Quantum Winter Hackathon

QUESTION 1

Team Quantico | 08-12-2020

1 Solution of 1D Wave Equation using Finite Difference Method

The wave equation given is

$$\frac{\partial u}{\partial t} + C\frac{\partial u}{\partial x} = 0$$

Our goal is to discretize the above equation in 2^{nd} order spatial accuracy and 1^{st} order time accuracy with the implicit time integration scheme.

$$\frac{\partial u}{\partial t} = -C\frac{\partial u}{\partial x}$$

Let $\frac{\partial u}{\partial x} = U_x$ and $\frac{\partial u}{\partial t} = U_t$ and the PDE becomes $U_x + CU_t = 0. ------eq\,1$

2 Spatial Discretization

Let the space step be Δx where

$$\Delta x = \frac{b-a}{N-1}$$

N is the number of points and $x \,\epsilon\, [a\ , b]$.
We find the points on x axis such that $x_{i+1} = x_i + \Delta x$ where $i = 0, 1, 2....., 40$.
The central difference formula can be applied to approximate U_x as

$$U_x = \frac{\partial u}{\partial x} = \frac{U_{i+1}^{n+1} - U_{i-1}^{n+1}}{2\Delta x} ------\text{eq}\,2$$

at a time instant t_{n+1} using implicit time integration scheme
Subtituting eq 2 in eq 1 we get

$$U_t + C\frac{U_{i+1}^{n+1} - U_{i-1}^{n+1}}{2\Delta x} = 0 ------\text{eq}\,3$$

3 Time Discretization

Let the time step be Δt where $t_n = t_0 + \Delta t$
The forward difference formula can be applied to approximate U_t as

$$U_t = \frac{\partial u}{\partial t} = \frac{U_i^{n+1} - U_i^n}{\Delta t} ------\text{eq}\,4$$

Subtituting eq 4 in eq 3 we get

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + C\frac{U_{i+1}^{n+1} - U_{i-1}^{n+1}}{2\Delta x} = 0 ------\text{eq}\,5$$

which is the approximation of given PDE.
By rearranging eq 5 we get

$$U_i^{n+1} = U_i^n - \frac{C\Delta t}{\Delta x}(U_{i+1}^{n+1} - U_i^{n+1}) - - - - - - -eq\,6$$

eq 6 can be used to find U at any given point in space and time.