfies the equation
(01) $\boxed{A^T A x = A^T B}$
(02) The matrix $A^T A$ is invertible only if the equation $AX = B$ has one Least $\overset{\text{square}}{\text{Solution}}$, $\hat{x}$ and is given by $\boxed{\hat{x} = (A^T A)^{-1} A^T B}$

(Q) Find Least Square Solution $Ax = B$ where :

$$A = \begin{bmatrix} 1 & -3 & -3 \\ 1 & 5 & 1 \\ 1 & 7 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ -3 \\ -5 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} \cancel{1} & \cancel{-3} & \cancel{-7} \\ 1 & 5 & \\ & & \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 5 & 7 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -3 & -3 \\ 1 & 5 & 1 \\ 1 & 7 & 2 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 3 & 9 & 0 \\ 9 & 83 & 28 \\ 0 & 28 & 14 \end{bmatrix} \quad \text{--- (1)}$$

$$A^TB = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 5 & 7 \\ -3 & 1 & 2 \end{bmatrix}_{3\times 3} \begin{bmatrix} 5 \\ -3 \\ -5 \end{bmatrix}_{3\times 1}$$

$$= \begin{bmatrix} -3 \\ -65 \\ -28 \end{bmatrix}_{3\times 1} \quad \text{--- (2)}$$

$\therefore \cancel{A^TAx = A^TB}$

Thus,
From (1) and (2)

it is of the form $\underset{(A^TA)}{A}x = \underset{(A^TB)}{B}$

$$\Rightarrow (A:B) = \left[\begin{array}{ccc|c} 3 & 9 & 0 & -3 \\ 9 & 83 & 28 & -65 \\ 0 & 28 & 14 & -28 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$

$$\therefore (A:B) = \left[\begin{array}{ccc|c} 3 & 9 & 0 & -3 \\ 0 & 56 & 28 & -56 \\ 0 & 28 & 14 & -28 \end{array}\right]$$

$R_3 \to 2R_3 - R_2$

$$= \left[\begin{array}{ccc|c} 3 & 9 & 0 & -3 \\ 0 & 56 & 28 & -56 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = 2$; $\rho(A:B) = 2$ $\because n = 3$

$\therefore$ The system is consistent and has infinite many solutions.

$\therefore$ The ~~corres~~ equation from corresponding matrix are

$56y + 28z = -56$ —①

$3x + 9y = -3$ —②

From ①

~~deta~~ put $z = k$

$\Rightarrow 56y + 28k = -56$

$\Rightarrow y = \dfrac{-56 - 28k}{56}$

$y = -1 - \dfrac{k}{2}$

or $y = \dfrac{-2-k}{2}$

From ②

$3x + 9\left(\dfrac{-2-k}{2}\right) = -3$

$3x = -3 - \dfrac{9(-2-k)}{2}$

$$3x = \frac{-6+18+9k}{2}$$

$$3x = \frac{9k+12}{2}$$

$$\boxed{x = \frac{9k+12}{6}} \quad \boxed{y = \frac{-(2+k)}{2}} \quad \boxed{z = k}$$

Verification: Let $k=0$ then

$$x = \frac{12}{6} = \boxed{2} \qquad y = \frac{-2}{2} = \boxed{-1} \qquad \boxed{z = 0}$$

$\therefore \hat{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ is one of the LSS.

(ii) Find LSS of $Ax = B$ where

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}_{3\times 4 \ 4\times 3} \text{ and } B = \begin{bmatrix} 3 \\ 5 \\ 7 \\ -3 \end{bmatrix}$$

$$\therefore A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 5 & 0 & 2 & 3 \end{bmatrix}_{3\times 4} \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}_{4\times 3}$$

$$= \begin{bmatrix} 4 & 8 & 10 \\ 8 & 20 & 26 \\ 10 & 26 & 38 \end{bmatrix}_{3\times 3} \qquad RHS = A^TB = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 5 & 0 & 2 & 3 \end{bmatrix}_{3\times 4} \begin{bmatrix} 3 \\ 5 \\ 7 \\ -3 \end{bmatrix}_{4\times 1}$$

$$= \begin{bmatrix} 12 \\ 12 \\ 20 \end{bmatrix}_{3\times 1}$$

$\therefore (A:B) = \begin{bmatrix} 4 & 8 & 10 & : 12 \\ 8 & 20 & 26 & : 12 \\ 10 & 26 & 38 & : 20 \end{bmatrix}$

