习题1

基本题

1. 利用对角线法则计算下列二、三阶行列式

(1) $\begin{vmatrix} a+b & a-b \\ a-b & a+b \end{vmatrix}$

```
Det[{{a + b, a - b}, {a - b, a + b}}]
```

mathematica中的不同的显示模式。
传统显示复制到新单元会隐藏部分显示，上面和下面表达式其实是一样的，若要显示可以先设置新的单元样式为Input，再转化为标准显示，方便计算。

在Mathematica中有时做题时为了显示而输入的内容，也可以进行计算，不用重新输入，同样可以先输入能计算的标准形式，再转为显示的模式进行排版。

```
Det[{{a + b, a - b}, {a - b, a + b}}]
```

$4\,a\,b$

(2) $\begin{vmatrix} x-1 & 1 \\ x^3 & x^2+x+1 \end{vmatrix}$

```
Det[{{x - 1, 1}, {x^3, x^2 + x + 1}}]
```

-1

(3) $\begin{vmatrix} 2 & 0 & 1 \\ 1 & -4 & -1 \\ -1 & 8 & 3 \end{vmatrix}$

```
2 (-4) 3 + 1 × 8 × 1 + (-1) (-1) 0 - (-1) (-4) 1 - 1 × 0 × 3 - 2 × 8 (-1)
```

-4

```
Det[{{2, 0, 1}, {1, -4, -1}, {-1, 8, 3}}]
```

-4

(4) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

传统显示也能直接计算。

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$a^2 (-b) + a^2 c + a b^2 - a c^2 - b^2 c + b c^2$

```
Simplify[-a^2 b + a b^2 + a^2 c - b^2 c - a c^2 + b c^2]
```

$-(a-b)(a-c)(b-c)$

```
A = {{1, 1, 1}, {a, b, c}, {a^2, b^2, c^2}}
```

$$\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$$

```
A[[2]] = A[[2]] - a A[[1]]
```

$\{0, b-a, c-a\}$

```
A[[3]] = A[[3]] - a^2 A[[1]]
```

$\{0, b^2-a^2, c^2-a^2\}$

```
A
```

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{pmatrix}$$

```
A[[3]] = A[[3]] - (b + a) A[[2]]
```

$\{0, -a^2-(b-a)(a+b)+b^2, -a^2-(a+b)(c-a)+c^2\}$

```
A // Simplify
```

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (a-c)(b-c) \end{pmatrix}$$

2. 确定下列各排列的逆序数

(1)2413

```
Needs["Combinatorica`"]
```

General::compat : Combinatorica Graph and Permutations functionality has been superseded by preloaded functionality. The package now being loaded may conflict with this. Please see the Compatibility Guide for details.

Inversions::shdw : Symbol Inversions appears in multiple contexts {Combinatorica`, Global`}; definitions in context Combinatorica` may shadow or be shadowed by other definitions. >>

```
Inversions[{2, 4, 1, 3}]
```

3

```
inversion[number_] := IntegerDigits[number] // Inversions
```

```
inversion[2413]
```
3

```
0 + 0 + 2 + 1
```
3

(2)6427531

```
inversion[6 427 531]
```
15

若一个数之前还有

```
myInversion[number_] :=
 Table[Take[IntegerDigits[number], i], {i, 1, Length@IntegerDigits[number]}]
```

```
myInversion[1234]
```
{{1}, {1, 2}, {1, 2, 3}, {1, 2, 3, 4}}

```
myInversion[number_] := Module[{list = IntegerDigits[number]},
  Table[Select[Take[list, i], # > list[[i]] &], {i, 1, Length@list}]]
```

```
myInversion[213 457 698]
```
{{}, {2}, {}, {}, {}, {}, {7}, {}, {9}}

(3)134782695

```
inversion[134 782 695]
```
10

```
myInversion[134 782 695]
```
{{}, {}, {}, {}, {}, {3, 4, 7, 8}, {7, 8}, {}, {7, 8, 6, 9}}

(4)13...(2n-1)24...(2n)

```
list = Range[1, 9, 2] ~ Join ~ Range[2, 9, 2]
```
{1, 3, 5, 7, 9, 2, 4, 6, 8}

```
Inversions[list]
```
10

```
myInversion[FromDigits@list]
```
{{}, {}, {}, {}, {}, {3, 5, 7, 9}, {5, 7, 9}, {7, 9}, {9}}

先用特例计算，再猜想，再归纳法证明。

```
list1[n_] := Range[1, 2 n, 2] ~ Join ~ Range[2, 2 n, 2]
```

```
inversionList = Table[Inversions[list1[n]], {n, 1, 10}]
```
{0, 1, 3, 6, 10, 15, 21, 28, 36, 45}

```
Table[{n, list1[n], Inversions[list1[n]]}, {n, 1, 9}]
```

$$\begin{pmatrix} 1 & \{1, 2\} & 0 \\ 2 & \{1, 3, 2, 4\} & 1 \\ 3 & \{1, 3, 5, 2, 4, 6\} & 3 \\ 4 & \{1, 3, 5, 7, 2, 4, 6, 8\} & 6 \\ 5 & \{1, 3, 5, 7, 9, 2, 4, 6, 8, 10\} & 10 \\ 6 & \{1, 3, 5, 7, 9, 11, 2, 4, 6, 8, 10, 12\} & 15 \\ 7 & \{1, 3, 5, 7, 9, 11, 13, 2, 4, 6, 8, 10, 12, 14\} & 21 \\ 8 & \{1, 3, 5, 7, 9, 11, 13, 15, 2, 4, 6, 8, 10, 12, 14, 16\} & 28 \\ 9 & \{1, 3, 5, 7, 9, 11, 13, 15, 17, 2, 4, 6, 8, 10, 12, 14, 16, 18\} & 36 \end{pmatrix}$$

也可以对逆序数序列应用Mathematica函数求通项公式

```
FindSequenceFunction[inversionList, n]
```

$$\frac{1}{2}(n-1)\,n$$

1 3 6 10...是个二阶等差数列

实际上，本题很简单，直接观察计算即可。

13 ... (2 n - 1)
24 ... (2 n)
分别是自然排列。
非零的逆序对于24 ... (2 n)中的每一个数，前面分别有对应用的除去1的n-1项，除去1、3的n-2项，直到0
即逆序数为 $\sum_{i=1}^{n}(i-1) = \frac{1}{2}(n-1)\,n$

$$\sum_{i=1}^{n}(i-1)\ //\ \mathbf{Factor}$$

$$\frac{1}{2}(n-1)\,n$$

(5)13...(2n-1)(2n)(2n-2)...2

```
list2[n_] := Range[1, 2 n, 2] ~Join~ Range[2 n, 2, -2]
```

```
inversionList = Table[Inversions[list2[n]], {n, 1, 9}]
```

{0, 2, 6, 12, 20, 30, 42, 56, 72}

```
Table[{n, list2[n], Inversions[list2[n]]}, {n, 1, 9}]
```

$$\begin{pmatrix} 1 & \{1, 2\} & 0 \\ 2 & \{1, 3, 4, 2\} & 2 \\ 3 & \{1, 3, 5, 6, 4, 2\} & 6 \\ 4 & \{1, 3, 5, 7, 8, 6, 4, 2\} & 12 \\ 5 & \{1, 3, 5, 7, 9, 10, 8, 6, 4, 2\} & 20 \\ 6 & \{1, 3, 5, 7, 9, 11, 12, 10, 8, 6, 4, 2\} & 30 \\ 7 & \{1, 3, 5, 7, 9, 11, 13, 14, 12, 10, 8, 6, 4, 2\} & 42 \\ 8 & \{1, 3, 5, 7, 9, 11, 13, 15, 16, 14, 12, 10, 8, 6, 4, 2\} & 56 \\ 9 & \{1, 3, 5, 7, 9, 11, 13, 15, 17, 18, 16, 14, 12, 10, 8, 6, 4, 2\} & 72 \end{pmatrix}$$

也可以对逆序数序列应用Mathematica函数求通项公式

```
FindSequenceFunction[inversionList, n]
```

$(n-1)\,n$

0 2 6 12 20...是个二阶等差数列

直接计算逆序数 $\sum_{i=1}^{n} 2\,(i-1) = (n-1)\,n$

$$\sum_{i=1}^{n} 2\,(i-1)\ //\ \text{Factor}$$

$(n-1)\,n$

3 在6阶行列式中，下列各项应取什么符号？

(1) $a_{15}a_{23}a_{32}a_{44}a_{51}a_{66}$

152332445166

```
Subscript[a, FromDigits[#]] & /@
  (IntegerDigits@152 332 445 166 // Partition[#, 2] &) //
 Row[#, ""] & (*Row用于排列成一行显示*)
```

$a_{15}a_{23}a_{32}a_{44}a_{51}a_{66}$

```
subscriptList[number0_] := Module[{number = number0},
  Subscript[a, FromDigits[#]] & /@ (IntegerDigits@number // Partition[#, 2] &)]
```

```
subscriptList@152 332 445 166
```

$\{a_{15}, a_{23}, a_{32}, a_{44}, a_{51}, 5\}$

```
a66
```

5

```
Clear[a66] (*(下标变量清除不方便)*)
```

Clear::ssym : a_{66} 不是一个符号或者一个字符串. >>

```
a66 =.
```

```
a66
```

a_{66}

```
subscriptList@152 332 445 166
```

$\{a_{15}, a_{23}, a_{32}, a_{44}, a_{51}, a_{66}\}$

(2) $a_{33}a_{42}a_{14}a_{51}a_{66}a_{25}$

334 214 516 625

在Mathematica里显示一些公式，如带下标的变量等，有的是用于排版和显示的，有的是能应用于计算的，比如这里若使用Row等用于显示的函数就不太方便进行计算，若要计算的，可以直接生成一个带下标的变量列表比较方便。

```
subscriptList[334 214 516 625]
```

$\{a_{33}, a_{42}, a_{14}, a_{51}, a_{66}, a_{25}\}$

```
{a33, a42, a14, a51, a66, a25} =.
```

Unset::norep : 未找到对于 a_{51} 在 Subscript 上的赋值. ≫

Unset::norep : 未找到对于 a_{66} 在 Subscript 上的赋值. ≫

{Null, Null, Null, $Failed, $Failed, Null}

```
{a33, a42, a14, a51, a66, a25} = Range[6]
```

{1, 2, 3, 4, 5, 6}

```
a33
```

1

```
subscriptList[number0_] :=
 Module[{number = number0}, Subscript[a, FromDigits[#]] & /@
   (list = (IntegerDigits@number // Partition[#, 2] &))]
```

```
subscriptList@334 214 516 625
```

$\{a_{33}, a_{42}, a_{14}, a_{51}, a_{66}, a_{25}\}$

```
Power[-1, #] & /@ (Total /@ list)
```

{1, 1, -1, 1, 1, -1}

```
Total[%]
```

2

```
sign[number_] := (subscriptList[number];
  -1^Inversions[list[[All, 2]]])
```

```
sign[152 332 445 166]
```

-1

先按下标的行排序，排成自然排列

a33a42a14a51a66a25

排成

a14a25a33a42a51a66

然后计算排列

453216

的逆序数=9

所以此项是负号项

```
myInversion[453 216]
```

{{4}, {4, 5}, {4, 5, 3}, {4, 5, 3, 2}, {4, 5, 3, 2, 1}, {4, 5, 3, 2, 1, 6}}

```
Inversions[IntegerDigits@453 216]
```

9

5 写出5阶行列式中含有因子a12a24a35的项

```
subscriptList[1 224 354 351]
```

$\{a_{12}, a_{24}, a_{35}, a_{43}, a_{51}\}$

```
Inversions[IntegerDigits@24 531]
```

6

```
Row@subscriptList[1 224 354 351]
```

$a_{12}a_{24}a_{35}a_{43}a_{51}$

```
sign@1 224 354 351
```

-1

$\{\boldsymbol{a_{12}}, \boldsymbol{a_{24}}, \boldsymbol{a_{35}}, \boldsymbol{a_{43}}, \boldsymbol{a_{51}}\}$

```
subscriptList[1 224 354 153]
```

$\{a_{12}, a_{24}, a_{35}, a_{41}, a_{53}\}$

```
Inversions[IntegerDigits@24 513]
```

5

```
-Row@subscriptList[1 224 354 153]
```

$-a_{12}a_{24}a_{35}a_{41}a_{53}$

5 计算下列各行列式

(1) $\left|\begin{pmatrix} 4 & 1 & 2 & 4 \\ 1 & 2 & 0 & 2 \\ 10 & 5 & 2 & 0 \\ 0 & 1 & 1 & 7 \end{pmatrix}\right|$

```
A = {{4, 1, 2, 4}, {1, 2, 0, 2}, {10, 5, 2, 0}, {0, 1, 1, 7}}
```

$$\begin{pmatrix} 4 & 1 & 2 & 4 \\ 1 & 2 & 0 & 2 \\ 10 & 5 & 2 & 0 \\ 0 & 1 & 1 & 7 \end{pmatrix}$$

```
10 : 36
```

按第4行展开，计算

```
Det[A] =
 -Det[A[[1 ;; 3, {1, 2, 4}]]] + 1 Det[A[[1 ;; 3, {1, 3, 4}]]] + 7 Det[A[[1 ;; 3, 1 ;; 3]]]
```

```
-Det@A[[1 ;; 3, {1, 2, 4}]]
```

80

```
Det@A[[1 ;; 3, {1, 3, 4}]]
```
32

```
7 Det[A[[1 ;; 3, 1 ;; 3]]]
```
-112

```
-Det[A[[1 ;; 3, {1, 2, 4}]]] + 1 Det[A[[1 ;; 3, {1, 3, 4}]]] + 7 Det[A[[1 ;; 3, 1 ;; 3]]]
```
0

```
Det[A]
```
0

直接计算稍麻烦点，既然确定值为0，看看有什么规律，按照行列式的性质得出值为0

性质1：若行列式的两行对应元素成比例，则行列式值为0
性质2：若行列式互换两行，则变符号
推论：若行列式有两行完全相同，则行列式值为0，这个即可以从性质1推出。两行完全相同也是对应成比例，比值为1。也可以从性质2推出。

```
A = {{4, 1, 2, 4}, {1, 2, 0, 2}, {10, 5, 2, 0}, {0, 1, 1, 7}}
```
$$\begin{pmatrix} 4 & 1 & 2 & 4 \\ 1 & 2 & 0 & 2 \\ 10 & 5 & 2 & 0 \\ 0 & 1 & 1 & 7 \end{pmatrix}$$

```
(*(类似排序函数中有一个中间变量)*)互换函数Signed[matrix0_, row10_, row20_] := Module[
  {matrix = matrix0, rowTemp, row1 = row10, row2 = row20}, rowTemp = matrix[[row1]];
  matrix[[row1]] = matrix[[row2]];
  matrix[[row2]] = -rowTemp;
  matrix]
```

```
(*(对换后没有符号变换的情况)*)互换函数UnSigned[matrix0_, row10_, row20_] := Module[
  {matrix = matrix0, rowTemp, row1 = row10, row2 = row20}, rowTemp = matrix[[row1]];
  matrix[[row1]] = matrix[[row2]];
  matrix[[row2]] = rowTemp;
  matrix]
```

把第1行放到第4行，等价与对调3次，要乘以一个-1，因为不是矩阵，所以-1是乘在某一行里的，所以直接乘以-1不太方便。
所以暂时先在外面记录符号的变换。

```
A1 = RotateLeft[A, 1]
```
$$\begin{pmatrix} 1 & 2 & 0 & 2 \\ 10 & 5 & 2 & 0 \\ 0 & 1 & 1 & 7 \\ 4 & 1 & 2 & 4 \end{pmatrix}$$

第二行，第四行消第一列的0

```
A1[[2]] = A1[[2]] - 10 A1[[1]]
```
{0, −15, 2, −20}

```
A1[[4]] = A1[[4]] - 4 A1[[1]]
```
{0, −7, 2, −4}

```
A1
```
$$\begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & -15 & 2 & -20 \\ 0 & 1 & 1 & 7 \\ 0 & -7 & 2 & -4 \end{pmatrix}$$

对调第2行和第3行，符号数变化一次。

```
A2 = 互换函数Signed[A1, 2, 3]
```
$$\begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 7 \\ 0 & 15 & -2 & 20 \\ 0 & -7 & 2 & -4 \end{pmatrix}$$

消去第3行、第4行的第2列元素。

```
A2[[3]] = A2[[3]] - 15 A2[[2]]
A2[[4]] = A2[[4]] + 7 A2[[2]]
```
{0, 0, −17, −85}

{0, 0, 9, 45}

```
A2
```
$$\begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & -17 & -85 \\ 0 & 0 & 9 & 45 \end{pmatrix}$$

```
A2 // FactorInteger
```
$$\begin{pmatrix} (1\ \ 1) & (2\ \ 1) & (0\ \ 1) & (2\ \ 1) \\ (0\ \ 1) & (1\ \ 1) & (1\ \ 1) & (7\ \ 1) \\ (0\ \ 1) & (0\ \ 1) & \begin{pmatrix} -1 & 1 \\ 17 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 1 \\ 5 & 1 \\ 17 & 1 \end{pmatrix} \\ (0\ \ 1) & (0\ \ 1) & (3\ \ 2) & \begin{pmatrix} 3 & 2 \\ 5 & 1 \end{pmatrix} \end{pmatrix}$$

此时，第3行和第4行对应元素成比例，行列式值为0.
若感觉0比0不太好，则可以理解为行列式展开的子块中的对应的两行的对应元素成比例，行列式值为0.
当然也可以继续消去

```
A2[[4]] = 17 / 9 A2[[4]] + A2[[3]]
```
{0, 0, 0, 0}

```
A2
```

$$\begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & -17 & -85 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

上三角行列式的值为主对角线上的元素的乘积，有个0这样积就为0了。

现在直接使用有符号变换的对换函数简化一下步骤

原来是把第1行放到第4行，就是对换(1 2) (2 3) (3 4)，然后又对换了(2 3)
对换了4次

```
A3 = 互换函数Signed[
  互换函数Signed[互换函数Signed[互换函数Signed[A, 1, 2], 2, 3], 3, 4], 2, 3]
```

$$\begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 7 \\ -10 & -5 & -2 & 0 \\ -4 & -1 & -2 & -4 \end{pmatrix}$$

在求行列式时和在用矩阵消元法时，应该先把首项是1的行设为第1行，0的行为第2行，这样在手工计算时是方便的。我在做这题的时候显然没有一下子意识到这个事情。

当然，这整个过程有助于对换及其在Mathematica中的实现的理解。
同时，对于之前的函数的嵌套使用中有一个不方便之处。

```
A3[[3]] - A3
```

$$\begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 7 \\ -10 & -5 & -2 & 0 \\ -4 & -1 & -2 & -4 \end{pmatrix}$$

在A3的初值求解中，嵌套应用函数时，因为要输入不同的对换的行数，不太方便，可以修改对换函数的行数输入参数为一个列表，即互换函数的输入参数由三个改成两个。

这里，这个过程，可以理解为：反应应用对调函数，每次追加一个参数，而参数提前存储在参数序列中，现在要用Mathematica的函数与语句表达这个过程。这可以用Fold来实现

类似的问题有多元参数的函数嵌套问题。

$$\text{NestList}\left[\left\{\frac{\#[[1]]+\#[[2]]}{2}, \sqrt{\#[[1]]\,\#[[2]]}\right\}\&, \{0.5, 1.0\}, 4\right]$$

$$\begin{pmatrix} 0.5 & 1. \\ 0.75 & 0.707107 \\ 0.728553 & 0.728238 \\ 0.728396 & 0.728396 \\ 0.728396 & 0.728396 \end{pmatrix}$$

```
互换函数Signed[matrix0_, rows0_] :=
 Module[{matrix = matrix0, rowTemp, row1 = rows0[[1]], row2 = rows0[[2]]},
  rowTemp = matrix⟦row1⟧;
  matrix⟦row1⟧ = matrix⟦row2⟧;
  matrix⟦row2⟧ = -rowTemp;
  matrix]
```

```
对换序列 = {{1, 2}, {2, 3}, {3, 4}, {2, 3}}
```

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 2 & 3 \end{pmatrix}$$

```
FoldList[f, x, 对换序列]
```

$\{x, f(x, \{1, 2\}), f(f(x, \{1, 2\}), \{2, 3\}), f(f(f(x, \{1, 2\}), \{2, 3\}), \{3, 4\}), f(f(f(f(x, \{1, 2\}), \{2, 3\}), \{3, 4\}), \{2, 3\})\}$

```
Fold[互换函数Signed, A, 对换序列]
```

$$\begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 7 \\ -10 & -5 & -2 & 0 \\ -4 & -1 & -2 & -4 \end{pmatrix}$$

```
A3
```

$$\begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 7 \\ -10 & -5 & -2 & 0 \\ -4 & -1 & -2 & -4 \end{pmatrix}$$

$$(2)\begin{pmatrix} 2 & 1 & 4 & 1 \\ 3 & -1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 5 & 0 & 6 & 2 \end{pmatrix}$$

```
Det[A = {{2, 1, 4, 1}, {3, -1, 2, 1}, {1, 2, 3, 2}, {5, 0, 6, 2}}]
```

0

```
A
```

$$\begin{pmatrix} 2 & 1 & 4 & 1 \\ 3 & -1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 5 & 0 & 6 & 2 \end{pmatrix}$$

互换，在这里，也可以按第一列的元素的自然排列排序，然后求一个变换数，即得符号。

```
ASorted = Sort[A]
```

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 2 & 1 & 4 & 1 \\ 3 & -1 & 2 & 1 \\ 5 & 0 & 6 & 2 \end{pmatrix}$$

```
Signature[A[[All, 1]]]
```

1

```
对换序列 = {{1, 3}, {2, 3}}
```

$$\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}$$

```
A1 = Fold[互换函数Signed, A, 对换序列]
```

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ -2 & -1 & -4 & -1 \\ -3 & 1 & -2 & -1 \\ 5 & 0 & 6 & 2 \end{pmatrix}$$

```
A1[[2]] = A1[[2]] + 2 A1[[1]]
A1[[3]] = A1[[3]] + 3 A1[[1]]
A1[[4]] = A1[[4]] - 5 A1[[1]]
```

{0, 3, 2, 3}

{0, 7, 7, 5}

{0, −10, −9, −8}

```
A1
```

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 3 & 2 & 3 \\ 0 & 7 & 7 & 5 \\ 0 & -10 & -9 & -8 \end{pmatrix}$$

第二行加上第三行，就跟第四行对应元素成比例了。

```
A1[[3]] = A1[[3]] + A1[[2]]
```

{0, 10, 9, 8}

```
A1
```

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 3 & 2 & 3 \\ 0 & 10 & 9 & 8 \\ 0 & -10 & -9 & -8 \end{pmatrix}$$

(3) $\begin{matrix} -ab & ac & ae \\ bd & -cd & de \\ bf & cf & -ef \end{matrix}$

```
A = {{-a b, a c, a e}, {b d, -c d, d e}, {b f, c f, -e f}}
```

$$\begin{pmatrix} -a\,b & a\,c & a\,e \\ b\,d & -c\,d & d\,e \\ b\,f & c\,f & -e\,f \end{pmatrix}$$

```
Collect[Plus @@ A[[1]], a]
```

$a\,(-b+c+e)$

本来提取a,d,f有如下简洁的方式，可惜函数中对f进行了排序，使得{a,d,f}不在存储的树的同一层中。

```
Thread[Collect[Plus @@@ A, {a, d, f}]]
```

$\{a\,(-b+c+e),\, d\,(b-c+e),\, f\,(b+c-e)\}$

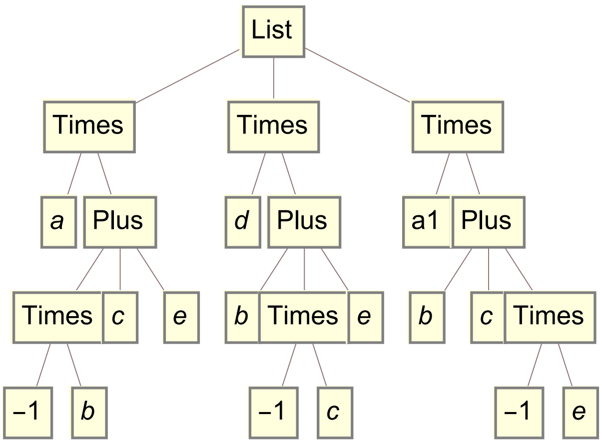
```
Thread[Collect[Plus @@@ A, {a, d, f}]][[All, 1]]
```

$\{a,\, d,\, b+c-e\}$

```
Thread[Collect[Plus @@@ A /. f → a1, {a, d, a1}]][[All, 1]]
```

$\{a,\, d,\, \mathrm{a1}\}$

```
TreeForm@(det = Thread[Collect[Plus @@@ A /. f → a1, {a, d, a1}]])
```




```
Position[det, Times]
```

$\{\{1,0\},\{1,2,1,0\},\{2,0\},\{2,2,2,0\},\{3,0\},\{3,2,3,0\}\}$

```
Times @@ det[[All, 1]].(det2 = det[[All, 2]] /. Plus → List)
```

$$(a\,\mathrm{a1}\,d).\begin{pmatrix} -b & c & e \\ b & -c & e \\ b & c & -e \end{pmatrix}$$

```
det2
```

$$\begin{pmatrix} -b & c & e \\ b & -c & e \\ b & c & -e \end{pmatrix}$$

```
det2[[2]] = det2[[2]] + det2[[1]]
det2[[3]] = det2[[3]] + det2[[1]]
(*互换2、3两行*)
互换函数Signed[det2, {2, 3}]
```

