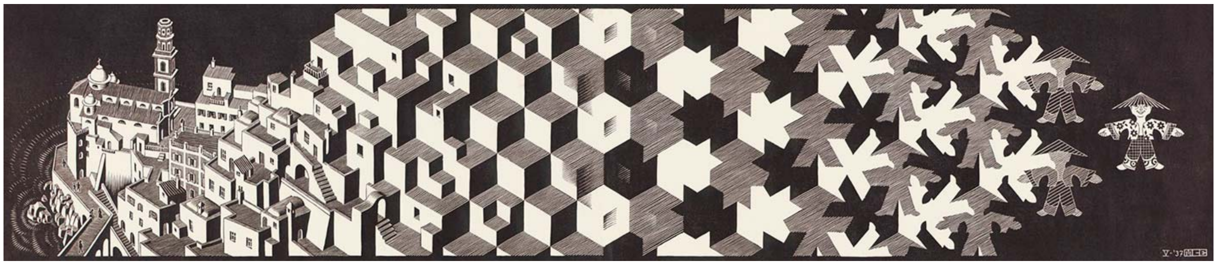

Theory of Differential Calculus

Math 115 Lecture notes

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Contents

Chapter 1 | Old Functions, New Functions

§1.1 A Catalog of Essential Functions

1.1.1 Polynomials

A function P is called a polynomial if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are constants called the coefficients of the polynomial. The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$. If the leading coefficient $a_n \neq 0$, then the degree of the polynomial is n. For example, the function

$$P(x) = 2x^6 - x^4 + \frac{2}{5}x^3 + \sqrt{2}$$

is a polynomial of degree 6.

If $n = 1$, we say that $P(x)$ is linear, if $n = 2$, it's a quadratic, and so on.

1.1.2 Power Functions

A function of the form $f(x) = x^a$, where a is a constant, is called a power function. Here a does not necessarily have to an integer, it can be any positive or negative real number (or 0).

1.1.3 Rational Function

A function which can be written as the quotient of two polynomials is called a **rational function**. The following are examples of rational functions

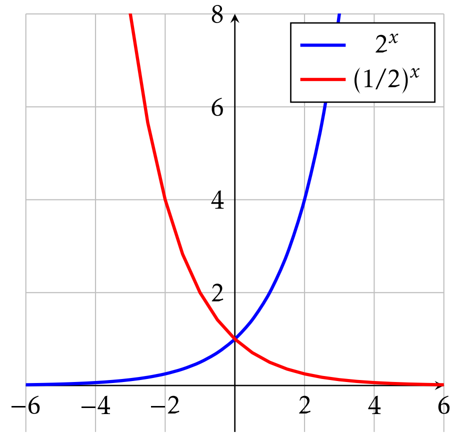
(i) $\dfrac{1}{x}$ (ii) $\dfrac{x^3 + 5x - 5}{x^8 - 3}$ (iii) $\dfrac{x^9 - 4x^4 + 2x}{x^7 - 3x^2 + 3}$

By definition, polynomials themselves are rational functions which have denominator 1. e.g. $x + 1 = \dfrac{x+1}{1} = \dfrac{x+1}{x^0}$, so it's a rational function.

1.1.4 Exponential Function

Exponential functions have the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$. When $b > 1$, an exponential function models rapid **growth**, while if $0 < b < 1$, the function models rapid **decay**. The graphs of $f(x) = 2^x$ and $f(x) = (0.5)^x$ are shown in the figure below. In both cases the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

Graphs of Exponential Functions

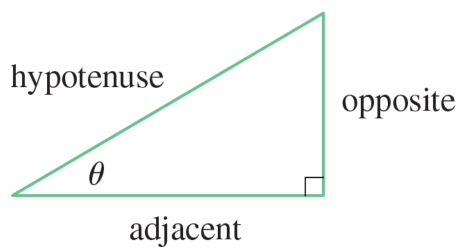


1.1.5 Logarithmic Function

The logarithmic functions $f(x) = \log_b x$, where the base b is a positive constant, are the inverse functions of the exponential functions. We will not go into too much details about inverses here. We will come back to these later.

§1.2 Trigonometry Review

You likely learned about trig functions via right triangles. For an acute angle θ, the six trigonometric functions are defined as ratios of lengths of sides of a right triangle as follows:



(a) $\sin(\theta) = \dfrac{opp}{hyp}$

(b) $\cos(\theta) =$

(c) $\tan(\theta) = \dfrac{opp}{adj}$

(d) $\csc(\theta) =$

(e) $\sec(\theta) = \dfrac{hyp}{adj}$

(f) $\cot(\theta) =$

The definition above doesn't apply to obtuse or negative angles, so for a general angle θ in standard position we let $P \equiv (x, y)$ be any point on the terminal side of θ and we let r be the distance $|OP|$ as in the figure below. Then we can algebraically define

(a) $\sin(\theta) = \dfrac{y}{r}$ (b) $\cos(\theta) = \dfrac{x}{r}$ (c) $\tan(\theta) = \dfrac{y}{x}$

> **Note:** This definition automatically takes care of the sign.

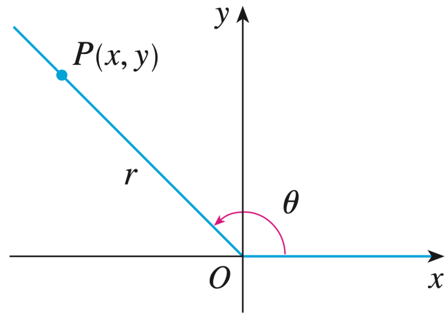


Figure 1.1: General Definition of Trigonometric Functions

The picture below, derived using Euclidean geometry results, will help you recall the trig values of some common angles. The circle in the picture has radius 1.

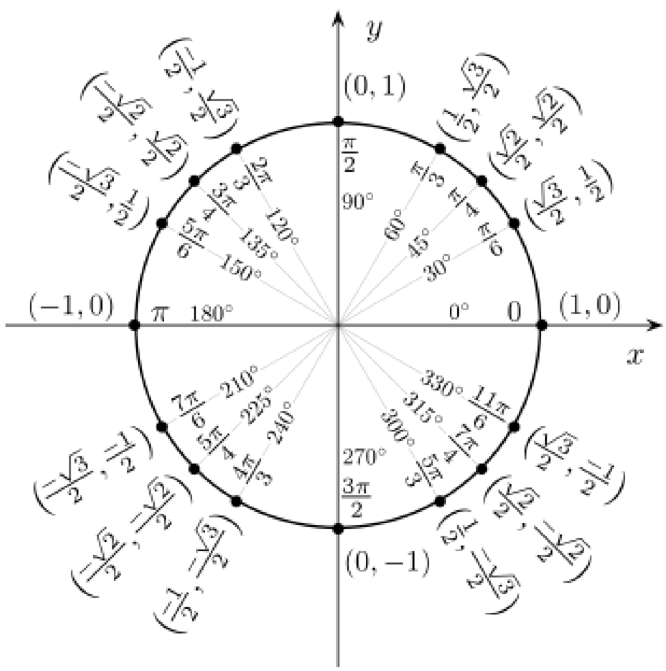


Figure 1.2: Common Trig Values

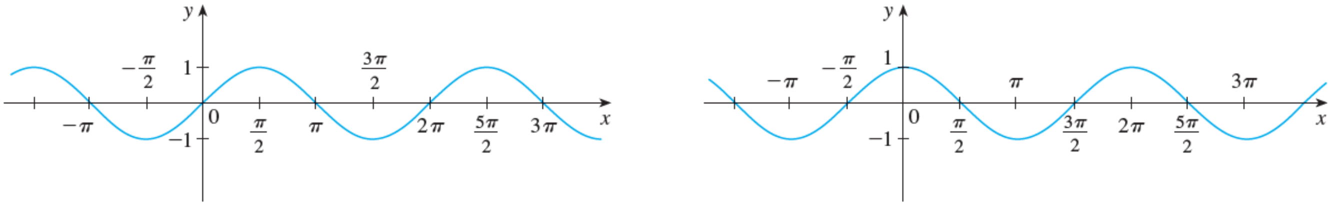
Example 1.2.1

Since the length of the radius is 1, we can use above picture to calculate

$$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2} \qquad \text{and} \qquad \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}.$$

1.2.1 Trigonometric Functions

In calculus, when we use the notation $f(x) = \sin(x)$ to mean the sine function, it is implicitly understood that $\sin x$ means the sine of the angle whose radian measure is x. Thus the graphs of the sine and cosine functions are given in fig. 1.3.



(a) $f(x) = \sin x$

(b) $g(x) = \cos x$

Figure 1.3: Graphs of $\sin x$ and $\cos x$

§1.3 Transformation of Functions

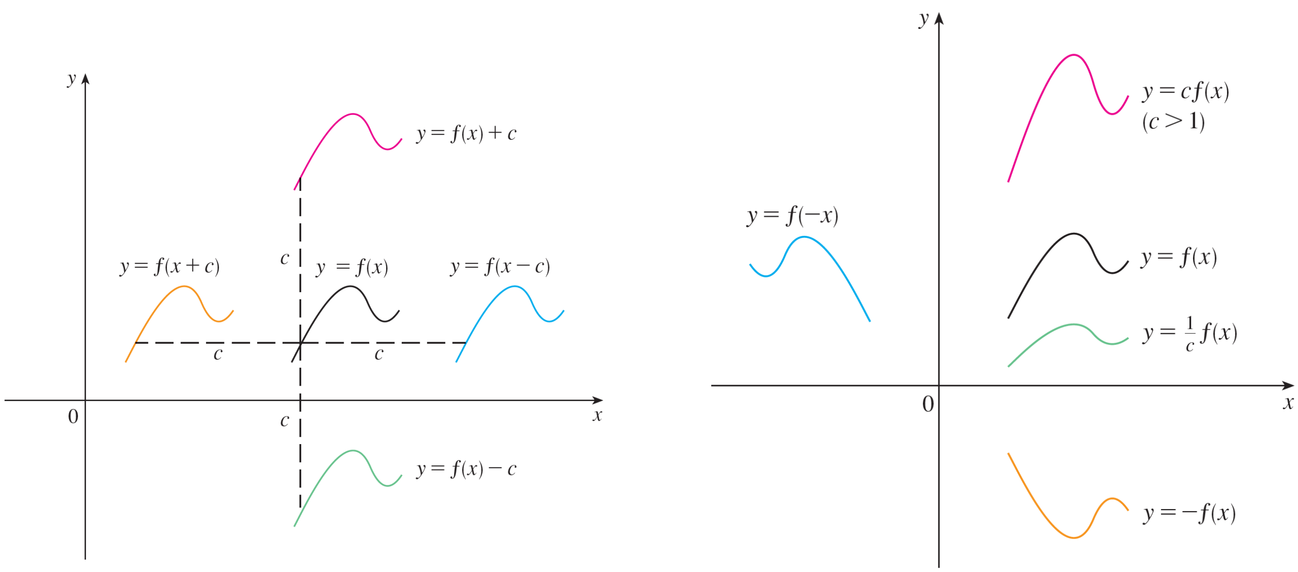
By applying certain transformations to the graph of a given function we can obtain the graphs of related functions. This will give us the ability to sketch the graphs of many functions quickly by hand. It will also enable us to write equations for given graphs.

1.3.1 Translations

See fig. 1.4a.

1.3.2 Vertical and Horizontal Stretching and Reflecting

See fig. 1.4b.



(a) Translating the graph of f

(b) Stretching and Reflecting the graph of f, where $c > 1$

Figure 1.4: Translation and Scaling

Question 1.

For the given functions, classify them as one of the following:

(i) *a single type of function from the list in section 1.1.*

(ii) *Product/Quotient of two different type of functions from above list.*

(iii) *Composition of two different type of functions from above list.*

(iv) *not from the above list.*

In each case, you must specify what type of function each one is.

(a) e^{x^3+5x}

(b) $\dfrac{x^3}{1+\ln x}$

(c) $\ln(3x^2+1)$

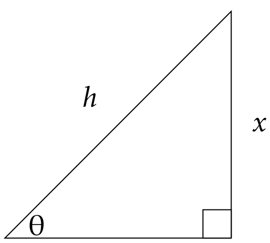
(d) e^x+4

(e) x^x

(f) $\sqrt{x+2}$

Question 2.

Use the right triangle below to write h as a function of x and θ.



Question 3.

Try to use the values in figure 1.2 above, and your knowledge of the unit circle, to compute the trig values below.

(a) $\sin\left(\dfrac{-5\pi}{6}\right)=$

(b) $\cos\left(-\dfrac{3\pi}{4}\right)=$

(c) $\tan\left(\dfrac{14\pi}{3}\right)=$

(d) $\sec\left(\dfrac{7\pi}{6}\right)=$

Question 4.

What are the values of x for which $\sin x = 0$? What about $\sin x = \frac{1}{2}$?

Question 5.

What is the domain of the function $f(x)=\dfrac{1}{1-2\cos x}$?

Question 6.

Starting from the graph of $\cos x$,

(a) *First try to draw the graph of* $\cos(x/2)$, $2\cos(x/2)$, $1-2\cos(x/2)$ *without use of any technology!*

(b) *Compare your answer to the pictures you get from* `DESMOS`.

Question 7.

Do the same for $\sqrt{x}$, $\sqrt{x}-2$, $\sqrt{x-2}$, $\sqrt{-x}$, $\sqrt{2-x}$.

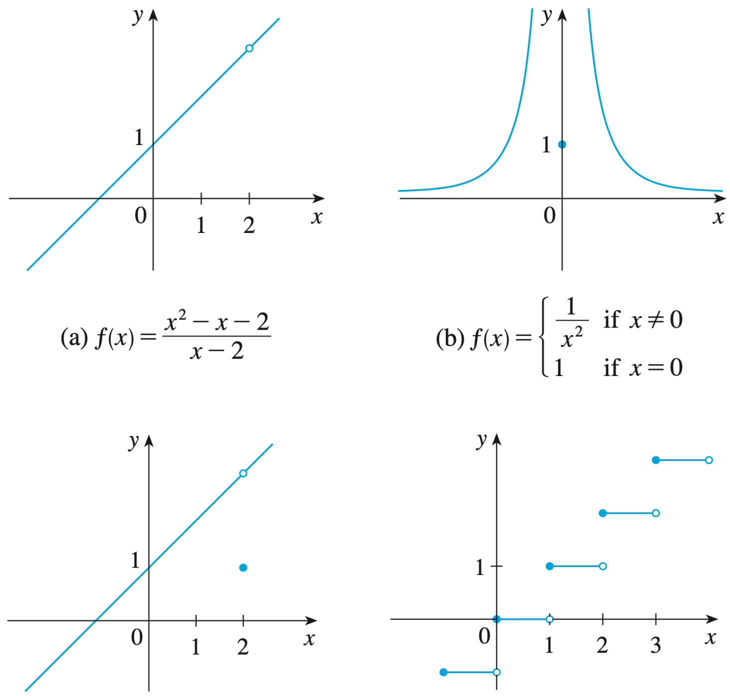
Question 8.

Same for x^2, x^2-2, *and* $|x^2-2|$. *How does the absolute value function*($|.|$) *change the graph?*

Chapter 2 | Limit and Continuity

§2.1 Continuity of a Function on an Interval: An Informal Recap

In your previous Calculus courses, you have probably learned how to visually identify points of discontinuity of a function and how to calculate limits of continuous functions. As a recap, consider graphs of four functions as follows:



(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

(d) $f(x) = [\![x]\!]$

Figure 2.1: Examples of Discontinuous Functions

Each of the four functions are discontinuous at one or more points. However, the reason for why discontinuity is different in each case.

Informally speaking, we say a function is **continuous at a point** if you can draw its graph without lifting the pencil from the paper at the point. Another way to think of this is nearby values of x give nearby values of y. In practical terms, this means small errors in the input leads to only small errors in output.

More formally, we will define continuity at a point as follows:

Definition 2.1.2

A function $f(x)$ is defined to be continuous at a point $x = a$ if

$$\lim_{x \to a} f(x) = f(a)$$

■ **Question 9.** □

From definition 2, note that there are three requirements for a function $f(x)$ to be continuous at $x = a$. Can you list the three things?

For each of the four examples in fig. 2.1, which one of the requirement is not satisfied by the function?

As we can see, the idea of continuity is inextricably linked with the idea of limit. So before we discuss different types of discontinuities in more details, let's reintroduce the idea of Limits formally once more.

§2.2 A Precise Definition of Limit

We will start with an informal definition of limit you may have seen in the past.

Definition 2.2.3

Assume that a function f is defined on an interval around a, except perhaps at the point $x = a$. We define the limit of the function $f(x)$ as x approaches a to be a number L (if one exists) such that $f(x)$ can be made to remain as close to L as we want by choosing x sufficiently close to a (but $x \neq a$). If L exists, we write

$$\lim_{x\to a} f(x) = \text{L}.$$

The intuitive definition of a limit given above is inadequate for some purposes because phrases such as "x approaches a"and "$f(x)$ can be made to remain as close to L" are vague at best (and misleading at worst). Indeed what does it precisely mean to say "close to" or "approach"? In order to conclusively prove any limit results (by algebraic or analytic means), we must make the definition of the word "approach" more precise.

To motivate the precise definition of a limit, let's consider the function

$$f(x) = \begin{cases} 2x-1 & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$$

Intuitively, it is clear that when x is close to 3 but $x \neq 3$, then $f(x)$ is close to 5, and so $\lim_{x\to 3} f(x) = 5$.

To obtain more detailed information about how $f(x)$ varies when x is close to 3, we ask the following question: Can I make $f(x)$ as close to 5 as I want?

How close to 3 does x have to be so that $f(x)$ differs from 5 by, say, less than 0.1?

The **distance** from x to 3 is $|x-3|$ and the **distance** from $f(x)$ to 5 is $|f(x)-5|$, so our problem is to find a number δ such that

$$|f(x)-5| < 0.1 \quad \text{if} \quad 0 < |x-3| < \delta$$

The $0 < |x-3|$ condition comes from the fact that we don't want $x = 3$.

Notice that

$$|f(x)-5| = |(2x-1)-5| = |2x-6| = 2|x-3|$$

So, if $0 < |x-3| < (0.1)/2 = 0.05$, then $|f(x)-5| < 2\times 0.05 = 0.1$. In other words,

$$|f(x)-5| < 0.1 \quad \text{if} \quad 0 < |x-3| < 0.05$$

Thus an answer to our question above is given by $\delta = 0.05$; that is, if x is within a distance of 0.05 from 3 , then $f(x)$ will be within a distance of 0.1 from 5 .

If we change the number 0.1 in our problem to the smaller number 0.01, then by using the same method we find that

$$|f(x) - 5| < 0.01 \quad \text{if} \quad 0 < |x - 3| < 0.005$$

Similarly,

$$|f(x) - 5| < 0.001 \quad \text{if} \quad 0 < |x - 3| < 0.0005$$

The numbers $0.1, 0.01$, and 0.001 that we have considered are **error tolerances** that we might allow. **For 5 to be the precise limit of $f(x)$ as x approaches 3, we must not only be able to bring the difference between $f(x)$ and 5 below each of these three numbers; we must be able to bring it below any positive number.** And, by similar reasoning as above, we can always do this for any error tolerance! If we write ε to denote the error tolerance, then we find that

$$|f(x) - 5| < \varepsilon \quad \text{if} \quad 0 < |x - 3| < \delta = \frac{\varepsilon}{2}$$

Using this example as a model, we give the following precise definition of limit.

