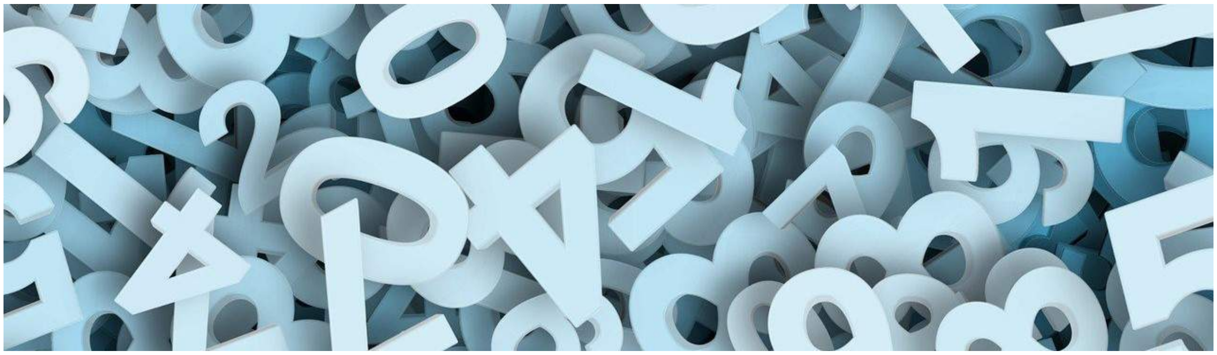

Applied Differential Calculus

Math 110 Lecture notes

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Contents

Part I

Functions

Chapter 1 | Function Fundamentals

§1.1 Function Placemat Activity

Before we define the fundamental terms related to functions, let's try an activity and review how much you remember from precalculus. The goal of this activity is to recall properties of functions, including:

- whether a relation gives a function;
- whether a function is even, odd, or neither;
- values of functions at particular points in the domain;
- domain and range of a function;
- multiple representations of functions;
- whether a function is one-to-one.

Most of these concepts are discussed in sections 1.1 and 1.2 of your course textbook.

1.1.1 Task

At your table, you will find a set of placemats and words that describe what is given in each space on the placemat. Work together in your group to place each sticker in an applicable position on the placemats. Each sticker will be used once and only once. There are some stickers that will appear more than once. This means, for example, that there are multiple functions where $f(1) = 2$.

1.1.2 Discussion Questions

When you are finished with your Powerpoint activity, see if you can answer the following questions.

- What was your strategy for choosing the correct matching?
- Which options were easiest for you to find, and why?
- Which options where hardest for you to find, and why?
- In your own words, what is a function?
- How can something fail to be a function?
- What is an odd function?
- What is an even function?
- What does $f(1) = 2$ mean?
- What does $f(x) = a$ mean?
- What is the domain of a function?
- What is the range of the function?
- What does it mean for a function to be periodic?
- Did you have anything that matched with option Q (“none of the above")? Why do you think that is the case?
- What did you learn about the absolute value of x?
- A function that has different formula on different part of its domain is called a piecewise-defined function. Which of the above were piecewise-defined functions?*
- Provide a graphical example of something that is not a function.
- Provide an example of a function described in words.
- Provide a table that gives an example of something that is not a function.
- What questions do you have after completing this activity?

*Check https://www.desmos.com/calculator/tk34ppdohh to see how to graph piecewise functions in Desmos.

Chapter 2 | A Catalog of Essential Functions

§2.1 Polynomials and Rational Functions

A function P is called a polynomial if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are constants called the coefficients of the polynomial. The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$. If the leading coefficient $a_n \neq 0$, then the degree of the polynomial is n. For example, the function

$$P(x) = 2x^6 - x^4 + \frac{2}{5}x^3 + \sqrt{2}$$

is a polynomial of degree 6.

If $n = 1$, we say that $P(x)$ is linear, if $n = 2$, it's a quadratic, and so on.

2.1.1 Power Functions

A function of the form $f(x) = x^a$, where a is a constant, is called a power function. Here a does not necessarily have to an integer, it can be any positive or negative real number (or 0).

2.1.2 Rational Function

A function which can be written as the quotient of two polynomials is called a **rational function**. The following are examples of rational functions

(i) $\frac{1}{x}$ (ii) $\frac{x^3 + 5x - 5}{x^8 - 3}$ (iii) $\frac{x^9 - 4x^4 + 2x}{x^7 - 3x^2 + 3}$

By definition, polynomials themselves are rational functions that have a denominator of 1. e.g. $x + 1 = \frac{x+1}{1} = \frac{x+1}{x^0}$, so it's a rational function.

§2.2 Inverse Function

Definition 2.2.1

We call a function f **one-to-one** if $f(a) \neq f(b)$ whenever $a \neq b$. That is, the function must send each input to a **different** output, i.e. no two outputs of the function can be the same.

Geometrically, when looking at the graph, a one-to-one function should pass a **horizontal line test** (does that sound familiar?).

■ **Question 1.** □

*Sketch a graph of a function that is **not** one-to-one.*

Question 2.

We can determine if a function is one-to-one by solving for x in the equation $f(x) = c$ for all c in the range of f. If there is always just one unique solution, then f is one-to-one. Try this for the following functions:

(a) *$f(x) = x^2$. Try to solve for x in $x^2 = c$, where c is any positive constant (or alternatively, draw a picture!). Is f a one-to-one function?*

(b) *$g(x) = \dfrac{1}{x+2}$. Try to solve $g(x) = c$ for x, where c is any constant such that $c \neq 0$.*
Is g a one-to-one function?

Definition 2.2.2

Functions that are one-to-one are precisely those that have **inverses**. For a function f, we say that f has an **inverse function**, denoted f^{-1}, if,

$$f(f^{-1}(y)) = y \text{ for all } y \text{ in the domain of } f^{-1}$$

and

$$f^{-1}(f(x)) = x \text{ for all } x \text{ in the domain of } f.$$

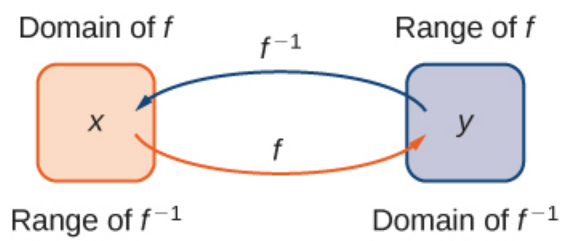


Figure 2.1: Domain and Range for a one-to-one function f and its inverse f^{-1}

Question 3.

Verify that for $f(x) = \dfrac{1}{x+2}$, the inverse function is $f^{-1}(x) = \dfrac{1-2x}{x}$. This means you will need to check the above definition and compose these two functions together (twice!).

Sketch both f and f^{-1} in DESMOS. Do you notice any symmetry between the two graphs?

Example 2.2.3

The function $f(x) = x^2$ is **not one-to-one**, so it doesn't have an inverse. If you want to give it one, you have to **restrict its domain** so that the graph is one-to-one.

For example, graph $f(x) = x^2$ on the domain $[0, \infty)$, i.e. where $x \geq 0$. Then with this restriction, we have that $f^{-1}(x) = \sqrt{x}$ is its inverse. See Figure 2.2.

So we conclude the following:

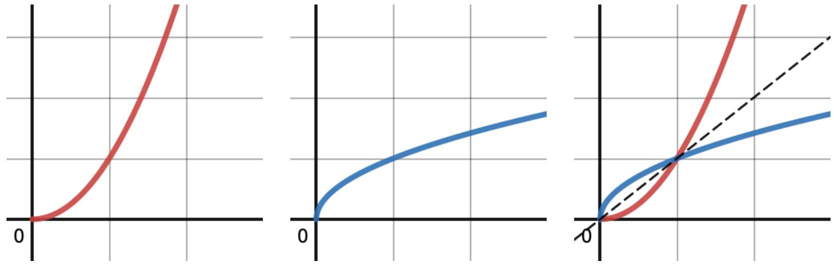


Figure 2.2: The function $y = x^2, x \geq 0$, its inverse $y = \sqrt{x}$, and both graph together. Note the symmetry around the line $y = x$

> **Observation:** The graphs of $f(x)$ and $f^{-1}(x)$ will always have a symmetry about the line $y = x$. If (a, b) is a point on the graph of $y = f(x)$, then (b, a) will be a point on the graph of $y = f^{-1}(x)$.

Question 4.

Click the link below to try a Desmos exercise similar to the above. You are given the graph of a (one-to-one) function and you want to try and plot the inverse. Drag the given green points to try and sketch the graph of the inverse function.

https://www.desmos.com/calculator/vwfs3pkl0b

§2.3 Exponential Functions

You likely first dealt with exponents algebraically when trying to solve some equations where part of the exponent involved a variable, like in the next two problems.

Question 5.

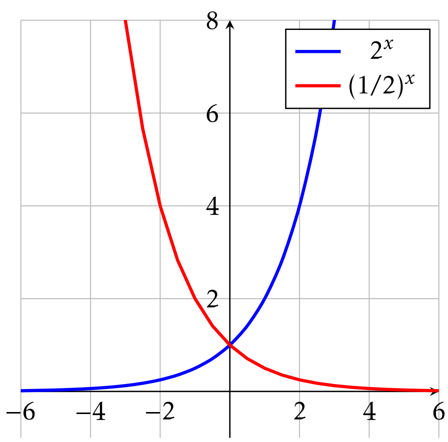
(a) *If* $4^{x+1} = 16$, *then what does x equal? (Don't do algebra, just think about what integer value will work...)*

(b) *If* $3^6 \times 3^x = 1$, *then what does x equal? (you can combine* 3^6 *and* 3^x *to make this look like the previous question...)*

Functions of the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$, are called Exponential functions. When $b > 1$, an exponential function models rapid **growth**, while if $0 < b < 1$, the function models rapid **decay**.

The graphs of $f(x) = 2^x$ and $f(x) = (0.5)^x$ are shown in the figure below. In both cases the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

Graphs of Exponential Functions



■ Question 6.

Find the equation of an exponential function of the form $f(x) = Ae^{Bx}$ whose graph looks like figure 2.3. □

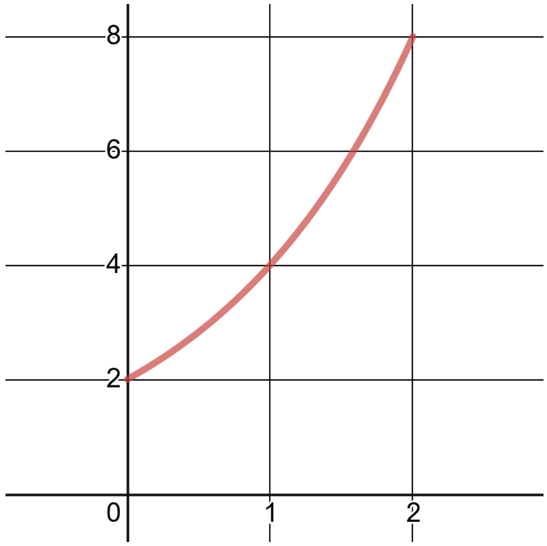


Figure 2.3: An Exponential Function

Examine the graph of the function $f(x) = \left(1 + \frac{1}{x}\right)^x$ via Desmos:

https://www.desmos.com/calculator/4pearyjab

Notice that the graph for $f(x)$ has a horizontal asymptote. That is, as x approaches ∞, the graph is approaching some definite value. This asymptote value is the irrational number e that you have probably heard of before. When we use $b = e$ as the base for an exponential function $y = b^x$, we get **the natural exponential function**.

§2.4 Logarithmic Function

Since exponential functions are one-to-one, they have inverses, called **logarithmic functions**. For the function $y = e^x$, the corresponding inverse function is called the **natural logarithm**, $y = \ln(x)$. In general, a **logarithm function** is the inverse of an exponential function:

$$f(x) = b^x \qquad f^{-1}(x) = \log_b(x)$$

So for example, if we have 2^x, then its inverse is $\log_2(x)$. We call b the **base** of the logarithm.

> **Note:** $\ln x$ is the same thing as $\log_e x$.

Example 2.4.4

To compute the inverse $\log_b(x)$, you are asking the question: for what value of y do we have $b^y = x$? Here are some examples of logarithm computations:

(a) $\log_4(16) = 2$ because $4^2 = 16$.

(b) $\log_3(1) = 0$ because $3^0 = 1$.

(c) $\log_{10}\left(\frac{1}{100}\right) = -2$ because $10^{-2} = \frac{1}{100}$.

Check out this Desmos link to play with general graphs of exponential functions and their inverse logarithms.

https://www.desmos.com/calculator/cxijzb5st6

■ Question 7.

Use the fact that the exponential function and natural logarithm are inverses to compute the following:

(i) $\ln(e^{\pi}) =$ *(ii)* $e^{\ln(2)} =$

Since exponential functions and logs are inverses, we can use them to solve equations involving exponents more easily. For example, to solve the equation $5^{x^2+1} = 5^3$, we can apply the function $\log_5(x)$ to both sides:

$$\log_5\left(5^{x^2+1}\right) = \log_5\left(5^3\right) \Rightarrow x^2 + 1 = 3.$$

Because 5^x and $\log_5(x)$ are inverses, they cancel each other out to leave us with the equation $x^2 + 1 = 3$, which we can easily solve. Give it a try below:

■ Question 8.

Solve for x.

(i) $7^{2x-1} = 7^{x+1}$ *(ii)* $\log_2(x^2) = \log_2(5)$

Note: Before we end the discussion, the two most important properties of log that you might need to use for the next section are

- $\log_c(ab) = \log_c a + \log_c b$
- $\log_c(a^b) = b \log_c a$

§2.5 Exponential Growth and Decay Word Problems

Definition 2.5.5

An exponential growth/decay process is given by the equation

$$Q(t) = Q_0 e^{kt}$$

k is called the (continuous) growth/decay rate.

- If $k > 0$, the process is a growth. If $k < 0$, the process is a decay.
- In an exponential decay process, the time it takes to reduce the starting amount by half is called the **half-life**, denoted $t_{1/2}$.

Example 2.5.6

In an exponential decay, can we figure out the half-life by solving some equation? Note that what we are really trying to do, is solve for a value of t such that $Q(t) = \frac{Q_0}{2}$:

$$\begin{aligned}\frac{Q_0}{2} &= Q(t) = Q_0 e^{kt}\\ \frac{1}{2} &= e^{kt}\\ \ln(1/2) &= \ln\left(e^{kt}\right)\\ \ln(1/2) &= kt\end{aligned}$$

See how in the third step we apply natural log to both sides? This then allows us to remove the e and obtain just the exponent value kt. Now we can easily solve for t to get that $t = \frac{\ln(1/2)}{k} = \frac{-\ln(2)}{k}$.

Question 9.

A biologist is researching a newly-discovered species of bacteria. At time $t = 0$ hours, she puts one hundred bacteria into what she has determined to be a favorable growth medium. Six hours later, she measures 450 bacteria. Assuming exponential growth, how long does it take for the bacteria population to become 1600?

Question 10.

The process of carbon dating involves evaluating the ratio of radioactive carbon-14 to stable carbon-12 isotope. Carbon-14 has a half-life of 5730 years and decays over time, whereas carbon-12 doesn't.

You are presented with a document that purports to contain the recollections of a Mycenaean soldier during the Trojan War. The city of Troy was finally destroyed about 3250 years ago. Given the amount of carbon-12 contained in a measured sample cut from the document, there would have been about 1.3×10^{-12} grams of carbon-14 in the sample when the parchment was new, assuming the proposed age is correct. According to your equipment, 1.0×10^{-12} grams of carbon-14 is remaining. Is there a possibility that this is a genuine document? Or is this instead a recent forgery?

Question 11.

We can use data to make an exponential population model via Desmos. See the link below for a table and model of the ***population of the US State of California, 1900 - 2000****.***:*

`https://www.desmos.com/calculator/iagxaytu3p`

(a) *Use the function modeled in the Desmos page to estimate the population of California in 2010 and 2020.*

(b) *How does the model compare to the actual population values in 2010 and 2020?*

*Source: `https://en.wikipedia.org/wiki/History_of_California_(1900-present)`

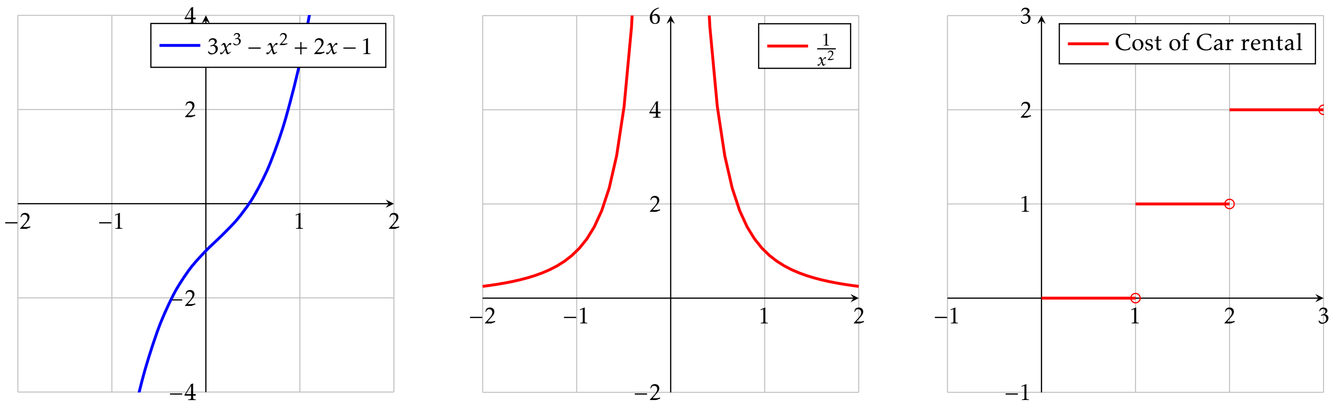
Part II

Limits & Continuity

Chapter 3 | An Informal Introduction to Limit and Continuity

§3.1 Continuity of a Function on an Interval: A Graphical Viewpoint

Consider graphs of the three functions as follows:



Informally, we will say that a function is called **continuous** if you can draw its graph without lifting the pencil from the paper. In the above three pictures, the first function is continuous everywhere; the second one is continuous on all intervals not containing 0, and the third function is continuous on intervals of the form $(n, n+1)$.

§3.2 Continuity of a Function at a Point: A Numerical Viewpoint

A function $y = f(x)$ is called **continuous at a point** if nearby values of x give nearby values of y. In practical terms, this means small errors in the input lead to only small errors in the output. For the same reason, the idea of continuity is important in real life.

A bit more formally, we can say that if $f(x)$ is continuous at $x = a$, the values of $f(x)$ approaches $f(a)$ as x approaches a. So to investigate whether or not a function f is continuous at a, we need to know what value is approached by $f(x)$ as x approaches a.

