UNIT-VI

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Recurrence Relation

* Generating functions and sequences
* Calculating coefficients of generating function
* Recurrence relation
* Solving recurrence relation by substitution and generating functions
* Characteristics roots
* Solution of inhomogeneous recurrence relation

Generating functions :-

Consider the sequence $a_0, a_1, a_2, a_3 \cdots a_n$ - if the function $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_{n-1} x^{n-1}$.

$f(x) = \sum_{r=0}^{\infty} a_r x^r$ there four $f(x)$ is called generating function for the sequence a_0, a_1, a_2 $a_3, \cdots a_n$

Note :-

i) $\boxed{1 + x + x^2 + x^3 + \cdots = (1-x)^{-1} = \sum_{r=0}^{\infty} x^r}$ $\frac{1}{(1-x)}$

$\therefore f(x) = (1-x)^{-1}$ is a generating function of the sequance $1, 1, 1, 1, \cdots$

ii) $1 - x + x^2 - x^3 + \cdots = (1+x)^{-1} = \sum_{r=0}^{\infty} (-1)^r x^r$

$\therefore f(x) = (1+x)^{-1}$ is a generating function of the sequance $1, -1, 1, -1 \cdots$

Eg :- Find the generating functions for the following sequences.

$1, 3, 9, 27$

i) $1, 2, 3, 4$ } Formula

ii) $1, -2, 3, -4$

iii) $0, 1, 2, 3$

iv) $0, 1, -2, 3, -4$

Sol i) $1, 2, 3, 4$

(ii) $1 + 2x + 3x^2 + 4x^3 + \cdots$

$= (1-x)^{-2}$

$f(x) = (1-x)^{-2}$ is the required generating function.

ii) $1, -2, 3, -4$

$= 1 - 2x + 3x^2 - 4x^3 + \cdots$

$= (1+x)^{-2}$

$f(x) = (1+x)^{-2}$ is required generating function

iii) $0, 1, 2, 3$ $a_0 \quad a_1 \quad a_2 \quad a_3$

$= 0 + x + 2x^2 + 3x^3 + \cdots$

$= (x + 2x^2 + 3x^3 + \cdots)$

$= x(1 + 2x + 3x^2 + \cdots)$

$= x[(1-x)^{-2}]$

$f(x) = \dfrac{x}{(1-x)^2}$ is a required generating function

iv) $0, 1, -2, 3, -4$

$= 0 + x - 2x^2 + 3x^3 - 4x^4 + \cdots$

$= x - 2x^2 + 3x^3 - 4x^4 + \cdots$

$= x[1 - 2x + 3x^2 - 4x^3 + \cdots]$

$= x[(1+x)^{-2}]$

$f(x) = \dfrac{x}{(1+x)^{-2}}$ is a required generating function

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2) Find the generating function for the following Sequences. (67)

i) $1^2, 2^2, 3^2, \ldots$

ii) $0^2, 1^2, 2^2, 3^2, \ldots$

Sol

Let i) $1^2, 2^2, 3^2 \ldots$

$$\boxed{0+1x+2x^2+3x^3+4x^4+\ldots = x[1-x]^{-2}}$$

Diff. w.r.to x

$$1+2(2x)+3(3x^2)+4(4x^3)+\ldots = \frac{d}{dx}\left[x(1-x)^{-2}\right]$$

$$1^2+2^2(x)+3^2(x^2)+4^2(x^3)+\ldots = \frac{d}{dx}\left[\frac{x}{(1-x)^2}\right]$$

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{d}{dx}(u)v - \frac{d}{dx}(v)u}{v^2}$

$u'v - v'u$

$$= \frac{2x(1-x)^2+1}{(1-x)^2}$$

$$= \frac{x \cdot \frac{d}{dx}[(1-x)^2] - (1-x)^2 \frac{d}{dx}(x)}{[(1-x)^2]^2}$$

$$\frac{x\,2(1-x)\frac{d}{dx}(1-x) - (1-x^2)}{(1-x)^4}$$

$$= \frac{2x(1-x)(-1) - (1-x)^2}{(1-x)^4}$$

$$= \frac{\cancel{(1-x)}(-2x-1+x)}{(1-x)^{\cancel{4}}}$$

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$

$$\frac{-x-1}{(1-x)^3}$$

$$\Rightarrow \frac{+x+1}{(1-x)^3} = \frac{+(1+x)}{(1-x)^3}$$

$$= \frac{(x+1)}{(1-x)^3}$$

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ii) $0^2, 1^2, 2^2, 3^2, \text{------}$

$$0 + 1^2x + 2^2x^2 + 3^2x^3 + \text{------}$$

$$x\left[1^2 + 2^2x + 3^2x^2 + \text{-----}\right]$$

$$\boxed{= x\left[\frac{(1+x)}{(1-x)^3}\right]}$$

3. $1^3, 2^3, 3^3$

$$= 1^3, 2^3, 3^3$$

Let

$$0^2 + 1^2x + 2^2x^2 + 3^2x^3 + \text{----} = \frac{x(1+x)}{(1-x)^3}$$

d.w.r.to x

$$1 + 2^2(2x) + 3^2(3x^2) + \text{----} = \frac{d}{dx}\left[\frac{x(1+x)}{(1-x)^3}\right]$$

$$1 + 2^3x + 3^3x^2 + \text{----} = \frac{d}{dx}\left[\frac{x+x^2}{(1-x)^3}\right] \quad 4$$

$$= \frac{(1-x)^3(1+2x) - (x+x^2)\,3(1-x)^2(-1)}{\left[(1-x)^3\right]^2}$$

$$= \frac{\cancel{(1-x)^2}\left[(1-x)(1+2x) + 3(x+x^2)\right]}{(1-x)^{\cancel{6}\,4}}$$

$$= \frac{(1-x)(1+2x) + 3x + 3x^2}{(1-x)^4}$$

$$\frac{d}{dx}(1-px)^n = n * (1-px)^{n-1}\frac{d}{dx}(px)$$

$p =$ constant

$$= \frac{1+2x-x-2x^2+3x+3x^2}{(1-x)^4}$$

$$= \frac{x^2+4x+1}{(1-x)^4}$$

4. $0^3, 1^3, 2^3, 3^3$

$0^3, 1^3, 2^3, 3^3$

$0^3 + 1^3 x + 2^3 x^2 + 3^3 x^3 + \cdots$

$x(1^3 + 2^3 x + 3^3 x^2 + \cdots)$

$$= \frac{x(-(x^2+4x+1))}{(1-x)^4}$$

5. find the generating function for the sequence $1, 1, 0, 1, 1, 1$

Sol $1, 1, 0, 1, 1, 1$

$= 1 + 1.x + 0.x^2 + 1.x^3 + 1 \cdot x^4 + 1.x^5 + \cdots$

$= 1 + x + x^3 + x^4 + x^5 + \cdots$

$= [1 + x + x^2 + x^3 + x^4 + x^5 + \cdots] - x^2$

$= (1-x)^{-1} - x^2$

Note:- ① $(1+x)^n = \sum_{r=0}^{\infty} {}^nC_r\, x^r$

$= \sum_{r=0}^{\infty} \binom{n}{r} x^r$

② $(1-x)^n = \sum_{r=0}^{\infty} {}^nC_r (-1)^r x^r$

③ $(1-x)^{-n} = \sum_{r=0}^{\infty} \left({}^{n+r-1}C_r\right) x^r$

(or)