Definition 2.2.4: Precise Definition of a Limit

Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the limit of $f(x)$ as x approaches a is L, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon$$

It is important to realize that the process as illustrated in the first two pictures below must work for every positive number ε, no matter how small it is chosen. The last picture below shows that if a smaller ε is chosen, then a smaller δ may be required.

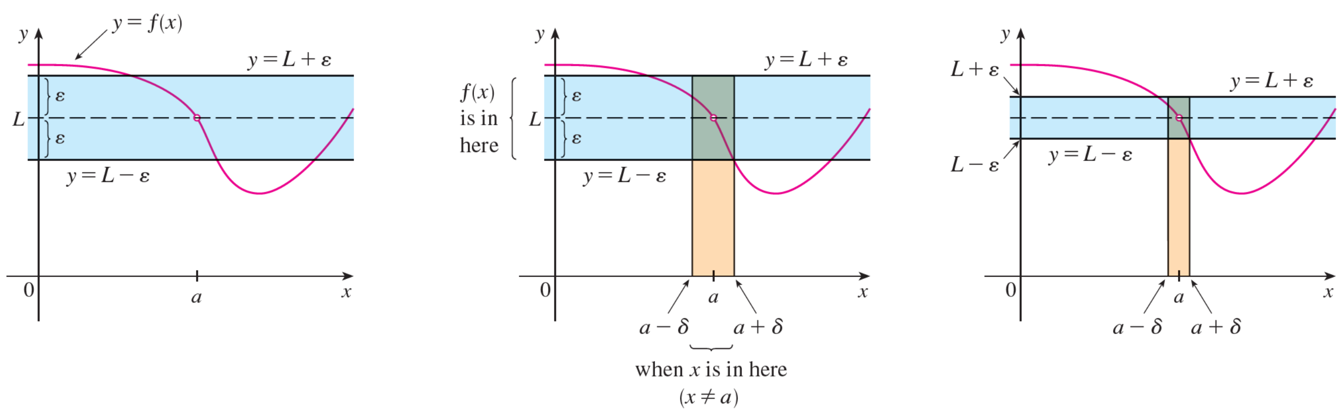


Figure 2.2: The Limit Process

■ Question 10. □

If the function f is defined by

$$f(x) = \begin{cases} 0 & \textit{if } x \textit{ is rational} \\ 1 & \textit{if } x \textit{ is irrational} \end{cases}$$

show that $\lim\limits_{x\to 0} f(x)$ *does not exist.*

2.2.1 Extending the Idea of Limits

Recall from our last class that the functional value at a point $f(a)$ and the limiting value at the same point $\lim\limits_{x\to a} f(x)$ may not be the same. In fact, one or both of these may or may not exist. In light of all these possibilities, we want to introduce two notations which will allow us to describe the behavior of a function near a point a in a bit more detail.

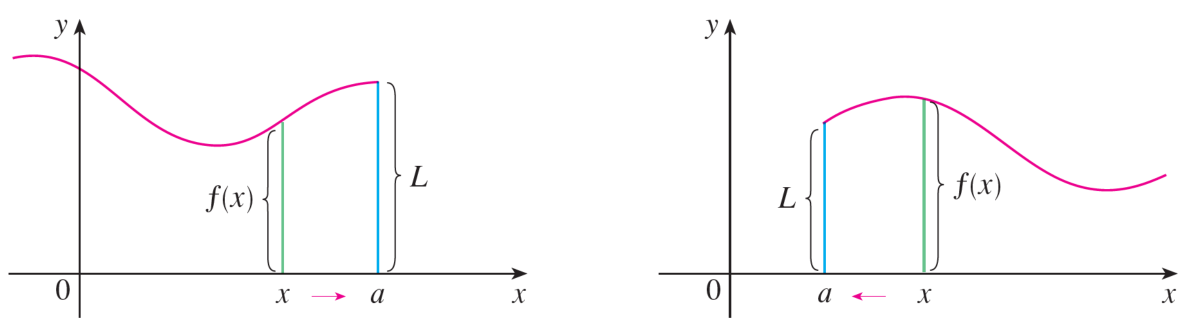
Definition 2.2.5: One Sided Limits

We write

$$\lim_{x\to a^-} f(x) = L$$

and say the **left-hand limit** of $f(x)$ as x approaches a (or the limit of $f(x)$ as x approaches a from the left) is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with *values of x less than a.*

Similarly, $\lim\limits_{x\to a^+} f(x)$ denotes the **right-hand limit**, where the notation $x \to a^+$ means that we consider only x *greater than a.*



(a) $\lim\limits_{x\to a^-} f(x) = L$ (b) $\lim\limits_{x\to a^+} f(x) = L$

Figure 2.3: One-sided Limits

More precisely, we can write

$$\lim_{x\to a^-} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad a-\delta < x < a \quad \text{then} \quad |f(x) - L| < \varepsilon$$

Notice that this is the same as the precise definition of limit except that x is restricted to lie in the left half $(a-\delta, a)$ of the interval $(a-\delta, a+\delta)$.

Question 11.

Give a precise definition of right-hand limit using ε and δ.

Theorem 2.2.6

$\lim_{x\to c} f(x) = L$ *if and only if both* $\lim_{x\to c^+} f(x)$ *and* $\lim_{x\to c^-} f(x)$ *exist, and they are both equal to* L.

Question 12.

Consider the fourth graph from fig. 2.1 in the last section. Use the picture to find

$\lim_{x\to 1^-} f(x) =$ $\lim_{x\to 1^+} f(x) =$ $\lim_{x\to 1} f(x) =$ $f(1) =$

Question 13.

Use Desmos to draw the graph of the function $f(x) = \dfrac{|x-1|}{x-1}$. *Use it to determine the given limits. Can you find these values without drawing a picture?*

$\lim_{x\to 1^-} \dfrac{|x-1|}{x-1} =$ $\lim_{x\to 1^+} \dfrac{|x-1|}{x-1} =$ $\lim_{x\to 1} \dfrac{|x-1|}{x-1} =$

Definition 2.2.7: Horizontal Asymptotes and Limits

We say $\lim_{x\to\infty} f(x) = L$ if the value of $f(x)$ can be be made to stay as close as L as we want by making x sufficiently large. More precisely,

$$\lim_{x\to\infty} f(x) = L$$

means that for every positive number ε there is a positive number M such that

$$\text{if } x > M \text{ then } |f(x) - L| < \varepsilon.$$

In that case we say the line $y = L$ is an **horizontal asymptote** to the graph of the function $f(x)$. We can similarly define $\lim_{x\to-\infty} f(x) = L$.

Example 2.2.8

Euler's constant e can be defined as

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

Question 14.

Discuss how we might find $\lim_{x\to\infty} \dfrac{x+3}{4-x}$ *algebraically. Make sure to justify why you might do any particular operation. We will come back to this question in the next section.*

Definition 2.2.9: Vertical Asymptotes and Limits

If one or both of the one-sided limit of a function $f(x)$ at a point a approach infinity or negative infinity, we say that the graph of $f(x)$ has a **vertical asymptote** at $x = a$.

Question 15.

Give a precise definition of what it means to write $\lim_{x\to a} f(x) = \infty$.

Note: The symbol ∞ is not a number. It is used to denote the adjective "not finite". However, there is a distinction between $+\infty$ and $-\infty$. Writing $\lim_{x\to a} f(x) = \infty$ means $f(x)$ **increases** without bound as x approaches a from **both sides.** Similarly, $\lim_{x\to a} f(x) = -\infty$ means $f(x)$ **decreases** without bound as x approaches a.

If $\lim_{x\to a^+} f(x) = \infty$ and $\lim_{x\to a^-} f(x) = -\infty$, then we will say $\lim_{x\to a} f(x)$ does not exist and avoid using the notation of infinity.

Question 16.

For the following graph, find

$\lim_{x\to -3^-} f(x) =$	$\lim_{x\to -1^-} f(x) =$	$\lim_{x\to 2^-} f(x) =$
$\lim_{x\to -3^+} f(x) =$	$\lim_{x\to -1^+} f(x) =$	$\lim_{x\to 2^+} f(x) =$
$\lim_{x\to -3} f(x) =$	$\lim_{x\to -1} f(x) =$	$\lim_{x\to 2} f(x) =$

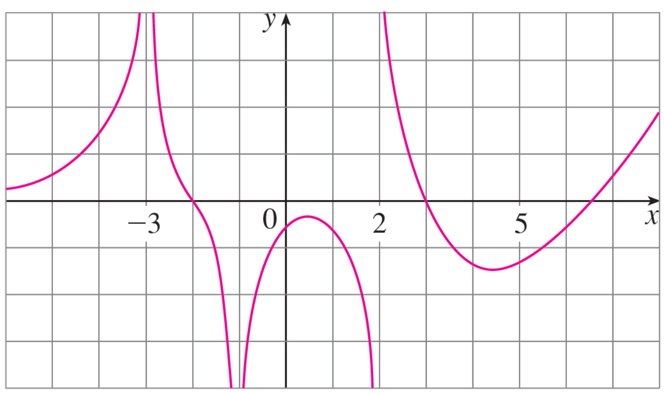


Figure 2.4: Infinite Limits

§2.3 Limits Laws

You might recall the following Limit laws from your previous Calculus courses.

Theorem 2.3.10: Arithmetic of Limits

If $\lim_{x\to c} f(x) = \mathrm{L}$ *and* $\lim_{x\to c} g(x) = \mathrm{K}$, *then*

(a) $\lim_{x\to c}[f(x) \pm g(x)] = L \pm K$

(b) $\lim_{x\to c}[f(x)g(x)] = LK$

(c) $\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{L}{K}$ *if* $K \neq 0$

(d) $\lim_{x\to c} a = a$

(e) $\lim_{x\to c} x = c$

Using the product law repeatedly, we can get $\lim_{x\to c} f(x)^n = L^n$ where n is a positive integer. This, along with the quotient law, gives $\lim_{x\to c} f(x)^n = L^n$ for negative integers n as long as L is nonzero.

Replacing $f(x)$ with x, we get that $\lim_{x\to c} x^n = c^n$ for positive integers n. Then $\lim_{x\to c} ax^n = ac^n$, which, by the addition law, implies $\lim_{x\to c} P(x) = P(c)$ for any polynomial function $P(x)$.

Infact, extending the idea further, if we have a rational function of the form $f(x) = \dfrac{P(x)}{Q(x)}$, then by quotient law, $\lim_{x\to c} f(x) = f(c)$ *for any* c *in the domain of* f.

> **Note:** The domain part is important as we don't want $Q(c) = 0$.

Thus we have found a collection of functions (polynomials and rational functions) where the limit can be calculated algebraically by direct substitution - these are precisely what we define to be **continuous** functions! Unfortunately, not all limits can be calculated by direct substitution (e.g. discontinuous functions).

2.3.1 Limits of Quotients

In calculus we often encounter limits of the form $\lim_{x\to c}\dfrac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are continuous. There are three types of behavior for this type of limit:

- When $g(c) \neq 0$, the limit can be evaluated by substitution (the quotient law).
- When $g(c) = 0$ but $f(c) \neq 0$, the limit is undefined.
- When $g(c) = 0$ and $f(c) = 0$, the limit may or may not exist and can take any value.

The third type is the most interesting and we will spend rest of this section learning different strategies to handle those limit.

Question 17. □

Find the following limits.

(a) $\lim_{x\to 3}\dfrac{x^2+2x+1}{x-1}$

(b) $\lim_{x\to 2}\dfrac{x+1}{x-2}$

Limits of the Form $\dfrac{0}{0}$

Example 2.3.11: Factorize and Cancel

$$\lim_{x\to -2}\frac{x^2-4}{x+2}=\lim_{x\to -2}\frac{(x-2)(x+2)}{x+2}=\lim_{x\to -2}(x-2)=\lim_{x\to -2}x-\lim_{x\to -2}2=-2-2=4$$

Question 18.

Find the following limits.

(a) $\displaystyle\lim_{x\to 3}\frac{x^2-3x}{2x^2-5x-3}$.

(b) $\displaystyle\lim_{x\to 0}\frac{(3+x)^2-9}{x}$

(c) $\displaystyle\lim_{x\to 1}\frac{x-1}{x^2-2x+1}$

Example 2.3.12: Multiply by Conjugate

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}=\lim_{x\to 4}\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)}=\lim_{x\to 4}\frac{x-4}{(x-4)(\sqrt{x}+2)}=\lim_{x\to 4}\frac{1}{\sqrt{x}+2}=\frac{1}{4}$$

Question 19.

Find the following limits.

(a) $\displaystyle\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{x-1}$.

(b) $\displaystyle\lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1}$

Limits of the Form $\dfrac{\infty}{\infty}$

Example 2.3.13: Divide by the highest power

$$\lim_{x\to\infty}\frac{3+4x^2}{x^2+3x+2}=\lim_{x\to\infty}\frac{\frac{3}{x^2}+4}{1+\frac{3}{x}+\frac{2}{x^2}}=\frac{0+4}{1+0+0}=4$$

Question 20.

Find the following limits.

(a) $\displaystyle\lim_{x\to\infty}\frac{3+4x}{x^2+3x+2}$

(b) $\displaystyle\lim_{x\to\infty}\frac{3+4x^2}{3x+2}$

2.3.2 The Squeeze Theorem

For the problems in this section, consider the function $f(x) = x^2 \cos\left(\frac{\pi}{x}\right)$.

(a) Sketch a graph of the function $f(x)$ below. (You may want to use a graphing utility e.g. DESMOS)

(b) Also graph the functions $-x^2$ and x^2 along with $f(x)$.

(c) Based off your sketch, do you agree that $-x^2 \leq f(x) \leq x^2$ for all $x \neq 0$?

(d) What is $\lim_{x\to 0} -x^2$? What is $\lim_{x\to 0} x^2$? What can you conclude about $\lim_{x\to 0} x^2 \cos\left(\frac{\pi}{x}\right)$?

Theorem 2.3.14: The Squeeze Theorem

If we have

$$b(x) \leq f(x) \leq a(x)$$

for all x close to $x = c$ except possibly at $x = c$, and if

$$\lim_{x\to c} b(x) = L = \lim_{x\to c} a(x),$$

then

$$\lim_{x\to c} f(x) = L.$$

The book uses the squeeze theorem to show that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, by using a geometric argument. I suggest you read it, but you do not need to memorize the exact proof. The key to applying the squeeze theorem is determining the inequality, which can come from geometry or an algebra identity. We are going to try to show the same limit as above for the function $f(x) = x^2 \cos\left(\frac{\pi}{x}\right)$, but this time without using graphs to justify our reasoning, by following the steps below.

(e) What is the range of the cosine function $y = \cos(\theta)$?

(f) Use your previous answer to find a constant k such that $-k \leq \cos(\theta) \leq k$ for every θ.

(g) Replace θ with $\frac{\pi}{x}$. For what x-values is the inequality still valid?

(h) Conclude that you have the same inequality as in (3) above, and apply the squeeze theorem to obtain the limit.

(i) Can you follow the same steps to show that $\lim_{x\to 0} x \sin\left(\frac{\pi}{x}\right) = 0$?

Example 2.3.15: Two Interesting Examples

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

■ Question 21.

□

Find $\lim_{x\to 0} \frac{\sin 3x}{2x}$.

§2.4 Back to Continuity

Using the idea of one-sided and infinite limits, we can now define the idea of continuity on an interval.

Definition 2.4.16: Continuity on an Interval

A function f is said to be continuous on an open interval (a,b) if it is continuous at every point in the interval.

A function f is said to be continuous from the right at c if $\lim_{x\to c^+} f(x) = f(c)$.

A function f is said to be continuous from the left at c if $\lim_{x\to c^-} f(x) = f(c)$.

A function f is said to be continuous on a closed interval $[a,b]$ if it is continuous on (a,b), and continuous from the right at a and continuous from the left at b.

We learned in the last section that

- any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.
- any rational function is continuous wherever it is defined; that is, it is continuous on its domain.
- Two other types of functions that are continuous on their domain are root functions and trigonometric functions.

Question 22. □

(a) *Explain why* $\lim_{x\to 1} \sqrt{x-1}$ *does not exist.*

(b) *On what interval is* $f(x) = \sqrt{x-1}$ *continuous?*

(c) *On what intervals is* $g(x) = \dfrac{x^2-10x}{x^2-16x+60}$ *continuous?*

(d) *On what intervals is* $f(x) = \dfrac{\sqrt{x-4}-1}{(x-5)(x-6)}$ *continuous?*

2.4.1 Limits of Combinations of Continuous Functions

The following basic arithmetic laws are straightforward from the definition of continuity using limit:

Theorem 2.4.17: Continuity of Sums, Products, and Quotients of Functions

Suppose that f and g are continuous on an interval and that k is a constant. Then, on that same interval,

(a) $kf(x)$ *is continuous.*

(b) $f(x)+g(x)$ *is continuous.*

(c) $f(x)g(x)$ *is continuous.*

(d) $f(x)/g(x)$ *is continuous, provided* $g(x)\neq 0$ *on the interval.*

The textbook uses the following **Composite Function Theorem** to show that the Trigonometric Functions are continuous on their respective domains, but we can also use it to evaluate some interesting limits.

Theorem 2.4.18: Composite Function Theorem

If $f(x)$ *is continuous at* L, *and* $\lim\limits_{x\to a} g(x) = \mathrm{L}$, *then*

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(\mathrm{L}).$$

As a consequence,

Theorem 2.4.19: Continuity of Composite and Inverse Functions

If f and g are continuous, then

(a) if the composite function $f(g(x))$ is defined on an interval, then $f(g(x))$ is continuous on that interval.

(b) if f has an inverse function f^{-1}, then f^{-1} is continuous.

Question 23.

□

For the following problems, evaluate the given limit using the above theorem. Feel free to graph the function to confirm your answer.

(a) $\lim\limits_{x\to \frac{2\pi}{3}} \sin\left(x-\frac{\pi}{3}\right)$.

(b) $\lim\limits_{x\to \frac{2\pi}{3}} \sin\left(1+\cos\left(x+\frac{\pi}{3}\right)\right)$.

(c) $\lim\limits_{x\to \pi^-} \ln(\sin(x))$. *(Recall that* $\lim\limits_{x\to 0^+} \ln(x) = -\infty$.*)*

(d) $\lim\limits_{x\to 1^+} \ln\left(\frac{(x-1)(x^2+2x-3)}{x^2-1}\right)$.

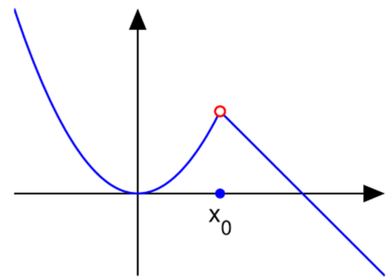
2.4.2 Types of Discontinuity

Recall from our previous discussion about continuity that a function can be discontinuous for several reasons. We can classify the types of discontinuities into three broad categories as follows:

- **Removable Discontinuity** - We say that f has a removable discontinuity at a if $\lim_{x\to a} f(x)$ exists but is not equal to $f(a)$ (which may or may not exist).

