Probability

Meaning of probability:

Probability is a measure of the likelihood of an event to occur. Many events cannot be predicted with total certainty.

Using probability, one can predict only the chance of an event to occur, i.e., how likely they are going to happen.

For example, when a coin is tossed, there is a probability to get heads or tails.

Properties:

- Probability of an impossible event is phi or a null set.
- The maximum probability of an event is its sample space (sample space is the total number of possible outcomes)
- Probability of any event exists between 0 and 1. (0 can also be a probability).
- There cannot be a negative probability for an event.
- If A and B are two mutually exclusive outcomes (Two events that cannot occur at the same time), then the probability of A or B occurring is the probability of A plus the probability of B.

The probability formula is the ratio of the possibility of occurrence of an outcome to the total number of outcomes.

Probability of occurrence of an event P(E) = Number of favorable outcomes/Total Number of outcomes.

It means chance of happening an event. Example: Getting a head when a coin is tossed, drawing a spade card from a pack of playing cards. Initially probability was a branch of mathematics.

Two Broad division of probability: Subjective probability and objective probability.

Subjective probability is basically dependent on personal judgment and experience and such as it may be influenced by the personal belief, attitude and bias of the person applying.

In the field of uncertainty, this would be quite helpful and it is being applied in the area of decision making management.

Objective probability does not depend on personal judgement. Example , Probability of getting a head when a coin is tossed , getting number four when a dice is thrown.

Some basic terms:

1. **Experiment(Trial):** An experiment may be described as a performance that produces certain results. Example: Tossing a coin, throwing a dice , drawing a card from a pack of playing cards.
2. **Random Experiment:** An experiment is defined to be random if the results of the experiment depend on chance only. For example: if a coin is tossed then we get two outcomes. Head(H) and

Tail(T). It is impossible to say in advance whether a Head or Tail would turn up when we toss the coin one. Thus, tossing a coin is an example of a random experiment.

3. **Events(Outcomes):** The result or outcomes of a random experiment are know as events. Head and Tail are two events when we toss a coin. Number 1,2,3,4,5,6 are events when we thrown a dice.
4. **Sample space:** Set of all possible event is called sample space and every evet is called sample point.

Calculate Probability: Sum of all possible outcomes = 1

$$P = \frac{number\ of\ favourable\ cases\ or\ events\ or\ outcomes}{Total\ number\ of\ cases\ or\ events\ or\ outcomes}$$

Simple event and composite event: An event which has only one sample point is called simple event.

Composite event is an event which can be split into two or more simple events. For example, getting an even no when a dice is tossed. Probability of a dice given even no: P(even no) = {2,4,6}

Equally likely events and not equally likely events:

If one of outcomes can not be expected to occur in performance to the other in an experiment the events are equally likely. Event that having equal probability chance to occurs. If P(A) = P(B), then the two events A and B are equally likely events. Examples:

Coin toss: Head probability ½ and also Tail probability ½ . that's mean equal chance to occur.

P(A) = 1/3 and P(B) = ¼ . Not equal chance to occur are not equally likely.

Impossible event: If P(A) = 0 then event is impossible event. Suppose next year will be 2026. It is not possible. That's mean it's a impossible event.

Sure or Certain Event: If P(A) = 1, then event is sure or certain event.

Mutually Exclusive and Not mutually exclusive event: If the occurrence of any one of them prevent the occurrence of the other then these events are called **mutually exclusive** events. Two events are mutually exclusive if the events cannot happen at the same time.. **Example:** Tossing a coin Head and Tail not occur at a time. That's mean If Head occur does not occur Tail.

When two events may not occur simultaneously then these events are known as **not mutually exclusive** events.

Independent and dependent event: If the happening of one event is not affected by the happening of others then events are called Independent.

Two or more events are said to be depended if the happening of one event affects other event.

Exhaustive event: The total number of possible outcomes of a random experiment is known as Exhaustive event.