$\{0, 0, 2e\}$

$\{0, 2c, 0\}$

$$\begin{pmatrix} -b & c & e \\ 0 & 2c & 0 \\ 0 & 0 & -2e \end{pmatrix}$$

```
Det@det2
```

$4bce$

故最终的值为4abcdef

```
Times @@ det[[All, 1]] Det@det2 /. a1 → f
```

$4abcdef$

$$(4)\ \begin{matrix} a_0 & b_1 & b_2 & \ldots & b_n \\ c_1 & a_1 & 0 & \ldots & 0 \\ c_2 & 0 & a_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_n & 0 & 0 & \ldots & a_n \end{matrix}\ (a_1 a_2 \ldots a_n \neq 0)$$

$$\begin{matrix} a_0 & b_1 & b_2 & \ldots & b_n \\ c_1 & a_1 & 0 & \ldots & 0 \\ c_2 & 0 & a_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_n & 0 & 0 & \ldots & a_n \end{matrix}$$

$$(5)\ \begin{matrix} 2 & 1 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 \\ 1 & 8 & 4 & 0 & 7 \\ 10 & -3 & -2 & 3 & 1 \\ 21 & 6 & 5 & -2 & 3 \end{matrix}$$

```
A = {{2, 1, 0, 0, 0}, {3, -1, 0, 0, 0},
  {1, 8, 4, 0, 7}, {10, -3, -2, 3, 1}, {21, 6, 5, -2, 3}}
```

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 \\ 1 & 8 & 4 & 0 & 7 \\ 10 & -3 & -2 & 3 & 1 \\ 21 & 6 & 5 & -2 & 3 \end{pmatrix}$$

```
Det[A]
```

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```
A1 = Transpose[A]
```

$$\begin{pmatrix} 2 & 3 & 1 & 10 & 21 \\ 1 & -1 & 8 & -3 & 6 \\ 0 & 0 & 4 & -2 & 5 \\ 0 & 0 & 0 & 3 & -2 \\ 0 & 0 & 7 & 1 & 3 \end{pmatrix}$$

```
A1[[2]] = -1 / 2 A1[[1]] + A1[[2]];
A1[[5]] = -7 / 4 A1[[3]] + A1[[5]];
```

```
A2 = A1
```

$$\begin{pmatrix} 2 & 3 & 1 & 10 & 21 \\ 0 & -\frac{5}{2} & \frac{15}{2} & -8 & -\frac{9}{2} \\ 0 & 0 & 4 & -2 & 5 \\ 0 & 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & \frac{9}{2} & -\frac{23}{4} \end{pmatrix}$$

```
A2[[5]] = -3 / 2 A2[[4]] + A2[[5]];
```

```
A2
```

$$\begin{pmatrix} 2 & 3 & 1 & 10 & 21 \\ 0 & -\frac{5}{2} & \frac{15}{2} & -8 & -\frac{9}{2} \\ 0 & 0 & 4 & -2 & 5 \\ 0 & 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & -\frac{11}{4} \end{pmatrix}$$

```
Eigenvalues[{{2, 3, 1, 10, 21}, {0, -5/2, 15/2, -8, -9/2},
  {0, 0, 4, -2, 5}, {0, 0, 0, 3, -2}, {0, 0, 0, 0, -11/4}}]
```

$\left\{4, 3, -\frac{11}{4}, -\frac{5}{2}, 2\right\}$

```
Times @@ {4, 3, -11/4, -5/2, 2}
```

165

```
Det[A2]
```

165

```
Det[{{2, 1, 0, 0, 0}, {3, -1, 0, 0, 0},
  {1, 8, 4, 0, 7}, {10, -3, -2, 3, 1}, {21, 6, 5, -2, 3}}]
```

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(6)

$$A = \begin{pmatrix} 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 3 & 0 & 2 \\ 0 & 0 & 2 & 4 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \end{pmatrix};$$

$$\text{det1} = \text{Det}\left[\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\right]$$

-5

有几个0时，直接对角线法则计算也方便。

$$\text{det2} = \text{Det}\left[\begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \\ 2 & 4 & 0 \end{pmatrix}\right];$$

```
det1 det2
```

-60

```
Det[A]
```

-60

补充题

例

例7

```
A =  2  0  3  5
     1  2  0  3
     3  1  2  1
    -1  1 -1  1
```

$$\begin{pmatrix} 2 & 0 & 3 & 5 \\ 1 & 2 & 0 & 3 \\ 3 & 1 & 2 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

```
Det[{{2, 0, 3, 5}, {1, 2, 0, 3}, {3, 1, 2, 1}, {-1, 1, -1, 1}}]
```

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```
A1 = 互换函数Signed[A, {1, 2}]
```

$$\begin{pmatrix} 1 & 2 & 0 & 3 \\ -2 & 0 & -3 & -5 \\ 3 & 1 & 2 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

```
A1[[2]] = 2 A1[[1]] + A1[[2]];
A1[[3]] = -3 A1[[1]] + A1[[3]];
A1[[4]] = 1 A1[[1]] + A1[[4]];
```

```
A2 = A1
```

$$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 4 & -3 & 1 \\ 0 & -5 & 2 & -8 \\ 0 & 3 & -1 & 4 \end{pmatrix}$$

```
A2[[3]] = 5 / 4 A1[[2]] + A1[[3]]; A2[[4]] = -3 / 4 A1[[2]] + A1[[4]];
```

```
A3 = A2
```

$$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 4 & -3 & 1 \\ 0 & 0 & -\frac{7}{4} & -\frac{27}{4} \\ 0 & 0 & \frac{5}{4} & \frac{13}{4} \end{pmatrix}$$

```
A3[[4]] = 5 / 7 A3[[3]] + A3[[4]]
```

$$\left\{0, 0, 0, -\frac{11}{7}\right\}$$

```
A3
```

$$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 4 & -3 & 1 \\ 0 & 0 & -\frac{7}{4} & -\frac{27}{4} \\ 0 & 0 & 0 & -\frac{11}{7} \end{pmatrix}$$

```
Det[{{1, 2, 0, 3}, {0, 4, -3, 1}, {0, 0, -7/4, -27/4}, {0, 0, 0, -11/7}}]
```

11

例8 计算行列式

$$\mathrm{Det}\left[\begin{matrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{matrix}\right]$$

```
A = {{a, b, b, b}, {b, a, b, b}, {b, b, a, b}, {b, b, b, a}}
```

$$\begin{pmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{pmatrix}$$

```
A[[2]] = -1 A[[1]] + A[[2]];
A[[3]] = -1 A[[1]] + A[[3]];
A[[4]] = -1 A[[1]] + A[[4]];
```

```
A1 = A
```

$$\begin{pmatrix} a & b & b & b \\ b-a & a-b & 0 & 0 \\ b-a & 0 & a-b & 0 \\ b-a & 0 & 0 & a-b \end{pmatrix}$$

```
A2 = (a - b)^3.(det = MapAt[# / (a - b) &, A, {{2}, {3}, {4}}] // Simplify)
```

$$(a-b)^3.\begin{pmatrix} a & b & b & b \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

```
det1 = Transpose@det
```

$$\begin{pmatrix} a & -1 & -1 & -1 \\ b & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ b & 0 & 0 & 1 \end{pmatrix}$$

第2行、第3行、第4行全加到第一行去

```
det1[[1]] = det1[[2]] + det1[[1]];
det1[[1]] = det1[[3]] + det1[[1]];
det1[[1]] = det1[[4]] + det1[[1]];
```

```
det1
```

$$\begin{pmatrix} a+3b & 0 & 0 & 0 \\ b & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ b & 0 & 0 & 1 \end{pmatrix}$$

```
(a - b)^3.Det[det1]
```

$(a-b)^3.(a+3b)$

```
det
```

$$\begin{pmatrix} a & b & b & b \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

```
detNew = 互换函数Signed[det, {1, 4}]
```

$$\begin{pmatrix} -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -a & -b & -b & -b \end{pmatrix}$$

按第一行展开计算，子式也是第二列、第三列全加到第一列中来计算方便。

$$\left\|\begin{pmatrix} \mathbf{-1} & \mathbf{1} & \mathbf{0} \\ \mathbf{-1} & \mathbf{0} & \mathbf{1} \\ \boldsymbol{-a} & \boldsymbol{-b} & \boldsymbol{-b} \end{pmatrix}\right\|$$

$-a-2b$

$$\left\|\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \boldsymbol{-b} & \boldsymbol{-b} & \boldsymbol{-b} \end{pmatrix}\right\|$$

$-b$

```
detNew
```

$$\begin{pmatrix} -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -a & -b & -b & -b \end{pmatrix}$$

例9 证明 $D = \mathrm{Det}\begin{bmatrix} a+b & b+c & c+a \\ a_1+b_1 & b_1+c_1 & c_1+a_1 \\ a_2+b_2 & b_2+c_2 & c_2+a_2 \end{bmatrix} = 2\,\mathrm{Det}\begin{bmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$

$$A = \begin{pmatrix} a+b & b+c & a+c \\ a_1+b_1 & b_1+c_1 & a_1+c_1 \\ a_2+b_2 & b_2+c_2 & a_2+c_2 \end{pmatrix};$$

-1乘第三列加到第1列

```
A[[All, 1]] = -1 A[[All, 3]] + A[[All, 1]];
A1 = A
```

$$\begin{pmatrix} b-c & b+c & a+c \\ b_1-c_1 & b_1+c_1 & a_1+c_1 \\ b_2-c_2 & b_2+c_2 & a_2+c_2 \end{pmatrix}$$

第一列加到第二列

```
A1[[All, 2]] = A1[[All, 1]] + A1[[All, 2]];
A2 = A1
```

$$\begin{pmatrix} b-c & 2\,b & a+c \\ b_1-c_1 & 2\,b_1 & a_1+c_1 \\ b_2-c_2 & 2\,b_2 & a_2+c_2 \end{pmatrix}$$

提取公因子2后，第三列加到第1列

```
A2[[All, 1]] = A2[[All, 3]] + A2[[All, 1]];
A3 = A2
```

$$\begin{pmatrix} a+b & 2\,b & a+c \\ a_1+b_1 & 2\,b_1 & a_1+c_1 \\ a_2+b_2 & 2\,b_2 & a_2+c_2 \end{pmatrix}$$

$$\mathbf{A4} = \begin{pmatrix} \boldsymbol{a+b} & \boldsymbol{b} & \boldsymbol{a+c} \\ \boldsymbol{a_1+b_1} & \boldsymbol{b_1} & \boldsymbol{a_1+c_1} \\ \boldsymbol{a_2+b_2} & \boldsymbol{b_2} & \boldsymbol{a_2+c_2} \end{pmatrix}$$

$$\begin{pmatrix} a+b & b & a+c \\ a_1+b_1 & b_1 & a_1+c_1 \\ a_2+b_2 & b_2 & a_2+c_2 \end{pmatrix}$$

-1乘第二列加到第一列，然后-1乘第一列加到第三列，得证。

```
A4[[All, 1]] = -1 A4[[All, 2]] + A4[[All, 1]];
A4[[All, 3]] = -1 A4[[All, 1]] + A4[[All, 3]];
A4
```

$$\begin{pmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$$

例10 计算行列式

$$\text{Det}\left[\begin{matrix} a & b & c & d \\ a & a+b & a+b+c & a+b+c+d \\ a & 2\,a+b & 3\,a+2\,b+c & 4\,a+3\,b+2\,c+d \\ a & 3\,a+b & 6\,a+3\,b+c & 10\,a+6\,b+3\,c+d \end{matrix}\right]$$

高斯消去法，轻松。

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} & \boldsymbol{d} \\ \boldsymbol{a} & \boldsymbol{a+b} & \boldsymbol{a+b+c} & \boldsymbol{a+b+c+d} \\ \boldsymbol{a} & \boldsymbol{2\,a+b} & \boldsymbol{3\,a+2\,b+c} & \boldsymbol{4\,a+3\,b+2\,c+d} \\ \boldsymbol{a} & \boldsymbol{3\,a+b} & \boldsymbol{6\,a+3\,b+c} & \boldsymbol{10\,a+6\,b+3\,c+d} \end{pmatrix};$$

```
A1 = {A[[1]], A[[2]] - A[[1]], A[[3]] - A[[1]], A[[4]] - A[[1]]}
```

$$\begin{pmatrix} a & b & c & d \\ 0 & a & a+b & a+b+c \\ 0 & 2a & 3a+2b & 4a+3b+2c \\ 0 & 3a & 6a+3b & 10a+6b+3c \end{pmatrix}$$

```
A2 = {A1[[1]], A1[[2]], A1[[3]] - 2 A1[[2]], A1[[4]] - 3 A1[[2]]} // Simplify
```

$$\begin{pmatrix} a & b & c & d \\ 0 & a & a+b & a+b+c \\ 0 & 0 & a & 2a+b \\ 0 & 0 & 3a & 7a+3b \end{pmatrix}$$

```
A3 = {A2[[1]], A2[[2]], A2[[3]], A2[[4]] - 3 A2[[3]]} // Simplify
```

$$\begin{pmatrix} a & b & c & d \\ 0 & a & a+b & a+b+c \\ 0 & 0 & a & 2a+b \\ 0 & 0 & 0 & a \end{pmatrix}$$

例11 计算n阶行列式

$$\mathrm{Det}\left[\begin{array}{ccccc} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 0 & \dots & 0 \\ 1 & 0 & 3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & n \end{array}\right]$$

箭形行列式

$$2\int_0^{\infty} e^{-x}\cos\lambda x\,dx = 2\sum_{k=0}^{\infty}\int_0^{\infty} e^{-x}(-1)^k\frac{(\lambda x)^{2k}}{(2k)!}\,dx$$

$$= 2\sum_{k=0}^{\infty}\frac{\lambda^{2k}(-1)^k}{(2k)!}\int_0^{\infty} e^{-x}x^{2k}\,dx = 2\sum_{k=0}^{\infty}\frac{\lambda^{2k}(-1)^k}{(2k)!}\Gamma(2k+1)$$

$$= 2\left(1+\left(-\lambda^2\right)+\lambda^4+\ldots\right) = \frac{2}{1+\lambda^2}$$

但要使得 $\left|\lambda^2\right| < 1$

或

$$2\int_0^{\infty} e^{-x}\cos\lambda x\,dx = \sum_{k=0}^{\infty}\frac{\left(i\lambda\right)^k}{k!}\int_0^{\infty} e^{-x}x^k\,dx + \sum_{k=0}^{\infty}\frac{\left(-i\lambda\right)^k}{k!}\int_0^{\infty} e^{-x}x^k\,dx$$

$$= \sum_{k=0}^{\infty}\left(i\lambda\right)^k + \sum_{k=0}^{\infty}\left(-i\lambda\right)^k = \frac{1}{1-i\lambda}+\frac{1}{1+i\lambda} = \frac{2}{1+\lambda^2}$$

1. 一动点M到A (3, 0) 的距离恒等于它到点B (-6, 0) 的距离的一半，求此动点M的轨迹方程，并作图。

$$\mathtt{MA} == \frac{1}{2}\mathtt{MB} == \sqrt{(x-3)^2+(y-0)^2} == \frac{1}{2}\sqrt{(x-(-6))^2+(y-0)^2} \;\texttt{// TraditionalForm}$$

$$\mathrm{MA} = \frac{\mathrm{MB}}{2} = \sqrt{(x-3)^2+y^2} = \frac{1}{2}\sqrt{(x+6)^2+y^2}$$

$$\mathtt{MA} = \sqrt{(x-3)^2+(y-0)^2};$$

```
MA^2
```

$$(-3+x)^2+y^2$$

$$\mathtt{a = Expand}\left[(-3+x)^2+y^2\right]$$

$$9-6x+x^2+y^2$$

$$\mathtt{b = Expand}\left[\left(\frac{1}{2}\sqrt{(x-(-6))^2+(y-0)^2}\right)^{\wedge}2\right]$$

$$9+3x+\frac{x^2}{4}+\frac{y^2}{4}$$

```
c = a - b
```

$$-9x+\frac{3x^2}{4}+\frac{3y^2}{4}$$

```
Simplify[c]
```

$$\frac{3}{4}\left(-12x+x^2+y^2\right)$$

Completing the Square

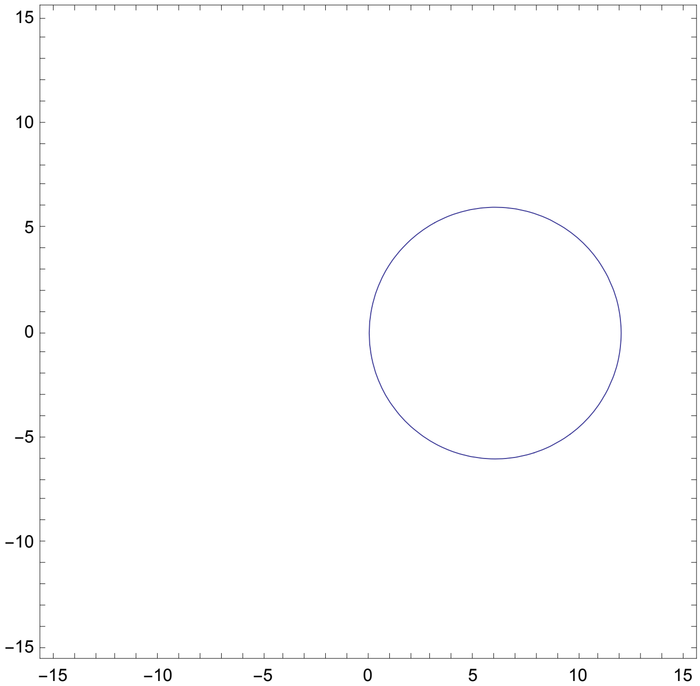
```
CompleteSquare[f_,x_]:=Module[{a,b,c},
{c,b,a}=CoefficientList[f,x];
a(x+b/2/a)^2+Simplify[(c-b^2/4/a)]
]
```

```
CompleteSquare[c, x]
```

$$\frac{3}{4}(-6+x)^2+\frac{3}{4}\left(-36+y^2\right)$$

$$(x-6)^2+y^2 == 36$$

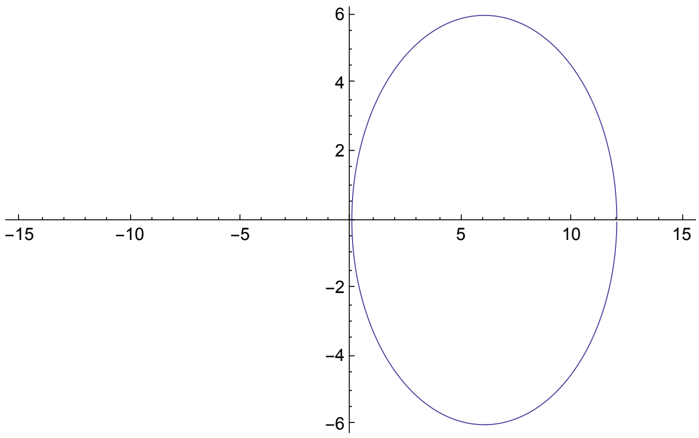
```
ContourPlot[(x - 6)^2 + y^2 == 36, {x, -15, 15}, {y, -15, 15}]
```




```
sols = Solve[(x - 6)^2 + y^2 == 36, {y}]
```

$\left\{\left\{y \to -\sqrt{12\,x - x^2}\right\}, \left\{y \to \sqrt{12\,x - x^2}\right\}\right\}$

```
Plot[y /. sols, {x, -15, 15}]
```



2. 有一长度为2a (a > 0) 的线段，它的两端点分别在x轴正半轴与y正半轴上移动，试求此线段中点的轨迹.

令A（x，0），B（0，y）为其两端点，其中点坐标为 （x / 2，y / 2），

$d = 2\,a$，$x^2 + y^2 = 4\,a^2$，$\frac{x^2}{4} + \frac{y^2}{4} = a^2$，

即 $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = a^2$，（$x \geq 0$，$y \geq 0$ 条件容易漏）

别解：直角三角形中，斜边中线长是斜边一半，$\sqrt{\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2} = \frac{1}{2}\sqrt{x^2 + y^2} = a$

3. 卡西尼卵形线轨迹方程.一动点到两定点（距离为2a）的距离乘积等于定值m^2.

设定点A（a1, b1）, B（c1, d1）.动点C（x, y）

$\sqrt{(x-a1)^2+(y-b1)^2}\times\sqrt{(x-c1)^2+(y-d1)^2}=m^2$,

$2a=\sqrt{(c1-a1)^2+(d1-b1)^2}$

令 $x-a1=\mathrm{sin}t$, $y-b1=\mathrm{cos}t$, $(\mathrm{sin}t+a1-c1)^2+(\mathrm{cos}t+b1-d1)^2=m^4$

Expand[(sint + a1 - c1)^2 + (cost + b1 - d1)^2]

$a1^2+b1^2-2\,a1\,c1+c1^2+2\,b1\,\mathrm{cost}+\mathrm{cost}^2-$
$2\,b1\,d1-2\,\mathrm{cost}\,d1+d1^2+2\,a1\,\mathrm{sint}-2\,c1\,\mathrm{sint}+\mathrm{sint}^2$

Expand[(c1 - a1)^2 + (d1 - b1)^2]

$a1^2+b1^2-2\,a1\,c1+c1^2-2\,b1\,d1+d1^2$

不知如何继续导出结果

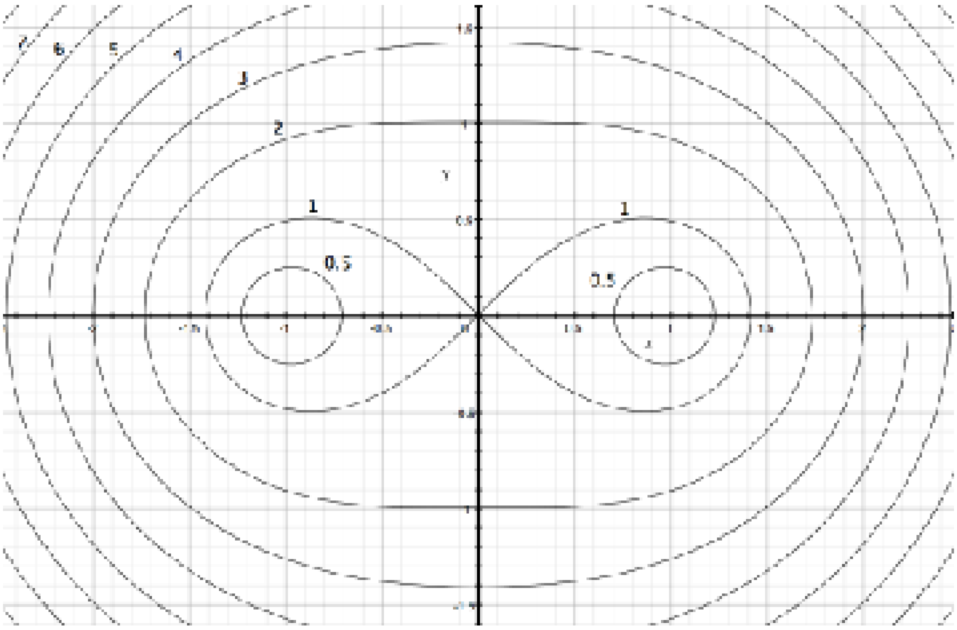
故参看解答，发现选取一合适的坐标非常有效，

取两定点的连线为x轴，两定点所连线段的中垂线为y轴，则其方程为

$(x^2+y^2)^2-2a^2(x^2-y^2)=m^4-a^4$

极坐标$r^4-2a^2r^2\cos 2\theta=m^4-a^4$

卵形线的形状与比值m/a有关。如果m/a大于1，则轨迹是一条闭曲线。如果m/a小于1，则轨迹是两条不相连的闭曲线。如果m/a等于1，则是伯努利双扭线。



4. 设P，Q, R是等轴双曲线上任意三点，求证ΔPQR的垂心H必在同一等轴双曲线上.

貌似一下子有点难

5. 过定点M_0（x_0, y_0）的直线与非零向量V =
{X, Y} 共线，试证直线l的向量式参数方程为 $r = \boldsymbol{r_0} + t\boldsymbol{V}$ $(-\infty < t < +\infty)$
其中$r_0 = \overrightarrow{OM_0}$, t为参数；坐标式参数方程为

$$\begin{cases} x = x_0 + Xt \\ y = y_0 + Yt \end{cases}$$

对称式方程为

$$\frac{x - x_0}{X} = \frac{x - y_0}{Y}$$

6. 旋轮线

$\begin{cases} x = t - \sin t \\ y = 1 - \cot \end{cases}$ $(0 \le t \le 2\pi)$ **的弧与直线**$y = \frac{3}{2}$ **的交点.**

$x = t - \sin t,\ 1 - \cos t = 3/2,\ \cos t = -1/2,\ t = \pi - \pi/3,\ t = \pi + \pi/3$

Sin[π - π / 3]

$$\frac{\sqrt{3}}{2}$$

Sin[π + π / 3]

$$-\frac{\sqrt{3}}{2}$$

$$x = \frac{2\pi}{3} - \frac{\sqrt{3}}{2},\ x = \frac{4\pi}{3} + \frac{\sqrt{3}}{2}$$

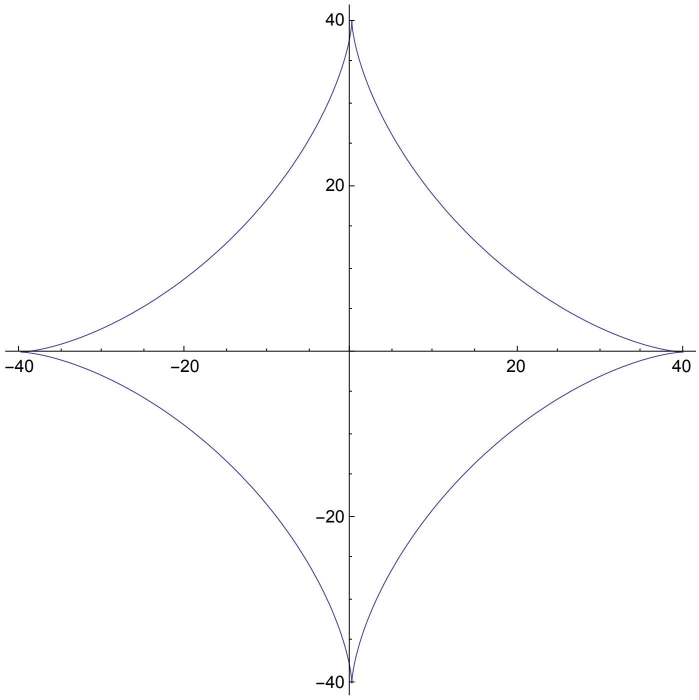
7. 化参数方程为普通方程.