Example 2.4.20

Consider the function $f(x) = \begin{cases} x^2 & \text{for } x < 1 \\ 0 & \text{for } x = 1 \\ 2 - x & \text{for } x > 1 \end{cases}$

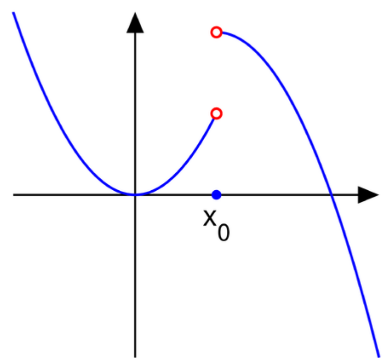


There is a removable discontinuity at $x = 1$. Both the left-hand and right-hand limits are 1.

- **Jump Discontinuity** - We say that f has a jump discontinuity at a if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist, but $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$

Example 2.4.21

Consider the function $f(x) = \begin{cases} x^2 & \text{for } x < 1 \\ 0 & \text{for } x = 1 \\ 2 - (x-1)^2 & \text{for } x > 1 \end{cases}$

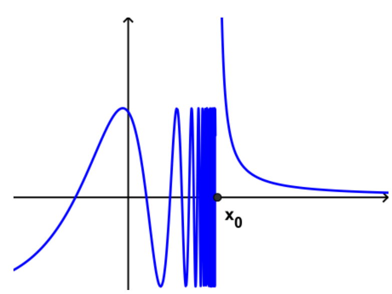


There is a jump discontinuity at $x = 1$. The left-hand limit is 1. The right-hand limit is 2.

- **Essential Discontinuity** - We say that f has an essential discontinuity at a if at least one of the one-sided limits, $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$, does not exist or is infinite.

Example 2.4.22

Consider the function $f(x) = \begin{cases} \sin \dfrac{5}{x-1} & \text{for } x < 1 \\ 0 & \text{for } x = 1 \\ \dfrac{1}{x-1} & \text{for } x > 1 \end{cases}$



There is an essential discontinuity at $x = 1$. The left-hand limit DNE. The right-hand limit is ∞.

2.4.3 Fixing Discontinuities & Gluing

If a function has a removable discontinuity, that implies that we can easily ‘‘fix" the discontinuity by filling in the hole. We can do this by using piecewise functions.

Question 24. ☐

The function $f(x) = \dfrac{x^3-8}{x-2}$ *has a removable discontinuity at* $x = 2$. *Determine the value of* k *necessary to make*

$$\mathrm{F}(x) = \begin{cases} \dfrac{x^3-8}{x-2} & \text{if } x \neq 2 \\ k & \text{if } x = 2 \end{cases}$$

continuous everywhere.

In the next problem, you are trying to determine where exactly you would ‘‘glue" the two lines in order to have a continuous function.

Question 25. ☐

Determine what value of k *will make the function continuous on the given interval.*

$$f(x) = \begin{cases} 2x+7 & \text{if } 0 \leq x < k \\ 4x-5 & \text{if } k \leq x \leq 10 \end{cases}.$$

In the next problem, there are removable discontinuities that need to be filled, but we also need to figure out what values of c and r will make sure the functions are ‘‘glued" together so as to be made continuous. Start by determining what k must be, then use that information to solve for c and then r.

■ Question 26. □

Determine the values of k, c, and r that make the given function continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x^4-16}{x^2-4} & \text{if } x < -2 \\ k & \text{if } x = -2 \\ -x^2+c & \text{if } -2 < x < 3 \\ \sqrt{rx} & \text{if } x \geq 3 \end{cases}.$$

■ Question 27. □

Find the constants a and b, so that the following piece-wise defined function is continuous everywhere.

$$f(x) = \begin{cases} a-bx & \text{if } x \leq 1 \\ x^2 & \text{if } 1 < x < 2 \\ b+ax & \text{if } x \geq 2 \end{cases}$$

2.4.4 The Intermediate Value Theorem

Let's work through a thought exercise.

(a) What was your height at birth? ____________

The average baby born measures about 20 inches at birth.

(b) What is your height today? ____________

(c) Sketch a graph of how your height has changed over the course of your lifetime.

(d) Was there a day in your life when you measured exactly 40 inches?

(e) A continuous function $y = f(x)$ is known to be negative at $x = 0$ and positive at $x = 1$. Explain why the equation $f(x) = 0$ must have at least one solution between $x = 0$ and $x = 1$? Illustrate with a sketch.

(f) How would your observation generalize if instead of $x = 0$ and $x = 1$, you had $x = a$ and $x = b$? Also what if we didn't require the function to be positive and negative at the two end points? Can we still conclude the existence of a root? If not a root, what can we conclude?

> **Theorem 2.4.23**
>
> *Suppose f is continuous on a closed interval $[a, b]$. If k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.*

(g) Draw a picture to explain the Intermediate Value Theorem (IVT) in your own words.

(h) Why do we need continuity for the Intermediate Value Theorem?

(i) Prove that the function $f(x) = x^{12345} + 2x^{6789} - 1$ has at least one zero between 0 and 2 (i.e. $f(x) = 0$).

(j) **True or False:** At some point since you were born your weight in pounds equaled your height in inches.

(k) Can you prove (without graphing!) that the equation $\cos(\theta) = \theta^3$ has at least one real solution?

(l) Suppose $g(x)$ is a continuous function with $g(0) = 3$, $g(1) = 8$, $g(2) = 4$.
True or False: $g(x)$ is an invertible function.

(m) **True or False:** Along the Equator, there are two diametrically opposite sites that have exactly the same temperature at the same time.

Chapter 3 | Derivatives - Definition and Basic Rules

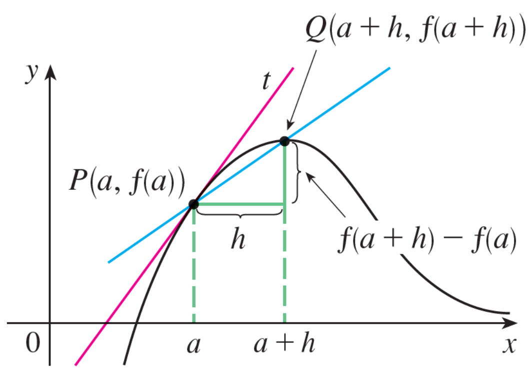
§3.1 Precise Definition of Derivative

Definition 3.1.24

The **derivative** of f at a, written as $f'(a)$, is defined as the instantaneous rate of change of f at a. In other words,

$$f'(a) := \lim_{b \to a} \frac{f(b) - f(a)}{b - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.



The process of finding the derivative of a function is called **differentiation**. A function $f(x)$ is said to be **differentiable** at a if its derivative exists at $x = a$. By now, we have seen lots of examples when a limit doesn't exist or is undefined. If the limit in the definition of the derivative doesn't exist at $x = a$, we say that the derivative $f'(a)$ doesn't exist. So in that case, we say that the function $f(x)$ is not differentiable at $x = a$. A function is said to be differentiable on an open interval (a, b) if its derivative exists at every x in the interval.

Question 28. ☐

Find the derivative of the function $f(x) = x^2 - 8x + 9$ at $x = a$.

Question 29. ☐

Find an equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at $(3, -6)$.

Question 30. ☐

A manufacturer produces bolts of a fabric with a fixed width. The cost of producing x yards of this fabric is $C = f(x)$ dollars.

(a) *What is the meaning of the derivative $f'(x)$? What are its units?*

(b) *In practical terms, what does it mean to say that $f'(1000) = 9$?*

3.1.1 The Derivative Function

Theorem 3.1.25

If $f' > 0$ on an interval, then the slope of the tangent to the graph of f is positive and consequently, f increasing over that interval.

Similarly, if $f' < 0$ on an interval, then f is decreasing over that interval. Finally, if $f' = 0$ on an interval, then f is constant over that interval.

Question 31. □

Sketch the derivative of the function $f(x)$ graphed below.

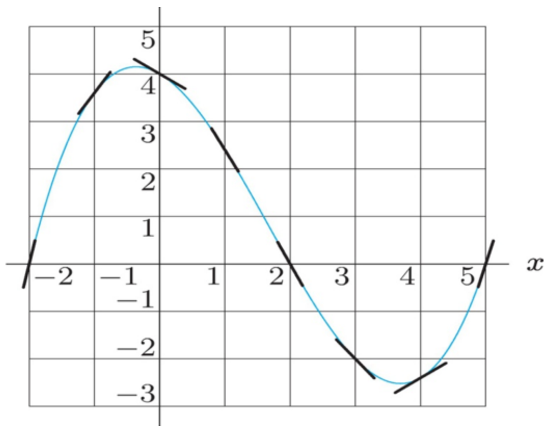


Figure 3.1: Estimating the derivative graphically as the slope of the tangent line

Question 32. □

In Moodle, you will find a collection of graphs of functions and their derivatives. Match the two collections.

3.1.2 Differentiability

Before moving forward, first a few words about notations. There are several different ways we can denote the derivative of f.

- If $f(x)$ is a function of x, then $f'(x)$ means the derivative of $f(x)$ with respect to x.
- We can forego the variable to denote the derivative of f by f'.
- If $y = f(x)$, we also write $f'(x) = \dfrac{dy}{dx}$ or $\dfrac{df(x)}{dx}$ or $\dfrac{\mathrm{d}}{\mathrm{d}x}f(x)$.

Warning: The significance of the third notation is to make you realise that $\dfrac{dy}{dx}$ is NOT a fraction. $\dfrac{\mathrm{d}}{\mathrm{d}x}$ is an operation we perform on $f(x)$ to produce $f'(x)$.

- For example if $f(x) = \sin x$, we can write $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin x$ to mean the derivative of the function $\sin x$.

Theorem 3.1.26

If f is differentiable at a, then f is continuous at a.

In other words, $\boxed{\text{Differentiability} \implies \text{Continuity.}}$ The converse is **not** true. Below are some examples where $f(x)$ is continuous at 0, but isn't differentiable at 0.

Question 33.

For each of the following functions, draw its graph in Desmos. Explain why f is continuous at $x = 0$ but $f'(0)$ is not defined.

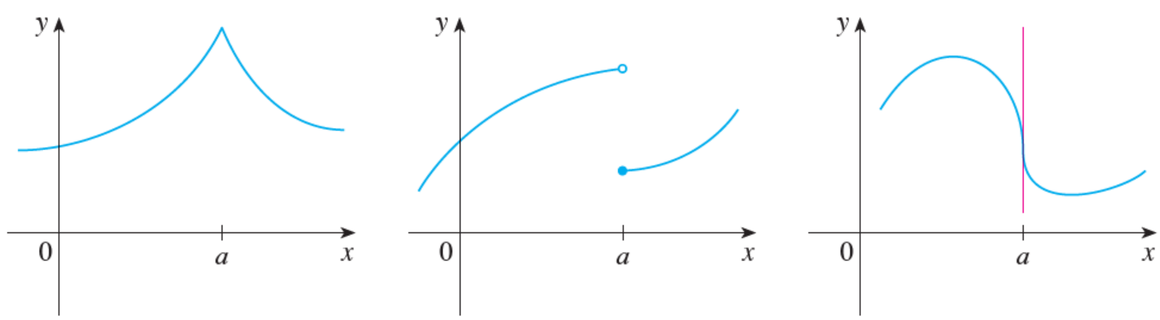
(a) $f(x) = |x|$.

(b) $f(x) = \sqrt[3]{x}$.

(c) $f(x) = \begin{cases} x\sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

□

In general, if the graph of a function f has a "corner" or "cusp" in it, then the graph of f has no tangent at this point and f is not differentiable there. Here are two other scenarios.



(a) A corner (b) A discontinuity (c) A vertical tangent

§3.2 Differentiation Formulas

Theorem 3.2.27: Basic Arithmetic

- *If c is a constant, then* $\dfrac{\mathrm{d}}{\mathrm{d}x}[c] = 0$
- *If c is a constant and $f(x)$ is differentiable, then* $\dfrac{\mathrm{d}}{\mathrm{d}x}[cf(x)] = cf'(x)$
- *If $f(x)$ and $g(x)$ are differentiable, then* $\dfrac{\mathrm{d}}{\mathrm{d}x}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

Question 34.

Here is a question to practice just these basic formulas. Using Table 3.1, compute the following:

(a) *Find $h'(1)$ if $h(x) = 5 - f(x)$.*

(b) *Find $k'(-2)$ if $k(x) = -\frac{1}{2}g(x)$.*

(c) *Find $p'(-2)$ if $p(x) = 2f(x) + 3g(x)$.*

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-2	-6	9	-10	16
1	5	-3	3	-2

Table 3.1: Table for Question 1

Theorem 3.2.28: Power Rule Part I

For any positive integer n,

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}.$$

Question 35.

The graph of $y = x^3 - 9x^2 - 16x + 1$ has a slope of 5 at two points. Find the coordinates of the points.

Definition 3.2.29

The derivative of a function is itself a function, so we can find the derivative of a derivative, called a second derivative. For example, derivative of displacement is velocity, and derivative of velocity is acceleration! Similarly, we can calculate a third derivative, fourth derivative and so on. We use the notations

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}f(x) = \frac{d^2}{dx^2}f(x) = \frac{d^2y}{dx^2}$$
$$f'''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}f(x) = \frac{d^3}{dx^3}f(x) = \frac{d^3y}{dx^3}$$
$$f^{(n)}(x) = \underbrace{\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\cdots\frac{\mathrm{d}}{\mathrm{d}x}}_{n\text{ times}}f(x) = \frac{d^n}{dx^n}f(x) = \frac{d^ny}{dx^n}$$

Question 36.

For the polynomial $f(x) = x^5 + x^3 + x + 1$, find

(a) $f''(x)$, *i.e. the second derivative of* f.

(b) $f^{(5)}(x)$, *i.e. the fifth derivative of* f.

(c) $f^{(6)}(x)$, *i.e. the sixth derivative of* f.

Theorem 3.2.30: Product Rule

If $f(x)$ and $g(x)$ are two differentiable functions, then

$$(fg)' = f'g + fg'.$$

In words: *The derivative of a product is the derivative of the first times the second plus the first times the derivative of the second.*

Question 37.

Let's practice this formula using a table of values like we did before. Use Table 3.1 to compute the given derivative value.

(a) *Find* $h'(1)$ *if* $h(x) = f(x)g(x)$.

(b) *Find* $k'(-2)$ *if* $k(x) = xf(x) - 2g(x)$.

(c) *Find* $p'(-2)$ *if* $p(x) = \dfrac{f(x)g(x)}{2} + x^2 f(x)$.

(d) *Find* $q'(1)$ *if* $q(x) = f(x)^2$.

Theorem 3.2.31: Quotient Rule

For $f(x)$ and $g(x)$ differentiable functions,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-2	4	5	-1	2
1	3	-1	2	-2

Table 3.2: Table for Problem 8

Question 38.

Let's practice! Use Table 3.2 to compute the given derivative value.

(a) *Find* $h'(1)$ *if* $h(x) = \dfrac{f(x)}{g(x)}$.

(b) *Find* $k'(-2)$ *if* $k(x) = \dfrac{xg(x)}{f(x)}$.

(c) *Find* $L'(1)$ *if* $L(x) = \dfrac{x^3+4}{f(x)+g(x)}$.

We can use the quotient rule to extend the power rule to negative integers.

Theorem 3.2.32: Power Rule Part II

For any non-zero integer n,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Question 39.

Find all values of $x = a$ *such that the tangent line to* $f(x) = \dfrac{x-1}{x+8}$ *at* $x = a$ *passes through the origin.*

3.2.1 Derivation of Product Rule and Quotient Rule

Why does the product rule look this way? If we start trying to write out $\frac{d}{dx}[f(x)g(x)]$ using the definition, we get the following limit:

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

To help us see how we can go from this limit to the product rule, we will use some geometry.

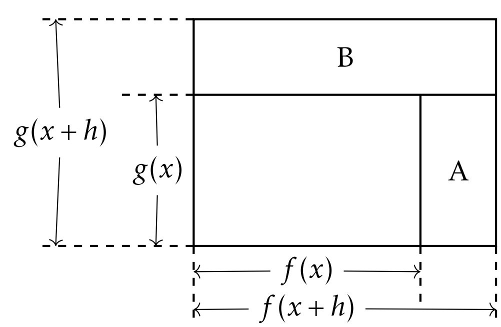


Figure 3.2: Rectangle Aid for Product Rule Formula

Question 40.

Use Figure 3.2 to aid in this problem. Write your answers in terms of $f(x), g(x), f(x+h)$, *and* $g(x+h)$.

(a) *What is the area of the rectangle denoted by* A?

(b) *What is the area of the rectangle denoted by* B?

(c) *What area in the figure does* $f(x+h)g(x+h) - f(x)g(x)$ *correspond to? Write this area in terms of* A *and* B.

Using the formulas from the previous problem, we can now see where the product rule comes from. The numerator $f(x+h)g(x+h) - f(x)g(x)$ can be replaced in our expression for $\frac{d}{dx}[f(x)g(x)]$ as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\ &= \lim_{h\to 0}\frac{(f(x+h)-f(x))g(x)+(g(x+h)-g(x))f(x+h)}{h}\end{aligned}$$

We can now break this fraction into pieces and use the fact that $f(x)$ and $g(x)$ are both differentiable to get our product formula. But note, we also use the fact that $\lim_{h\to 0} f(x+h) = f(x)$, which is true because f is necessarily continuous since it is differentiable.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] &= \lim_{h\to 0}\frac{(f(x+h)-f(x))g(x)+(g(x+h)-g(x))f(x+h)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}g(x)+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}f(x+h)\\ &= f'(x)g(x)+g'(x)f(x)\end{aligned}$$

■ Question 41. □

See if you can't derive the quotient rule formula using the help of Figure 3.3. Follow the steps below.

(a) *Firstly, write out* $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right]$ *using the definition of the derivative. Then, combine your fractions in the numerator.*

(b) *Write out the rectangles* C *and* D *in terms of* $f(x), g(x), f(x+h)$, *and* $g(x+h)$.

(c) *In part (1), did you get the difference* $f(x+h)g(x)-f(x)g(x+h)$? *Identify this difference in terms of areas in Figure 3.3.*

(d) *Put everything together and you've just proven the quotient rule!!*

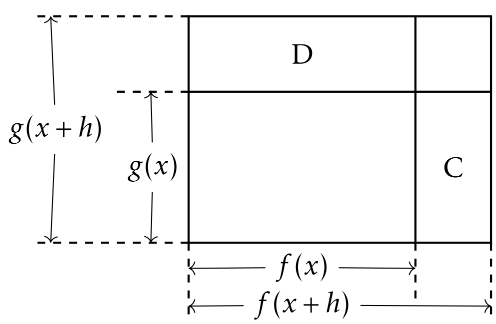


Figure 3.3: Rectangle Aid for Quotient Rule Formula

§3.3 Derivative of Trigonometric Functions

First, let's look at the graph of $f(x) = \sin x$. We might ask, where is the derivative equal to zero? Then ask where the derivative is positive and where it is negative. In exploring these answers, we get something like the following graph.