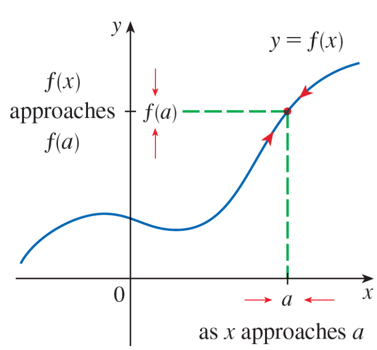


Figure 3.1

§3.3 The Idea of a Limit

The notation

$$\lim_{x\to a} f(x) = L$$

means the values of $f(x)$ approaches L as x approaches a. We read it as:

"the limit of $f(x)$ as x goes to (or tends to) a is L".

Using this new notation, a function $f(x)$ is defined to be continuous at a point $x = a$ if the above limit $L = f(a)$. In other words, if $\lim_{x\to a} f(x)$ is not the same as $f(a)$ then f is not continuous at a. Here's an example of how that might happen.

Example 3.3.7

Consider the function $f(x) = \dfrac{x^2-4}{x-2}$. What is $f(2)$? Draw the graph of $f(x)$ vs. x. Can you guess $\lim_{x\to 2} f(x)$ from the picture?

The most important thing to note from this example is that even when $f(a)$ does not exist, the limit $\lim_{x\to a} f(x)$ might. Here's a second example where $f(a)$ exists but $\lim_{x\to a} f(x)$ doesn't.

Example 3.3.8

Consider the function from the third picture above. Check that $f(1) = 1$. But $\lim_{x\to 1} f(x)$ doesn't exist. Can you explain why not?

■ Question 12. □

Come up with an example or a scenario where neither $f(a)$ nor $\lim_{x\to a} f(x)$ exists.

■ Question 13. □

Come up with an example or a scenario where both $f(a)$ and $\lim_{x\to a} f(x)$ exist but they are not equal to each other.

Chapter 4 | Limits of Functions

§4.1 Exploring Limits

Continuing our discussion from last class, let's jump directly into an example and see if we can recall the ideas of limits. For a function $f(x)$, we write $\boxed{\lim_{x\to a} f(x) = L}$ to mean that the values $f(x)$ get closer and closer to L as x gets closer and closer (but not equal) to a.

Consider the graph of $g(x)$ given in Figure 4.1. Let us use this to show some examples of limits.

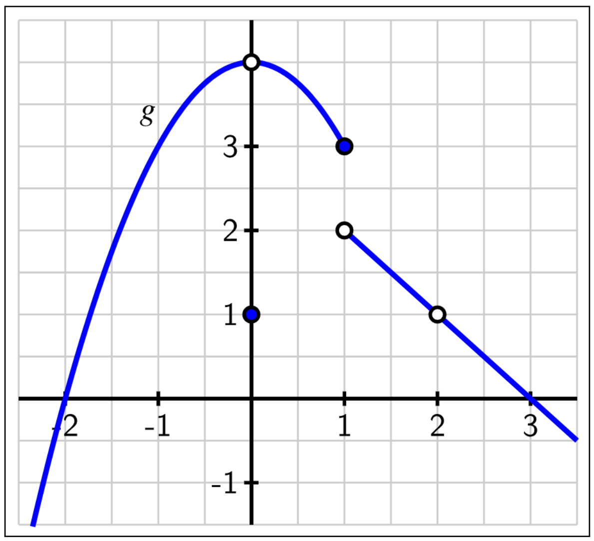


Figure 4.1: Graph of a function $y = g(x)$

- For x-values like $x = -2, -1$, or $x = 3$, the values of $g(x)$ approach $g(-2)$, $g(-1)$, and $g(3)$, respectively. No strange behavior is happening right by these points. Hence, we could write things like:

$$\lim_{x\to(-2)} g(x) = 0 \qquad \lim_{x\to(-1)} g(x) = 3 \qquad \lim_{x\to 3} g(x) = 0.$$

- Consider $x = 0$. Notice that $g(0)$ is defined and $g(0) = 1$. But, which y-value does $g(x)$ approach as x gets close to (but not equal to) 0? We can see in the graph that $g(x)$ is approaching 4, hence,

$$\lim_{x\to 0} g(x) = 4.$$

  Limits tell us about **where a function is going**. It doesn't matter that $g(0) = 1$; that has no bearing on the value of the limit, because the limit is trying to capture the behavior of g **around** $x = 0$ and not actually at $x = 0$.

- Consider $x = 2$. We can see on the graph that $g(2)$ is undefined, but everywhere around $x = 2$, the function $g(x)$ is defined and appears linear. Hence, we would say that

$$\lim_{x\to 2} g(x) = 1.$$

- Now consider $x = 1$. Notice that the function g has a jump in it's graph. For values $x < 1$, the function $g(x)$ is approaching the y-value of 3. But for values $x > 1$, $g(x)$ is approaching 2. Because $g(x)$ is trying to approach two different values from each side of $x = 1$, we say here that

$$\lim_{x\to 1} g(x) \text{ does not exist.}$$

You can also abbreviate "does not exist" with the letters DNE.

- Returning again to $x = 1$, we actually have notation to help us describe situations like this one.

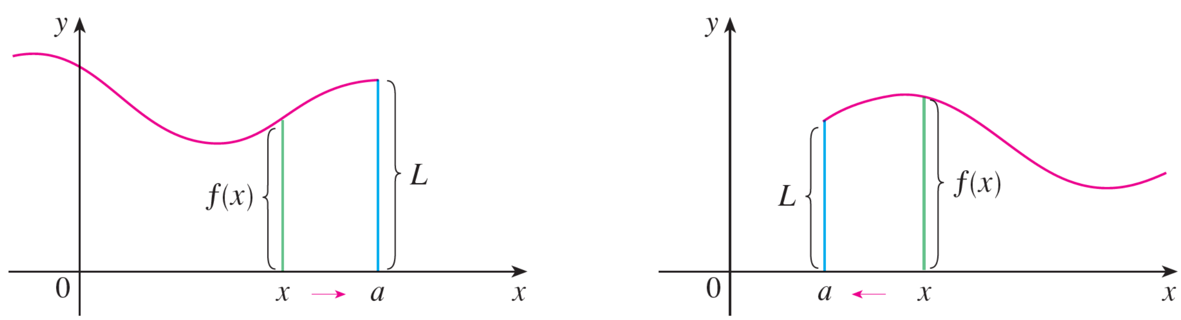
Definition 4.1.9: One Sided Limits

We write

$$\lim_{x\to a^-} f(x) = \mathrm{L}$$

and say the **left-hand limit** of $f(x)$ as x approaches a is equal to L if $f(x)$ approaches L as $x \to a$ with $x < a$ (i.e., **approaching from the left**)

Similarly, $\lim\limits_{x\to a^+} f(x)$ denotes the **right-hand limit** where the notation $x \to a^+$ means that we consider only $x < a$ (i.e., **approaching from the right**).



(a) $\lim\limits_{x\to a^-} f(x) = L$ (b) $\lim\limits_{x\to a^+} f(x) = L$

Figure 4.2: One-sided Limits

So for the function $g(x)$ above in Figure 4.1, we could write:

$$\lim_{x\to 1^-} g(x) = 3 \quad \text{and} \quad \lim_{x\to 1^+} g(x) = 2.$$

Question 14.

For the functions $f(x) = \frac{x^2-9}{x-3}$, $g(x) = \frac{|x-3|}{x-3}$, and $h(x) = \frac{4}{x-3}$, find the following limits.

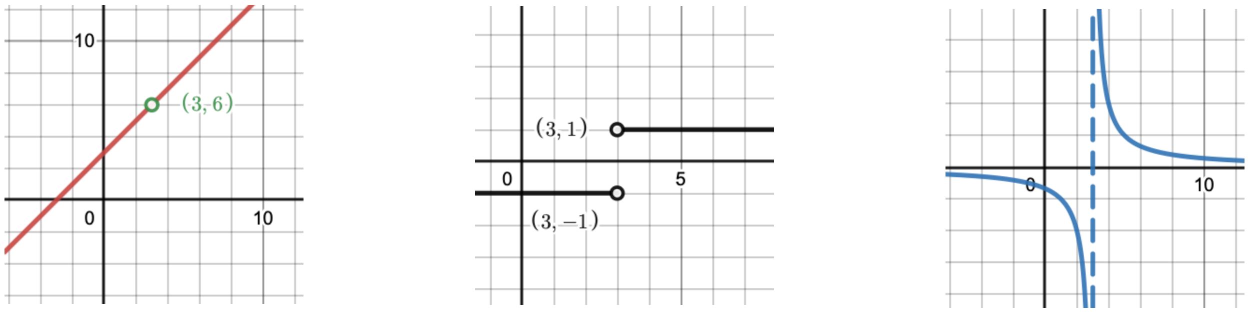
(a) $\displaystyle\lim_{x\to 3} \frac{x^2-9}{x-3} =$ ____________________

(b) $\displaystyle\lim_{x\to 3^-} \frac{|x-3|}{x-3} =$ ____________________

(c) $\displaystyle\lim_{x\to 3^+} \frac{|x-3|}{x-3} =$ ____________________

(d) $\displaystyle\lim_{x\to 3^-} \frac{4}{x-3} =$ ____________________

(e) $\displaystyle\lim_{x\to 3^+} \frac{4}{x-3} =$ ______________________________



(a) $f(x) = \frac{x^2-9}{x-3}$ (b) $g(x) = \frac{|x-3|}{x-3}$ (c) $h(x) = \frac{4}{x-3}$

Figure 4.3: Graphs for $f(x)$, $g(x)$, $h(x)$

Definition 4.1.10: Horizontal Asymptotes and Limits

We say $\lim\limits_{x\to\infty} f(x) = \mathrm{L}$ if the value of $f(x)$ approches L as x keeps increasing. In that case, we say the line $y = \mathrm{L}$ is an **horizontal asymptote** to the graph of the function $f(x)$. We can similarly define $\lim\limits_{x\to-\infty} f(x) = \mathrm{L}$.

Example 4.1.11

Recall from the last week, we defined

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

■ Question 15. □

Use DESMOS to guess the value of $\displaystyle\lim_{x\to\infty}\frac{x+3}{4-x}$. *Could you have guessed this value without drawing the graph?*

Definition 4.1.12: Vertical Asymptotes and Limits

If one or both of the one-sided limits of a function $f(x)$ at a point a approach infinity or negative infinity, we say that the graph of $f(x)$ has a **vertical asymptote** at $x = a$.

Warning: The symbol ∞ is not a number. It is used to denote the adjective "not finite". However, there is a distinction between $+\infty$ and $-\infty$. Writing $\lim\limits_{x\to a} f(x) = \infty$ means $f(x)$ **increases** without bound as x approaches a from **both sides.** Similarly, $\lim\limits_{x\to a} f(x) = -\infty$ means $f(x)$ **decreases** without bound as x approaches a.

If $\lim\limits_{x\to a^+} f(x) = \infty$ and $\lim\limits_{x\to a^-} f(x) = -\infty$, then we will say $\lim\limits_{x\to a} f(x)$ does not exist and avoid using the notation of infinity.

■ Question 16. □

For the following graph, find

$\lim\limits_{x\to-3^-} f(x) =$

$\lim\limits_{x\to-3^+} f(x) =$

$\lim\limits_{x\to-3} f(x) =$

$\lim\limits_{x\to-1^-} f(x) =$

$\lim\limits_{x\to-1^+} f(x) =$

$\lim\limits_{x\to-1} f(x) =$

$\lim\limits_{x\to2^-} f(x) =$

$\lim\limits_{x\to2^+} f(x) =$

$\lim\limits_{x\to2} f(x) =$

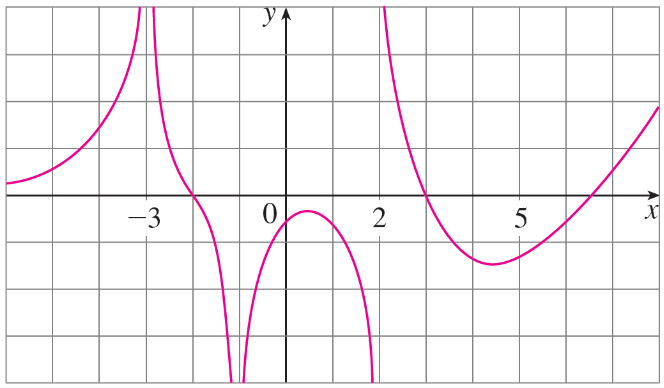
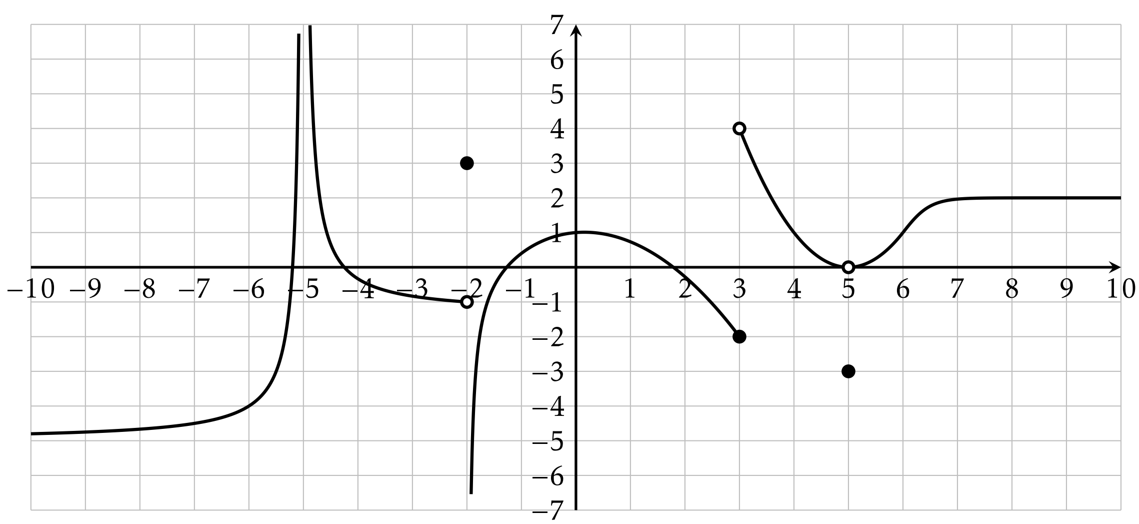


Figure 4.4: Infinite Limits

■ Question 17.

*The graph of a function $f(x)$ is given below. Try to answer the given questions using the "**answer bank**" provided. Each **answer** goes with exactly one statement below.*

Graph of $f(x)$



(a) $\lim\limits_{x\to-\infty} f(x) =$

(b) $\lim\limits_{x\to\infty} f(x) =$

(c) $\lim\limits_{x\to-5^-} f(x) =$

(d) $\lim\limits_{x\to-5^+} f(x) =$

(e) $\lim\limits_{x\to-5} f(x) =$

(f) $f(-5) =$

(g) $\lim\limits_{x\to-2^-} f(x) =$

(h) $\lim\limits_{x\to-2^+} f(x) =$

(i) $\lim\limits_{x\to-2} f(x) =$

(j) $f(-2) =$

(k) $\lim\limits_{x\to0^-} f(x) =$

(l) $\lim\limits_{x\to0^+} f(x) =$

(m) $\lim\limits_{x\to0} f(x) =$

(n) $f(0) =$

(o) $\lim\limits_{x\to3^-} f(x) =$

(p) $\lim_{x\to 3^+} f(x) =$

(q) $\lim_{x\to 3} f(x) =$

(r) $f(3) =$

(s) $\lim_{x\to 5^-} f(x) =$

(t) $\lim_{x\to 5^+} f(x) =$

(u) $\lim_{x\to 5} f(x) =$

(v) $f(5) =$

Answer Bank

(i) 1	(vi) undefined	(xi) ∞	(xvi) 0	(xxi) 1
(ii) -2	(vii) ∞	(xii) -1	(xvii) -3	(xxii) 1
(iii) 4	(viii) -5	(xiii) DNE	(xviii) 0	
(iv) DNE	(ix) 2	(xiv) $-\infty$	(xix) 3	
(v) -2	(x) ∞	(xv) 0	(xx) 1	

■ Question 18. □

*(**Sketch a Graph I**) Sketch a graph for a function $f(x)$ that satisfies the following:*

- $f(-2) = 2$ *and* $\lim_{x\to(-2)} f(x) = 1$
- $f(-1) = 3$ *and* $\lim_{x\to(-1)} f(x) = 3$
- $f(1)$ *is not defined and* $\lim_{x\to 1} f(x) = 0$
- $f(2) = 1$ *and* $\lim_{x\to 2} f(x)$ *does not exist.*

■ Question 19. □

*(**Sketch a Graph II**) Sketch a graph for a function $f(x)$ that satisfies the following:*

- $\lim_{x\to(-2)^-} f(x) = +\infty$
- $\lim_{x\to(-2)^+} f(x) = -\infty$
- $f(0) = 2$ *and* $\lim_{x\to 0} f(x) = 1$
- $\lim_{x\to 3} f(x)$ *does not exist but* $\lim_{x\to 3^+} f(x) = -4$

Chapter 5 | Continuity

Let's start by formally restating the definition of continuity we have already seen.

Definition 5.0.13: Continuity at a Point

The function $f(x)$ is said to be **continuous** at a point $x = a$ if

- f is defined at $x = a$, and
- if $\lim\limits_{x \to a} f(x)$ exists, and
- if $\lim\limits_{x \to a} f(x) = f(a)$

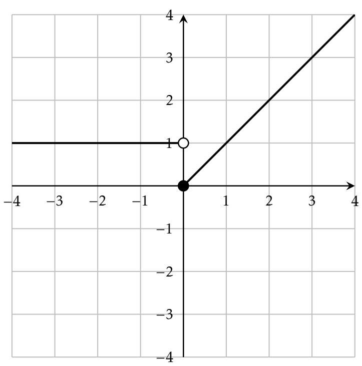
In other words, $f(x)$ can be made to remain as close as we want to $f(c)$ provided x is chosen close enough to c.

Warning: An important difference between limits and continuity: a limit is only concerned with what happens **near** a point, but continuity depends on what happens **near** a point and **at** that point.

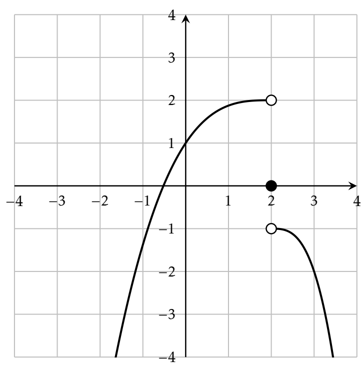
Question 20.