$= \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$

6. Find the co-efficient of x^{12} of $x^3(1-2x)^{10}$

Sol

$= x^3(1-2x)^{10}$

$(2x)$

$2^r \cdot x^r$

$x^n \cdot x^m = x^{n+m}$

$= x^3 \times \sum_{r=0}^{\infty} 10_{C_r} (-1)^r (2x)^r$

$= x^3 \times \sum_{r=0}^{\infty} 10_{C_r} (+2)^r x^r$

$= \sum_{r=0}^{\infty} 10_{C_r} (+2)^r x^{r+3}$

$\boxed{r=9}$

Coeff x^{12} is

$\boxed{= {}^{10}C_9 (+2)^9}$

$= 10 \times +2^9$

$= 10 \times +512$

$x^{12} = +5120$

7. Find the Co-efficient of x^5 of $(1-2x)^{-7}$ (9)

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sol

$$= (1-2x)^{-7}$$

$$= \sum_{r=0}^{\infty} \left({}^{n+r-1}C_r \right) 2^r x^r$$

$$= \sum_{r=0}^{\infty} \left({}^{7+r-1}C_r \right) 2^r x^r$$

$7+r-1=$

$$\boxed{r = 5}$$

$7+r-1$

$r-1=-7$

$r=-8$

$$\text{coeffi of } x^5 = {}^{7+5-1}C_5 \times 2^5$$

$$= {}^{11}C_5 \times 2^5$$

$$= \frac{11!}{(11-1)!\,5!}$$

$$= \frac{11!}{10!\,5!}$$

8. Find the coefficient of x^{12} of $x^2/(1-x)^{10}$

sol

$$= x^2/(1-x)^{10}$$

$$= x^2 (1-x)^{-10}$$

$$= x^2 . \sum_{r=0}^{\infty} \left({}^{10+r-1}C_r \right) x^r$$

$$= \sum_{r=0}^{\infty} \left({}^{10+r-1}C_r \right) x^{2+r}$$

$$\boxed{r = 10}$$

coeff x^{12} $= {}^{10+10-1}C_{10} = {}^{19}C_{10}$

9) Find the Co-efficient of x^{12} of $(1-4x)^{-5}$

Sol

$$= (1-4x)^{-5}$$

$$= \sum_{r=0}^{\infty} \left({}^{n+r-1}C_r \right) x^r 4^r$$

$$= \sum_{r=0}^{\infty} \left({}^{5+r-1}C_r \right) x^r 4^r$$

$$r = 12$$

coeffi of x^{12}

$$= {}^{5+12-1}C_{12} \times 4^{12}$$

$$= {}^{16}C_{12} \times 4^{12}$$

10) find the coefficient of x^{12} of $(1+x)^{-20}$

Sol

$$= (1+x)^{-20}$$

$$= \left({}^{20+r-1}C_r \right) x^r$$

$$\boxed{r = 12}$$

$$=$$

Coffeic of x^{12}

$$= {}^{20+12-1}C_{12}$$

$$= {}^{13}C_{12}$$

$$(1+x)^{-20}$$

$$= \sum_{r=0}^{\infty} {}^{-20}C_r \, x^r$$

$$= \sum_{r=0}^{\infty} {}^{20+r-1}C_r \, x^r (-1)^r$$

$$r = 12$$

$$= {}^{20+12-1}C_{12} (-1)^{12}$$

$$= {}^{31}C_{12}$$

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Note :-

We note that the co-efficient of $-^{n}C_r\,(-x)^r$

$$-^{n}C_r = (-1)^r \left({}^{n+r-1}C_r \right)$$

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Find the coefficient of x^{27} of

i) $(x^4 + x^5 + x^6 + \cdots)^5$

ii) $(x^4 + 2x^5 + 3x^6 + \cdots)^5$

Sol i) $(x^4 + x^5 + x^6 + \cdots)$

$$= (x^4)^5 \left[(1 + x + x^2 + \cdots)^5 \right]$$

$$= x^{20} (1 + x + x^2 + \cdots)^5$$

$$= x^{20} \left[(1-x)^{-1} \right]^5$$

$$= x^{20} (1-x)^{-5}$$

$$= x^{20} \sum_{r=0}^{\infty} \left({}^{5+r-1}C_r \right) x^r$$

$$= \sum_{r=0}^{\infty} \left({}^{5+r-1}C_r \right) x^{20+r}$$

$$r = 7$$

$$= {}^{5+7-1}C_7$$

$$= {}^{11}C_7$$

$$=$$

ii) $(x^4 + 2x^5 + 3x^6 + \cdots)^5$

$$(x^4)^5\left[(1+2x+3x^2+\cdots)\right]^5$$

$$= x^{20}(1+2x+3x^2+\cdots)^5$$

$$= x^{20}\left((1-x)^{-2}\right)^5$$

$$= x^{20}(1-x)^{-10}$$

$$= x^{20}\sum_{r=0}^{\infty}\left({}^{10+r-1}C_r\right)x^r$$

$$= \sum_{r=0}^{\infty}\left({}^{10+r-1}C_r\right)x^{20+r}$$

$$r = 7$$

$$= {}^{10+7-1}C_7$$

$$= {}^{16}C_7$$

12) Solve reccurance relation $a_n = a_{n-1} + n$, $n \geq 1$ where $a_0 = 2$ by substitution method.

Sol

Given $a_n = a_{n-1} + n$ where $a_0 = 2$

$n=1 \quad a_1 = a_0 + 1$

$n=2 \quad a_2 = a_1 + 2 = a_0 + 1 + 2$

$n=3 \quad a_3 = a_2 + 3 = a_0 + 1 + 2 + 3$

$$a_n = a_0 + \underbrace{1+2+3+4+\cdots n}$$

$$a_n = a_0 + \sum_1 n$$

$$a_n = a_0 + \frac{n(n+1)}{2} \quad / \quad \therefore \sum_1 n = \frac{n(n+1)}{2}$$

$$a_n = \frac{2 + n(n+1)}{2}$$

$$a_n = \frac{4 + n(n+1)}{2}$$

13) Solve recurrence relation $a_n = a_{n-1} + n^3$, $n \geq 1$ where $a_0 = 5$ by substitution method

Sol Given

$$a_n = a_{n-1} + n^3 \text{ where } a_0 = 5$$

$n=1 \quad a_1 = a_0 + 1^3$

$n=2 \quad a_2 = a_1 + 2^3 = a_0 + 1^3 + 2^3$

$n=3 \quad a_3 = a_2 + 3^3 = a_0 + 1^3 + 2^3 + 3^3$

$$a_n = a_0 + 1^3 + 2^3 + 3^3 + \cdots n^3$$

$$a_n = a_0 + \sum_1 n^3$$

$$= a_0 + \left(\frac{n(n+1)}{2}\right)^2 \qquad \sum_1 n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$= 5 + \left(\frac{n(n+1)}{2}\right)^2$$

14) Solve recurrance relation $a_n = a_{n-1} + 3n^2 + 3n + 1$, $n \geq 1$ where $a_0 = 1$ by substitution method.

Sol

Given

$$a_n = a_{n-1} + 3n^2 + 3n + 1$$

$n=1$ $\quad a_1 = a_0 + 3(1)^2 + 3(1) + 1$

$$a_2 = a_1 + 3(2)^2 + 3(2) + 1 = a_0 + 3(1)^2 + 3(1) + 1 + 3(2)^2 + 3(2) + 1$$

$$= a_0 + 3\left[1^2 + 2^2\right] + 3\left[1+2\right] + \left[1+1\right]$$

$$a_3 = a_0 + 3\left[1^2 + 2^2 + 3^2\right] + 3\left[1+2+3\right] + \cancel{\left[1+1+1\right]}$$

$\vdots$

$$a_n = a_0 + 3\left[1^2 + 2^2 + \cdots n^2\right] + 3\left[1+2+3+\cdots n\right] + \left[1+1+1 \cdots n\right]$$

$$a_n = a_0 + 3\sum_1 n^2 + 3\sum_1 n + n$$

$$= a_0 + \cancel{3}\,\frac{n(n+1)(2n+1)}{\cancel{6}_{2}} + 3\,\frac{n(n+1)}{2} + n$$

$$= 1 + \frac{n(n+1)(2n+1)}{2} + \frac{3(n(n+1))}{2} + n$$

$$= \frac{2 + n(n+1)(2n+1) + 3n(n+1) + 2n}{2}$$

$$= \frac{2 + (n^2+n)(2n+1) + 3n^2 + 3n + 2n}{2}$$

$$= \frac{2 + 2n^3 + n^2 + 2n^2 + n + 3n^2 + 5n}{2}$$

$$= \frac{2n^3 + 6n^2 + 6n + 2}{2}$$

$$= \frac{\cancel{2}(n^3 + 3n^2 + 3n + 1)}{\cancel{2}}$$

$$\boxed{a_n = n^3 + 3n^2 + 3n + 1}$$

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(15) Solve recurrance relation $a_n = a_{n-1} + n^2$, $n \geq 1$ $a_0 = 7$ by substitution method.

Sol Given

$$a_n = a_{n-1} + n^2$$

$n=1$ $a_1 = a_0 + 1^2$

$a_2 = a_1 + 2^2 = a_0 + 1^2 + 2^2$

$a_3 = a_2 + 3^2 = a_0 + 1^2 + 2^2 + 3^2$

⋮

$a_n = a_0 + 1^2 + 2^2 + 3^2 + \cdots n^2$

$$a_n = a_0 + \sum_1 n^2$$

$$= a_0 + \frac{n(n+1)(2n+1)}{6}$$

$$= 7 + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{42 + (n^2+n)(2n+1)}{6}$$

$$= \frac{42 + 2n^3 + n^2 + 2n^2 + n}{6}$$

$$a_n = \frac{42 + 2n^3 + 3n^2 + n}{6}$$

$$a_n = \frac{2n^3 + 3n^2 + n + 42}{6}$$

16) Solve recurrance relation $a_n = a_{n-1} + 3^n, n \geq 1$, $a_0 = 1$ by substitution method.

Sol

$$a_n = a_{n-1} + 3^n$$

$n=1$ $\quad a_1 = a_0 + 3^1$

$n=2$ $\quad a_2 = a_1 + 3^2 = a_0 + 3^1 + 3^2$

$n=3$ $\quad a_3 = a_2 + 3^3 = a_0 + 3^1 + 3^2 + 3^3$

$$a_n = a_0 + 3^1 + 3^2 + 3^3 + \cdots 3^n$$

$3^1, 3^2, 3^3, \ldots 3^n$ G.P

$a_{n}{}^{*}$ $\quad G.P = t_n = \left(\frac{a(r^n-1)}{(r-1)}\right)$ 73

$$= \frac{3(3^n-1)}{3-1}$$

$$= \frac{3(3^n-1)}{2}$$

$$a_n = a_0 + \frac{3(3^n-1)}{2}$$

$$= 1 + \frac{3(3^n-1)}{2}$$

$$= \frac{2+3(3^n-1)}{2}$$

$$= \frac{2+3.3^n-3}{2}$$

$$\boxed{a_n = \frac{3^{n+1}-1}{2}}$$

Solution of first order (or) direct method :-

Let the recurrance relation

$$a_n = c \cdot a_{n-1} + f(n), \quad n \geq 1$$

here c = constant

$f(n)$ = known function

$\therefore$ The general solution of above recurrance relation is $\boxed{a_n = c^n a_0 + \sum_1^n f(n)}$