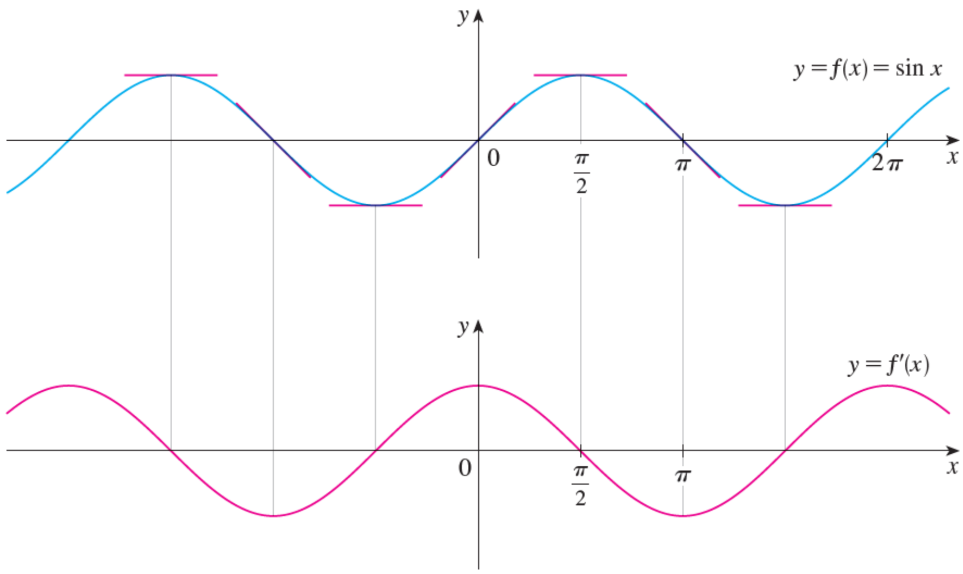


Figure 3.4: The sine function and its derivative

The graph of the derivative in Figure 3.4 looks suspiciously like the graph of the cosine function. This might lead us to conjecture, quite correctly, that the derivative of the sine is the cosine. For a more formal proof, we need the following trigonometric identity:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

Then we can write

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x(\cos h - 1) + \cos x \sin h}{h}\\
&= \sin x\left(\lim_{h\to 0}\frac{(\cos h - 1)}{h}\right) + \cos x\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\\
&= \sin(x)\times 0 + \cos x \times 1\\
&= \cos x
\end{aligned}$$

where we used the fact that $\lim\limits_{h\to 0}\dfrac{\sin h}{h} = 1$ and $\lim\limits_{h\to 0}\dfrac{(\cos h - 1)}{h} = 0$. A proof of the second result is at the end of this section. Similarly, we can calculate derivative of $\cos x$ using the identity

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

Theorem 3.3.33: Trig Derivatives

For x in radians,

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x \quad \textit{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$$

Question 42.

Use the quotient rule to find derivatives of the following trigonometric functions.

(a) $\frac{\mathrm{d}}{\mathrm{d}x}[\tan x]$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}[\cot x]$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}[\sec x]$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}[\csc x]$

□

3.3.1 Proof of a certain result

To calculate $\lim_{h\to 0}\frac{(\cos h - 1)}{h}$, we can use a sort of 'multiply by the conjugate' method.

$$\begin{aligned}
\lim_{h\to 0}\frac{(\cos h-1)}{h} &= \lim_{h\to 0}\frac{(\cos h-1)(\cos h+1)}{h(\cos h+1)} \\
&= \lim_{h\to 0}\frac{\cos^2 h-1}{h(\cos h+1)} \\
&= \lim_{h\to 0}\frac{-\sin^2 h}{h(\cos h+1)} \\
&= -\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{\sin h}{\cos h+1}\right) \\
&= -(1)\left(\frac{\sin 0}{\cos 0+1}\right) \\
&= -(1)\left(\frac{0}{1+1}\right) \\
&= 0
\end{aligned}$$

§3.4 Chain Rule

Chain rule says

Rate of change of composite function	=	Rate of change of outside function	×	Rate of change of inside function

Theorem 3.4.34: Chain Rule

If f and g are differentiable functions, then

$$\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = f'(g(x)) \cdot g'(x)$$

Warning: The derivative of the outside function must be evaluated at the inside function.

This is called the **chain rule** because, if you have multiple compositions (i.e. several functions stuffed inside of each other) then you will end up with a "chain" of products in the derivative. The Leibniz notation is very suggestive and helpful for remembering how the chain rule works: for the function $y = f(u) = f(g(x))$, meaning $u = g(x)$, we have

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Warning: $\frac{dy}{du}$ and $\frac{du}{dx}$ are NOT fractions.

Question 43.

Suppose $f(x)$ and $g(x)$ and their derivatives have the values given in the table.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	1	5	2	-5
1	3	-2	0	1
2	0	2	3	1
3	2	4	1	-6

Let $h(x) = f(g(x))$. Find the following.

(i) $h'(0)$ (ii) $h'(1)$ (iii) $h'(2)$ (iv) $h'(3)$

Example 3.4.35: Chain Rule with more than two functions

Using the chain rule twice we can similarly write

$$f(g(h(x))) = f'(g(h(x)))g'(h(x))h'(x)$$

As an example, consider the function

$$\mathrm{P}(x) = \frac{1}{\sin\left(\frac{x^2}{5}\right) - 1}$$

The order of operation here is as follows

$$x \longrightarrow \frac{x^2}{5} \longrightarrow \sin\left(\frac{x^2}{5}\right) - 1 \longrightarrow \frac{1}{\sin\left(\frac{x^2}{5}\right) - 1}$$

We broke our function into exact steps as above because we want to be able to take derivative at each step using the simpler rules we have learned so far. We would like to write $\mathrm{P}(x)$ as $f(g(h(x)))$. Note that h is applied first to x, and then g and then f. So in the above sequence of steps, we can identify f, g and h as follows:

$$x \xrightarrow{h} \underbrace{\frac{x^2}{5}}_{h(x)} \xrightarrow{g} \underbrace{\sin\left(\frac{x^2}{5}\right) - 1}_{g(h(x))} \xrightarrow{f} \underbrace{\frac{1}{\sin\left(\frac{x^2}{5}\right) - 1}}_{f(g(h(x)))}$$

where

$$h(x) = \frac{x^2}{5} \implies h'(x) = \frac{2x}{5}$$
$$g(x) = \sin x - 1 \implies g'(x) = \cos x \implies g'(h(x)) = \cos\left(\frac{x^2}{5}\right)$$
$$f(x) = \frac{1}{x} = x^{-1} \implies f'(x) = -x^{-2} \implies f'(g(h(x))) = -\left(\sin\left(\frac{x^2}{5}\right) - 1\right)^{-2}$$

So,

$$\begin{aligned} \mathrm{P}'(x) &= f'(g(h(x)))g'(h(x))h'(x) \\ &= -\left(\sin\left(\frac{x^2}{5}\right) - 1\right)^{-2} \cos\left(\frac{x^2}{5}\right)\frac{2x}{5} \end{aligned}$$

■ Question 44.

Find the derivative of

$$y = \frac{1}{(\tan(\sin(x)))^2}$$

§3.5 Practice Problems

Question 45.

☐

Differentiate the given functions.

(a) $y = \tan(x + x^{-1})$

(b) $f(x) = \left(\dfrac{x-1}{x+1}\right)^3$

(c) $g(y) = (y^3 - y^2 - y - 1 - y^{-1} - y^{-2})^3$

(d) $y = \sin(x)\sec^3(x)$

(e) $y = \sin(\cos(\sin(x)))$

(f) $w = \sqrt{2 + \sqrt{4 + z^2}}$

(g) $y = \theta^2 \sin\left(\dfrac{4}{\theta}\right)$

Question 46.

☐

Calculate the specified derivative.

(a) $\dfrac{d^2}{d\theta^2}[\sin(\theta)\cos(\theta)]$

(b) $y^{(4)}$ *for* $y = 11(1-x)^{-1}$.

(c) $\dfrac{d^{100}}{dx^{100}}[\sqrt{2}x^{100} - 88x^{99} + 87x^{64} + 17x^{36}]$

(d) $y^{(32)}$ *for* $y = \sin(x)$?

Question 47.

☐

Find the equation of the tangent line at the given point.

(a) $y = \dfrac{\sin x - \cos x}{x}, \quad x = \frac{\pi}{6}$

(b) $y = \csc x - \cot x, \quad x = \frac{\pi}{4}$

(c) *Is it possible for the graph of* $y = \tan(x)$ *to have a horizontal tangent? Explain.*

Question 48.

☐

Suppose f *and* g *are functions with* $g(3) = 2$, $f'(3) = -1$, *and* $g'(3) = 0$. *What is the derivative of* $h(x) = \dfrac{f(x)}{g(x)}$ *at* $x = 3$?

Question 49.

☐

Let f *be a function with* $f(5) = 2$ *and* $f'(5) = -1$. *Let* $g(x) = x^2 f(x)$. *Find* $g'(5)$.

Question 50.

☐

Suppose $h(x) = f(x)g(x)$ *and* $g'(x) = xg(x)$. *If* $g(2) = 1$, $f'(2) = 3$, *and* $f(2) = 4$, *then find* $h'(2)$.

Question 51.

Suppose f, g, and h are nonzero differentiable functions with $h(x) = f(x)g(x)$ for all real x. Suppose also that

$$h'(1) = 12h(1), \quad f'(1) = 4f(1), \quad g'(1) = \lambda g(1)$$

Then find the value of λ.

Question 52.

Consider a function $f(x)$ defined as follows:

$$f(x) = \begin{cases} b + ax - x^2 & \text{for } x < 2 \\ ax^2 + bx + 2 & \text{for } x \geq 2 \end{cases}$$

If $f(x)$ is both continuous and differentiable at $x = 2$, then find a and b.

Question 53.

If $f(x) = x^2 + x$ and $g(x) = x^3 + \lambda$, for what value of λ do we have $f(\lambda) = g(\lambda)$ and $f'(\lambda) = g'(\lambda)$?

Question 54.

Let $\mathrm{P}(x) = ax^3 + bx^2 + cx + d$. If $\mathrm{P}(0) = \mathrm{P}(1) = -2$, $\mathrm{P}'(0) = -1$, and $\mathrm{P}''(0) = 10$, what is $\mathrm{P}'''(0)$?

Question 55.

Suppose $f(x) = \sin x$ and $g(x) = ax^2 + bx + c$. If $f(0) = g(0)$, $f'(0) = g'(0)$, and $f''(0) = g''(0)$, then find a, b, and c.

Question 56.

Let $f(x)$ be a continuous and differentiable function defined as

$$f(x) = a\sin^2 x + b\cos x$$

where a and b are real numbers. If $f(\pi/2) = 2$ and $f'(\pi/2) = 3$, what are the values of a and b?

Question 57.

Let $f(x)$ and $g(x)$ be continuous and differentiable functions such that

$$f(x) = \sin(g(x)) + \cos x$$

If $g(0) = \pi$ and $g'(0) = \dfrac{\pi}{4}$, find the value of $f'(0)$.

Question 58.

Let a and b be real numbers such that $f(x) = ax\sin x + b\cos x$ and $f'(x) = x\cos x$ for every real number x. What are the values of a and b?

Chapter 4 | Derivatives involving Inverse Functions

§4.1 Inverse Function

Definition 4.1.36

We call a function f **one-to-one** if it never takes on the same value twice; that is $f(a) \neq f(b)$ whenever $a \neq b$.

Geometrically, when looking at the graph, a one-to-one function should pass a **horizontal line test** (does that sound familiar?). For example, the function in fig. 4.1 is **not one-to-one**.

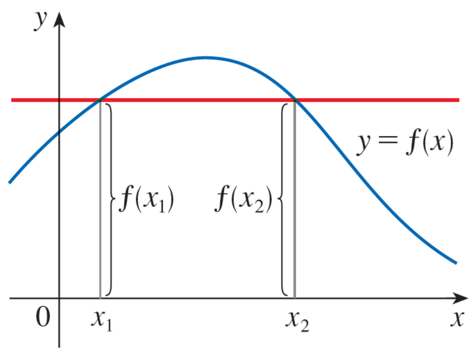


Figure 4.1: This function is not one-to-one because $f(x_1) = f(x_2)$

Question 59. ☐

Consider $f(x) = x^2$. Is f a one-to-one function?

If we are unable to draw the graph, or if we do not have an explicit formula for the function, we can determine if a function is one-to-one by solving for x in the equation $f(x) = y$. For a fixed value of y, if there is only one unique solution, then f is one-to-one. Try this for the following functions:

Question 60. ☐

Consider $g(x) = \dfrac{1}{x+2}$. Try to solve $g(x) = c$ for x, where c is some nonzero constant.

Is g a one-to-one function?

Functions that are one-to-one are precisely those that have **inverses** according to the following definition:

Definition 4.1.37

If f is a one-to-one function with domain A and range B, then its inverse function f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

for any y in B.

Essentially this definition says that f^{-1} reverses the effect of f. That means

$$\text{Domain of } f^{-1} = \text{Range of } f\text{, and}$$
$$\text{Range of } f^{-1} = \text{Domain of } f$$

The letter x is traditionally used as the independent variable, so when we concentrate on f^{-1} rather than on f, we usually reverse the roles of x and y in above definition and write

$$\boxed{f^{-1}(x) = y \iff f(y) = x}$$

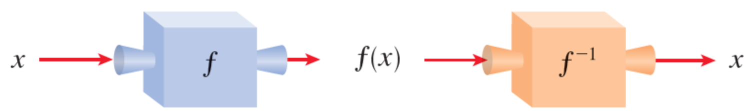


Figure 4.2: $f^{-1}(f(x)) = x$

So the general algortithm for finding the inverse function is to solve the equation $y = f(x)$ for x (in terms of y); and then interchange x and y in the final expression, resulting in $y = f^{-1}(x)$.

Question 61.

Find the inverse function for $f(x) = \dfrac{1}{x+2}$.

If you have a graphing utility, sketch both f and f^{-1}. Do you notice any symmetry between the two graphs?

□

Question 62.

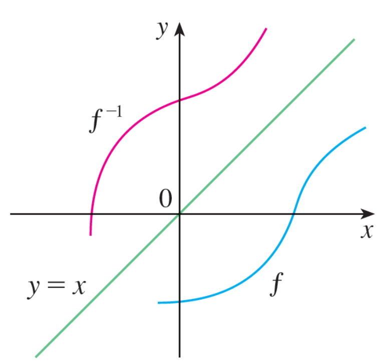
Try to solve for x in the function $y = x^2$. What does this tell you about the inverse of the function?

Is there a way to ***restrict the domain*** *of f so that the square root function, $y = \sqrt{x}$, will be its inverse?*

□

4.1.1 Graph of the Inverse Function

Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}. But we get the point (b, a) from (a, b) by reflecting about the line $y = x$.



4.1.2 Exponential and Logarithmic Functions

Firstly, recall that exponential functions have the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$. When $b > 1$, an exponential function models rapid **growth**, while if $0 < b < 1$, the function models rapid **decay**. See the graphs below in Figure 1.

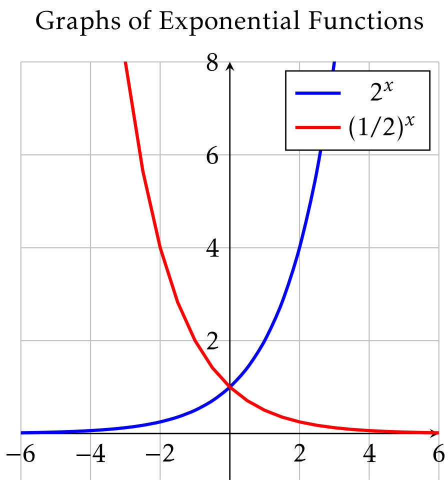


Figure 4.3: Graphs of $y = 2^x$ and $y = (\frac{1}{2})^x$

Since the graph is strictly increasing or strictlyt decreasing, it is one-to-one. Therefore the function exponential function $f(x) = b^x$ has an inverse f^{-1}, which is called **logarithmic function with base** b. Since

$$f^{-1}(x) = y \iff f(y) = x$$

we have

$$\log_b x = y \iff b^y = x$$

Question 63.

What is the domain and range of $f(x) = log_b x$?

§4.2 Derivative of Exponential Function

First of all, note that exponential functions are not power functions, and as such,

Warning: the power rule does not apply to exponential functions!

So we need to find the fornula for the derivative from the limit definition. We have,

$$\begin{aligned} f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}[b^x] &= \lim_{h\to 0} \frac{b^{x+h}-b^x}{h} \\ &= b^x \left(\lim_{h\to 0} \frac{b^h - 1}{h} \right) \end{aligned}$$

Note that the quantity $\boxed{\lim_{h\to 0} \frac{b^h-1}{h}}$ does not depend on x. Most likely, it is some constant that depends on b, let's call it $m(b)$. We will find the formula for $m(b)$ in a moment.

However, we can still conclude that

the derivative of the exponential function is proportional to the function itself!

No function we have seen thus far has had this property.

Numerically, we can check that that for $b = 2$, we have $m(b) \approx 0.693$ and for $b = 3$, we have $m(b) \approx 1.099$. So by IVT, there must be a value between 2 and 3 such that $m(b) = 1$. It is traditional to denote this value by the letter e, called Euler's constant. So we have the following definition:

Definition 4.2.38

We define **the number** e, Euler's constant, as the unique number b for which $\lim_{h\to 0} \frac{b^h-1}{h} = 1$.

In fact, we can also check that if $\frac{b^h-1}{h} = 1$, for very small values of h, we can rewrite this as

$$\frac{b^h-1}{h} \approx 1 \iff b^h \approx 1+h \iff b \approx (1+h)^{\frac{1}{h}}$$

In other words, b must be the limit of $(1+h)^{\frac{1}{h}}$ as h approaches 0. So we get a second alternate definition, and a limit formula:

Definition 4.2.39

We define **the number** e, Euler's constant, as the value of the limit

$$e = \lim_{h\to 0}(1+h)^{\frac{1}{h}}$$

Exploration Activity

Both of these can be used as **equivalent** definitions of e, meaning, you can assume one is true and use it to show that the other limit holds. There are actually several ways to define the number e. But in terms of Calculus, the above definition is the most natural one to give.