In the following pictures, if you believe the function is discontinuous at a point, discuss ***why*** *you think it's discontinuous at that point. Which of the three parts in the definition does it fail to satisfy (if discontinuous)?*

(a)



(b)



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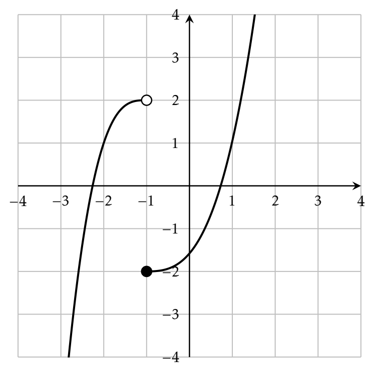

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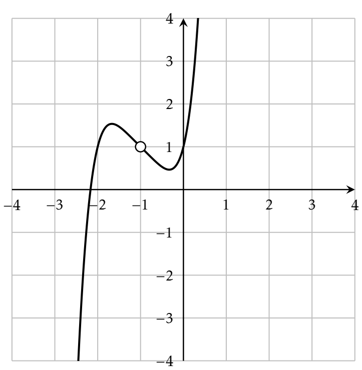

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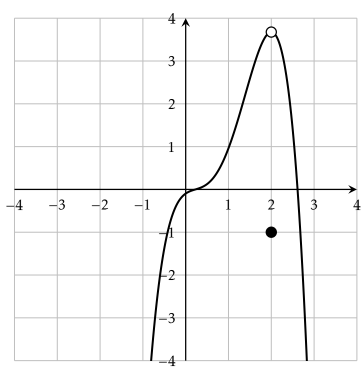

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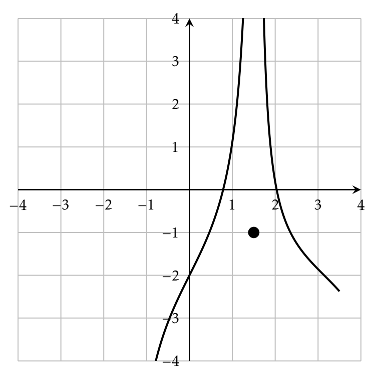
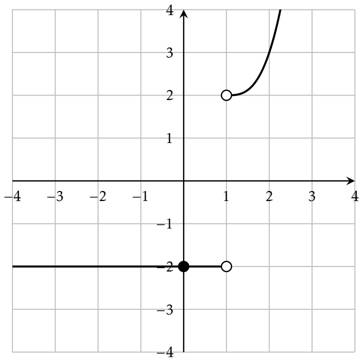
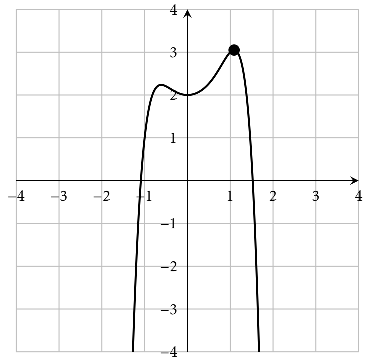


(g) 

(h) 

Definition 5.0.14: Continuity on an Interval

A function f is said to be continuous on an open interval (a,b) if it is continuous at every point in the interval.

Most of the functions you have studied prior to Calculus are continuous functions. Or at least, functions that are continuous on their domains. Here is a summary:

- Polynomials are continuous for all real numbers
- Even fractional powers like $y=\sqrt{x}$ or $y=x^{1/4}$ are continuous on their domains $[0,\infty)$
- Rational functions are continuous on their domain, e.g. $y=\frac{x^2+1}{x-3}$ is continuous on the intervals $(-\infty,3)$ and $(3,\infty)$.
- Exponential and logarithm functions are continuous on their domains.
- Adding, subtracting, and multiplying continuous functions will yield continuous functions.
- Dividing continuous functions will give you a continuous function, where defined (i.e. where you aren't dividing by zero).
- Composing continuous functions together will give you a continuous function.

§5.1 Using Continuity to evaluate a Limit

The last condition of continuity, that $\lim\limits_{x\to a} f(x)=f(a)$, tells us how to compute limits for continuous functions.

Example 5.1.15

(a) The function $p(x)=2x^3-6x^2+5$ is a polynomial, which is continuous for any real number. Hence, if we wanted to know, say, $\lim\limits_{x\to 2} p(x)$, we simply have to **evaluate** $p(x)$ **at** $x=2$:

$$\lim_{x\to 2}\left(2x^3-6x^2+5\right)=2(2)^3-6(2)^2+5=16-24+5=-3.$$

(b) Even a somewhat scary-looking limit is easy to evaluate if the function is continuous at the point in question.

$$\lim_{x\to 25}\left(xe^{\sqrt{x}+1}-(x-20)^2\right)=(25)e^6-(5)^2=25(e^6-1)\approx 10060.7198$$

(c) We can use what we know about continuity to help us determine if a piecewise function is continuous without even needing to graph it (although it's still fun to graph just about anything). For

$$f(x)=\begin{cases}5x+2, & \text{if } 0\leq x<2\\ 6, & \text{if } x=2\\ x^2+x+6, & \text{if } x>2\end{cases}$$

we can evaluate left and right-hand limits at $x=2$:

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-} 5x+2=12$$

and

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+} x^2+x+6=12.$$

From what we have learned previously, we know that $\lim_{x\to 2} f(x)=12$ because the left and right hand limits are the same. However, $f(2)=6\neq 12$, so $f(x)$ is **not continuous** at $x=2$.

In what way could we adjust the function f so that it is continuous at $x=2$? ________________

■ Question 21.

Determine a value of c that will make the function g continuous for all real numbers.

$$g(x)=\begin{cases}cx & \textit{if } 0\leq x<2\\ 3x^2 & \textit{if } 2\leq x\end{cases}$$

□

■ Question 22.

Determine a value of k that will make the function f continuous on the interval $[0,8]$.

$$g(x)=\begin{cases}e^{kx} & \textit{if } 0\leq x<4\\ x+3 & \textit{if } 4\leq x\leq 8\end{cases}$$

□

Part III

The Meaning of Derivative

Chapter 6 | Introduction to Derivatives

§6.1 The Velocity Problem

Suppose you drop a ball from the top of a tower: how long does it take it to reach the ground? How fast is the ball moving at any given point in time?

Questions like the above are known as "the velocity problem", and are central to the study of differential calculus. Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. (This model for free fall neglects air resistance.) If the height of the tower is H (in meters), and the current height of the ball after t seconds is denoted by $s(t)$ (measured in meters), then Galileo's observation is expressed by the equation

$$s(t) = \mathrm{H} - 4.9t^2$$

The graph of this quadratic function is clearly a parabola. You can see a general figure below in 6.1.

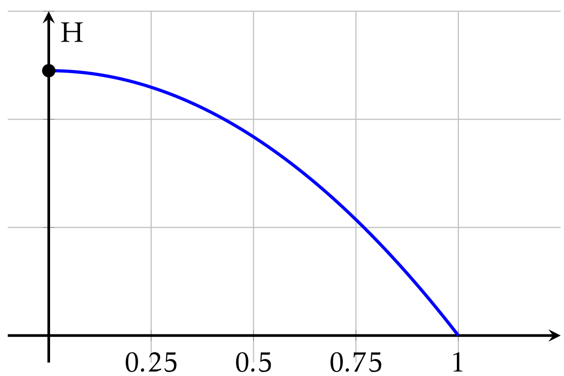


Figure 6.1: Position model of a ball falling from a height H

Example 6.1.16

In the above scenario, suppose we wish to find the velocity of the ball after 0.5 seconds. You may have seen before that velocity as displacement over time. However, the difficulty in finding the velocity at 0.5 seconds is that we are dealing with a single instant of time $t = 0.5$, so no time interval is involved.

However, we can approximate the desired quantity by computing the average velocity over the brief time interval of a hundredth of a second from $t = 0.5$ to $t = 0.51$.

$$\begin{aligned}
\text{Average Velocity} = v_{avg} &= \frac{\text{change in position}}{\text{time elapsed}} \\
&= \frac{s(0.51) - s(0.5)}{0.01} \\
&= \frac{[\mathrm{H} - 4.9(0.51)^2] - [\mathrm{H} - 4.9(0.5)^2]}{0.01} \\
&\approx -4.949
\end{aligned}$$

The negative denotes the fact that the ball is moving downward.

Now, if we were to reduce the time interval even further, we would probably get a better approximation. Go to

https://www.desmos.com/calculator/wbxi7lj3av

and try it yourself.

In general,

Definition 6.1.17

For an object moving in a straight line with position function $s(t)$, the **average velocity** of the object on the interval from $t = a$ to $t = b$, denoted by v_{avg}, is given by

$$v_{avg} = \frac{s(b) - s(a)}{b - a} \quad \text{on interval} \quad [a, b]$$

Question 23.

What is v_{avg} on the time interval $[0.5, 0.501]$? What about $[0.5, 0.50001]$? Can you guess what value v_{avg} is approaching?

Fortunately, we already have developed some terminology to formally describe exact situations like this.

Definition 6.1.18

For an object moving in a straight line with position function $s(t)$, the **instantaneous velocity** of the object at $t = a$, denoted by $v_{instant}$, is given by

$$v_{instant} = \lim_{b \to a} \frac{s(b) - s(a)}{b - a}$$

Example 6.1.19

Going back to the situation with the ball, at $t = 0.5$,

$$v_{instant} = \lim_{b \to 0.5} \frac{[\mathrm{H} - 4.9b^2] - [\mathrm{H} - 4.9(0.5)^2]}{b - 0.5} = -4.9 \lim_{b \to 0.5} \frac{b^2 - (0.5)^2}{b - 0.5} = -4.9 \lim_{b \to 0.5} (b + 0.5) = -4.9$$

Note: It may not be always easy to calculate the limit for a different function $s(t)$. But that's a question for another chapter, more specifically, when we study differentiation formulas.

§6.2 The Tangent Problem

Another way to interpret the average velocity is to observe that the difference quotient in the example above is also the slope of the secant joining the point $\mathrm{P} = (a, s(a))$ and the point $\mathrm{Q} = (b, s(b))$. Then the idea of shortening the interval gradually is the same as moving Q closer to P. However, we observe from the picture below that as Q moves closer to P, the secant lines are approaching towards the tangent line at P.

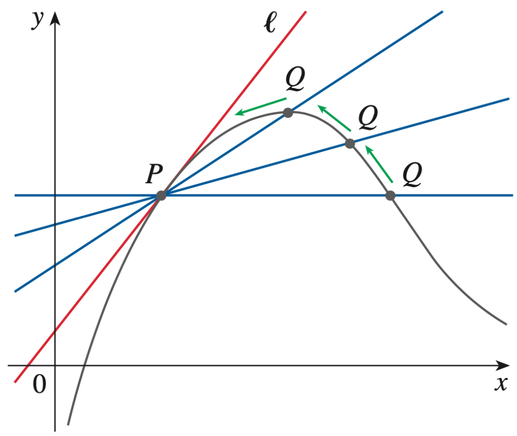


Figure 6.2

In other words, the instantaneous velocity must correspond to the slope of the tangent!

Replacing $s(t)$ with a general function $f(x)$, we arrive at the following definition:

Definition 6.2.20

The tangent line to the curve $y = f(x)$ at the point P$(a, f(a))$ is the line through P with slope

$$m = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

provided that this limit exists.

Note: Recall that the equation of a straight line through a point (x_0, y_0) with slope m is given by

$$y - y_0 = m(x - x_0).$$

Question 24.

Use DESMOS to approximate the slope of the tangent to the graph of $f(x) = \sqrt{x}$ at $x = 4$. Use this to find the equation of the tangent line.

§6.3 Derivative and Rate of Change

So far, we have the same type of limit in both the velocity problem and the tangent problem. It turns out that a similar limit arises whenever we calculate a *rate of change* in any of the sciences or engineering, such as a rate of reaction in chemistry or a marginal cost in economics.

Suppose y is a quantity that depends on another quantity x, i.e. $y = f(x)$ is a function of x. If x changes from a to b, then the change in x is $\Delta x = b - a$ and the corresponding change in y is $\Delta y = f(b) - f(a)$. The difference quotient we have seen multiple times so far, $\frac{\Delta y}{\Delta x}$ is called the **average rate of change** of y with respect to x over the interval $[a, b]$ and the limit of the quotient (as $b \to a$) is defined as the **instantaneous rate of change** of y with respect to x at $x = a$.

Since this type of limit occurs so widely, it is given a special name and notation.

Definition 6.3.21

The derivative of a function $f(x)$ at a point $x = a$, denoted $f'(a)$, is

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

if this limit exists.

In other words, the derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

Question 25.

A manufacturer produces bolts of a fabric with a fixed width. The cost of producing x yards of this fabric is $C = f(x)$ *dollars.*

(a) *What is the meaning of the derivative $f'(x)$? What are its units?*

(b) *In practical terms, what does it mean to say that $f'(1000) = 9$?*

Question 26.

Recall when we looked at the California data in Desmos, we were able to make a function model. This function model was found to be:

$$f(t) = 3206500e^{0.02428t}$$

We can use this model to estimate average rate of change a bit better than the table, since we can approximate values closer than just 10 *year intervals.*

Use this Desmos page to more easily compute rates of change for our population model.

https://www.desmos.com/calculator/bjd0ci5g2h

(a) *Use the above Demsos page to help you compute the average rate of change for $f(t)$ on the interval* $[100, 101]$.

(b) *Use the above Desmos page to estimate the instantaneous rate of change at* $a = 100$.

Chapter 7 | The Derivative Function

In the last chapter, we defined the derivative of a function $f(x)$ at $x = a$. Now if we let a vary over all possible real numbers, we could ask the same question at every possible point: what is the derivative of f at $x = a$? Changing our point of view and replacing a by a variable x, we obtain

$$f'(x) = \lim_{b \to x} \frac{f(b) - f(x)}{b - x}$$

which we can think of as the **derivative function**.

Example 7.0.22

Open the DESMOS link below to find the graph of a function $f(x)$ is given below. Can we use it to give a rough sketch of the graph of the derivative f'?

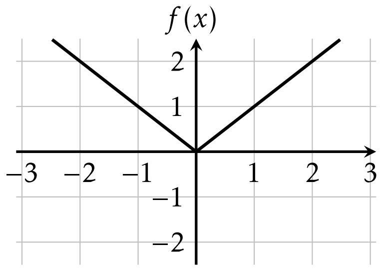
https://www.desmos.com/calculator/mwatkatvza

■ Question 27.

□

On the next two sheets are pictures of graphs. On the first sheet, there are 15 functions numbered $1-15$ and their graphs. On the second sheet, there are also 15 graphs of functions labeled A through O. Each graph labeled with a letter is the derivative graph for a graph labeled with a number. Your task is to match the appropriate graph $f(x)$ with its derivative $f'(x)$.

- *Remember, the derivative $f'(x)$ gives the slope of the tangent line at x on the graph of $f(x)$. Use this geometry to help you match the graphs!*
- *Keep track of ideas, thoughts, or questions you have while working on the matching in your group. What strategies did you employ to identify the correct match?*
- *Write down any notes or questions as you work on the matching activity*



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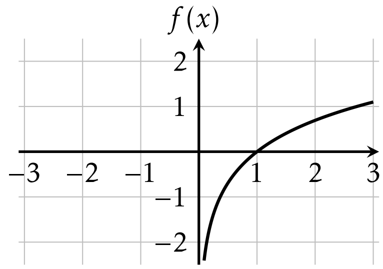

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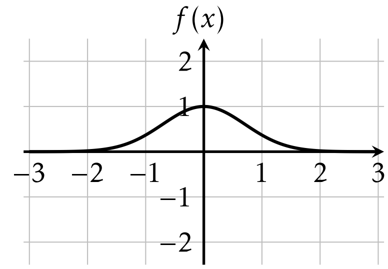

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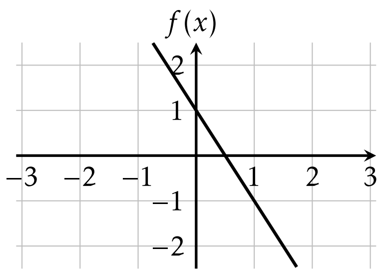

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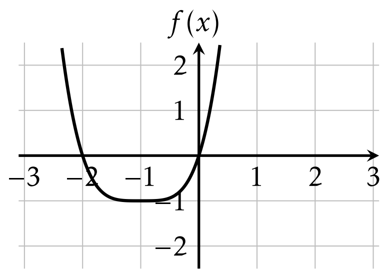

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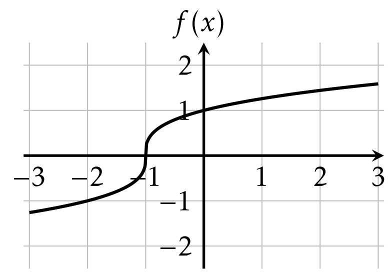

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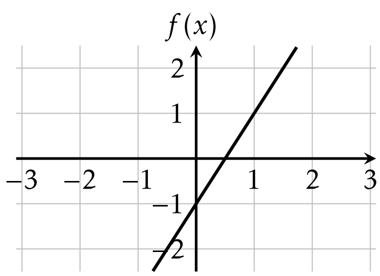

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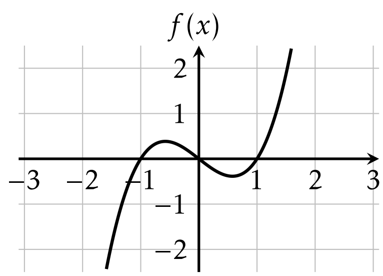

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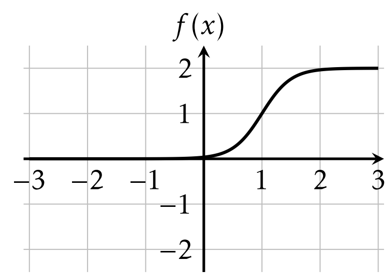

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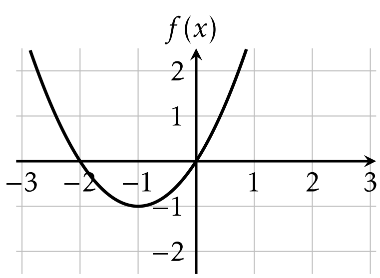

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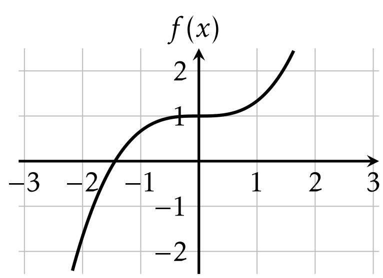

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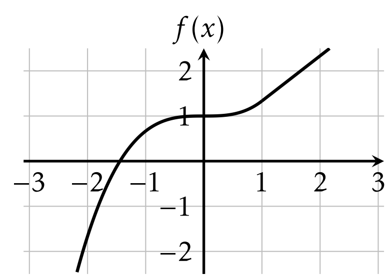

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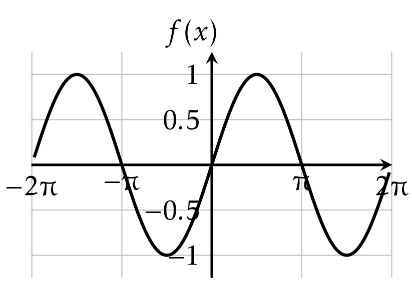

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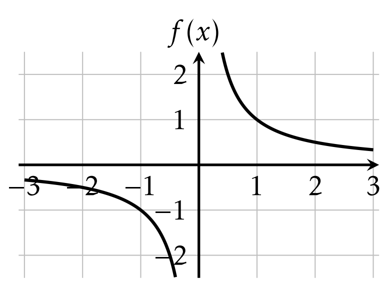

