Bowdoin College, Math 1800

Multivariable Calculus Lecture Notes

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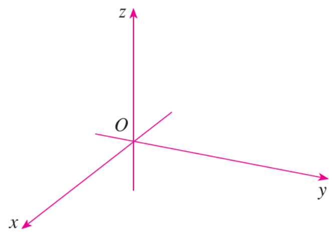
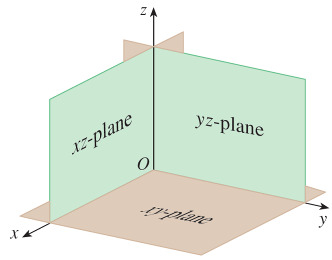
Contents

§1 | Three Dimensional Coordinate Geometry

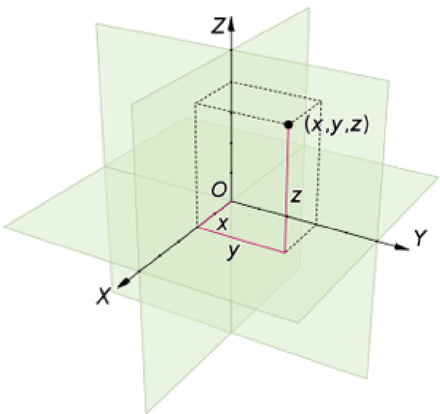
1.1 Coordinate Axes and Points in 3-space

The three coordinate axes in 3-space are drawn using a right-hand-thumb rule as follows. It is important that you understand how to draw the axes in different orientations.





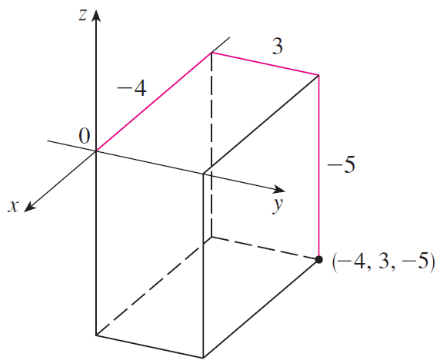
A point in three dimension has three coordinates (x, y, z) that denote how far in/out, left/right, up/down a point is from the origin.

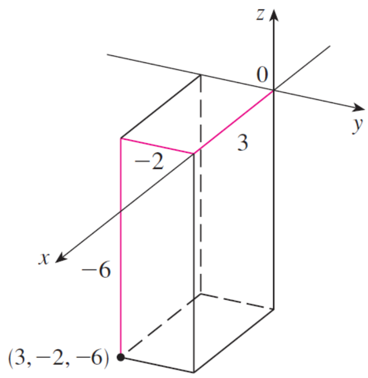


Question 1.1

Draw $P(-4, 3, -5)$ and $Q(3, -2, -6)$ on a graph.

Answer



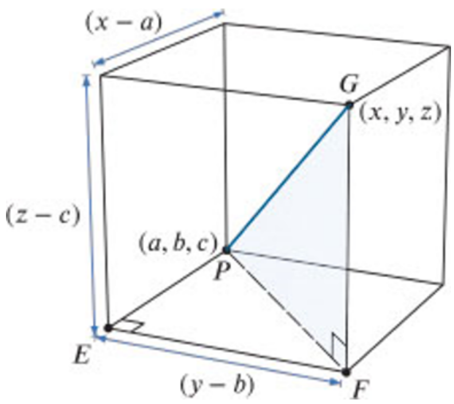


□

1.2 Distance between two points

The distance between two points (a, b, c) and (x, y, z) in space is given by

$$\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}.$$



Question 1.2

What is the distance from $(1, 2, 1)$ to $(-3, 1, 2)$?

1.2.1 Equation of a Sphere

We can define a sphere to be collection (locus) of points that are equidistant to a fixed point called the *center*. Suppose the center is at (a, b, c) and the radius is r. Then the distance formula tells us that the required points (x, y, z) satisfy

$$\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r \iff (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

1.3 Sets of points described using coordinates

Question 1.3

Assume the origin is at the bottom, left, front corner of the classroom. Describe the set of points with

- $x > 0, y > 0$ and $z \leq 0$
- $z = 3$
- $x = 0, y = 0$

Question 1.4

Describe/draw the set of points with $0 \leq x \leq 1, 0 \leq y \leq 1 - x$, and $z \leq 0$

Question 1.5

Find any points where the sphere $(x-1)^2 + (y+3)^2 + (z-2)^2 = 4$ intersects the y -axis.

Question 1.6

An equilateral triangle is standing vertically in 3-space with a vertex above the xy-plane and its two other vertices at $(7, 0, 0)$ and $(9, 0, 0)$. What are the coordinates of the third vertex?

Question 1.7

How would you describe the points making up the *solid* cube with sides of length 2 and centered at the origin?

§2 | Vectors in 3D

2.1 Definition

Vectors in two or three dimensions are quantities that have both *magnitude* and *direction*. Quantities that are not vectors are called *scalars*.

Question 2.8

Which of the following are vectors?

(a) The cost of a movie ticket.

(b) The volume of the Bowdoin polar bear.

(c) The weight of the Bowdoin polar bear.

(d) The number of students in our class.

(e) The velocity of a car.

(f) The speed of a car.

2.2 Graphical representation of vectors

The vector $\vec{v} = \overrightarrow{\mathbf{PQ}}$ can be drawn as an arrow with "tail" at the point P and "tip" at the point Q. It represents a vector whose magnitude is equal to the length of PQ and its direction is from Q towards P. This vector is defined entirely by its direction and length, and can be moved around the space.

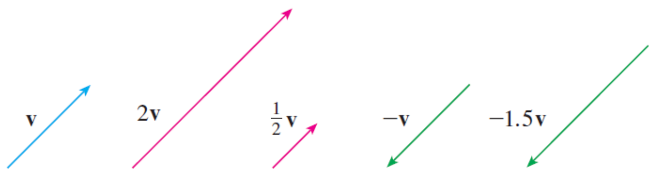
2.3 Vector Arithmetic

2.3.1 Addition

To add two vectors $\vec{u}$ and $\vec{v}$, draw them one after the other, tip to tail. The vector from the base of the pair to the final tip is the sum of the vectors.

2.3.2 Scalar Multiplication

If λ is a scalar i.e. a real number, then $\lambda\vec{v}$ is a vector whose magnitude is λ times the magnitude of $\vec{v}$. It has the same direction as $\vec{v}$ if $\lambda > 0$ and the opposite direction if $\lambda < 0$.



If $\lambda = 0$, we get the *Zero Vector*, a vector whose magnitude is zero and is *omnidirectional*!

2.3.3 Subtraction

$\vec{u} - \vec{v}$ is defined as $\vec{u} + (-1) \times \vec{v}$.

Question 2.9

Suppose the three sides of a triangle $\triangle ABC$ are denoted by vectors as $\vec{c} = \overrightarrow{\mathbf{AB}}$, $\vec{a} = \overrightarrow{\mathbf{BC}}$, and $\vec{b} = \overrightarrow{\mathbf{CA}}$. What is $\vec{a} + \vec{b} + \vec{c}$?

Question 2.10

Consider $\triangle ABC$ as above. Let D be the mid-point of BC and let $\vec{m} = \overrightarrow{\mathbf{AD}}$ be one of the medians of the triangle. Find $\vec{m}$ in terms of $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Digression

It is in fact also possible to find the angle bisector vector in terms of the sides. However that requires the law of sines in a triangle. In case you are interested, here is the precise problem. Suppose D is a point on BC such that $\angle BAD = \angle CAD$. Find $\overrightarrow{\mathbf{AD}}$ in terms of $\vec{a}$, $\vec{b}$ and $\vec{c}$.

2.4 Symbolic representation of vectors

A vector of magnitude 1 is called a *unit vector*. The unit vectors along X−, Y− and Z−axes are called $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ respectively.

Consider the vector $\vec{v} = \overrightarrow{\mathbf{PQ}}$ where P $= (x_1, y_1, z_1)$ and Q $= (x_2, y_2, z_2)$. Then it can be also written as

$$\vec{v} = \underbrace{(x_2 - x_1)}_{\Delta x}\hat{\imath} + \underbrace{(y_2 - y_1)}_{\Delta y}\hat{\jmath} + \underbrace{(z_2 - z_1)}_{\Delta z}\hat{k} = \langle \Delta x, \Delta y, \Delta z \rangle$$

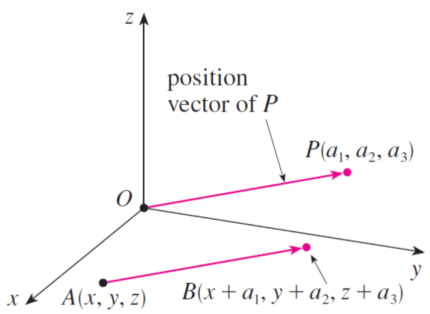
The projection of a vector on to the axes are called its *components*. Thus the X−component of the vector $\vec{v}$ above is $(\Delta x)\hat{\imath}$ etc. Clearly a vector is the sum of its components.

Addition or scalar multiplication of a vector can be done component-wise.

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle \qquad c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

2.4.1 Position vector - components vs coordinates

If we write $\vec{v} = \langle a_1, a_2, a_3 \rangle$ or equivalently $\vec{v} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$, we mean a vector with tail at $(0,0,0)$ and tip at P(a_1, a_2, a_3). In this way every point (a_1, a_2, a_3) corresponds to the unique vector $\langle a_1, a_2, a_3 \rangle$ called its *position vector*. The length of the components of the position vector are equal to the coordinates of the point.



2.5 Magnitude

The magnitude of a vector $\vec{v} = \langle a, b, c \rangle$ is denoted by $\|\vec{v}\|$, pronounced “norm" or $\vec{v}$, and is equal to $\sqrt{a^2 + b^2 + c^2}$. This is an easy consequence of Pythagoras theorem!

Question 2.11

How can you use $\|\vec{v}\|$ to create a unit vector in the same direction as $\vec{v}$?

2.6 Calculating components in 2D

Suppose a vector $\vec{v}$ makes an angle θ with the positive X–axis. Then we can the magnitude $\|\vec{v}\|$ to express a vector $\vec{v}$ in terms of trigonometric functions as follows:

$$\vec{v} = (\Delta x)\hat{i} + (\Delta y)\hat{j} = (\|\vec{v}\|\cos\theta)\hat{i} + (\|\vec{v}\|\sin\theta)\hat{j} = \|\vec{v}\|\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right)$$

Question 2.12

Suppose a three dimensional vector $\vec{v}$ makes angle α, β and γ with positive X−, Y− and Z−axis respectively. Then show that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

2.7 Word problems

Question 2.13

Bert and Ernie are trying to drag a large box on the ground. Bert pulls the box toward the north with a force of 30 N, while Ernie pulls the box toward the east with a force of 40 N. What is the resultant force on the box?

Question 2.14

An airplane at altitude is flying NE with airspeed 700 km/hr, with wind from the West at 60 km/hr. Use vectors to determine the resulting direction and ground speed of the plane.

§3 | Dot Product of Vectors

3.1 Definition

3.1.1 Algebraic

The *dot product* or scalar product of two vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is defined to be the sum of the product of the components.

$$\vec{u} \cdot \vec{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle := u_1 v_1 + u_2 v_2 + u_3 v_3$$

Question 3.15

What is $\langle 1, 2, 3 \rangle \cdot \langle 4, -5, 6 \rangle$? How about $\langle 1, 2 \rangle \cdot \langle 3, 4, 5 \rangle$?

3.1.2 Geometric

If the angle between $\vec{u}$ and $\vec{v}$ is θ then

$$\vec{u} \cdot \vec{v} := \|\vec{u}\| \|\vec{v}\| \cos\theta$$

Question 3.16

Does it matter whether the angle θ is calculated from $\vec{u}$ to $\vec{v}$ or in the other order?