(1) $\begin{cases} x = at^2 \\ y = 2at \end{cases}$ $(-\infty < t < +\infty)$； $y^2 = 4a^2t^2 = 4ax$

(2) $\begin{cases} x = \sin t + 5 \\ y = -2\cos t - 1 \end{cases}$ $(0 <= t < 2\pi)$； $(x-5)^2 - \left(\frac{y+1}{2}\right)^2 = 1$

(3) $\begin{cases} x = r(3\cos t + \cos 3t) \\ y = r(3\sin t - \sin 3t) \end{cases}$ $(0 < t < 2\pi)$； $x^2 + y^2 = r^2$ ()

```
ParametricPlot[{10 (3 Cos[t] + Cos[3 t]), 10 (3 Sin[t] - Sin[3 t])}, {t, 0, 2 π}]
```




```
x = r (3 cost + cos3t)
```

(cos3t + 3 cost) r

```
y = r (3 sint - sin3t)
```

r (- sin3t + 3 sint)

```
Expand[x^2 + y^2] =
 cos3t^2 r^2 + 6 cos3t cost r^2 + 9 cost^2 r^2 + r^2 sin3t^2 - 6 r^2 sin3t sint + 9 r^2 sint^2 =
  r^2 (10 + 6 cos3t cost - 6 sin3t sint) = r^2 (10 + 6 cos4t)
```

```
Collect[cos3t^2 r^2 + 6 cos3t cost r^2 + 9 cost^2 r^2 + r^2 sin3t^2 - 6 r^2 sin3t sint + 9 r^2 sint^2, r]
```

$r^2 (cos3t^2 + 6 cos3t cost + 9 cost^2 + sin3t^2 - 6 sin3t sint + 9 sint^2)$

```
Expand[x^2 - y^2] =
 cos3t^2 r^2 + 6 cos3t cost r^2 + 9 cost^2 r^2 - r^2 sin3t^2 + 6 r^2 sin3t sint - 9 r^2 sint^2
```

```
Collect[cos3t^2 r^2 + 6 cos3t cost r^2 + 9 cost^2 r^2 - r^2 sin3t^2 + 6 r^2 sin3t sint - 9 r^2 sint^2, r]
```

$r^2 (cos3t^2 + 6 cos3t cost + 9 cost^2 - sin3t^2 + 6 sin3t sint - 9 sint^2)$

```
Expand[x y]
```

$- cos3t\ r^2 sin3t - 3 cost\ r^2 sin3t + 3 cos3t\ r^2 sint + 9 cost\ r^2 sint$

```
Collect[-cos3t r^2 sin3t - 3 cost r^2 sin3t + 3 cos3t r^2 sint + 9 cost r^2 sint, r]
```

$r^2 (- cos3t\ sin3t - 3 cost\ sin3t + 3 cos3t\ sint + 9 cost\ sint)$

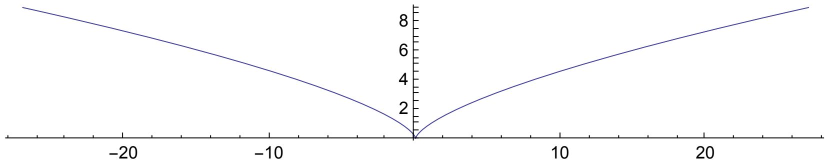
看来不简单，注$cost^2$，$sint^2$即为$cos^2 t$，$sin^2 t$，以下类同

8. 普通方程化为参数方程

（1）$y^2 = x^3$；

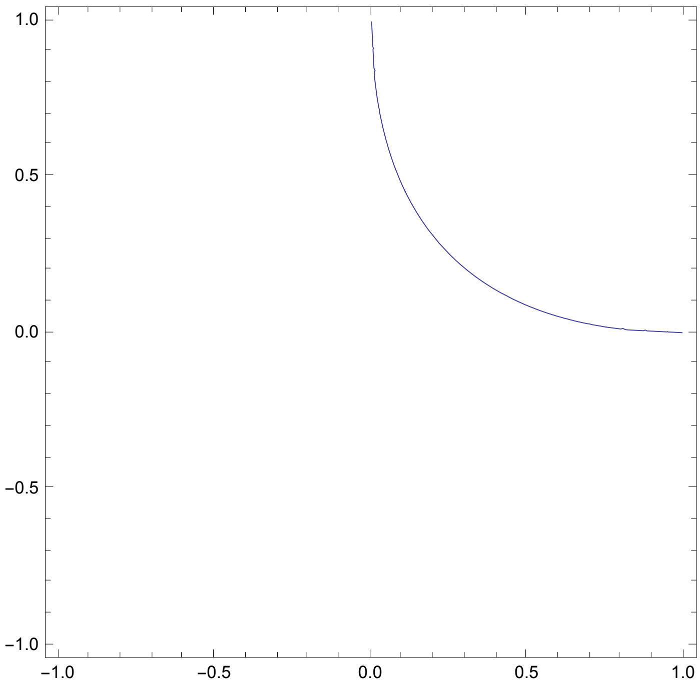
令$y = t^3$，$x = t^2$，两边取对数：$2\,\mathrm{Log}[y] = 6\,\mathrm{Log}[t] = 3\,\mathrm{Log}[x] = 6\,\mathrm{Log}[t]$，$\frac{\mathrm{Log}[y]}{\mathrm{Log}[x]} = \frac{3}{2} = \mathrm{Log}[x, y]$ ... 看答案突然发现一开始便已给出，其后不知在做什么。

```
ParametricPlot[{t^3, t^2}, {t, -3, 3}]
```

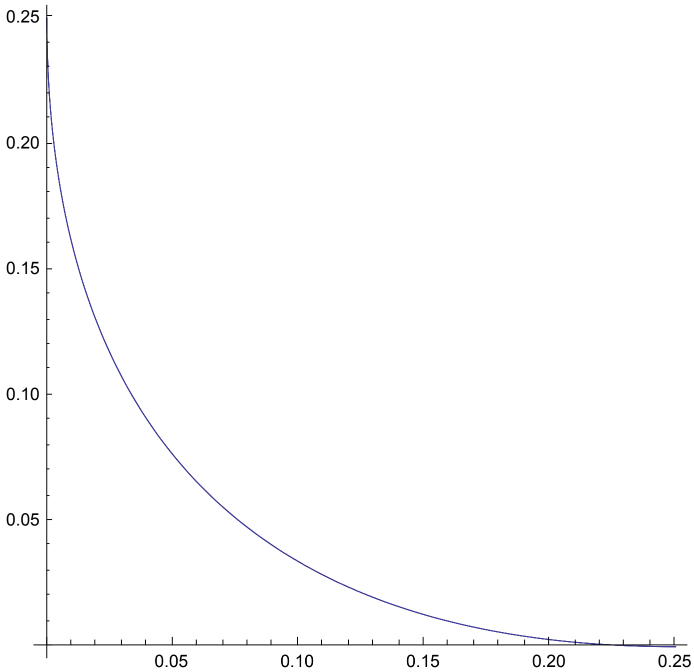


(2) $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ $(a > 0)$；令 $\begin{cases} x = \frac{a}{4}\sin^4 t \\ y = \frac{a}{4}\cos^4 t \end{cases}$ $(0 < t < 2\pi)$；

```
ContourPlot[x^(1/2) + y^(1/2) == 1, {x, -1, 1}, {y, -1, 1}]
```



ParametricPlot$\left[\left\{\frac{1}{4}(\mathrm{Sin}[t])^4, \frac{1}{4}(\mathrm{Cos}[t])^4\right\}, \{t, 0, 2\pi\}\right]$



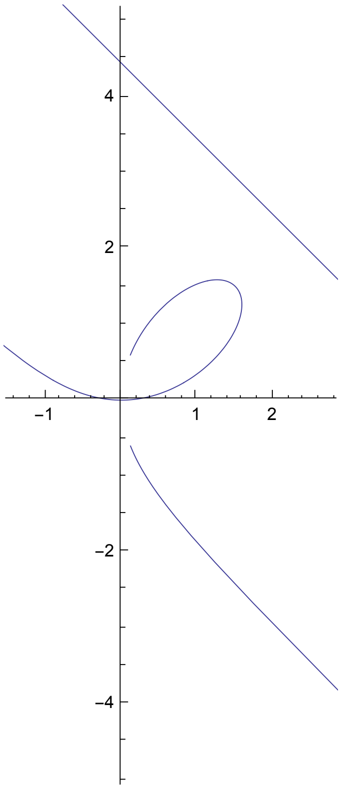
(3) $x^3 + y^3 - 3axy = 0$ ($a > 0$)；参看解答，设$y = tx$.

$$x^3 + t^3 x^3 - 3ax \cdot tx = 0,\ x^2 \left(x + xt^3 - 3at\right) = 0,\ x = \frac{3at}{1+t^3},\ y = \frac{3at^2}{1+t^3}$$

如果不设$y = tx$又如何得出结果呢？因为这上步和最后结果是一样的。

```
ParametricPlot[{3 t/(1 + t^3), 3 t^2/(1 + t^3)}, {t, -5, 5}]
```

$$\text{ParametricPlot}\left[\left\{\frac{3\,t}{1+t^3}, \frac{3\,t^2}{1+t^3}\right\}, \{t, -5, 5\}\right]$$



9. 外摆线的参数方程

此题可参照例4，内摆的就方法,

10. 箕舌线的方程

模糊数学作业第三章节部分习题

模糊识别： 设A1=“不热”， A2=“不冷”， U=[0,∞], x 表示温度（单位：°C）, A1和A2相应的隶属函数如下，问世A3=A1∩A2表示“暖和”. 试问： x=20°C时气温属于哪种状态?

```
f = (Partition[Cases[ToExpression[Characters[ToString[#1]]], Except[Null]],
      #2]) / 10 /. a → 10 & // N;
```

```
R1 = f[5 421 264 519 a7, 4];
R2 = f[411 659 756 876, 4];
R3 = f[6578 a923, 2];
R4 = f[73 645 546, 2];
R5 = f[237 648, 3];
R6 = f[269 321, 3];
```

```
IntegerDigits[Range[6]]
```

```
{{1}, {2}, {3}, {4}, {5}, {6}}
```

```
Table[StringJoin["R", ToString[k]], {k, 6}] // ToExpression
```

```
{R1, R2, R3, R4, R5, R6}
```

```
v = ToExpression["R" <> ToString[#]] & /@ Range[6]
```

```
{R1, R2, R3, R4, R5, R6}
```

```
ToExpression /@ StringJoin /@ Thread[Append[{"R"}, ToString /@ Range[6]]]
```

```
{{{0.5, 0.4, 0.2, 0.1}, {0.2, 0.6, 0.4, 0.5}, {0.1, 0.9, 1., 0.7}},
 {{0.4, 0.1, 0.1, 0.6}, {0.5, 0.9, 0.7, 0.5}, {0.6, 0.8, 0.7, 0.6}},
 {{0.6, 0.5}, {0.7, 0.8}, {1., 0.9}, {0.2, 0.3}},
 {{0.7, 0.3}, {0.6, 0.4}, {0.5, 0.5}, {0.4, 0.6}},
 {{0.2, 0.3, 0.7}, {0.6, 0.4, 0.8}}, {{0.2, 0.6, 0.9}, {0.3, 0.2, 0.1}}}
```

```
R12 = Table[Max[R1[[i, j]], R2[[i, j]]], {i, 3}, {j, 4}];
% // MatrixForm
```

$$\begin{pmatrix} 0.5 & 0.4 & 0.2 & 0.6 \\ 0.5 & 0.9 & 0.7 & 0.5 \\ 0.6 & 0.9 & 1. & 0.7 \end{pmatrix}$$

```
{R1 // MatrixForm, R2 // MatrixForm}
```

$$\left\{\begin{pmatrix} 0.5 & 0.4 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.4 & 0.5 \\ 0.1 & 0.9 & 1. & 0.7 \end{pmatrix}, \begin{pmatrix} 0.4 & 0.1 & 0.1 & 0.6 \\ 0.5 & 0.9 & 0.7 & 0.5 \\ 0.6 & 0.8 & 0.7 & 0.6 \end{pmatrix}\right\}$$

```
R1 ⋃ R2 = R12
```

```
R3 // MatrixForm
R5 // MatrixForm
```

$$\begin{pmatrix} 0.6 & 0.5 \\ 0.7 & 0.8 \\ 1. & 0.9 \\ 0.2 & 0.3 \end{pmatrix}$$

$$\begin{pmatrix} 0.2 & 0.3 & 0.7 \\ 0.6 & 0.4 & 0.8 \end{pmatrix}$$

```
4 * 2 <> 2 * 3
```

```
str2mat[.5 × .4 × .6 ×
   .6 × .4 × .8 ×
   .6 × .4 × .8 ×
   .3 × .3 × .3
  , 3] // MatrixForm
```

$$\begin{pmatrix} 0.5 & 0.4 & 0.6 \\ 0.6 & 0.4 & 0.8 \\ 0.6 & 0.4 & 0.8 \\ 0.3 & 0.3 & 0.3 \end{pmatrix}$$

```
Table[Min[R3[[m, k]], R5[[k, n]]], {m, 4}, {n, 3}, {k, 2}] // MatrixForm
```

$$\begin{pmatrix} \begin{pmatrix} 0.2 \\ 0.5 \end{pmatrix} & \begin{pmatrix} 0.3 \\ 0.4 \end{pmatrix} & \begin{pmatrix} 0.6 \\ 0.5 \end{pmatrix} \\ \begin{pmatrix} 0.2 \\ 0.6 \end{pmatrix} & \begin{pmatrix} 0.3 \\ 0.4 \end{pmatrix} & \begin{pmatrix} 0.7 \\ 0.8 \end{pmatrix} \\ \begin{pmatrix} 0.2 \\ 0.6 \end{pmatrix} & \begin{pmatrix} 0.3 \\ 0.4 \end{pmatrix} & \begin{pmatrix} 0.7 \\ 0.8 \end{pmatrix} \\ \begin{pmatrix} 0.2 \\ 0.3 \end{pmatrix} & \begin{pmatrix} 0.2 \\ 0.3 \end{pmatrix} & \begin{pmatrix} 0.2 \\ 0.3 \end{pmatrix} \end{pmatrix}$$

```
dot1 = Max /@ Table[Min[R3[[m, k]], R5[[k, n]]], {m, 4}, {n, 3}, {k, 2}][[#]] &;
```

```
Table[dot1[k], {k, 4}] // MatrixForm
```

$$\begin{pmatrix} 0.5 & 0.4 & 0.6 \\ 0.6 & 0.4 & 0.8 \\ 0.6 & 0.4 & 0.8 \\ 0.3 & 0.3 & 0.3 \end{pmatrix}$$

```
Table[Min[R3[[m, k]], R5[[k, n]]], {m, 4}, {n, 3}, {k, 2}]
```

```
{{{0.2, 0.5}, {0.3, 0.4}, {0.6, 0.5}}, {{0.2, 0.6}, {0.3, 0.4}, {0.7, 0.8}},
 {{0.2, 0.6}, {0.3, 0.4}, {0.7, 0.8}}, {{0.2, 0.3}, {0.2, 0.3}, {0.2, 0.3}}}
```

```
dot = Table[Max /@ Table[Min[#1[[m, k]], #2[[k, n]]],
       {m, Dimensions[#1][[1]]}, {n, Dimensions[#2][[2]]},
       {k, Dimensions[#1][[2]]}][[i]], {i, 1, Dimensions[#1][[1]]}] &;
```

```
R35 = dot[R3, R5];
% // MatrixForm
```

$$\begin{pmatrix} 0.5 & 0.4 & 0.6 \\ 0.6 & 0.4 & 0.8 \\ 0.6 & 0.4 & 0.8 \\ 0.3 & 0.3 & 0.3 \end{pmatrix}$$

R3∘R5 = R35

计算 **R12∘R35**

```
R12 // MatrixForm
```

$$\begin{pmatrix} 0.5 & 0.4 & 0.2 & 0.6 \\ 0.5 & 0.9 & 0.7 & 0.5 \\ 0.6 & 0.9 & 1. & 0.7 \end{pmatrix}$$

```
dot[R12, R35] // MatrixForm
```

$$\begin{pmatrix} 0.5 & 0.4 & 0.5 \\ 0.6 & 0.4 & 0.8 \\ 0.6 & 0.4 & 0.8 \end{pmatrix}$$

```
R1C = 1 - R1;
% // MatrixForm
```

$$\begin{pmatrix} 0.5 & 0.6 & 0.8 & 0.9 \\ 0.8 & 0.4 & 0.6 & 0.5 \\ 0.9 & 0.1 & 0. & 0.3 \end{pmatrix}$$

R3⋂R4 = R3i4

```
R3i4 = Table[Min[R3[[i, j]], R4[[i, j]]], {i, 4}, {j, 2}];
% // MatrixForm
```

$$\begin{pmatrix} 0.6 & 0.3 \\ 0.6 & 0.4 \\ 0.5 & 0.5 \\ 0.2 & 0.3 \end{pmatrix}$$

(R1C∘(R3⋂R4))∘R6

```
dot[dot[R1C, R3i4], R6]
```

```
{{0.3, 0.6, 0.6}, {0.3, 0.6, 0.6}, {0.3, 0.6, 0.6}}
```

```
Fold[dot, R1C, {R3i4, R6}]
```

```
{{0.3, 0.6, 0.6}, {0.3, 0.6, 0.6}, {0.3, 0.6, 0.6}}
```

模糊关系的合成
在中医诊断中存在模糊关系：
建立{寒,热,虚,实}等症状与{肺,心}之间的模糊关系 R1∘R2

```
a = {寒, 热, 虚, 实}; b = {肺, 心}; c = {自汗, 恶寒, 咳嗽, 喘};
```

```
R1 = f[3 337 333 383 884 220, 4];
Append[Table[Append[R1[[k]], a[[k]]], {k, 1, 4}], c] // TableForm
```

0.3	0.3	0.3	0.7	寒
0.3	0.3	0.3	0.3	热
0.8	0.3	0.8	0.8	虚
0.4	0.2	0.2	0.	实
自汗	恶寒	咳嗽	喘	

```
TableForm[R1, TableHeadings → {a, c}]
```

	自汗	恶寒	咳嗽	喘
寒	0.3	0.3	0.3	0.7
热	0.3	0.3	0.3	0.3
虚	0.8	0.3	0.8	0.8
实	0.4	0.2	0.2	0.

```
R2 = f[53 338 263, 2];
Append[Table[Append[R2[[k]], a[[k]]], {k, 1, 4}], b] // TableForm
```

0.5	0.3	寒
0.3	0.3	热
0.8	0.2	虚
0.6	0.3	实
肺	心	

```
TableForm[R2, TableHeadings → {c, b}]
```

	肺	心
自汗	0.5	0.3
恶寒	0.3	0.3
咳嗽	0.8	0.2
喘	0.6	0.3

```
R = dot[R1, R2];
Append[Table[Append[R[[k]], a[[k]]], {k, 1, 4}], b] // TableForm
```

0.6	0.3	寒
0.3	0.3	热
0.8	0.3	虚
0.4	0.3	实
肺	心	

```
TableForm[R, TableHeadings → {a, b}]
```

	肺	心
寒	0.6	0.3
热	0.3	0.3
虚	0.8	0.3
实	0.4	0.3

已知模糊相似关系R $\in \mathcal{F}$ (X*X), X = {x1, x2, ⋯, x7}, 求R的传递闭包，并作聚类分析。

```
R = f[a9759a49a5454175a5714545a5799575a71a4177a5414915a, 7];
```

```
% // MatrixForm
```

$$\begin{pmatrix} 1. & 0.9 & 0.7 & 0.5 & 0.9 & 1. & 0.4 \\ 0.9 & 1. & 0.5 & 0.4 & 0.5 & 0.4 & 0.1 \\ 0.7 & 0.5 & 1. & 0.5 & 0.7 & 0.1 & 0.4 \\ 0.5 & 0.4 & 0.5 & 1. & 0.5 & 0.7 & 0.9 \\ 0.9 & 0.5 & 0.7 & 0.5 & 1. & 0.7 & 0.1 \\ 1. & 0.4 & 0.1 & 0.7 & 0.7 & 1. & 0.5 \\ 0.4 & 0.1 & 0.4 & 0.9 & 0.1 & 0.5 & 1. \end{pmatrix}$$

```
R2 = dot[R, R]; R4 = dot[R2, R2]; R8 = dot[R4, R4]; R16 = dot[R8, R8];
{"R²", R2, "", "R⁴", R4, "" , "R⁸", R8, "", "R¹⁶", R16} // TableForm
```

R^2

1.	0.9	0.7	0.7	0.9	1.	0.5
0.9	1.	0.7	0.5	0.9	0.9	0.4
0.7	0.7	1.	0.5	0.7	0.7	0.5
0.7	0.5	0.5	1.	0.7	0.7	0.9
0.9	0.9	0.7	0.7	1.	0.9	0.5
1.	0.9	0.7	0.7	0.9	1.	0.7
0.5	0.4	0.5	0.9	0.5	0.7	1.

R^4

1.	0.9	0.7	0.7	0.9	1.	0.7
0.9	1.	0.7	0.7	0.9	0.9	0.7
0.7	0.7	1.	0.7	0.7	0.7	0.7
0.7	0.7	0.7	1.	0.7	0.7	0.9
0.9	0.9	0.7	0.7	1.	0.9	0.7
1.	0.9	0.7	0.7	0.9	1.	0.7
0.7	0.7	0.7	0.9	0.7	0.7	1.

R^8

1.	0.9	0.7	0.7	0.9	1.	0.7
0.9	1.	0.7	0.7	0.9	0.9	0.7
0.7	0.7	1.	0.7	0.7	0.7	0.7
0.7	0.7	0.7	1.	0.7	0.7	0.9
0.9	0.9	0.7	0.7	1.	0.9	0.7
1.	0.9	0.7	0.7	0.9	1.	0.7
0.7	0.7	0.7	0.9	0.7	0.7	1.