Summarizing, we have a derivative formula for the **very special** exponential function $f(x) = e^x$ as follows:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}[e^x] = e^x}. \tag{4.1}$$

So $f(x) = f'(x)$ for this function. This is one of the reasons why e^x is so special! In general, using chain rule, we can modify above statement to give the general rule:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{g(x)}\right] = e^{g(x)}g'(x)}. \tag{4.2}$$

4.2.1 General Exponential Function

To find the general formula for $m(b) = \lim\limits_{h\to 0}\dfrac{b^h - 1}{h}$, we need to use the inverse function of e^x. The inverse function of $f(x) = e^x$ is called the **natural logarithm** and is written as $f^{-1}(x) = \ln x = \log_e x$. It follows that

$$\ln(e^b) = b = e^{\ln b}$$

for all real numbers $b > 0$. So we can replace the b in above limit and rewrite it as

$$\begin{aligned}
m(b) &= \lim_{h\to 0}\frac{b^h - 1}{h} \\
&= \lim_{h\to 0}\frac{\left(e^{\ln b}\right)^h - 1}{h} \\
&= \lim_{h\to 0}\frac{e^{h\ln b} - 1}{h} \\
&= \lim_{k\to 0}\frac{e^k - 1}{\left(\frac{k}{\ln b}\right)}, \qquad \text{where } k = h\ln b \\
&= \ln b\left(\lim_{k\to 0}\frac{e^k - 1}{k}\right) \\
&= \ln b, \qquad \text{because } \lim_{k\to 0}\frac{e^k - 1}{k} = \lim_{h\to 0}\frac{e^h - 1}{h} = 1 \text{ by definition}
\end{aligned}$$

Thus we have proved the general result that for any $b > 0$, we have

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}[b^x] = b^x\ln b}. \tag{4.3}$$

Try your hand at using this derivative formula for the next several examples.

Question 64. **Exponential Derivative Practice**

Compute the derivative $\dfrac{dy}{dx}$.

(i) $y = e^{x^2}$

(ii) $y = e^{\sin(x)}$

(iii) $y = x^2 e^{2x}$ *(remember the product rule!)*

(iv) $y = e^{e^x}$

(v) $y = 3^x - 2^x$

(vi) $y = 2^{\sin(x)}$

§4.3 Derivative of Logarithm Function

Next we would like to find the derivative of $f(x) = \log_b(x)$. Let's start with $\ln x = \log_e(x)$. Since $e^{\ln x} = x$, we can write

$$\frac{\mathrm{d}}{\mathrm{d}x} e^{\ln x} = \frac{\mathrm{d}}{\mathrm{d}x}[x] \implies e^{\ln x}\frac{\mathrm{d}}{\mathrm{d}x}[\ln x] = 1 \implies x\frac{\mathrm{d}}{\mathrm{d}x}[\ln x] = 1$$

Here in the second step, we used the chain rule and the formula from (4.2). So we conclude that,

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}[\ln x] = \frac{1}{x}}. \tag{4.4}$$

In general, using chain rule, we can modify above statement to give the general rule:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}[\ln g(x)] = \frac{1}{g(x)} g'(x)}. \tag{4.5}$$

With this result, here's a second way to prove result (4.3). We will start with the identity $\ln(b^x) = x \ln b$. Then differentiating both sides with respect to x and using chain rule, we get

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln b^x = \frac{\mathrm{d}}{\mathrm{d}x}[x \ln b] \implies \frac{1}{b^x}\frac{\mathrm{d}}{\mathrm{d}x} b^x = \ln b \implies \frac{\mathrm{d}}{\mathrm{d}x} b^x = b^x \ln b.$$

What about the derivative of the general function $f(x) = \log_b x$. Fortunately, this is easier to differentiate than the general exponential function, since $\log_b x = \dfrac{\ln x}{\ln b}$ using a property of logarithm. Hence we can conclude that

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} \log_b x = \frac{1}{x \ln b}}. \tag{4.6}$$

Question 65. **Logarithm Derivative Practice**

Compute the derivative $\dfrac{dy}{dx}$.

(i) $y = \dfrac{x}{\ln(x)}$

(ii) $y = \tan(x) \cdot \ln(x)$

(iii) $y = \ln(9x^2 - 8)$

(iv) $y = \ln(\sec(x))$

(v) $y = \ln(\ln(v))$

(vi) $y = \log_{10}(z^2 - 10z + 5)$

(vii) $y = \log_7(\sqrt{6^x + 1})$

§4.4 Inverse Trignonometric Functions

First of all, we will have a brief review of inverse trigonometric functions. Trigonometric functions are periodic, so they fail to be one-to-one, and thus do not have inverse functions. However, we can restrict the domain of each trigonometric function so that it is one-to-one on that domain.

For instance, consider the sine function on the domain $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. Because no output of the sine function is repeated on this interval, the function is one-to-one and thus has an inverse. Thus, the function $f(x)=\sin x$ with domain $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and range $[-1,1]$ has an inverse function f^{-1} such that

$$f^{-1}:[-1,1]\to\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

We call f^{-1} the **arcsine** function and write $f^{-1}(y)=\arcsin(y)$. It is especially important to remember that

$$y=\sin(x) \text{ and } x=\arcsin(y)$$

say the same thing. "The arcsine of y" means "the **angle** whose sine is y." For example, $\arcsin\left(\frac{1}{2}\right)=\frac{\pi}{6}$ means that $\frac{\pi}{6}$ is the angle whose sine is $\frac{1}{2}$ which is equivalent to writing $\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$.

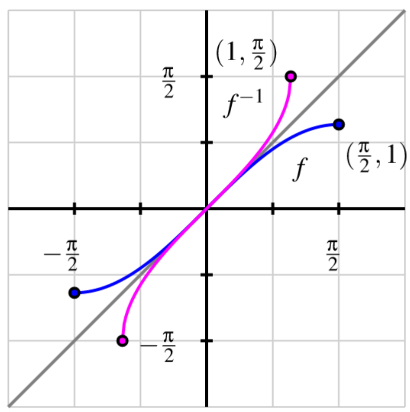


Figure 4.4: A graph of $f(x)=\sin x$ (in blue), restricted to the domain $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, along with its inverse, $f^{-1}(x)=\arcsin(x)$ (in magenta).

4.4.1 Derivative of arcsin

Next, we determine the derivative of the arcsine function. Letting $g(x)=\arcsin(x)$, our goal is to find $g'(x)$. since $g(x)$ is the angle whose sine is x it is equivalent to write

$$\sin(g(x))=x$$

Differentiating both sides of the previous equation, we have

$$\frac{d}{dx}[\sin(g(x))]=\frac{d}{dx}[x]$$

The right hand side is simply 1 , and by applying the chain rule applied to the left side,

$$\cos(g(x))g'(x)=1$$

Solving for $g'(x)$, it follows that

$$g'(x) = \frac{1}{\cos(g(x))}$$

Finally, we recall that $g(x) = \arcsin(x)$, so the denominator of $g'(x)$ is the function $\cos(\arcsin(x))$, or in other words, "the cosine of the angle whose sine is x." A bit of right triangle trigonometry allows us to simplify this expression considerably.

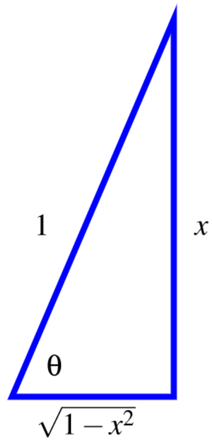


Figure 4.5: The right triangle that corresponds to the angle $\theta = \arcsin(x)$.

Let's say that $\theta = \arcsin(x)$, so that θ is the angle whose sine is x. We can picture θ as an angle in a right triangle with hypotenuse 1 and a vertical leg of length x, as shown in figure (4.5). The horizontal leg must be $\sqrt{1-x^2}$ by the Pythagorean Theorem.

Now, because $\theta = \arcsin(x)$, the expression for $\cos(\arcsin(x))$ is equivalent to $\cos(\theta)$. From the figure,

$$\cos(\arcsin(x)) = \cos(\theta) = \sqrt{1-x^2}.$$

Substituting this expression into our formula, $g'(x) = \dfrac{1}{\cos(\arcsin(x))}$, we have now shown that

$$g'(x) = \frac{1}{\sqrt{1-x^2}}$$

Theorem 4.4.40

For all real numbers x such that $-1 < x < 1$

$$\boxed{\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}}$$

4.4.2 Derivative of arctan

The derivative of $\arctan x$ is given by

$$\boxed{\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2}}$$

Question 66. Deriavtion of the formula

The following prompts will lead you to develop the derivative of the inverse tangent function yourself!

(a) *Let* $r(x) = \arctan(x)$. *Use the relationship between the arctangent and tangent functions to rewrite this equation using only the tangent function.*

(b) *Differentiate both sides of the equation you found in (a). Solve the resulting equation for* $r'(x)$, *writing* $r'(x)$ *as simply as possible in terms of a trigonometric function evaluated at* $r(x)$.

(c) *Recall that* $r(x) = \arctan(x)$. *Update your expression for* $r'(x)$ *so that it only involves trigonometric functions and the independent variable* x.

(d) *Introduce a right triangle with angle* θ *so that* $\theta = \arctan(x)$. *What are the three sides of the triangle?*

(e) *In terms of only* x *and* 1, *what is the value of* $\cos(\arctan(x))$ *?*

(f) *Use the results of your work above to find an expression involving only* 1 *and* x *for* $r'(x)$.

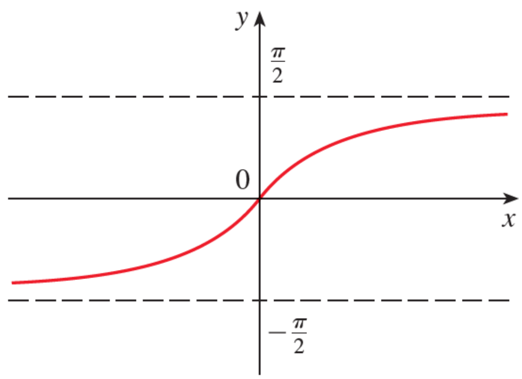


Figure 4.6: $f(x) = \arctan x$

Question 67. Derivative Practice

Compute the derivative of the following functions.

(i) $f(x) = x^3 \arctan(x) + e^x \ln(x)$

(ii) $p(t) = 2^{t \arcsin(t)}$

(iii) $h(z) = (\arcsin(5z) + \arctan(4-z))^{27}$

(iv) $s(y) = \cot(\arctan(y))$

(v) $m(v) = \ln\left(\sin^2(v) + 1\right)$

(vi) $g(w) = \arctan\left(\frac{\ln(w)}{1+w^2}\right)$

§4.5 Derivative of the Inverse Function

Suppose f and g are differentiable functions that are inverses of each other, i.e. $y = f(x)$ if and only if $x = g(y)$. Then we can write $f(g(x)) = x$ for every x in the domain of f^{-1}. Differentiating both sides of this equation, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = \frac{\mathrm{d}}{\mathrm{d}x}[x]$$

and by the chain rule,

$$f'(g(x))g'(x) = 1$$

Solving for $g'(x)$, we have $g'(x) = \dfrac{1}{f'(g(x))}$. In other words,

Theorem 4.5.41

Suppose that the domain of a function f is an open interval I and that f is differentiable and one-to-one on this interval. Then f^{-1} is differentiable at any point x in the range of f at which $f'(f^{1-}(x)) \neq 0$, and its derivative is

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}}$$

The formula for the derivatives of arcsin and arctan obtained above are just applications of this result.

Question 68.

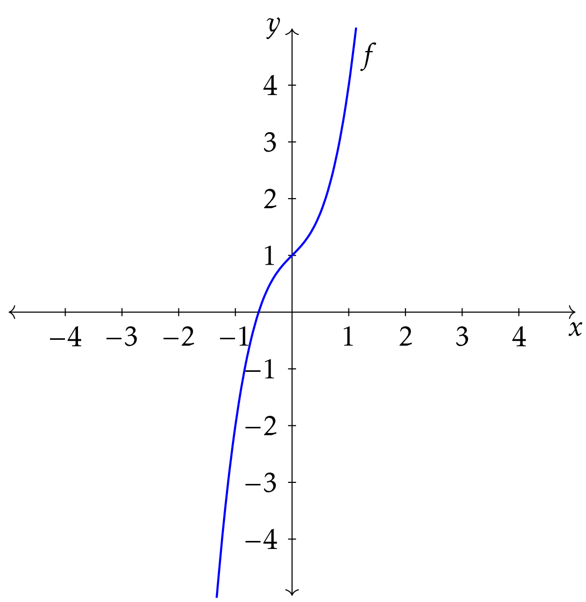
Let g denote the inverse function of f. Suppose

$$f(3) = -6, \quad f'(3) = 2/3, \quad f(-6) = 2, \quad f'(2) = 1,$$
$$f'(-6) = 3, \quad f'(-1) = -6, \quad f'(-6) = 5$$

What is $g'(-6)$? □

Question 69.

Let $g(x)$ be the inverse function of $f(x) = 2x^3 + x + 1$. What is $g'(4)$? □



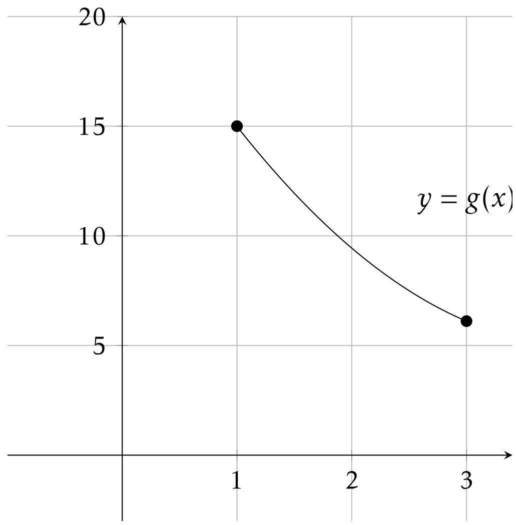
Chapter 5 | Application of Derivatives Part I - MVT and L'Hôpital's Rule

§5.1 Mean Value Theorem

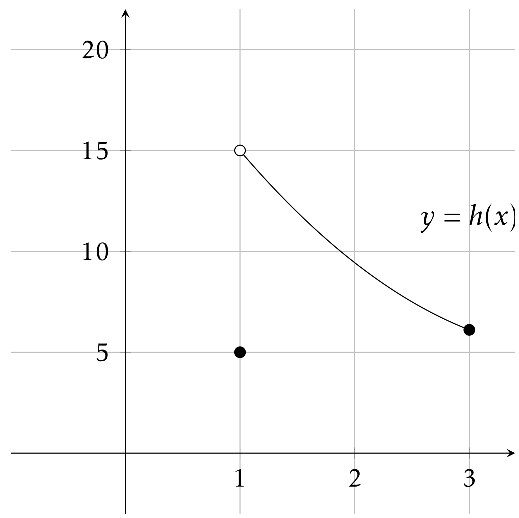
Question 70.

Draw the secant line between the endpoints for the given interval $[a, b]$. Can you identify a point c, with $a < c < b$, such that the slope of the tangent line to the graph at $x = c$ is equal to the slope of the secant line between a and b?

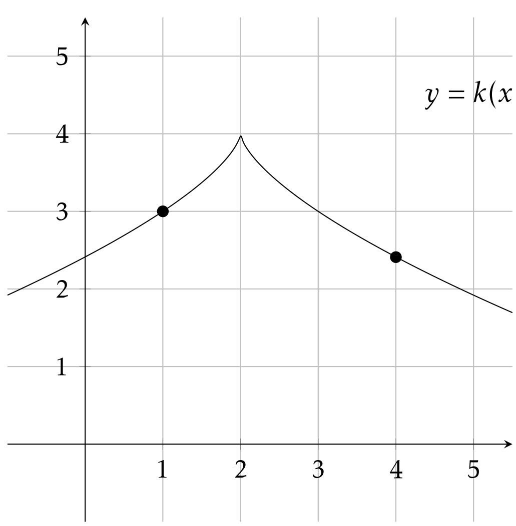
$g(x)$ on the interval $[1, 3]$



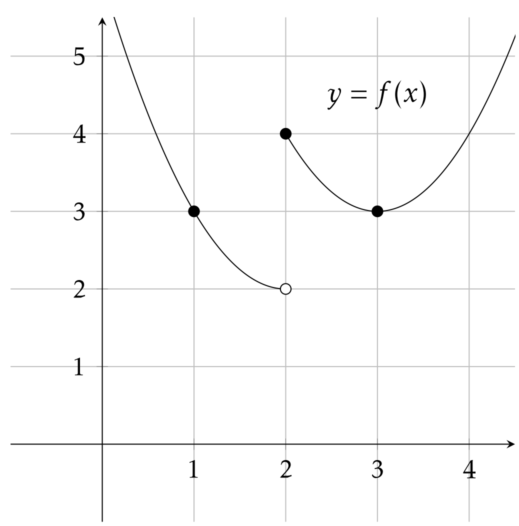
$h(x)$ on the interval $[1, 3]$



$k(x)$ on the interval $[1, 4]$



$f(x)$ on the interval $[1, 3]$



Question 71.

Using your observations from these four cases, make a conjecture regarding when it is possible to find such a point c. In other words, what properties does the function need to have?

Theorem 5.1.42

If f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then there exists at least one number c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

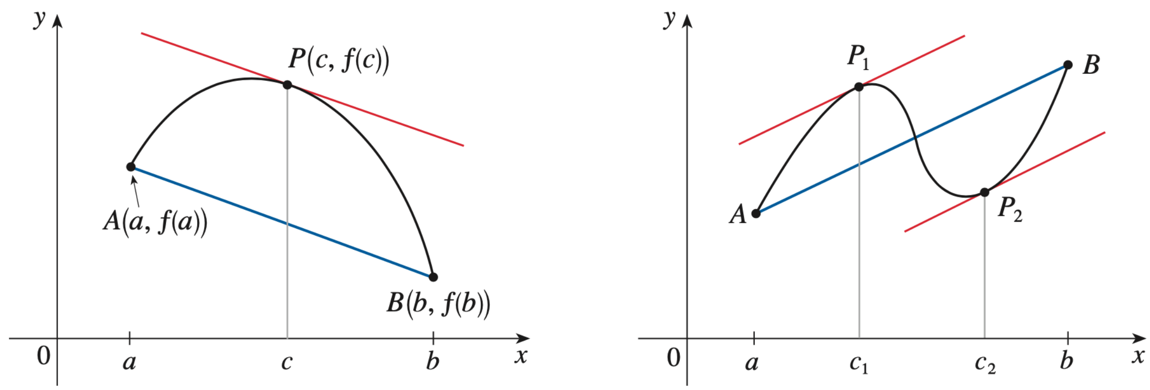


Figure 5.1: The function f attains the slope of the secant between a and b as the derivative at the point(s) $c \in (a,b)$.

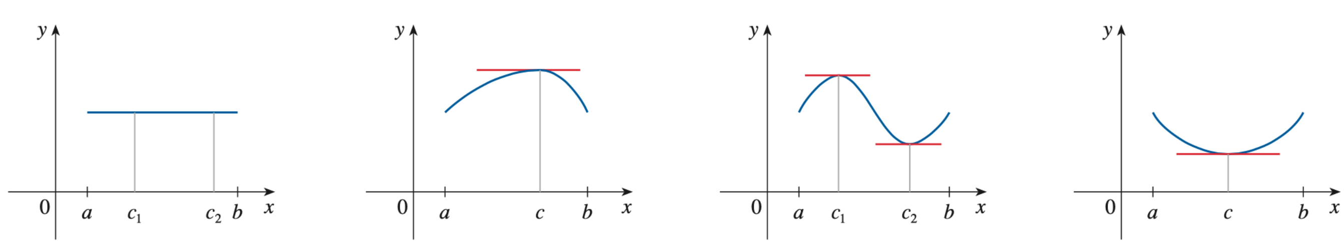
Question 72.