$= \begin{bmatrix} 4 & 8 & 10 & : & 12 \\ 8 & 20 & 26 & : & 12 \\ 10 & 26 & 38 & : & 20 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$

$R_3 \to 4R_3 - 10R_1$ $\qquad 4(10) - 10(4)$

$= \begin{bmatrix} 4 & 8 & 10 & : & 12 \\ 0 & 4 & 6 & : & -12 \\ 0 & 24 & 52 & : & -40 \end{bmatrix}$

$R_3 \to R_3 - 6R_2$

$= \begin{bmatrix} 4 & 8 & 10 & : & 12 \\ 0 & 4 & 6 & : & -12 \\ 0 & 0 & 16 & : & 32 \end{bmatrix}$

$\therefore \rho(A) = 3 \; ; \rho(A:B) = 3 \; ; n = 3$

$\therefore$ The system is consistent & has unique soln.

$\therefore$ The eq^n from corres^g matrix are

$16z = 32$

$\boxed{z = 2}$

$4y + 6z = -12$

$4y = -12 - 6(2)$

$\boxed{y = -6}$

$4x + 8y + 10z = 12$

$4x + 8(-6) + 10(2) = 12$

$\boxed{x = 10}$

$\therefore \hat{x} = \begin{bmatrix} 10 \\ -6 \\ 2 \end{bmatrix}$ are the LSS

Q. Find LSS of $AX = B$ where

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix}$$

$$LHS = A^T A = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}}_{4\times 6} \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}}_{6\times 4}$$

$$A^T A = \underbrace{\begin{bmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}}_{4\times 4}$$

R.H.S:

$$A^TB = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}_{4\times 6} \begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix}_{6\times 1}$$

$$A^TB = \begin{bmatrix} 4 \\ -4 \\ 2 \\ 6 \end{bmatrix}_{4\times 1}$$

$$\therefore \left[A^TA : A^TB\right] = \begin{bmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}_{4\times 4} : \begin{bmatrix} 4 \\ -4 \\ 2 \\ 6 \end{bmatrix}_{4\times 1}$$

$[A : B]$

$$= \begin{bmatrix} 6 & 2 & 2 & 2 & : & 4 \\ 2 & 2 & 0 & 0 & : & -4 \\ 2 & 0 & 2 & 0 & : & 2 \\ 2 & 0 & 0 & 2 & : & 6 \end{bmatrix}$$

$R_2 \to 3R_2 - R_1$

$R_3 \to 3R_3 - R_1$

$R_4 \to 3R_4 - R_1$

$$= \begin{bmatrix} 6 & 2 & 2 & 2 & : & 4 \\ 0 & 4 & -2 & -2 & : & -16 \\ 0 & -2 & 4 & -2 & : & 2 \\ 0 & -2 & -2 & 4 & : & 14 \end{bmatrix}$$

$R_3 \to 2R_3 + R_2$
$R_4 \to 2R_4 + R_2$

$$\left[\begin{array}{ccccc} 6 & 2 & 2 & 2 & 4 \\ 0 & 4 & -2 & -2 & -16 \\ 0 & 0 & 6 & -6 & -12 \\ 0 & 0 & -6 & 6 & 12 \end{array}\right]$$

$R_4 \to R_4 + R_3$

$$= \left[\begin{array}{cccc|c} 6 & 2 & 2 & 2 & 4 \\ 0 & 4 & -2 & -2 & -16 \\ 0 & 0 & 6 & -6 & -12 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \rho(A) = \rho(A:B) = 3 < 4(n)$

∴ The system is consistent and has infinite solutions

∴ The eq from corres matrix,

$6y - 6z = -12$
$4x - 2y - 2z = -16$
$6w + 2x + 2y + 2z = 4$

Verification: ~~put~~ let $\boxed{z=0}$

$\Rightarrow 6y = -12$

$\boxed{y = -2}$

$\Rightarrow 4x - 2(-2) = -16$

$\boxed{x = -5}$

$\Rightarrow 6w + 2(-5) + 2(-2) = 4$

$\boxed{w = 3}$

∴ L.S.S. are

$$\hat{x} = \begin{bmatrix} 3 \\ -5 \\ -2 \\ 0 \end{bmatrix}$$

(H.W) Find L.S.S

(Q4) $A = \begin{bmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 6 \end{bmatrix}$