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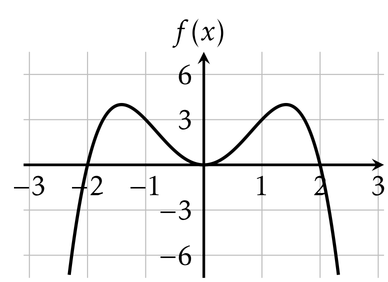

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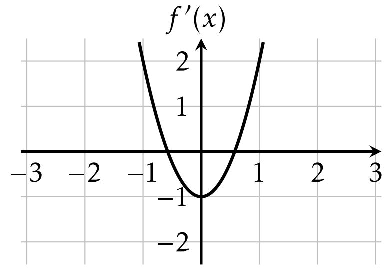

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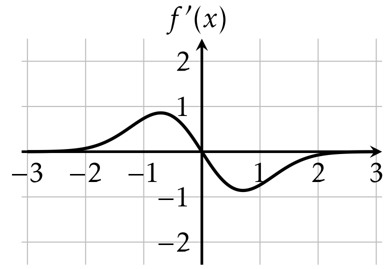

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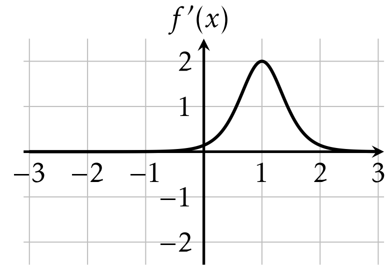

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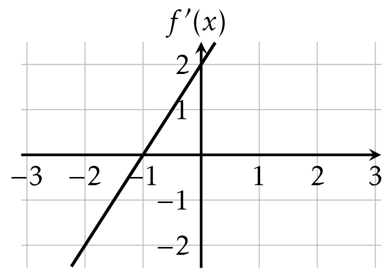

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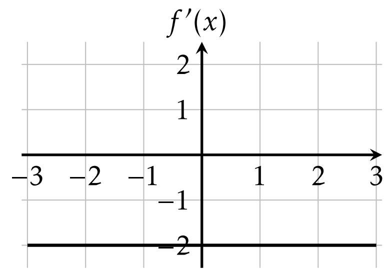

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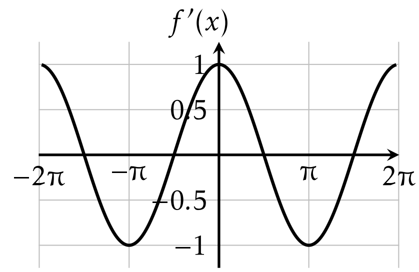

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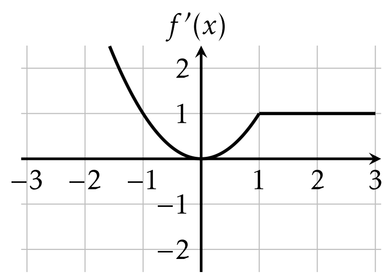

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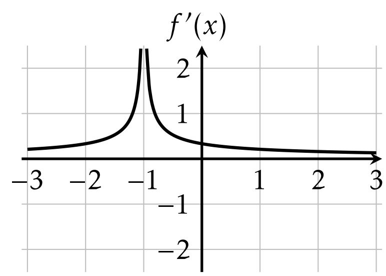

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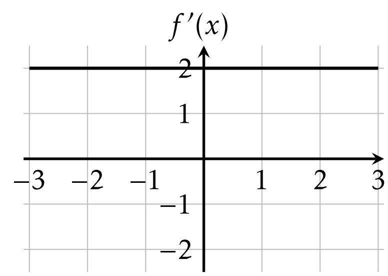

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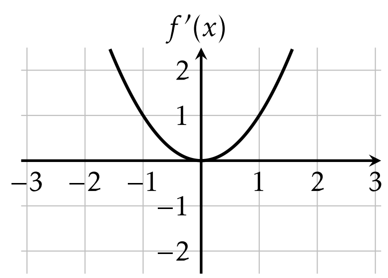

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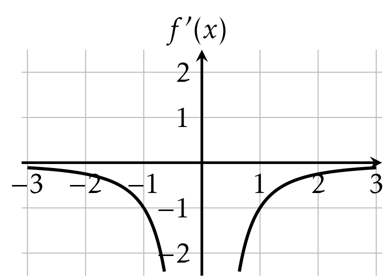

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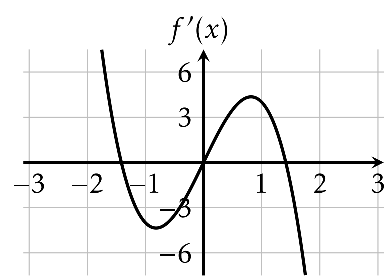

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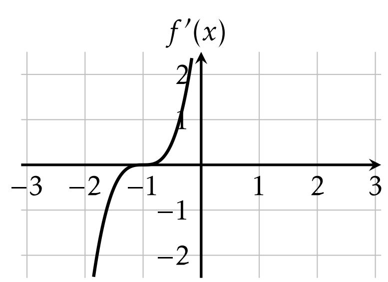

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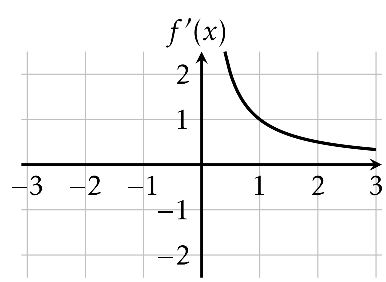

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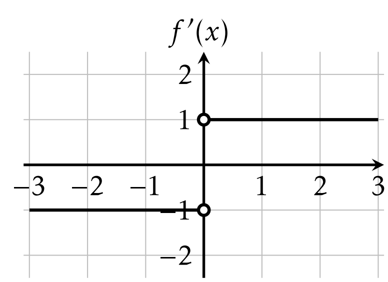


O

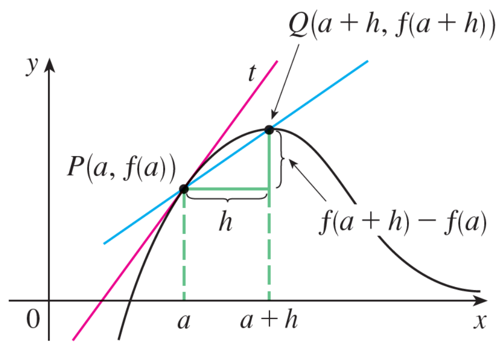
Chapter 8 | Differentiability

In the last chapter we defined the derivative of a function $f(x)$ as

$$f'(x) = \lim_{b \to x} \frac{f(b) - f(x)}{b - x}$$

Now if we write $b = x + h$, then we have $b - x = h$ and b approaches x if and only if h approaches 0. Therefore an equivalent way of stating the definition of the derivative is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

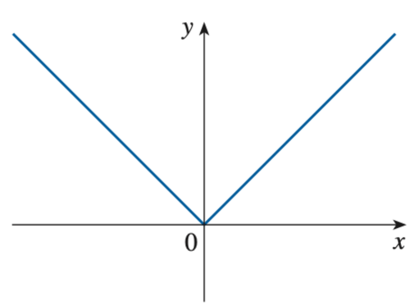


> **Note:** If P is the point $(a, f(a))$ on the graph of $f(x)$, sometime we will use the derivative $f'(\mathrm{P})$ interchangeably with $f'(a)$ to mean the **instantaneous rate of change** of $f(x)$ at the point P, and this is the same as the slope of the tangent line at that point.

The process of finding the derivative of a function is called **differentiation**. A function $f(x)$ is said to be **differentiable** at a if its derivative exists at $x = a$. By now, we have seen lots of examples when a limit doesn't exist or is undefined. If the limit in the definition of the derivative doesn't exist at $x = a$, we say that the derivative $f'(a)$ doesn't exist. So in that case, we say that the function $f(x)$ is not differentiable at $x = a$. A function is said to be differentiable on an open interval (a, b) if its derivative exists at every x in the interval.

Example 8.0.23

Consider the function $f(x) = |x|$ graphed below.



We already matched this function with a graph of its derivative in the last activity. Nonetheless, let's go over it once more and try to evaluate the derivative at $x = 0$ using the limit definition from above.

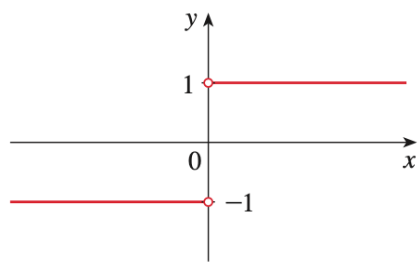
The function is defined and continuous at $x = 0$. In fact $f(0) = |0| = 0$. Now what can we say about the quantity

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{|0+h|-|0|}{h} = \underline{\hspace{10cm}}$$

Similarly,

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{|0+h|-|0|}{h} = \underline{\hspace{10cm}}$$

Another way to confirm these two values is to calculate the slopes of tangents to the graph from the left and right as x approaches 0.



Either way, we find that the left hand limit and the right hand limit does not match! Hence the overall limit doesn't exist. We conclude that $f(x) = |x|$ is NOT differentiable at $x = 0$.

In general, if the graph of a function f has a "corner" or "cusp" in it, then the graph of f has no tangent at this point and f is not differentiable there. In trying to compute $f'(x)$, we find that the left and right limits are different. This is the case in the first picture below. Here are two other scenarios.

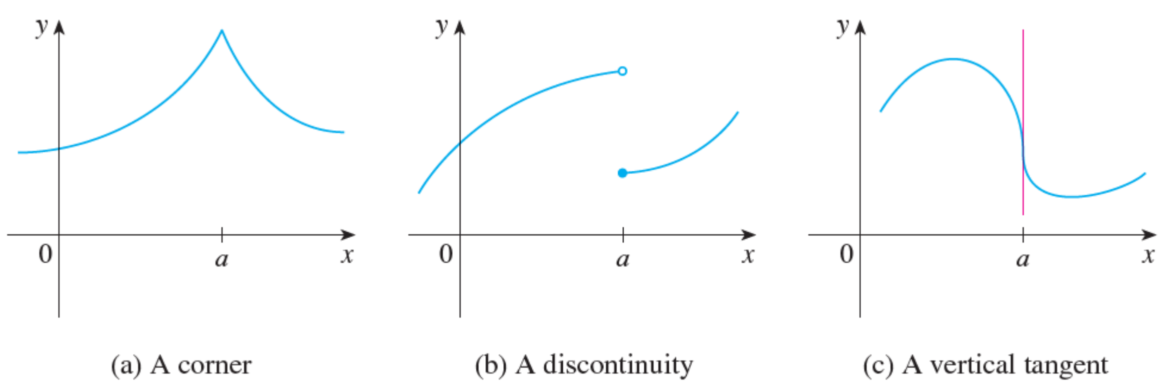


Figure 8.1: Three ways for f not to be differentiable at a

Note: From example 23 and fig. 8.1, we also observe the following relation between continuity and differentiability. A continuous function may not be always differentiable. A discontinuous function is definitely not differentiable. However, a differentiable function has to be continuous. We will not prove these results in this course.

Question 28.

The function $y = p(x)$ is pictured in Figure 8.2 below. Each portion of the graph is a straight line.

(a) *For what values of a, does $\lim_{x\to a} p(x)$ not exist?*

(b) *For which x-values is $f(x)$* **not** *continuous?*

(c) *For which x-values is $f(x)$* **not** *differentiable?*

(d) *Sketch an accurate graph of $p'(x)$.*

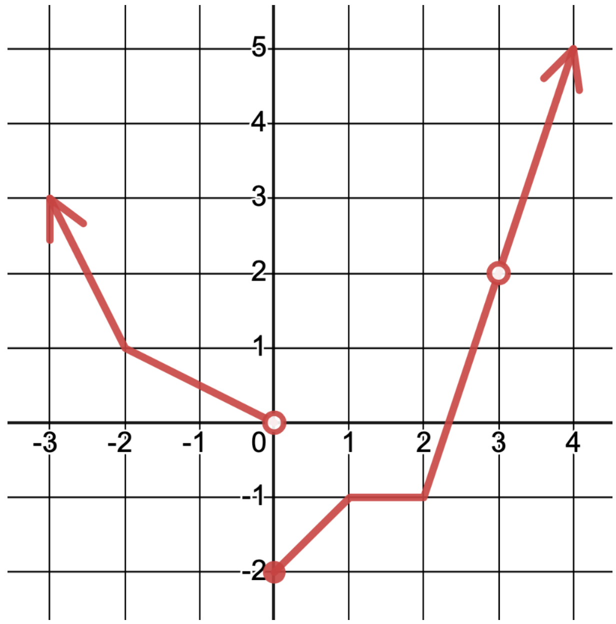
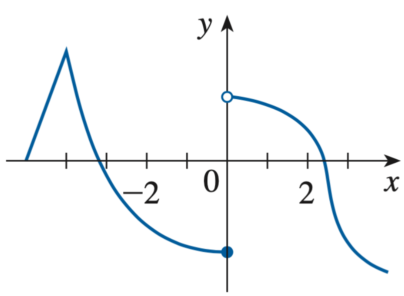


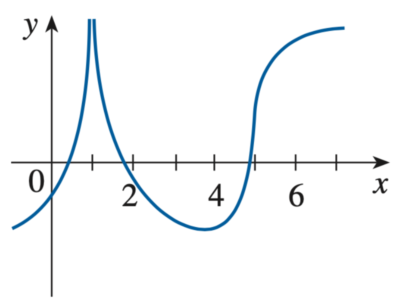
Figure 8.2: Piecewise linear function, $y = p(x)$

Question 29.

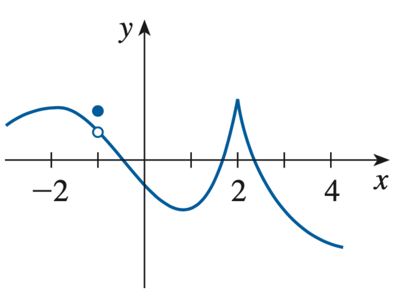
For each of the functions $f(x)$ given below, determine, with reasons, the numbers at which $f(x)$ is not differentiable.



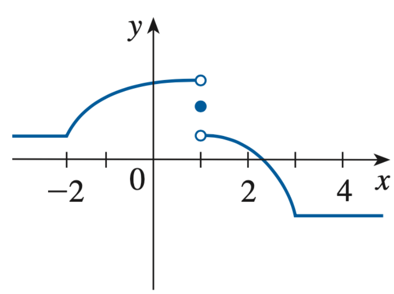
(a)



(c)



(b)



(d)

Part IV

Differentiaton Formulas

Chapter 9 | Essential Derivative Formulas

§9.1 Some Key Notation

Before we introduce our derivative formulas, here is a quick word on derivative notation. We have been using f' to denote the derivative of f, but there are other ways to do this that you will see in your text or elsewhere online.

If we write $y = f(x)$, then we can write the expressions

$$\frac{\mathrm{d}y}{\mathrm{d}x} \quad \text{or} \quad \frac{\mathrm{d}f}{\mathrm{d}x}$$

instead of f'. We read these expressions as "dee-y dee-x" or "dee-f dee-x." This notation is referred to as **Leibnitz notation**.

The notation $\frac{\mathrm{d}y}{\mathrm{d}x}$ comes from the notation $\frac{\Delta y}{\Delta x}$ that is often used to denote the slope of a line. Although we read $\frac{\Delta y}{\Delta x}$ as "change in y over change in x," we view $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a single symbol, not as a quotient of two quantities.

Since we can use $\frac{\mathrm{d}y}{\mathrm{d}x}$ in place of f', to write out an expression like $f'(2)$ with Leibnitz notation, we would write something like $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=2}$, which says "evaluate $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = 2$."

Lastly, we use a variation of the Leibnitz notation as a command for taking the derivative. That is, the following

$$\frac{\mathrm{d}}{\mathrm{d}x}[\square]$$

says to "take the derivative of $\square$ with respect to x." For example, $\frac{\mathrm{d}}{\mathrm{d}x}[7x+1] = 7$.

§9.2 Constant, Multiple and Sum of Functions

Let's start with the simplest of all functions, the constant function $f(x) = c$. Its graph is a horizontal line with slope zero at every point. Thus, its derivative should be zero everywhere. We summarize this with the following rule.

Theorem 9.2.24

For any real number c, if $f(x) = c$, then $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)] = 0$.

It is also easy to check this fact from the limit definition, but we will leave that as a homework. Next, we have two rules regarding sum and constant multiples.

Theorem 9.2.25

- *If c is a constant and $f(x)$ is differentiable, then $\frac{\mathrm{d}}{\mathrm{d}x}[cf(x)] = cf'(x)$.*

- *If $f(x)$ and $g(x)$ are differentiable, then* $\dfrac{\mathrm{d}}{\mathrm{d}x}[f(x) \pm g(x)] = f'(x) \pm g'(x)$.

A visual justification for the first fact can be observed from the picture below.

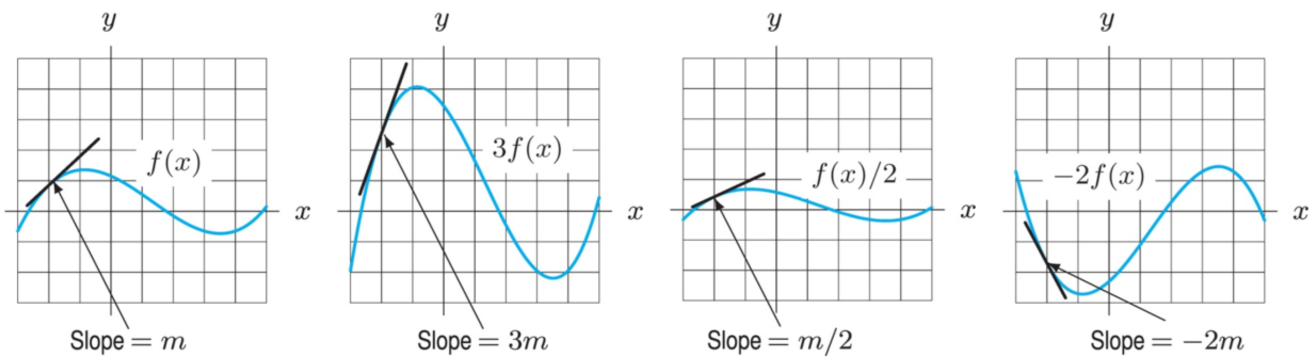


Figure 9.1: Derivative of multiple is multiple of derivative

These facts can be also checked using the limit definition.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}[f(x)+g(x)] &= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= f'(x)+g'(x)\end{aligned}$$

Question 30.