An elevator starts at ground level at time $t = 0$ seconds. At $t = 20$ seconds, the elevator has risen 100 feet. What does the Mean Value Theorem tell you about this situation? (Be specific to this case.)

A special case of the Mean Value Theorem is called Rolle's Theorem.

Theorem 5.1.43: Rolle's Theorem

Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). If $f(a) = f(b)$, then there is at least one number c in (a,b) such that $f'(c) = 0$.



Sketch of Proof: The proof of Rolle's theorem follows from the Extreme Value Theorem which says that a continuous function on a closed interval must attain its extremum at some c. This fact along with the fact that local extrema are critical points gives us $f'(c) = 0$ at such points. ■

Question 73.

Like other Math theorems, the MVT is a "If-Then" statement. There are some hypotheses and there is a conclusion. Can you identify which part is which?

5.1.1 Applications

Question 74.

Let $g(x) = |x^2 - 1|$. Graph this function using Desmos and answer the questions below.

(a) *Do the hypotheses of the MVT hold on $[0,3]$? Does the conclusion hold? Explain.*

(b) *Do the hypotheses of the MVT hold on $[1,3]$? Does the conclusion hold? Explain.*

(c) *Do the hypotheses of the MVT hold on $[-1,3]$? Does the conclusion hold? Explain.*

Exploration Activity

It is important to think about the chain of logic for a theorem. Let me use the Cats analogy. Consider the statement: If we have a cat, then we have a mammal. Note that the converse isn't true. Just because an animal is a mammal, it doesn't necessarily mean it's a cat. Indeed a dog is also a mammal. So the conclusion can be valid even when the hypothesis isn't. Relating to the problem above, in part (a), the conclusion is valid, even when the hypothesis isn't.

Similarly, when the hypothesis doesn't hold, we can't really say whether the conclusion holds or not. For example, if your animal is not a cat, we do not know if it is a mammal or not, it could be an octopus, or it could be a dog. Relating to the problem above, the hypothesis doesn't hold in both (a) and (c); but for one of them the conclusion holds, for the other it doesn't.

Question 75.

Does the MVT apply to $g(x) = x^{1/3}$ on $[0,8]$? Why or why not? If so, find all values of c that satisfy the theorem.

Question 76.

Explain why $h(x) = x^3 + 6x + 2$ satisfies the hypotheses of the MVT on the interval $[-1,3]$. Then find all values of c in $[-1,3]$ guaranteed by the theorem.

Question 77.

Show a Write-up

Considering the following situation. You are driving a car on a highway, traveling at the speed limit of 55 mph. At 10 : 17am, you pass a police car on the side of the road, presumably checking for speeders. At 10 : 53am, 39 miles from the first police car, you pass another police car. You are of course obeying the speed limit and traveling exactly 55 mph. However, you are shocked when the police turn on their lights and pull you over. The officer claims you were speeding at some point in the last 39 miles. Is the officer telling the truth, or needlessly pulling you over?

Question 78.

Show a Write-up

Let $f(x) = \dfrac{1}{x^2}$. Show analytically why there cannot exist a number c in $(-1,1)$ such that

$$f(1) - f(-1) = 2f'(c).$$

Does this contradict the MVT? Explain.

■ Question 79.

Use Rolle's theorem to show that the graph of $f(x) = x\cos x$ *has a horizontal tangent in the interval* $\left[0, \frac{\pi}{2}\right]$. □

■ Question 80.

Show that $|\sin x - \sin y| \le |x - y|$ *for all* x *and* y. □

■ Question 81.

Suppose f *is a polynomial. Which of the following statements are correct? There may be more than one correct answer.* □

(I) Between any two consecutive roots of f, there must be at least one root of f'.

(II) Between any two consecutive roots of f', there must be at least one root of f.

(III) Between any two consecutive roots of f, there can be at most one root of f'.

(IV) Between any two consecutive roots of f', there can be at most one root of f.

We end with a wild MVT spotted on a bridge in Beijing!

§5.2 L'Hôpital's Rule

Differential calculus is based on derivative, and the definition of the derivative involves a limit, so one can say that all of calculus rests on limits. Recall that the definition of a derivative is given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

An interesting thing to note here is that the limit on the right hand side has a $\frac{0}{0}$ form, not only does $h \to 0$, but also $f(x+h) - f(x) \to 0$ since f is continuous. Remember, saying that a limit has an indeterminate form only means that we don't yet know its value and have more work to do: indeed, limits of the form $\frac{0}{0}$ can take on any value.

Example 5.2.44

Consider a very simple example where $f(x) = 2x^2$ and $g(x) = x^2$. Note that as $x \to 0$, both f and g approach 0. However, clearly $f(x)$ is always twice the amount of $g(x)$. So when $x \to 0$, the quantity $\frac{f(x)}{g(x)} \to 2$.

We learned different ways of calculating limits for functions in indeterminate form by factorizing, multiplying by conjugate etc. Now let's take a look at an example where those strategies may not work.

Example 5.2.45

Consider the following limit which has a $\frac{0}{0}$ form

$$\lim_{x\to 0} \frac{\sin(2x)}{e^x - 1}.$$

There is no further simplification to be done here, so how do we calculate the limit?

Let's try to look at the graphs of the two functions $f(x)$ and $g(x)$ near the point $x = 0$. Open the `DESMOS` link at

https://www.desmos.com/calculator/upwozqfmno

The idea here is that that we can evaluate an indeterminate limit of the form $\frac{0}{0}$ by replacing each of the numerator and denominator with their 'tangents' at $x = 0$.

That clearly shows that the limit of the quotient is equal to 2 as $x \to 0$.

In general, we can use the tangent line formula at $x = a$ to replace both $f(x)$ and $g(x)$ as follows. Suppose we need to calculate $\lim_{x\to a} \frac{f(x)}{g(x)}$ where $f(a) = g(a) = 0$. Recall that the equation of the tangent line to a function $y = f(x)$ at $x = a$ is given by $y = f(a) + f'(a)(x-a)$. Then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(a) + f'(a)(x-a)}{g(a) + g'(a)(x-a)} = \lim_{x\to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} = \lim_{x\to 0} \frac{f'(a)}{g'(a)} = \frac{f'(a)}{g'(a)}$$

This result holds as long as $g'(a)$ is not equal to zero. The formal name of the result is L'Hôpital's Rule.

Theorem 5.2.46

Let f and g be differentiable at $x = a$, and suppose that $f(a) = g(a) = 0$ and that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Note: Technically, since we are calculating a limit, the values $f(a)$ and $g(a)$ do not have to be defined. In that case, we can replace the conditions $f(a) = g(a) = 0$ with $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ instead.

Find the following limits:

(a) $\lim_{x \to 0} \dfrac{\sin x}{x}$

(b) $\lim_{x \to 1} \dfrac{x^5 + x - 2}{x^2 - 1}$

(c) $\lim_{x \to 1} \dfrac{2\ln(x)}{1 - e^{x-1}}$

(d) $\lim_{x \to 1} \dfrac{\sin(\pi x)}{\ln(x)}$

What if both $f'(a)$ and $g'(a)$ are also 0? Then we continue the process. A more general form of L'Hôpital's rule says,

Theorem 5.2.47

Let f and g be differentiable at $x = a$, and suppose that $f(a) = g(a) = 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided that the limit on the right-hand side exists.

Find the following limits:

(a) $\lim_{x \to 0} \dfrac{\sin(x) - x}{\cos(2x) - 1}$

(b) $\lim_{x \to 0} \dfrac{e^x - 1 - x}{x^2}$

5.2.1 L'Hôpital's Rule - Other Indeterminate Forms

The expressions $\frac{\infty}{\infty}$, $\infty - \infty$, $0 \times \infty$, 1^∞, ∞^0, and 0^0 are all considered indeterminate forms. These expressions are not real numbers. Rather, they represent forms that arise when trying to evaluate certain limits.

L'Hôpital's rule can be also applied to these indeterminate forms.

Question 82. □

Find the following limits:

(a) $\lim_{x \to \frac{\pi}{2}^-} \dfrac{\tan(x)}{x - \frac{\pi}{2}}$

(b) $\lim_{x \to 0^+} x \ln(x)$

(c) $\lim_{x \to 0} x \cot(x)$

(d) $\lim_{x \to 0} \left(\dfrac{1}{x} - \dfrac{1}{\sin x} \right)$

(e) $\lim_{x \to \infty} \dfrac{e^x + x}{2e^x + x^2}$

(f) $\lim\limits_{x\to\infty} \dfrac{(\ln x)^3}{x^2}$

■ Question 83. Indeterminate form 0^0

Consider the function $g(x) = x^{2x}$, which is defined for all $x > 0$. Observe that $\lim\limits_{x\to 0^+} g(x)$ is indeterminate due to its form of 0^0. (Think about how we know that $0^k = 0$ for all $k > 0$, while $b^0 = 1$ for all $b \neq 0$, but that neither rule can apply to 0^0.)

(a) *Let $h(x) = \ln(g(x))$. Explain why $h(x) = 2x\ln(x)$.*

(b) *Use L'Hôpital's Rule to compute $\lim\limits_{x\to 0^+} h(x)$.*

(c) *Based on the value of $\lim\limits_{x\to 0^+} h(x)$, determine $\lim\limits_{x\to 0^+} g(x)$.*

■ Question 84. Indeterminate form ∞^0

Find $\lim\limits_{x\to\infty} x^{1/x}$.

5.2.2 Growth and Dominance

Suppose the functions f and g both approach infinity as $x \to \infty$. Although the values of both functions become arbitrarily large as the values of x become sufficiently large, sometimes one function is growing more quickly than the other.

Definition 5.2.48

We say that a function g **dominates** a function f provided that $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} g(x) = \infty$, and

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

Question 85. □

(a) *Which function dominates the other:* $\ln(x)$ *or* $\sqrt{x}$?

(b) *Which function dominates the other:* $\ln(x)$ *or* $\sqrt[n]{x}$ *? (n can be any positive integer)*

(c) *Explain why* e^x *will dominate any polynomial function.*

(d) *Compare the growth rates of* x^{100} *and* 2^x.

(e) *Explain why* x^n *will dominate* $\ln(x)$ *for any positive integer* n.

(f) *Give any example of two nonlinear functions such that neither dominates the other.*

Chapter 6 | Implicit Differentiation

§6.1 What is an Implicit Function

In all of our studies with derivatives so far, we have worked with functions whose formula is given explicitly in the form of $y = f(x)$. But not all planar curves are graphs of functions of the form $y = f(x)$. For example, take the case of a circle

$$x^2 + y^2 = 4.$$

Is this graph of a function? Since there are x–values that correspond to two different y–values, y is not a function of x on the whole circle. In fact, if we solve for y in terms of x, we quickly find that y is a function of x on the top half, and y is a different function of x on the bottom half.

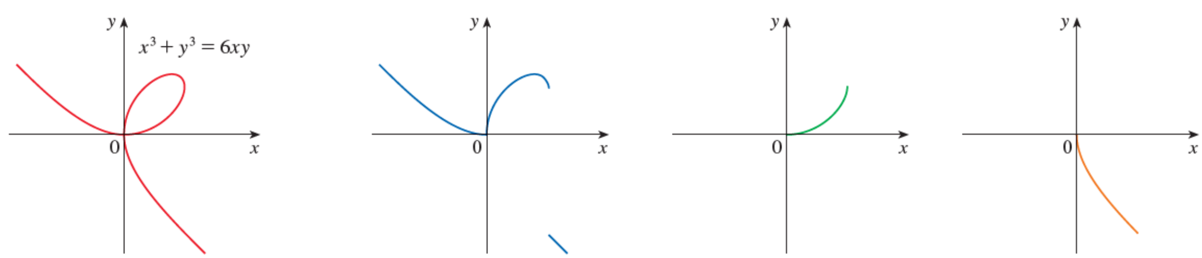
Question 86.

Use DESMOS *to draw the following curves on the plane and conclude that none of them can be written in the form $y = f(x)$.*

(i) $y^4 + xy = x^3 - x + 2$ *(ii)* $x^3 + y^3 = 6xy$

However, we can always draw **tangents** to a curve at any point! Thus, it makes sense to wonder if we can compute $\dfrac{dy}{dx}$ at any point on such curves, even though we cannot write y explicitly as a function of x.

In such cases, we say that the equation of the curve, $x^2 + y^2 = 4$ for example, defines y **implicitly** as a function of x. **An implicitly defined curve can be broken into pieces where each piece can be defined by an explicit function of x.**



§6.2 The Implicit Differentiation Process

As it should become clear by looking at the examples from question 1, It is often rather difficult to solve expressions involving x and y to obtain the explicitly defined functions (the circle is a rare exception where the calculation is easy). However, by viewing y as an implicit function of x, we can still think of y as some function of x whose formula $f(x)$ is unknown, but which we **can** differentiate.

Finding $\dfrac{dy}{dx}$ in such scenario involves the method of **implicit differentiation**. This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for $\dfrac{dy}{dx}$. Let's showcase with the earlier example of the circle

$$x^2 + y^2 = 4$$

Differentiating both sides of the equation with respect to x gives

$$\frac{d}{dx}\left[x^2+y^2\right]=\frac{d}{dx}[4].$$

On the right, the derivative of the constant 4 is 0 , and on the left we can apply the sum rule, so it follows that

$$\frac{d}{dx}\left[x^2\right]+\frac{d}{dx}\left[y^2\right]=0$$

Note carefully the different roles being played by x and y. Because x is the independent variable, $\frac{d}{dx}\left[x^2\right]=2x$. But y is the dependent variable and y is an implicit function of x. Recall from last week where we computed $\frac{d}{dx}\left[f(x)^2\right]$. Computing $\frac{d}{dx}\left[y^2\right]$ is the same, and requires **the chain rule**, by which we find that $\frac{d}{dx}\left[y^2\right]=2y\frac{dy}{dx}$.

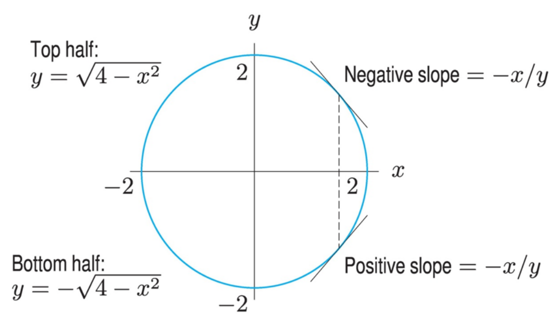
We now have that

$$2x+2y\frac{dy}{dx}=0$$

We solve this equation for $\frac{dy}{dx}$ by subtracting $2x$ from both sides and dividing by $2y$

$$\frac{dy}{dx}=-\frac{2x}{2y}=-\frac{x}{y}$$

The most important thing to observe here is that this expression for the derivative involves both x and y. This makes sense because there are two corresponding points on the circle for each value of x between -2 and 2, and the slope of the tangent line is different at each of these points.



Another interesting thing is to note that the formula doesn't work when $y=0$, which makes sense since the tangents are vertical there. In general, this process of implicit differentiation leads to a derivative whenever the expression for the derivative does not have a zero in the denominator.

Example 6.2.49

Here is another example to help you see the chain rule. How would you find $\frac{\mathrm{d}}{\mathrm{d}x}\sin(y)$?

Just think of y as $g(x)$. Then with the chain rule, you would get:

$$\frac{d}{dx}\sin(g(x))=\cos(g(x))\cdot g'(x).$$

Now change $g(x)$ back to y and note that $g'(x) = y' = \dfrac{dy}{dx}$. So,

$$\frac{d}{dx}\sin(y) = \cos(y)\frac{dy}{dx}.$$

Question 87.

Differentiate each expression as indicated. Assume that the variables x, y, and t may mutually depend on each other.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}[y^3]$

(b) $\dfrac{d}{dy}[y^3]$

(c) $\dfrac{d}{dt}[y^3]$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}[\sin y]$

(e) $\dfrac{d}{dy}[\sin y]$

(f) $\dfrac{d}{dt}[\sin y]$

(g) $\dfrac{\mathrm{d}}{\mathrm{d}x}[x^2y^2]$

(h) $\dfrac{d}{dy}[x^2y^2]$

(i) $\dfrac{d}{dt}[x^2y^2]$

Warning: There is a big difference between writing $\dfrac{d}{dx}$ and $\dfrac{dy}{dx}$. For example,

$$\frac{d}{dx}\left[x^2+y^2\right]$$

gives an instruction to take the derivative with respect to x of the quantity x^2+y^2, presumably where y is a function of x. On the other hand,

$$\frac{dy}{dx}\left(x^2+y^2\right)$$

means the product of the derivative of y with respect to x with the quantity x^2+y^2. Make sure to use the correct notation for the correct purpose.

Question 88.

For each of the following curves,

- *first find a formula for* $\dfrac{dy}{dx}$.
- *Then evaluate* $\dfrac{dy}{dx}$ *at the specified (a,b) point.*
- *Then, write the equation of the tangent line at the given point.*

(a) $\sqrt{x}-\sqrt{y}=-1$, $(1,4)$

(b) $x^3+y^2-2xy=2$, $(-1,1)$

(c) $xy^2+3x^3y-y=3$, $(1,1)$

(d) $\tan(y)+y^2=x^2$, (π,π)

(e) $\sin(x+y)+\cos(x-y)=1$, $(\pi/2,\pi/2)$

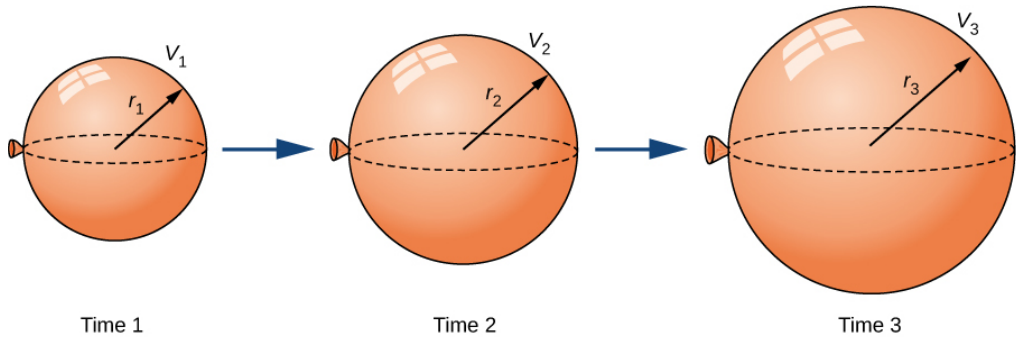
(f) $x\ln y+y^3=3\ln x+1$, $(1,1)$

§6.3 Related Rates

In most of our applications of the derivative so far, we have been interested in the instantaneous rate at which one variable, say y, changes with respect to another, say x, leading us to compute and interpret $\frac{dy}{dx}$. We next consider situations where several variable quantities are related, but where each quantity is implicitly a function of time, which will be represented by the variable t. Through knowing how the quantities are related, we will be interested in determining how their respective rates of change with respect to time are related. We call these **related rates problems**.