R^{16}

1.	0.9	0.7	0.7	0.9	1.	0.7
0.9	1.	0.7	0.7	0.9	0.9	0.7
0.7	0.7	1.	0.7	0.7	0.7	0.7
0.7	0.7	0.7	1.	0.7	0.7	0.9
0.9	0.9	0.7	0.7	1.	0.9	0.7
1.	0.9	0.7	0.7	0.9	1.	0.7
0.7	0.7	0.7	0.9	0.7	0.7	1.

```
λ = { 1, 0.9, 0.7};
```

```
Grid[Table[If[R8[[i, j]] ≥ #, True, ""], {i, 7}, {j, 7}], Frame → All] & /@ λ //
 TableForm
```

True					True	
	True					
		True				
			True			
				True		
True					True	
						True

True	True			True	True	
True	True			True	True	
		True				
			True			True
True	True			True	True	
True	True			True	True	
			True			True

True	True	True	True	True	True	True
True	True	True	True	True	True	True
True	True	True	True	True	True	True
True	True	True	True	True	True	True
True	True	True	True	True	True	True
True	True	True	True	True	True	True
True	True	True	True	True	True	True

```
X = ToExpression["X" <> ToString[#]] & /@ Range[7];
Partition[X, 1];
```

现在开始分类:

$\lambda = 1$ 时, 分为6类 : {{X1, X6}, {X2}, {X3}, {X4}, {X5}, {X7}}

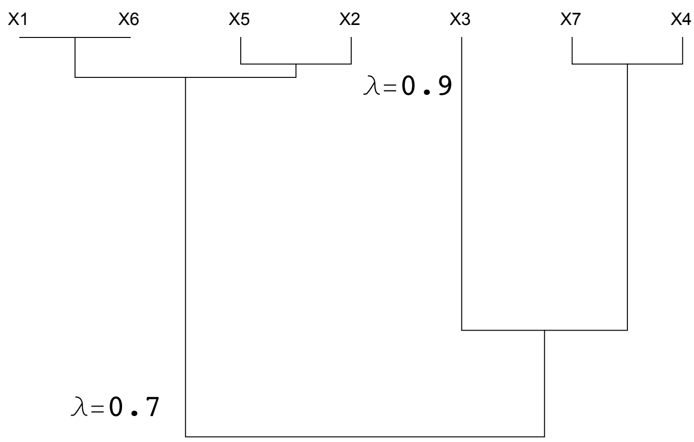
$\lambda = 0.9$ 时, 分为3类 : {{X1, X2, X5, X6}, {X3}, {X4, X7}}

$\lambda = 0.7$ 时, 分为1类: {X1, X2, X3, X4, X5, X6, X7}

画动态聚类图:

```
Needs["HierarchicalClustering`"]
XLabel = "X" <> ToString[#] & /@ Range[7];
```

```
p1 = DendrogramPlot[R8, LeafLabels → XLabel, Orientation → Bottom]
```



预报： 某地历史上虫害情况分为I(轻)， II(中)， III(重)3类，今年的测报资料为N， 其相似关系矩阵为R， 问今年的虫害情况如何?

```
R = str2mat[1 w w w 0.39 × 1 w w 0.16 × 0.55 × 1 w 0.59 × 0.41 × 0.26 × 1, 4];
rate = ToString /@ {I, II, III, N}; R = R /. w → □;
TableForm[R, TableHeadings → ("rate" * ConstantArray[1, 2] // ToExpression)]
```

	I	II	III	N
I	1	□	□	□
II	0.39	1	□	□
III	0.16	0.55	1	□
N	0.59	0.41	0.26	1

```
R = str2mat[1 × 0.39 × 0.16 × 0.59 × 0.39 × 1 ×
   0.55 × 0.41 × 0.16 × 0.55 × 1 × 0.26 × 0.59 × 0.41 × 0.26 × 1, 4];
```

```
% // MatrixForm
```

$$\begin{pmatrix} 1 & 0.39 & 0.16 & 0.59 \\ 0.39 & 1 & 0.55 & 0.41 \\ 0.16 & 0.55 & 1 & 0.26 \\ 0.59 & 0.41 & 0.26 & 1 \end{pmatrix}$$

```
R2 = dot[R, R]; R4 = dot[R2, R2]; R8 = dot[R4, R4];
{"R²", R2, "", "R⁴", R4, "" , "R⁸", R8} // TableForm
```

R^2

1	0.41	0.39	0.59
0.41	1	0.55	0.41
0.39	0.55	1	0.41
0.59	0.41	0.41	1

R^4

1	0.41	0.41	0.59
0.41	1	0.55	0.41
0.41	0.55	1	0.41
0.59	0.41	0.41	1

R^8

1	0.41	0.41	0.59
0.41	1	0.55	0.41
0.41	0.55	1	0.41
0.59	0.41	0.41	1

```
λ = { 1, 0.59, 0.55, 0.41};
```

```
Grid[Table[If[R4[[i, j]] ≥ #, True, ""], {i, 4}, {j, 4}], Frame → All] & /@ λ //
 TableForm
```

True			
	True		
		True	
			True

True			True
	True		
		True	
True			True

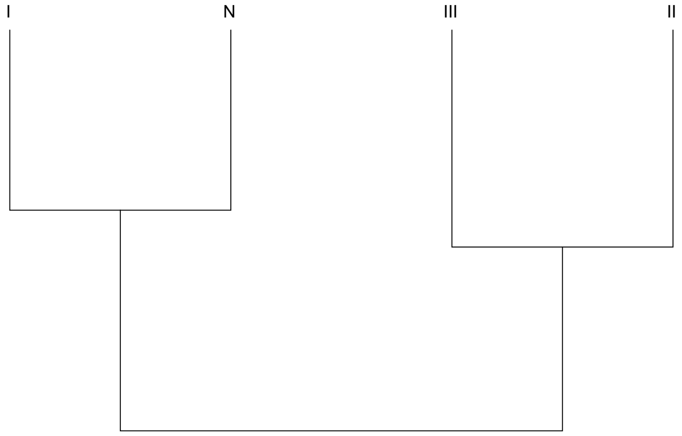
True			True
	True	True	
	True	True	
True			True

True	True	True	True
True	True	True	True
True	True	True	True
True	True	True	True

$\lambda = 0.59$ 时, 分为3类 : {{N, I}, {II, III}}

故今年虫害较轻。

```
p1 = DendrogramPlot[R4, LeafLabels → rate, Orientation → Bottom]
```



模糊数学作业第二章节部分习题

先写个小函数来减小输入数据的工作量。因为ToString[]括号中同时有数字和字母时，会产生一个空格，而且双引号则没有，但是双引号中的字符串不能用来做变量，故稍麻烦了点。

```
f = (Partition[Cases[ToExpression[Characters[ToString[#1]]], Except[Null]],
      #2]) / 10 /. a → 10 & // N;
```

```
R1 = f[5 421 264 519 a7, 4];
R2 = f[411 659 756 876, 4];
R3 = f[6578 a923, 2];
R4 = f[73 645 546, 2];
R5 = f[237 648, 3];
R6 = f[269 321, 3];
```

```
IntegerDigits[Range[6]]
```

```
{{1}, {2}, {3}, {4}, {5}, {6}}
```

```
Table[StringJoin["R", ToString[k]], {k, 6}] // ToExpression
```

```
{R1, R2, R3, R4, R5, R6}
```

```
v = ToExpression["R" <> ToString[#]] & /@ Range[6]
```

```
{R1, R2, R3, R4, R5, R6}
```

```
ToExpression /@ StringJoin /@ Thread[Append[{"R"}, ToString /@ Range[6]]]
```

```
{{{0.5, 0.4, 0.2, 0.1}, {0.2, 0.6, 0.4, 0.5}, {0.1, 0.9, 1., 0.7}},
 {{0.4, 0.1, 0.1, 0.6}, {0.5, 0.9, 0.7, 0.5}, {0.6, 0.8, 0.7, 0.6}},
 {{0.6, 0.5}, {0.7, 0.8}, {1., 0.9}, {0.2, 0.3}},
 {{0.7, 0.3}, {0.6, 0.4}, {0.5, 0.5}, {0.4, 0.6}},
 {{0.2, 0.3, 0.7}, {0.6, 0.4, 0.8}}, {{0.2, 0.6, 0.9}, {0.3, 0.2, 0.1}}}
```

```
R12 = Table[Max[R1[[i, j]], R2[[i, j]]], {i, 3}, {j, 4}];
% // MatrixForm
```

$$\begin{pmatrix} 0.5 & 0.4 & 0.2 & 0.6 \\ 0.5 & 0.9 & 0.7 & 0.5 \\ 0.6 & 0.9 & 1. & 0.7 \end{pmatrix}$$

```
{R1 // MatrixForm, R2 // MatrixForm}
```

$$\left\{\begin{pmatrix} 0.5 & 0.4 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.4 & 0.5 \\ 0.1 & 0.9 & 1. & 0.7 \end{pmatrix}, \begin{pmatrix} 0.4 & 0.1 & 0.1 & 0.6 \\ 0.5 & 0.9 & 0.7 & 0.5 \\ 0.6 & 0.8 & 0.7 & 0.6 \end{pmatrix}\right\}$$

```
R1 ⋃ R2 = R12
```

```
R3 // MatrixForm
R5 // MatrixForm
```

$$\begin{pmatrix} 0.6 & 0.5 \\ 0.7 & 0.8 \\ 1. & 0.9 \\ 0.2 & 0.3 \end{pmatrix}$$

$$\begin{pmatrix} 0.2 & 0.3 & 0.7 \\ 0.6 & 0.4 & 0.8 \end{pmatrix}$$

```
4 * 2 <> 2 * 3
```

```
str2mat[.5 × .4 × .6
    .6 × .4 × .8
    .6 × .4 × .8
    .3 × .3 × .3
   , 3] // MatrixForm
```

$$\begin{pmatrix} 0.5 & 0.4 & 0.6 \\ 0.6 & 0.4 & 0.8 \\ 0.6 & 0.4 & 0.8 \\ 0.3 & 0.3 & 0.3 \end{pmatrix}$$

```
Table[Min[R3[[m, k]], R5[[k, n]]], {m, 4}, {n, 3}, {k, 2}] // MatrixForm
```

$$\begin{pmatrix} \begin{pmatrix} 0.2 \\ 0.5 \end{pmatrix} & \begin{pmatrix} 0.3 \\ 0.4 \end{pmatrix} & \begin{pmatrix} 0.6 \\ 0.5 \end{pmatrix} \\ \begin{pmatrix} 0.2 \\ 0.6 \end{pmatrix} & \begin{pmatrix} 0.3 \\ 0.4 \end{pmatrix} & \begin{pmatrix} 0.7 \\ 0.8 \end{pmatrix} \\ \begin{pmatrix} 0.2 \\ 0.6 \end{pmatrix} & \begin{pmatrix} 0.3 \\ 0.4 \end{pmatrix} & \begin{pmatrix} 0.7 \\ 0.8 \end{pmatrix} \\ \begin{pmatrix} 0.2 \\ 0.3 \end{pmatrix} & \begin{pmatrix} 0.2 \\ 0.3 \end{pmatrix} & \begin{pmatrix} 0.2 \\ 0.3 \end{pmatrix} \end{pmatrix}$$

```
dot1 = Max /@ Table[Min[R3[[m, k]], R5[[k, n]]], {m, 4}, {n, 3}, {k, 2}][[#]] &;
```

```
Table[dot1[k], {k, 4}] // MatrixForm
```

$$\begin{pmatrix} 0.5 & 0.4 & 0.6 \\ 0.6 & 0.4 & 0.8 \\ 0.6 & 0.4 & 0.8 \\ 0.3 & 0.3 & 0.3 \end{pmatrix}$$

```
Table[Min[R3[[m, k]], R5[[k, n]]], {m, 4}, {n, 3}, {k, 2}]
```

```
{{{0.2, 0.5}, {0.3, 0.4}, {0.6, 0.5}}, {{0.2, 0.6}, {0.3, 0.4}, {0.7, 0.8}},
 {{0.2, 0.6}, {0.3, 0.4}, {0.7, 0.8}}, {{0.2, 0.3}, {0.2, 0.3}, {0.2, 0.3}}}
```

```
dot = Table[Max /@ Table[Min[#1[[m, k]], #2[[k, n]]],
      {m, Dimensions[#1][[1]]}, {n, Dimensions[#2][[2]]},
      {k, Dimensions[#1][[2]]}][[i]], {i, 1, Dimensions[#1][[1]]}] &;
```

```
R35 = dot[R3, R5];
% // MatrixForm
```

$$\begin{pmatrix} 0.5 & 0.4 & 0.6 \\ 0.6 & 0.4 & 0.8 \\ 0.6 & 0.4 & 0.8 \\ 0.3 & 0.3 & 0.3 \end{pmatrix}$$

R3 ∘ R5 = R35

计算 **R12 ∘ R35**

```
R12 // MatrixForm
```

$$\begin{pmatrix} 0.5 & 0.4 & 0.2 & 0.6 \\ 0.5 & 0.9 & 0.7 & 0.5 \\ 0.6 & 0.9 & 1. & 0.7 \end{pmatrix}$$

```
dot[R12, R35] // MatrixForm
```

$$\begin{pmatrix} 0.5 & 0.4 & 0.5 \\ 0.6 & 0.4 & 0.8 \\ 0.6 & 0.4 & 0.8 \end{pmatrix}$$

```
R1C = 1 - R1;
% // MatrixForm
```

$$\begin{pmatrix} 0.5 & 0.6 & 0.8 & 0.9 \\ 0.8 & 0.4 & 0.6 & 0.5 \\ 0.9 & 0.1 & 0. & 0.3 \end{pmatrix}$$

```
R3 ⋂ R4 = R3i4
```

```
R3i4 = Table[Min[R3[[i, j]], R4[[i, j]]], {i, 4}, {j, 2}];
% // MatrixForm
```

$$\begin{pmatrix} 0.6 & 0.3 \\ 0.6 & 0.4 \\ 0.5 & 0.5 \\ 0.2 & 0.3 \end{pmatrix}$$

```
(R1C ∘ (R3 ⋂ R4)) ∘ R6
```

```
dot[dot[R1C, R3i4], R6]
```

{{0.3, 0.6, 0.6}, {0.3, 0.6, 0.6}, {0.3, 0.6, 0.6}}

```
Fold[dot, R1C, {R3i4, R6}]
```

{{0.3, 0.6, 0.6}, {0.3, 0.6, 0.6}, {0.3, 0.6, 0.6}}

模糊关系的合成
在中医诊断中存在模糊关系:
建立{寒,热,虚,实}等症状与{肺,心}之间的模糊关系 R1∘R2

```
a = {寒, 热, 虚, 实}; b = {肺, 心}; c = {自汗, 恶寒, 咳嗽, 喘};
```

```
R1 = f[3 337 333 383 884 220, 4];
Append[Table[Append[R1[[k]], a[[k]]], {k, 1, 4}], c] // TableForm
```

0.3	0.3	0.3	0.7	寒
0.3	0.3	0.3	0.3	热
0.8	0.3	0.8	0.8	虚
0.4	0.2	0.2	0.	实
自汗	恶寒	咳嗽	喘	

```
TableForm[R1, TableHeadings → {a, c}]
```

	自汗	恶寒	咳嗽	喘
寒	0.3	0.3	0.3	0.7
热	0.3	0.3	0.3	0.3
虚	0.8	0.3	0.8	0.8
实	0.4	0.2	0.2	0.

```
R2 = f[53 338 263, 2];
Append[Table[Append[R2[[k]], a[[k]]], {k, 1, 4}], b] // TableForm
```

0.5	0.3	寒
0.3	0.3	热
0.8	0.2	虚
0.6	0.3	实
肺	心	

```
TableForm[R2, TableHeadings → {c, b}]
```

	肺	心
自汗	0.5	0.3
恶寒	0.3	0.3
咳嗽	0.8	0.2
喘	0.6	0.3

```
R = dot[R1, R2];
Append[Table[Append[R[[k]], a[[k]]], {k, 1, 4}], b] // TableForm
```

0.6	0.3	寒
0.3	0.3	热
0.8	0.3	虚
0.4	0.3	实
肺	心	

```
TableForm[R, TableHeadings → {a, b}]
```

	肺	心
寒	0.6	0.3
热	0.3	0.3
虚	0.8	0.3
实	0.4	0.3

已知模糊相似关系$R \in \mathcal{F}(X*X)$, $X = \{x1, x2, \cdots, x7\}$, 求R的传递闭包，并作聚类分析。

```
R = f[a9759a49a5454175a5714545a5799575a71a4177a5414915a, 7];
```

```
% // MatrixForm
```

$$\begin{pmatrix} 1. & 0.9 & 0.7 & 0.5 & 0.9 & 1. & 0.4 \\ 0.9 & 1. & 0.5 & 0.4 & 0.5 & 0.4 & 0.1 \\ 0.7 & 0.5 & 1. & 0.5 & 0.7 & 0.1 & 0.4 \\ 0.5 & 0.4 & 0.5 & 1. & 0.5 & 0.7 & 0.9 \\ 0.9 & 0.5 & 0.7 & 0.5 & 1. & 0.7 & 0.1 \\ 1. & 0.4 & 0.1 & 0.7 & 0.7 & 1. & 0.5 \\ 0.4 & 0.1 & 0.4 & 0.9 & 0.1 & 0.5 & 1. \end{pmatrix}$$

```
R2 = dot[R, R]; R4 = dot[R2, R2]; R8 = dot[R4, R4]; R16 = dot[R8, R8];
{"R^2", R2, "", "R^4", R4, "" , "R^8", R8, "", "R^16", R16} // TableForm
```

R^2

1.	0.9	0.7	0.7	0.9	1.	0.5
0.9	1.	0.7	0.5	0.9	0.9	0.4
0.7	0.7	1.	0.5	0.7	0.7	0.5
0.7	0.5	0.5	1.	0.7	0.7	0.9
0.9	0.9	0.7	0.7	1.	0.9	0.5
1.	0.9	0.7	0.7	0.9	1.	0.7
0.5	0.4	0.5	0.9	0.5	0.7	1.

R^4

1.	0.9	0.7	0.7	0.9	1.	0.7
0.9	1.	0.7	0.7	0.9	0.9	0.7
0.7	0.7	1.	0.7	0.7	0.7	0.7
0.7	0.7	0.7	1.	0.7	0.7	0.9
0.9	0.9	0.7	0.7	1.	0.9	0.7
1.	0.9	0.7	0.7	0.9	1.	0.7
0.7	0.7	0.7	0.9	0.7	0.7	1.

R^8

1.	0.9	0.7	0.7	0.9	1.	0.7
0.9	1.	0.7	0.7	0.9	0.9	0.7
0.7	0.7	1.	0.7	0.7	0.7	0.7
0.7	0.7	0.7	1.	0.7	0.7	0.9
0.9	0.9	0.7	0.7	1.	0.9	0.7
1.	0.9	0.7	0.7	0.9	1.	0.7
0.7	0.7	0.7	0.9	0.7	0.7	1.

R^{16}

1.	0.9	0.7	0.7	0.9	1.	0.7
0.9	1.	0.7	0.7	0.9	0.9	0.7
0.7	0.7	1.	0.7	0.7	0.7	0.7
0.7	0.7	0.7	1.	0.7	0.7	0.9
0.9	0.9	0.7	0.7	1.	0.9	0.7
1.	0.9	0.7	0.7	0.9	1.	0.7
0.7	0.7	0.7	0.9	0.7	0.7	1.

```
λ = { 1, 0.9, 0.7};
```

```
Grid[Table[If[R8[[i, j]] ≥ #, True, ""], {i, 7}, {j, 7}], Frame → All] & /@ λ //
 TableForm
```

True					True	
	True					
		True				
			True			
				True		
True					True	
						True

True	True			True	True	
True	True			True	True	
		True				
			True			True
True	True			True	True	
True	True			True	True	
			True			True

True	True	True	True	True	True	True
True	True	True	True	True	True	True
True	True	True	True	True	True	True
True	True	True	True	True	True	True
True	True	True	True	True	True	True
True	True	True	True	True	True	True
True	True	True	True	True	True	True

```
X = ToExpression["X" <> ToString[#]] & /@ Range[7];
Partition[X, 1];
```

现在开始分类:

$\lambda = 1$ 时, 分为6类 : {{X1, X6}, {X2}, {X3}, {X4}, {X5}, {X7}}

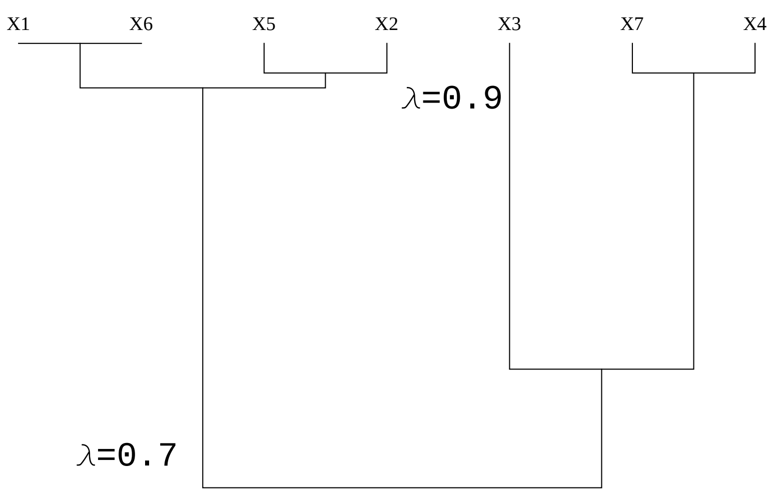
$\lambda = 0.9$ 时, 分为3类 : {{X1, X2, X5, X6}, {X3}, {X4, X7}}

$\lambda = 0.7$ 时, 分为1类:{X1, X2, X3, X4, X5, X6, X7}

画动态聚类图:

```
Needs["HierarchicalClustering`"]
XLabel = "X" <> ToString[#] & /@ Range[7];
```

```
p1 = DendrogramPlot[R8, LeafLabels → XLabel, Orientation → Bottom]
```



预报：某地历史上虫害情况分为I(轻)，II(中)， III(重)3类，今年的测报资料为N，其相似关系矩阵为R，问今年的虫害情况如何？

```
R = str2mat[1 w w w 0.39 × 1 w w 0.16 × 0.55 × 1 w 0.59 × 0.41 × 0.26 × 1, 4];
rate = ToString /@ {I, II, III, N}; R = R /. w → □;
TableForm[R, TableHeadings → ("rate" * ConstantArray[1, 2] // ToExpression)]
```

	I	II	III	N
I	1	□	□	□
II	0.39	1	□	□
III	0.16	0.55	1	□
N	0.59	0.41	0.26	1

```
R = str2mat[1 × 0.39 × 0.16 × 0.59 × 0.39 × 1
    0.55 × 0.41 × 0.16 × 0.55 × 1 × 0.26 × 0.59 × 0.41 × 0.26 × 1, 4];
```

```
% // MatrixForm
```

$$\begin{pmatrix} 1 & 0.39 & 0.16 & 0.59 \\ 0.39 & 1 & 0.55 & 0.41 \\ 0.16 & 0.55 & 1 & 0.26 \\ 0.59 & 0.41 & 0.26 & 1 \end{pmatrix}$$

```
R2 = dot[R, R]; R4 = dot[R2, R2]; R8 = dot[R4, R4];
{"R^2", R2, "", "R^4", R4, "" , "R^8", R8} // TableForm
```

R^2

1	0.41	0.39	0.59
0.41	1	0.55	0.41
0.39	0.55	1	0.41
0.59	0.41	0.41	1

R^4

1	0.41	0.41	0.59
0.41	1	0.55	0.41
0.41	0.55	1	0.41
0.59	0.41	0.41	1

R^8

1	0.41	0.41	0.59
0.41	1	0.55	0.41
0.41	0.55	1	0.41
0.59	0.41	0.41	1

```
λ = {1, 0.59, 0.55, 0.41};
```

```
Grid[Table[If[R4[[i, j]] ≥ #, True, ""], {i, 4}, {j, 4}], Frame → All] & /@ λ //
 TableForm
```

True			
	True		
		True	
			True

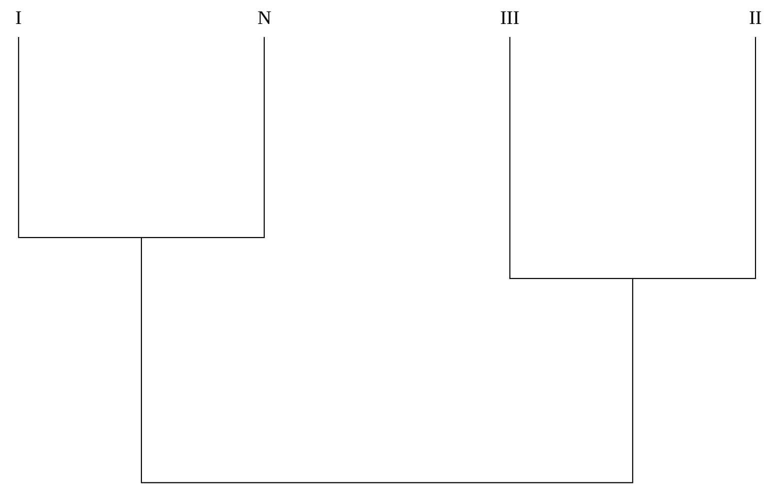
True			True
	True		
		True	
True			True

True			True
	True	True	
	True	True	
True			True

True	True	True	True
True	True	True	True
True	True	True	True
True	True	True	True

λ = 0.59 时, **分**为3类 : {{N, I}, {II, III}}
故今年虫害较轻。

```
p1 = DendrogramPlot[R4, LeafLabels → rate, Orientation → Bottom]
```



模糊数学作业第三章节部分习题

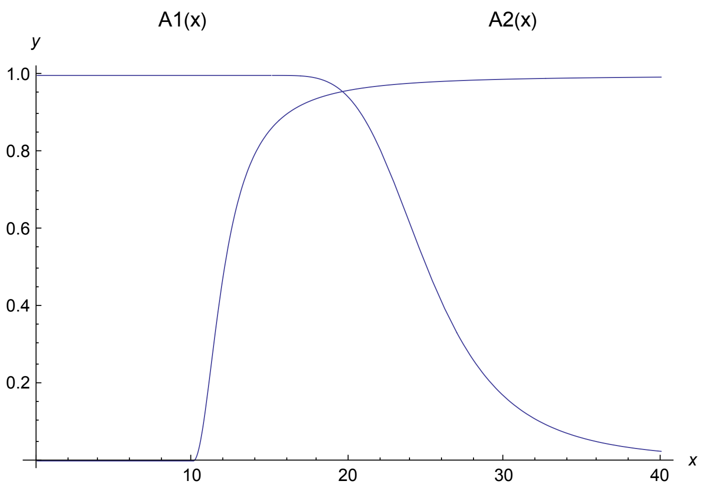
模糊识别： 设A1=“不热”，A2=“不冷”，U=[0,∞], x表示温度（单位: °C），A1和A2相应的隶属函数如下，A3=A1∩A2表示“暖和”. 试问： x=20 °C 时气温属哪种状态?

```
f = (Partition[Cases[ToExpression[Characters[ToString[#1]]], Except[Null]],
        #2]) / 10 /. a → 10 & // N;
```

$$A1\ (x)\ =\ \begin{cases} 1 & x \le 15 \\ \left(1+\left(\frac{x-15}{10}\right)^{4}\right)^{-1} & x > 15 \end{cases}$$

$$A2\ (x)\ =\ \begin{cases} 0 & x \le 10 \\ \left(1+\left(\frac{x-10}{2}\right)^{-2}\right)^{-1} & x > 10 \end{cases}$$

```
A1[x_] := Piecewise[{{1, x ≤ 15}, {(1 + ((x - 15)/10)^4)^-1, x > 15}}];

A2[x_] := Piecewise[{{0, x ≤ 10}, {(1 + ((x - 10)/2)^-2)^-1, x > 10}}];

p1 = Plot[A1[x], {x, 0, 40}, AxesLabel → {x, y}];
p2 = Plot[A2[x], {x, 0, 40}];

Show[p1, p2, PlotLabel → "A1(x)\t\t\tA2(x)"]
```




```
{A1@20, A2@20} // N
```

```
{0.941176, 0.961538}
```