$\rightarrow$ If $f(n) = 0$ then the recurance relation is "homogeneous" and the solution is

$$\boxed{a_n = c^n a_0}$$

1. Solve the recurrance relation $a_{n+1} = 8a_n, \; n \geq 0$ where $a_0 = 4$

Sol Given that $a_{n+1} = 8a_n$

Replace $n = n-1$

$$a_{n-\cancel{1}+\cancel{1}} = 8\, a_{n-1} \qquad n \geq 1$$

$$a_n = 8a_{n-1}$$

$$\boxed{a_n = c\, a_{n-1}}$$

$$c = 8, \quad a_0 = 4$$

$\therefore$ The general solution is

$$a_n = c^n a_0$$

$$\boxed{a_n = 8^n 4}$$

2. Solve the Recurrance relation $4a_n - 5a_{n-1} = 0$ $n \geq 1$, $a_0 = 1$.

Sol

Given that

$$4a_n - 5a_{n-1} = 0$$

$$4a_n = 5a_{n-1}$$

$$a_n = \frac{5}{4} a_{n-1}$$

$$a_n = c\, a_{n-1}$$

$$c = \frac{5}{4}, \quad a_0 = 1$$

∴ The general solution is

$$a_n = c^n a_0$$

$$= \left(\frac{5}{4}\right)^n (1)$$

$$\boxed{a_n = \left(\frac{5}{4}\right)^n}$$

Solution of Second and higher order relation :-

A relation of form $c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} = f(n)$ $n \geq 2$ where $c_0, c_1, c_2 ...$ are constants such type of relation is called second order linear reccurance relation with constant co-efficient.

Substitute $a_n = \alpha^n$, in the above reccurance relation, we get.

$c_0 \alpha^n + c_1 \alpha^{n-1} + c_2 \alpha^{n-2} = 0$ is a characteristic equation (or) accelary equation.

* If $\alpha_1, \alpha_2, \alpha_3 ---- \alpha_n$ are different roots then the homogeneous solution is

$$\alpha_n^H = c_1 \alpha_1^n + c_2 \alpha_2^n + ----- c_m \alpha_m^n$$

* If $\alpha_1, \alpha_1, \alpha_2, ------ \alpha_n --- \alpha_m$ are two roots are equal then the homogeneous solution is

$$\alpha_n^H = (c_1 + c_2 n) \alpha_1^n + c_3 \alpha_3^n + -----$$

* Three roots are equal

$$\alpha_n^H = (c_1 + c_2 n + c_3 n^2) \alpha_1^n + c_4 \alpha_4^n + ----$$

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Problems :-

(1) Solve the recurrance relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$

$n \geq 2$, $a_0 = 10$, $a_1 = 41$

Sol Given that

$$a_n - 7a_{n-1} + 10a_{n-2} = 0$$

(1) The auxilary equation is

$$\alpha^2 - 7\alpha + 10 = 0$$

10

-2 -5

$$\alpha^2 - 2\alpha - 5\alpha + 10 = 0$$

$$\alpha(\alpha - 2) - 5(\alpha - 2) = 0$$

$$(\alpha - 2)(\alpha - 5) = 0$$

$$\alpha = 5, \alpha = 2$$

The homogeneous solution of a reccurance relation is

$$\boxed{a_n = \alpha_1^n c_1 + \alpha_2^n c_2}$$

$$a_n = 5^n c_1 + 2^n c_2 \longrightarrow (1)$$

$a_0 = 10$

$n = 0$

$$a_0 = 5^0 c_1 + 2^0 c_2$$

$$10 = c_1 + c_2 \longrightarrow (2)$$

$a_1 = 41$

$n = 1$

$$a_1 = 5^1 c_1 + 2^1 c_2$$

$$41 = 5c_1 + 2c_2 \longrightarrow (3)$$

~~$\alpha_1, \alpha_2, \alpha_3 \ldots \alpha^i$~~

~~$a_n = c_1\alpha_1^n + c_2\alpha_2^n + c_3\alpha_3^n + \ldots$~~

$$\cancel{a_0 = c_1\alpha^0 + c_2\alpha^0}$$

$$\cancel{10 = c_1 5^0 + c_2 2^0}$$

$$\cancel{10 = c_1 + c_2 \rightarrow (4)}$$

$$\cancel{a_1 = c_1 5^1 + c_2 2^1}$$

$$\cancel{41 = c_1 5^1 + c_2 2 \rightarrow (5)}$$

$$c_1 + c_2 - 10 = 0 - (2)$$

$$5c_1 + 2c_2 - 41 = 0 - (3)$$

c_1	c_2	1	
1	-10	1	1
2	-41	5	2

(P)

$$\frac{c_1}{-41+20} = \frac{c_2}{-50+41} = \frac{1}{2-5}$$

$$\frac{c_1}{-21} = \frac{c_2}{-9} = \frac{1}{-3}$$

$$c_1 = \frac{-21}{-3} \qquad c_2 = \frac{-9}{-3}$$

$$\boxed{c_1 = 7 \qquad c_2 = 3}$$

Sub c_1 and c_2 in equ (4)

$$\cancel{c_1 2^n + c_2 5}$$

$$a_n = c_1 5^n + c_2 \cdot 2^n$$

$$\boxed{a_n = 7 \cdot 5^n + 3 \cdot 2^n}$$

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Solve recurrance relation $a_n - 9a_{n-1} + 26a_{n-2} - 24a_{n-3} = 0$

$n \geq 3$ $a_0 = 0$, $a_1 = 1$, $a_2 = 10$

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sol Given that

$$a_n - 9a_{n-1} + 26a_{n-2} - 24a_{n-3} = 0$$

$$\alpha^3 - 9\alpha^2 + 26\alpha - 24 = 0$$

7

2	1	-9	26	-24
	0	2	-14	24
	1	-7	12	0

$$(\alpha-2)(\alpha^2 - 7\alpha + 12) = 0$$

$$(\alpha-2)(\alpha^2 - 3\alpha - 4\alpha + 12) = 0$$

$$(\alpha-2)(\alpha(\alpha-3) - 4(\alpha-3)) = 0$$

$$(\alpha-2)(\alpha-3)(\alpha-4) = 0$$

$$\boxed{\alpha = 2, 3, 4.}$$

The homogeneous solution of recurrance relation is

$$a_n^H = C_1 2^n + C_2 3^n + C_3 4^n \rightarrow (1)$$

$n=0$ $a_0 = C_1 2^0 + C_2 3^0 + C_3 4^0$

$$0 = C_1 + C_2 + C_3 \rightarrow (2)$$

$n=1$ $a_1 = C_1 2^1 + C_2 3^1 + C_3 4^1$

$$1 = 2C_1 + 3C_2 + 4C_3 \rightarrow (3)$$

$$a_2 = C_1 2^2 + C_2 3^2 + C_3 4^2$$

$$10 = 4C_1 + 9C_2 + 16C_3 \rightarrow (4)$$

$$\left.\begin{array}{l} c_1+c_2+c_3=0 \\ 2c_1+3c_2+4c_3=1 \\ 4c_1+9c_2+16c_3=10 \end{array}\right\}$$

$\therefore\ c_1 = 1.5 = 3/2$

$c_2 = -4$

$c_3 = 2.5 = 5/2$

$\therefore\ a_n = \frac{3}{2} 2^n - 4 \cdot 3^n + \frac{5}{2} 4^n$

3. Solve the recurrance relation $a_n - 6a_{n-1} + 8a_{n-2} = 0$

$a_0 = 3$, $a_1 = 7$.

(9)

Sol Given $a_n - 6a_{n-1} + 8a_{n-2} = 0$

$\alpha^2 - 6\alpha + 8 = 0$

$\alpha^2 - 2\alpha - 4\alpha + 8 = 0$

$\alpha(\alpha-2) - 4(\alpha-2) = 0$

$(\alpha-2)(\alpha-4) = 0$

$\alpha = 2, 4$

$a_n^H = c_1 2^n + c_2 4^n \longrightarrow (1)$

$a_0 = c_1 2^0 + c_2 4^0$

$3 = 2c_1 + 4c_2 \longrightarrow (2)$

$c_1 + c_2 - 3 = 0$

$2c_1 + 4c_2 - 7 = 0$ (77)

$c_1 = 5/2$

$c_2 = 1/2$

$a_n = \frac{5}{2} 2^n + \frac{1}{2} 4^n$

4. Solve the R.R $a_n - 6a_{n-1} + 9a_{n-2} = 0$ $a_0 = 5$, $a_1 = 12$

5. Solve the R.R $a_n + 3a_{n-1} - 10a_{n-2} = 0$ $a_0 = 4$, $a_1 = 3$.

4 Sol

Given that (3)