Using Table 9.1, compute the following:

(a) *Find $h'(1)$ if $h(x) = 5 - f(x)$.*

(b) *Find $k'(-2)$ if $k(x) = -\frac{1}{2}g(x)$.*

(c) *Find $p'(-2)$ if $p(x) = 2f(x) + 3g(x)$.*

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-2	-6	9	-10	16
1	5	-3	3	-2

Table 9.1

§9.3 Powers and Polynomials

Using the limit defintion and the following factorization*

$$y^n - x^n = (y-x)(y^{n-1} + y^{n-2}x + \cdots + yx^{n-2} + x^{n-1}) \qquad \text{for any positive integer} n$$

we can prove the following theorem

*We are including the process for the sake of completeness and in case any student wants to justify the algebra on their own. A complete proof is outside the scope of the syllabus.

Theorem 9.3.26: Power Rule

If n is any real number then

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^n] = nx^{n-1}$$

Once we know the derivative of any power of x, we can use the constant multiple and the sum/difference rules to find derivatives of any polynomials.

Question 31.

Let $f(x) = x^5 - 2x^4 + 3x + 12$. Use the power rule and the basic derivative formulas to compute $f'(x)$.

Question 32.

Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

Question 33.

Homework

The graph of $y = x^3 - 9x^2 - 16x + 1$ has a slope of 5 at two points. Find the coordinates of the points.

§9.4 Derivative of Exponential Functions

Lastly, we record here a formula for the derivatives of exponential functions, $y = b^x$.

Theorem 9.4.27

For any $b > 0$, $\frac{\mathrm{d}}{\mathrm{d}x}[b^x] = \ln(b) \cdot b^x$

So for example, if $f(x) = 7^x$, then $f'(x) = \ln(7)7^x$. Note that for the function $y = e^x$, since $\ln(e) = 1$ we have the very special derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}[e^x] = e^x.$$

Question 34.

Use the constant, power, and exponential function formulas to compute some derivatives quickly below. State your answers using full and proper notation, to get into the habit of doing so. For instance, if you are given a function $f(x)$, then you should write "$f'(x) = \ldots$" or "$\frac{\mathrm{d}f}{\mathrm{d}x} = \ldots$" as part of your answer. Don't just put an equal sign.

(a) $f(x) = \pi$

(b) $p(x) = \sqrt{3}$

(c) $m(t) = t^{11}$

(d) $h(x) = 10^x$

(e) $y(z) = z^{-4}$

(f) $f(t) = \frac{1}{t^4}$

(g) $h(w) = w^{3/4}$

(h) $r(t) = (\sqrt{3})^t$

(i) $F(x) = 2^{-x}$

■ Question 35.

☐

Using the Sum and Constant Multiple rules and the rules for Power, Constant, and Exponential functions, compute the derivative for each function with respect to the given independent variable. Make sure to write your answer using proper derivative notation.

(a) $f(x) = x^{5/3} - x^4 + 2^x$

(b) $g(x) = 14e^x + 3x^5 - x$

(c) $h(z) = \sqrt{z} + \dfrac{1}{z^4} + 5^z$

(d) $r(t) = \sqrt{53} \cdot t^7 - \pi e^t + e^4$

(e) $s(y) = (y^2 + 1)(y^2 - 1)$ *(distribute first)*

(f) $p(a) = 3a^4 - 2a^3 + 7a^2 - a + 12$

(g) $q(x) = \dfrac{x^3 - x + 2}{x}$ *(break it into three fractions first)*

Chapter 10 | Product and Quotient Rule

So far, we can differentiate power functions (x^n) and exponential functions (b^x). With the sum rule and constant multiple rules, we can also compute the derivative of their sums as follows:

Example 10.0.28

Differentiate

$$f(x) = 7x^{11} - 4 \cdot 9^x$$

Answer: Because f is a sum of basic functions, we can now quickly say that $f'(x) =$ ________________.

While the derivative of a sum is the sum of the derivatives, it turns out that the rules for computing derivatives of products and quotients are more complicated.

§10.1 Product Rule

Question 36. ☐

Let $f(x) = x^5$ and $g(x) = x^2 - 1$.

(a) $f'(x) =$ ______________________________ *and* $g'(x) =$ ______________________________.

(b) $f'(x)g'(x) =$ ____.

(c) *Form the product* $h(x) = f(x)g(x) =$ ____________________. *Then* $h'(x) =$ ____________________.

(d) *Is it the case that* $f'(x)g'(x) = h'(x)$ ____?

Here is the actual product rule formula:

Theorem 10.1.29: Product Rule

For $f(x)$ and $g(x)$ differentiable functions,

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Example 10.1.30

Consider $h(x) = 4^x\left(x^{3/2} - 2x^5\right)$. This is a product of the functions $f(x) = 4^x$ and $g(x) = x^{3/2} - 2x^5$. Hence, the derivative $h'(x)$ will have the derivative of the first function times the second function, plus the first function times the derivative of the second:

$f'(x) =$ ______________________________ $g'(x) =$ ______________________________

$h'(x) = f'(x)g(x) + f(x)g'(x) =$ ____

There is some algebraic simplification that we **could do** to this derivative, but it generally is not really worth it to bother simplifying the derivative that results from the product rule (or the quotient rule that we explore in a bit). The only times it can be advantageous to simplify a derivative would be if you need to set the derivative equal to something and solve for x, or if you want to make taking another derivative easier.

Usually, we will want to know the derivative at a point, and in that case we can just evaluate the derivative there using a calculator. For instance, if we wanted to know $h'(1)$, we could now calculate that:

$$\begin{aligned} h'(1) &= \ln(4)\cdot 4^1((1)^{3/2} - 2(1)^5) + 4^1\left(\frac{3}{2}(1)^{1/2} - 10(1)^4\right) \\ &= -4\ln(4) + 4\left(\frac{-17}{2}\right) \\ &= -4\ln(4) - 34 \approx -39.545 \end{aligned}$$

Question 37.

Find the derivative of the given function by recognizing it as a product and applying the Product Rule for derivatives. Then determine the specific derivative value using your derivative equation.

(a) $f(x) = x^4 e^x$

$f'(x) =$

$f'(1) =$

(b) $g(x) = \left(x^9 - (x^{1/3})\right)2^x$

$g'(x) =$

$g'(-1) =$

(c) $z = \dfrac{\pi^x}{x^\pi}$

$\dfrac{\mathrm{d}z}{\mathrm{d}x} =$

$\left.\dfrac{\mathrm{d}z}{\mathrm{d}x}\right|_{x=1} =$

■ Question 38.

☐

Use table 10.1 to compute the given derivative value.

(a) Find $h'(1)$ if $h(x) = f(x)g(x)$.

(b) Find $k'(-2)$ if $k(x) = xf(x) - 2g(x)$.

(c) Find $p'(-2)$ if $p(x) = \dfrac{f(x)g(x)}{2} + x^2 f(x)$.

(d) Find $q'(1)$ if $q(x) = f(x)^2$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-2	-6	9	-10	16
1	5	-3	3	-2

Table 10.1

§10.2 Quotient Rule

As with the product rule, the derivative formula for the derivative of a quotient of two differentiable functions is anything but what we would expect.

■ Question 39.

☐

Come up with an example of two functions $f(x)$ and $g(x)$ such that $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] \neq \dfrac{f'(x)}{g'(x)}$.

Here is the actual quotient rule formula:

Theorem 10.2.31: Quotient Rule

For $f(x)$ and $g(x)$ differentiable functions,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Example 10.2.32

Consider a simple rational function like $R(x) = \dfrac{2x+1}{3x^2+x+1}$. Since $R(x)$ is naturally defined as a quotient, we must use the quotient rule to determine its derivative. Sometimes, it can be helpful to identify the **top** and the **bottom** of the quotient, write out their derivatives separately, and then combine everything into the quotient rule formula.

Taking our own advice, let's identify our rational function as $R(x) = \dfrac{f(x)}{g(x)}$, so that $f(x) = 2x+1$ and $g(x) = 3x^2+x+1$. Then $f'(x) =$ ____________________ and $g'(x) =$ ____________________.

Substituting into the quotient rule formula, we have:

$R'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} =$ ____

For the purposes of computation, this is a **perfectly acceptable answer**! If we needed to take another derivative or determine something else specific related to $R(x)$'s rate of change, we might need to do some algebra to simplify $R'(x)$. However, for determining, say, the slope of the tangent line to $R(x)$ at the point $(1, R(1))$, we can just use the formula we have and a calculator to determine the slope of the tangent line:

$$R'(1) = \frac{(2)(5) - (3)(7)}{5^2} = \frac{-11}{25}$$

Question 40.

Find the derivative of the given function by recognizing it as a quotient and applying the Quotient Rule for derivatives. Then determine the specific derivative value using your derivative equation.

(a) $r(x) = \dfrac{3^x}{x^4+1}$

$r'(x) =$

$r'(2) =$

(b) $F(x) = \dfrac{x^2-2x-8}{x^2-9}$

$F'(x) =$

$F'(0) =$

(c) $y = \dfrac{t^4}{e^t}$

$\dfrac{dy}{dt} =$

$\left.\dfrac{dy}{dt}\right|_{t=1} =$

(d) $y = \dfrac{x \cdot 7^x}{x^2+1}$

$\dfrac{dy}{dx} =$

$\left.\dfrac{dy}{dx}\right|_{x=1} =$

Question 41.

Use table 10.2 to compute the given derivative value.

(a) *Find* $h'(1)$ *if* $h(x) = \dfrac{f(x)}{g(x)}$.

(b) *Find* $k'(-2)$ *if* $k(x) = \dfrac{xg(x)}{f(x)}$.

(c) *Find* $\mathrm{L}'(1)$ *if* $\mathrm{L}(x) = \dfrac{x^3+4}{f(x)+g(x)}$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-2	4	5	-1	2
1	3	-1	2	-2

Table 10.2

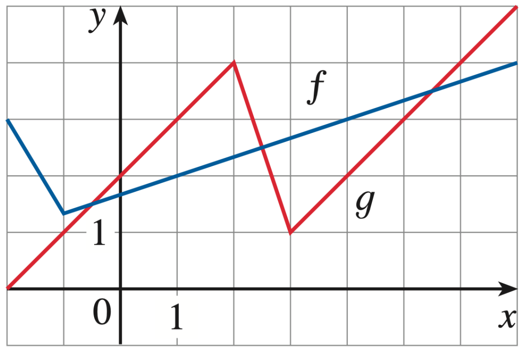


Figure 10.1

■ Question 42.

☐

*Use the graphs of functions $f(x)$ (in blue) and $g(x)$ (in red) in fig. 10.1 to determine the given derivative values (**if they exist!**)*

(a) Let $h(x) = 3f(x) - g(x)$.

(i) $h'(1)$

(ii) $h'(3)$

(iii) $h'(4)$

(b) Let $h(x) = \frac{1}{2}f(x)g(x)$.

(i) $h'(1)$

(ii) $h'(3)$

(iii) $h'(4)$

(c) Let $h(x) = \frac{g(x)}{f(x)}$

(i) $h'(1)$

(ii) $h'(3)$

(iii) $h'(4)$

Chapter 11 | Chain Rule

Imagine we are moving straight upward in a hot air balloon. Let y be our distance from the ground. The air pressure, P, is changing as a function of altitude, so $P = f(y)$. How does our air pressure change with time?

Since air pressure is a function of height, $P = f(y)$, and height is a function of time, $y = g(t)$, we can think of air pressure as a composite function of time, $H = f(g(t))$, with f as the outside function and g as the inside function. The example suggests the following result, which turns out to be true:

Rate of change of composite function	=	Rate of change of outside function	×	Rate of change of inside function

Theorem 11.0.33

If f and g are differentiable functions, then

$$\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = f'(g(x)) \cdot g'(x)$$

Warning: The derivative of the outside function must be evaluated at the inside function.

This is called the **chain rule** because, if you have multiple compositions (i.e. several functions stuffed inside of each other) then you will end up with a "chain" of products in the derivative. The Leibniz notation is very suggestive and helpful for remembering how the chain rule works: for the function $y = f(u) = f(g(x))$, meaning $u = g(x)$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

Example 11.0.34

Consider the functions $f(x) = \sqrt{x}$ and $g(x) = 3x-4$. The composite function here is $f(g(x)) = \sqrt{3x-4} = (3x-4)^{1/2}$. Notice the structure of this function:

$$\underbrace{(3x-4)^{1/2}}_{\text{outside function is the 1/2 power}} \qquad \sqrt{\underbrace{3x-4}_{\text{inside function is the polynomial}}}$$

We will always try to identify the **inside function** $g(x)$ and the **outside function** $f(x)$ before starting the chain rule process. Learning to spot insides and outsides correctly is the key to using chain rule.

The derivative of the **outside function** is $f'(x) = \frac{1}{2}x^{-1/2}$. Keeping $g(x)$ inside of $f'(x)$ looks like $f'(g(x)) = \frac{1}{2}(3x-4)^{-1/2}$.

The derivative of the **inside function** is $g'(x) = 3$.

Putting all these pieces of the chain rule formula together, we get:

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = \frac{1}{2}(3x-4)^{-1/2} \cdot 3.$$

So to compute the derivative $\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))]$, we take the derivative of the outside, put the inside function back inside the derivative of the outside, and then multiple by the derivative of the inside.

Question 43.

Differentiate the following functions using the chain rule:

(a) $\mathrm{F}(x) = 2^{x^2+3x+1}$

(b) $h(x) = (2x^3+2x+1)^4$

Question 44.

Let $f(x)$ be a function with

$$f(1)=1, \quad f(2)=2, \quad f'(1)=3, \quad f'(2)=5.$$

If $g(x) = 2f(2x)+f(x)$, what is $g'(1)$?

Question 45.

Suppose $f(x)$ and $g(x)$ and their derivatives have the values given in the table.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	1	5	2	-5
1	3	-2	0	1
2	0	2	3	1
3	2	4	1	-6

Let $h(x) = f(g(x))$. Find the following.

(i) $h'(0)$ *(ii)* $h'(1)$ *(iii)* $h'(2)$ *(iv)* $h'(3)$

§11.1 Derivative Formula for Logarithmic Functions

Before we do more examples of Chain Rule, let's introduce one more basic differentiation formula to our toolkit.

Theorem 11.1.35

Derivative of the natural logarithm function is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln x] = \frac{1}{x}$$

If a logarithm is composed with a function $g(x)$, then by chain rule, our formula becomes:

$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln(g(x))] = \frac{g'(x)}{g(x)}$$

Example 11.1.36

For $y = \ln(x + 11x^2 - 3x^{1/3})$, the derivative using chain rule is

$$\frac{dy}{dx} = \frac{\frac{d}{dx}[x + 11x^2 - 3x^{1/3}]}{x + 11x^2 - 3x^{1/3}} = \frac{1 + 22x - x^{-2/3}}{x + 11x^2 - 3x^{1/3}}$$

Chain Rule with more than two functions

Using the chain rule twice we can similarly write

$$f(g(h(x))) = f'(g(h(x)))g'(h(x))h'(x)$$

As an example, consider the function

$$P(x) = \frac{1}{\ln(3^x + 1)}$$

The order of operation here is as follows

$$x \longrightarrow 3^x + 1 \longrightarrow \ln(3^x + 1) \longrightarrow \frac{1}{\ln(3^x + 1)}$$

We broke our function into exact steps as above because we want to be able to take derivative at each step using the simpler rules we have learned so far. We would like to write $P(x)$ as $f(g(h(x)))$. Note that h is applied first to x, and then g and then f. So in the above sequence of steps, we can identify f, g and h as follows:

$$x \xrightarrow{\ h\ } \underbrace{3^x + 1}_{h(x)} \xrightarrow{\ g\ } \underbrace{\ln(3^x + 1)}_{g(h(x))} \xrightarrow{\ f\ } \underbrace{\frac{1}{\ln(3^x + 1)}}_{f(g(h(x)))}$$

where

$$\begin{aligned}
&h(x) = 3^x + 1 && \Longrightarrow h'(x) = 3^x \ln 3 && \\
&g(x) = \ln x && \Longrightarrow g'(x) = \frac{1}{x} && \Longrightarrow g'(h(x)) = \frac{1}{3^x + 1} \\
&f(x) = \frac{1}{x} = x^{-1} && \Longrightarrow f'(x) = -x^{-2} && \Longrightarrow f'(g(h(x))) = -(\ln(3^x + 1))^{-2}
\end{aligned}$$

So,

$$\begin{aligned}
P'(x) &= f'(g(h(x)))g'(h(x))h'(x) \\
&= -(\ln(3^x + 1))^{-2} \frac{1}{3^x + 1} 3^x \ln 3
\end{aligned}$$

Question 46.

Compute the derivative for each given function.

(a) $y = \ln(x^3 + 3x - 4)$

(b) $f(t) = \ln(\ln(t))$

(c) $g(x) = (\ln(x^2 + x + 1))^5$

§11.2 Practice Problems for all Differentiation Rules Combined

Question 47.

Sometimes, we might have to differentiate using a combination of the chain rule and the product rule, or the chain rule and the quotient rule. Try that in the two problems below.

(a) $y = x^3 e^{-4x^5}$ *(use the product rule, then chain rule!)*

Solution.

$$\begin{aligned}\frac{dy}{dx} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right)\cdot e^{-4x^5} + x^3\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-4x^5}\right)\\ &= 3x^2e^{-4x^5} + x^3e^{-4x^5}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(-4x^5\right)\\ &= 3x^2e^{-4x^5} + x^3e^{-4x^5}\cdot\left(-20x^4\right)\\ &= e^{-4x^5}\left(3x^2 - 20x^7\right)\end{aligned}$$

■

(b) $g(x) = x^2\ln(9x)$ *(use the product rule!)*

Solution.

$$\begin{aligned}g'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\right)\ln(9x) + x^2\frac{\mathrm{d}}{\mathrm{d}x}(\ln(9x))\\ &= 2x\ln(9x) + x^2\cdot\frac{1}{9x}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(9x)\\ &= 2x\ln(9x) + x^2\cdot\frac{1}{9x}\cdot 9\\ &= 2x\ln(9x) + x\end{aligned}$$

■

(c) $f(t) = \dfrac{\ln(t)}{e^t+1}$ *(quotient rule!)*

Solution.

$$\begin{aligned}f'(t) &= \frac{\frac{\mathrm{d}}{\mathrm{d}t}(\ln t)\cdot\left(e^t+1\right) - (\ln t)\frac{\mathrm{d}}{\mathrm{d}t}\left(e^t+1\right)}{(e^t+1)^2}\\ &= \frac{\left(\frac{1}{t}\right)\left(e^t+1\right) - (\ln t)\left(e^t\right)}{(e^t+1)^2}\\ &= \frac{\frac{1}{t}(e^t+1) - e^t\ln t}{(e^t+1)^2}\end{aligned}$$

■

(d) $f(x) = \dfrac{\ln(3x+1)}{(2x)^{3/2}-x}$ *(use the quotient rule! Each function here has an* ***inside****!)*

Solution.

$$f'(x) = \frac{\frac{\mathrm{d}}{\mathrm{d}x}(\ln(3x+1)) \cdot \left((2x)^{3/2} - x\right) - \ln(3x+1) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left((2x)^{3/2} - x\right)}{\left((2x)^{3/2} - x\right)^2}$$
$$= \frac{\frac{1}{3x+1} \cdot 3 \cdot \left((2x)^{3/2} - x\right) - \ln(3x+1) \cdot \left(\frac{3}{2}(2x)^{1/2}(2) - 1\right)}{\left((2x)^{3/2} - x\right)^2}$$

■

■ Question 48.