由最大隶属原则知属于不冷状态。

通货膨胀问题： 设论域 $R^{+} = \{x \in R : x \geq 0\}$ 表示价格指数的集合.将通货状态分成5个类型，x表示物价上涨x %，试按最大隶属原则判断：
当x1 = 6，x2 = 21 属于通货膨胀哪一种状态?

```
Thread[(A = ToExpression["A" <> ToString[#]] & /@ Range[5]) ↔
   {"通货稳定", "轻度", "中度", "重度", "恶性"}] // List // TableForm
```

A1 ↔ 通货稳定　　A2 ↔ 轻度　　A3 ↔ 中度　　A4 ↔ 重度　　A5 ↔ 恶性

$$A1[x\_] := \text{Piecewise}\left[\left\{\{1, 0 \leq x < 5\}, \left\{\text{Exp}\left[-\left(\frac{x-5}{3}\right)^2\right], x \geq 5\right\}\right\}\right];$$

$$A2[x\_] := \text{Exp}\left[-\left(\frac{x-10}{5}\right)^2\right];$$

$$A3[x\_] := \text{Exp}\left[-\left(\frac{x-20}{7}\right)^2\right];$$

$$A4[x\_] := \text{Exp}\left[-\left(\frac{x-30}{7}\right)^2\right];$$

$$A5[x\_] := \text{Piecewise}\left[\left\{\left\{\text{Exp}\left[-\left(\frac{x-50}{15}\right)^2\right], 0 \leq x \leq 50\right\}, \{1, x > 50\}\right\}\right];$$

```
A
```

{A1, A2, A3, A4, A5}

```
Ax = Table[A[[k]] /@ {6, 21.7}, {k, 5}] // N;
TableForm[Ax, TableHeadings -> {x /@ A, {"x1=6", "x2=21.7"}}]
```

	x1=6	x2=21.7
x[A1]	0.894839	3.48481×10^{-14}
x[A2]	0.527292	0.00418772
x[A3]	0.0183156	0.942726
x[A4]	7.84918×10^{-6}	0.245142
x[A5]	0.000183289	0.0284527

```
Ax[[All, 1]] // Max
```

0.894839

```
Ax[[All, 2]] // Max
```

0.942726

故x1属于通货稳定，x2属于中度通货膨胀

设论域 U = {x1, x2, x3, x4, x5, x6} 上的标准模型库如下，给定一个待识别的模糊集B, 试用贴近度σ判别B与标准模型库中的哪个模型最贴近?

$$\sigma(A, B) = \frac{\sum_{i=1}^{n} [A(x_i) \wedge B(x_i)]}{\sum_{i=1}^{n} [A(x_i) \vee B(x_i)]}$$

```
A1 = f[a85401, 6];
A2 = f[528 a60, 6];
A3 = f[za2758, 6] /. z → 0;
A4 = f[40 a965, 6];
A5 = f[8205 a7, 6];
A6 = f[57 805 a, 6];
B = f[7214 a8, 6] // Flatten;
```

这里在输入有0开头的矩阵时有一个问题， 因为ToString函数会增加一个空格，而空格被识别为乘法然后提前计算把0后面的输入 "吃掉了", 不过这个问题已经被研究过了
这里暂且输入z代表0。

```
A = Flatten[{A1, A2, A3, A4, A5, A6}, 1];
```

$$\sigma[A\_, B\_] := \frac{\sum_{i=1}^{6} \mathrm{Min}[A[[k]][[i]], B[[i]]]}{\sum_{i=1}^{6} \mathrm{Max}[A[[k]][[i]], B[[i]]]}$$

```
Table[σ[A, B], {k, 1, 6}]
% // Max
```

```
{0.333333, 0.4, 0.454545, 0.434783, 0.882353, 0.456522}
```

```
0.882353
```

B与A5最贴近

下面B与哪个Ai最贴近?

$$\sigma(A, B) = \frac{2\sum_{i=1}^{n} [A(x_i) \wedge B(x_i)]}{\sum_{i=1}^{n} A(x_i) + \sum_{i=1}^{n} A(x_i)}$$

```
A1 = f[2451, 4];
A2 = f[1531, 4];
A3 = f[2341, 4];
B = f[6310, 4] // Flatten;
```

```
A = Flatten[{A1, A2, A3}, 1];
Thread[{"A1", "A2", "A3", "B"} ==
    (Plus @@ Thread[Append[A, B][[#]] / {x1, x2, x3, x4}] & /@ Range[4])] //
  MatrixForm // TraditionalForm
```

$$\begin{pmatrix} \mathrm{A1} = \frac{0.2}{\mathrm{x1}} + \frac{0.4}{\mathrm{x2}} + \frac{0.5}{\mathrm{x3}} + \frac{0.1}{\mathrm{x4}} \\ \mathrm{A2} = \frac{0.1}{\mathrm{x1}} + \frac{0.5}{\mathrm{x2}} + \frac{0.3}{\mathrm{x3}} + \frac{0.1}{\mathrm{x4}} \\ \mathrm{A3} = \frac{0.2}{\mathrm{x1}} + \frac{0.3}{\mathrm{x2}} + \frac{0.4}{\mathrm{x3}} + \frac{0.1}{\mathrm{x4}} \\ \mathrm{B} = 0. + \frac{0.6}{\mathrm{x1}} + \frac{0.3}{\mathrm{x2}} + \frac{0.1}{\mathrm{x3}} \end{pmatrix}$$

```
A
```

```
{{0.2, 0.4, 0.5, 0.1}, {0.1, 0.5, 0.3, 0.1}, {0.2, 0.3, 0.4, 0.1}}
```

$$\sigma[A\_, B\_] := \frac{2\sum_{i=1}^{3} \mathrm{Min}[A[[k]][[i]], B[[i]]]}{\sum_{i=1}^{3} A[[k]][[i]] + \sum_{i=1}^{3} B[[i]]}$$

```
Table[σ[A, B], {k, 1, 3}]
% // Max
```

{0.571429, 0.526316, 0.631579}

0.631579

B与A3最贴近

心理素质识别: 在运动员中心理选材中，以"内-克"表的9个指标为论域，即X
已知某类优秀运动员 E1, 及两名选手 A1,A2 试按贴近度 σ对两名运动员作一心理教材.

```
X = {m1, m2, r1, r2, s1, s2, v, n, t};
E1 = (str2mat[83 × 84 × 95 × 96 × 94 × 93 × 99 × 97 × 99, 9] // Flatten) / 100 // N;
A1 = (str2mat[86 × 96 × 78 × 100 × 84 × 95 × 65 × 94 × 86, 9] // Flatten) / 100 // N;
A2 = (str2mat[99 × 99 × 89 × 90 × 93 × 92 × 88 × 77 × 99, 9] // Flatten) / 100 // N;
```

$$\sigma(A, B) = \frac{\sum_{i=1}^{n} [A(x_i) \wedge B(x_i)]}{\sum_{i=1}^{n} [A(x_i) \vee B(x_i)]}$$

```
Thread[{"E1", "A1", "A2"} == (Plus @@ Thread[{E1, A1, A2}[[#]] / X] & /@ Range[3])] //
  MatrixForm // TraditionalForm
```

$$\begin{pmatrix} E1 = \frac{0.83}{m1} + \frac{0.84}{m2} + \frac{0.97}{n} + \frac{0.95}{r1} + \frac{0.96}{r2} + \frac{0.94}{s1} + \frac{0.93}{s2} + \frac{0.99}{t} + \frac{0.99}{v} \\ A1 = \frac{0.86}{m1} + \frac{0.96}{m2} + \frac{0.94}{n} + \frac{0.78}{r1} + \frac{1.}{r2} + \frac{0.84}{s1} + \frac{0.95}{s2} + \frac{0.86}{t} + \frac{0.65}{v} \\ A2 = \frac{0.99}{m1} + \frac{0.99}{m2} + \frac{0.77}{n} + \frac{0.89}{r1} + \frac{0.9}{r2} + \frac{0.93}{s1} + \frac{0.92}{s2} + \frac{0.99}{t} + \frac{0.88}{v} \end{pmatrix}$$

```
A = {A1, A2}; TableForm[A]
```

0.86	0.96	0.78	1.	0.84	0.95	0.65	0.94	0.86
0.99	0.99	0.89	0.9	0.93	0.92	0.88	0.77	0.99

$$\sigma[A\_, B\_] := \frac{\sum_{i=1}^{9} \mathrm{Min}[A[[k]][[i]], E1[[i]]]}{\sum_{i=1}^{9} \mathrm{Max}[A[[k]][[i]], E1[[i]]]}$$

```
Table[σ[A, B], {k, 1, 2}]
% // Max
```

{0.886179, 0.912744}

0.912744

E1与A2更贴近，故A2的心理更优秀

模糊数学作业第三章节部分习题

模糊识别：设A1="不热"，A2="不冷"，U=[0,∞], x表示温度(单位: °C), A1和A2相应的隶属函数如下，A3=A1⋂A2表示"暖和". 试问：x=20 °C 时气温属哪种状态？

```
f = (Partition[Cases[ToExpression[Characters[ToString[#1]]], Except[Null]],
      #2]) / 10 /. a → 10 & // N;
```

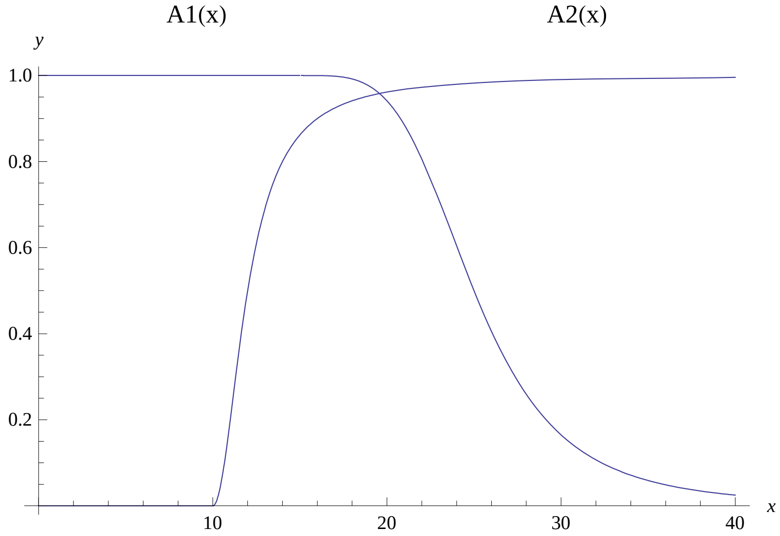
$$A1(x) = \begin{cases} 1 & x \le 15 \\ \left(1+\left(\frac{x-15}{10}\right)^4\right)^{-1} & x > 15 \end{cases}$$

$$A2(x) = \begin{cases} 0 & x \le 10 \\ \left(1+\left(\frac{x-10}{2}\right)^{-2}\right)^{-1} & x > 10 \end{cases}$$

$$A1[x\_] := \text{Piecewise}\left[\left\{\{1,\ x \le 15\},\ \left\{\left(1+\left(\frac{x-15}{10}\right)^4\right)^{-1},\ x > 15\right\}\right\}\right];$$

$$A2[x\_] := \text{Piecewise}\left[\left\{\{0,\ x \le 10\},\ \left\{\left(1+\left(\frac{x-10}{2}\right)^{-2}\right)^{-1},\ x > 10\right\}\right\}\right];$$

```
p1 = Plot[A1[x], {x, 0, 40}, AxesLabel → {x, y}];
p2 = Plot[A2[x], {x, 0, 40}];

Show[p1, p2, PlotLabel → "A1(x)\t\t\t\tA2(x)"]
```




```
{A1@20, A2@20} // N

{0.941176, 0.961538}
```

由最大隶属原则知属于不冷状态。

通货膨胀问题: 设论域 $R^+ = \{x \in R : x \geq 0\}$ 表示价格指数的集合.将通货状态分成5个类型,
x表示物价上涨x %, 试按最大隶属原则判断:
当x1 = 6, x2 = 21 属于通货膨胀哪一种状态?

```
Thread[(A = ToExpression["A" <> ToString[#]] & /@ Range[5]) ↔
   {"通货稳定", "轻度", "中度", "重度", "恶性"}] // List // TableForm
```

A1 ↔ 通货稳定　A2 ↔ 轻度　A3 ↔ 中度　A4 ↔ 重度　A5 ↔ 恶性

$$A1[x\_] := \text{Piecewise}\left[\left\{\{1, 0 \le x < 5\}, \left\{\text{Exp}\left[-\left(\frac{x-5}{3}\right)^2\right], x \ge 5\right\}\right\}\right];$$

$$A2[x\_] := \text{Exp}\left[-\left(\frac{x-10}{5}\right)^2\right]; A3[x\_] := \text{Exp}\left[-\left(\frac{x-20}{7}\right)^2\right]; A4[x\_] := \text{Exp}\left[-\left(\frac{x-30}{7}\right)^2\right];$$

$$A5[x\_] := \text{Piecewise}\left[\left\{\left\{\text{Exp}\left[-\left(\frac{x-50}{15}\right)^2\right], 0 \le x \le 50\right\}, \{1, x > 50\}\right\}\right];$$

```
A
```

{A1, A2, A3, A4, A5}

```
Ax = Table[A[[k]] /@ {6, 21.7}, {k, 5}] // N;
TableForm[Ax, TableHeadings -> {x /@ A, {"x1=6", "x2=21.7"}}]
```

	x1=6	x2=21.7
x[A1]	0.894839	3.48481×10^{-14}
x[A2]	0.527292	0.00418772
x[A3]	0.0183156	0.942726
x[A4]	7.84918×10^{-6}	0.245142
x[A5]	0.000183289	0.0284527

```
Ax[[All, 1]] // Max
```

0.894839

```
Ax[[All, 2]] // Max
```

0.942726

故x1属于通货稳定, x2属于中度通货膨胀

设论域 U = {x1, x2, x3, x4, x5, x6} 上的标准模型库如下, 给定一个待识别的模糊集B, 试用贴近度σ判别B与标准模型库中的哪个模型最贴近?

$$\sigma(A, B) = \frac{\sum_{i=1}^{n} [A(x_i) \wedge B(x_i)]}{\sum_{i=1}^{n} [A(x_i) \vee B(x_i)]}$$

```
A1 = f[a85401, 6];
A2 = f[528 a60, 6];
A3 = f[za2758, 6] /. z → 0;
A4 = f[40 a965, 6];
A5 = f[8205 a7, 6];
A6 = f[57 805 a, 6];
B = f[7214 a8, 6] // Flatten;
```

这里在输入有0开头的矩阵时有一个问题,　因为ToString函数会增加一个空格, 而空格被识别为乘法然后提前计算把0后面的输入 "吃掉了", 不过这个问题已经被研究过了
这里暂且输入z代表0。

```
A = Flatten[{A1, A2, A3, A4, A5, A6}, 1];
```

$$\sigma[A\_, B\_] := \frac{\sum_{i=1}^{6} \mathrm{Min}[A[[k]][[i]], B[[i]]]}{\sum_{i=1}^{6} \mathrm{Max}[A[[k]][[i]], B[[i]]]}$$

```
Table[σ[A, B], {k, 1, 6}]
% // Max
```

{0.333333, 0.4, 0.454545, 0.434783, 0.882353, 0.456522}

0.882353

B与A5最贴近

下面B与哪个Ai最贴近？

$$\sigma\,(A, B) = \frac{2\sum_{i=1}^{n}[A\,(x_i) \wedge B\,(x_i)]}{\sum_{i=1}^{n} A\,(x_i) + \sum_{i=1}^{n} A\,(x_i)}$$

```
A1 = f[2451, 4];
A2 = f[1531, 4];
A3 = f[2341, 4];
B = f[6310, 4] // Flatten;
```

```
A = Flatten[{A1, A2, A3}, 1];
Thread[{"A1", "A2", "A3", "B"} ==
   (Plus @@ Thread[Append[A, B][[#]] / {x1, x2, x3, x4}] & /@ Range[4])] //
  MatrixForm // TraditionalForm
```

$$\begin{pmatrix} A1 = \frac{0.2}{x1} + \frac{0.4}{x2} + \frac{0.5}{x3} + \frac{0.1}{x4} \\ A2 = \frac{0.1}{x1} + \frac{0.5}{x2} + \frac{0.3}{x3} + \frac{0.1}{x4} \\ A3 = \frac{0.2}{x1} + \frac{0.3}{x2} + \frac{0.4}{x3} + \frac{0.1}{x4} \\ B = 0. + \frac{0.6}{x1} + \frac{0.3}{x2} + \frac{0.1}{x3} \end{pmatrix}$$

```
A
```

{{0.2, 0.4, 0.5, 0.1}, {0.1, 0.5, 0.3, 0.1}, {0.2, 0.3, 0.4, 0.1}}

$$\sigma[A\_, B\_] := \frac{2\sum_{i=1}^{3} \mathrm{Min}[A[[k]][[i]], B[[i]]]}{\sum_{i=1}^{3} A[[k]][[i]] + \sum_{i=1}^{3} B[[i]]}$$

```
Table[σ[A, B], {k, 1, 3}]
% // Max
```

{0.571429, 0.526316, 0.631579}

0.631579

B与A3最贴近

心理素质识别：在运动员中心理选材中，以 "内-克"表的9个指标为论域，即X
已知某类优秀运动员 E1，及两名选手 A1,A2 试按贴近度 σ对两名运动员作一心理教材.

```
X = {m1, m2, r1, r2, s1, s2, v, n, t};
E1 = (str2mat[83 × 84 × 95 × 96 × 94 × 93 × 99 × 97 × 99, 9] // Flatten) / 100 // N;
A1 = (str2mat[86 × 96 × 78 × 100 × 84 × 95 × 65 × 94 × 86, 9] // Flatten) / 100 // N;
A2 = (str2mat[99 × 99 × 89 × 90 × 93 × 92 × 88 × 77 × 99, 9] // Flatten) / 100 // N;
```

$$\sigma(A, B) = \frac{\sum_{i=1}^{n}[A(x_i) \bigwedge B(x_i)]}{\sum_{i=1}^{n}[A(x_i) \vee B(x_i)]}$$

```
Thread[
  {"E1", "A1", "A2"} == (Plus @@ Thread[{E1, A1, A2}[[#]] / X] & /@ Range[3])] //
 MatrixForm // TraditionalForm
```

$$\begin{pmatrix} E1 = \frac{0.83}{m1} + \frac{0.84}{m2} + \frac{0.97}{n} + \frac{0.95}{r1} + \frac{0.96}{r2} + \frac{0.94}{s1} + \frac{0.93}{s2} + \frac{0.99}{t} + \frac{0.99}{v} \\ A1 = \frac{0.86}{m1} + \frac{0.96}{m2} + \frac{0.94}{n} + \frac{0.78}{r1} + \frac{1.}{r2} + \frac{0.84}{s1} + \frac{0.95}{s2} + \frac{0.86}{t} + \frac{0.65}{v} \\ A2 = \frac{0.99}{m1} + \frac{0.99}{m2} + \frac{0.77}{n} + \frac{0.89}{r1} + \frac{0.9}{r2} + \frac{0.93}{s1} + \frac{0.92}{s2} + \frac{0.99}{t} + \frac{0.88}{v} \end{pmatrix}$$

```
A = {A1, A2}; TableForm[A]
```

0.86	0.96	0.78	1.	0.84	0.95	0.65	0.94	0.86
0.99	0.99	0.89	0.9	0.93	0.92	0.88	0.77	0.99

$$\sigma[A\_, B\_] := \frac{\sum_{i=1}^{9} \text{Min}[A[[k]][[i]], E1[[i]]]}{\sum_{i=1}^{9} \text{Max}[A[[k]][[i]], E1[[i]]]}$$

```
Table[σ[A, B], {k, 1, 2}]
% // Max
```

{0.886179, 0.912744}

0.912744

E1与A2更贴近，故A2的心理更优秀

定理"3.1.4

$\mathcal{F}$是定义在集合X上的非空集族，若集族 $\mathcal{A}$:={A|A为$\mathcal{F}$中有限个两两不交的元素之并}

如果$\mathcal{F}$满足

(1)若E，F∈$\mathcal{F}$，则E⋂F∈$\mathcal{F}$

(2)若E∈$\mathcal{F}$，则E^c∈$\mathcal{F}$

则K($\mathcal{F}$)=$\mathcal{A}$

即，若$\mathcal{F}$是代数，则$\mathcal{F}$中有限个两两不交的元素的并集组成的集族$\mathcal{A}$是由$\mathcal{F}$生成的代数

例如：

```
X = Range[1, 4]
```

```
{1, 2, 3, 4}
```

```
Subsets[X]
```

```
{{}, {1}, {2}, {3}, {4}, {1, 2}, {1, 3}, {1, 4}, {2, 3},
 {2, 4}, {3, 4}, {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}, {1, 2, 3, 4}}
```

若 {}, {1}, {2}, {1, 2, 3, 4} ∈ $\mathcal{F}$

```
𝓕1 = {{}, {1}, {2}, {1, 2, 3, 4}};
```

则 $\{1\}^c = \{2, 3, 4\}, \{2\}^c = \{1, 3, 4\} \in \mathcal{F}$

```
𝓕2 = {{2, 3, 4}, {1, 3, 4}};
```

则 $\{1\}^c = \{2, 3, 4\}, \{2\}^c = \{1, 3, 4\} \in \mathcal{F}$

```
Table[Intersection[𝓕1[[m]], 𝓕2[[n]]], {m, 1, 4}, {n, 1, 2}] // MatrixForm
```

$$\begin{pmatrix} \{\} & \{\} \\ \{\} & \{1\} \\ \{2\} & \{\} \\ \{2, 3, 4\} & \{1, 3, 4\} \end{pmatrix}$$

```
Complement[X, {1}]
```

```
{2, 3, 4}
```

```
Complement[X, {2}]
```

```
{1, 3, 4}
```

```
𝓕 = Union[𝓕1, 𝓕2]
```

```
{{}, {1}, {2}, {1, 3, 4}, {2, 3, 4}, {1, 2, 3, 4}}
```

取$\mathcal{F}$为 {{}, {1}, {2}, {1, 3, 4}, {2, 3, 4}, {1, 2, 3, 4}}

```
Table[Union[𝓕[[m]], 𝓕[[n]]], {m, 1, 6}, {n, 1, 6}] // MatrixForm
```

$$\begin{pmatrix} \{\} & \{1\} & \{2\} & \{1,3,4\} & \{2,3,4\} & \{1,2,3,4\} \\ \{1\} & \{1\} & \{1,2\} & \{1,3,4\} & \{1,2,3,4\} & \{1,2,3,4\} \\ \{2\} & \{1,2\} & \{2\} & \{1,2,3,4\} & \{2,3,4\} & \{1,2,3,4\} \\ \{1,3,4\} & \{1,3,4\} & \{1,2,3,4\} & \{1,3,4\} & \{1,2,3,4\} & \{1,2,3,4\} \\ \{2,3,4\} & \{1,2,3,4\} & \{2,3,4\} & \{1,2,3,4\} & \{2,3,4\} & \{1,2,3,4\} \\ \{1,2,3,4\} & \{1,2,3,4\} & \{1,2,3,4\} & \{1,2,3,4\} & \{1,2,3,4\} & \{1,2,3,4\} \end{pmatrix}$$

```
𝒜 = {{}, {1}, {2}, {1, 2}, {1, 3, 4}, {2, 3, 4}, {1, 2, 3, 4}}
```

定理 "3.1.5

若$\mathcal{A}$为集合X上的代数, 则M($\mathcal{A}$)=S($\mathcal{A}$)

命题C$\Longleftrightarrow$命题A成立的充要条件是命题B$\Longleftrightarrow$(B$\Longrightarrow$A)$\wedge$(A$\Longrightarrow$B)
(B$\Longrightarrow$A)$\Longleftrightarrow$B是A的充分条件 (A$\Longrightarrow$B)$\Longleftrightarrow$B是A的必要条件
(A$\Longrightarrow$B)$\Longleftrightarrow$A是B的充分条件 (B$\Longrightarrow$A)$\Longleftrightarrow$A是B的必要条件
$\Longrightarrow$(命题C的充分性$\Longleftrightarrow$(B$\Longrightarrow$A))$\vee$(命题C的必要性$\Longleftrightarrow$(A$\Longrightarrow$B))

1. $(X, \mathcal{A})$ 为一可测空间, $E \in \mathcal{A}$, 证明：f为E上可测函数的充要条件是对于任一$r \in Q$均有$E(f > r) \in \mathcal{A}$.

证明：充分性显然.$\forall a \in R, E(f > a) \in \mathcal{A} \Longrightarrow \forall r \in Q, E(f > r) \in \mathcal{A}$.
必要性.$\forall r \in Q, E(f > r) \in \mathcal{A}$, 则$\forall a_i \in$,
$r_{i0}(r_{i0}, r_{i1}) < r_{i1} \in Q, E(f > r_{i0}) \in \mathcal{A}$,
故$E(f > a_i > r_{i0}) \in \mathcal{A}$
由a_i的任意性$\Longrightarrow \forall a \in R, E(f > a) \in \mathcal{A}$, 即f为E上可测函数.

必要性：或$E(f > r_{i0}) \bigcap E(f < r_{i1}) \in \mathcal{A}$, 故$E_i(f \in (r_{i0}, r_{i1})) \in \mathcal{A}$

$\mathcal{A}$是σ代数$\Longrightarrow \forall a \in R = \bigcup_{i=1}^{\infty} E_i, E(f > a) \in \mathcal{A}$, 即f为E上可测函数

2. $(X, \mathcal{A})$ 为一可测空间, (f_n) 为$E \in \mathcal{A}$上的可测函数列.证明：
(f_n) 的收敛点集与发散点集均为可测集.

证明：若$\lim_{n\to\infty} f_n = f$, 则f可测 (由定理 4.1.2 的推论2)
即收敛点集E_1可测, 因发散点集$E_0 = E \setminus E_1 \in \mathcal{A}$, 故其也可测.