$a_n - 6a_{n-1} + 9a_{n-2} = 0$

$\alpha^2 - 6\alpha + 9 = 0$

9
3 3

$\alpha^2 - 3\alpha - 3\alpha + 9 = 0$

$\alpha(\alpha - 3) - 3(\alpha - 3) = 0$

$(\alpha - 3)(\alpha - 3) = 0$

$\alpha = 3, 3$

$a_n^H = (c_1 + c_2 n)\alpha^n \rightarrow (1)$

$a_n = (c_1 + c_2 n) 3^n$

$a_0 = 5$ $a_0 = (c_1 + c_2(0)) 3^0$

$5 = c_1$

$a_1 = 12 \quad \alpha_1 = (c_1 + c_2(1))3^1$

$12 = (c_1 + c_2)3$

$3c_1 + 3c_2 = 12 \rightarrow (3)$

$3(5) + 3c_2 = 12$

$15 + 3c_2 = 12$

$\cancel{3c_2 = 15 - 12}$

$15 + 3c_2 = 12$

$15 - 12 + 3c_2 = 0$

$3 + 3c_2 = 0$

$\cancel{3}c_2 = -\cancel{3}$

$c_2 = -1$

$\alpha_n = (c_1 + c_2 n)3^n$

$\alpha_n = (5 + (-1)n)3^n \Rightarrow \alpha_n = (5 - n)3^n$

5) Sol

$a_n + 3a_{n-1} - 10a_{n-2}$

$\alpha^2 + 3\alpha - 10 = 0$

4

$\alpha^2 + 3\alpha - 10 = 0$

$\alpha^2 + 5\alpha + 2\alpha - 10$

$\alpha = 2, -5$

$a_n^H = c_1(2)^n + c_2(-5)^n \rightarrow (1)$

$a_0 = 4 \quad \alpha_0 = c_1 2^0 + c_2(-5)^0$

$\alpha_0 = c_1 + c_2 \rightarrow (2)$

$a_1 = -5 \quad \alpha_1 = c_1(2)^1 + c_2(-5)^1$

$-5 = 2c_1 + (-5)c_2 \rightarrow (3)$

$4 = c_1 + c_2$

$-5 = 2c_1 - 5c_2$

$c_1 + c_2 = 4$

$2c_1 - 5c_2 = -5$

$c_1 = \frac{25}{7}, \quad c_2 = \frac{3}{7}$

$\alpha_n^H = c_1(2)^n + c_2(-5)^n$

$\alpha_n^H = \frac{25}{7}(2)^n + \frac{3}{7}(-5)^n$

$\alpha_n^H = \frac{25}{7} 2^n - \frac{3}{7} 5^n$

No pending

$\alpha^2 + 2\alpha - 5\alpha - 10 = 0$ $\quad (\alpha-2)(\alpha-5)$

$\alpha(\alpha+2) - 5(\alpha+2)$ $\quad \alpha(\alpha-2) - 5(\alpha-2)$

6. Solve $2a_{n+2} - 11a_{n+1} + 5a_n = 0$ $\quad a_0 = 2,\ a_1 = -8$

Sol:-

$$2a_{n+2} - 11a_{n+1} + 5a_n = 0 \rightarrow (1)$$

Sub $n = n-2$ in equ(1) we get

$$2a_{n-2+2} - 11a_{n-2+1} + 5a_{n-2} = 0$$

5

$$2a_n - 11a_{n-1} + 5a_{n-2} = 0$$

$$2\alpha^2 - 11\alpha + 5 = 0$$

$$2\alpha^2 - 10\alpha - 1\alpha + 5 = 0$$

$$2\alpha(\alpha - 5) - 1(\alpha - 5) = 0$$

$$(2\alpha - 1)(\alpha - 5) = 0$$

$$\alpha = 1/2,\ \alpha = 5$$

$$\alpha = 1/2, 5$$

$$\alpha_n^H = c_1\left(\frac{1}{2}\right)^n + c_2(5)^n \longrightarrow (2)$$

$$a_0 = 2,\ \alpha_0 = c_1(1/2)^0 + c_2 5^0$$

$$2 = c_1 + c_2 \longrightarrow (3)$$

$$a_1 = -8.\ \alpha_1 = c_1\left(\frac{1}{2}\right)^1 + c_2(5)^1$$

$$-8 = c_1\left(\frac{1}{2}\right) + c_2(5) \longrightarrow (4)$$

$c_1 + c_2 - 2 = 0$

$\frac{1}{2}c_1 + 5c_2 + 8 = 0$

$1 \quad -2 \quad 1 \quad 1$

$5 \quad 8 \quad \frac{1}{2} \quad 5$

$$\frac{c_1}{8+10} = \frac{c_2}{-1-8} = \frac{1}{5-\frac{1}{2}}$$

$$c_1 = \frac{18}{5-\frac{1}{2}} \qquad c_2 = \frac{-9}{5-\frac{1}{2}}$$

$$c_1 = \frac{18}{\frac{10-1}{2}} \qquad c_2 = \frac{-9}{\frac{10-1}{2}}$$

$$c_1 = \frac{18}{9/2} \qquad c_2 = \frac{-9}{9/2}$$

$$c_1 = \frac{\cancel{18}^{2} \times 2}{\cancel{9}} \qquad c_2 = \frac{-\cancel{9} \times 2}{\cancel{9}}$$

$$c_1 = 4 \qquad c_2 = -2$$

$$\alpha_n = 4\left(\tfrac{1}{2}\right)^n + (-2)(5)^n$$

$$\alpha_n = 4\left(\tfrac{1}{2}\right)^n - 2(5)^n$$

7. Solve $2a_n = 7a_{n-1} - 3a_{n-2}$, $n \geq 2$, $a_0 = 2$, $a_1 = 5$

sol

$$2a_n = 7a_{n-1} - 3a_{n-2}$$ 6

$$2a_n - 7a_{n-1} + 3a_{n-2} = 0$$

$$2\alpha^2 - 7\alpha + 3 = 0$$

$$\alpha = 3, \tfrac{1}{2}$$

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$$\alpha_n^H = c_1(3)^n + c_2\left(\tfrac{1}{2}\right)^n \rightarrow (1)$$

$a_0 = 2$ $\quad \alpha_0 = c_1(3)^0 + c_2\left(\tfrac{1}{2}\right)^0$

$$2 = c_1 + c_2 \rightarrow (2)$$

$a_1 = 5$ $\quad \alpha_1 = c_1(3)^1 + c_2\left(\tfrac{1}{2}\right)^1$

$$5 = c_1\, 3 + c_2\left(\tfrac{1}{2}\right)$$

$$5 = 3c_1 + \tfrac{1}{2}c_2 \rightarrow (3)$$

$$c_1 = 8/5 \quad , \quad c_2 = 2/5$$

$$c_1 = 1.6 \quad , \quad c_2 = 0.4$$

$$a_n = 8/5(3)^n + 2/5\left(\tfrac{1}{2}\right)^n$$

8) Solve R.R $a_n + a_{n-1} - 8a_{n-2} - 12a_{n-3} = 0$ $\quad n \geq 3,\ a_0 = 1$
$a_1 = 5$
$a_2 = 1$

sol

$$a_n + a_{n-1} - 8a_{n-2} - 12a_{n-3} = 0$$

$$\alpha^3 + 1\alpha^2 - 8\alpha - 12 = 0$$

$$\alpha = -2, -2, 3$$

$$\alpha_n^H = (c_1 + c_2 n)(-2)^n + c_3(3)^n \rightarrow (1)$$

$a_0 = 1,$ $\quad \alpha_0 = [c_1 + c_2(0)](-2)^0 + c_3(3)^0$

$$1 = c_1 + c_3 \rightarrow (2)$$

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$a_1 = 5 \quad \alpha_1 = [c_1 + c_2(1)](-2)^1 + c_3(3)^1$ (30)

$5 = (c_1 + c_2)(-2) + 3c_3$

$5 = -2c_1 - 2c_2 + 3c_3 \rightarrow (3)$

$a_2 = 1 \quad \alpha_2 = [c_1 + c_2(2)](-2)^2 + c_3(3)^2$

$1 = 4(c_1 + 2c_2) + 9c_3 \rightarrow (4)$

$1 = 4c_1 + 8c_2 + 9c_3$

$c_1 = 0, \quad c_2 = -1, \quad c_3 = 1$

$\alpha_n = (c_1 + c_2 n)(-2)^n + c_3(3)^n$

$= ((0 + (-1)n))(-2)^n + 1(3)^n$

$= (0 - n)(-2)^n + 3^n$

$\alpha n = -n(-2)^n + 3^n$

9) solve R.R $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$ $\quad a_0 = 1, \; a_1 = 4, \; a_2 = 28$

$a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$

$a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = 0$

$\alpha^3 - 6\alpha^2 + 12\alpha - 8 = 0$

$\alpha = 2, 2, 2$

$a_n^H = (c_1 + c_2 n + c_3 n^2)\, 2^n$

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$a_0 = 1 \quad \alpha_0 = [c_1 + c_2(0) + c_3(0)^2] \times 2^0$