Question 3.17

What is $\vec{u} \cdot \vec{v}$ if

(a) $\vec{u} \perp \vec{v}$?

(b) $\vec{u} = \vec{v}$?

(c) $\vec{\boldsymbol{u}} \parallel \vec{\boldsymbol{v}}$?

Theorem 3.1

Two vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\vec{u} \cdot \vec{v} = 0$.

3.2 Basic properties

From the definitions the following basic properties of the dot product are easy to prove. If $\vec{u}, \vec{v}$ and $\vec{w}$ are vectors of the same dimension and c is a scalar, then

1. $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
3. $(c\vec{v}) \cdot \vec{w} = c(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (c\vec{w})$

3.3 Angle between two vectors

We can use both definitions of dot product together to calculate angle between two given vectors.

Question 3.18

Find the angle between $\langle 1, 2, 1 \rangle$ and $\langle 1, -1, 1 \rangle$.

Question 3.19

In a molecule of Methane (CH_4) you have four hydrogen atoms bonded to a carbon atom. The four hydrogen atoms form the corner of a regular tetrahedron and the carbon atom lies in the center. What is the angle between any two of the C – H bonds?

Hint: Think of the four hydrogen atoms lying on four corners of a cube.

3.4 Vector projections

Take a vector $\vec{u}$ and resolve it into two components, one along another given vector $\vec{v}$ and another perpendicular to $\vec{v}$. We call these two components $\vec{u}_{\parallel}$ and $\vec{u}_{\perp}$ respectively. Thus

$$\vec{u} = \vec{u}_{\parallel} + \vec{u}_{\perp}$$

The component $\vec{u}_{\|}$ is called the *projection* of $\vec{u}$ on to $\vec{v}$, and is denoted $\text{Proj}_{\vec{v}}\,\vec{u}$.

Geometrically speaking, if we take a screen along $\vec{v}$ and shine a light perpendicular to it from above, the shadow cast by $\vec{u}$ would be $\text{Proj}_{\vec{v}}\,\vec{u}$.

Theorem 3.2: Projection Formula

$$\text{Proj}_{\vec{v}}\,\vec{u} = \vec{u}_{\|} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v}$$

Question 3.20

Prove above theorem.

HINT: Use the dot product formula for $\cos\theta$.

Question 3.21 **3 points**

The vertices of a triangle $\triangle ABC$ are $A = (4,3,2), B = (1,3,1)$, and $C = (-5,5,-2)$. Let D be the foot of the perpendicular from A to the side $\overline{BC}$. Find the vector $\overrightarrow{\mathbf{AD}}$.

HINT: Find $\overrightarrow{\mathbf{BD}}$ first.

§4 | Cross Product of Vectors

4.1 Definition

4.1.1 Geometric

The cross product of two vectors $\vec{u}$ and $\vec{v}$ is the *vector* $\vec{u} \times \vec{v}$ whose

- magnitude is equal to $\|\vec{u}\|\|\vec{v}\| \sin\theta$, where θ is the angle from $\vec{u}$ to $\vec{v}$ and
- direction is perpendicular to the both $\vec{u}$ and $\vec{v}$, determined by the right-hand-thumb rule.

Question 4.22

Does it matter whether the angle θ is calculated from $\vec{u}$ to $\vec{v}$ or in the other order?

Question 4.23

What is $\hat{i} \times \hat{j}$? What is $\hat{i} \times \hat{i}$? What is $\hat{j} \times \hat{i}$?

4.1.2 Algebraic

If $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$, then we define $\vec{u} \times \vec{v}$ to be

$$\langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle.$$

There is a handy way of remembering this definition: the cross product $\vec{u} \times \vec{v}$ is equal to the determinant

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{k}$$

The cross product is only defined for three-dimensional vectors.

4.2 Basic Properties

From the definitions the following basic properties of the cross product are easy to prove. If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors of the same dimension and c is a scalar, then

1. $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$
2. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
3. $(c\vec{v}) \times \vec{w} = c(\vec{v} \times \vec{w}) = \vec{v} \times (c\vec{w})$

Theorem 4.1

Two vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u} \times \vec{v} = 0$.

4.3 Cross product as area

From the geometric definition, it is easy to see that $\|\vec{u} \times \vec{v}\|$ is equal to the area of the parallelogram whose two adjacent sides are given by $\vec{u}$ and $\vec{v}$.

Question 4.24

Show this. Use the fact that the area of the parallelogram is twice the area of the triangle whose two sides are given by $\vec{u}$ and $\vec{v}$. Then use the area formula for a triangle.

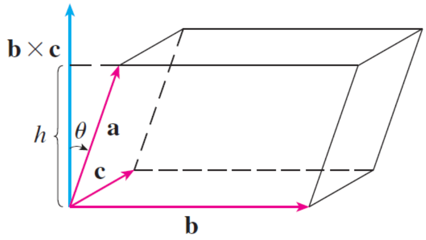
4.3.1 Volume of a parallelepiped - Coplanarity

Extending this geometric idea, we can show that the volume of a parallelepiped whose three adjacent sides are given by $\vec{a}$, $\vec{b}$, and $\vec{c}$ is equal to $\left|\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)\right|$.

Consequently, we have the following theorem

Theorem 4.2

Three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are coplanar iff $\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = 0$.



Digression

$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)$ is called the *scalar triple product* or *box product* of the three vectors, as also denoted by $[\vec{a}, \vec{b}, \vec{c}]$. An interesting observation here is that since the box product relates to the volume of the parallelepiped, we get

$$[\vec{a}, \vec{b}, \vec{c}] = [\vec{b}, \vec{c}, \vec{a}] = [\vec{c}, \vec{a}, \vec{b}]$$

So if our goal is to only calculate the volume, it doesn't matter which vectors we choose as $\vec{a}, \vec{b}$ and $\vec{c}$ as long they are the three adjacent sides.

Question 4.25 **3 points**

Suppose λ and μ are real numbers such that

- the three vectors

$$\begin{aligned}\vec{u} &= 2\hat{\imath} + 3\hat{\jmath} + \hat{k},\\ \vec{v} &= \hat{\imath} + \lambda\hat{\jmath} + \mu\hat{k},\\ \vec{w} &= 7\hat{\imath} + 3\hat{\jmath} + 2\hat{k}\end{aligned}$$

  are coplanar, and

- The vector $\vec{v}$ has magnitude $\sqrt{2}$.

Find all possible values of λ and μ.

Project 1 | Distances

In this project, we will use Dot product and Cross product of vectors to derive formula for calculating distances between points, lines and planes. We will use the notation $d(\cdot, \cdot)$ to denote distance.

4.1 Distance between two points

To begin with, the distance between two points P and Q with position vectors $\vec{\mathbf{P}}$ and $\vec{\mathbf{Q}}$ is simply given by

$$d(\mathrm{P}, \mathrm{Q}) = \|\vec{\mathbf{Q}} - \vec{\mathbf{P}}\| = \|\overrightarrow{\mathbf{PQ}}\|$$

where $\|\cdot\|$ denotes the magnitude of a vector.

Question 4.26

Find the distance between $(-5, 2, 4)$ and $(-2, 2, 0)$.

4.2 Distance from a point to a plane

The distance of a point P from a plane Σ is defined as the length of the perpendicular from P to Σ. Suppose the plane Σ passes through a point Q and has normal vector $\vec{\boldsymbol{n}}$.

Question 4.27

Explain using a picture why the distance from P to Σ is the length of the projection of $\overrightarrow{\mathbf{PQ}}$ onto $\vec{\boldsymbol{n}}$. Then

$$d(\mathrm{P}, \Sigma) = \frac{|\overrightarrow{\mathbf{PQ}} \cdot \vec{\boldsymbol{n}}|}{\|\vec{\boldsymbol{n}}\|}$$

Question 4.28

Find the distance of the point $(7,1,4)$ from the plane $2x+4y+5z=9$.

Question 4.29

Without the absolute sign in the numerator of the distance formula, your answer in question (2) would have been negative. What does the negative sign signify here?

4.3 Distance from a point to a line

The distance of a point P from a line $\mathcal{L}$ is defined as the length of the perpendicular from P to $\mathcal{L}$. Suppose the line $\mathcal{L}$ passes through a point Q and is parallel to a vector $\vec{u}$ (i.e. its parametric equation looks like $\vec{r}(t)=\vec{\mathrm{Q}}+t\vec{u}$).

Question 4.30

Use the definition of cross product to derive the following formula:

$$d(\mathrm{P},\mathcal{L})=\frac{\|\overrightarrow{\mathbf{PQ}}\times\vec{u}\|}{\|\vec{u}\|}$$

Question 4.31

Find the distance of the point $(2,3,1)$ from the straight line $\vec{r}(t)=(1,1,2)+t\langle 5,0,1\rangle$.

Question 4.32

What is the equation of the plane which contains the point P and the line $\mathcal{L}$?

4.4 Distance between two straight lines

Suppose the two straight lines $\mathcal{L}_1$ and $\mathcal{L}_2$ are given by

$$\vec{r}_1(t) = \vec{\mathbf{P}} + t\vec{u} \quad \text{and} \quad \vec{r}_2(t) = \vec{\mathbf{Q}} + t\vec{v}$$

i.e. the straight lines pass through P (and Q respectively) and is parallel to $\vec{u}$ (and $\vec{v}$ respectively).

Question 4.33

Draw a picture and explain using geometry why the distance between the two straight lines is given by

$$d(\mathcal{L}_1, \mathcal{L}_2) = \frac{|\overrightarrow{\mathbf{PQ}} \cdot (\vec{u} \times \vec{v})|}{\|\vec{u} \times \vec{v}\|}$$

Question 4.34

Find the distance between the lines $\vec{r}_1(t) = (2,1,4) + t\langle -1,1,0\rangle$ and $\vec{r}_2(t) = (-1,0,2) + t\langle 5,1,2\rangle$.

4.5 Distance between two planes

Before deriving the formula, observe that the distance between two planes is non-zero iff the two planes are parallel to each other, in which case they have the same normal vector $\vec{n} = \langle a,b,c\rangle$. Suppose the two planes Σ_1 and Σ_2 are given by

$$ax + by + cz = d \quad \text{and} \quad ax + by + cz = e$$

Question 4.35

Show that the distance formula is given by

$$d(\Sigma_1, \Sigma_2) = \frac{|d - e|}{\|\vec{n}\|}$$

Question 4.36

Find the distance between the planes $5x + 4y + 3z = 8$ and $5x + 4y + 3z = 1$.

Question 4.37

Find the distance between the planes $x + 3y - 2z = 2$ and $5x + 15y - 10z = 30$.

§5 | Lines and Planes

5.1 Lines in space

The equation of a line through a point (x_0, y_0, z_0) and parallel to the vector $\vec{u} = \langle a, b, c\rangle$ can be expressed in many ways:

- as parametric scalar equations:
$$x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc;$$
- as a parametric vector equation:
$$\vec{r}(t) = \vec{r}_0 + t\vec{u}, \quad \text{where } \vec{r} = \langle x, y, z\rangle \text{ and } \vec{r}_0 = \langle x_0, y_0, z_0\rangle$$
- or by symmetric equations:
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Observe that the symmetric equation form doesn't make sense in case one of a, b or c is zero.

Question 5.38

Let $\mathcal{L}$ be the line which passes through the points $(1, -2, 3)$ and $(4, -4, 6)$. Find its equation in all three forms.

5.2 Planes in space

The equation of a plane through the point (x_0, y_0, z_0) and perpendicular (or normal or orthogonal) to the vector $\vec{n} = \langle a, b, c\rangle$ also has many (equivalent) equations:

- $$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \quad \text{or} \quad \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$
  where again $\mathbf{r} = \langle x, y, z\rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$; or equivalently

- $$a(x-x_0)+b(y-y_0)+c(z-z_0)=0 \quad \text{or} \quad ax+by+cz=d$$
  where $d=ax_0+by_0+cz_0$ is a constant.