If A,B and C are mutually exclusive and exhaustive events then P(A) + P(B) + P(C) = 1

Example: If an unbiased coin is tossed once, then the two events Head and Tail are:

i. Mutually Exclusive
ii. Exhaustive
iii. Equally likely

Complementary Event: $P(A^c) = 1 - P(A)$

Whenever an event is the complement of another event, specifically, if A is an event, then P(not A)=1−P(A) or P(A') = 1 - P(A'). P(A)+P(A′)=1

Addition Rule: Whenever an event is the union of two other events, say A and B, then

- **i. When events are mutually exclusive:**
  - i. P(AUB) = P(A) + P(B)
  - ii. P(AUBUC) = P(A)+P(B)+P(C)
- **ii. When events are not mutually Exclusive:**
  - i. $P(AUB) + P(A) + P(B) - P(A \cap B)$
  - ii. $P(AUBUC) + P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$
  - iii.

Identify as addition rule:

- i. P(A+B)
- ii. P(AUB)
- iii. P(A or B)
- iv. P(either A or B)
- v. P(any of them)
- vi. P(at least one)

Examples -01: Suppose a bag contains 20 balls. 10 of the balls are white, 7 of the balls are red, and 3 of the balls are blue. Suppose one ball is selected at random from the bag.

1. Are the events "selecting a white ball" and "selecting a red ball" mutually exclusive? Why?
2. What is the probability of selecting a white or red ball?

Solution:

1. The events "selecting a white ball" and "selecting a red ball" are **mutually exclusive** because the events cannot happen at the same time. It is not possible for the selected ball to be both white and red.
2. P(white or red)=P(white)+P(red)=10/20+720=0.85
3. P(white and red)=0

Example – 02: At a local college, 60% of the students are taking a math class, 50% of the students are taking a science class, and 30% of the students are taking both a math and a science class.

1. Are the events "taking a math class" and taking a science class" mutually exclusive? Explain.
2. What is the probability that a randomly selected student is taking a math class or a science class?

Solution

1. The events “taking a math class” and “taking a science class” are not mutually exclusive because the events can happen at the same time (i.e. a student can be taking both a math class and a science class). As stated in the question, P(math and science) = 0.3 ≠0 P(math and science) = 0.3 ≠ 0.
2. P(math or science) = P(math) + P(science) − P(math and science) = 0.6 + 0.5 − 0.3=0.8

Example -03:

You roll a fair die one time.

1. Are the events “rolling a 4” and “rolling an even number” mutually exclusive?
2. Are the events “rolling a 4” and “rolling an odd number” mutually exclusive?
3. What is the probability of rolling a 4 or rolling an odd number.

Solution:

1. The events “rolling a 4” and “rolling an even number” are not mutually exclusive because the events can happen at the same time (i.e. 4 is an even number).
2. The events “rolling a 4” and “rolling an odd number” are mutually exclusive because the events cannot happen at the same time. It is not possible to roll a die and get a 4 (an even number) and an odd number on the top face at the same time
3. P(4 or odd)=P(4)+P(odd)=1/6+3/6=4/6

Multiplication Rule:

Multiplication rule of probability states that whenever an event is the intersection of two other events, that is, events A and B need to occur simultaneously. Then, P(A and B)=P(A)·P(B). The set A∩B denotes the simultaneous occurrence of events A and B, that is the set in which both events A and event B have occurred. Event A∩B can be written as AB. The probability of event AB is obtained by using the properties of conditional probability, which is given as P(A ∩ B) = P(A) P(B | A).

i. When events are mutually exclusive:

i. P(A∩B) = P(A) × P(B)

ii. P(A∩B∩C) = P(A)×P(B)×P(C)

ii. When events are not mutually Exclusive:

ii. $P(A \cap B) = P(A) \times P\left(\frac{B}{A}\right), Probability\ of\ event\ B, wehn\ event\ A\ has\ already\ occured.$

iii. $P(A \cap B\ \cap C) =\ P(A) \cap P\left(\frac{B}{A}\right)\ \times P\left(\frac{C}{A \cap B}\right)$

Identify as addition rule:

vii. $P(A \cap B)$

viii. P(A and B)