Question 89. □

Consider the following scenario. A spherical balloon is being inflated at a constant rate of 20 cubic inches per second. How fast is the radius of the balloon changing at the instant the balloon's diameter is 12 inches? Is the radius changing more rapidly when $d = 12$ or when $d = 16$? Why?



(a) *Recall that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$. Note that in the setting of this problem, both V and r are changing as time t changes, and thus both V and r may be viewed as implicit functions of t, with respective derivatives $\frac{dV}{dt}$ and $\frac{dr}{dt}$. Differentiate both sides of the equation $V = \frac{4}{3}\pi r^3$ with respect to t (using the chain rule on the right) to find a formula for $\frac{dV}{dt}$ that depends on both r and $\frac{dr}{dt}$.*

(b) *At this point in the problem, by differentiating we have **related the rates** of change of V and r. Recall that we are given in the problem that the balloon is being inflated at a constant rate of 20 cubic inches per second. Is this rate the value of $\frac{dr}{dt}$ or $\frac{dV}{dt}$? Why?*

(c) *From part (b), we know the value of $\frac{dV}{dt}$ at every value of t. Next, observe that when the diameter of the balloon is 12, we know the value of the radius. In the equation from part (a), substitute these values for the relevant quantities and solve for the remaining unknown quantity, which is $\frac{dr}{dt}$. How fast is the radius changing at the instant $d = 12$?*

(d) *How is the situation different when $d = 16$? When is the radius changing more rapidly, when $d = 12$ or when $d = 16$?*

6.3.1 The Problem Solving Strategy

Algorithm for Solving Related Rates Problems

Step 1. Identify the quantities in the problem that are changing and choose clearly defined variable names for them. Draw one or more figures that clearly represent the situation.

Step 2. State, in terms of the variables, all rates of change that are known or given and identify the rate(s) of change to be found.

Step 3. Find an equation that relates the variables whose rates of change are known to those variables whose rates of change are to be found.

Step 4. Using implicit differentiation, differentiate both sides of the equation found in step 3 with respect to t to relate the rates of change of the involved quantities.

Step 5. Substitute all known values into the equation from step 4, then solve for the unknown rate of change.

Warning: When solving a related-rates problem, it is crucial not to substitute known values too soon. For example, if the value for a changing quantity is substituted into an equation before both sides of the equation are differentiated, then that quantity will behave as a constant and its derivative will not appear in the new equation found in step 4.

6.3.2 Practice Problems

Question 90. □

Suppose $z^2 = x^2 + y^2$. *Find* $\dfrac{dz}{dt}$ *at* $(x, y) = (1, 3)$ *if* $\dfrac{dx}{dt} = 4$ *and* $\dfrac{dy}{dt} = 3$.

Question 91. □

You and a friend are riding your bikes to a restaurant that you think is East; your friend thinks the restaurant is North. You both leave from the same point, with you riding at 16 km/hr *east and your friend riding* 12 km/hr *north. After you traveled* 4 km, *at what rate is the distance between you changing?*

Question 92. □

Imagine a rectangle with whose length x is increasing at a rate of 0.2 m/s *and whose width y is decreasing at a rate of* 0.1 m/s. *How fast is the area of rectangle changing at the moment when $x = 3$ m and $y = 2$ m.*

Question 93. □

The radius of a circle increases at a rate of 2 m/sec. *Find the rate at which the area of the circle increases when the radius is* 5 m.

Question 94. □

A cylindrical tank is leaking water at a constant rate. The cylinder has a height of 2 m and a radius of 2 m. We notice that the rate at which water level is decreasing is 10 cm/min *when the water level is* 1 m. *Find the rate at which the water is leaking out.*

Question 95. □

Gravel is being unloaded from a truck and falls into a pile shaped like a cone at a rate of 10 ft^3/min. *The radius of the cone base is three times the height of the cone. Find the rate at which the height of the gravel changes when the pile has a height of 5 ft.*

Question 96. □

A 6 ft tall person walks away from a 10 ft lamppost at a constant rate of 3 ft/sec. What is the rate that the tip of the shadow moves away from the pole when the person is 10 ft away from the pole?

■ Question 97. □

A baseball diamond is a square with side 90 ft. *A batter hits the ball and runs toward first base with a speed of* 24 ft/s.

(a) *At what rate is his distance from second base decreasing when he is halfway to first base?*

(b) *At what rate is his distance from third base increasing at the same moment?*

■ Question 98. □

A rocket is launched so that it rises vertically. A camera is positioned 5000 *ft from the launch pad. When the rocket is* 1000 *ft above the launch pad, its velocity is* 600 *ft/sec. Find the necessary rate of change of the camera's angle as a function of time so that it stays focused on the rocket.*

■ Question 99. □

You are stationary on the ground and are watching a bird fly horizontally at a rate of 10 m/sec. *The bird is located* 40 m *above your head. How fast does the angle of elevation change when the horizontal distance between you and the bird is* 9 *m?*

■ Question 100. □

A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 *m higher than the bow of the boat. If the rope is pulled in at a rate of* 1 *m/s, how fast is the boat approaching the dock when it is* 8 *m from the dock?*

■ Question 101. □

If the minute hand of a clock has length r (in centimeters), find the rate at which it sweeps out area as a function of r.

■ Question 102. □

The minute hand on a watch is 8 *mm long and the hour hand is* 4 *mm long. How fast is the distance between the tips of the hands changing at one o'clock?*

§6.4 Logarithmic Differentiation

You now know how to differentiate polynomials and exponential functions. But what about a function that looks like the following:

$$f(x) = x^x.$$

This is **not** an exponential function, nor is it a polynomial (do you see why?). If we try to use the definition of the derivative, we get this horrible thing:

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^{x+h} - x^x}{h}$$

and, yeah - how do we compute that limit?

The method of **logarithmic differentiation** will allow us to find the derivative of x^x. This utilizes the rules of logarithms, the derivatives we have recently discovered, and implicit differentiation.

Example 6.4.50

Let $y = f(x) = x^x$. Then

$$\ln(y) = \ln(x^x) = x\ln(x).$$

Now we differentiate:

$$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \ln(x) = 1 + \ln(x).$$

Solving for $\dfrac{dy}{dx}$,

$$\frac{dy}{dx} = y(1+\ln(x)) = x^x(1+\ln(x)).$$

Example 6.4.51

Let $y = f(x) = x^{\sin(x)}$. Now take the natural log of both sides like so:

$$\ln(y) = \ln\left(x^{\sin(x)}\right) = \sin(x)\cdot\ln(x).$$

Apply differentiation (implicitly):

$$\frac{1}{y}\frac{dy}{dx} = \cos(x)\ln(x) + \frac{\sin(x)}{x}.$$

Solving for $\dfrac{dy}{dx}$:

$$\frac{dy}{dx} = y\left(\cos(x)\ln(x) + \frac{\sin(x)}{x}\right) = x^{\sin(x)}\left(\cos(x)\ln(x) + \frac{\sin(x)}{x}\right).$$

Question 103. □

In your own words, summarize the steps involved in a logarithmic differentiation problem.

Now try your hand at these practice problems. Each is a function of the form $f(x)^{g(x)}$.

■ Question 104. □

Compute $\dfrac{dy}{dx}$ *for each function.*

(i) $y = x^{x^2}$ (ii) $y = x^{\tan(x)}$ (iii) $y = x^{e^x}$

(iv) $y = (\sin x)^x$ (v) $y = (\ln x)^{(e^x)}$ (vi) $y = 2^{x^x}$

Chapter 7 | Application of Derivatives Part II - Shape of a Graph

§7.1 Recap on Definitions

Definition 7.1.52

Given a function f, we say that $f(c)$ is a **absolute** or **global maximum** of f on an interval I if $f(c) \geq f(x)$ for all x in I.

Similarly we call $f(c)$ a **absolute** or **global minimum** of f on an interval I whenever $f(c) \leq f(x)$ for all x in I.

We say that $f(c)$ is a **local maximum** of f provided that $f(c) \geq f(x)$ for all x *near* c. Similarly, $f(c)$ is called a **local minimum** of f whenever $f(c) \leq f(x)$ for all x near c.

Note: If I is not specified, we take I to be $(-\infty, \infty)$, the set of all real numbers.

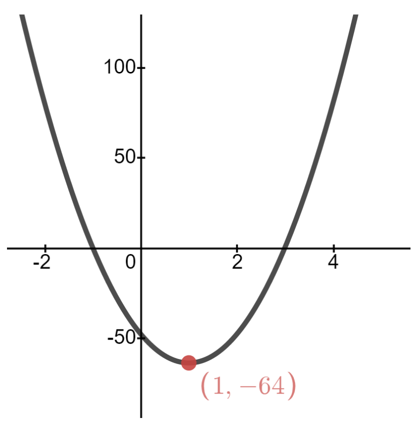
Question 105.

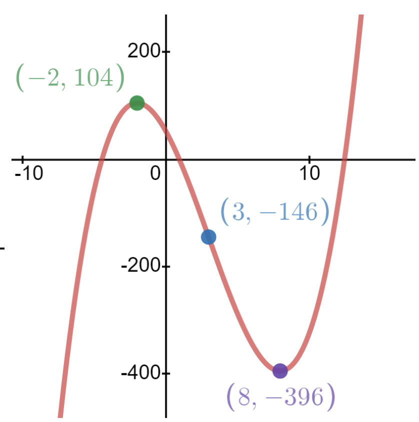
For each of the functions, answer the following.

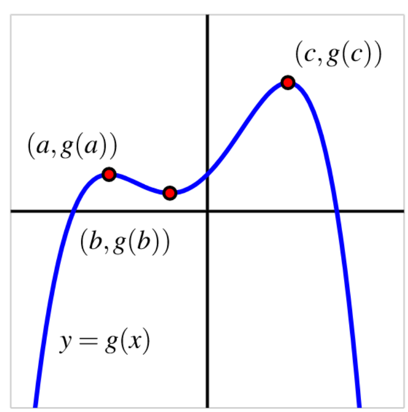
(a) *For what values of x, is the function increasing and decreasing? What is the sign of the derivative on the corresponding intervals?*

(b) *Which of the above three functions has a global maximum or minimum? What are the values?*

(c) *Which of the above three functions has a local maximum or minimum? What are the values?*







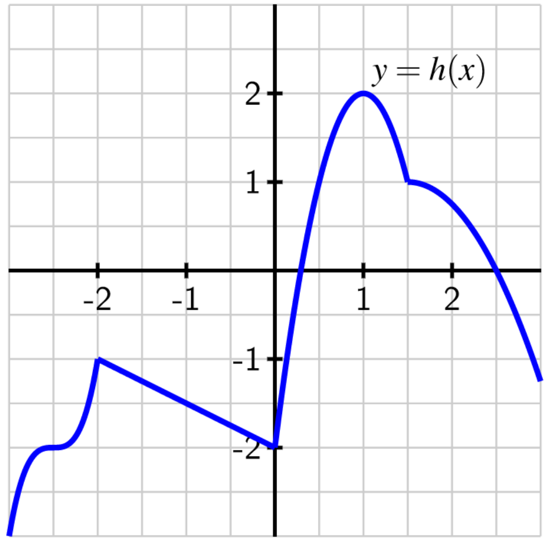
Question 106.

Consider the function h given by the graph in the figure below.

(a) *Identify all of the values of c in $(-3, 3)$ for which $h(c)$ is a local maximum of h*

(b) *Identify all of the values of c in $(-3, 3)$ for which $h(c)$ is a local minimum of h.*

(c) *Does h have a global maximum on the interval $[-3, 3]$? If so, what is the value of this global maximum?*



(d) *Does h have a global minimum on the interval* $[-3, 3]$ *? If so, what is its value?*

(e) *Identify all values of c for which* $h'(c) = 0$.

(f) *Identify all values of c for which* $h'(c)$ *does not exist.*

(g) *True or false: every relative maximum and minimum of h occurs at a point where* $h'(c)$ *is either zero or does not exist.*

(h) *True or false: at every point where* $h'(c)$ *is zero or does not exist, h has a relative maximum or minimum.*

From the two examples above, we observe functions seem to have extreme values when the domain is restricted to a closed interval. In fact, this is due to following theorem gives conditions under which a function is guaranteed to possess extreme values.

Theorem 7.1.53: Extreme Value Theorem

If f *is continuous on a closed interval* $[a, b]$, *then* f ***attains*** *an absolute maximum value* $f(c)$ *and an absolute minimum value* $f(d)$ *at some numbers c and d in* $[a, b]$.

Warning: Note the usage of the word ‘attain’ here. It specifically says that not only the global max and min exist, but also they are attainable! In other words, we can find some numbers c and d in $[a, b]$ such that $f(c)$ is the global min and $f(d)$ is the global max.

The theorem does not tell us where these extreme values occur, but rather only that they must exist. The first theorem in the next section tells us how to find these values!

§7.2 Critical Points and the First Derivative Test

Theorem 7.2.54: Fermat's Theorem

If f has a local extremum at $x = c$ and f is differentiable at c then $f'(c) = 0$.

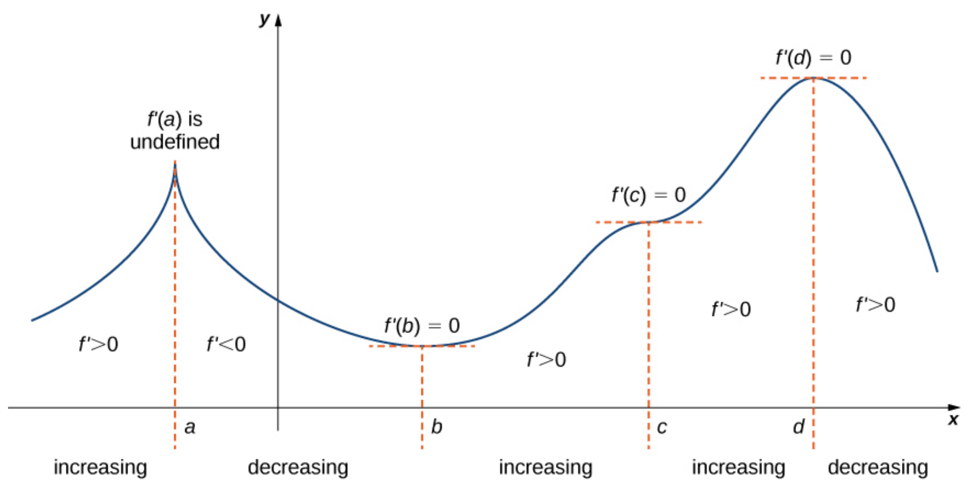
Definition 7.2.55: Critical Point

For any function f, a point c in the *domain* of f where $f'(c)$ is either 0 or undefined, is called a **critical point** of the function. In addition, the point $(c, f(c))$ on the graph of f is also called a critical point. A critical value of f is the value, $f(c)$, at a critical point, c.

Thus Fermat's theorem can be restated as

If f has a local extremum at $x = c$, then c is a critical point.

Warning: The converse of Fermat's theorem is not true. Not every critical point is a local extremum. Consider for example, a function $f(x)$ whose graph is as follows:



The function f has four critical points: a, b, c, and d. The function has local maxima at a and d, and a local minimum at b, The function does not have a local extremum at c.

Perhaps, the most interesting observation to make here is that the sign of f' changes at all local extrema.

Theorem 7.2.56: First Derivative Test

If c is a critical point of a continuous function f that is differentiable near c (except possibly at $x = c$), then

(i) f has a relative maximum at c if and only if f' changes sign from positive to negative at c, and

(ii) f has a relative minimum at c if and only if f' changes sign from negative to positive at c.

Question 107.

Find and classify the local extrema of the function

$$g(x) = x + 2\sin x, \qquad 0 \leq x \leq 2\pi$$

Finding the global maximum and minimum of a continuous function f on the interval $[a, b]$

Step 1. Find the critical points $f(x)$ that lie inside the interval (a, b). you do not need to check if these are local max/min.

Step 2. Find the value of the function $f(p)$ for every critical point p above.

Step 3. Find the values of $f(x)$ at the endpoints of the interval, i.e. find $f(a)$ and $f(b)$.

Step 4. The largest of the values from Steps 2 and 3 is the global maximum value; the smallest of these values is the global minimum value.

Question 108.

Suppose that $g(x)$ is a function continuous for every value of $x \neq 2$ whose first derivative is

$$g'(x) = \frac{(x+4)(x-1)^2}{x-2}.$$

Further, assume that it is known that g has a vertical asymptote at $x = 2$.

(a) *Determine all critical points of g.*

(b) *By developing a carefully labeled first derivative sign chart, decide whether g has as a local maximum, local minimum, or neither at each critical point.*

(c) *Does g have a global maximum? global minimum? Justify your claims.*

(d) *What is the value of $\lim_{x\to\infty} g'(x)$? What does the value of this limit tell you about the long-term behavior of g ?*

(e) *Sketch a possible graph of $y = g(x)$.*

Question 109.

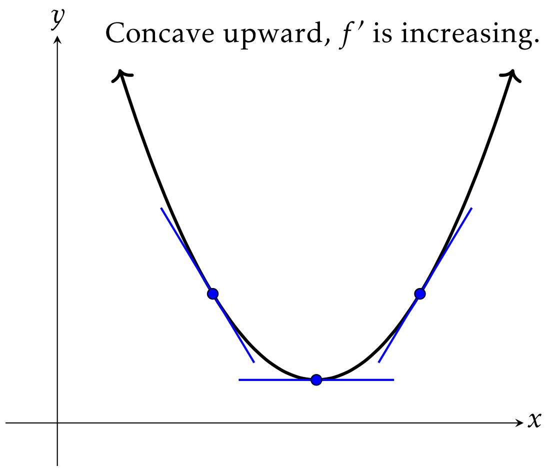
*Find the **exact** global maximum and minimum for the function below on the stated interval.*

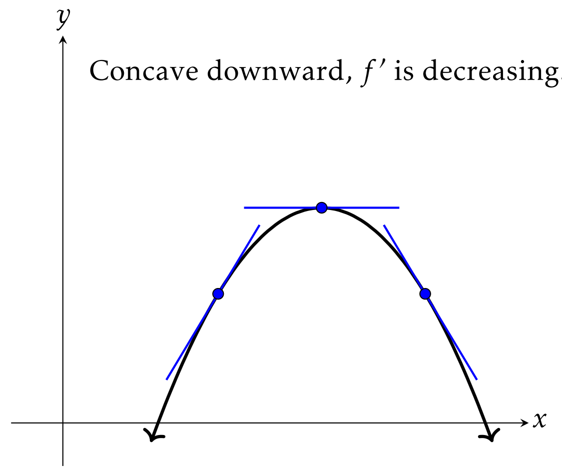
$$p(x) = \sin(x) + \cos(x), \quad \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

§7.3 Concavity of a graph

Definition 7.3.57

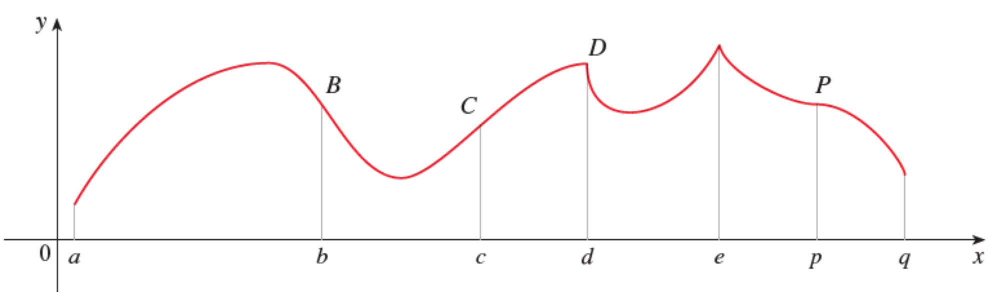
Let f be a differentiable function on an open interval I. Then f is said to be **concave up** on I if and only if f' is increasing on I, and f is said to be **concave down** on I if and only if f' is decreasing on I,





Question 110.

