MATH 111 - Calculus and Analytic Geometry I

Lecture 28 Worksheet

Fall 2020 | **Subhadip Chowdhury** | **Oct 21-23**

Title: Related Rates

Summary: If two quantities that are related, and are both changing as implicit functions of time, how are their rates of change related?

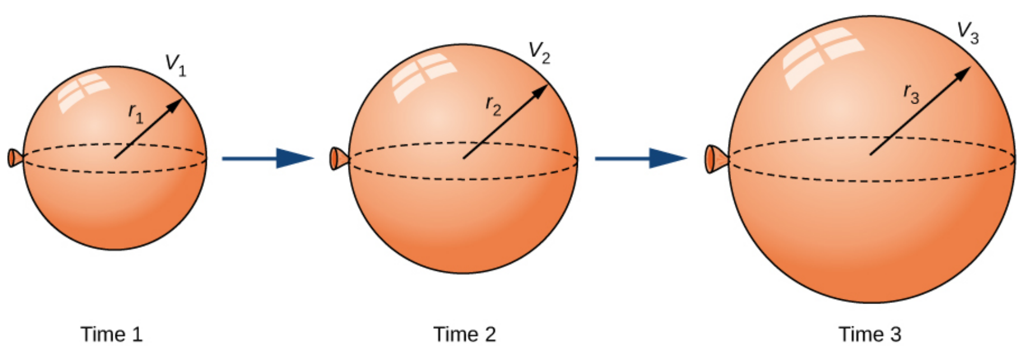
Related Reading: Chapter 4.1 from the textbook.

§A. Preview Example

In most of our applications of the derivative so far, we have been interested in the instantaneous rate at which one variable, say y, changes with respect to another, say x, leading us to compute and interpret $\dfrac{dy}{dx}$. We next consider situations where several variable quantities are related, but where each quantity is implicitly a function of time, which will be represented by the variable t. Through knowing how the quantities are related, we will be interested in determining how their respective rates of change with respect to time are related. We call these **related rates problems**.

Question 1.

Consider the following scenario. A spherical balloon is being inflated at a constant rate of **20** cubic inches per second. How fast is the radius of the balloon changing at the instant the balloon's diameter is **12** inches? Is the radius changing more rapidly when $d = 12$ or when $d = 16$? Why?



(a) Recall that the volume of a sphere of radius r is $\mathbf{V} = \frac{4}{3}\pi r^3$. Note that in the setting of this problem, both $\mathbf{V}$ and r are changing as time t changes, and thus both $\mathbf{V}$ and r may be viewed as implicit functions of t, with respective derivatives $\frac{d\mathbf{V}}{dt}$ and $\frac{dr}{dt}$. Differentiate both sides of the equation $\mathbf{V} = \frac{4}{3}\pi r^3$ with respect to t (using the chain rule on the right) to find a formula for $\frac{d\mathbf{V}}{dt}$ that depends on both r and $\frac{dr}{dt}$.

(b) At this point in the problem, by differentiating we have **related the rates** of change of $\mathbf{V}$ and r. Recall that we are given in the problem that the balloon is being inflated at a constant rate of **20** cubic inches per second. Is this rate the value of $\frac{dr}{dt}$ or $\frac{d\mathbf{V}}{dt}$? Why?

(c) From part (b), we know the value of $\frac{d\mathbf{V}}{dt}$ at every value of t. Next, observe that when the diameter of the balloon is **12**, we know the value of the radius. In the equation from part (a), substitute these values for the relevant quantities and solve for the remaining unknown quantity, which is $\frac{dr}{dt}$. How fast is the radius changing at the instant $d = 12$?

(d) How is the situation different when $d = 16$? When is the radius changing more rapidly, when $d = 12$ or when $d = 16$?

§B. The Problem Solving Strategy

Algorithm for Solving Related Rates Problems

Step 1. Identify the quantities in the problem that are changing and choose clearly defined variable names for them. Draw one or more figures that clearly represent the situation.

Step 2. State, in terms of the variables, all rates of change that are known or given and identify the rate(s) of change to be found.

Step 3. Find an equation that relates the variables whose rates of change are known to those variables whose rates of change are to be found.

Step 4. Using implicit differentiation, differentiate both sides of the equation found in step 3 with respect to t to relate the rates of change of the involved quantities.

Step 5. Substitute all known values into the equation from step 4, then solve for the unknown rate of change.

When solving a related-rates problem, it is crucial not to substitute known values too soon. For example, if the value for a changing quantity is substituted into an equation before both sides of the equation are differentiated, then that quantity will behave as a constant and its derivative will not appear in the new equation found in step 4.

§C. Practice Problems

Question 2.

Book Problem 4.1.3

Suppose $z^2 = x^2 + y^2$. Find $\dfrac{dz}{dt}$ at $(x, y) = (1, 3)$ if $\dfrac{dx}{dt} = 4$ and $\dfrac{dy}{dt} = 3$.

Question 3.

Book Problem 4.1.8

You and a friend are riding your bikes to a restaurant that you think is East; your friend thinks the restaurant is North. You both leave from the same point, with you riding at 16 km/hr east and your friend riding 12 km/hr north. After you traveled 4 km, at what rate is the distance between you changing?

Question 4.

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Imagine a rectangle with whose length x is increasing at a rate of 0.2 m/s and whose width y is decreasing at a rate of 0.1 m/s. How fast is the area of rectangle changing at the moment when $x = 3$ m and $y = 2$ m.

■ Question 5. **Book Problem 4.1.18**

The radius of a circle increases at a rate of $\mathbf{2}$ m/sec. Find the rate at which the area of the circle increases when the radius is $\mathbf{5}$ m.

■ Question 6. **Book Problem 4.1.29**

A cylindrical tank is leaking water. rate. The cylinder has a height of $\mathbf{2}$ m and a radius of $\mathbf{2}$ m. We notice that the rate at which water level is decreasing is $\mathbf{10}$ cm/min when the water level is $\mathbf{1}$ m. Find the rate at which the water is leaking out.

■ Question 7. **Book Problem 4.1.35**

Gravel is being unloaded from a truck and falls into a pile shaped like a cone at a rate of $\mathbf{10}$ ft^3/min. The radius of the cone base is three times the height of the cone. Find the rate at which the height of the gravel changes when the pile has a height of $\mathbf{5}$ ft.

■ Question 8. **Book Problem 4.1.10**

A $\mathbf{6}$ ft tall person walks away from a $\mathbf{10}$ ft lamppost at a constant rate of $\mathbf{3}$ ft/sec. What is the rate that the tip of the shadow moves away from the pole when the person is $\mathbf{10}$ ft away from the pole?

■ Question 9. **Book Example 4.3**

A rocket is launched so that it rises vertically. A camera is positioned $\mathbf{5000}$ ft from the launch pad. When the rocket is $\mathbf{1000}$ ft above the launch pad, its velocity is $\mathbf{600}$ ft/sec. Find the necessary rate of change of the camera's angle as a function of time so that it stays focused on the rocket.

■ Question 10. **Book Problem 4.1.37**

You are stationary on the ground and are watching a bird fly horizontally at a rate of $\mathbf{10}$ m/sec. The bird is located $\mathbf{40}$ m above your head. How fast does the angle of elevation change when the horizontal distance between you and the bird is $\mathbf{9}$ m?