MATH 111 - Calculus and Analytic Geometry I

Lecture 24 Worksheet

Fall 2020 **Subhadip Chowdhury** **Oct 12**

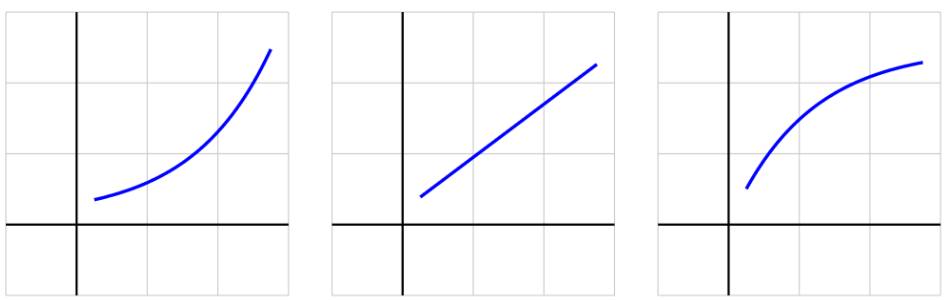
Title: Shape of a Graph II - Concavity and SDT

Summary: We will learn about how the second derivative of a function reveals important information about the behavior of the function, including the function's concavity.

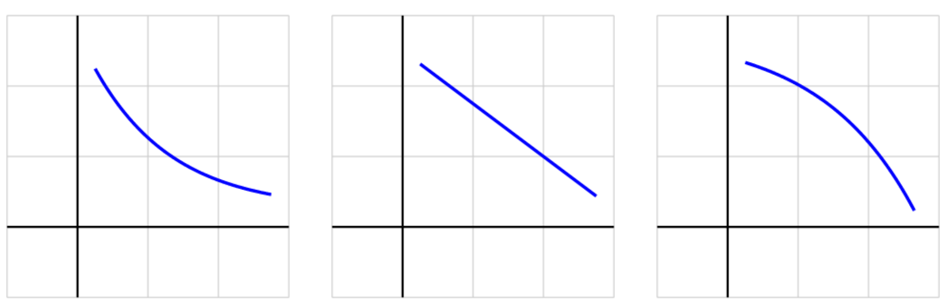
Related Reading: Chapters 4.3 and 4.5 from the textbook.

§A. Motivation

Last lecture, we learned that the first derivative tells us when a function is increasing or decreasing and that leads us to finding the maximum or minimum. In addition to asking whether a function is increasing or decreasing, it is also natural to inquire how a function is increasing or decreasing. There are three basic behaviors that an increasing function can demonstrate on an interval: the function can increase more and more rapidly, it can increase at the same rate, or it can increase in a way that is slowing down.



Similarly for the decreasing case.