Question 5.39

Find an equation describing the plane which goes through the point $(1,3,5)$ and is perpendicular to the vector $\langle 2,1,-3\rangle$.

Question 5.40

Find an equation describing the plane which passes through the points $P(2,2,1), Q(3,1,0)$, and $R(0,-2,1)$.

5.3 Types of problems

One of the fundamental topic in Multivariable Calculus is to learn how to find equations of straight lines and planes in three dimensions using ideas from vectors (dot and cross product).The following lists an incomplete but fairly diverse types of problems that you should be able to solve using these ideas.

5.3.1 Finding a Plane

You should know how to find equation of a plane from the following data.

1. Plane through a given point and perpendicular to a given vector.
2. Plane through a given point and parallel to a given plane.
3. Plane containing a given line and parallel to a given plane.
4. Plane passing through three given points.
5. Plane with specified $x-$, $y-$ and $z-$intercepts.
6. Plane through a given point and containing a given straight line.
7. Plane through a given point and containing the line of intersection of two other given planes.
8. Plane through a given point and perpendicular to two other given planes.
9. Plane passing through two points and perpendicular to a given plane.

10. Plane containing the line of intersection of two other given planes and perpendicular to a given plane.

5.3.2 Finding a Line

You should know how to find equation of a straight line from the following data.

1. Line through two given points.
2. Line through one given point and in the direction of a given vector.
3. Line through one given point and parallel to a given straight line.
4. Line of intersection of two given planes.
5. Line through one given point and perpendicular to a given plane.
6. Line through a given point, that is perpendicular to a given straight line and intersects this second line.
7. Line through a given point, that is parallel to (i.e. lies in) a given plane and perpendicular to a given straight line.

5.4 Practice problems

Question 5.41

Below is a list of vectors and a list of properties. Match the two sets in such a way that each entry in left column matches a different entry in right column.

A. $\langle 3, -2, 8\rangle$	I. is parallel to the straight line $\frac{x-1}{2} = y - 3 = z$
B. $\langle 4, 2, 2\rangle$	II. is perpendicular to the plane $z - 2y - x = 3$
C. $\langle 3, 1, -1\rangle$	III. is perpendicular to both $\langle 2, 3, 0\rangle$ and $\langle -2, 5, 2\rangle$
D. $\langle 1, 2, -1\rangle$	IV. is parallel to the plane $x - y + 2z = 3$

Question 5.42

(a) Find parametric equations for the line through the points $(6, 1, 1)$ and $(9, 1, 4)$. Call this line L_1.

(b) Find parametric equations for the line through the points $(-4, 4, 0)$ and $(-6, 5, 1)$. Call this line L_2.

(c) Find parametric equations for the line through the points $(6,-1,-5)$ and $(2,1,-3)$. Call this line L_3.

(d) Verify that L_2 and L_3 are parallel. (Their direction vectors should be parallel.) Are they the same line? How could you tell?

(e) Do lines L_1 and L_2 intersect? If so, where?

(f) Find the intersection of L_1 with the plane given by the equation $2x + y + 3z = 7$.

(g) (3 points) Find the point on the plane $2x + y + 3z = 7$ which is closest to the origin.

(h) (3 points) Find the point on L_2 closest to the origin.

Question 5.43

Find a vector parallel to the intersection of the two planes $2x - 3y + 5z = 2$ and $4x + y - 3z = 7$.

Find the equation of the line of intersection.

Question 5.44

Find the distance of the point $P = (1,0,1)$ from the plane $x + y - z = 1$.

Question 5.45

Let L_1 be the line with parametric vector equation $\vec{r}_1(t) = \langle 7,1,3\rangle + t\langle 1,0,-1\rangle$ and L_2 be the line described parametrically by $x = 5, y = 1 + 3t, z = t$. How many planes are there that contain L_2 and are parallel to L_1? Find an equation describing one such plane.

Question 5.46

Find an equation for the plane that contains the line in the XY-pane where $y = 1$, and the line in the XZ-pane where $z = 2$.

§6 | Functions of Two Variables

6.1 Examples of functions of two variables

- Household lobster consumption is a function of income and the price of lobster.
- The density of cars along a highway is a function of position and time.
- The daily temperature across the United States is a function of latitude and longitude.
- Volume of a cylinder is a function of its radius and height.

6.2 Representations of two-variable functions

6.2.1 Numerical

The body mass index (BMI) is a value that attempts to quantify a person's body fat based on their height h and weight w.

		Weight w (lbs)				
		120	140	160	180	200
Height h (inches)	60	23.4	27.3	31.2	35.2	39.1
	63	21.3	24.8	28.3	31.9	35.4
	66	19.4	22.6	25.8	29.0	32.3
	69	17.7	20.7	23.6	26.6	29.5
	72	16.3	19.0	21.7	24.4	27.1
	75	15.0	17.5	20.0	22.5	25.0

Question 6.47

What is the BMI of a person who is 72 inches tall and weighs 160 lb?

6.2.2 Algebraic

A solid cylinder with closed ends has radius r and height h. Its volume V is given by

$$V = f(r,h)$$

Question 6.48

What does $V(10,h)$ mean? What does $V(r,10)$ mean?

Question 6.49

Can you give a formula for the function $f(r,h)$? What about the surface area $A = g(r,h)$?

6.2.3 Visual - Using Graphs

The *graph* of a function of two variables, f, is the set of all points (x,y,z) such that $z = f(x,y)$. In general, the graph of a function of two variables is a surface in 3–space.

Question 6.50

Sketch the graph of the function $f(x,y) = 6 - 3x - 2y$.

Question 6.51

Sketch the graph of $g(x,y) = \sqrt{9 - x^2 - y^2}$.

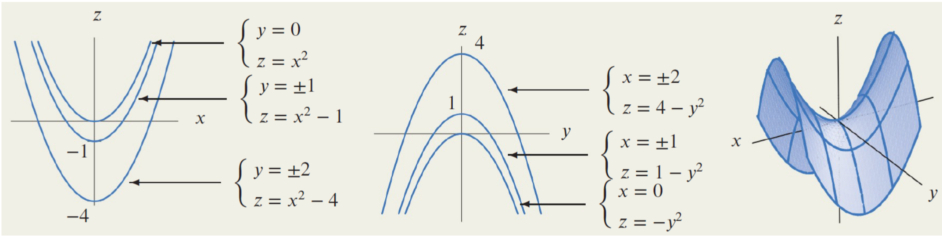
6.3 Analyzing Graphs using Cross-sections

One way to visualize surfaces that are three-dimensional graphs is to view pieces of them as two-dimensional graphs. If we intersect the graph of $z = f(x, y)$ with a plane (such as $x = k$ or $y = k$), we get a graph in a two-dimensional plane (the kind we're used to). This is called a *cross-section* (or a trace).

Question 6.52

Describe the cross-sections of the function $g(x, y) = x^2 - y^2$ with y fixed and then with x fixed. Use these cross-sections to describe the shape of the graph of g.

Answer



□

6.4 Group Problems

Question 6.53

Here are six functions whose graphs we ought to be able to visualize in space:

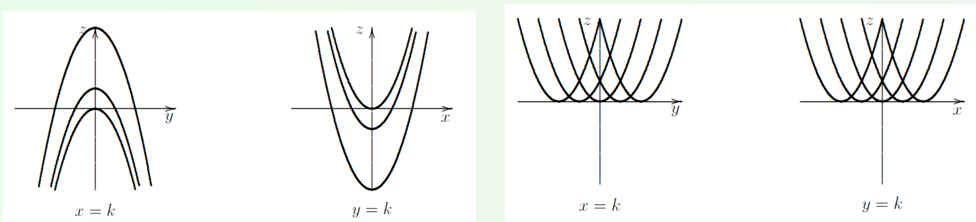
1. $z = f(x,y) = 6 - 3x - 2y$
2. $z = f(x,y) = x^2 + y^2$
3. $z = f(x,y) = x^2 - y^2$
4. $z = f(x,y) = x^2 + y + 1$
5. $z = f(x,y) = (x-y)^2$
6. $z = f(x,y) = \frac{1}{1+x^2+y^2}$

Here are cross-sections for two of the surfaces above. Your job is to:

(a) identify which graph these cross-sections belong with,

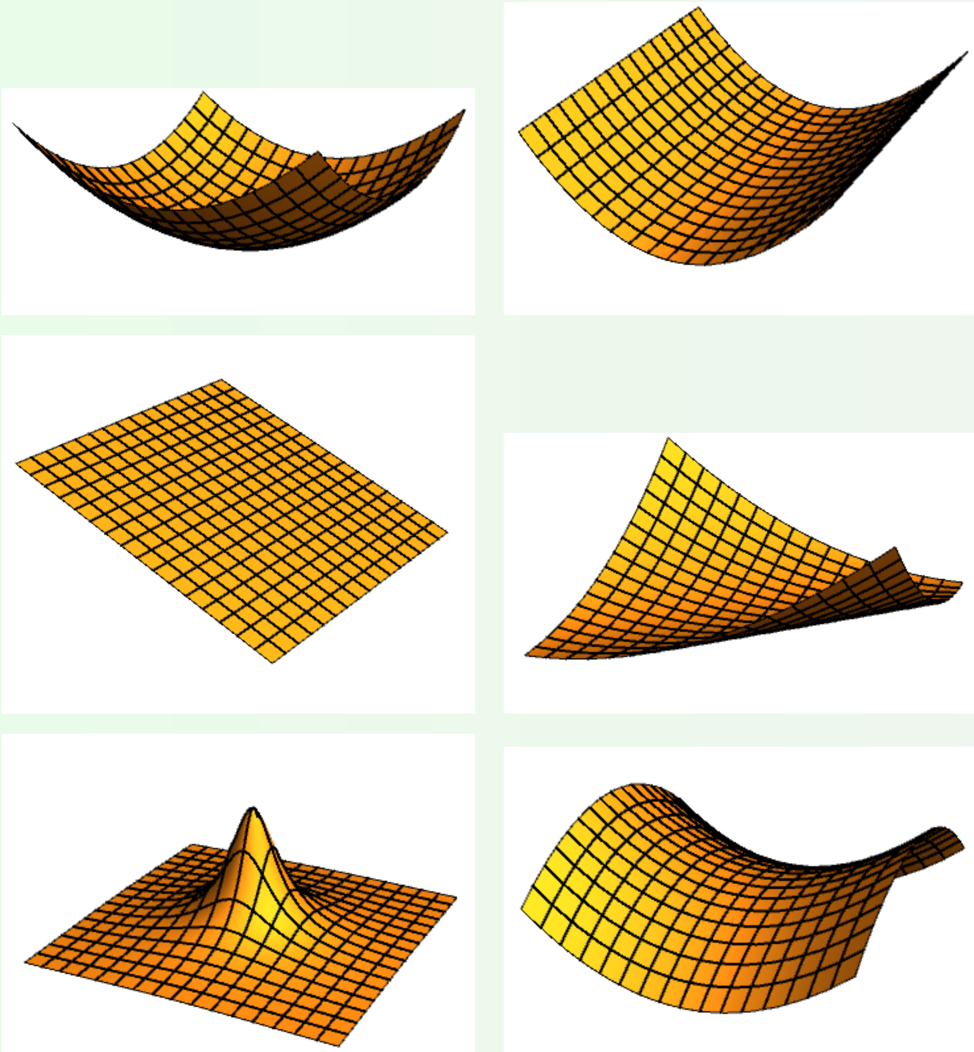
(b) graph cross-sections of the remaining graphs of surfaces, and

(c) try to visualize the original surface.