$1 = c_1$

$c_1 = 1 \rightarrow (2)$

$a_1 = 4$

$\alpha_1 = [c_1 + c_2(1) + c_3(1)^2] \times 2^1$

$\cancel{4}^{2} = (c_1 + c_2 + c_3)\, \cancel{2}$

$c_1 + c_2 + c_3 = 2 \rightarrow (3)$

$a_2 = 28$

$\alpha_2 = [c_1 + c_2(2) + c_3(2)^2] \times 2^2$

$28 = [c_1 + 2c_2 + 4c_3]\, 4$

$c_1 + 2c_2 + 4c_3 = 7 \rightarrow (4)$

$c_1 = 1, \quad c_2 = -1, \quad c_3 = 2$

$\alpha_n = (c_1 + c_2 n + c_3 n^2) \times 2^n$

$\alpha_n = (1 + (-1)n + 2n^2) \times 2^n$

$\alpha_n = (1 - 1n + 2n^2)\, 2^n$

10) i) $a_{n+3} = 3a_{n+2} + 4a_{n+1} - 12a_n \quad n \geq 0 \quad a_0 = 0, a_1 = -11, a_2 = -15$

ii) $a_n - 7a_{n-1} + 16a_{n-2} - 12a_{n-3} = 0 \quad n \geq 3 \quad a_0 = 1, a_1 = 4, a_2 = 8$

iii) $a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n = 0 \quad n \geq 0$

v) $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0 \quad n \geq 3$

81

10) i) $a_{n+3} = 3a_{n+2} + 4a_{n+1} - 12a_n$

$a_{n+3} - 3a_{n+2} - 4a_{n+1} + 12a_n = 0$

$n = n-3$

$a_{n-3+3} - 3a_{n-3+2} - 4a_{n-3+1} + 12a_{n-3} = 0$

$a_n - 3a_{n-1} - 4a_{n-2} + 12a_{n-3} = 0$

$\alpha^3 - 3\alpha^2 - 4\alpha + 12 = 0$

$\alpha = -2, 3, 2$

$\alpha_n^H = C_1(-2)^n + C_2(3)^n + C_3(2)^n \rightarrow (1)$

$a_0 = 0 \quad \alpha_0 = C_1(-2)^0 + C_2(3)^0 + C_3(2)^0$

$0 = C_1 + C_2 + C_3 \longrightarrow (2)$

$a_1 = -11 \quad \alpha_1 = C_1(-2)^1 + C_2(3)^1 + C_3(2)^1$

$-11 = -2C_1 + 3C_2 + 2C_3 \rightarrow (3)$

$a_2 = -15 \quad \alpha_2 = C_1(-2)^2 + C_2(3)^2 + C_3(2)^2$

$-15 = 4C_1 + 9C_2 + 4C_3$

$C_1 = 2, C_2 = -3, C_3 = 1$

$\alpha_n = 2(-2)^n + (-3)(3)^n + (1)2^n$

$\alpha_n = 2(-2)^n + 2^n - 3^{n+1}$

ii) $a_n - 7a_{n-1} + 16a_{n-2} - 12a_{n-3} = 0$

Sol

$$\alpha^3 - 7\alpha^2 + 16\alpha - 12 = 0$$

$$\alpha = 3, 2, 2$$

$$a_n^H = (c_1 + c_2 n)(2)^n + c_3(3)^n \rightarrow (1)$$

$a_0 = 1$ $\quad a_0 = (c_1 + c_2(0))(2)^0 + c_3(3)^0$

$$1 = (c_1 + c_3) \rightarrow (2)$$

$a_1 = 4$ $\quad a_1 = (c_1 + c_2(1))(2)^1 + c_3(3)^1$

$$4 = (c_1 + c_2)2 + 3c_3$$

$a_2 = 8$

$$a_2 = (c_1 + c_2(2))(2)^2 + c_3(3)^2$$

$$8 = (c_1 + 2c_2)4 + 9c_3$$

$$c_1 = 5, \quad c_2 = -4, \quad c_3 = 3$$

$$a_n = (5 + 3n)2^n - 4 \times 3^n$$

iii) $a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n = 0$

$$n = n-3$$

$$a_{n-3+3} - 3a_{n-3+2} + 3a_{n-3+1} - a_{n-3} = 0$$

$$a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3} = 0$$

$$\alpha^3 - 3\alpha^2 + 3\alpha - 1 = 0$$

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$$\alpha_n = [c_1 + c_2 n + c_3 n^2]\,, n$$

iv) $a_n - 9a_{n-1} + 27a_{n-2} - 27a_{n-3} = 0$

$$\alpha^3 - 9\alpha^2 + 27\alpha - 27 = 0$$

$$\alpha = 3, 3, 3$$

$$\alpha_n = [c_1 + c_2 n + c_3 n^2]\, 3^n$$

v) $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0$

$$\alpha^3 - 8\alpha^2 + 21\alpha - 18 = 0$$

$$\alpha = 2, 3, 3$$

$$\alpha_n = (c_1 + c_2 n) 3^n + c_3 2^n$$

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11) i) $4a_n - 5a_{n-1} = 0 \quad n \geq 1,\ a_0 = 1$

ii) $3a_{n+1} - 4a_n = 0 \quad n \geq 0,\ a_1 = 5$

iii) $2a_n - 3a_{n-1} = 0 \quad n \geq 1,\ a_4 = 81$

i) Sol

$$4a_n - 5a_{n-1} = 0$$

$$a_n = \frac{5}{4} a_{n-1}$$

$$a_n = c^n a_0$$

$$a_n = \left(\frac{5}{4}\right)^n \times 1$$

ii)

Sol

$$3a_{n+1} - 4a_n = 0$$

$$n = n-1$$

$$3a_{n+1-1} - 4a_{n-1} = 0$$

$$3a_n - 4a_{n-1} = 0$$

$$3a_n = 4a_{n-1}$$

$$a_n = \left(\frac{4}{3}\right) a_{n-1}$$

$$a_n = c^n a_{n-1}$$

$$a_n = \left(\frac{4}{3}\right)^n (a_0) \rightarrow (1)$$

$$a_1 = \left(\frac{4}{3}\right)^1 a_0$$

$$5 = \frac{4}{3} a_0$$

$$\frac{15}{4} = a_0$$

$$a_n = \left(\frac{4}{3}\right)^n \left(\frac{15}{4}\right)$$

iii)

Sol

$$2a_n - 3a_{n-1} = 0, \quad a_4 = 81.$$

$$2a_n = 3a_{n-1}$$

$$a_n = \frac{3}{2} a_{n-1}$$

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$$a_n = c\, a_0$$

$$a_n = \left(\frac{3}{2}\right)^n a_0 \rightarrow 1$$

$$a_1 = \left(\frac{3}{2}\right)^1 a_0 \rightarrow 2$$

$$a_2 = \left(\frac{3}{2}\right)^2 a_1 \rightarrow 3$$

$$a_3 = \left(\frac{3}{2}\right)^3 a_2 \rightarrow 4$$

$$a_4 = \left(\frac{3}{2}\right)^4 a_3 \rightarrow 5$$

$$\frac{81 \times 2}{16\ 2}$$

$$81 = \left(\frac{3}{2}\right)^4 a_3$$

$$\cancel{81} = \frac{\cancel{81}}{16} a_3$$

$$\boxed{a_3 = 16}$$

$$a_3 = \left(\frac{3}{2}\right)^3 a_2$$

$$16 = \frac{27}{8} a_2$$

$$\frac{81 \times 16}{81}$$

$$\frac{16 \times 8}{27} = a_2$$

$$\boxed{a_2 = \frac{128}{27}}$$

$$a_2 = \left(\frac{3}{2}\right)^2 a_1$$

$$\frac{128}{27} = \frac{9}{4} a_1$$

$$\boxed{a_1 = \frac{512}{243}}$$

$$a_1 = \frac{3}{2} a_0$$

$$\frac{512}{243} = \frac{3}{2} a_0$$

$$\boxed{a_0 = \frac{1024}{729}}$$

$$a_n = \left(\frac{3}{2}\right)^n a_0$$

$$a_n = \left(\frac{3}{2}\right)^n \left(\frac{1024}{729}\right)$$

12) Solve $a_n - 7a_{n-2} + 10a_{n-4} = 0 \quad n \geq 4$

$$a_n - 7a_{n-2} + 10a_{n-4} = 0$$

$$\alpha^4 - 7\alpha^2 + 10 = 0$$

Let $\alpha^2 = p$

$$(\alpha^2)^2 - 7\alpha^2 + 10 = 0$$

$$p^2 - 7p + 10 = 0$$

$$p^2 - 7p + 10 = 0$$

$$p^2 - 2p - 5p + 10 = 0$$

$$p(p-2) - 5(p-2) = 0$$

$$(p-2)(p-5) = 0$$

$$p = 2, 5$$

10

$-5 \times -2 = 0$

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$P = \alpha^2$ $\qquad$ $P = \alpha^2$

$2 = \alpha^2$ $\qquad$ $5 = \alpha^2$

$\alpha = \pm\sqrt{2}$ $\qquad$ $\alpha^2 = 5$

$\alpha = +\sqrt{2}, -\sqrt{2}$ $\qquad$ $\alpha = \pm\sqrt{5}$

$\alpha = +\sqrt{5}, -\sqrt{5}$

$$\alpha = +\sqrt{2}, -\sqrt{2}, +\sqrt{5}, -\sqrt{5}$$

$$x_n = (+\sqrt{2})^n c_1 + (-\sqrt{2})^n c_2 + (+\sqrt{5})^n c_3 + (-\sqrt{5})^n c_4$$

Non-homogeneous first & higher order recurrence

Method for finding perticular Solution :-

1) If $f(n) = c$, where c is constant then the perticular solution, if take $a_n = d$

2) If $f(n) = a+bn$ (or n) then we assume that the perticular solution, if take $a_n = c+dn$

3) If $f(n) = a+bn+cn^2$ (or n^2) then we assume that the perticular solution, if take $a_n = c_1 + c_2 n + c_3 n^2$

4) If $f(n) = a^n$ then we assume that the perticular Solution, if take $a_n = cd^n$

i) If 'a' have one root of auxilary equation then the perticular solution $a_n = n(d a^n)$

ii) If 'a' repeated roots (k times) then the perticular solution is $a_n = n^k(d a^n)$

problems :-

1) Solve the perticular Solution of non-homogeneous R.R

$a_n = 6a_{n-1} - 9a_{n-2} + f(n)$ where $f(n)$ is

i) 3^n ii) $n \cdot 3^n$ iii) $n^2 \cdot 2^n$ and $K = 2$ roots

Sol:- Given that

$$a_n = 6a_{n-1} - 9a_{n-2} + f(n)$$

$$a_n - 6a_{n-1} + 9a_{n-2} = f(n)$$

The A.E is

$$\alpha^2 - 6\alpha + 9 = 0$$

$3, 3$

i) $f(n) = 3^n$

$= n^k (d\, a^n)$

$f(n) = n^2 (d . 3^n)$

2 85

ii) $f(n) = n! \, 3^n$

$= (c+dn)\, n^2 d\, 3^n$

iii) $f(n) = 1\, n^2 . 2^n$

$[c_1 + c_2 n + c_3 n^2]\, n^2 d\, 2^n$.