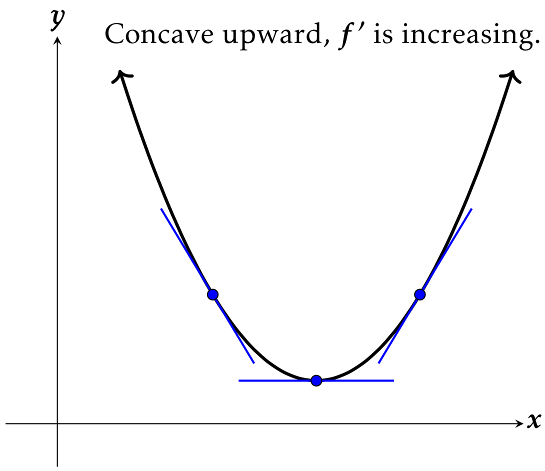


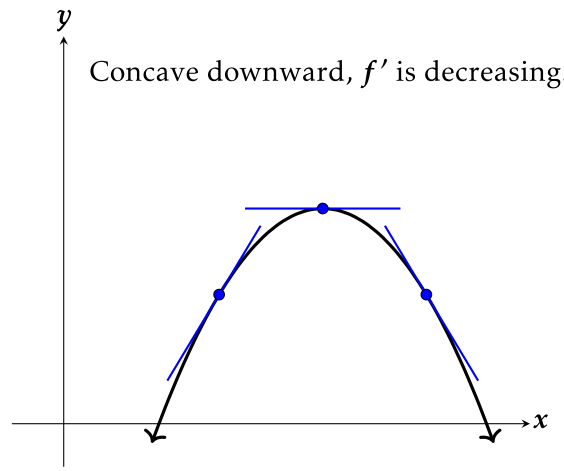
The notion of concavity provides a simpler language to describe these behaviors.

§B. Concavity of a graph

Definition B.1

Let f be a differentiable function on an open interval $\mathbf{I}$. Then f is said to be **concave up** on $\mathbf{I}$ if and only if f' is increasing on $\mathbf{I}$, and f is said to be **concave down** on $\mathbf{I}$ if and only if f' is decreasing on $\mathbf{I}$,





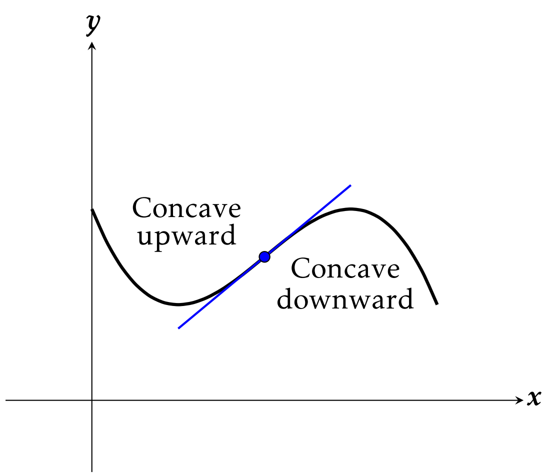
■ Question 1. ☐

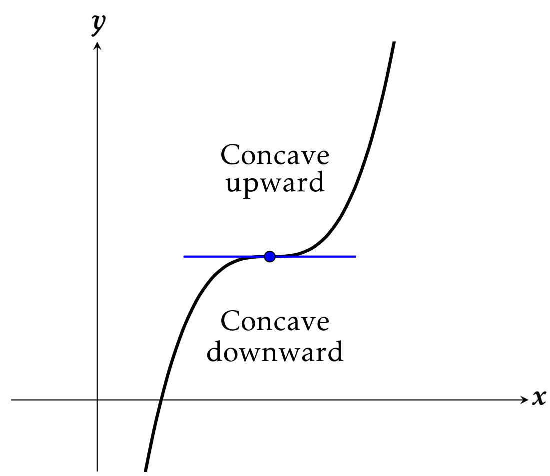
(a) If $f'(x)$ is increasing, what can we conclude about $f''(x)$?

(b) If $f'(x)$ is decreasing, what can we conclude about $f''(x)$?

Definition B.2

A point p, at which the graph of a continuous function, f, changes concavity is called an **inflection point** of f.





Similar to critical points, these inflection points **may** occur when $f''(x) = 0$ or when $f''(x)$ is undefined. To test whether p is an inflection point, check whether f'' changes sign at p.

■ Question 2. ☐

Consider $f(x) = x^3 - 3x^2 - 9x - 1$. Determine the intervals where $f(x)$ is concave up and concave down, and list any points of inflection.

(a) Just like with increasing and decreasing, start by determining the important points where concavity could change. That is, compute $f''(x)$ and solve for when $f''(x) = 0$ or when $f''(x)$ is undefined.

(b) You should only get one value $x = p$ in the previous step, and so there are two subintervals to consider:

$$(-\infty, p) \text{ and } (p, \infty).$$

We can again use a table of some kind (or whatever organizational device you choose), to determine the sign of $f''(x)$ and make conclusions about the graph of f.

Intervals	$(-\infty, p)$	(p, ∞)
Test Points		
Sign of $f''(x)$		
Conclusion		

Question 3.