3. (X, τ) 为一拓扑空间, $\mathcal{A}$为X上的σ代数, 且$\mathcal{A} \supset \tau$, 证明：X上连续函数必为X上的可测函数.

证明：$f :=$
$X \to Y$连续, Y也是一个拓扑空间, 故开集$U \subset Y$, $f^{-1}(U) = V \in \tau \subset \mathcal{A}$.A是$\sigma$代数.故f为X上的$\mathcal{A}$可测函数.

4. 证明：L - S可测集E上的单调函数必为E上的L - S可测函数.

证明：$f := X \to Y$连续, 若Y也是一个拓扑空间, 则开集$U \subset Y$, $f^{-1}(U) =$
$V \in \tau \subset \mathcal{A}$.故$Y = R$时也成立.$\mathcal{A}$是$\sigma$代数.故f为X上的$\mathcal{A}$可测函数.

5. f为可测集E上的有界可测函数, 证明：存在一致有界的简单函数列(φ_n)在E上一致收敛于f.

证明：由定理4.1.5---设f为E$\in\mathcal{A}$上的可测函数, 则存在E上的简单函数列(φ_n)满足
$f(x)=\lim_{n\to\infty}\varphi_n(x)$, $|\varphi_n(x)| \uparrow |f(x)|$, $x\in E$.
$|f(x)|\le M$, $M>0$, $|\varphi_n(x)|\le|f(x)|\le M$, 因此$\varphi_n(x)$一致有界

6. (f_n)为完备测度空间中的可测集E上的可测函数序列,
$f: E \to R^*$ 为(f_n)的a.e.收敛的极限函数, 证明：f为E上的可测函数.

证明：由定理4.2.1---设$(X, \mathcal{A}, \mu)$为一测度空间, (f_n)为E$\in\mathcal{A}$上的可测函数列, 若E上的函数f为(f_n)的a.e.收敛的极限函数, 则存在E上的可测函数g, 使得f(x)=g(x) a.e.于E.
记E1=E(f=g), E2=E(f≠g), 则μ(E2)=0, E1、E2都是可测集.
要证明∀a∈R, E(f>a)为可测集.

7. 设$(X, \mathcal{A}, \mu)$为一测度空间, $E \in \mathcal{A}$, f, $f_n (n = 1, 2, \ldots)$为E上的可测函数, 若对任一$\delta >$
0, 存在$E_\delta \in \mathcal{A}$使得$\mu(E - E_\delta) < \delta$且$(f_n)$在$E_\delta$上一致收敛于f, 证明：
(1) (f_n)在E上依测度收敛于f;
(2) (f_n)在E上a.e.收敛于f.

证明：

（1）$f_n \to \to f$，故 $\forall \varepsilon > 0$，$\exists$ 一个N使得对所有 $x \in E_\delta$ 当 $n \geq N$ 时有 $|f_n - f| < \varepsilon$

可测函数序列 (f_n) 依测度收敛与f：若给定 $\varepsilon >$

0 存在一个N使得对所有 $n \geq N$ 有 $\mu\{x : |f(x) - f_n(x)| \geq \varepsilon\} < \varepsilon$

当 $n \geq N$ 时，$\{x : |f - f_n| \geq \varepsilon\} \subset E \setminus E_\delta$

或记 $E1 = E(|f - f_n| \geq \varepsilon) \subset (E \setminus E_\delta)$，$\mu(E1) \leq \mu(E \setminus E_\delta) < \delta$

$\lim\limits_{n\to+\infty} \mu(E1) = 0$，$\mu(E(|f - f_n| \geq \varepsilon)) < \varepsilon$，故 $f_n \Rightarrow f$ 于E

（2）取 $\delta = 1/k > 0$，$\exists E_{1/k} \in \mathcal{A}$，使得 $\mu(E - E_{1/k}) < 1/k$ 且 (f_n) 在 $E_{1/k}$ 上一致收敛于f，$(k = 1, 2, \ldots)$

$$E2 = \bigcup_{k=1}^{+\infty} E_{1/k},\ \lim_{n\to+\infty} f_n = f,\ 0 \leq \mu(E \setminus E2) = \mu\left(\bigcap_{k=1}^{+\infty}(E \setminus E_{1/k})\right) = \mu(E \setminus E_k) \leq \frac{1}{k}$$

$\lim\limits_{k\to+\infty} \mu(E \setminus E2) = 0$. 即 $(E \setminus E2)$ 为零测集，$\lim\limits_{n\to+\infty} f_n = f$ a.e于E

8. 设 $(X, \mathcal{A}, \mu)$ 为一测度空间，$E \in \mathcal{A}$，$f_n \Rightarrow f(x)$ 于E，$g : E \to R^*$，证明：若 $f_n(x) \leq g(x)$ a.e.于E，$(n = 1, 2, \ldots)$，则 $f(x) \leq g(x)$ a.e.于E.

证明：由Riesz定理，存在子列 (f_{n_j})，

$f_{n_j} \to f$ a.e.于E.但 $f_{n_j}(x) \leq g(x)$ a.e.于E，故 $\lim\limits_{j\to+\infty} f_{n_j}(x) = f(x) \leq g(x)$ a.e. 于E.

9. 设 $(X, \mathcal{A}, \mu)$ 为一测度空间，$E \in \mathcal{A}$，$f_n(x) \leq f_{n+1}(x)$ a.e. 于E，$(n = 1, 2, \ldots)$，且 $f_n(x) \Rightarrow f(x)$ 于E，证明：$f_n(x) \to f(x)$ a.e.于E.

证明：由Riesz定理，存在子列 (f_{n_j})，$f_{n_j} \to f$ a.e.于E，存在 $E1 \subset E$，$\mu(E1) = 0$，

$x \in (E \setminus E1)$ 时，$\lim\limits_{j\to+\infty} f_{n_j}(x) = f(x)$.

$f_n(x) < f_{n+1}(x)$ a.e.于E，存在 $E2 \subset E$，$\mu(E2) = 0$，

$x \in (E \setminus (E1 \cup E2))$ 时，若 $n > n_j$，则 $f_{n_j}(x) \leq f_n(x) \leq f(x)$

故 $\lim\limits_{j\to+\infty} f_{n_j}(x) = \lim\limits_{n\to+\infty} f_n(x) = f(x)$.

$0 \leq \mu(E1 \cup E2) \leq \mu(E1) + \mu(E2) = 0$

即 $f_n(x) \to f(x)$ a.e.于E.　$f_n(x) \xrightarrow{\text{a.e.于E}} f(x)$

10. 设f为L – S可测集E上的a.e.有限的L – S可测函数，证明：存在R上的连续函数列 (φ_n) 在E上几乎处处收敛于f.

证明：

11. 设f为L – S可测集E上的a.e有限的函数.证明：若对于任一 $\delta >$ 0，存在闭集 $E_\delta \subset E$ 使得 $m_g(E \setminus E_\delta) < \delta$ 且f在 E_δ 上的限制是有限值连续函数，则f为E上L – S可测函数.

证明：

12. 设 $(X, \mathcal{A}, \mu)$ 为一测度空间，$E \in \mathcal{A}$，f, g, f_n, g_n $(n = 1, 2, \ldots)$ 均为E上的a.e有限的可测函数，且 $f_n(x) \Rightarrow f(x)$，$g_n(x) \Rightarrow g(x)$ 于E.证明：在E上成立

（1）$|f_n(x)| \Rightarrow |f(x)|$

（2）$\forall a \in R : \alpha f_n(x) \Rightarrow \alpha f(x)$

（3）$(f_n + g_n)(x) \Rightarrow (f + g)(x)$

（4）$\min\{f_n\{x\}, g_n\{x\}\} \Rightarrow \min\{f(x), g(x)\}$

$\max\{f_n\{x\}, g_n\{x\}\} \Rightarrow \max\{f(x), g(x)\}$

（5）若设 $\mu(E) < +\infty$，则 $f_n(x) g_n(x) \Rightarrow f(x) g(x)$

证明：(1)

1. $(X, \mathcal{A})$ is a measurable space, $\mathrm{E} \in \mathcal{A}$, Proof: necessary and sufficient condition of f is measurable function on E is $\forall\, r \in Q$ have $E(f > r) \in \mathcal{A}$.

Proof: sufficiency is obvious. $\forall\, a \in R,\ E(f > a) \in \mathcal{A} \Longrightarrow \forall\, r \in Q,\ E(f > r) \in \mathcal{A}$.

necessity1 : $\forall\, r \in Q,\ E(f > r) \in \mathcal{A}$, then $\forall\, a_i \in \left(r_{\mathrm{i}0},\ r_{\mathrm{i}1}\right),\ r_{\mathrm{i}0} < r_{\mathrm{i}1} \in Q,\ E\left(f > r_{\mathrm{i}0}\right) \in \mathcal{A}$, so $E\left(f > a_i > r_{\mathrm{i}0}\right) \in \mathcal{A}$ for randomicity of $a_i \Longrightarrow \forall\, a \in R,\ E(f > a) \in \mathcal{A}$, ie. f is measurable function on E.

necessity2: or $E\left(f > r_{\mathrm{i}0}\right) \bigcap E(f < r_{\mathrm{i}1}) \in \mathcal{A}$, so $E_i\left(f \in \left(r_{\mathrm{i}0},\ r_{\mathrm{i}1}\right)\right) \in \mathcal{A}$

$\mathcal{A}$ is σ algebra $\Longrightarrow \forall\, a \in R = \bigcup_{i=1}^{\infty} E_i,\ E(f > a) \in \mathcal{A}$, namely f is measurable function on E.

2. $(X, \mathcal{A})$ is a measurable space, (f_n) is measurable function sequence on $E \in \mathcal{A}$. Proof: set of points of convergence and set of points of divergence are both measurable sets.

Proof: if $\lim\limits_{n \to \infty} f_n = f$, then f is measurable. namely setE_1 of convergence

points measurable , for set of points of divergence $E_0 = E \backslash E_1 \in \mathcal{A}$, so it is measurable.

$$\left\{\xi(\mathrm{x\_},\,\mathrm{y\_}) := \frac{x}{y},\ \eta(\mathrm{x\_},\,\mathrm{y\_}) = y\right\};$$

$$\left\{\frac{\partial \xi(x,\,y)}{\partial x},\ \frac{\partial \xi(x,\,y)}{\partial y},\ \frac{\partial \eta(x,\,y)}{\partial x},\ \frac{\partial \eta(x,\,y)}{\partial y},\ \frac{\partial^2 \xi(x,\,y)}{\partial x\,\partial y}\right\}$$

$$\left\{\frac{1}{y},\ -\frac{x}{y^2},\ 0,\ 1,\ -\frac{1}{y^2}\right\}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{1}{y}\frac{\partial u}{\partial \xi}$$

$$\frac{\partial^2 u}{\partial x\,\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y}\frac{\partial u}{\partial \xi}\right) = \frac{1}{y^2}\frac{\partial^2 u}{\partial \xi\,\partial \xi}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = -\frac{x}{y^2}\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial y\,\partial y} = \frac{\partial}{\partial y}\left(-\frac{x}{y^2}\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) = \frac{\partial}{\partial y}\left(-\frac{x}{y^2}\right)\frac{\partial u}{\partial \xi} - \frac{x}{y^2}\frac{\partial^2 u}{\partial \xi\,\partial \xi}\frac{\partial \xi}{\partial y} - \frac{x}{y^2}\frac{\partial^2 u}{\partial \xi\,\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta\,\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta\,\partial \xi}\frac{\partial \xi}{\partial y} =$$
$$\frac{2\,x}{y^3}\frac{\partial u}{\partial \xi} + \frac{x^2}{y^4}\frac{\partial^2 u}{\partial \xi\,\partial \xi} - \frac{x}{y^2}\frac{\partial^2 u}{\partial \xi\,\partial \eta} + \frac{\partial^2 u}{\partial \eta\,\partial \eta} - \frac{x}{y^2}\frac{\partial^2 u}{\partial \eta\,\partial \xi}$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{1}{y}\frac{\partial u}{\partial \xi}\right) = \frac{1}{y}\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial}{\partial y}\left(\frac{1}{y}\right)\frac{\partial u}{\partial \xi} = \frac{1}{y}\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} - \frac{1}{y^2}\frac{\partial u}{\partial \xi}$$

$$x^2\,u_{\mathrm{xx}} + y^2\,u_{\mathrm{yy}} + 2\,x\,y\,u_{\mathrm{xy}} = 0$$

习题2

2.3

（2）$\begin{cases} u''+\lambda u,\ 0<x<1 \\ u'[0]=u'[l]+\sigma u[l]=0 \end{cases}$

对应的特征方程为$a^2+\lambda=0$

$\lambda\le 0$时不是特征值

$\lambda>0$时, $a^2=-\lambda$, $a=\pm\sqrt{-\lambda}=0\pm\sqrt{\lambda}\,i$

$u=e^{0x}\left(C1\,Cos\left[\sqrt{\lambda}\right]+C2\,Sin\left[\sqrt{\lambda}\,x\right]\right)=C1\,Cos\left[\sqrt{\lambda}\right]+C2\,Sin\left[\sqrt{\lambda}\,x\right]$

$u[x\_]:=C1\,Cos\left[\sqrt{\lambda}\,x\right]+C2\,Sin\left[\sqrt{\lambda}\,x\right]$

$u'[x]=C2\sqrt{\lambda}\,Cos\left[x\sqrt{\lambda}\right]-C1\sqrt{\lambda}\,Sin\left[x\sqrt{\lambda}\right]$

$C2\sqrt{\lambda}\,Cos\left[x\sqrt{\lambda}\right]-C1\sqrt{\lambda}\,Sin\left[x\sqrt{\lambda}\right]$ /. $x\to 0$

$C2\sqrt{\lambda}$

$u'[0]=C2\sqrt{\lambda}=0$, 故$C2=0$

$u'[l]+\sigma u[l]=C1\,\sigma\,Cos\left[l\sqrt{\lambda}\right]-C1\sqrt{\lambda}\,Sin\left[l\sqrt{\lambda}\right]=C1\left(\sigma\,Cos\left[l\sqrt{\lambda}\right]-\sqrt{\lambda}\,Sin\left[l\sqrt{\lambda}\right]\right)=0,\ \sigma>0$

$Collect\left[C1\,\sigma\,Cos\left[l\sqrt{\lambda}\right]-C1\sqrt{\lambda}\,Sin\left[l\sqrt{\lambda}\right],\ C1\right]$

$C1\left(\sigma\,Cos\left[l\sqrt{\lambda}\right]-\sqrt{\lambda}\,Sin\left[l\sqrt{\lambda}\right]\right)$

$\sigma\,Cos\left[l\sqrt{\lambda}\right]-\sqrt{\lambda}\,Sin\left[l\sqrt{\lambda}\right]=0$

解$Cot\left[l\sqrt{\lambda}\right]=\dfrac{\sqrt{\lambda}}{\sigma}$

记$r=\sqrt{\lambda}\,l$, r_n是方程$Cot[r]=\dfrac{1}{\sigma l}r$的第n个正根, $\lambda_n=\dfrac{r_n^2}{l^2}$, $n=1,2,\ldots$

对就的特征函数为$u_n=Cos\left[\sqrt{\lambda_n}\,x\right]$, $n=1,2,\ldots$

验算了一下page 43

Solve[{u[1] == 0, u[e] == 0}, {C[1], C[2]}]

$DSolve\left[R''[s]-n^2R[s]==0,\ R[s],\ s\right]$

$\{\{R[s]\to e^{ns}C[1]+e^{-ns}C[2]\}\}$

$e^{ns}C[1]+e^{-ns}C[2]$ /. s -> Log[ρ]

$\rho^nC[1]+\rho^{-n}C[2]$

（4）$\begin{cases} x^2u''+3xu'+\lambda u=0,\ 0<x<e \\ u[1]=u[e]=0 \end{cases}$

欧拉方程，作代换令$x=e^t$, $t=ln$

$xu'[x]=u'[t]$, $xu''[x]=u''[t]-u'[t]$

$u''[t]+2u'[t]+\lambda u[t]=0$

```
DSolve[u''[t] + 2 u'[t] + λ u[t] == 0, u[t], t]; % /. t → Log[x]
```

$\left\{\left\{u[\mathrm{Log}[x]] \to x^{-1-\sqrt{1-\lambda}}\, C[1] + x^{-1+\sqrt{1-\lambda}}\, C[2]\right\}\right\}$

```
u[x_] := x^(-1-Sqrt[1-λ]) C[1] + x^(-1+Sqrt[1-λ]) C[2]
u[e]
```

$e^{-1-\sqrt{1-\lambda}}\, C[1] + e^{-1+\sqrt{1-\lambda}}\, C[2]$

```
Solve[a^2 + 2 a + λ == 0, a]
```

$\left\{\left\{a \to -1 - \sqrt{1-\lambda}\right\}, \left\{a \to -1 + \sqrt{1-\lambda}\right\}\right\}$

$\Delta = 4 - 4\lambda > 0$时，$\lambda < 1$两实根

$u[t] = C1\, e^{-1-\sqrt{1-\lambda}\ t} + C2\, e^{-1+\sqrt{1-\lambda}\ t}$

$u[x] = x^{-1-\sqrt{1-\lambda}}\, C[1] + x^{-1+\sqrt{1-\lambda}}\, C[2]$

由条件 $u[1] = u[e] = 0$得 $C1, C2$ 为 0

$\lambda = 1$时，$u = (C1 + C2\, t)\, e^{-t}$，$\dfrac{C1 + C2\,\mathrm{Log}[x]}{x}$ 也得 $C1, C2$ 为

故$\lambda \le 1$时不是特征值

$\lambda > 1$, $a = -1 \pm \sqrt{\lambda - 1}\ i$

$u = e^{-t}\left(C1\, \mathrm{Cos}\left[\sqrt{\lambda-1}\ t\right] + C2\, \mathrm{Sin}\left[\sqrt{\lambda-1}\ t\right]\right)$

$$e^{-t}\left(C1\cos\left[\sqrt{\lambda-1}\ t\right]+C2\sin\left[\sqrt{\lambda-1}\ t\right]\right)\ /.\ t\to \mathrm{Log}[x]$$

$$\frac{C1\cos\left[\sqrt{-1+\lambda}\ \mathrm{Log}[x]\right]+C2\sin\left[\sqrt{-1+\lambda}\ \mathrm{Log}[x]\right]}{x}$$

$$u[x\_] := \frac{C1\cos\left[\sqrt{-1+\lambda}\ \mathrm{Log}[x]\right]+C2\sin\left[\sqrt{-1+\lambda}\ \mathrm{Log}[x]\right]}{x};$$

$$u[1] = C1 = 0,\ u[e] = \frac{C1\cos\left[\sqrt{-1+\lambda}\right]+C2\sin\left[\sqrt{-1+\lambda}\right]}{e} = 0$$

故$\sin\left[\sqrt{-1+\lambda}\right] = 0$

$\sqrt{-1+\lambda} = n\pi,\ \lambda_n = 1+(n\pi)^2,\ n = 1, 2, \ldots$

$u_n(x) = \frac{1}{x}\sin(n\pi\ln x),\ n = 1, 2, \ldots$

验算**page 38**

$$\boldsymbol{\alpha[n\_] := (2n-1)/2;\ \varphi[x\_] := \sin\left[\frac{x}{2}\right];\ \sin[n\_, x\_] := \sin\left[\frac{2n-1}{2}x\right];\ f[x\_, t\_] := \frac{1}{2}x\,t}$$

$$\boldsymbol{c[n\_] := \frac{2}{\pi}\mathrm{Integrate}[\varphi[x]\sin[\alpha[n]\,x], \{x, 0, \pi\}];}$$

$$\boldsymbol{f[n\_, t\_] := \frac{2}{\pi}t\,\mathrm{Integrate}\left[\frac{1}{2}x\sin[\alpha[n]\,x], \{x, 0, \pi\}\right];}$$

$$\boldsymbol{f[n, t] = \frac{2t(-2\cos[n\pi]+(1-2n)\pi\sin[n\pi])}{(1-2n)^2\pi}}$$

$$\boldsymbol{c[n] = \frac{(1-2n)\sin[n\pi]}{(-1+n)\,n\pi} = 0,\ n \ge 2,\ c[1] = 1}$$

Table[f[n, t], {n, 1, 10}]

$$\left\{\frac{4t}{\pi}, -\frac{4t}{9\pi}, \frac{4t}{25\pi}, -\frac{4t}{49\pi}, \frac{4t}{81\pi}, -\frac{4t}{121\pi}, \frac{4t}{169\pi}, -\frac{4t}{225\pi}, \frac{4t}{289\pi}, -\frac{4t}{361\pi}\right\}$$

$$\boldsymbol{\mathrm{Table}\left[(-1)^{n-1}t\frac{1}{\pi(\alpha[n])^2}, \{n, 1, 10\}\right]}$$

$$\left\{\frac{4t}{\pi}, -\frac{4t}{9\pi}, \frac{4t}{25\pi}, -\frac{4t}{49\pi}, \frac{4t}{81\pi}, -\frac{4t}{121\pi}, \frac{4t}{169\pi}, -\frac{4t}{225\pi}, \frac{4t}{289\pi}, -\frac{4t}{361\pi}\right\}$$

暂时不知道如何 让**f[n, t]** 变成这种形式 $\boldsymbol{(-1)^{n-1}t\frac{1}{\pi(\alpha[n])^2}}$

公式**2 - 4 - 3**

$$\boldsymbol{u[x\_, t\_] := \cos[\alpha[1]\,t]\sin[\alpha[1]\,x] + \mathrm{Sum}\left[\frac{1}{\alpha[n]}\sin[\alpha[n]\,x]\,\mathrm{Integrate}[f[n, \tau]\sin[\alpha[n]\,(t-\tau)], \{\tau, 0, t\}], \{n, 1, \infty\}\right]}$$

```
u[x, t]
```

$$
\begin{aligned}
&\frac{1}{8\pi} e^{-\frac{i t}{2}-\frac{i x}{2}} \Big(-2 e^{i x}\,\mathrm{LerchPhi}\left[-e^{-i (t-x)}, 5, \frac{1}{2}\right] - 2 e^{i t}\,\mathrm{LerchPhi}\left[-e^{i (t-x)}, 5, \frac{1}{2}\right] + \\
&\quad 2 i e^{\frac{i t}{2}} t\,\mathrm{LerchPhi}\left[e^{-i \pi - i x}, 4, \frac{1}{2}\right] + i e^{\frac{i t}{2}} t\,\mathrm{LerchPhi}\left[e^{-i \pi - i x}, 5, \frac{1}{2}\right] + \\
&\quad 2 i e^{\frac{i t}{2}} t\,\mathrm{LerchPhi}\left[e^{i \pi - i x}, 4, \frac{1}{2}\right] + i e^{\frac{i t}{2}} t\,\mathrm{LerchPhi}\left[e^{i \pi - i x}, 5, \frac{1}{2}\right] - \\
&\quad 2 i e^{\frac{i t}{2}+i x} t\,\mathrm{LerchPhi}\left[e^{-i \pi + i x}, 4, \frac{1}{2}\right] - i e^{\frac{i t}{2}+i x} t\,\mathrm{LerchPhi}\left[e^{-i \pi + i x}, 5, \frac{1}{2}\right] - \\
&\quad 2 i e^{\frac{i t}{2}+i x} t\,\mathrm{LerchPhi}\left[e^{i \pi + i x}, 4, \frac{1}{2}\right] - i e^{\frac{i t}{2}+i x} t\,\mathrm{LerchPhi}\left[e^{i \pi + i x}, 5, \frac{1}{2}\right] - \\
&\quad 2 i e^{\frac{i t}{2}} t\,\mathrm{LerchPhi}\left[-e^{-i x}, 5, \frac{1}{2}\right] + 2 i e^{\frac{i t}{2}+i x} t\,\mathrm{LerchPhi}\left[-e^{i x}, 5, \frac{1}{2}\right] + \\
&\quad 2\,\mathrm{LerchPhi}\left[-e^{-i (t+x)}, 5, \frac{1}{2}\right] + 2 e^{i t + i x}\,\mathrm{LerchPhi}\left[-e^{i (t+x)}, 5, \frac{1}{2}\right]\Big) + \mathrm{Cos}\left[\frac{t}{2}\right] \mathrm{Sin}\left[\frac{x}{2}\right]
\end{aligned}
$$

```
k[n_, t_] := Integrate[f[n, τ] Sin[α[n] (t - τ)], {τ, 0, t}]
```

$$\texttt{Table}\left[\frac{\texttt{k[n, t]}}{\alpha\texttt{[n]}}, \{\texttt{n}, 1, 3\}, \{\texttt{t}, 1, 5\}\right] \texttt{ // N}$$

```
{{0.20957, 1.61476, 5.11847, 11.1098, 19.3688},
 {-0.0210637, -0.119837, -0.229604, -0.263216, -0.275062},
 {0.00619802, 0.0194231, 0.0213888, 0.0343682, 0.0409598}}
```

这个积分值跟题目结果中的和式内的 进行对比发现是一样的，故一定程度上验证完毕。

$$\texttt{Table}\left[(-1)^{n-1}\left(t - \frac{\mathrm{Sin}[\alpha[n]\ t]}{\alpha[n]}\right) \Big/ \left(\pi\ (\alpha[n])^4\right), \{n, 1, 3\}, \{t, 1, 5\}\right] \texttt{ // N}$$

```
{{0.20957, 1.61476, 5.11847, 11.1098, 19.3688},
 {-0.0210637, -0.119837, -0.229604, -0.263216, -0.275062},
 {0.00619802, 0.0194231, 0.0213888, 0.0343682, 0.0409598}}
```

k[n, t]

$$-\frac{1}{(1-2n)^4\pi}4\left(-2\,\mathrm{Cos}[n\pi]+(1-2n)\pi\,\mathrm{Sin}[n\pi]\right)\left(t-2nt+2\,\mathrm{Sin}\left[\left(-\frac{1}{2}+n\right)t\right]\right)$$