Condition of Multiplication:

1. If two event A and B independent:

i. A and B^c

ii. A^c and B^c

iii. A^c and B

iv. $If\ P\left(\frac{A}{B}\right) = P(A)$

v. $If\ P\left(\frac{B}{A}\right) = P(B)$

2. **Without replacement:** $P(A \cap B) = P(A) \times P\left(\frac{B}{A}\right)$
3. **With replacement:** P(A∩B) = P(A) × P(B)

<u>Conditional Probability Formula</u>:

i. $P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)}$

ii. $P\left(\frac{B}{A^c}\right) = \frac{P(A^c \cap B)}{P(A^c)}$

iii. $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$

$A^c \cap B = P(B) - P(A \cap B)$

Example 1: What is the probability of getting a 5 and then a 2 with the normal 6-sided die?

Solution:

Sample space = {1, 2, 3, 4, 5, 6}

Total events = 6

Probability of getting a 5 = 1/6

Probability of getting a 6 = 1/6

Applying the multiplication rule of probability for independent events,

P(getting a 5 and then a 2) = (1/6).(1/6) = 1/36.

Therefore, the probability of getting a 5 and then a 2 with the normal 6-sided die is 1/36.

Example 2: Two cards are selected without replacing the first card from the deck. Find the probability of selecting a king and then selecting a queen.

Solution:

Total events = 52

Since the first card is not replaced, the events are dependent.

Probability of selecting a king = P(K) = 4/52

Probability of getting a queen = P(Q) = 4/51 (one card drawn first has not been replaced)

P(a king & a then queen) = P(K).P(Q|K) =4/52 . 4/51 = 16/2652 = 1/166.

Therefore, the probability of selecting a king and then selecting a queen is 1/166.

Bayes Theorem

Bayes theorem is a theorem in probability and statistics, named after the Reverend Thomas Bayes, that helps in determining the probability of an event that is based on some event that has already occurred. Bayes rule has many applications such as Bayesian interference, in the healthcare sector - to determine the chances of developing health problems with an increase in age and many others.

The Bayes theorem is based on finding P(A | B) when P(B | A) is given. Here, we will aim at understanding the use of the Bayes rule in determining the probability of events, its statement, formula, and derivation with the help of examples.

What is Bayes Theorem?

Bayes theorem, in simple words, determines the conditional probability of event A given that event B has already occurred based on the following:

- Probability of B given A
- Probability of A
- Probability of B

Bayes Law is a method to determine the probability of an event based on the occurrences of prior events. It is used to calculate **conditional probability** . Bayes theorem calculates the probability based on the hypothesis. Now, let us state and prove Bayes Theorem. Bayes rule states that the conditional probability of an event A, given the occurrence of another event B, is equal to the product of the likelihood of B, given A and the probability of A divided by the probability of B. It is given as:

P(A|B)=P(B|A)P(A)P(B)P(A|B)=P(B|A)P(A)P(B)

Here, P(A) = how likely A happens(Prior knowledge)- The probability of a hypothesis is true before any evidence is present.

P(B) = how likely B happens(Marginalization)- The probability of observing the evidence.

P(A|B) = how likely A happens given that B has happened(Posterior)-The probability of a hypothesis is true given the evidence.

P(B|A) = how likely B happens given that A has happened(Likelihood)- The probability of seeing the evidence if the hypothesis is true.

Terms Related to Bayes Theorem

As we have studied about Bayes theorem in detail, let us understand the meanings of a few terms related to the concept which have been used in the Bayes theorem formula and derivation:

- **Conditional Probability** - Conditional Probability is the probability of an event A based on the occurrence of another event B. It is denoted by P(A|B) and represents the probability of A given that event B has already happened.
- **Joint Probability** - Joint probability measures the probability of two more events occurring together and at the same time. For two events A and B, it is denoted by P(A∩B)P(A∩B).