Question 6.54

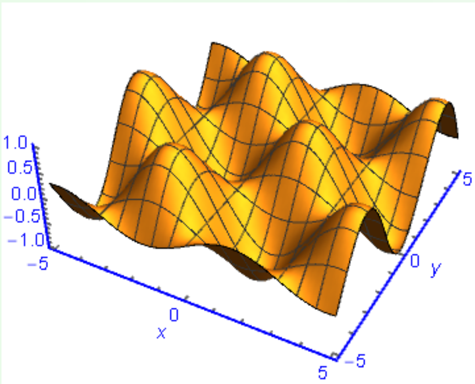
Now that you have a good idea of what each of these graphs look like, you should have no problem identifying which of the following (axes-less) graphs go with each equation for the previous page. Your reasoning should involve the cross-sections you drew as well.

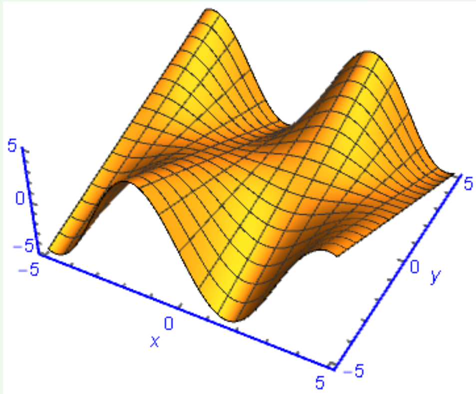


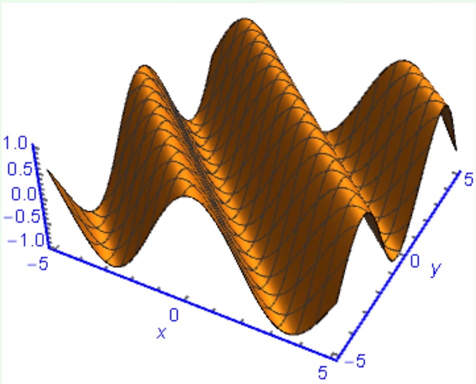
Question 6.55

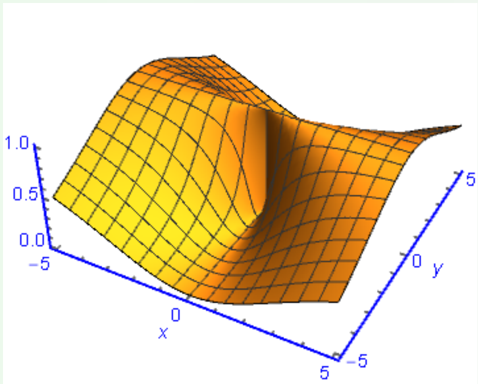
3 Match each function with its graph.

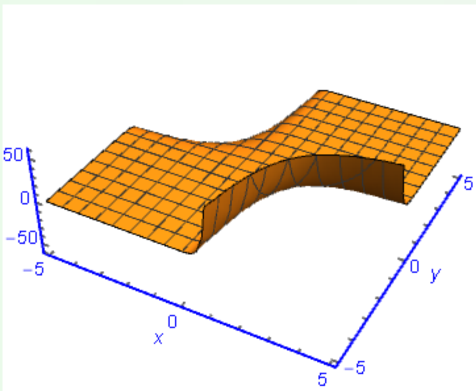
1. $f(x,y) = \frac{x^2}{x^2+y^2}$
2. $f(x,y) = y \sin x$
3. $f(x,y) = \sin(x+y)$
4. $f(x,y) = \sin x \cos y$
5. $f(x,y) = xe^{-xy}$











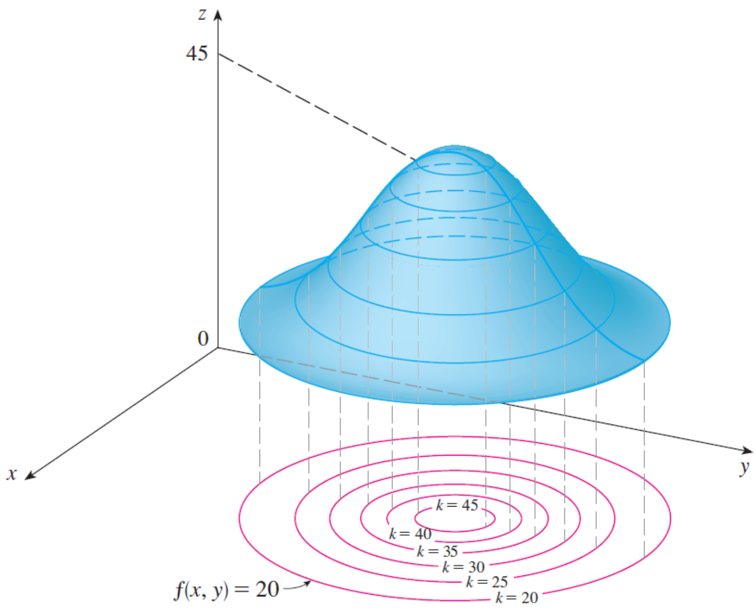
§7 | Contour Plots - Level Curves and Level Surfaces

7.1 Level Curves

7.1.1 Definition

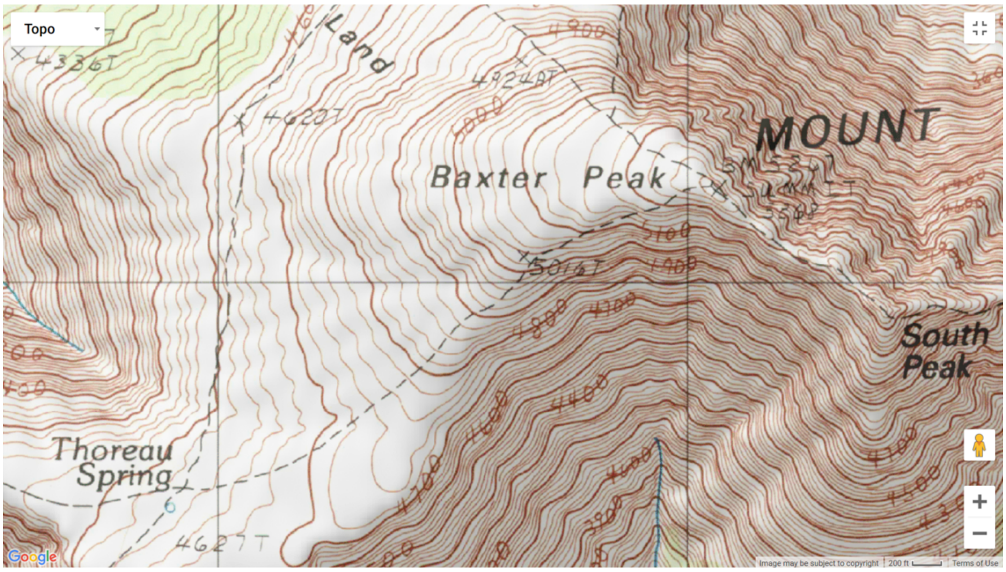
The *level curves* of a function f of two variables are the curves with equations $f(x,y) = c$, where c is a constant (in the range of f).

These are essentially the z–cross-sections of the graph of $f(x,y)$. The collection of all the level curves is called a *contour plot*.



7.1.2 Basic Facts

- The level curve for fixed output c is called the c -level of $f(x,y)$.
- unless otherwise indicated, contours are drawn at regular z–increments.
- We can build contour plots by hand following the same procedure we used for x and y cross-sections, but now fixing the z-values and plotting in the xy-plane.

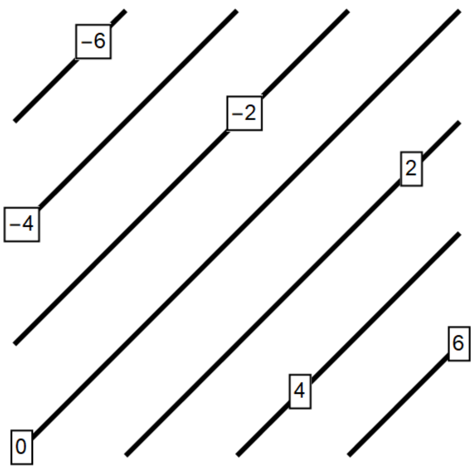


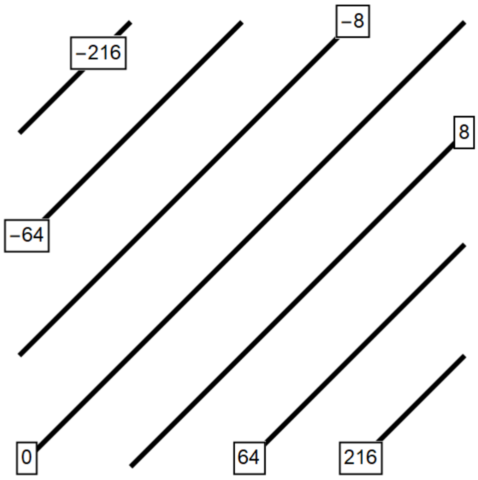
A height contour plot of Mount Katahdin

- Two different level curves cannot cross.

7.2 Practice Problems

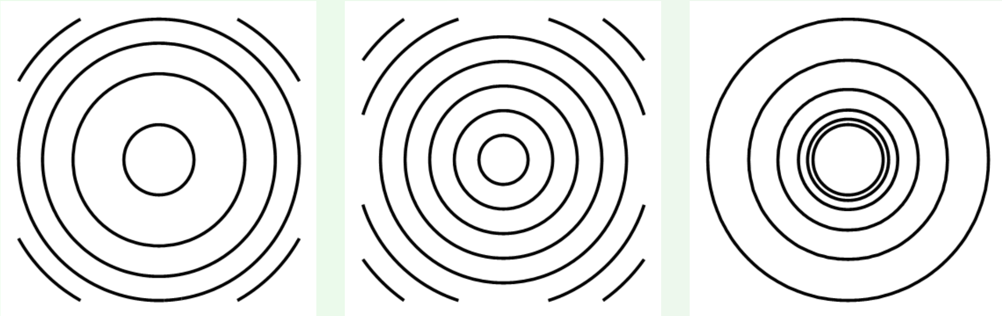
Consider two functions $f(x,y) = x - y$ and $g(x,y) = (x-y)^3$. Observe the difference in the contour plots for each of the graphs below.





Question 7.56

Assuming regular z–increments, describe what the corresponding graphs might look like.



7.3 Functions of three variables - Level surface

It's very difficult to visualize a function f of three variables by its graph, since that would lie in a four-dimensional space. However, we do gain some insight into f by examining its *level surfaces*, which are the surfaces with equations $f(x,y,z) = c$, where c is a constant. If the point (x,y,z) moves along a level surface, the value of $f(x,y,z)$ remains fixed.

7.3.1 Level surfaces of $f(x,y,z) = x^2 + y^2 + z^2$

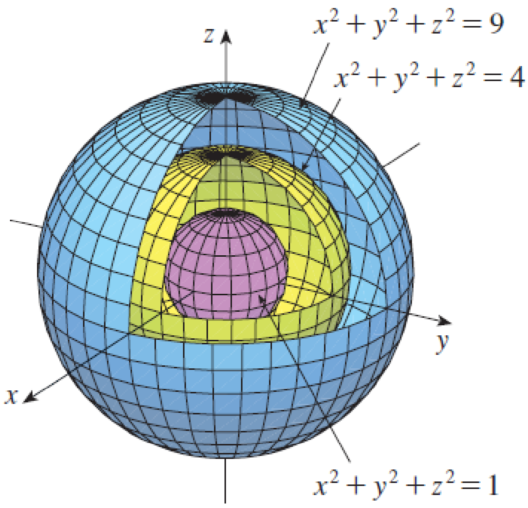
The level surfaces are $x^2 + y^2 + z^2 = k$, where $k \geq 0$. These form a family of concentric spheres with radius $\sqrt{k}$. Thus, as (x,y,z) varies over any sphere with center O the value of $f(x,y,z)$ remains fixed.

