Calculus & Analytical Geometry II

Lecture 32-34 Worksheet

Spring 2021 **Subhadip Chowdhury** **Math 112**

§A. Introduction to Power Series

In the last section, we saw that

$$\text{Series} \iff \text{Infinite sum of numbers}$$

We now extend this idea by introducing variables, that is, we ask what happens when we add up an infinite number of functions of x? In particular, we are going to add up terms that involve **powers** of x and write it as an infinite sum. This gives us something called a power series.

$$\text{Power Series} \iff \text{Infinite sum of polynomials}$$

As a result, a power series can be used to represent common functions and also to define new functions. We will see that functions like $e^x, \sin(x)$, and $\ln(x+1)$ can all be realized as some power series.

Example A.1

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Note: One might ask what is the benefit of writing e^x as a series? Since e is an irrational number, there is no way to get the exact numerical vale of, say, e^2, by multiplying e twice. Instead to get the value of e^2, your calculator (or a computer) evaluates the sum on the right hand side when $x = 2$, up to finitely many terms, depending on the precision required.

$$e^2 = \sum_{n=0}^{\infty} \frac{2^n}{n!} = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \ldots$$

This is much easier to do since a finite partial sum on the right hand side involves only evaluating some finite powers of 2.

So our goal in this section is twofold:

- Understand when an infinite sum of polynomials makes sense. E.g. given an expression such as

  $$1 + 2x + 3x^2 + 4x^3 + \ldots$$

  we seek the values of x for which the expression **converges**.

- Try to find constants c_0, c_1, c_2, etc. so that we can express a function $f(x)$ as a power series

  $$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

  This way we can approximate the value of $f(x)$ numerically at some point $x = a$ up to arbitrary precision.

§B. Definition and Examples

Let's start with an example of power series you are already familiar with. Consider a **geometric series** of the form

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + cr^4 + \dots$$

Recall that we know how to get the sum here:

$$\sum_{n=0}^{\infty} cr^n = \begin{cases} \dfrac{c}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$$

So if we consider a function $F(x) = \sum_{n=0}^{\infty} cx^{n-1}$, we can say that the domain of $F(x)$ is ______________________.

Let's assume $c = 1$. Then we can write

$$F(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots = \frac{1}{1-x} \text{ for } |x| < 1$$

How do we interpret above identity? We are saying that we can *approximate* the numerical value of the function $\frac{1}{1-x}$ on the interval $(-1, 1)$ up to arbitrary precision by evaluating truncated partial sums that are polynomials of larger and larger degree. In the limit, the infinite polynomial is *equal to* the function on this interval. See

https://www.desmos.com/calculator/wkwkrwtdsk

for some graphical evidence.

Question 1.

□

Once we know that a given function can be written via a power series, we can create many more examples via simple substitutions. Find the domain of the following power series, and find the function it converges to on that domain.

(i) $\sum_{n=0}^{\infty} x^{2n}$

(ii) $\sum_{n=0}^{\infty} 2^n x^n$

You can see the examples here:

https://www.desmos.com/calculator/0qyqwojain

Definition B.2

A **power series** is a **function** of the form

$$F(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots + c_n(x-a)^n + \dots$$

We call a the **center** of the series. A power series centered at $a = 0$ is of the form:

$$F(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots + c_n x^n + \ldots$$

The **domain** of a power series $F(x)$ is all values of x for which the series converges.

Note: In the definition of a power series, x is a variable, a is a constant, and the c_n's are called coefficients.

Example B.3

Let's take a look at a power series that's not a geometric series. Consider

$$F(x) = \sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$$

This power series is centered at $x =$ ________. We wish to find the domain of $F(x)$, i.e. the values of x for which the series converges.

We will do this in two steps.

Step 1. Use the **Ratio Test**. Let $a_n = \dfrac{(x-3)^n}{n}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad} \longrightarrow |x-3| \text{ as } n \to \infty$$

By the Ratio Test, the given series is absolutely convergent, and therefore convergent, when $|x-3| < 1$ and divergent when $|x-3| > 1$. Now

$$|x-3| < 1 \iff \underline{\qquad\qquad\qquad\qquad} \iff \underline{\qquad\qquad\qquad\qquad}$$

So the series converges when $2 < x < 4$ and diverges when $x < 2$ or $x > 4$.

Note that the Ratio Test gives no information when $|x-3| = 1$ so we must consider $x = 2$ and $x = 4$ separately.

Step 2. **Check the endpoints** of the interval for convergence using series tests.

If we put $x = 4$ in the series, it becomes $\sum \frac{1}{n}$, the harmonic series, which is ____________.

If $x = 2$, the series is $\sum \frac{(-1)^n}{n}$, which __________________ by the Alternating Series Test.

Thus the domain of the given power series is ____________________________.

Question 2.

Find the values of x for which the series $\sum_{n=0}^{\infty} n! x^n$ converges.

Question 3.

Define $P(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$.

(a) *The first couple of partial sums are plotted in the above* `DESMOS` *link. Can you guess the function the series appears to be converging towards?*

(b) *Use the Ratio Test to determine the values of x for which $P(x)$ converges.*

§C. Radius and Interval of Convergence

For the power series that we have looked at so far, the set of values of x for which the series is convergent has always turned out to be an interval (a finite interval for the geometric series and the series in example 3), the infinite interval (in question 3), or a collapsed interval $[0,0] = \{0\}$ (in question 2). The following theorem, which we will not prove, says that this is true in general.

Theorem C.4

For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, *there are* ***only three possibilities****:*

(i) *The series converges only when $x = a$.*

(ii) *The series converges for all x.*

(iii) *There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.*

The number R in case (iii) is called the **radius of convergence** of the power series. By convention, we will say that the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii).

The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges. In case (i) the interval consists of just a single point $\{a\}$. In case (ii) the interval is $(-\infty, \infty)$.

in case (iii), note that the theorem doesn't say what happens when x is an endpoint of the interval, that is, $x = a \pm R$,. In fact, depending on the example, anything can happen - the series might converge at one or both endpoints or it might diverge at both endpoints. Thus in case (iii) there are four possibilities for the interval of convergence:

$$(a-R, a+R) \qquad (a-R, a+R] \qquad [a-R, a+R) \qquad [a-R, a+R]$$

The situation is illustrated in fig. 1.

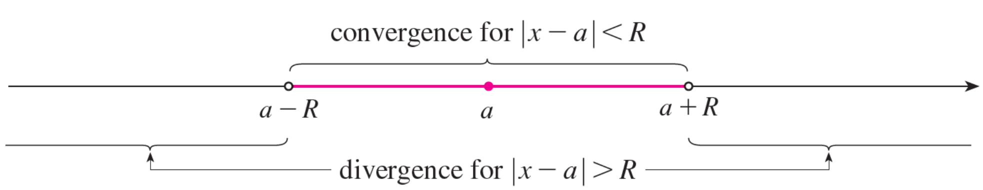

Figure 1

Here's a summary of the radius and interval of convergence for examples we have seen so far:

Series	Radius of Convergence	Interval of Convergence
$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
$\sum_{n=0}^{\infty} (2x)^n$	$R = \frac{1}{2}$	$(-1/2, 1/2)$
$\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
$\sum_{n=0}^{\infty} n! x^n$	$R = 0$	$\{0\}$
$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$R = \infty$	$(-\infty, \infty)$

Question 4.

□

Find the radius and interval of convergence of the following series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-3)^n}{\sqrt{n}}$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$.