MATH 111 - Calculus and Analytic Geometry I

Lecture 3 Worksheet

Fall 2020 | **Subhadip Chowdhury** | **Aug 24**

Title: Chapter 1 - Review of Functions III

Summary: Next in our list of Transcendental functions are Exponential and Logarithm. They are both one-to-one and inverses of each other.

§A. Exponential Function

You likely first dealt with exponents algebraically when trying to solve some equations where part of the exponent involved a variable, like in the next two problems.

Question 1.

(a) If $4^{x+1} = 16$, then what does x equal?

(b) If $3^6 \times 3^x = 1$, then what does x equal? (**Hint:** you can combine 3^6 and 3^x.)

Question 2.

Sketch a graph of a general exponential function $f(x) = b^x$ where $b > 1$. What would the graph look like if $b = 1$? What about a graph of $f(x) = b^x$, where $0 < b < 1$?

If you look at the graph of the function $f(x) = \left(1 + \frac{1}{x}\right)^x$, you'll notice that the graph appears to have a horizontal asymptote as $x \to \infty$. This asymptote value is the irrational number e that you have probably heard of before. When we use $b = e$ as the base for an exponential function $y = b^x$, we get **the natural exponential function**.

§B. Logarithmic Function

Since exponential functions are one-to-one, they have inverses, called **logarithmic functions**. For the function $y = e^x$, the corresponding inverse function is called the **natural logarithm**, $y = \ln(x)$.

- In general, $b^x = a \implies x = \log_b a$.
- $\ln x$ is the same thing as $\log_e x$.
- The two most important properties of $\log$ that you might need to use are
  - $\log_c(ab) = \log_c a + \log_c b$
  - $\log_c(a^b) = b \log_c a$

Question 3.

Use the fact that the exponential function and natural logarithm are inverses to compute the following:

a) $\ln(e^{\pi}) =$

b) $e^{\ln(2)} =$

■ **Question 4.**

Find equation of an exponential function of the form $\mathbf{A}e^{\mathbf{B}x}$ whose graph looks like figure 1.

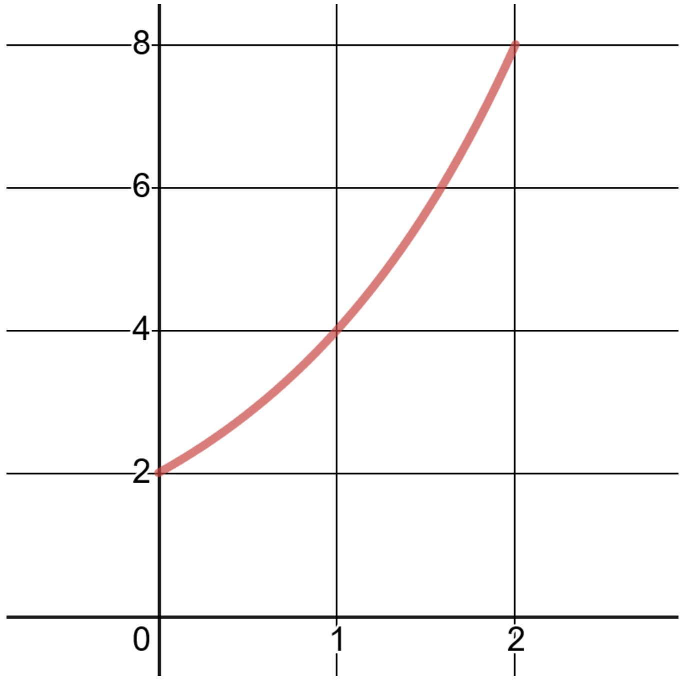

Figure 1

§C. Exponential Growth and Decay Word Problems

- An exponential growth/decay process is given by the equation
$$Q(t) = Q_0 e^{kt}$$
  k is called the (continuous) growth/decay rate.
- If $k > 0$, the process is a growth. If $k < 0$, the process is a decay.
- In an exponential decay process, the time it takes to reduce the starting amount by half is called the **half-life**, denoted $t_{1/2}$.

■ **Question 5.**

Show that $t_{1/2} = \dfrac{\ln 2}{|k|}$.

■ **Question 6.**

A biologist is researching a newly-discovered species of bacteria. At time $t = 0$ hours, she puts one hundred bacteria into what she has determined to be a favorable growth medium. Six hours later, she measures **450** bacteria. Assuming exponential growth, how long does it take for the bacteria population to become **1600**?

■ **Question 7.**

The process of carbon-dating involves evaluating the ratio of radioactive carbon-14 to stable carbon-12 isotope. Carbon-14 has a half-life of **5730** years and decays over time whereas carbon-12 doesn't.

You are presented with a document which purports to contain the recollections of a Mycenaean soldier during the Trojan War. The city of Troy was finally destroyed about $\mathbf{3250}$ years ago. Given the amount of carbon-12 contained a measured sample cut from the document, there would have been about $\mathbf{1.3 \times 10^{-12}}$ grams of carbon-14 in the sample when the parchment was new, assuming the proposed age is correct. According to your equipment, there remains $\mathbf{1.0 \times 10^{-12}}$ grams. Is there a possibility that this is a genuine document? Or is this instead a recent forgery? Justify your conclusions.