Question 7.57 **3 points**

What do the level surfaces of $f(x,y,z) = x^2 + y^2 - z^2$ look like? Finish project 2 to find out.

Digression

Functions of any number of variables can be considered. A function of n variables is a rule that assigns a number $z = f(x_1, x_2, \ldots, x_n)$ to an n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers. We denote by $\mathbb{R}^n$ the set of all such n-tuples.



Recall that there is a one-to-one correspondence between points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ and their position vectors $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$. So we have three ways of looking at a function f defined on a subset of $\mathbb{R}^n$:

1. As a function of n real variables $x_1, x_2, \ldots, x_n$
2. As a function of a single point variable $(x_1, x_2, \ldots, x_n)$
3. As a function of a single vector variable $\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle$

We will see that all three points of view are useful.

Project 2 | Conic Sections and Quadric Surfaces

7.1 Conic Sections

A *conic section* (or simply conic) is a curve obtained as the intersection of the surface of a cone with a plane. The three types of conic sections are the *hyperbola*, the *parabola*, and the *ellipse*. The circle is type of ellipse, and is sometimes considered to be a fourth type of conic section.

A cone has two identically shaped parts called *nappes*. One nappe is what most people mean by "cone", and has the shape of a dunce hat. It can be thought of as the surface of revolution of a straight line around an axis.

- If the intersecting plane is parallel to the axis of revolution of the cone, then the conic section is a hyperbola.
- If the plane is parallel to the generating line, the conic section is a parabola.
- If the plane is perpendicular to the axis of revolution, the conic section is a circle.
- If the plane intersects one nappe at an angle to the axis (other than 90°), then the conic section is an ellipse.

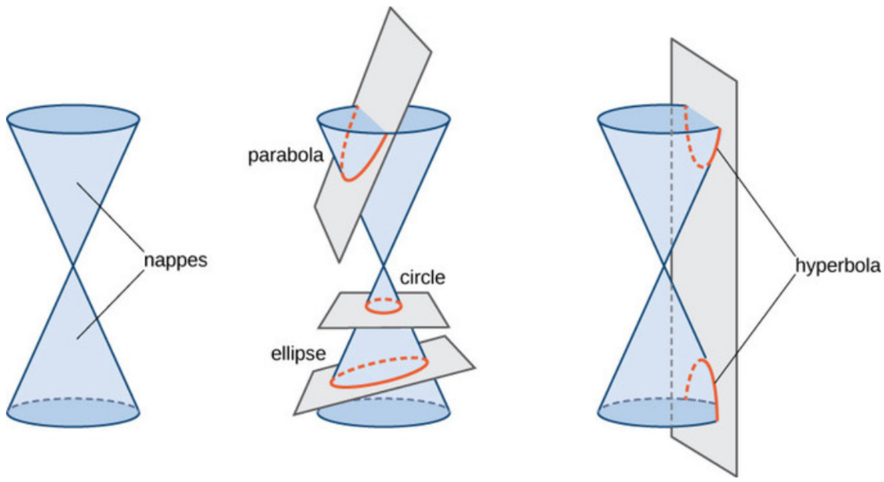


Figure 7.1: Conic Sections

Observe that when the intersecting plane is parallel to the axis of revolution and passes through the vertex of the cone, the conic section becomes a pair of straight lines (also known as a degenerate hyperbola).

7.1.1 Standard Equations in Cartesian Coordinates

The **Major Axis** is the chord between the two vertices: the longest chord of an ellipse, the shortest chord between the branches of a hyperbola. The **Minor Axis** is the shortest chord of an ellipse.

Conic Type	Standard Equation	Major Axis	Minor Axis
Circle	$x^2 + y^2 = r^2, \quad r \geq 0$	$2r$	$2r$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a, b > 0$	$2a$	$2b$
Parabola	$y^2 = 4ax$	N/A	N/A
	$x^2 = 4ay$	N/A	N/A
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad a, b > 0$	$2a$	N/A
	$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1, \quad a, b > 0$	$2b$	N/A
Pair of Straight Lines	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0, \quad a, b > 0$	N/A	N/A
	$\iff y = \pm\frac{b}{a}x$	N/A	N/A

- In each of the above cases the center of the conic is at the origin. If the curve is translated h units horizontally and k units vertically, its new equation is obtained by replacing x with $(x-h)$ and y with $(y-k)$.

Question 7.58

Use your precalculus memory or your favorite computer graphics software (e.g. Desmos) to draw a picture of each of the above conic sections. Clearly denote the center, radius, major axis, minor axis etc. and specify their lengths in terms of a, b etc. as applicable.

Question 7.59

For each of the following curves, find out what kind of conic section it is.

(a) $(x-3)^2 + (y-4)^2 = y^2$

(b) $(x-3)^2+(y-4)^2=2y^2$

(c) $(x-3)^2+(y-4)^2=\frac{y^2}{2}$

(d) $(x-3)^2+(y-4)^2=1$

Question 7.60

Find the value(s) of p for which does the curve $\frac{x^2}{9-p}+\frac{y^2}{p-3}=1$ looks like a

(a) Circle.

(b) Ellipse.

(c) Hyperbola.

7.2 Quadric Surface

Equations of surfaces in three dimension are of the form $f(x,y,z)=c$. One special case of interest is when $f(x,y,z)$ is a polynomial which is not linear and quadratic at most, containing terms involving x,y,z,x^2,y^2, and z^2 only. This kind of surface is called a *Quadric Surface*. Quadric surfaces are often used as example surfaces since they are relatively simple. There are nine different basic quadric surfaces listed below. A catalog of the equations and pictures of the quadric surfaces is available on page 691 of your textbook.

- **Cylinders:** A cylinder basically has no control over one of the variables. Take some sort of a curve in the plane, and draw a family of parallel lines so that each of the lines intersects the curve in a point. For example, a cylinder over the line $(0,t,t)$ would be all points of the form (s,t,t) for any values of s and t. This is a plane (a plane is a cylinder over a line!), and has the equation $y=z$. Some basic variations are
  - **elliptical cylinder:** A cylinder over an ellipse
  - **parabolic cylinder:** A cylinder over a parabola
  - **hyperbolic cylinder:** A cylinder over a hyperbola
- **ellipsoid**, the three-dimensional analogue of the ellipse. A sphere is an uniform ellipsoid.
- **elliptic paraboloid**, a sort of cup or a bowl
- **hyperbolic paraboloid**, looks like a horse-saddle or a pringle

- **cone**, take a straight line intersecting the z–axis and consider its surface of revolution around the z–axis
- **hyperboloids:** In three dimensions there are two different analogs of hyperbolas. The word "sheet" is used in an antique, specialized sense with surfaces: it means one connected "piece" of a surface. So a hyperboloid with one sheet is a surface with one (connected) piece, and a hyperboloid with two sheets is a surface with two (connected) pieces.
  - **hyperboloid of one sheet**, obtained by revolving a hyperbola around its minor axis. The surface is connected but there is a hole in it.
  - **hyperboloid of two sheet**, obtained by revolving a hyperbola around its major axis. This surface has two pieces.

In each case, note the direction of the axes relative to the surfaces and how the corresponding variables show up in the equation. For example, $z = 2y^2 - x^2$ is a hyperbolic paraboloid that goes downward in the x–axis direction and upwards in y–axis direction. The equation $y = 2x^2 - z^2$ is also a hyperbolic paraboloid that goes downward in the z–axis direction and upwards in x–axis direction. Similarly, $y = x^2 + z^2$ is an elliptical paraboloid that opens in the y–axis direction.

Question 7.61

For each of the following quadric surfaces, describe what kind of conic sections are obtained by taking its cross-sections parallel to the YZ–plane, XZ–plane and XY–plane. Then use the catalog to pick the term from the list above which seems to most accurately describe the surface and draw a rough picture of the surface (mark the origin and axis ticks) without using any graphing tool.

(a) $\frac{x^2}{9} - \frac{y^2}{16} = z$.

- Intersection with $x = k$:
- Intersection with $y = k$:
- Intersection with $z = k$:

(b) $\frac{x^2}{4} + \frac{y^2}{25} + \frac{z^2}{9} = 1$.

- Intersection with $x = k$:
- Intersection with $y = k$:
- Intersection with $z = k$:

(c) $\frac{x^2}{4} + \frac{y^2}{9} = \frac{z}{2}$.

- Intersection with $x = k$:
- Intersection with $y = k$:
- Intersection with $z = k$:

(d) $\frac{z^2}{4} - x^2 - \frac{y^2}{4} = 1$.

- Intersection with $x = k$:
- Intersection with $y = k$:
- Intersection with $z = k$:

(e) $x^2 + \frac{y^2}{9} = \frac{z^2}{16}$.

- Intersection with $x = k$:
- Intersection with $y = k$:
- Intersection with $z = k$:

(f) $\frac{x^2}{9} + y^2 - \frac{z^2}{16} = 1$.

- Intersection with $x = k$:
- Intersection with $y = k$:
- Intersection with $z = k$:

Question 7.62

Identify and sketch the following surfaces.

(a) $9y^2 + 4z^2 = 36$

(b) $y^2 + 2y + z^2 = x^2$

(c) $4x^2 - y^2 + z^2 + 9 = 0$

Question 7.63

(A non-basic Quadric Surface) Google Geogebra 3D calculator. Use the website to plot the surface

$$x^2 - 17z^2 - 2y^2 - 2xz - 12yz - 1 = 0$$

Which of the nine basic quadric surface does this resemble most closely? Can you explain how the equation of the surface might tell us what kind of surface it is, without using a graphing software?

[HINT: Complete the squares.]

§8 | Linear Functions

8.1 Definition

A function of two variables of the form

$$f(x,y) = mx + ny + d$$

where m, n and d are fixed constants is called a linear functions.

Recall that the equation $z = mx + ny + d$ represents a plane in three-dimensions. Thus clearly, the graph of a linear function looks like a plane. The constant m and n respectively represent the slope of the graph in x–direction and y–direction.

8.2 Graphical Representation

What does the contour plot of a linear function look like? If we set $f(x,y) = c$, we can rewrite the equation as

$$y = -\frac{m}{n}x + \frac{c-d}{n}$$

which is a line with slope $-\frac{m}{n}$. So no matter what level c we choose, the lines remain parllel. Thus the contour plot of a linear function is a set of evenly-spaced parallel lines (assuming regular c–increments).

8.3 Practice Problems

Question 8.64

Consider the function $z = f(x,y) = 8x - 4y + 2$.

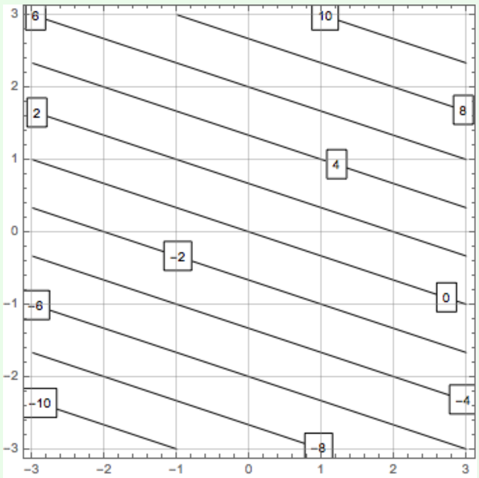
1. Sketch the contour plot for the graph with z-increment value of 1.
2. a) Starting at any point (x,y), what is the slope of the surface in the x–direction?

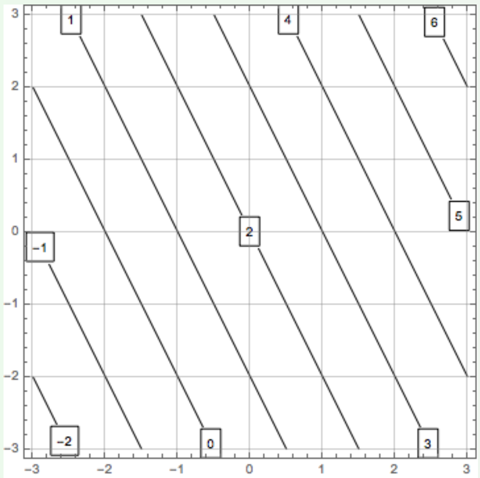
   b) What is the slope in y–direction?

   c) What is the slope along the line $x = y$?