Identify the intervals on which f is concave up or concave down.



When $f'(x)$ is increasing, its derivative $f''(x)$ is positive. Similarly, when f' decreases, f'' is negative. So we get the following test;

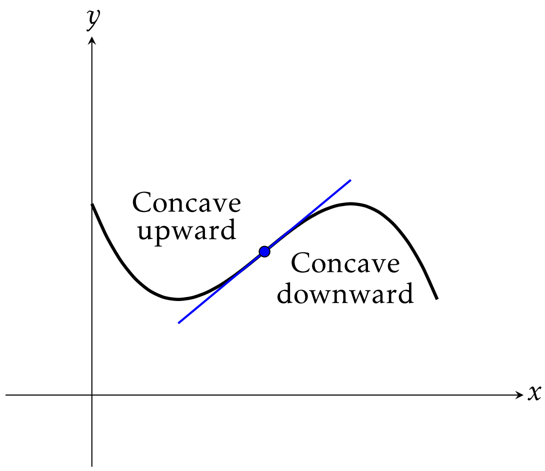
Theorem 7.3.58: Concavity Test

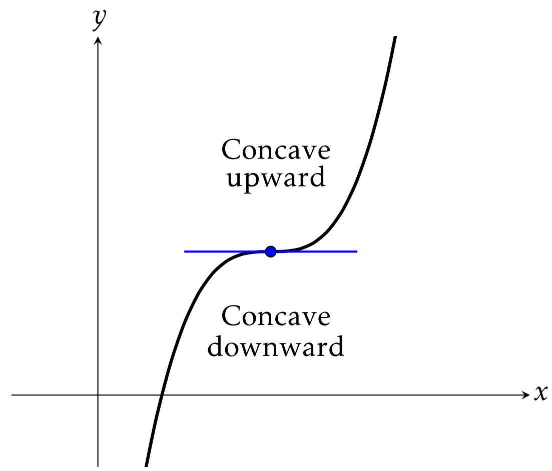
(a) If $f''(x) > 0$ on an interval I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ on an interval I, then the graph of f is concave downward on I.

Definition 7.3.59: Inflection Point

A point p, at which the graph of a continuous function, f, changes concavity is called an **inflection point** of f.





Similar to critical points, these inflection points **may** occur when $f''(x) = 0$ or when $f''(x)$ is undefined. To test whether p is an inflection point, we need to check whether f'' changes sign at p.

■ Question 111.

Consider $f(x) = x^3 - 3x^2 - 9x - 1$. Determine the intervals where $f(x)$ is concave up and concave down, and list any points of inflection. □

■ Question 112.

Sketch a possible graph of a function f that satisfies the following conditions: □

(a) $f'(x) > 0$ *on* $(-\infty, 1)$, $f'(x) < 0$ *on* $(1, \infty)$

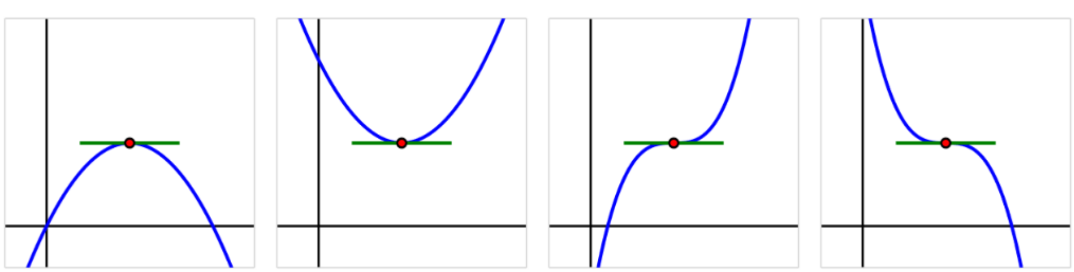
(b) $f''(x) > 0$ *on* $(-\infty, -2)$ *and* $(2, \infty)$, $f''(x) < 0$ *on* $(-2, 2)$

(c) $\lim_{x\to-\infty} f(x) = -2$, $\lim_{x\to\infty} f(x) = 0$

§7.4 The Second Derivative Test

We have seen how to use the first derivative to determine whether a critical point corresponds to a local extrema. This was the **First Derivative Test**. We have just examined how the second derivative can be used to understand the concavity of a function. But, we can also use the second derivative to verify if a critical point is a local extrema. This is called the **Second Derivative Test**.

In the last section, we saw that there are four possibilities for the graph of a function f with a horizontal tangent line at a critical point.



From the pictures, we can conclude the following.

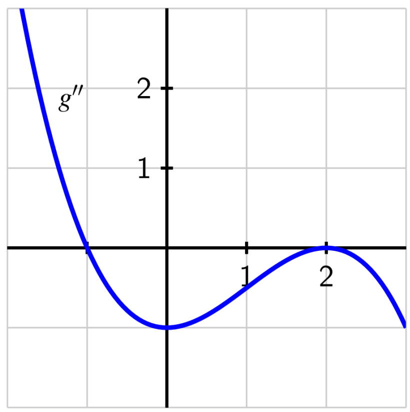
Theorem 7.4.60: Second Derivative Test

If p is a critical point of a continuous function f such that $f'(p) = 0$ and $f''(p) \neq 0$, then f has a local maximum at p if and only if $f''(p) < 0$, and f has a local minimum at p if and only if $f''(p) > 0$.

Warning: In the event that $f''(p) = 0$, the second derivative test is inconclusive. That is, the test doesn't provide us any information. This is because if $f''(p) = 0$, it is possible that f has a local minimum, local maximum, or neither.

Question 113.

Consider a function $g(x)$ whose second derivative g'' is given by the following graph.



(a) *Find the x -coordinates of all points of inflection of g.*

(b) *Fully describe the concavity of g by making an appropriate sign chart.*

(c) *Suppose you are given that $g'(-1.6) = 0$. Is there a local maximum, local minimum, or neither (for the function g) at this critical point of g, or is it impossible to say? Why?*

(d) *Assuming that $g''(x)$ is a polynomial (and that all important behavior of g'' is seen in the graph above), what degree polynomial do you think $g(x)$ is? Why?*

§7.5 Summary of Curve Sketching

The following checklist is intended as a guide to sketching a curve $y = f(x)$ by hand. Not every item is relevant to every function. (For instance, a given curve might not have an asymptote or possess symmetry.) But the guidelines provide all of the information you need to make a sketch that displays the **most important aspects** of the function.

A. Domain. It's often useful to start by determining the domain D of f, that is, the set of values of x for which $f(x)$ is defined.

B. Intercepts. The y-intercept is $f(0)$ and this tells us where the curve intersects the y-axis. To find the x-intercepts, we set $y = 0$ and solve for x. (You can omit this step if the equation is difficult to solve.)

C. Symmetry.

(i) If $f(-x) = f(x)$ for all x in D, that is, the equation of the curve is unchanged when x is replaced by $-x$, then f is an **even function** and the curve is symmetric about the y-axis. This means that our work is cut in half. If we know what the curve looks like for $x > 0$, then we need only reflect about the y-axis to obtain the complete curve. Here are some examples: $y = x^2, y = x^4, y = |x|$, and $y = \cos x$.

(ii) If $f(-x) = -f(x)$ for all x in D, then f is an **odd function** and the curve is symmetric about the origin. Again we can obtain the complete curve if we know what it looks like for $x > 0$. [Rotate 180° about the origin] Some simple examples of odd functions are $y = x, y = x^3, y = x^5$, and $y = \sin x$.

(iii) If $f(x+p) = f(x)$ for all x in D, where p is a positive constant, then f is called a periodic function and the smallest such number p is called the period. For instance, $y = \sin x$ has period 2π and $y = \tan x$ has period π. If we know what the graph looks like in an interval of length p, then we can use translation to sketch the entire graph.

D. Asymptotes.

(i) A line $y = \mathrm{L}$ is a horizontal asymptote of a function $f(x)$ if $f(x) \to \mathrm{L}$ as $x \to \infty$ or $x \to -\infty$. Note that a function may have different **horizontal asymptotes** as it goes towards $+\infty$ or $-\infty$. If it turns out that $f(x) \to \infty$ or $-\infty$ as $x \to \infty$, then we do not have an asymptote to the right, but this fact is still useful information for sketching the curve.

(ii) A line $x = \mathrm{K}$ is a vertical asymptote of a function $f(x)$ if $f(x) \to \infty$ or $-\infty$ as $x \to \mathrm{K}+$ and $x \to \mathrm{K}-$. For rational functions you can locate the **vertical asymptotes** by equating the denominator to 0 after canceling any common factors. But for other functions this method does not apply.

E. Intervals of Increase or Decrease. Find the intervals on which $f'(x)$ is positive (f is increasing) and the intervals on which $f'(x)$ is negative (f is decreasing).

F. Local Maximum and Minimum Values. Follow the steps from last section.

G. Concavity and Points of Inflection. Compute $f''(x)$. The curve is concave upward where $f''(x) \geq 0$ and concave downward where $f''(x) \leq 0$. Inflection points occur where the direction of concavity changes.

H. Drawing the Graph. Using the information in items A-G, draw the graph. Sketch the asymptotes as dashed lines. Plot the intercepts, maximum and minimum points, and inflection points. Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes. If additional accuracy is desired near any point, you can compute the value of the derivative there. The tangent indicates the direction in which the curve proceeds.

Question 114.

Sketch the graph of the function $f(x) = xe^x$.

Question 115.

Sketch the graph of the function $f(x) = x^{2/3}(6-x)^{1/3}$.

Question 116.

Sketch the graph of a function that satisfies ***all*** *of the following conditions.*

- $f'(0) = f'(4) = 0, \quad f'(x) = 1$ *if* $x < -1$
- $f'(x) > 0$ *if* $0 < x < 2$
- $f'(x) < 0$ *if* $-1 < x < 0$ *or* $2 < x < 4$ *or* $x > 4$
- $\lim_{x\to 2^-} f'(x) = \infty$, $\lim_{x\to 2^+} f'(x) = -\infty$
- $f''(x) > 0$ *if* $-1 < x < 2$ *or* $2 < x < 4$
- $f''(x) < 0$ *if* $x > 4$

Chapter 8 | Application of Derivatives Part III - Optimization

§8.1 The Problem Solving Strategy

The basic idea of the optimization problems that follow is the same. We have a particular quantity that we are interested in maximizing or minimizing. However, we also have some auxiliary condition that needs to be satisfied. The problem is that the function to optimize may not be explicitly provided. We may need to understand the problem by drawing a figure, introducing variables, and then by developing a formula for a function that models the quantity to be optimized. While there is no single algorithm that works in every situation where optimization is used, in most of the problems we consider, the following steps are helpful:

Algorithm for Solving Applied Optimization Problems

Step 1. **Draw a picture. Introduce and label variables.**

It is essential to first understand what quantities are allowed to vary in the problem and then to represent those values with variables. Constructing a figure with the variables labeled is almost always an essential first step. Sometimes drawing several diagrams can be especially helpful to get a sense of the situation.

Step 2. **Make sure that you know what quantity or function is to be optimized.**

Write down a formula for this quantity algebraically using the variables you introduced in the last step. This function is called the **Objective Function**.

Step 3. **Using information given in the problem, re-write your formula from Step 2 as a function of ONE variable.**

The information given in the problem regarding the relationship among the variables should aid you in making the necessary substitutions or eliminations in this step. The information given is usually in the form of other equations; we refer to this as a **constraint equations**. you have to eliminate all but one variable.

Step 4. **Decide the domain on which to optimize your Objective Function.**

Often the physical constraints of the problem will limit the possible values that the variables can take on. Thinking back to the diagram describing the overall situation and any relationships among variables in the problem often helps identify the smallest and largest values of the input variable.

Step 5. **Apply the techniques you know to identify the Max/Min(s).**

This always involves finding the critical numbers of the function first. Then evaluate the function at the endpoints and critical numbers to find the global max and/or min.

Step 6. **Finally, bring all your information together, and answer whatever questions were posed by the problem.**

Make sure that you have answered the correct question: does the question seek the absolute maximum of a quantity, or the values of the variables that produce the maximum? Also make sure to answer all asked questions! (Many problems have multiple parts!)

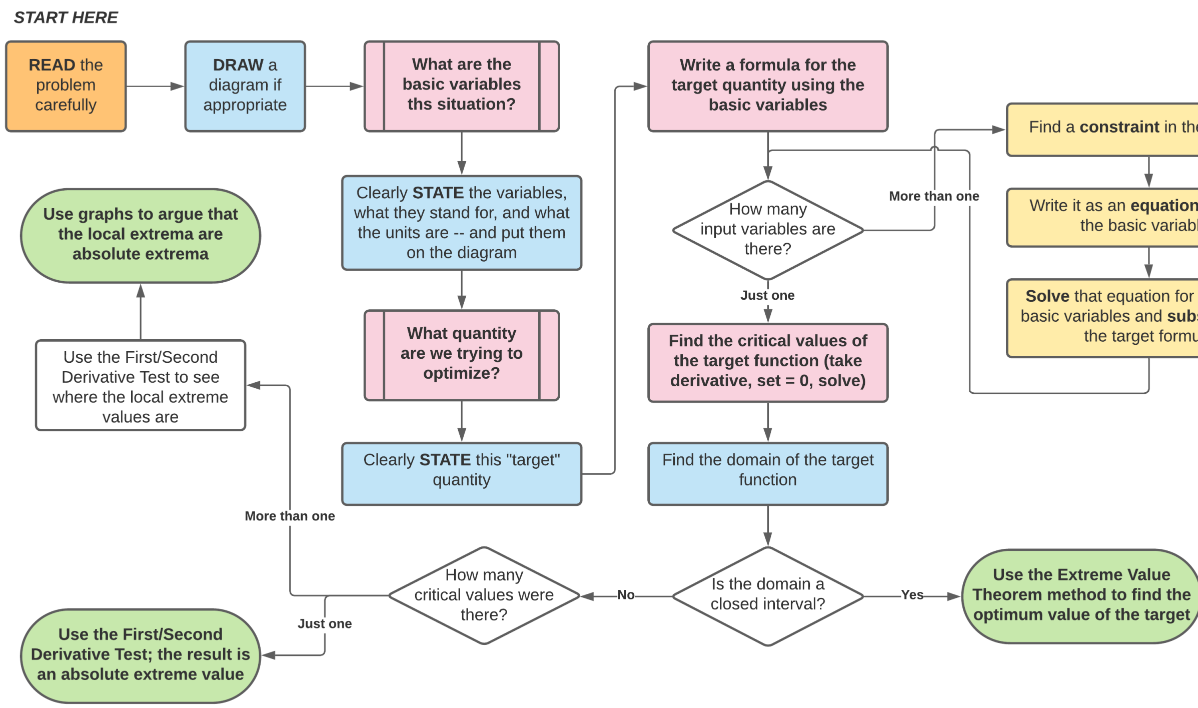


Figure 8.1: Flowchart for Optimization Problems (Picture Courtesy: Robert Talbert)

8.1.1 Practice Problems

Question 117.

A box with an open top of fixed volume $V = 4000\ \mathrm{m}^3$ *with a square base is to be constructed. Find the dimensions of the box that minimize the amount of material used in its construction.* □

Question 118.

According to U.S. postal regulations, the girth plus the length of a parcel sent by mail may not exceed 108 inches, where by "girth" we mean the perimeter of the smallest end. What is the largest possible volume of a rectangular parcel with a square end that can be sent by mail? What are the dimensions of the package of largest volume? □

Question 119.

A metal can manufacturer needs to build cylindrical cans with volume 300 cubic centimeters. The material for the side of a can costs 0.03 cents per cm^2*, and the material for the bottom and top of the can costs 0.06 cents per* cm^2*. What is the cost of the least expensive can that can be built?* □

Question 120.

Consider your typical piece of notebook paper. The dimensions are likely to be 8.5 by 11 inches. Suppose we remove a square of side length x from the corners of the paper, and we then fold up each newly formed flap to make an open-top box. How large must the removed squares be in order to achieve a box with the largest possible volume? □

Question 121.

Consider the region in the $x-y$ *plane that is bounded by the x axis and the function* $f(x) = 25 - x^2$*. Construct a* □

rectangle whose base lies on the x-axis and is centered at the origin, and whose sides extend vertically until they intersect the curve $y = 25 - x^2$. Which such rectangle has the maximum possible area?

■ Question 122. □

Two vertical poles of heights 60 ft and 80 ft stand on level ground, with their bases 100 ft apart. A cable that is stretched from the top of one pole to some point on the ground between the poles, and then to the top of the other pole. What is the minimum possible length of cable required? Justify your answer completely using calculus.

■ Question 123. □

A 20 cm piece of wire is cut into two pieces. One piece is used to form a square and the other to form an equilateral triangle. How should the wire be cut to maximize the total area enclosed by the square and triangle? to minimize the area?

■ Question 124. □

Consider an isosceles triangle that circumscribes a circle of radius 1. What is the smallest possible area of the triangle?

■ Question 125. □

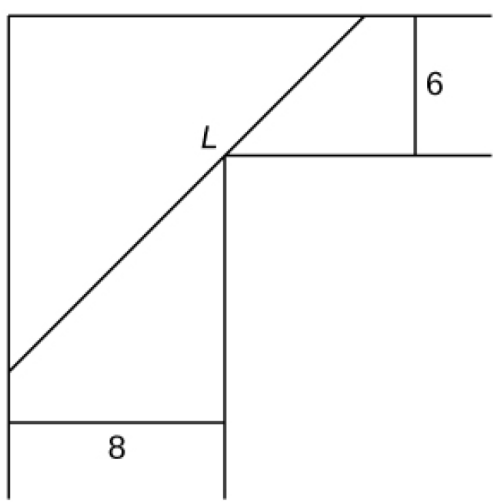
Find the volume of the largest right circular cylinder that fits in a sphere of radius 1.

■ Question 126. □

Find the dimensions of a right circular cone with surface area $S = 4\pi$ that has the largest volume.

■ Question 127. □

You are moving into a new apartment and notice there is a corner where the hallway narrows from 8 ft to 6 ft. What is the length of the longest item that can be carried horizontally around the corner?



■ Question 128. □

A window is composed of a semicircle placed on top of a rectangle. If you have 20 ft of window-framing materials for the outer frame (three sides of the rectangle and the half-circle), what is the maximum size (i.e. area) of the window you can create?