*** Imp
2) Solve the R.R $a_n = 3a_{n-1} + 2n$ $a_1 = 3$

Sol:- Given that

$a_n = 3a_{n-1} + 2n$

$$\boxed{a_n - 3a_{n-1} = 2n} \rightarrow (1)$$

The A.E is

$\alpha - 3 = 0$

$\boxed{\alpha = 3}$

The homgeneous Solution is

$$\boxed{\alpha_n^H = c.3^n} \rightarrow (2)$$

Assume that perticular solution is

$f(n) = 2n$

$a_n = a + bn \rightarrow (3)$

$a_{n-1} = a + b(n-1) \rightarrow (4)$

$b(n-1)$

Sub (3),(4) in equ (1) we get

bn

$a_n - 3a_{n-1} = 2n$

$a + bn - 3(a + b(n-1)) = 2n$

$a + bn - 3a - 3b(n-1)$

Compare n co-efficient

$$b - 3b = 2$$

$$-2b = 2$$

$$\boxed{b = -1}$$

Compare constant

$$a - 3a + 3b = 0$$

$$-2a + 3b = 0$$

$$-2a + 3(-1) = 0$$

$$-2a - 3 = 0$$

$$-2a = 3$$

$$\boxed{a = -3/2}$$

Sub $b = -1$, $a = -3/2$ in equ (3)

$$a_n^p = a + bn$$

$$a_n^p = -3/2 + (-1)n$$

$$\boxed{a_n^p = -3/2 - n}$$

The General solution of recurance relation is

$$\alpha_n = \alpha_n^H + \alpha_n^p$$

$$\alpha_n = C.3^n + [-3/2 - n]$$

$$\boxed{\alpha_n = C.3^n - 3/2 - n} \rightarrow (5)$$

$n=1$

$$\alpha_1 = C.3^1 - 3/2 - 1$$

$$3 = \frac{6C - 3 - 2}{2}$$

$6 = 6c - 5$

$6c = 11$

$c = 11/6$

sub c in equ 5

$$\boxed{\alpha_n = \frac{11}{6} 3^n - 3/2 - n}$$

∴ The required general ~~of~~ R.R is

$$\alpha_n = \frac{11}{6} 3^n - 3/2 - n.$$

3. Solve $a_n + 6a_{n-1} + 12a_{n-2} + 8a_{n-3} = 3^n$

Sol Given that

$$a_n + 6a_{n-1} + 12a_{n-2} + 8a_{n-3} = 3^n$$

A.E

$$\alpha^3 + 6\alpha^2 + 12\alpha + 8 = 0$$

$$\alpha = -2, -2, -2$$

The Homogeneous solution is

$$\alpha_n^H = [c_1 + c_2 n + c_3 n^2](-2)^n \rightarrow (1)$$

P.S of $f(n) = 3^n$

Assume that

$$a_n^P = C \cdot 3^n$$

$$a_n = C \cdot 3^n$$

$$a_{n-1} = C \cdot 3^{n-1}$$

$$a_{n-2} = C \cdot 3^{n-2}$$

$$a_{n-3} = C \cdot 3^{n-3}$$

$n-2)^2) = 16n$

$16a_n + 16bn^2 - 24\left[an - a + bn^2 + b - 2nb\right] + 8\left[an - 2a + n^2b + 4b - 4nb\right] = n \times 16$

$16a_n + 16bn^2 - 24a_n - 24a + 24bn^2 + 24b - 48nb + 8an - 16a + 8n^2b$
$+ 32b - 32nb = 16n$

$16a_n - 24a_n + 8a_n + 16bn^2 - 24bn^2 + 8bn^2 - 24a - 16a + 24b + 32b$
$+ 48nb - 32nb = 16n$

$\cancel{24bn^2} - \cancel{24bn^2} + \cancel{24an} - \cancel{24an} + 16bn + 8a + 8b = 16n$

$16bn + 8a + 8b = 16n$

$n(16b) + (8a + 8b) = 16n$

$16b = 16$

$\boxed{b = 1}$

$8a + 8b = 0 \Rightarrow 8a + 8 = 0$

$8a = -8 \Rightarrow \boxed{a = -1}$

$a_n = (a + bn)n \cdot 4^n$

$a_n^P = (-1 + n)n \cdot 4^n$

The general solution of R.R is

$\alpha_n = \alpha_n^H + \alpha_n^P$

$\alpha_n = c_1 2^n + c_2 4^n + (n-1)n4^n \rightarrow (5)$

$a_0 = 3$ $\quad \alpha_0 = c_1 2^0 + c_2 4^0 + (0-1) \times 0 \times 4^0$

$3 = c_1 + c_2 \rightarrow (6)$

$a_1 = 7$ $\quad \alpha_1 = c_1 2^1 + c_2 4^1 + (1-1) \times 1 \times 4^1$

$7 = 2c_1 + 4c_2 \rightarrow (7)$

$c_1 + c_2 - 3 = 0$

$2c_1 + 4c_2 - 7 = 0$

$$\begin{array}{cccc} 1 & -3 & 1 & 1 \\ 4 & -7 & 2 & 4 \end{array}$$

$$\frac{c_1}{-7+12} = \frac{c_2}{-6+7} = \frac{1}{4-2}$$

$$c_1 = \frac{-7+12}{4-2} \qquad c_2 = \frac{-6+7}{4-2}$$

$$c_1 = \frac{5}{2} \qquad c_2 = \frac{1}{2}$$

Sub c_1, c_2 in eq (3)

$$a_n = \left(\frac{5}{2}\right)2^n + \left(\frac{1}{2}\right)4^n + (n-1)n.4^n$$

5. Solve $a_n + 6a_{n-1} + 12a_{n-2} + 8a_{n-3} = 3^n$ $a_0 = 1$, $a_1 = 2$

6. Solve $a_n - 2a_{n-1} + 8a_{n-2} = 7.3^n$

Generating functions :-

Solution of recurance relation by the method of generating function with 1st order R.R :-

$a_n = C \cdot a_{n-1} + f(n)$, $n \geq 1$ this is equals to

$a_{n+1} = C \cdot a_n + \phi(n)$, $n \geq 0$ Then the generating function above R.R is

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

where

a_n = The solution of R.R

Now formula for finding the generating function

$$\boxed{f(x) = \frac{a_0 + x \cdot g(x)}{1 - cx}}$$ where.

$$\boxed{g(x) = \sum \phi(n) x^n}$$

* If $\phi(n) = 0$, $g(x) = 0$, The generating function of homogeneous R.R is

$$f(x) = \frac{a_0}{1 - cx}$$

Note :-

1) $1 + x + x^2 + x^3 - \cdots = (1-x)^{-1} = \sum x^n$
2) $1 - x + x^2 - x^3 \cdots = (1+x)^{-1} = \sum (-1)^n x^n$
3) $1 + 2x + 3x^2 + 4x^3 + \cdots = (1-x)^{-2} = \sum (n+1) x^n = \sum {}^{-2}C_n x^n$
4) $1 - 2x + 3x^2 - 4x^3 + \cdots = (1+x)^{-2} = \sum (-1)(n+1) x^n$

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problems:-

1) Solve $a_n - 3a_{n-1} = n$; $n \geq 1$ $a_0 = 1$ by using generating function. (89)

Sol: Given that

$$a_n - 3a_{n-1} = n \quad , n \geq 1$$

Sub. $n = n+1$

$$a_{n+1} - 3a_n = n+1 \quad n \geq 0$$

$$a_{n+1} = 3a_n + (n+1)$$

$$a_{n+1} = c \cdot a_n + \phi(n)$$

where $c = 3$

$\phi(n) = n+1$

Now $g(x) = \sum \phi(n) x^n$

$$= \sum (n+1) x^n$$

$$\boxed{g(x) = (1-x)^{-2}}$$

$$g(x) = \frac{1}{(1-x)^2}$$

Now generating function of R.R.