Question 8.65

Find the linear functions whose contour plots are shown below.





Question 8.66

Fill in the blank with "certainly", "possibly", or "certainly not": If $f(x, y)$ is a linear function, then the graph of f is ________________ parallel to the xz-plane.

§9 | Parametrized Curves

9.1 Definition

We can think of any curve in 3–space (or plane) as a *vector valued function* of a real variable t. Then we can describe it either as a collection of points

$$(x(t), y(t), z(t))$$

for t in some interval (possibly infinite) or as the trace of the heads of the position vectors (which start at the origin)

$$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$$

Example 1 (Circle)**.** A circle in the xy-plane can be described as the set of pairs $(x(t), y(t))$ generated by the *parameterization*

$$x(t) = \cos(t) \text{ and } y(t) = \sin(t) \quad \text{for } 0 \le t \le 2\pi$$

associated with the position vectors

$$\vec{r}(t) = \cos(t)\hat{\imath} + \sin(t)\hat{\jmath} \quad \text{for } 0 \le t \le 2\pi$$

where the parameter t is the angle from the x–axis.

When curves are generated in this manner, they are called parametrized curves. Notice that parameterization carries more information than simply the final curve shape in the xy-plane. In particular, the example above indicates that the curve shape (circle) in the xy-plane is traced exactly once around, and that this curve shape is traced counterclockwise as t increases through its range.

9.2 Different ways of parametrizing the same curve

We can parametrize the same curve in different ways and interpret each parametrization as the motion of a particle with the parameter t being time.

Question 9.67

Explain why all of the parametrized curves below should look like a circle. In each of the parametrization, determine how many times around, and in which direction (clockwise or counterclockwise) the curve is traced.

- $x(t) = \cos(2t) \quad \text{and} \quad y(t) = \sin(2t) \quad \text{for} \quad 0 \le t \le 2\pi$
- $x(t) = \cos(t^2) \quad \text{and} \quad y(t) = \sin(t^2) \quad \text{for} \quad 0 \le t \le 2\pi$
- $x(t) = \cos(t) \quad \text{and} \quad y(t) = -\sin(t) \quad \text{for} \quad 0 \le t \le 2\pi$
- $x(t) = -\cos(t) \quad \text{and} \quad y(t) = \sin(t) \quad \text{for} \quad 0 \le t \le 2\pi$
- $x(t) = -\cos(t) \quad \text{and} \quad y(t) = -\sin(t) \quad \text{for} \quad 0 \le t \le 2\pi$

Question 9.68

Recall that $\vec{r}(t) = \vec{r}_0 + t\vec{v}$ (for $t \in \mathbb{R}$) gives the parametrization of a straight line passing through $\vec{r}_0$ and parallel to $\vec{v}$. What kind of curve are the following:

- $\vec{r}(t) = \vec{r}_0 + t^2\vec{v}$ for $t \in \mathbb{R}$
- $\vec{r}(t) = \vec{r}_0 + t\vec{v}$ for $a \le t \le b$

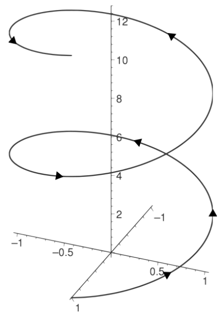
9.3 3D Curves

9.3.1 Helix

The parametric equation of a helix is given by

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$$

When looked on from above (from the positive z direction), this curve is simply a circle in the xy-plane. The $z = t$ component lifts the circle into the helix spinning above the circle in the plane. If we visualize this as a particle at the tip of the position vector $\vec{r}(t)$, then from above it looks like the particle is simply spinning in a circle. But we also know that $z = t$, so the particle is rising at a constant rate. Hence we get the picture below



Question 9.69

Match each vector-valued function to the curve it parametrizes.

(a) $\vec{r}(t) = \langle t\cos t, t\sin t, t\rangle$

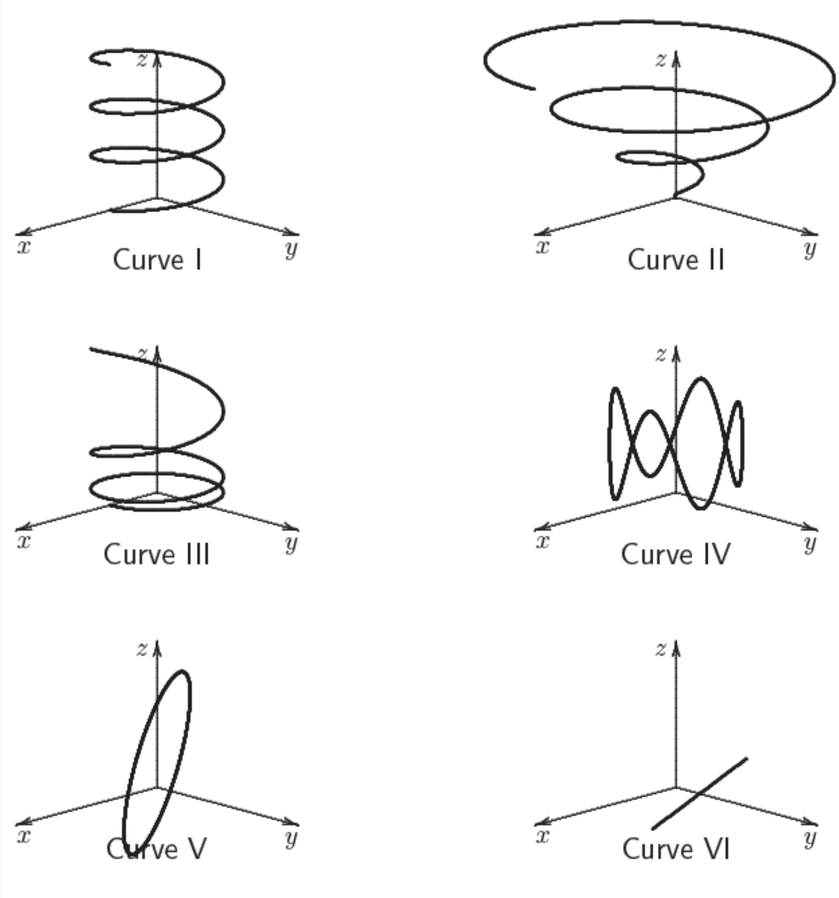
(b) $\vec{r}(t) = \left\langle \cos t, \sin t, t^3 \right\rangle$

(c) $\vec{r}(t) = \left\langle \cos(t^3), \sin(t^3), t^3 \right\rangle$

(d) $\vec{r}(u) = \langle \cos u, \sin u, 1 + \sin 4u\rangle$

(e) $\vec{r}(u) = \langle \cos u, \sin u, 1 + 4\sin u\rangle$

(f) $\vec{r}(t) = \langle 2\cos t, 1 + 4\cos t, 3\cos t\rangle$



9.4 Parametrization from equation of curve

Example 2. A parametrization of the parabola $x = 1 - y^2$ in xy- plane can be given by

$$y(t) = t, \quad x(t) = 1 - t^2, \quad t \in \mathbb{R}$$

Question 9.70

The surfaces $z = \sin(x - y)$ and $y = 2x$ intersect in a curve. Find a parameterization of the curve.

Question 9.71

The surfaces $x^2 + \frac{y^2}{4} = 1$ and $z = \sin(x - y)$ intersect in a curve. Find a parameterization of the curve.

§10 | Motion, Velocity, Speed and Distance

Think of a parametrized curve as the trajectory of a point that is moving on the curve. At time t, its position vector is given by $\vec{r}(t)$. Then considering the appropriate vector difference quotients, we can build a concept of velocity vector of a parametrized curve.

Question 10.72

A particle travels along the line $x = 1 + t, y = 5 + 2t, z = -7 + t$. *When* and *where* does the particle hit the plane $x + y + z = 1$?

10.1 Velocity and Acceleration

The idea is to consider the difference vector $\vec{r}(t+\Delta t) - \vec{r}(t)$ between the position vector $\vec{r}(t+\Delta t)$, a little time Δt beyond t, and the position vector $\vec{r}(t)$ at t. By taking $\Delta t \to 0$, the instantaneous *velocity vector* at time t is given by

$$\begin{aligned}
\vec{r}\,'(t) &= \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \left(\frac{x(t+\Delta t) - x(t)}{\Delta t} \right) \hat{i} + \lim_{\Delta t \to 0} \left(\frac{y(t+\Delta t) - y(t)}{\Delta t} \right) \hat{j} + \lim_{\Delta t \to 0} \left(\frac{z(t+\Delta t) - z(t)}{\Delta t} \right) \hat{j} \\
&= x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}
\end{aligned}$$

Similarly *acceleration vector* is given by

$$\vec{r}\,''(t) = x''(t)\hat{i} + y''(t)\hat{j} + z''(t)\hat{k}$$

Question 10.73

A fly is sitting on the wall at the point $(0, 1, 3)$. At time $t = 0$, he starts flying; his velocity at time t is given by $\vec{v}(t) = \left\langle \cos 2t, e^t, \sin t \right\rangle$. Find the fly's location at time t.

10.2 Distance - Length of the curve

The magnitude $\|\vec{r}'(t)\|$ of the velocity vector is the *speed*. Hence the value $\|\vec{r}'(t)\|$ approximates the distance traveled from t to $t+\Delta t$ along the curve parameterized by $\vec{r}(t)$. Adding such approximations from $t=a$ to $t=b$ approximates the length of the curve

$$\sum_{t=a}^{t=b} \left\|\vec{r}'(t)\right\| \Delta t \approx \text{ Length of curve } \vec{r}(t) \text{ from } t=a \text{ to } t=b$$

Taking $\Delta t \to 0$ gives a perfect approximation in terms of the following integral:

$$\int_{t=a}^{t=b} \left\|\vec{r}'(t)\right\| dt = \text{ Length of curve } \vec{r}(t) \text{ from } t=a \text{ to } t=b$$

Question 10.74

(a) What is the speed of an object on the circle parameterized by $\vec{r}(t) = \cos(t)\hat{\imath} + \sin(t)\hat{\jmath}$?

(b) Compute the length of this $\vec{r}(t)$ from $0 \le t \le 2\pi$.

(c) Use a dot product to find the orientation of the circle's velocity vectors $\vec{r}'(t)$ relative to its position vectors $\vec{r}(t) = \cos(t)\hat{\imath} + \sin(t)\hat{\jmath}$.

(d) Use a dot product to find the orientation of the circle's acceleration vectors $\vec{r}''(t)$ relative to its velocity vectors.

10.3 Equation of motion

10.3.1 Projectile motion

Suppose x measures horizontal distance in meters, and y measures distance above the ground in meters. At time $t=0$ in seconds, a projectile starts from a point h meters above the origin with speed v meters/sec at an angle θ to the horizontal. Its path is given by

$$x = (v\cos\theta)t, \quad y = h + (v\sin\theta)t - \frac{1}{2}gt^2$$

Question 10.75

(Book problem 17.2.33) Suppose a ball thrown off the top of a cliff travels along the path

$$x = 20t, \quad y = 2 + 25t - 4.9t^2$$

(a) When and where does the ball hit the ground?

(b) At what height above the ground does the ball start?

(c) What is the value of g, the acceleration due to gravity?