☐

(a) *If* $p(x) = f(g(x))$*, compute* $p'(1)$*.*

Solution.

$$p'(x) = f'(g(x)) \cdot g'(x)$$
$$p'(1) = f'(g(1)) \cdot g'(1)$$
$$p'(1) = f'(2) \cdot g'(1) = (-2)(-2) = 4$$

■

(b) *If* $q(x) = g(f(x))$*, compute* $q'(-2)$*.*

Solution.

$$q'(x) = g'(f(x)) \cdot f'(x)$$
$$q'(-2) = g'(f(-2)) \cdot f'(-2)$$
$$q'(-2) = g'(4) \cdot f'(-2)$$

■

(c) *If* $\mathrm{H}(x) = \dfrac{1}{(x^4 + g(x))^2}$*, compute* $\mathrm{H}'(1)$*.*

Solution.

$$\mathrm{H}(x) = \left(x^4 + g(x)\right)^{-2}$$
$$\mathrm{H}'(x) = -2\left(x^4 + g(x)\right)^{-3} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(x^4 + g(x)\right)$$
$$\mathrm{H}'(x) = -\frac{2}{(x^4 + g(x))^3} \cdot \left(4x^3 + g'(x)\right)$$
$$\mathrm{H}'(1) = -\frac{2}{(1^4 + g(1))^3} \cdot \left(4 \cdot 1^3 + g'(1)\right)$$
$$\mathrm{H}'(1) = \frac{-2}{(1+2)^3} \cdot (4 + (-2)) = \frac{-2 \cdot 2}{3^3} = -\frac{4}{27}$$

■

(d) *If* $\mathrm{R}(x) = \left(\dfrac{f(x)}{g(x)}\right)^2$*, compute* $\mathrm{R}'(2)$*.*

Solution.

$$\begin{aligned}
R'(x) &= 2\left(\frac{f(x)}{g(x)}\right)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right)\\
R'(x) &= \frac{2f(x)}{g(x)}\cdot\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}\\
R'(2) &= \frac{2f(2)\cdot(f'(2)g(2)-f(2)g'(2))}{[g(2)]^3}\\
R'(2) &= \frac{2\cdot 1\cdot((-2)(-3)-(1)(1))}{(-3)^3} = \frac{2\cdot(6-1)}{-3^3} = \frac{-10}{27}
\end{aligned}$$

■

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-2	4	5	-1	2
1	-2	-1	2	-2
2	1	-2	-3	1

Table 11.1: Table for Problem 48

■ Question 49. □

Compute the derivative of the given function.

(a) $y = \ln(x+2)$

Solution.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{x+2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(x+2)\\
&= \frac{1}{x+2}\cdot(1)\\
&= \frac{1}{x+2}
\end{aligned}$$

■

(b) $f(x) = e^{10x-2}$

Solution.

$$\begin{aligned}
f'(x) &= e^{10x-2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(10x-2)\\
&= 10e^{10x-2}
\end{aligned}$$

■

(c) $g(x) = x^2\ln(x) + 2x$

Solution.

$$\begin{aligned}
g'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\right)\cdot\ln(x) + x^2\cdot\frac{\mathrm{d}}{\mathrm{d}x}(\ln(x)) + \frac{\mathrm{d}}{\mathrm{d}x}(2x)\\
&= 2x\ln(x) + x^2\cdot\frac{1}{x} + 2\\
&= 2x\ln(x) + x + 2
\end{aligned}$$

■

(d) $h(x) = \dfrac{x}{\sqrt{1-x}}$

Solution.

$$\begin{aligned}
h(x) &= \frac{x}{\sqrt{1-x}} = x(1-x)^{-1/2} \\
h'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(x)\cdot(1-x)^{-1/2} + x\cdot\frac{\mathrm{d}}{\mathrm{d}x}(1-x)^{-1/2} \\
&= 1\cdot(1-x)^{-1/2} + x\left(-\frac{1}{2}\right)(1-x)^{-3/2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(1-x) \\
&= (1-x)^{-1/2} + \frac{1}{2}x(1-x)^{-3/2}
\end{aligned}$$

■

(e) $y = \sqrt{x+\sqrt{x+\sqrt{x}}}$

Solution.

$$\begin{aligned}
y &= (x+(x+(x)^{1/2})^{1/2})^{1/2} \\
y' &= \frac{1}{2}\left(x+\left(x+x^{1/2}\right)^{1/2}\right)^{-1/2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(x+\left(x+x^{1/2}\right)^{1/2}\right) \\
&= \frac{1}{2}\left(x+\left(x+x^{1/2}\right)^{1/2}\right)^{-1/2}\cdot\left(1+\frac{1}{2}\left(x+x^{1/2}\right)^{-1/2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(x+x^{1/2}\right)\right) \\
&= \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\cdot\left(1+\frac{1}{2\sqrt{x+\sqrt{x}}}\cdot\left(1+\frac{1}{2}x^{-1/2}\right)\right) \\
&= \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\cdot\left(1+\frac{1}{2\sqrt{x+\sqrt{x}}}\cdot\left(1+\frac{1}{2\sqrt{x}}\right)\right) \\
&= \frac{\dfrac{\dfrac{1}{2\sqrt{x}}+1}{2\sqrt{x+\sqrt{x}}}+1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}
\end{aligned}$$

■

Part V

Application of Derivatives

Chapter 12 | Maximum and Minimum Values

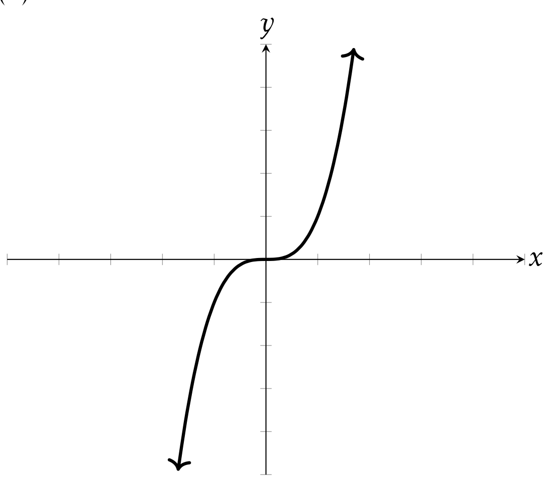
In many different settings, we are interested in knowing where a function achieves its least and greatest values. These can be important in applications — say to identify a point at which maximum profit or minimum cost occurs — or in theory to characterize the behavior of a function or a family of related functions.

In the rest of this course we are going to learn how to use Derivatives to find such maximum or minimum values. To begin the process, let's recall how to find the graph of the derivative from a given graph of a function.

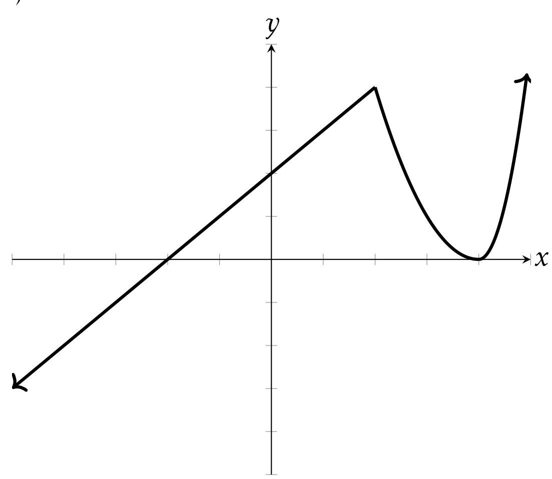
Question 50.

Given the graph of $f(x)$, sketch the graph of $f'(x)$ on the same graph.

(a)



(b)



We make the following observation from these examples.

A function f is said to be **increasing** on an interval (a,b) if $f(x) < f(y)$ whenever $x < y$ on the given interval. Similarly, a function f is **decreasing** on an interval (a,b) if $f(x) > f(y)$ whenever $x < y$ on the given interval. In other words: a function is **increasing** if it the y-values are always getting smaller on an interval. A function is **decreasing** if the y-values are always getting bigger on an interval.

Theorem 12.0.37

If $f'(x) > 0$ for all x in an interval (a,b), then f is increasing on (a,b). Similarly, if $f'(x) < 0$ for all x in an interval (a,b), then f is decreasing on (a,b).

§12.1 Local Extrema

Definition 12.1.38

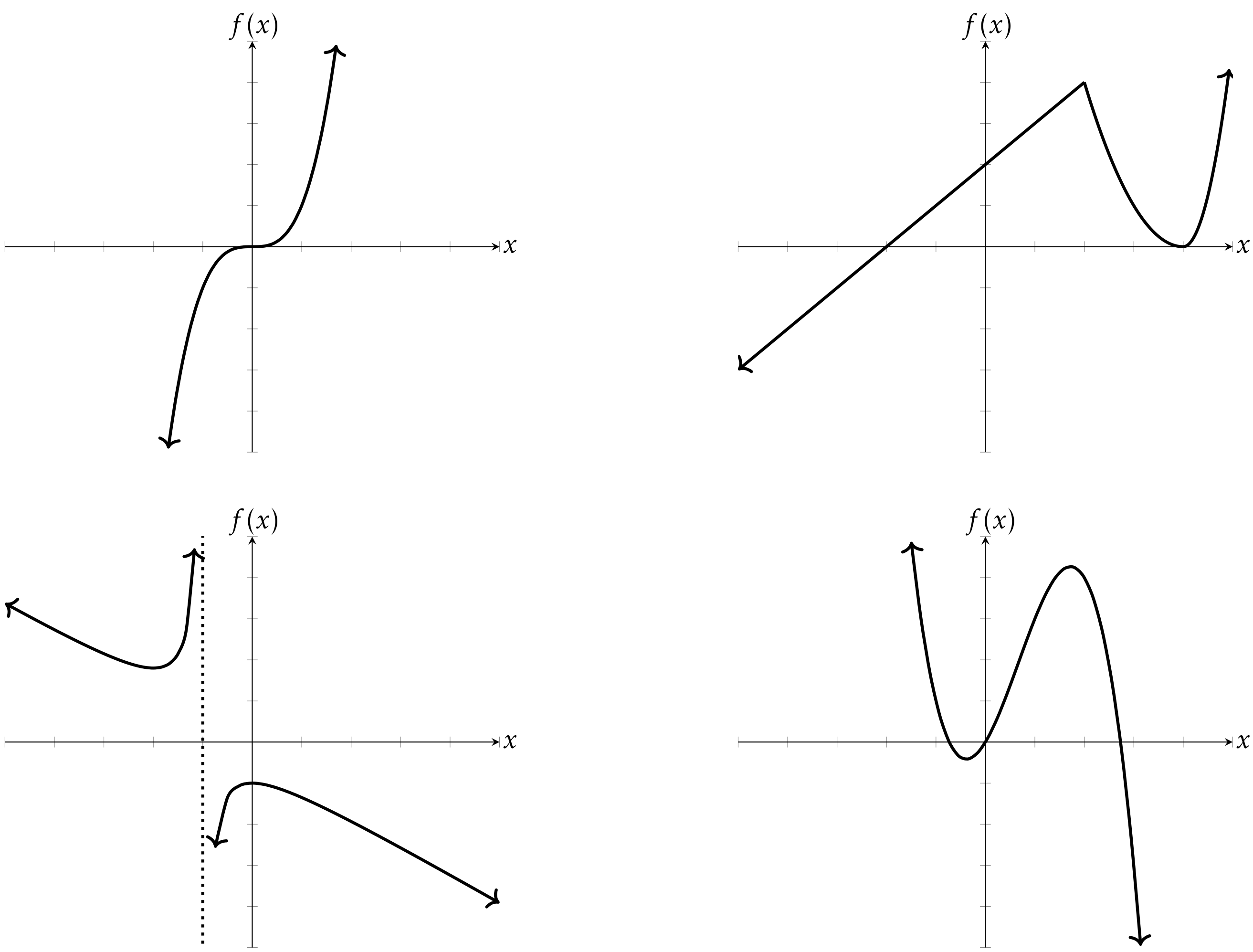
We say that that a function $f(x)$ has a **local maximum** at $x = c$ provided that $f(c) \geq f(x)$ for all x *near* c. Similarly, c is called a **local minimum** of f whenever $f(c) \leq f(x)$ for all x near c.

For a function defined on a domain D, we say that $f(x)$ has a **absolute** or **global maximum** at $x = c$, if $f(c) \geq f(x)$ for all x in D.

Note: We will sometimes restrict the domain of $f(x)$ and define it only on a closed interval $[a, b]$.

Question 51.

Identify points corresponding to local maximums or local minimums in the given graphs (if any exist). Note that the third graph has an asymptote at $x = -1$.



Question 52.

Next, in each of the above four graphs identify the points where the derivative $f'(x)$ is either 0 or does not exist.

■ Question 53. □

Consider the function h given by the graph in fig. 12.1 below.

(a) *Identify all of the values of c in* $(-3, 3)$ *where* $h(x)$ *has a local maximum.*

(b) *Identify all of the values of c in* $(-3, 3)$*where* $h(x)$ *has a local maximum.*

(c) *Does h have a global maximum on the interval* $[-3, 3]$ *? If so, where is it?*

(d) *Does h have a global minimum on the interval* $[-3, 3]$ *? If so, where is it?*

(e) *Identify all values of c for which* $h'(c) = 0$.

(f) *Identify all values of c for which* $h'(c)$ *does not exist.*

(g) *True or false: every local maximum and minimum of h occurs at a point where* $h'(c)$ *is either zero or does not exist.*

(h) *True or false: at every point where* $h'(c)$ *is zero or does not exist, h has a local maximum or minimum.*

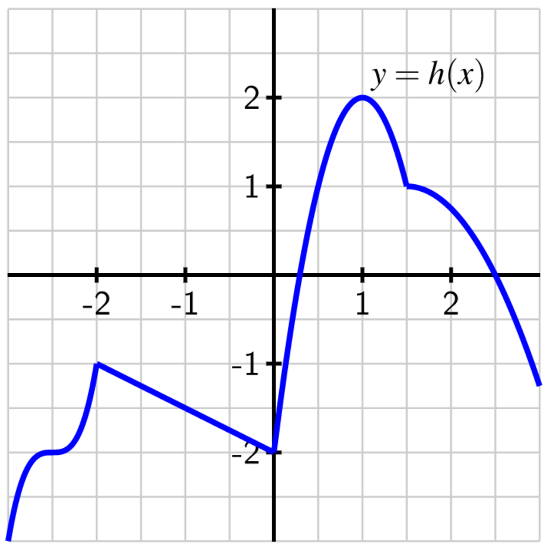


Figure 12.1

§12.2 Critical Points and the First Derivative Test

Definition 12.2.39: Critical Point

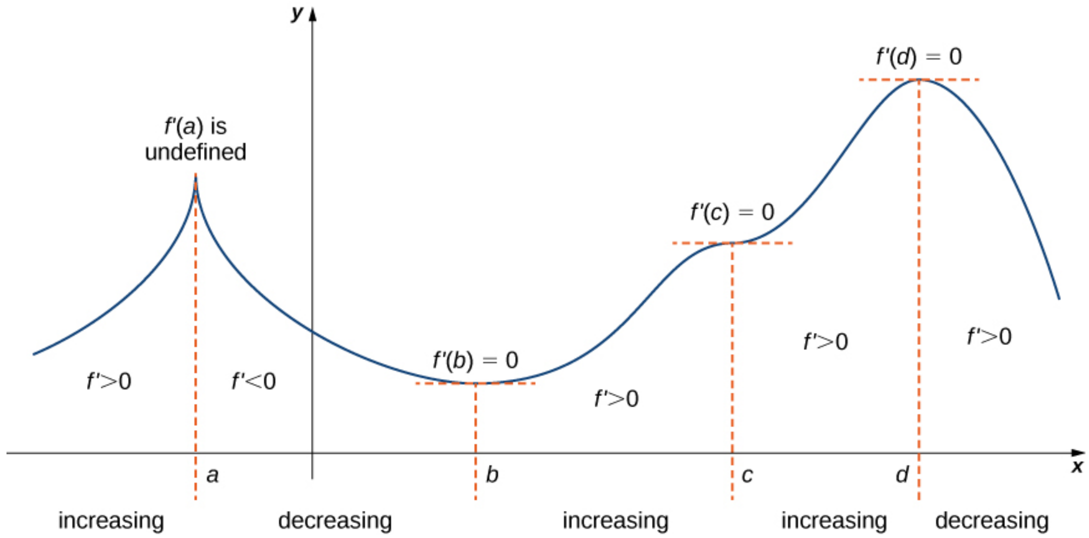
For any function f, a point c in the *domain* of f where $f'(c)$ is either 0 or undefined, is called a **critical point** of the function. In addition, the point $(c, f(c))$ on the graph of f is also called a critical point. A critical value of f is the value, $f(c)$, at a critical point, c.

From our previous examples, we conclude that

Theorem 12.2.40: Fermat's theorem

If f has a local extremum at $x = c$, then c is a critical point.

Warning: The converse of Fermat's theorem is not true. Not every critical point is a local extremum. Consider for example, a function $f(x)$ whose graph is as follows:



The function f has four critical points: a, b, c, and d. The function has local maxima at a and d, and a local minimum at b, The function does not have a local extremum at c.

Perhaps, the most interesting observation to make here is that the sign of f' changes at all local extrema.

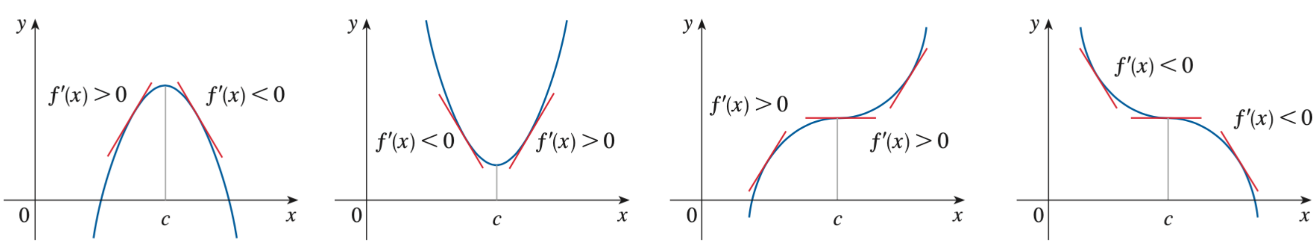
Theorem 12.2.41: First Derivative Test

If c is a critical point of a continuous function f that is differentiable near c (except possibly at $x = c$), then

(i) f has a local maximum at c if and only if f' changes sign from positive to negative at c, and

(ii) f has a local minimum at c if and only if f' changes sign from negative to positive at c.

(iii) If f' has the same sign for $x < c$ and $x > c$, then $f(c)$ is neither a local max nor a local min.