2.7

$$(2)\begin{cases} u_{tt}-u_{xx}=x+\mathrm{Sin}[x], & 0<x<\pi,\ t>0 \\ u[0,t]=u[\pi,t]=0, & t\geq 0 \\ u[x,0]=\frac{1}{2}\mathrm{Sin}[2x],\ u_t[x,0]=0, & 0\leq x\leq\pi \end{cases}$$

与之对应的定解 问题对应 的特征 值问题 是

$$\begin{cases} X''[x]+\lambda X[x]=0, & 0<x<\pi \\ X[0]=X'[\pi]=0 & \square \end{cases}$$

其对应的特征函数系是 $\{\mathrm{Sin}[\alpha_n x]\}_{n=1}^{\infty}$, 其中$\alpha_n=(2n-1)/2$ 把函数$\varphi[x]=\frac{1}{2}\mathrm{Sin}[2x]$,

$\psi[x]=0$ 和$f[x,t]=x+\mathrm{Sin}[x]$ 关于x按特征函数 $\{\mathrm{Sin}[\alpha_n x]\}_{n=1}^{\infty}$

可以算出 $c_1=-\frac{8}{15\pi}$, $c_n=\frac{8\cos(\pi n)}{\pi\left(-4n^2+4n+15\right)}$, $d_n=0$, $f_n(t)=\frac{4\left(\frac{8\,\mathrm{Cos}[n\pi]}{-3-4n+4n^2}+(1-2n)\pi\,\mathrm{Sin}[n\pi]\right)}{(1-2n)^2\pi}$

```
α[n_] := (2 n - 1) / 2; φ[x_] := 1/2 Sin[2 x]; sin[n_, x_] := Sin[(2 n - 1)/2 x];
c[n_] := 2/π Integrate[φ[x] sin[n, x], {x, 0, π}]
f[x_, t_] := x + Sin[x]
f[n_, t_] := 2/π Integrate[(x + Sin[x]) Sin[α[n] x], {x, 0, π}];
```

$$f[n,t]=\frac{4\left(\frac{8\,\mathrm{Cos}[n\pi]}{-3-4n+4n^2}+(1-2n)\pi\,\mathrm{Sin}[n\pi]\right)}{(1-2n)^2\pi}$$

$$c[n]=\frac{8\,\mathrm{Cos}[n\pi]}{\left(15+4n-4n^2\right)\pi}$$

$$\text{Integrate}[f[n,\tau]\,\text{Sin}[\alpha[n]\,(t-\tau)],\{\tau,0,t\}]$$

$$\left(16\left(8\,\text{Cos}[n\,\pi]+\left(-3+2\,n+12\,n^2-8\,n^3\right)\pi\,\text{Sin}[n\,\pi]\right)\text{Sin}\left[\frac{1}{4}(-1+2\,n)\,t\right]^2\right)\Big/\left((-1+2\,n)^3\left(-3-4\,n+4\,n^2\right)\pi\right)$$

公式**2 – 4 – 3**

$$u[x\_,t\_]:=\text{Sum}\left[\frac{8\,\text{Cos}[n\,\pi]}{\left(15+4\,n-4\,n^2\right)\pi}\text{Cos}[\alpha[n]\,t]\,\text{Sin}[\alpha[n,x]],\{n,1,2\}\right]+$$
$$\text{Sum}\left[\frac{1}{\alpha[n]}\text{Sin}[\alpha[n]\,x]\,\text{Integrate}[f[n,\tau]\,\text{Sin}[\alpha[n]\,(t-\tau)],\{\tau,0,t\}],\{n,1,2\}\right]+$$
$$\text{Sum}\left[\frac{8\,\text{Cos}[n\,\pi]}{\left(15+4\,n-4\,n^2\right)\pi}\text{Cos}[\alpha[n]\,t]\,\text{Sin}[\alpha[n,x]],\{n,3,\infty\}\right]+$$
$$\text{Sum}\left[\frac{1}{\alpha[n]}\text{Sin}[\alpha[n]\,x]\,\text{Integrate}[f[n,\tau]\,\text{Sin}[\alpha[n]\,(t-\tau)],\{\tau,0,t\}],\{n,3,\infty\}\right]$$

$$u[x,t]$$

有空再算。

$$\text{Sum}\left[\frac{8\,\text{Cos}[n\,\pi]}{\left(15+4\,n-4\,n^2\right)\pi}\text{Cos}[\alpha[n]\,t]\,\text{Sin}[\alpha[n]\,x],\{n,1,2\}\right]+$$
$$\text{Sum}\left[\frac{1}{\alpha[n]}\text{Sin}[\alpha[n]\,x]\,\text{Integrate}[f[n,\tau]\,\text{Sin}[\alpha[n]\,(t-\tau)],\{\tau,0,t\}],\{n,1,2\}\right]$$

$$-\frac{8\,\text{Cos}\left[\frac{t}{2}\right]\text{Sin}\left[\frac{x}{2}\right]}{15\,\pi}+\frac{256\,\text{Sin}\left[\frac{t}{4}\right]^2\text{Sin}\left[\frac{x}{2}\right]}{3\,\pi}+$$
$$\frac{8\,\text{Cos}\left[\frac{3t}{2}\right]\text{Sin}\left[\frac{3x}{2}\right]}{7\,\pi}+\frac{256\,\text{Sin}\left[\frac{3t}{4}\right]^2\text{Sin}\left[\frac{3x}{2}\right]}{405\,\pi}\ //\ \text{FullSimplify}$$

$$\frac{-1512\left(-80+81\,\text{Cos}\left[\frac{t}{2}\right]\right)\text{Sin}\left[\frac{x}{2}\right]+8\left(112+293\,\text{Cos}\left[\frac{3t}{2}\right]\right)\text{Sin}\left[\frac{3x}{2}\right]}{2835\,\pi}\ /.\ \{t\to1,x\to1\}\ //\ N$$

0.844576

$$3\,(1-\text{Cos}[t])\,\text{Sin}[x]+\frac{1}{4}(3\,\text{Cos}[2\,t]-1)\,\text{Sin}[2\,x]\ /.\ \{t\to1,x\to1\}\ //\ N$$

0.649342

本来想从答案 凑出非无 穷和式，结果发现错 了，要么是之前的就有 错，要么是无穷和式里 还可以拿出来部分放到前 面的非无 穷和式

习题1

1.1　有一长度为 l 且两端固定的均匀而柔软的细弦做微小横振动，在振动过程中不计重力但计阻力，阻力的大小与速度成正比，比例常数为 R ．试导出此弦的微小横振动所满足的偏微分方程.

$$u_{tt} - a^2 u_{xx} = f(x, t),\ \frac{-f(x, t)}{\rho} = \frac{-R}{\rho} v = \frac{-R}{\rho} u_t = u_{tt} - a^2 u_{xx},$$

$$u_{tt} = a^2 u_{xx} - \frac{R}{\rho} u_t,\ a = \sqrt{T_0/\rho} > 0$$

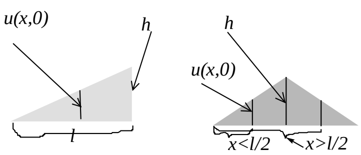
1.2　一均匀细杆的表面绝热，内部热源是 $f_0(x, t)$ ．试导出杆的温度分布所满足的偏微分方程.

$$u_t - a^2 \Delta u = f(x, t)$$
$$u_t = a^2 u_{xx} + f(x, t)$$
$$a^2 = k/(c\rho) > 0,\ f = f_0/c$$

常数 k, c 和 ρ分别为热传导系数 ,比热和密度

1.3　一根长度为 l 的均匀细弦的左端固定，右端自由滑动．在右端点把弦垂直提起高度为 h ，等弦静止后放手任其自由振动．试推导弦振动满足的定解问题.

$$\begin{cases} u_{tt} = a^2 u_{xx} & 0 < x < l, t > 0 \\ u(x, 0) = hx/l,\ u_t(x, 0) = 0 & 0 \le x \le l \\ u(0, t) = 0,\ u_x(l, t) = 0 & t > 0 \end{cases}$$



1.6　一根长度为 l 的均匀细弦，两端固定，沿弦的中点垂直提起高度 h ，等弦稳定后再放开任其自由振动，导出弦的振动满足的定解问题.

$$\begin{cases} u_{tt} = a^2 u_{xx} & 0 < x < l, t > 0 \\ u(x, 0) = \begin{cases} 2hx/l & 0 \le x < l/2 \\ h - 2hx/l & l/2 < x < l \end{cases},\ u_t(x, 0) = 0 & \square \\ u(0, t) = 0,\ u_x(l, t) = 0 & t > 0 \end{cases}$$

1.8　判断下述方程的类型：

（1）$y^2 u_{xx} + x^3 u_{yy} = 0$ → xy = 0, 抛物型 ;$y^2 \ge 0$,x>0时, Δ=-det(A1)<0, 椭圆型 ;x<0时, Δ=-det(A1)>0,双曲型

（2）$u_{xx} + (x+y) u_{yy} = 0$ → x + y=0,抛物型 ;x+y>0,椭圆型 ;x+y<0,双曲型

（3）$u_{xx} + (x^2+y) u_{yy} = 0$

（4）$x u_{xx} + u_{yy} = f(x, y)$

3,4同上

```
A = {{{y^2, 0}, {0, x^3}}, {{1, 0}, {0, x + y}}, {{1, 0}, {0, x^2 + y}}, {{x, 0}, {0, 1}}};
Table[Det[A[[k]]], {k, 1, 4}]
```

$\{x^3 y^2,\ x + y,\ x^2 + y,\ x\}$

1. $\phi = \mathrm{span}\left\{1, x, x^2\right\}$

```
x1 = Table[n, {n, 0.00, 1.00, 0.25}];
```

```
f1 = {1.0000, 1.2840, 1.6487, 2.1170, 2.7183};
```

```
data = Table[{x1[[k]], f1[[k]]}, {k, 1, 5}]
```

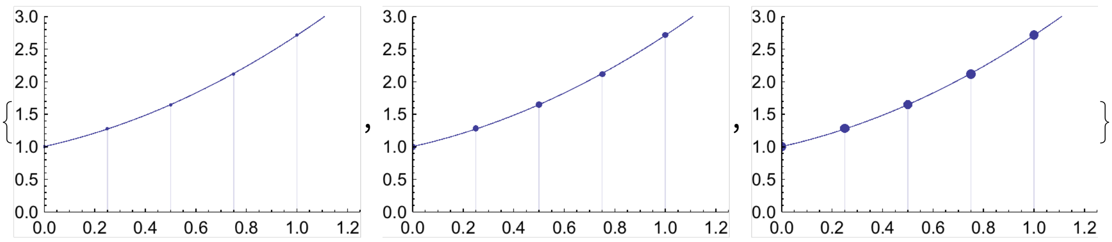
{{0., 1.}, {0.25, 1.284}, {0.5, 1.6487}, {0.75, 2.117}, {1., 2.7183}}

```
parabola = Fit[data, {1, x, x^2}, x]
```

$1.00514 + 0.864183\, x + 0.843657\, x^2$

```
p1 = Plot[parabola, {x, 0, 1.25}, PlotRange → {{0, 1.25}, {0, 3}}];
p = Table[
   ListPlot[data, PlotStyle → PointSize[k], Filling → 0], {k, 0.01, 0.03, 0.01
   }];
```

```
Table[Show[p1, p[[k]]], {k, 1, 3}]
```



```
f[x_] := a0 + a1 x + a2 x;
```

Construct the normal equation.

```
A = {Table[Sum[x_i^k, {i, 1, 5}], {k, 0, 2}], Table[Sum[x_i^k, {i, 1, 5}], {k, 1, 3}],
   Table[Sum[x_i^k, {i, 1, 5}], {k, 2, 4}]} // MatrixForm
```

$$\begin{pmatrix} 5 & x_1 + x_2 + x_3 + x_4 + x_5 & x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \\ x_1 + x_2 + x_3 + x_4 + x_5 & x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 & x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 & x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 & x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 \end{pmatrix}$$

```
b = {{a0}, {a1}, {a2}} // MatrixForm
```

$$\begin{pmatrix} a0 \\ a1 \\ a2 \end{pmatrix}$$

```
A.b // MatrixForm
```

$$\begin{pmatrix} 5 & x_1 + x_2 + x_3 + x_4 + x_5 & x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \\ x_1 + x_2 + x_3 + x_4 + x_5 & x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 & x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 & x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 & x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 \end{pmatrix} . \begin{pmatrix} a0 \\ a1 \\ a2 \end{pmatrix}$$

```
Y = {{Sum[y_i, {i, 1, 5}]}, {Sum[x_i * y_i, {i, 1, 5}]},
   {Sum[(x_i)^2 * y_i, {i, 1, 5}]}} // MatrixForm
```

$$\begin{pmatrix} y_1 + y_2 + y_3 + y_4 + y_5 \\ x_1\, y_1 + x_2\, y_2 + x_3\, y_3 + x_4\, y_4 + x_5\, y_5 \\ x_1^2\, y_1 + x_2^2\, y_2 + x_3^2\, y_3 + x_4^2\, y_4 + x_5^2\, y_5 \end{pmatrix}$$

```
b
```

$$\begin{pmatrix} a0 \\ a1 \\ a2 \end{pmatrix}$$

```
LinearSolve[A, Y] /. {x1 → 0, x2 → 0.25, x3 → 0.50, x4 → 0.75,
  x5 → 1, y1 → 1, y2 → 1.284, y3 → 1.6487, y4 → 2.117, y5 → 2.7183}
```

$$\text{LinearSolve}\left[\begin{pmatrix} 5 & 2.5\grave{} & 1.875\grave{} \\ 2.5\grave{} & 1.875\grave{} & 1.5625\grave{} \\ 1.875\grave{} & 1.5625\grave{} & 1.3828125\grave{} \end{pmatrix}, \begin{pmatrix} 8.768\grave{} \\ 5.4514\grave{} \\ 4.4015375\grave{} \end{pmatrix}\right]$$

```
{{1.00514}, {0.864183}, {0.843657}}
```

2.$y = b\, e^{ax}$

```
x2 = Table[n, {n, 1.00, 2.00, 0.25}];
```

```
f2 = {5.1, 5.79, 6.53, 7.45, 8.46};
```

```
data = Table[{x2[[k]], f2[[k]]}, {k, 1, 5}]
```

```
{{1., 5.1}, {1.25, 5.79}, {1.5, 6.53}, {1.75, 7.45}, {2., 8.46}}
```

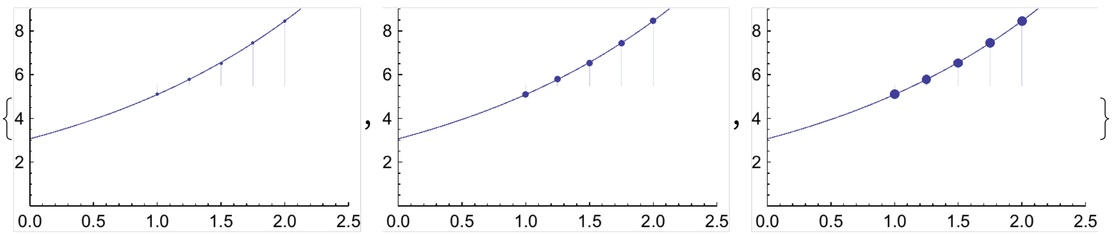
```
FindFit[data, b e^(a x), {a, b}, x]
```

```
{a → 0.506955, b → 3.06658}
```

```
g[x_] := b e^(a x) /. {a → 0.5069548151177287`, b → 3.0665759310803415`}
```

```
p1 = Plot[g[x], {x, 0, 2.5}, PlotRange → {{0, 2.5}, {0, 9}}];
p = Table[ListPlot[data, PlotStyle → PointSize[k], Filling → Automatic],
   {k, 0.01, 0.03, 0.01
   }];
```

```
Table[Show[p1, p[[k]]], {k, 1, 3}]
```



$y = b\, e^{ax}$,ln y=ln a+b x,Let A=ln a,B=b,u=A+Bx

```
A = Log[b]; B = a; u[x_] := A + B x;
```

Construct the normal equation.

```
A1 = {Table[Sum[x_i^k, {i, 1, 5}], {k, 0, 1}],
   Table[Sum[x_i^k, {i, 1, 5}], {k, 1, 2}]} // MatrixForm
```

$$\begin{pmatrix} 5 & x_1 + x_2 + x_3 + x_4 + x_5 \\ x_1 + x_2 + x_3 + x_4 + x_5 & x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \end{pmatrix}$$

```
B1 = {{A}, {B}} // MatrixForm
```

$$\begin{pmatrix} \text{Log}[b] \\ a \end{pmatrix}$$

```
A1.B1 // MatrixForm
```

$$\begin{pmatrix} 5 & x_1+x_2+x_3+x_4+x_5 \\ x_1+x_2+x_3+x_4+x_5 & x_1^2+x_2^2+x_3^2+x_4^2+x_5^2 \end{pmatrix} . \begin{pmatrix} \text{Log}[b] \\ a \end{pmatrix}$$

```
Y1 = {{Sum[Log[yᵢ], {i, 1, 5}]}, {Sum[xᵢ * Log[yᵢ], {i, 1, 5}]}} // MatrixForm
```

$$\begin{pmatrix} \text{Log}[y_1]+\text{Log}[y_2]+\text{Log}[y_3]+\text{Log}[y_4]+\text{Log}[y_5] \\ \text{Log}[y_1]\,x_1+\text{Log}[y_2]\,x_2+\text{Log}[y_3]\,x_3+\text{Log}[y_4]\,x_4+\text{Log}[y_5]\,x_5 \end{pmatrix}$$

```
LinearSolve[A1, Y1] /. {x1 → 1, x2 → 1.25, x3 → 1.50,
  x4 → 1.75, x5 → 2, y1 → 5.1, y2 → 5.79, y3 → 6.53, y4 → 7.45, y5 → 8.46}
```

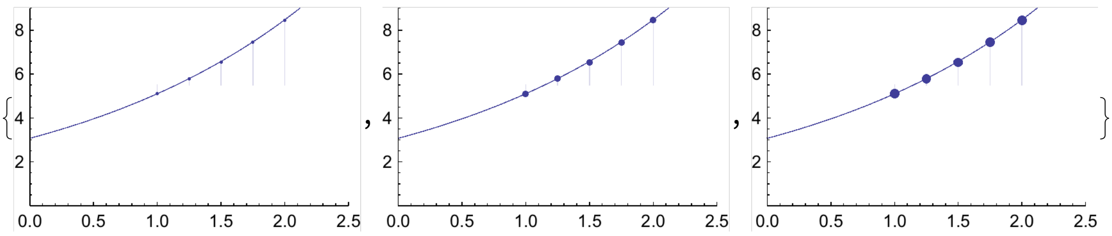
$$\text{LinearSolve}\left[\begin{pmatrix} 5 & 7.5\text{`} \\ 7.5\text{`} & 11.875\text{`} \end{pmatrix}, \begin{pmatrix} 9.405342980613124\text{`} \\ 14.424089223065254\text{`} \end{pmatrix}\right]$$

{{1.12249}, {0.50572}}

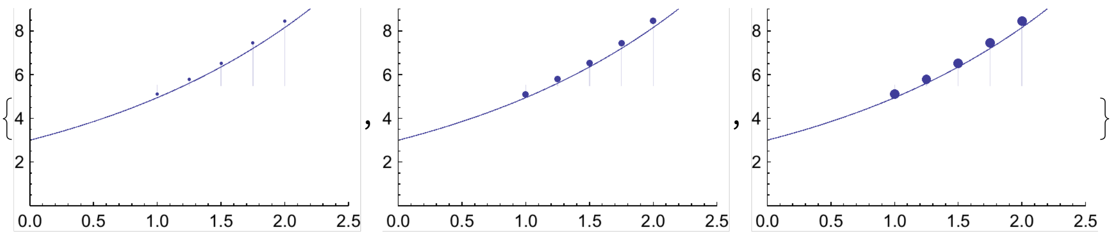
```
e^1.122489190973264
```

3.07249

```
g[x_] := 3.07249 e^(0.50725 x);

p1 = Plot[g[x], {x, 0, 2.5}, PlotRange → {{0, 2.5}, {0, 9}}];
p = Table[ListPlot[data, PlotStyle → PointSize[k], Filling → Automatic],
   {k, 0.01, 0.03, 0.01
   }];

Table[Show[p1, p[[k]]], {k, 1, 3}]
```



```
g[x_] := 3 e^(0.5 x);

p1 = Plot[g[x], {x, 0, 2.5}, PlotRange → {{0, 2.5}, {0, 9}}];
p = Table[ListPlot[data, PlotStyle → PointSize[k], Filling → Automatic],
   {k, 0.01, 0.03, 0.01
   }];

Table[Show[p1, p[[k]]], {k, 1, 3}]
```



指派问题，**有4个工人**，**要指派他**们**去完成4**种**工作**，

每**人做各**种**工作所消耗的**时间**如下表所示**，问**指派**哪**个人去完成**哪种**工作**，**可使**总**的消耗**时间为**最小？**

```
mat = str2mat[15 × 18 × 21 × 24 × 19 × 23 × 22 × 18 × 26 × 17 × 16 × 19 × 19 × 21 × 23 × 17, 4]
```

```
{{15, 18, 21, 24}, {19, 23, 22, 18}, {26, 17, 16, 19}, {19, 21, 23, 17}}
```

```
天干 = ToString /@ {甲, 乙, 丙, 丁, 戊, 己, 庚, 辛, 壬, 癸}
```

```
{甲, 乙, 丙, 丁, 戊, 己, 庚, 辛, 壬, 癸}
```

```
天干Range[x_] := Fold[Take, 天干, Flatten@Position[天干, ToString@x]]
```

```
天干Range[丁]
```

```
{甲, 乙, 丙, 丁}
```

```
TableForm[mat, TableHeadings -> {天干Range[4], CharacterRange["A", "D"]}]
```

	A	B	C	D
甲	15	18	21	24
乙	19	23	22	18
丙	26	17	16	19
丁	19	21	23	17

```
mat1 = mat - Min /@ mat;
% // MatrixForm
```

$$\begin{pmatrix} 0 & 3 & 6 & 9 \\ 1 & 5 & 4 & 0 \\ 10 & 1 & 0 & 3 \\ 2 & 4 & 6 & 0 \end{pmatrix}$$

```
ReplacePart[mat1, {{1, 1} → Framed[0], {2, 4} → Framed[0], {3, 3} → Framed[0]}] //
 MatrixForm
```

$$\begin{pmatrix} \boxed{0} & 3 & 6 & 9 \\ 1 & 5 & 4 & \boxed{0} \\ 10 & 1 & \boxed{0} & 3 \\ 2 & 4 & 6 & 0 \end{pmatrix}$$

```
mat2 = ReplacePart[mat1, {{{1, 1}, {2, 4}, {3, 3}} → Framed[0], {4, 4} → ϕ}];
% // MatrixForm
```

$$\begin{pmatrix} \boxed{0} & 3 & 6 & 9 \\ 1 & 5 & 4 & \boxed{0} \\ 10 & 1 & \boxed{0} & 3 \\ 2 & 4 & 6 & \phi \end{pmatrix}$$

```
TableForm[mat2, TableHeadings -> {{"", √"", "", √""}, {"", "", "", √""}}]
```

				√
	$\boxed{0}$	3	6	9
√	1	5	4	$\boxed{0}$
	10	1	$\boxed{0}$	3
√	2	4	6	ϕ

直接在这上面划线**好像有困**难，**可以在新矩**阵**里**划线**或做**标记**，**
事实**上不**划线**也没事**，这**里要求得的是**划线**覆盖所有零的直**线**数**，这**里**显**然是3，**
没有被直线**覆盖的元素中的最小元素**为**1。**

2 行4行减去1，4 列加上1

```
mat3 = ReplacePart[mat1, {2 → mat1[[2]] - 1, 4 → mat1[[4]] - 1}];
mat4 = ReplacePart[Transpose[mat3], {-1 → mat3[[All, 4]] + 1}] // Transpose;
% // MatrixForm
```

$$\begin{pmatrix} 0 & 3 & 6 & 10 \\ 0 & 4 & 3 & 0 \\ 10 & 1 & 0 & 4 \\ 1 & 3 & 5 & 0 \end{pmatrix}$$

标记

```
mat5 =
  ReplacePart[mat4, {{{1, 1}, {2, 4}, {3, 3}} → Framed[0], {{4, 4}, {2, 1}} → ϕ}];
TableForm[mat5, TableHeadings -> {{√"", √"", "", √""}, {√"", "", "", √""}}]
```

	√			√
√	$\boxed{0}$	3	6	10
√	ϕ	4	3	$\boxed{0}$
	10	1	$\boxed{0}$	4
√	1	3	5	ϕ

1, 2, 4 行减去3，1 列4列加上3

```
mat6 = ReplacePart[mat4 - 3, {3 → mat4[[3]]}];(*全部减去3，三行加上3*)
mat7 = ReplacePart[Transpose[mat6],
    {-1 → mat6[[All, 4]] + 3, 1 → mat6[[All, 1]] + 3}] // Transpose;
% // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & 3 & 10 \\ 0 & 1 & 0 & 0 \\ 13 & 1 & 0 & 7 \\ 1 & 0 & 2 & 0 \end{pmatrix}$$

现**在框出独立的零**，**求出**结**果指派。**