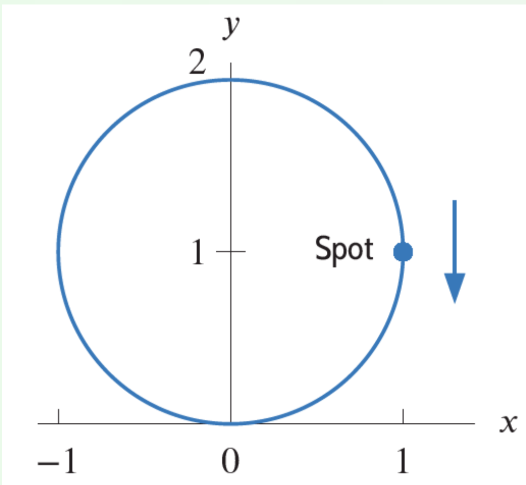
(d) What are the values of v and θ?

10.3.2 Cycloid

A *cycloid* is the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slipping.

Question 10.76

(Book problem 17.2.41) A wheel of radius 1 meter rests on the x-axis with its center on the y-axis. There is a spot on the rim at the point $(1, 1)$. At time $t = 0$ the wheel starts rolling on the x-axis in the direction shown at a rate of 1 radian per second.



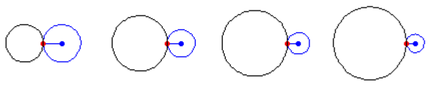
(a) Find parametric equations describing the motion of the center of the wheel.

(b) Find parametric equations describing the motion of the spot on the rim. Plot its path.

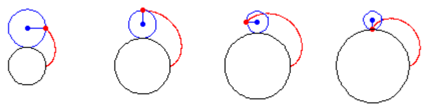
Project 3 | Epicycloids and the Rotary Engine

10.1 The Epicyloid

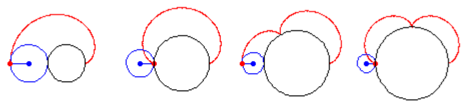
Consider a (black) circle of radius R with its center at the origin O. A (blue) circle of radius r rolls around the *outside* of the circle of radius R. See figure 10.1 for diagrams of different values of r. A (red) point P is located on the circumference of the rolling circle. The path traced out by P is called an epicycloid.



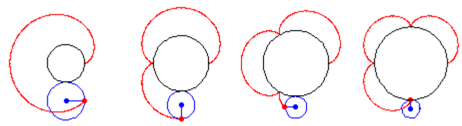
(a) $t = 0$



(b) $t = \pi/2$



(c) $t = \pi$



(d) $t = 3\pi/2$

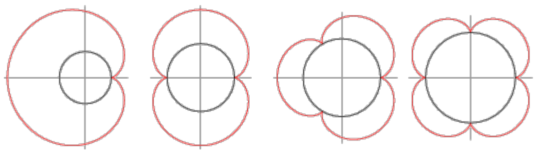


Figure 10.1

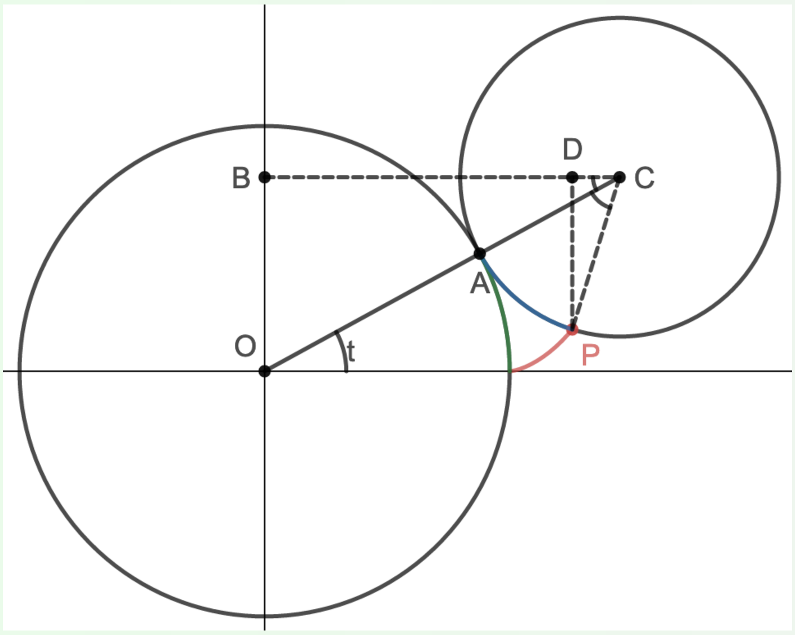
Assume that initially at time $t = 0$, the rolling circle sits to the right of the fixed circle and the point P is located at $(R, 0)$. After the rolling circle has moved a bit, draw a line from the center O of the large circle to the point of contact with the rolling circle and let t be the angle the line makes with the positive X−axis. If the location of P at this moment is given by $(x(t), y(t))$ (see figure 10.1), then the parametric equation of the *epicycloid* is given by

$$x(t) = (R + r)\cos t - r\cos\frac{(R + r)t}{r}$$

$$y(t) = (R + r)\sin t - r\sin\frac{(R + r)t}{r}$$

If the initial location of P and the rolling circle is chosen differently, you will get the same shape with a different orientation and the form of the parametric equations will change slightly. For example, if the sin and cos functions are interchanged (sin $\leftrightarrow$ cos), we get a vertically oriented epicycloid. If we replace t with $t + \phi$ we get an epicycloid that has been rotated by angle ϕ.

Question 10.77



Use the diagram above to analyze the geometry of the epicycloid. Write the detailed steps for the derivation of the parametric form of the epicycloid.

[**Hints:** The length of the blue arc and the green arc are equal, why? Use this to derive the angle $\angle$PCA in terms of t. The x–coordinate of P is given by BD which is equal to BC – CD. The y–coordinate is equal to OB – PD.]

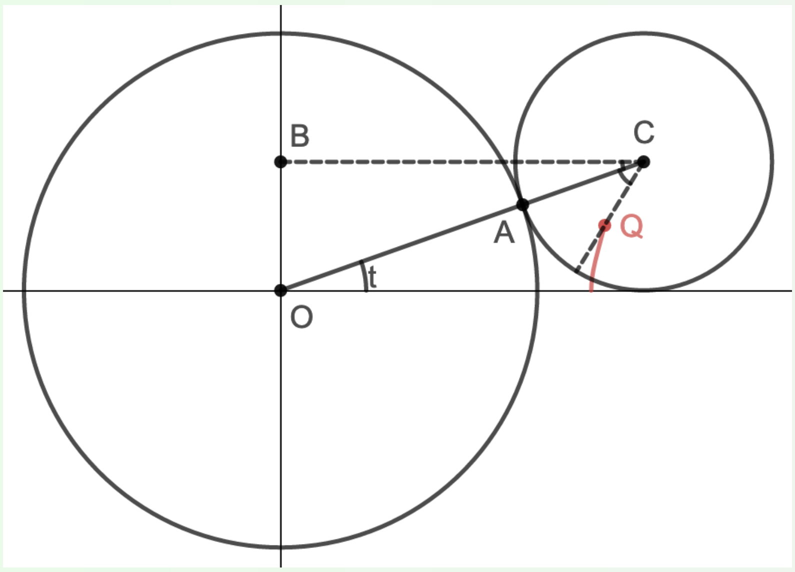
Question 10.78

In figure 10.1, you have picture of 4 epicycloids. You are told that each of the picture corresponds to a different value of $\frac{R}{r}$. Can you identify the values?

10.2 The Wankel Rotary Engine

We are especially interested in a special case from among the epicycloids as it pertains to the revolutionary rotary engine. First let's slightly modify the starting location of the point we are tracing. Instead of being situated on the circumference of the rolling circle, suppose we trace the point Q located at a distance h from the center of the rolling circle. The curve traced out by Q is called an epitrochoid. Epicycloids are epitrochoids with $h = r$.

Question 10.79



Use this figure to show that the parametric equation of the *epitrochoid* is given by

$$x(t) = (R + r)\cos t - h\cos\frac{(R+r)t}{r}$$

$$y(t) = (R + r)\sin t - h\sin\frac{(R+r)t}{r}$$

where $CQ = h$.

Now that you have the general form of an epitrochoid, consider the case where the fixed circle is twice the size of the rolling one ($R = 2r$) and $h = 4r/9$. This is the geometry of the Wankel rotary engine.

Question 10.80

Give the simplified parametric equations for this special case.

What makes this geometry especially useful is that an equilateral triangle fits perfectly within the epitrochoids no matter how the triangle it is rotated. We will show that an *equilateral triangle* can be inscribed in this epitrochoid independent of the value of t. See figure 10.2 below for a demonstration.

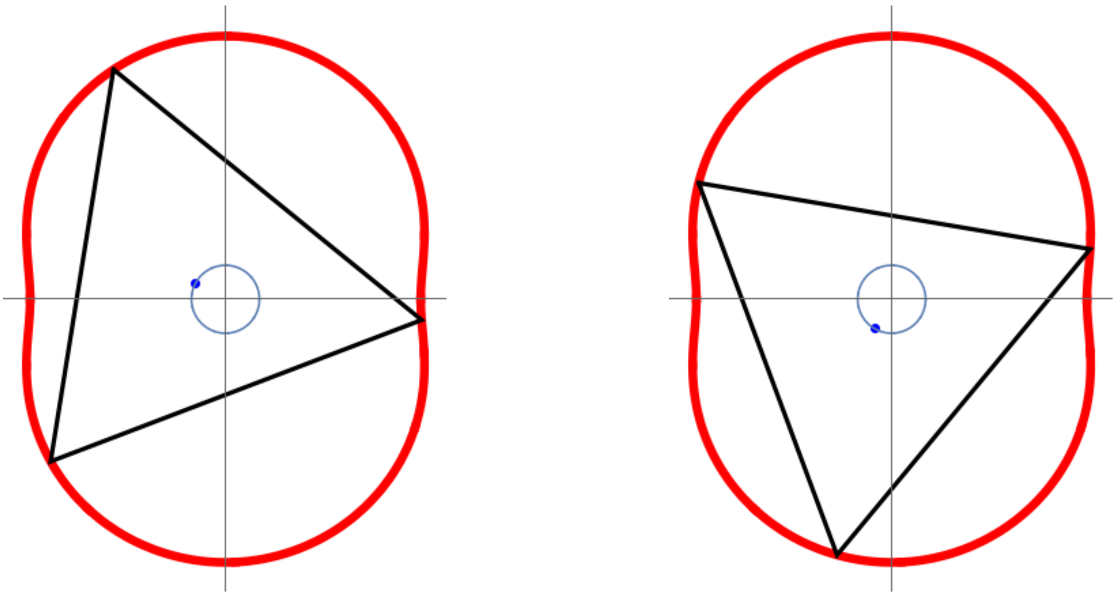


Figure 10.2

We will derive this in the following steps. Let

$$Q_0 = (x(t), y(t)), \quad Q_1 = (x(t + 2\pi/3), y(t + 2\pi/3)), \quad Q_2 = (x(t - 2\pi/3), y(t - 2\pi/3))$$

Question 10.81

(a) Show that the epitrochoid makes one revolution for $t \in [0, 2\pi]$.

(b) Since the three points Q_0, Q_1, Q_2 are $2\pi/3$ apart they are evenly spaced in $[0, 2\pi]$. Thus, if their pairwise distances are equal then they must form an equilateral triangle. Show that

$$\operatorname{dist}(Q_0, Q_1) = \operatorname{dist}(Q_1, Q_2) = \operatorname{dist}(Q_2, Q_0)$$

(c) Furthermore, the centroid of the equilateral triangle always lies on a circle of radius h. Show that the average of Q_0, Q_1 and Q_2 is always at a distance h from the origin.

This is the principle of the Wankel rotary engine. When the equilateral triangle rotates with its vertices on the epitrochoid, its centroid sweeps out a circle whose center is at the center of the epitrochoid. The space between the triangle and the epitrochoid is the firing chamber. A gif of the engine in action can be found here.