(a) Local maximum at c (b) Local minimum at c (c) No maximum or minimum at c (d) No maximum or minimum at c

Example 12.2.42

onsider $f(x) = x^3 - 6x^2 + 15$. Follow the steps below to identify the local extrema for $f(x)$.

(a) Start by finding the possible candidates for local extrema. That is, determine all critical points of $f(x)$. You should find two critical points.

(b) The domain of $f(x)$ is the interval $(-\infty, \infty)$ because f is a polynomial. For each critical point c_1 and c_2 you found, we divide the domain up into three subintervals:

$$(-\infty, c_1), (c_1, c_2), (c_2, \infty).$$

Write down the three intervals.

To determine the sign of $f'(x)$ on each subinterval, we only need to pick one **test value** in each subinterval and plug it into $f'(x)$. Can you explain why $f'(x)$ is either always positive or always negative in each of these intervals.

(c) It's beneficial to make a table or some other organizational device to gather your information about the sign of f'. Fill out the table and make a conclusion about whether or not f is **increasing** or **decreasing** on each subinterval interval.

Intervals			
Test Points			
Sign of $f'(x)$			
Behavior of f			

(d) Using your table of information, can you conclude where f has a local maximum and a local minimum? Graph $f(x)$ to confirm your answer.

■ Question 54. □

For each function below, use the First Derivative Test to determine all local extrema.

(a) $f(x) = -x^3 + \frac{3}{2}x^2 + 18x$

(b) $f(x) = (x-1)^{1/3}$

(c) $f(x) = (x^2 - 1)e^{-x}$ ***Note:*** *Use the quadratic formula to help find the critical points here.*

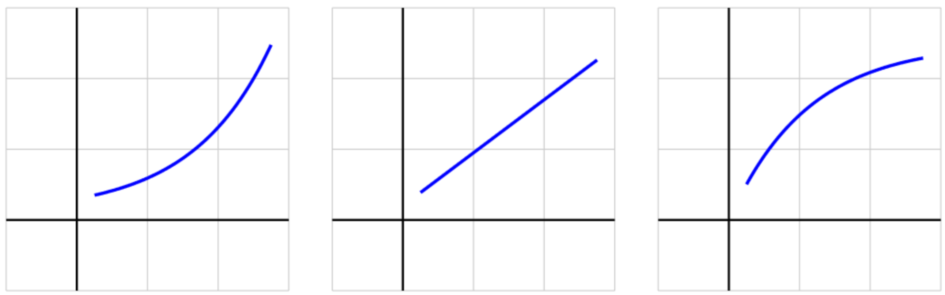
(d) $f(x) = \dfrac{x^2}{x-9}$

Note: *The domain of this function is $(-\infty, 9) \cup (9, \infty)$. So when you make your table, take into account $x = 9$ as you break up your intervals across the critical points (if any), but remember that 9* ***is not*** *a critical point.*

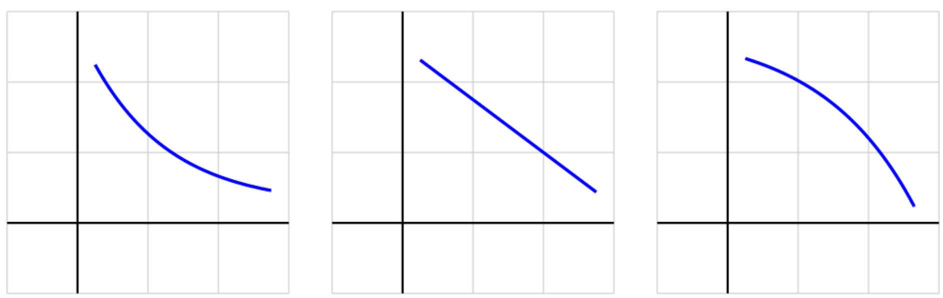
(e) $f(x) = \sqrt{x}\ln(x)$

Chapter 13 | Concavity and the Second Derivative Test

Last chapter, we learned that the first derivative tells us when a function is increasing or decreasing and that leads us to finding the maximum or minimum. In addition to asking whether a function is increasing or decreasing, it is also natural to inquire whether it changes at a constant rate. There are three basic behaviors that an increasing function can demonstrate on an interval: the function can increase more and more rapidly, it can increase at the same rate, or it can increase in a way that is slowing down.



Similarly for the decreasing case.



The notion of concavity provides a simpler language to describe these behaviors.

§13.1 Concavity of a graph

Definition 13.1.43

Let f be a differentiable function on an open interval I. Then f is said to be **concave up** on I if and only if f' is increasing on I, and f is said to be **concave down** on I if and only if f' is decreasing on I,

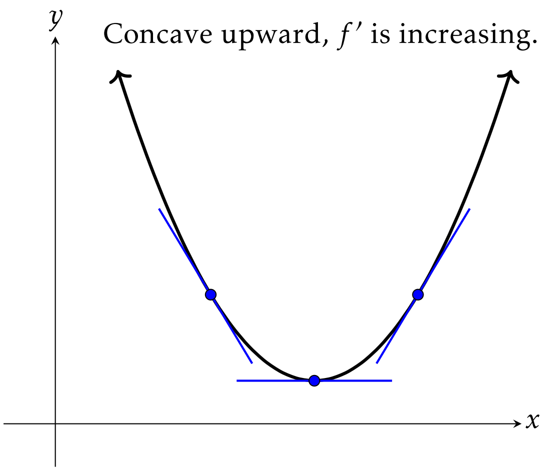
A geometric description of concavity is as follows. If the graph of f lies above all of its tangents on an interval I, then f is called concave upward on I. If the graph of f lies below all of its tangents on I, then f is called concave downward on I.

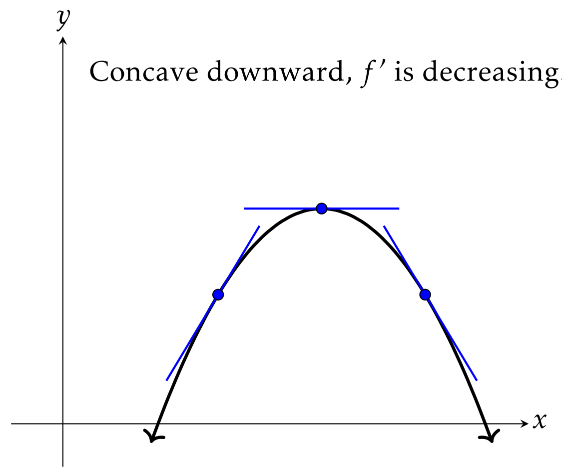
Question 55. □

Use the pictures below to answer the following questions:

(a) If $f'(x)$ is increasing, what can we conclude about $f''(x)$?

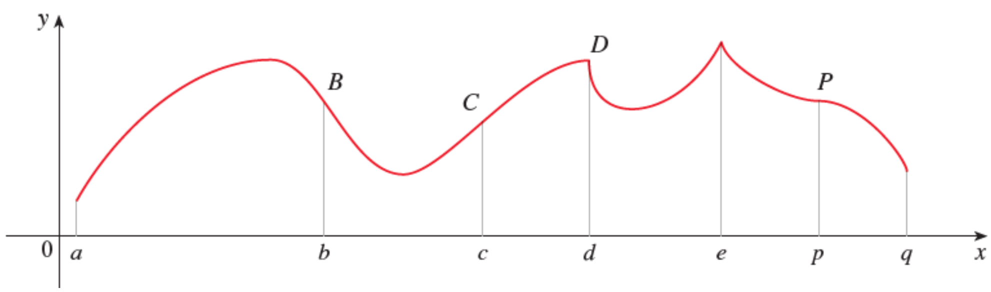
(b) If $f'(x)$ is decreasing, what can we conclude about $f''(x)$?





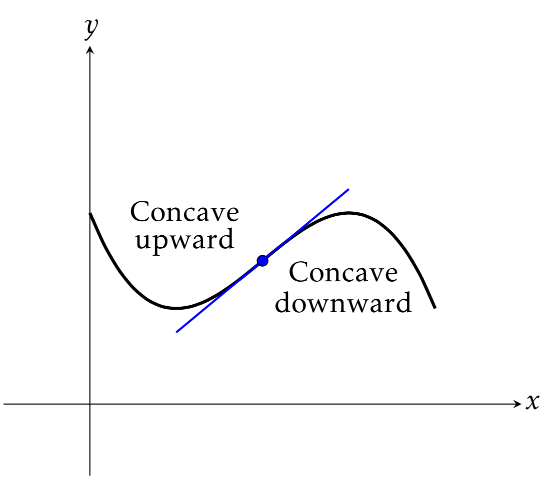
Question 56.

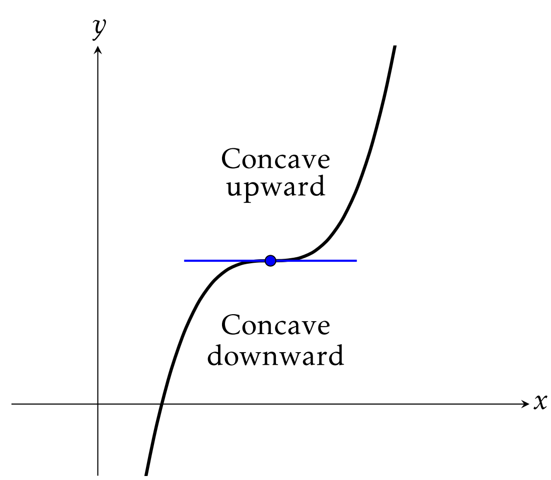
Identify the intervals on which f is concave up or concave down.



Definition 13.1.44

A point p, at which the graph of a continuous function, f, changes concavity is called an **inflection point** of f.





Note: Similar to critical points, these inflection points **may** occur when $f''(x) = 0$ or when $f''(x)$ is undefined. To test whether p is an inflection point, check whether f'' changes sign at p.

Question 57.

Consider $f(x) = x^3 - 3x^2 - 9x - 1$. Determine the intervals where $f(x)$ is concave up and concave down, and list any points of inflection.

(a) *Just like with increasing and decreasing, start by determining the important points where concavity could change. That is, compute $f''(x)$ and solve for when $f''(x) = 0$ or when $f''(x)$ is undefined.*

(b) *You should only get one value $x = p$ in the previous step, and so there are two subintervals to consider:*

$$(-\infty, p) \text{ and } (p, \infty).$$

We can again use a table of some kind (or whatever organizational device you choose), to determine the sign of $f''(x)$ and make conclusions about the graph of f.

Intervals	$(-\infty, p)$	(p, ∞)
Test Points		
Sign of $f''(x)$		
Conclusion		

Question 58.

For each function below, determine the following:

- *Interval where f is concave up*
- *Interval where f is concave down*
- *Inflection Points*

(a) $f(x) = \frac{1}{4}x^4 + 2x^3$

(b) $f(x) = (x-1)e^x$

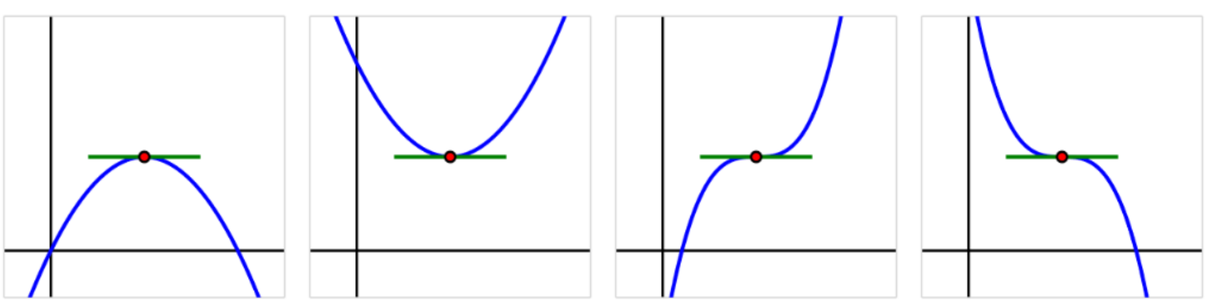
(c) $f(x) = \dfrac{x}{x-5}$

(d) $f(x) = \sqrt{x}\ln(x)$

§13.2 The Second Derivative Test

We have seen how to use the first derivative to determine whether a critical point corresponds to a local extrema. This was the **First Derivative Test**. We have just examined how the second derivative can be used to understand the concavity of a function. But, we can also use the second derivative to verify if a critical point is a local extrema. This is called the **Second Derivative Test**.

Last chapter we saw that there are four possibilities for the graph of a function f with a horizontal tangent line at a critical point.



From the pictures, we can conclude the following.

Theorem 13.2.45: Second Derivative Test

Suppose p is a critical point of a continuous function f such that $f'(p) = 0$ and $f''(p) \neq 0$. Then,

(a) *f has a local maximum at p if and only if $f''(p) < 0$.*

(b) *f has a local minimum at p if and only if $f''(p) > 0$.*

Warning: In the event that $f''(p) = 0$, the second derivative test is inconclusive. That is, the test doesn't provide us any information. This is because if $f''(p) = 0$, it is possible that f has a local minimum, local maximum, or neither.

Question 59.

For each of the following functions, you are provided with the critical points. Use the second derivative test to classify the critical point as a local maximum, local minimum, or mention if the test is inconclusive.

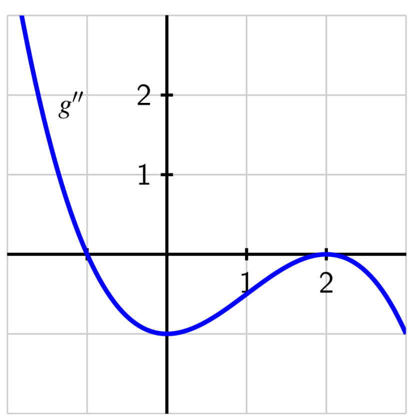
(a) $f(x) = x^3 - 6x^2 + 15$ *has critical points* $x = 0$ *and* $x = 4$.

(b) $f(x) = (x-1)e^x$ *has critical points* $x = 0$.

(c) $f(x) = \sqrt{x}\ln(x)$ *has critical points* $x = e^{-2} = \dfrac{1}{e^2}$.

Question 60.

Consider a function $g(x)$ whose second derivative g'' is given by the following graph.



(a) *Find the x -coordinates of all points of inflection of g.*

(b) *Fully describe the concavity of g by making an appropriate sign chart.*

(c) *Suppose you are given that $g'(-1.6) = 0$. Is there a local maximum, local minimum, or neither (for the function g) at this critical point of g, or is it impossible to say? Why?*

§13.3 Finding Global Maximum and Minimum

We briefly defined global maximum and minimu before. Now let's see how we can find one in practice.

Example 13.3.46

Find the global maximum and minimum of $f(x) = x^4 - 8x^2$ on the interval $[-100, 1]$.

(a) What are the critical points of f on the interval $(-100, 1)$?

(b) What is the value of f (the y-value) at each critical points?

Note: we only care about the values at critical points in the interval $[-100, 1]$. (Why?)

(c) What is the value of f (the y-value) at each endpoint of $[-100, 1]$?

Why do we need to check the endpoints?

(d) Order all of these values (the y-values) from least to greatest. The least of these values is the *global minimum*. The greatest of these values is the *global maximum*.

Here's a summary of the general procedure:

Finding the global maximum and minimum of a continuous function f on the interval $[a, b]$

(a) Find the critical points $f(x)$ that lie inside the interval (a, b). you do not need to check if these are local max/min.

(b) Find the value of the function $f(p)$ for every critical point p above.

(c) Find the values of $f(x)$ at the endpoints of the interval, i.e. find $f(a)$ and $f(b)$.

(d) The largest of the values from Steps 2 and 3 is the global maximum value; the smallest of these values is the global minimum value.

Question 61.

Find the ***exact*** *global maximum and minimum for each of the functions below on the stated intervals.*

(a) $r(x) = 2x^3 - 3x^2 - 12x + 1$ *on* $[-2, 3]$

(b) $s(x) = x + \frac{3}{x}$ *on* $[1, 4]$

(c) $f(x) = |x + 4|$ *on* $[-7, 1]$

(d) $y = x^2 - 8\ln x$ *on* $[1, 5]$

(e) $y = (x - x^2)^2$ *on* $[-1, 1]$

(f) $y = \sqrt{9 - x}$ *on* $[1, 9]$

Chapter 14 | STEM Applications

Recall our derivative function definition is given by the limit

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Physically, we know that for a value $x = a$, the derivative $f'(a)$ equals the slope of the tangent line to $y = f(x)$ at $x = a$. When we use our alternative derivative notation, like $\frac{\mathrm{d}f}{\mathrm{d}x}$ or $\frac{\mathrm{d}y}{\mathrm{d}x}$, this helps to remind us of the units (and meaning) of the derivative as the **instantaneous rate of change of f with respect to x**.

Now if we add/subtract two quantities of a certain unit, then their result will be of the same units. Hence, the units of the difference quotient in the definition of derivative is simply the quotient of the units on top and bottom. When we take the limit and get the derivative, the units do not change, hence

$$\text{Units of } f'(x) = \frac{[\text{output units}]}{[\text{input units}]}.$$

Example 14.0.47

Suppose $P(t)$ is a function measuring the air pressure (pounds per square inch) in a truck tire at a time t (hours). What are the units and meaning of $P(5) = 33$? What are the units and meaning of $P'(6) = -0.48$?

Solution: When $t = 5$ hours, $P(5) = 33$ lbs per square inch or $\frac{\text{lbs}}{\text{in}^2}$ (or PSI). So after 5 hours, this tire has 33 PSI.

The rate of change $P'(6) = -0.48$ has units $\frac{\text{PSI}}{\text{hour}}$ or lbs per square inch per hour. We could also write this as $\frac{\frac{\text{lbs}}{\text{in}^2}}{\text{hr}}$. The value of -0.48 tells us that the tire is **losing** 0.48 PSI every hour after 6 hours. We know tires slowly lose air pressure when in use/over time, so this negative rate of change makes sense.

■ **Question 62.** □

The cost, C (in dollars) to produce g gallons of ice cream can be expressed as a function $C = f(g)$.

(a) In the expression $f(100) = 250$, what is the meaning and what are the units of 100 and of 250?

(b) In the expression $f'(100) = 1.2$, what is the meaning and what are the units of 100 and of 1.2?