```
solmat1 = ReplacePart[mat7, {{{1, 1}, {4, 2}, {3, 3}, {2, 4}} → Framed[0]}];
% // MatrixForm
```

$$\begin{pmatrix} \boxed{0} & 0 & 3 & 10 \\ 0 & 1 & 0 & \boxed{0} \\ 13 & 1 & \boxed{0} & 7 \\ 1 & \boxed{0} & 2 & 0 \end{pmatrix}$$

相应指派为:

```
Thread[天干Range[丁] → ToString /@ {A, D, C, B}]
```

```
solmat2 = ReplacePart[mat7, {{{2, 1}, {1, 2}, {3, 3}, {4, 4}} → Framed[0]}];
% // MatrixForm
```

$$\begin{pmatrix} 0 & \boxed{0} & 3 & 10 \\ \boxed{0} & 1 & 0 & 0 \\ 13 & 1 & \boxed{0} & 7 \\ 1 & 0 & 2 & \boxed{0} \end{pmatrix}$$

相应指派为:

```
Thread[天干Range[丁] → ToString /@ {B, A, C, D}]
```

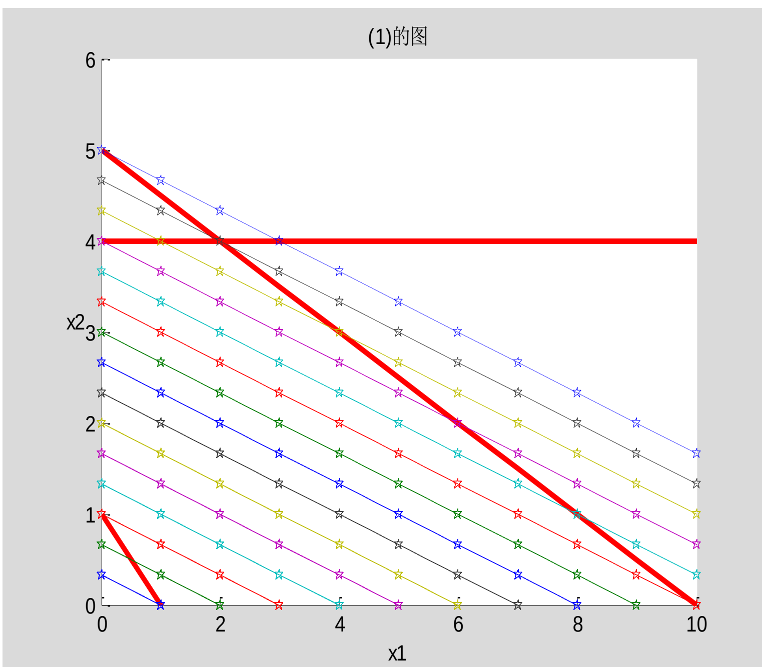
{甲 → B, 乙 → A, 丙 → C, 丁 → D}

1.1 用图解法,并指出解的情况

$$\max z = x_1 + 3x_2$$

（1） $\begin{cases} 5x_1 + 10x_2 \le 50 \\ x_1 + x_2 \ge 1 \\ x_2 \le 4 \\ x_1, x_2 \ge 0 \end{cases}$ x2=(z-x1)/3

```
clear;clf;x1=0:10;x2=[(50-5*x1)/10;1-x1;repmat(4,1,11)];plot(x1,x2)
line(x1,x2,'Color','r','LineWidth',2.5)
hold on
for z=1:15
x3(:,z)=(z-x1)/3;
plot(x1,x3,'--p')
end
xlabel('x1');box off;axis([0 10 0 6]);ylabel('x2');
set(get(get(gcf,'Children'),'YLabel'),'Rotation',0);title('(1)的图')
hold off
```

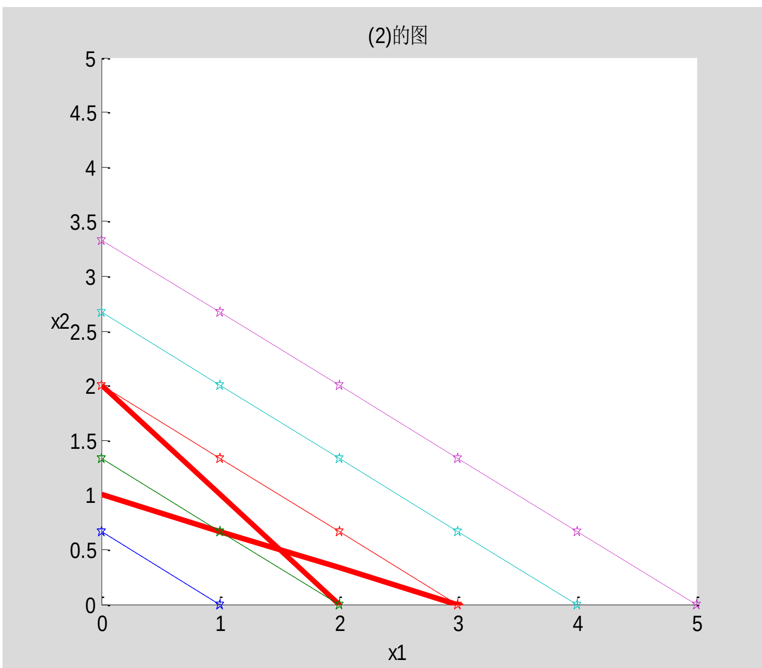


故有唯一最优解 x1=2,x2=4,z=14

$$\min z = x_1 + 1.5x_2$$

（2） $\begin{cases} x_1 + 3x_2 \geq 3 \\ x_1 + x_2 \geq 2 \\ x_1, x_2 \geq 0 \end{cases}$ x2=(z-x1)/1.5

```
clear;clf;clf;x1=0:5;x2=[(3-x1)/3;2-x1];plot(x1,x2)
line(x1,x2,'Color','r','LineWidth',2.5)
hold on
for z=1:5
x3(:,z)=(z-x1)/1.5;
plot(x1,x3,'--p')
end
xlabel('x1');box off;axis([0 5 0 5]);ylabel('x2');
set(get(get(gcf,'Children'),'YLabel'),'Rotation',0);title('(2)的图')
```



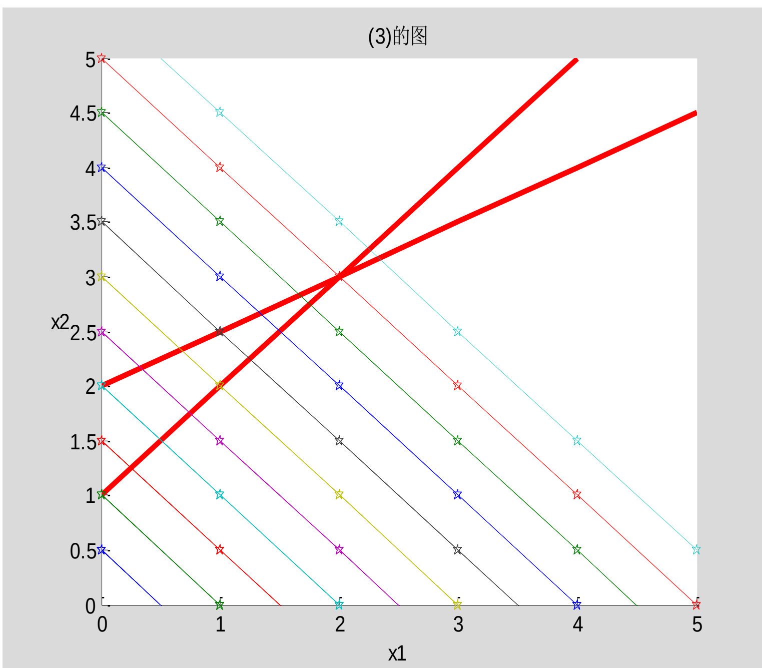

```
syms x1
x1=solve((3-x1)/3-(2-x1))

x1 =
                                        3/2
x2=3-x1

x2 =
                                        3/2

x1+1.5*x2

ans =
                                    3.750000000
```

故有唯一最优解 x1=3,x2=0,z=3

$$\max z = 2x_1 + 2x_2$$

(3) $\begin{cases} x_1 - x_2 \geq -1 \\ -0.5x_1 + x_2 \leq 2 \\ x_1, x_2 \geq 0 \end{cases}$ x2=(z-2*x1)/2

```
clear;clf;clf;x1=0:5;x2=[1+x1;2+0.5*x1];plot(x1,x2)
line(x1,x2,'Color','r','LineWidth',2.5)
hold on
for z=1:11
x3(:,z)=(z-2*x1)/2;
plot(x1,x3,'--p')
end
xlabel('x1');box off;axis([0 5 0 5]);ylabel('x2');
set(get(get(gcf,'Children'),'YLabel'),'Rotation',0);title('(3)的图')
```



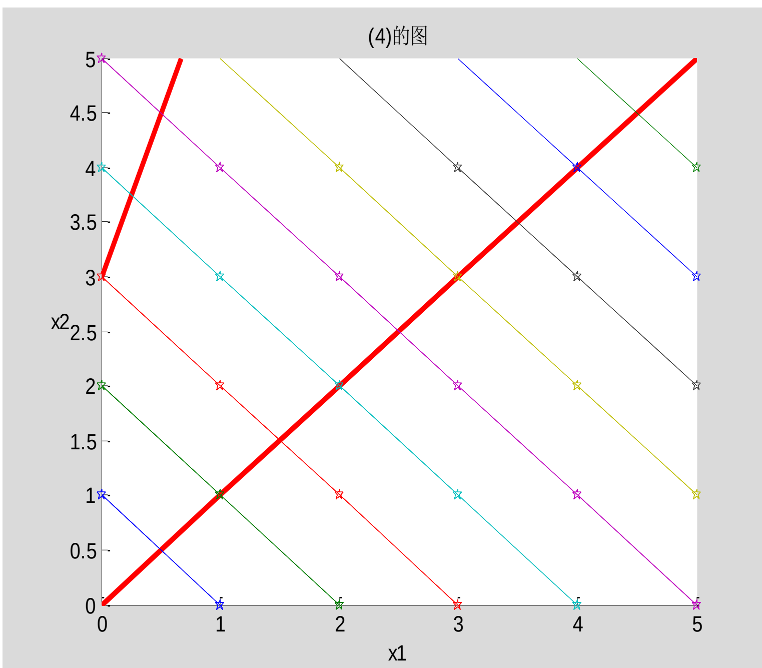
故有可行解 x1=2,x2=3,z=10 但无界,

```
x1=100;x2=50;
x1-x2,-0.5*x1+x2

ans =
    50
ans =
     0
```

$$\max z = x_1 + x_2$$

(3) $\begin{cases} x_1 - x_2 \ge 0 \\ 3x_1 - x_2 \le -3 \\ x_1, x_2 \ge 0 \end{cases}$ x2=z-x1

```
 clear;clf;clf;x1=0:5;x2=[x1;3+3*x1];plot(x1,x2)
line(x1,x2,'Color','r','LineWidth',2.5)
hold on
for z=1:11
x3(:,z)=z-x1;
plot(x1,x3,'--p')
end
xlabel('x1');box off;axis([0 5 0 5]);ylabel('x2');
set(get(get(gcf,'Children'),'YLabel'),'Rotation',0);title('(4)的图')
```



故无可行解

1.2 将下列 LP 问题变成标准型, 并列出初始单纯形表

(1)

$$
\min z = -3x_1 + 4x_2 - 2x_3 + 5x_4
$$

$$
\begin{cases}
4x_1 - x_2 + 2x_3 - x_4 = -2 \\
x_1 + x_2 + 3x_3 - x_4 \le 14 \\
-2x_1 + 3x_2 - x_3 + 2x_4 \ge 2 \\
x_1, x_2, x_3 \ge 0, x_4 \text{无约束}
\end{cases}
$$

标准形

$$
\max z' = 3x_1 - 4x_2 + 2x_3 - 5\left(x_5 - x_6\right) + 0x_7 + 0x_8
$$

$$
\begin{cases}
-4x_1 + x_2 - 2x_3 + \left(x_5 - x_6\right) = 2 \\
x_1 + x_2 + 3x_3 - \left(x_5 - x_6\right) + x_7 = 14 \\
-2x_1 + 3x_2 - x_3 + 2\left(x_5 - x_6\right) - x_8 = 2 \\
x_1, x_2, x_3, x_5, x_6, x_7, x_8 \ge 0
\end{cases}
$$

增加一个人工变量

$$
\max z' = 3x_1 - 4x_2 + 2x_3 - 5\left(x_5 - x_6\right) - Mx_4 + 0x_7 + 0x_8 - Mx_9
$$

$$
\begin{cases}
-4x_1 + x_2 - 2x_3 + \left(x_5 - x_6\right) + x_4 = 2 \\
x_1 + x_2 + 3x_3 - \left(x_5 - x_6\right) + x_7 = 14 \\
-2x_1 + 3x_2 - x_3 + 2\left(x_5 - x_6\right) - x_8 + x_9 = 2 \\
x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \ge 0
\end{cases}
$$

初始单纯形表

	$c_j \to$		3	-4	2	-5	5	0	0	$-M$	$-M$	
C_B	X_B	b	x_1	x_2	x_3	x_5	x_6	x_7	x_8	x_4	x_9	θ
5	x_4	2	-4	1	-2	1	-1	0	0	1	0	
0	x_7	14	1	1	3	-1	1	1	0	0	0	
0	x_9	2	-2	3	-1	2	-2	0	-1	0	1	
	$c_j - z'_j$	0	0	0	0	0	0	0	0			

Charpter1-LP

1.1 Convert the following LP problems to standard form:

(1) $$\begin{cases} \max & z = 3x1 - 5x2 + 4x3 - 6x4, \\ & x1 - 2x2 + 3x3 - 4x4 = -5, \\ \text{s.t.} & x1 + x2 - x3 + 2x4 \le 20, \\ & -3x1 + 5x2 + 2x3 - x4 \ge 3, \\ & x1,\ x2,\ x3 \ge 0;\ x4\ \text{unconstrained}. \end{cases}$$ (2)

$$\begin{cases} \max & z = -3x1 - x2 + 5x3 + 2x4, \\ & x1 + 7x2 + 4x3 - 2x4 \ge -9, \\ \text{s.t.} & 2x1 - x2 + 4x3 + 3x4 \le 10, \\ & 6x1 + 2x2 + x3 + x4 \ge 5, \\ & x1 \le 0;\ x2,\ x3 \ge 0;\ x4\ \text{unconstrained}. \end{cases}$$

```
(1) Minimize[z = 3 x1 - 5 x2 + 4 x3 - 6 x4,
  {x1 - 2 x2 + 3 x3 - 4 x4 == -5, x1 + x2 - x3 + 2 x4 ≤ 20, -3 x1 + 5 x2 + 2 x3 - x4 ≥ 3 , x1 ≥ 0,
   x2 ≥ 0, x3 ≥ 0}, {x1, x2, x3}]
```

$$\left\{\begin{cases} -\infty & x1 \ge 0 \\ \infty & \text{True} \end{cases},\ \{x1 \to \text{Indeterminate},\ x2 \to \text{Indeterminate},\ x3 \to \text{Indeterminate}\}\right\}$$

```
(2) Maximize[z = -3 x1 - x2 + 5 x3 + 2 x4,
  {x1 + 7 x2 + 4 x3 - 2 x4 ≥ -9, 2 x1 - x2 + 4 x3 + 3 x4 ≤ 10, 6 x1 + 2 x2 + x3 + x4 ≥ 5,
   x1 ≤ 0, x2 ≥ 0, x3 ≥ 0}, {x1, x2, x3, x4}]
```

```
{∞, {2 (x1 → Indeterminate), 2 (x2 → Indeterminate),
  2 (x3 → Indeterminate), 2 (x4 → Indeterminate)}}
```

1.2 Graph method:

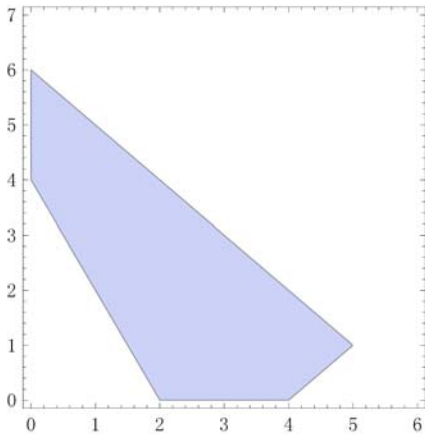
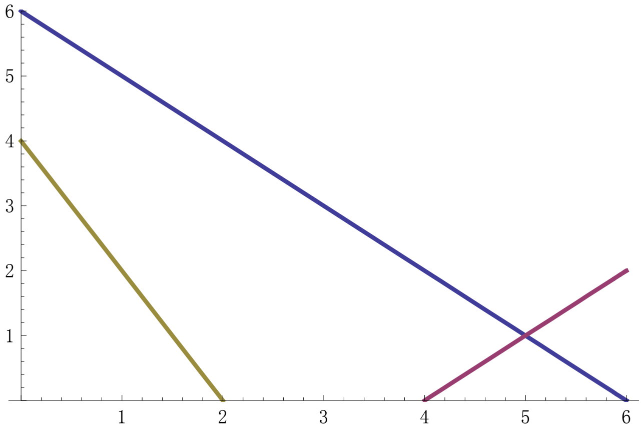
(1) $$\begin{cases} \max & z = 2x1 + 5x2, \\ & x1 + x2 \le 6, \\ \text{s.t.} & x1 - x2 \le 4, \\ & 2x1 + x2 \ge 4, \\ & x1,\ x2 \ge 0. \end{cases}$$

(2) $$\begin{cases} \min & f = x1 + 4x2, \\ & -x1 + x2 \ge 1, \\ \text{s.t.} & x1 + x2 \le 3, \\ & x1,\ x2 \ge 0. \end{cases}$$

```
(1) Maximize[z = 2 x1 + 5 x2, {x1 + x2 ≤ 6, x1 - x2 ≤ 4, 2 x1 + x2 ≥ 4, x1 ≥ 0, x2 ≥ 0}, {x1, x2}]
```

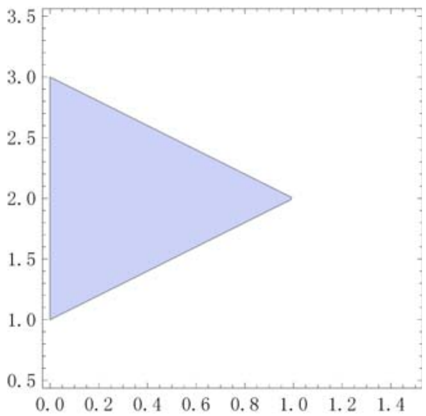
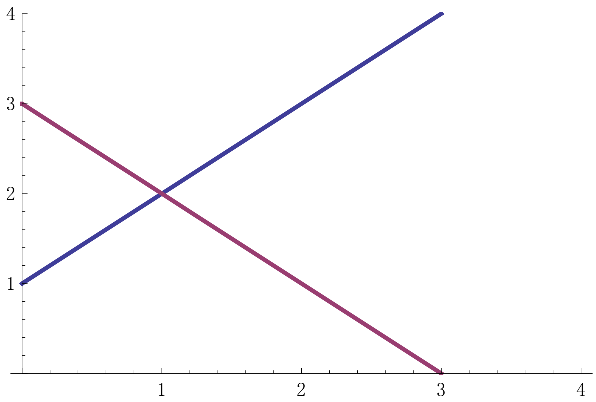
```
{30, {x1 → 0, x2 → 6}}
```

```
RegionPlot[x + y ≤ 6 && x - y ≤ 4 && 2 x + y ≥ 4 && x ≥ 0 && y ≥ 0, {x, 0, 6}, {y, 0, 7}]
Plot[{6 - x, x - 4, 4 - 2 x, x ≥ 0}, {x, 0, 6}, PlotRange → {0, 6}, PlotStyle → Thick]
```

```
(2) Minimize[f = x1 + 4 x2, {-x1 + x2 ≥ 1, x1 + x2 ≤ 3, x1 ≥ 0, x2 ≥ 0}, {x1, x2}]
{4, {x1 → 0, x2 → 1}}
```

```
RegionPlot[-x + y ≥ 1 && x + y ≤ 3 && x ≥ 0 && y ≥ 0, {x, 0, 1.5}, {y, 0.5, 3.5}]
region = Plot[{1 + x, 3 - x, x ≥ 0}, {x, 0, 4}, PlotRange → {0, 4}, PlotStyle → Thick]
```

1.3 Simplexmethod:

(1) $\begin{cases} \min & f = -4x1 + 5x2 - 3x3 + 6x4, \\ & 5x1 - 2x2 + 3x3 - 2x4 = -1, \\ \text{s.t.} & x1 + 2x2 + 3x3 - x4 \le 15, \\ & -3x1 + x2 - x3 + x4 \ge 3, \\ & xj \ge 0,\ j = 1, 2, 3, 4. \end{cases}$

(2) $\begin{cases} \min & f = 4x1 + 8x2 + 12x3, \\ & x1 + 3x3 \ge 4, \\ \text{s.t.} & 3x2 + 2x3 \ge 7, \\ & xj \ge 0,\ j = 1, 2, 3. \end{cases}$

(1) Minimize[f = -4 x1 + 5 x2 - 3 x3 + 6 x4,
{5 x1 - 2 x2 + 3 x3 - 2 x4 == -1, x1 + 2 x2 + 3 x3 - x4 ≤ 15, -3 x1 + x2 - x3 + x4 ≥ 3,
x1 ≥ 0, x2 ≥ 0, x3 ≥ 0, x4 ≥ 0}, {x1, x2, x3, x4}]

$$\left\{\frac{91}{3}, \left\{x1 \to 0, x2 \to \frac{8}{3}, x3 \to 5, x4 \to \frac{16}{3}\right\}\right\}$$

(2) Minimize[f = 4 x1 + 8 x2 + 12 x3, {x1 + 3 x3 ≥ 4, 3 x2 + 2 x3 ≥ 7, x1 ≥ 0, x2 ≥ 0, x3 ≥ 0},
{x1, x2, x3}]

$$\left\{\frac{496}{9}, \left\{2\ (x1 \to 0), 2\left(x2 \to \frac{13}{9}\right), 2\left(x3 \to \frac{4}{3}\right)\right\}\right\}$$

1.4 Big-M method:

$$(1)\ \begin{cases} \min & f = x1 + 4\,x2 + 7\,x3, \\ & x1 + 3\,x2 + 2\,x3 \ge 5, \\ \text{s.t.} & 3\,x1 + 2\,x2 \ge 7, \\ & xj \ge 0,\ j = 1, 2, 3. \end{cases} \qquad (2)\ \begin{cases} \max & f = 3\,x1 - x2 + 3\,x3, \\ & x1 + x2 + x3 \le 8, \\ \text{s.t.} & -3\,x1 + x3 \le 2, \\ & 4\,x2 - x3 \ge 1, \\ & xj \ge 0,\ j = 1, 2, 3. \end{cases}$$

(1) Minimize[f = x1 + 4 x2 + 7 x3, {x1 + 3 x2 + 2 x3 ≥ 5, 3 x1 + 2 x2 ≥ 7, x1 ≥ 0, x2 ≥ 0, x3 ≥ 0},
{x1, x2, x3}]

{5, {x1 → 5, x2 → 0, x3 → 0}}

(2) Maximize[f = 3 x1 - x2 + 3 x3,
{x1 + x2 + x3 ≤ 8, -3 x1 + x3 ≤ 2, 4 x2 - x3 ≥ 1, x1 ≥ 0, x2 ≥ 0, x3 ≥ 0}, {x1, x2, x3}]

$$\left\{46, \left\{2\left(x1 \to \frac{31}{4}\right), 2\left(x2 \to \frac{1}{4}\right), 2\ (x3 \to 0)\right\}\right\}$$

1.5 Two-phase method :

$$(1)\ \begin{cases} \min & f = 8\,x1 + 6\,x2 + 3\,x3 + 2\,x4, \\ & x1 + 2\,x2 + x4 \ge 3, \\ & 3\,x1 + x2 + x3 + x4 \ge 6, \\ \text{s.t.} & 2\,x3 + x4 \ge 2, \\ & x1 + x3 \ge 2, \\ & xj \ge 0,\ j = 1, 2, 3, 4. \end{cases} \qquad (2)$$

$$\begin{cases} \max & f = 5\,x1 + 10\,x2 + 7\,x3, \\ & 3\,x1 + 2\,x2 + x3 \le 7, \\ \text{s.t.} & -3\,x1 + 4\,x2 + 10\,x3 \le 12, \\ & 5\,x1 + 3\,x2 + x3 \ge 6, \\ & xj \ge 0,\ j = 1, 2, 3. \end{cases}$$

(1) Minimize[f = 8 x1 + 6 x2 + 3 x3 + 2 x4,
{x1 + 2 x2 + x4 ≥ 3, 3 x1 + x2 + x3 + x4 ≥ 6, 2 x3 + x4 ≥ 2, x1 + x3 ≥ 2, x1 ≥ 0, x2 ≥ 0,
x3 ≥ 0, x4 ≥ 0}, {x1, x2, x3, x4}]

{14, {x1 → 0, x2 → 0, x3 → 2, x4 → 4}}

```
(2) Maximize[f = 5 x1 + 10 x2 + 7 x3,
  {3 x1 + 2 x2 + x3 ≤ 7, -3 x1 + 4 x2 + 10 x3 ≤ 12, 5 x1 + 3 x2 + x3 ≥ 6, x1 ≥ 0, x2 ≥ 0, x3 ≥ 0},
  {x1, x2, x3}]
```

$$\left\{\frac{590}{9},\ \left\{2\left(x1\to\frac{2}{9}\right),\ 2\left(x2\to\frac{19}{6}\right),\ 2\,(x3\to 0)\right\}\right\}$$