LHS = $A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -6 & -2 & 1 & 7 \end{bmatrix}_{2\times 4} \begin{bmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix}_{4\times 2}$

$A^T A = \begin{bmatrix} \cancel{2}\,4 & 0 \\ 0 & 90 \end{bmatrix}$

$A^T B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -6 & -2 & 1 & 7 \end{bmatrix}_{2\times 4} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 6 \end{bmatrix}_{4\times 1}$

$= \begin{bmatrix} 10 \\ 33 \end{bmatrix}_{2\times 1}$

$\therefore [A^T A : A^T B] = \left[\begin{array}{cc:c} 4 & 0 & 10 \\ 0 & 90 & 33 \end{array}\right]$

$\therefore$ The system is consistent and has unique solution.

$\therefore$ The eq from corres. eqn's

$90y = 33$ $4x = 10$

$\boxed{y = \frac{11}{30}}$ $\boxed{x = \frac{5}{2}}$

$\therefore \hat{x} = \begin{bmatrix} 5/2 \\ 11/30 \end{bmatrix}$

Isomorphism

If V and W are vectorspace over field F, any one-one LT of V onto W is called as Isomorphism denoted by $V \cong W$

P.T every finite Dimensional vectorspace V is isomorphic to Field F

Proof:

Let V be the vectorspace over the field F and $B = \{v_1, v_2, v_3, \ldots, v_n\}$ be Basis for V then,

Every vectors $v \in V$ can be written as Linear Combination of the elements of the Basis B

ie, $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$ where $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n \in F$

Consider a mapping $T : V \rightarrow F$ such that $T \cdot v = (\alpha_1, \alpha_2, \ldots, \alpha_n)$

(i) <u>T</u> <u>is</u> <u>Linear</u>:

Let $u, v \in V$. We define $u = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$

$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

Now, $T(cu+v) = T\cancel{(c(\beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n) + \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n)}$

$= \cancel{c\beta_1 T(T}$

$= \cancel{c\beta_1 T(u_1) + c\beta_2 T(u_2) + \cdots + c\beta_n T(u_n) + \alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n)}$

$= \cancel{cT(\beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n) + T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n)}$

$\boxed{\cancel{T(cu+v) = cT(u) + T(v)}}$

$cT(\alpha)\ cT(\beta)$

$= T(c(\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n) + \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n)$

$= \cancel{T(c(\beta_1 + \alpha_1) v_1 + c((\beta_2) + \alpha_2) v_2 + \cdots + c(\beta}$

$= T((c\beta_1 + \alpha_1) v_1 + (c\beta_2 + \alpha_2) v_2 + \cdots + [c(\beta_n) + \alpha_n] v_n)$

$= \cancel{cT((\beta_1 + \alpha_1) v_1 + (\beta_2 + \alpha_2) v_2 + \cdots + (\beta_n + \alpha_n) v_n)}$

$= cT(\beta_1 v_1 + \beta_2 v_2 + \cdots) + T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n)$

$= cT(u) + T(v) = \boxed{R.H.S}$

(ii) $\boxed{\text{T is one-one}}$

Suppose $T.v = T.u$

i.e. $(\alpha_1, \alpha_2, \ldots, \alpha_n) = (\beta_1, \beta_2, \ldots, \beta_n)$

$\Rightarrow \alpha_1 = \beta_1 ; \alpha_2 = \beta_2 ; \ldots ; \beta_n = \alpha_n$ ~~For any~~

$\Rightarrow \alpha_i = \beta_i$ For any i

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n$

$\Rightarrow \boxed{v = u}$

$\therefore$ T is one-one

(iii) $\boxed{\text{T is onto}}$

Corresponding to each α_i's i.e. $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$
such that $T.v = T(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n)$
$= (\alpha_1, \alpha_2, \ldots, \alpha_n)$

$\therefore$ T is onto

$\boxed{\therefore v \cong F}$ Hence proved.