■ Question 63. □

A veterinary study investigating the relationship between diet and weight in dogs found that the weight W *(measured in lbs) was a function* $W = f(c)$*, of the average number of Calories,* c*, consumed by the dog in a day.*

(a) *In the expression* $f(950) = 50$*, what is the meaning and what are the units of* 950 *and of* 50*?*

(b) *In the expression* $f'(1125) = 0$*, what is the meaning and what are the units of* 1125 *and of* 0*?*

(c) *In the expression* $f^{-1}(70) = 1480$*, what is the meaning and what are the units of* 1480 *and of* 70*?*

(d) *What are the units of* $f'(c) = \frac{dW}{dc}$*?*

(e) *Suppose Melanie reads this vet study and concludes the following: If Melanie's dog increases her average calorie intake from* 950 *to* 1010 *Calories per day, then her weight will increase by approximately* 0.8 *pounds.*

Fill in the blanks so that the equation below supports Melanie's conclusion:

$$f'(\_\_\_\_\_\_\_\_\_\_\_\_) = \_\_\_\_\_\_\_\_\_\_\_\_$$

§14.1 Change Formula

The amount of change formula is less of an equation but an idea about how we can use the derivative of a function to **estimate** values of the function. For very small values of $h \neq 0$, the value of $f'(a)$ is approximately equal to $\dfrac{f(a+h)-f(a)}{h}$. Thus, if we know a function value of $f(a)$, and we need to approximate a nearby value of $f(a+h)$, then we set the approximate value of $f(a+h)$ to be

$$\boxed{f(a+h) \approx f(a) + f'(a)h}.$$

14.1.1 Application in Economics - Marginal Cost

Suppose $C(x)$ is the total cost that a company incurs in producing x units of a certain commodity. The function C is called a cost function. If the number of items produced is increased from x_1 to x_2, then the additional cost is $\Delta C = C(x_2) - C(x_1)$, and the average rate of change of the cost is

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1} = \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}$$

The limit of this quantity as $\Delta x \to 0$, that is, the instantaneous rate of change of cost with respect to the number of items produced, is called the marginal cost by economists:

$$\text{marginal cost} = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}$$

Since x often takes on only integer values, it may not make literal sense to let Δx approach 0. Taking $\Delta x = 1$ and n large (so that Δx is small compared to n), we have

$$C'(n) \approx C(n+1) - C(n)$$

Thus the marginal cost of producing n units is approximately equal to the cost of producing one more unit [the $(n+1)$ st unit].

Example 14.1.48

Suppose a company has estimated that the cost (in dollars) of producing a total of x gallons of some chemical is

$$C(x) = 10000 + 5x + 0.01x^2$$

(a) Then the marginal cost function is $C'(x) =$ ________________ dollars/gallon.

(b) The marginal cost at the production level of 500 gallons is $C'(500) =$ ____________

This gives the rate at which costs are increasing with respect to the production level when $x = 500$ and predicts the cost of the 501st gallon.

(c) The actual cost of producing the 501st gallon is

$C(501) - C(500) =$ ________________________________.

(d) Check that your answers from parts (c) and part (b) are approximately equal.

Question 64.

The cost function, in dollars, of a company that manufactures food processors is given by

$$C(x) = 200 + \frac{7}{x} + \frac{x^2}{7},$$

where x is the number of food processors manufactured.

(a) Find the marginal cost.

(b) Find the marginal cost of manufacturing 12 food processors.

(c) Approximately how much does it cost to produce one more food processor after the 12?

(d) Find the actual cost of producing the 13th food processor and compare how far off the approximation is from the actual cost.

□

■ Question 65. □

A profit is earned when revenue exceeds cost. Suppose the profit function for a skateboard manufacturer is given by $P(x) = 30x - 0.3x^2 - 250$, *where x is the number of skateboards sold. Estimate the profit gained from the sale of the* 30*th skateboard using marginal profit. How far off is this from the exact profit of the* 30*th skateboard?*

14.1.2 Application in Biology - Population Models

Let $n = P(t)$ be the number of individuals in an animal or plant population at time t. The change in the population size between the times $t = t_1$ and $t = t_2$ is $\Delta n = P(t_2) - P(t_1)$, and so the average rate of growth during the time period $t_1 \leqslant t \leqslant t_2$ is

$$\text{average rate of growth } = \frac{\Delta n}{\Delta t} = \frac{P(t_2) - P(t_1)}{t_2 - t_1}$$

The instantaneous rate of growth is obtained from this average rate of growth by letting the time period Δt approach 0 :

$$\text{growth rate } = \lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}$$

Strictly speaking, this is not quite accurate because the actual graph of a population function $n = f(t)$ would be a step function that is discontinuous whenever a birth or death occurs and therefore not differentiable. However, for a large animal or plant population, we can replace the graph by a smooth approximating curve.

■ Question 66. □

Suppose the population of a country $P(t)$ *is counted in millions where t is counted in years. What is the unit of the instantaneous growth rate?*

■ Question 67. □

The current population of a mosquito colony is 3,000*; that is,* $P(0) = 3{,}000$. *If* $P'(0) = 100$, *estimate the size of the mosquito population after* 2 *days, where t is measured in days.*

■ Question 68. □

A small town in Ohio commissioned an actuarial firm to conduct a study that modeled the rate of change of the town's population. The study found that the town's population (measured in thousands of people) can be modeled by the function $P(t) = -\frac{1}{3}t^3 + 64t + 3000$, *where t is measured in years.*

(a) *Find* $P(1), P(2), P(3)$, *and* $P(4)$ *and interpret what the results mean for the town.*

(b) *Find* $P'(1), P'(2), P'(3)$, *and* $P'(4)$ *and interpret what the results mean for the town.*

14.1.3 Application in Physics - Motion along a line

As we have seen previously, we can use the derivative to think about the velocity of an object in motion. We summarize these facts below:

Definition 14.1.49

Let $s(t)$ denote the position of an object at time t.

- The velocity function $v(t)$ of the object at time T is given by $v(t) = s'(t)$.
- The speed of the object at time t is $|v(t)|$, the absolute value of $v(t)$.
- The acceleration function $a(t)$ of the object at time t is given by $a(t) = v'(t) = s''(t)$.

Question 69. □

The position function $s(t) = t^2 - 2t - 4$ *represents the position of the back of a car backing out of a driveway, and then driving in a straight line, with s in feet and t in seconds. In this case,* $s(t) = 0$ *represents the time at which the back of the car is at the garage door, so* $s(0) = -4$ *is the starting position of the car, 4 feet inside the garage.*

What can you say about the velocity at the moment when $s(t) = 4$*?*

Question 70. □

A potato is launched vertically upward from a potato gun, with an initial velocity of 100 *ft/s, and from atop an 85-foot-tall building. The distance in feet that the potato travels is given by* $s(t) = -16t^2 + 100t + 85$.

(a) *When does the potato reach it's maximum height?*

(b) How long is the potato in the air?

(c) What is the speed of the potato when it hits the ground?

We can tell if an object is **slowing down** or **speeding up** by examining the signs of the velocity and acceleration functions:

Slowing down: $a(t)$ has opposite sign from $v(t)$.

Speeding Up: $a(t)$ and $v(t)$ have the same sign.

Alternately, we can also think about the speed, $|v(t)|$. If $|v(t)|$ is increasing, then we are speeding up, and if $|v(t)|$ is decreasing, then we are slowing down.

Question 71.

The function $s(t) = 2t^3 - 3t^2 - 12t + 8$ *gives the position of a particle moving along a horizontal line. When is the particle slowing down or speeding up?*

Chapter 15 | Applied Optimization

The goal of this chapter (and our final course standard) is to directly apply what we have learned about finding extrema of functions to applications. We want to see how the techniques of Calculus can be used to take function models and find precise optimization solutions.

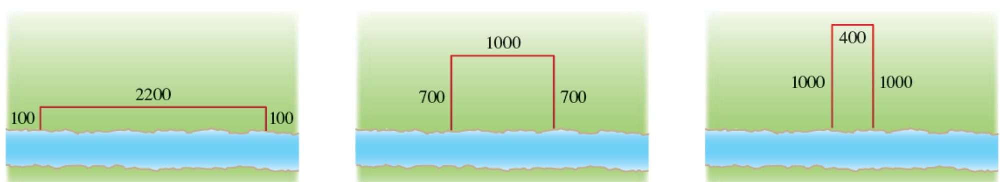
In solving such practical problems the greatest challenge is often to convert the word problem into a mathematical optimization problem by setting up the function that is to be maximized or minimized. Let's work through an example to understand the steps required:

Example 15.0.50

A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Step 1. **Draw a picture.** *We will always try to start by drawing a picture to have a visual representation of our problem. Having a picture let's us experiment with some specific cases to get a feeling for what is happening.*

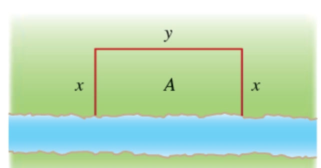
The figure below (not to scale) shows three possible ways of laying out the 2400 ft of fencing.



Area = 100 · 2200 = 220,000 ft^2 Area = 700 · 1000 = 700,000 ft^2 Area = 1000 · 400 = 400,000 ft^2

We see that when we try shallow, wide fields or deep, narrow fields, we get relatively small areas. It seems plausible that there is some intermediate configuration that produces the largest area.

Step 2. **Introduce and label variables.** *Next step is to understand what quantities are allowed to vary in the problem and then to represent those values with variables. So we will construct a general figure with the variables labeled.*



Let x and y be the depth and width of the rectangle (in feet) in the general case.

Step 3. **Make sure that you know what quantity or function is to be optimized.**

Write down a formula for this quantity algebraically using the variables you introduced in the last step. This function is called the ***Objective Function****.*

We wish to maximize the area A of the rectangle. So next we express A in terms of x and y.

$$\text{A} = \text{Area} = \text{V} = \text{length} \times \text{width} = xy$$

In order to do apply the techniques we have learned in this course so far, we need a function of **one** variable. So we go to the next step.

Step 4. **Using information given in the problem, re-write your formula from Step 3 as a function of ONE variable.** *The information given in the problem regarding the relationship among the variables should aid you in making the necessary substitutions or eliminations in this step. The information given is usually in the form of other equations; we refer to this as a* ***constraint equations****. Remember, you have to eliminate all but one variable.*

We want to express A as a function of just one variable, so we eliminate y by expressing it in terms of x. To do this we use the given information that the total length of the fencing is 2400 ft. Thus

$$2x + y = 2400$$

From this equation we have $y = 2400 - 2x$, which gives

$$A = xy = x(2400 - 2x) = 2400x - 2x^2$$

Step 5. **Decide the domain on which to optimize your Objective Function.** *Often the physical constraints of the problem will limit the possible values that the variables can take on. Thinking back to the diagram describing the overall situation and any relationships among variables in the problem often helps identify the smallest and largest values of the input variable.*

Note that the largest x can be is 1200 (this uses all the fence for the depth and none for the width) and x can't be negative, so the function that we wish to maximize is

$$A(x) = 2400x - 2x^2, \qquad 0 \le x \le 1200$$

We know that any continuous function on a closed interval will always have an absolute maximum and an absolute minimum. These are exactly what we wish to find.

Step 6. **Apply the techniques you know to identify the Max/Min(s).** *This always involves finding the critical numbers of the function first. Then evaluate the function at the endpoints and critical numbers to find the global max and/or min.*

The derivative is $A'(x) = 2400 - 4x$, so to find the critical numbers we solve the equation

$$2400 - 4x = 0$$

which gives $x = 600$.

The maximum value of A must occur either at this critical number or at an endpoint of the interval. Since $A(0) = 0$, $A(1200) = 0$ and $A(600) = 720000$, the maximum value of A must be $A(600) = 720000$.

[We could have also done above step using the second derivative test instead. How would that work?]

Step 7. **Finally, bring all your information together, and answer whatever questions were posed by the problem.** *Make sure that you have answered the correct question: does the question seek the absolute maximum of a quantity, or the values of the variables that produce the maximum? Also make sure to answer all asked questions! (Many problems have multiple parts!)*

The example asks for the dimensions of the field that gives the maximum. So we still need to find out y. The corresponding y-value is $y = 2400 - 2x = 2400 - 1200 = 1200$. So the rectangular field should be 600 ft deep and 1200 ft wide.

Here's a flowchart to summarize the process and help you with examples below:

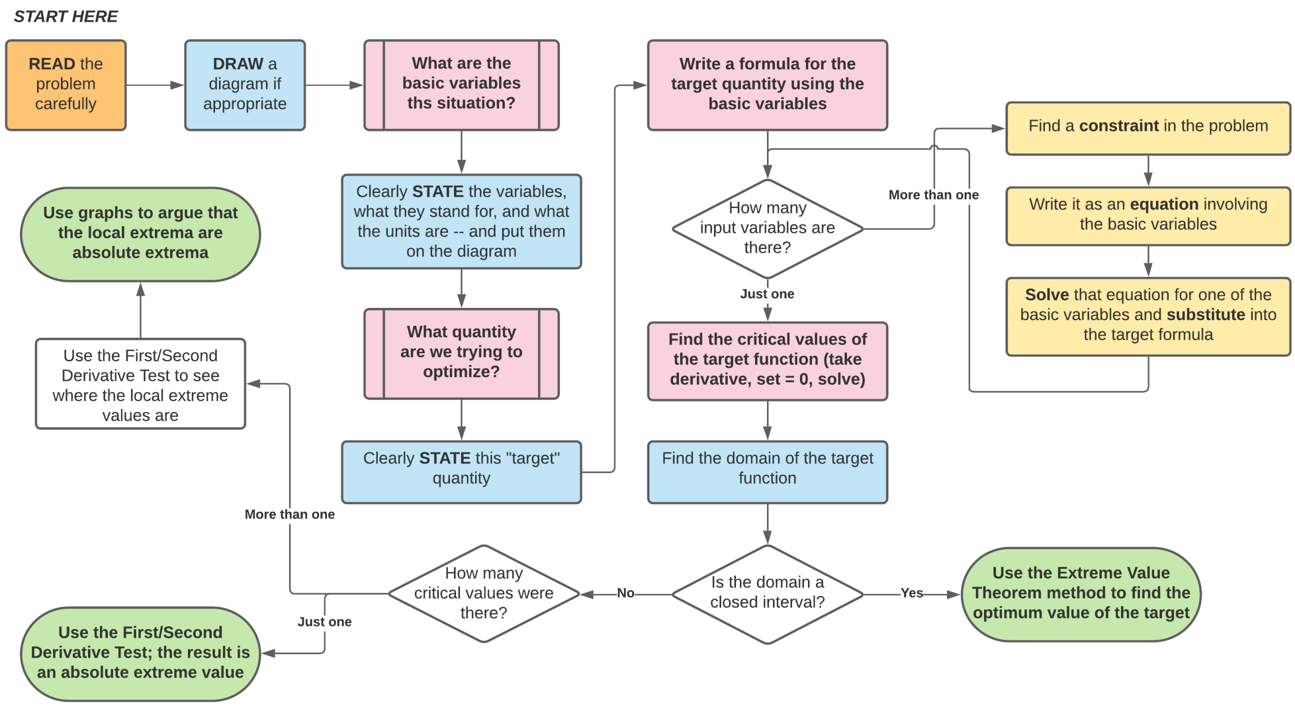


Figure 15.1: Flowchart for Optimization Problems (Picture Courtesy: Robert Talbert)

Warning: Familiarity with common geometric formulas is particularly essential in solving optimization problems. Sometimes those involve perimeter, area, volume, or surface area of geometric objects. At other times, the constraints of a problem introduce right triangles (where the Pythagorean Theorem apply) or other shapes whose formulas provide relationships among the variables. So it is highly recommended that you brush up on those result as we progress further.

Question 72.

You want to build a box for your cat to sit in. The box will have no top, a square base, and rectangular sides. Looking at the size of your cat, you want the box to have a volume of 150 in^3. What should the dimensions of the box in order to ***minimize the surface area*** *(uses the least amount of material) of the box?* *

(a) *Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.*

(b) *Write an expression for the total area.*

(c) *Use the given information to write an equation that relates the variables.*

(d) *Use part (c) to write the total area as a function of one variable.*

*__Answer:__ For x the length of the square base and y the height of the side, the minimal surface area occurs when $x \approx 6.69$in and $y \approx 3.35$in.

(e) *Finish solving the problem using Calculus.*

Question 73.

Suppose we have a piece of cardboard that is 10 inches by 15 inches. We need to remove squares of side length x from the four corners of the cardboard, and then fold up each newly formed flap to make an open-top box. Additionally, the box must be at least 1 inch deep, but no more than 3 inches deep. What is the maximum possible volume of a box that we can make? What is the minimum volume that we can make?

Question 74.

Consider a wire of length 4 feet. Suppose the wire is cut into two pieces and denote one of the pieces with length x. One piece is formed into a square, and the other is formed into a circle. What value of x will maximize the sum of the areas of the square and the circle? †

Question 75.

Find the area of the largest rectangle that can be inscribed in a semicircle of radius 10.

Question 76.

An oil refinery is located on the north bank of a straight river that is 2 km wide. A pipeline is to be constructed from the refinery to storage tanks located on the south bank of the river 10 km downstream of the refinery. The cost of laying pipe is \$10,000/km over land to a point P along the north bank and then \$40,000/km under the river to the tanks. To minimize the cost of pipeline, where should P be located?‡

Question 77.

A nurse takes a patient's pulse three times and measures 70 bpm, 80 bpm, and 120 bpm (bpm stands for "beats per minute").

(a) *To identify a more accurate reading of the patient's pulse, the nurse wants to minimize the function*

$$P(x) = (x-70)^2 + (x-80)^2 + (x-120)^2.$$

What value of x will be a minimum for P?

> ***Note:*** *This kind of function is similar to how one finds a "line of best fit" or regression line in statistics*

(b) *Suppose the nurse noticed the patient was nervous for the third reading. They decide to adjust $P(x)$ to give less weight to the third reading like so:*

$$P(x) = (x-70)^2 + (x-80)^2 + \frac{1}{2}(x-120)^2.$$

Now what value of x minimizes the function?§

---

†**Answer:** The maximum occurs when you just make a circle.

‡**Answer:** P should be located $10 - \frac{2}{\sqrt{15}}$km downriver from the refinery.

§**Answer:** For the first function, the minimum is $x = 90$bpm. For the weighted function, the minimum is $x = 74$bpm.

Question 78.

A retailer has been selling 1200 *tablet computers a week at* \$350 *each. The marketing department estimates that if the price is lowered by* \$10, *an additional* 80 *tablets will sell each week.*¶

(a) *What should the price be set at in order to maximize revenue?*

(b) *If the retailer's weekly cost function is* $C(x) = 35000 + 120x$ *what price should it choose in order to maximize its profit?*

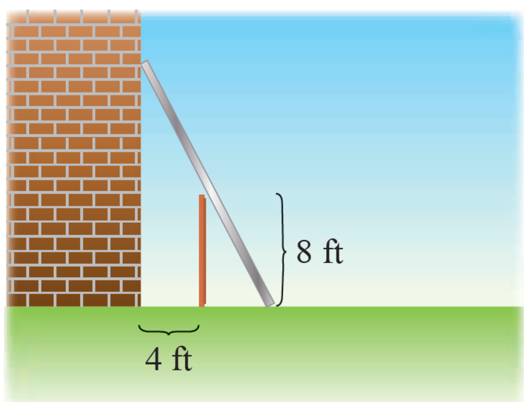
Question 79.

Two posts, on 12 *feet high and the other* 28 *feet high stand* 30 *feet apart. They are to be stayed by two wires, attached to a single stake, running from ground level to the top of each post. Where should the stake be placed to use the least amount of wire?*‖

Question 80.

Challenge Problem!

A fence 8 *ft tall runs parallel to a tall building at a distance of* 4 *ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?*



¶**Answer:** (a) \$250, (b)\$310

‖**Answer:** 9 ft from the 12 ft post.