$$f(x) = \frac{a_0 + x \cdot g(x)}{1 - cx}$$

$$= \frac{1 + x \cdot \frac{1}{(1-x)^2}}{1-3x}$$

$$\frac{(1-x)^2 + x}{(1-x)^2 (1-3x)} = \frac{1^2 + x^2 - 2x + x}{(1-x)^2 (1-3x)}$$

$$f(x) = \frac{x^2 - x + 1}{(1-x)^2(1-3x)}$$

$$\frac{x^2 - x + 1}{(1-x)^2(1-3x)} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(1-3x)} \rightarrow \text{I}$$

$$x^2 - x + 1 = A(1-x)(1-3x) + B(1-3x) + C(1-x)^2$$

$$x^2 - x + 1 = A\left[1 - 3x - x + 3x^2\right] + B(1-3x) + C(1-x)^2$$

$$x^2 - x + 1 = A\left[1 - 4x + 3x^2\right] + B(1-3x) + C(1-x)^2$$

$1 + x^2 - 2x$

$$1 = A + B + C \longrightarrow 1$$ x^2 co-effs

$$-1 = -4A - 3B - 2C \longrightarrow 2$$

$$1 = 3A + C \longrightarrow 3$$

$$A = -1/4$$

$$B = -1/2$$

$$C = 7/4$$

Sub A, B, C in equ I

$$\frac{x^2 - x + 1}{(1-x)^2(1-3x)} = \frac{-1/4}{1-x} + \frac{-1/2}{(1-x)^2} + \frac{7/4}{(1-3x)}$$

$$= -\frac{1}{4}(1-x)^{-1} - \frac{1}{2}(1-x)^{-2} + \frac{7}{4}(1-3x)^{-1}$$

$$= -\frac{1}{4}\sum_1 x^n - \frac{1}{2}\sum_1 (n+1)x^n + \frac{7}{4}\sum_1 (3x)^n$$

$$f(x) = -\frac{1}{4}\sum_1 x^n - \frac{1}{2}\sum_1 (n+1)x^n + \frac{7}{4}\sum_1 (3x)^n$$

$$f(x) = \left[-\frac{1}{4} - \frac{1}{2}(n+1) + \frac{7}{4} 3^n\right] \sum_1 x^n$$

(90)

$$f(x) = \sum_1 x^n \left[-\frac{1}{4} - \frac{1}{2}(n+1) + \frac{7}{4} 3^n\right]$$

$$f(x) = \sum_1 a^n \cdot x^n$$

$$\therefore a_n = -\frac{1}{4} - \frac{1}{2}(n+1) + \frac{7}{4} 3^n$$

$\therefore$ it is required general solution of R.R.

2) Solve R.R $a_{n+1} - a_n = 3^n \quad n \geq 0 \quad a_0 = 1$

sol

$$a_{n+1} = a_n + 3^n$$

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$$a_{n+1} = c \cdot a_n + \phi(n)$$

$$c = 1$$

$$\phi(n) = 3^n$$

$$g(x) = \sum_1 \phi(n) x^n$$

$$= \sum_1 3^n \cdot x^n$$

$$= \sum_1 (3x)^n$$

$$= (1-3x)^{-1}$$

$$\boxed{g(x) = \frac{1}{1-3x}}$$

$$f(x) = \frac{a_0 + x g(x)}{1 - cx}$$

$$f(x) = \frac{1 + x \frac{1}{1-3x}}{1 - 1x}$$

$$\frac{1-3x+x}{(1-x)(1-3x)} = \frac{1-2x}{(1-x)(1-3x)}$$

$$\frac{1-2x}{(1-x)(1-3x)} = \frac{A}{(1-x)} + \frac{B}{(1-3x)} \rightarrow (1)$$

$$1-2x = A(1-3x) + B(1-x)$$

$$1 = A + B$$

$$-2 = -3A - B$$

$$\begin{aligned} A + \cancel{B} &= 1 \\ -3A - \cancel{B} &= -2 \\ \hline -2A &= -1 \end{aligned}$$

$$\cancel{-}A = \cancel{-}1/2$$

$$A = 1/2$$

$$1 = 1/2 + B$$

$$1 - 1/2 = B$$

$$B = \frac{1}{2}$$

sub A, B in f(x) i.e eq (1)

$$\frac{1-2x}{(1-x)(1-3x)} = \frac{1/2}{(1-x)} + \frac{1/2}{(1-3x)}$$

$$= \frac{1}{2}(1-x)^{-1} + \frac{1}{2}(1-3x)^{-1}$$

$$= \frac{1}{2}\sum x^n + 1/2 \sum (3x)^n$$

$$f(x) = \left[\frac{1}{2} + \frac{1}{2}3^n\right]x^n$$

$$f(x) = \sum_{n=0}^{\infty}{}' a_n x^n$$

$$a_n = \frac{1}{2} + \frac{1}{2}3^n$$

$\therefore$ requried G.S of given R.R

3. Solve R.R $a_n = a_{n-1} + 2n$. $n \geq 1$ $a_0 = 1$ by using generating function.

Sol

Given that

$$a_n = a_{n-1} + 2n$$

Sub $n = n+1$.

$$a_{n+1} = a_{n+\cancel{1}-\cancel{1}} + 2(n+1)$$

$$a_{n+1} = a_n + 2(n+1)$$

$$a_{n+1} = ca_n + \phi(n)$$

$$c = 1$$

$$\phi(n) = 2(n+1)$$

$$g(x) = \sum{}' \phi(n)x^n$$

$$= \sum{}' 2(n+1)x^n$$

$$= 2\sum{}'(n+1)x^n$$

$$= 2(1-x)^{-2}$$

$$\boxed{g(x) = \frac{2}{(1-x)^2}}$$

$$f(x) = \frac{a_0 + x\, g(x)}{1 - c \cdot x}$$

$$= \frac{1 + x \cdot \frac{2}{(1-x)^2}}{1-x}$$

$$= \frac{(1-x)^2 + 2x}{(1-x)(1-x)^2} = \frac{(1-x)^2 + 2x}{(1-x)^3}$$

$$= \frac{1 + x^2 - \cancel{2x} + \cancel{2x}}{(1-x)^3}$$

$$= \frac{1+x^2}{(1-x)^3}$$

$$\frac{1+x^2}{(1-x)^3} = \frac{A}{(1-x)} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} \rightarrow (1)$$

$$1+x^2 = A(1-x)^2 + B(1-x) + C$$

$$x^2+1 = A(x^2-2x+1) + B(1-x) + C$$

$$1 = A + B + C$$

$$0 = -2A - B$$

$$\boxed{1 = A}$$

$$-2(1) - B = 0 \qquad 1 = 1 + (-2) + C$$

$$-2 - B = 0 \qquad 1 = 1 - 2 + C$$

$$\boxed{B = -2} \qquad 1 = -1 + C$$

$$C = 1 + 1$$

sub A, B, C in above equ ①

$$\frac{1+x^2}{(1-x)^3} = \frac{1}{(1-x)} + \frac{-2}{(1-x)^2} + \frac{2}{(1-x)^3}$$

$$= (1-x)^{-1} - 2(1-x)^{-2} + 2(1-x)^{-3}$$

$$= \sum x^n - 2\sum (n+1)x^n + \cancel{2}\sum \frac{(n+1)(n+2)}{\cancel{2}} x^n$$

$$= \sum x^n \left[1 - 2(n+1) + (n+1)(n+2)\right]$$

$$= \sum x^n \left[1 - 2n - 2 + (n^2 + 2n + n + 2)\right]$$

$$= \sum x^n \left[1 - \cancel{2n} - \cancel{2} + n^2 + \cancel{2n} + n + \cancel{2}\right]$$

$$= \sum x^n \left[n^2 + n + 1\right]$$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\therefore a_n = n^2 + n + 1$$

4. Solve R.R $a_n - 4a_{n-1} = 0$ $n \geq 1$ $a_0 = 1$ by using generation function.

Sol Given

$$a_n - 4a_{n-1} = 0$$

$$n = n+1$$

$$a_{n+1} - 4a_n = 0$$

$$a_{n+1} = 4a_n$$

$$c = 4, \quad \phi(n) = 0$$

$$g(x) = \sum \phi(n) x^n$$

$$= \sum 0 x^n$$

$$f(x) = \frac{a_0}{1-cx}$$

$$= \frac{1}{1-4x}$$

$$= (1-4x)^{-1}$$

$$f(x) = \sum (4x)^n$$

a_{n+1} ... a_n ... (cut off)

Solution of second order R.R by the method of generating function.

Let

$$a_n + A_1 a_{n-1} + A_2 a_{n-2} = f(n) \qquad n \geq 2$$

this can be written as

$$a_{n+2} + A_1 a_{n+1} + A_2 a_n = \phi(n) \quad , n \geq 0$$

Then $\boxed{f(x) = \dfrac{a_0 + (a_1 + a_0 A_1)x + x^2 g(x)}{1 + A_1 x + A_2 x^2}}$

where $\boxed{g(x) = \sum_1 \phi(n) x^n}$

If $\phi(n) = 0$, $g(x) = 0$, then Generating function of homogeneous

$$\boxed{f(x) = \frac{a_0 + (a_1 + a_0 A_1)x}{1 + A_1 x + A_2 x^2}}$$

Problems:-

1) Solve R.R $a_{n+2} - 2a_{n+1} + a_n = 2^n$, $n \geq 0$, $a_0 = 1$, $a_1 = 2$ by using generating function.