For each function below, use the idea of First Derivative Test and the Concavity Test to determine the following:

- Critical Points
- Interval where f is increasing
- Interval where f is decreasing
- Local Max
- Local Min
- Interval where f is concave up
- Interval where f is concave down
- Inflection Points

(a) $f(x) = x^3 - 6x^2 + 15$

(b) $f(x) = \frac{1}{4}x^4 + 2x^3$

(c) $f(x) = (x-1)e^x$

(d) $f(x) = xe^{-x}$

(e) $f(x) = \dfrac{x}{x-5}$

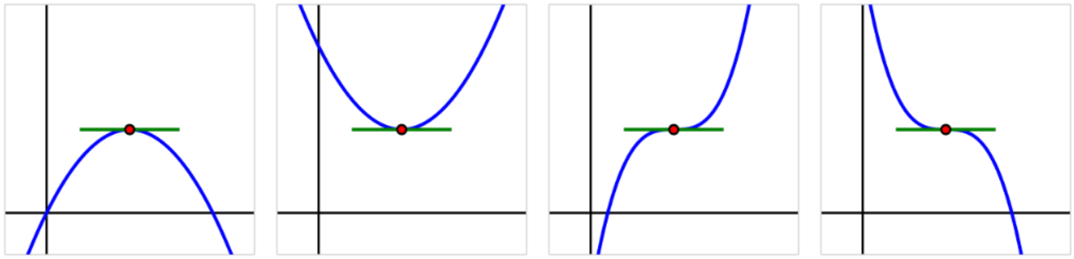
(f) $f(x) = \sin(x)e^x$ on the interval $[-\pi, \pi]$.

(g) $f(x) = \sqrt{x}\ln(x)$

§C. The Second Derivative Test

We have seen how to use the first derivative to determine whether a critical point corresponds to a local extrema. This was the **First Derivative Test**. We have just examined how the second derivative can be used to understand the concavity of a function. But, we can also use the second derivative to verify if a critical point is a local extrema. This is called the **Second Derivative Test**.

Last lecture we saw that there are four possibilities for the graph of a function f with a horizontal tangent line at a critical point.



From the pictures, we can conclude the following.

Theorem C.1: Second Derivative Test

If p is a critical point of a continuous function f such that $f'(p) = 0$ and $f''(p) \neq 0$, then f has a local maximum at p if and only if $f''(p) < 0$, and f has a local minimum at p if and only if $f''(p) > 0$.

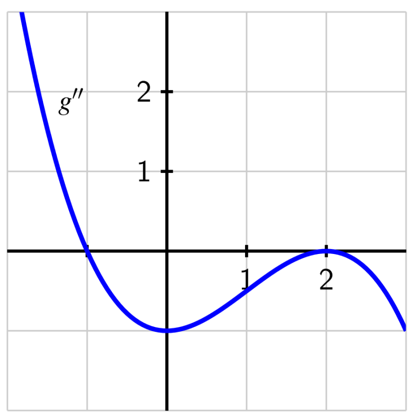
In the event that $f''(p) = 0$, the second derivative test is inconclusive. That is, the test doesn't provide us any information. This is because if $f''(p) = 0$, it is possible that f has a local minimum, local maximum, or neither.

Question 4.

Check that the second derivative test can be used to identify the local extrema for the functions in Question 3.

Question 5.

Consider a function $g(x)$ whose second derivative g'' is given by the following graph.



(a) Find the x -coordinates of all points of inflection of g.

(b) Fully describe the concavity of g by making an appropriate sign chart.

(c) Suppose you are given that $g'(-1.6) = 0$. Is there a local maximum, local minimum, or neither (for the function g) at this critical point of g, or is it impossible to say? Why?

(d) Assuming that $g''(x)$ is a polynomial (and that all important behavior of g'' is seen in the graph above), what degree polynomial do you think $g(x)$ is? Why?