- **Random Variables** - Random variable is a real-valued variable whose possible values are determined by a random experiment. The probability of such variables is also called the experimental probability.
- **Posterior Probability** - Posterior probability is the probability of an event that is calculated after all the information related to the event has been accounted for. It is also known as conditional probability.
- **Prior Probability** - Prior probability is the probability of an event that is calculated before considering the new information obtained. It is the probability of an outcome that is determined based on current knowledge before the experiment is performed.

Important Notes on Bayes Law:

- Bayes theorem is used to determine conditional probability.
- When two events A and B are independent, $P(A|B) = P(A)$ and $P(B|A) = P(B)$
- Conditional probability can be calculated using the Bayes theorem for continuous random variables.

Example 1: Amy has two bags. Bag I has 7 red and 4 blue balls and bag II has 5 red and 9 blue balls. Amy draws a ball at random and it turns out to be red. Determine the probability that the ball was from the bag I.

Solution: Assume A to be the event of drawing a red ball. We know that the probability of choosing a bag for drawing a ball is 1/2, that is,

$P(X) = P(Y) = 1/2$

Let X and Y be the events that the ball is from the bag I and bag II, respectively. Since there are 7 red balls out of a total of 11 balls in the bag I, therefore, P(drawing a red ball from the bag I) = $P(A|X) = 7/11$

Similarly, P(drawing a red ball from bag II) = $P(A|Y) = 5/14$

We need to determine the value of P(the ball drawn is from the bag I given that it is a red ball), that is, $P(X|A)$. To determine this we will use Bayes Theorem. Using Bayes theorem, we have the following:

$P(X|A)=P(A|X)P(X)P(A|X)P(X)+P(A|Y)P(Y)P(X|A)=P(A|X)P(X)P(A|X)P(X)+P(A|Y)P(Y)$

$= [((7/11)(1/2))/(7/11)(1/2)+(5/14)(1/2)]$

$= 0.64$

Example 2: Assume that the chances of a person having a skin disease are 40%. Assuming that skin creams and drinking enough water reduces the risk of skin disease by 30% and prescription of a certain drug reduces its chance by 20%. At a time, a patient can choose any one of the two options with equal probabilities. It is given that after picking one of the options, the patient selected at random has the skin disease. Find the probability that the patient picked the option of skin screams and drinking enough water using the Bayes theorem.

Solution: Assume E1: The patient uses skin creams and drinks enough water; E2: The patient uses the drug; A: The selected patient has the skin disease

P(E1) = P(E2) = 1/2

Using the probabilities known to us, we have

P(A|E1) = 0.4 × (1-0.3) = 0.28

P(A|E2) = 0.4 × (1-0.2) = 0.32

Using Bayes rule, the probability that the selected patient uses skin creams and drinks enough water is given by,

P(E1|A)=P(A|E1)P(E1)P(A|E1)P(E1)+P(A|E2)P(E2)P(E1|A)=P(A|E1)P(E1)P(A|E1)P(E1)+P(A|E2)P(E2)

= (0.28 × 0.5)/(0.28 × 0.5 + 0.32 × 0.5)

= 0.14/(0.14 + 0.16)

= 0.47

Example 3: A man is known to speak the truth 3/4 times. He draws a card and reports it is king. Find the probability that it is actually a king.

Solution:

Let E be the event that the man reports that king is drawn from the pack of cards

A be the event that the king is drawn

B be the event that the king is not drawn.

Then we have P(A) = probability that king is drawn = 1/4

P(B) = probability that king is drawn = 3/4

P(E/A) = Probability that the man says the truth that king is drawn when actually king is drawn = P(truth) = 3/4

P(E/B)= Probability that the man lies that king is drawn when actually king is drawn = P(lie) = 1/4

Then according to Bayes formula, the probability that it is actually a king = P(A/E)

=P(A)P(E|A)P(A)P(E|A)+P(B)P(E|B)P(A)P(E|A)P(A)P(E|A)+P(B)P(E|B)

= [1/4 × 3/4] ÷[(1/4 × 3/4) + (1/4 × 3/4)]

= 3/16 ÷12/16

= 3/16 × 16/12

=1/2 = 0.5

Resources: Youtube + google different websites