Sol Given that

$$a_{n+2} - 2a_{n+1} + a_n = 2^n$$

$$a_{n+2} + A_1 a_{n+1} + A_2 a_n = \phi(n)$$

$$A_1 = -2, \quad A_2 = 1, \quad \phi(n) = 2^n$$

$$g(x) = \sum_1 \phi(x) x^n$$

$$= \sum 2^n x^n$$

$$= \sum (2x)^n$$

$$= (1-2x)^{-1}$$

$$g(x) = \frac{1}{1-2x}$$

$$f(x) = \frac{a_0 + (a_1 + a_0 A_1)x + x^2 g(x)}{1 + A_1 x + A_2 x^2}$$

$$= \frac{1 + (2 + 1(-2))x + x^2 \frac{1}{1-2x}}{1 + (-2)x + 1x^2}$$

$$= \frac{1 + (2-2)x + x^2 \frac{1}{1-2x}}{1 - 2x + x^2}$$

$$= \frac{1 + x^2 \frac{1}{1-2x}}{1-2x+x^2}$$

$$= \frac{\cancel{(1-2x+x^2)}}{\cancel{(1-2x+x^2)}(1-2x)}$$

$$= \frac{1}{1-2x}$$

$$f(x) = (1-2x)^{-1}$$

$$f(x) = \sum (2x)^n$$

$$f(x) = \sum 2^n x^n$$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\boxed{a_n = 2^n}$$

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2) Find the generating function of R.R

$a_{n+2} - 3a_{n+1} + 2a_n = 0 \;,\; n \geq 0 \;,\; a_0 = 1 \;,\; a_1 = 6$

Sol:-

Given that

$$a_{n+2} - 3a_{n+1} + 2a_n = 0$$

$$A_1 = -3 \;,\; A_2 = 2 \;,\; \phi(n) = 0$$

$$g(x) = \sum \phi(n)\, x^n$$

$$= \sum 0\, x^n$$

$$g(x) = 0$$

$$f(x) = \frac{a_0 + (a_1 + a_0 A_1)x}{1 + A_1 x + A_2 x^2}$$

$$= \frac{1 + (6 + 1(-3))x}{1 + (-3)x + (2)x^2}$$

$$= \frac{1 + (6-3)x}{1 - 3x + 2x^2}$$

$$= \frac{1+3x}{2x^2 - 3x + 1}$$

$$= \frac{1+3x}{(1-2x)(1-x)}$$

$$\frac{1+3x}{(1-2x)(1-x)} = \frac{A}{1-2x} + \frac{B}{1-x}$$

(94)

$$1+3x = A(1-x) + B(1-2x)$$

$$A = 5$$

$$B = -4$$

$$f(x) = \frac{5}{(1-2x)} - \frac{4}{(1-x)}$$

$$= 5((1-2x)^{-1} - 4(1-x)^{-1}$$

$$= 5\sum (2x)^n - 4\sum x^n$$

$$f(x) = \sum (5 2^n - 4)x^n$$

$$f(x) = a_n x^n$$

$$\boxed{a_n = 5 \cdot 2^n - 4}$$

3) find the generating function of R.R

$$a_n - 7a_{n-1} + 10a_{n-2} = 4^n \qquad a_0 = 8, a_1 = 36$$

Sol

$$a_n - 7a_{n-1} + 10a_{n-2} = 4^n$$

$$n = n+2$$

$$a_{n+2} - 7a_{n+2-1} + 10a_{n+2-2} = 4^{n+2}$$

$$a_{n+2} - 7a_{n+1} + 10a_n = 4^{n+2}$$

$$A_1 = -7, \ A_2 = 10, \ \phi(n) = 4^{n+2}$$

$$g(x) = \sum \phi(n) x^n$$

(ii)

$$g(x) = \sum_1 \phi(n)\, x^n$$

$$= \sum_1 4^{n+2} \cdot x^n$$

$$= \sum_1 (4x)^n \cdot 4^2$$

$$= (1-4x)^{-1} \cdot 4^2$$

$$g(x) = \frac{16}{1-4x}$$

$$f(x) = \frac{a_0 + (a_1 + a_0 A_1)x + x^2 g(x)}{1 + A_1 x + A_2 x^2}$$

$$= \frac{8 + (36 + 8(-7))x + x^2 \frac{1}{1-4x}}{1 + (-7)x + 10x^2}$$

$$= \frac{8 + (36 - 56)x + x^2 \frac{1}{1-4x}}{1 - 7x + 10x^2}$$

$$= \frac{8 - 20x + x^2 \frac{1}{1-4x}}{10x^2 - 7x + 1}$$

$$= \frac{96x^2 - 52x + 8}{(10x^2 - 7x + 1)(1-4x)}$$

$$= \frac{96x^2 - 52x + 8}{(1-2x)(1-5x)(1-4x)}$$

$$\frac{96x^2 - 52x + 8}{(1-2x)(1-5x)(1-4x)} = \frac{A}{1-2x} + \frac{B}{1-5x} + \frac{C}{1-4x}$$

$$96x^2 - 52x + 8 = A(1-5x)(1-4x) + B(1-2x)(1-4x) + C(1-2x)(1-5x)$$

$$20A + 8B + 10C = 96$$

$$-9A - 6B - 7C = -52$$

$$A + B + C = 8$$

$$A = 4, \quad B = 12, \quad C = -8$$

$$\frac{96x^2 - 52x + 8}{(1-2x)(1-5x)(1-4x)} = \frac{4}{1-2x} + \frac{12}{1-5x} + \frac{(-8)}{1-4x}$$

$$= 4(1-2x)^{-1} + 12(1-5x)^{-1} - 8(1-4x)^{-1}$$

$$= 4\sum (2x)^n + 12\sum (5x)^n - 8\sum (4x)^n$$

$$f(x) = \left[4 \cdot 2^n + 12 \cdot 5^n - 8 \cdot 4^n\right] \sum x^n$$

$$f(x) = \sum_{n=0}^{\infty} x^n a_n$$

$$\boxed{\therefore a_n = 4 \cdot 2^n + 12 \cdot 5^n - 8 \cdot 4^n}$$

4. Solve $a_{n+2} - 7a_{n+1} + 10a_n = 0$ $\quad a_0 = 3,\ a_1 = 7$

Sol :-

$$a_{n+2} - 7a_{n+1} + 10a_n = 0$$

$$A_1 = -7,\ A_2 = 10,\ \phi(n) = 0$$

$$g(x) = 0$$

$$f(x) = \frac{3 + [7 + 3(-7)]x}{1 - 7x + 10x^2}$$

$$= \frac{3 + (7 - 21)x}{1 - 7x + 10x^2}$$

$$= \frac{3 - 14x}{1 - 7x + 10x^2}$$

$$= \frac{3 - 14x}{1 - 2x - 5x + 10x^2}$$

$$= \frac{3 - 14x}{(1 - 2x)(1 - 5x)}$$

$$\frac{3 - 14x}{(1 - 2x)(1 - 5x)} = \frac{A}{1 - 2x} + \frac{B}{1 - 5x}$$

$$3 - 14x = A(1 - 5x) + B(1 - 2x)$$

$$3 = A + B \rightarrow \times 2$$

$$-14 = -5A - 2B$$

$$6 = 2A + \cancel{2B}$$

$$-14 = -5A - \cancel{2B}$$

$$\cancel{-}8 = \cancel{-}3A$$

$$\boxed{A = 8/3}$$

$$3 = A + B$$

$$3 = 8/3 + B$$

$$3 - 8/3 = B$$

$$\boxed{B = 1/3}$$

$\alpha_n = \alpha_n^H + \alpha_n^P$

$$= \frac{8/3}{1-2x} + \frac{1/3}{1-5x}$$

$$= \frac{8}{3}(1-2x)^{-1} + \frac{1}{3}(1-5x)^{-1}$$

$$= \frac{8}{3}\sum_1 (2x)^n + \frac{1}{3}\sum_1 (5x)^n$$

$$= \left(\frac{8}{3}2^n + 1/3\, 5^n\right)\sum_1 x^n$$

$$= \frac{1}{3}(8.2^n + 5^n)x^n$$

$$a_n = \frac{1}{3}\left[8 \cdot 2^n + 5^n\right]$$

5. Solve $a_n - 9a_{n-1} + 20a_{n-2} = 0 \quad n \geq 2 \quad a_0 = -3,\ a_1 = -10$

$n = n+2$

$$a_{n+2} - 9a_{n+1} + 20a_n = 0$$

$$A_1 = -9, \quad A_2 = 20, \quad \phi(n) = 0$$

$$g(x) = 0$$

$$f(x) = \frac{a_0 + (a_1 + a_0 A_1)x}{1 + A_1 x + A_2 x^2}$$

$$= \frac{-3 + (-10 + (-3)(-9))x}{1 + (-9)x + 20x^2}$$

$$= \frac{-3 + (-10 + 27)x}{1 - 9x + 20x^2}$$

$$= \frac{-3 + 17x}{1 - 9x + 20x^2}$$

$$= \frac{-3 + 17x}{20x^2 - 9x + 1}$$

$$= \frac{-3 + 17x}{20x^2 - 5x - 4x + 1}$$

$$= \frac{-3 + 17x}{5x(4x-1) - 1(4x-1)}$$

$$= \frac{-3 + 17x}{(4x-1)(5x-1)}$$

$$\frac{-3 + 17x}{(4x-1)(5x-1)} = \frac{A}{4x-1} + \frac{B}{5x-1}$$

$$-3 + 17x = A(5x-1) + B(4x-1)$$

$$-3 = -A - B$$

$$3 = A + B \rightarrow (1) \times 4$$

$$17 = 5A + \cancel{4B} \rightarrow (2)$$
$$12 = 4A + \cancel{4B}$$
$$5 = A$$
$$\boxed{A = 5}$$
$$-3 = -A - B$$
$$-3 = -5 - B$$
$$2 = -B$$
$$\boxed{B = -2}$$
$$= \frac{5}{4x-1} + \frac{-2}{5x-1}$$
$$= 5(4x-1)^{-1} - 2(5x-1)^{-1}$$
$$= 5\sum(-1)(4x)^n - 2\sum(-1)(5x)^n$$
$$= 5\cdot4^n x^n(-1) - 2\cdot5^n x^n(-1)$$
$$= (5\cdot4^n + 2\cdot5^n)\,x^n$$
$$a_n = 5\cdot4^n + 2\cdot5^n$$