Question 10.82 **Bonus 4 points**

How would the shape of the epicycloids (and epitrochoids) change if R/r is a rational number but not an integer? Try plotting the curve with $R = 3, r = 2$. Does it complete a full revolution for $t \in [0, 2\pi]$?

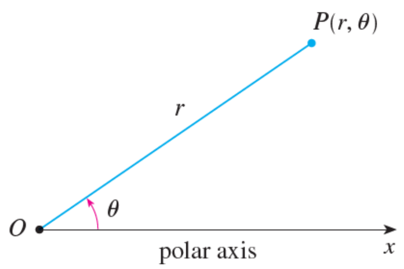
What happens if R/r is an irrational number. Try plotting the curve with $R = e, r = 1$. How long does it take to complete a full revolution?

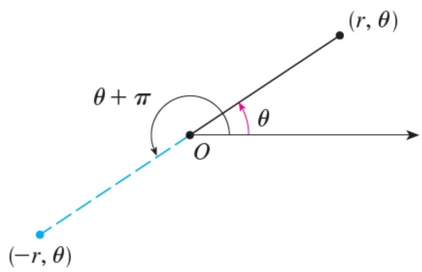
§11 | Polar Coordinate System and Some Interesting Parametric Curves

11.1 Definition

A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates. Usually we use Cartesian coordinates, which are directed distances from two perpendicular axes. In this chapter, we look at a coordinate system introduced by Newton, called the *polar coordinate system*, which is more convenient for many purposes.

We choose a point in the plane that is called the pole (or origin) and is labeled O. Then we draw a ray (half-line) starting at O called the polar axis. This axis is usually drawn horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates.





If P is any other point in the plane, let r be the distance from O to P and let θ be the angle (usually measured in radians) between the polar axis and the line OP as in the figure above. Then the point P is represented by the ordered pair (r, θ) and r, θ are called polar coordinates of P. We use the convention that an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction. If P = O, then $r = 0$ and we agree that $(0, \theta)$ represents the pole for any value of θ.

The convention is to regard r as a 'signed' radius. That means if for some value of θ, the value of r is negative, we go across origin and take the diametrically opposite point. In the second figure above, the points $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance $|r|$ from O, but on opposite sides of O. If $r > 0$, the point (r, θ) lies in the same quadrant as θ; if $r < 0$, it lies in the quadrant on the opposite side of the pole. Notice that $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$.

Question 11.83

(a) Plot the points $(2, -2\pi/3)$ and $(-3, 3\pi/4)$.

(b) Find the Cartesian coordinates of the points above.

(c) In polar coordinates, what shapes are described by $r = k$ and $\theta = k$, where k is a constant?

(d) Draw $r = 0, r = \frac{2\pi}{3}, r = \frac{4\pi}{3}, r = 2\pi, \theta = 0, \theta = \frac{2\pi}{3}$, and $\theta = \frac{4\pi}{3}$.

(e) Find the equations of above curves in Cartesian coordinates.

11.2 Vector perspective

Another way to think about this is that the point (r, θ) in polar coordinate corresponds to the position vector $\vec{r}$ that makes an angle θ with the positive x–axis and has magnitude r. Thus $(-r, \theta)$ corresponds to the vector $-\vec{r}$.

This is most useful when thinking about parametrized curves of the form $r = f(\theta)$ in polar coordinates where θ is the parameter. Although we can think of $r = f(\theta)$ as a scalar valued function of one scalar variable, in polar coordinates it describes the position vector $\vec{r}$ as a function of parameter θ.

Note that the polar point (r, θ) corresponds to the cartesian point $(r\cos\theta, r\sin\theta)$. So how do we plot a curve whose equation is of the form $r = f(\theta)$? We make the change of variable

$$x = r\cos\theta, \quad y = r\sin\theta$$

or equivalently,

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\frac{y}{x}$$

to change it into cartesian coordinates and plot that instead.

Question 11.84

1. Sketch the curve with polar equation $r = 2\cos\theta$.
2. Find the polar equation of the straight line $x = 5$.

11.2.1 Symmetry

When we sketch polar curves it is sometimes helpful to take advantage of symmetry.

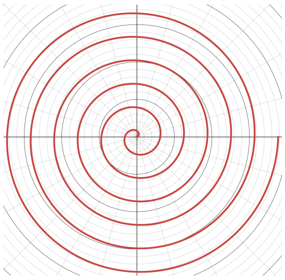
(a) If a polar equation is unchanged when θ is replaced by $-\theta$, the curve is symmetric about the polar axis.

(b) If the equation is unchanged when r is replaced by $-r$, or when θ is replaced by $\theta+\pi$, the curve is symmetric about the pole. (This means that the curve remains unchanged if we rotate it through 180° about the origin.)

(c) If the equation is unchanged when θ is replaced by $\pi-\theta$, the curve is symmetric about the vertical line $\theta=\pi/2$.

11.3 A list of interesting 2D curves

11.3.1 Spiral of Archimedes



The curve is a never-ending spiral. The parametric representation is

$$x(t) = t\cos t, \quad y(t) = t\sin t, \quad t \geq 0$$

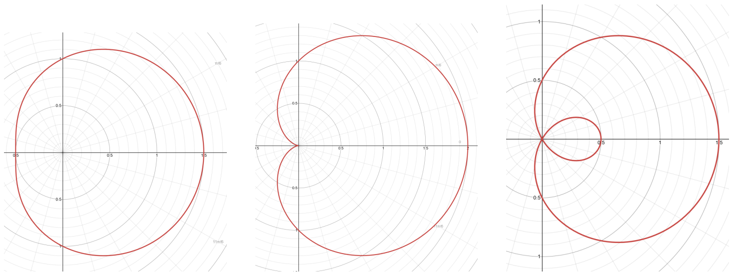
Question 11.85

Find it's equation in polar coordinates.

11.3.2 Limaçons

Recall the *epitrochoids* from project 3. When the two circles in the construction have equal radii, we get a family of curves called Limaçons. After some simplifications, their parametric representations look like

$$r = a + b\cos\theta, \quad \theta \in [0, 2\pi]$$

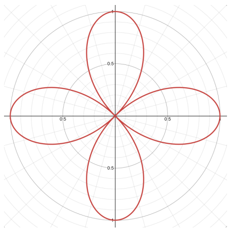


Question 11.86

1. In the above figure, the three pictures corresponds to $a < b$, $a = b$ and $a > b$. Find out which one is which.
2. When $a = b$, the Limaçon is called a *Cardioid*. Find it's equation in cartesian coordinates.

11.3.3 Roses

Consider the curve $r = \cos(2\theta)$. It looks like a four-petalled rose as below.



Question 11.87

(a) What is the equation of the right petal only?

(b) What is the equation of the bottom petal only?

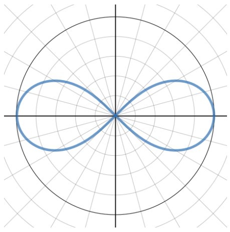
(c) Draw the part of the curve that corresponds to $\theta \in [0, \pi/2]$.

11.3.4 Lemniscate

The lemniscate of Bernoulli is a curve whose cartesian equation is given by

$$\left(x^2 + y^2\right)^2 = 2c^2\left(x^2 - y^2\right)$$

for some constant c.



Question 11.88

Find it's equation in polar coordinates.

§12 | Partial Derivatives

12.1 Motivation

Consider the contour plot for the function $f(x,y) = x^2 + y$.

1. Sketch the cross-section of the graph with the plane $x = 4$.
2. Compute the rate of change of z with respect to y as (x,y) moves towards increasing y-value, along the line $x = 4$.
3. What happens to the rate of change of z with respect to x as you move from $(4,5)$ towards increasing x-value along the line $y = 5$.

12.2 Definition

For a two-variable function $f(x,y)$ we define the *partial derivative* with respect to x by

$$f_x(a,b) = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

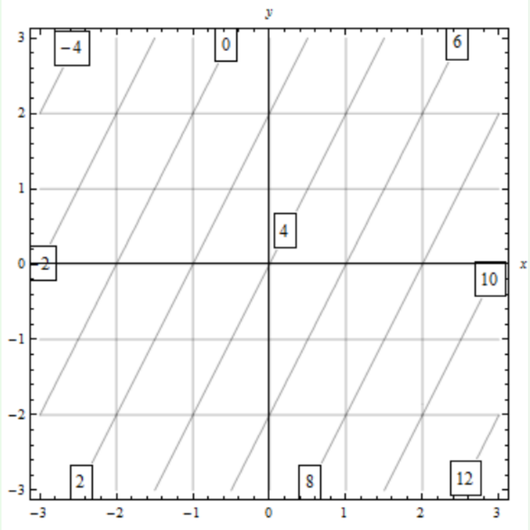
and similarly the *partial derivative* with respect to y by

$$f_y(a,b) = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

In other words, $f_x(a,b)$ is the derivative at a of the single variable function $x \mapsto f(x,b)$ with $y = b$ fixed.

Question 12.89

What are the values $f_x(0,0)$ and $f_y(0,0)$ for the linear function $f(x,y)$ whose contour plot looks like



12.3 Partial derivative function

We can think of partial derivatives as functions of the base point.

- $f_x(x,y)$ is the function giving f's rate of change in the x-direction from any point (x,y).
- $f_y(x,y)$ is the function giving f's rate of change in the y-direction from any point (x,y).

Example 3. If $f(x,t)$ is the vertical height of a guitar string at time t and position x units from one end of the guitar,

- $f_x(x,t)$ measures the rate of change of string height as position changes, or the slope of the string "snapshot" at time t.
- $f_t(x,t)$ measures the rate of change of string height as time passes, or the velocity of the string at position x.

Question 12.90

For each of the following functions, compute both first partial derivatives f_x and f_y (or f_t):

1. $f(x,y) = e^x \cos y$
2. $f(x,y) = x^3 - 3xy^2$
3. $f(x,t) = e^{-(x+t)^2}$

4. $f(x,t) = \sin(x-t) + \sin(x+t)$

Question 12.91

Let $F(M,r) = \frac{GM}{r^2}$ denote the gravitational force experienced by a unit mass at distance r from the center of a planet with mass M. What does the partial derivative $\frac{\partial F}{\partial M}$ say about how the gravitational force changes when the mass of the planet increases? What does the partial derivative $\frac{\partial F}{\partial r}$ say about how the gravitational force changes when the distance from the planet increases?

12.4 Higher order partials

We can compute higher order derivatives by simply repeating the process. For example,

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

Theorem 12.1: Clairout's Theorem

If f_{xy} and f_{yx} are both continuous, then $f_{xy} = f_{yx}$.

Question 12.92

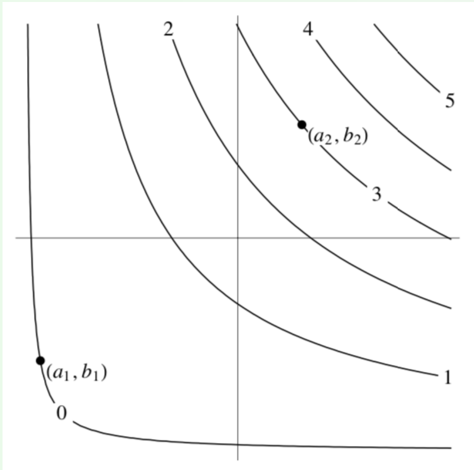
Use Clairaut's Theorem to compute the requested derivatives of the following functions:

(a) f_{xyxyxy} if $f(x,y) = x^2\cos(e^y + y^2)$

(b) f_{xxxyy} if $f(x,y) = x^3y^2 - \frac{y}{x+\ln(x)}$

Question 12.93

Below is the contour plot of a function $f(x,y)$. The value of f on each level set is labeled.



Based on the diagram, decide whether each of the statements should be true or false. (For which can you be totally sure, and for which would you need more information to be totally sure?)

(a) $f_x(a_1,b_1) \geq 0$

(b) $f_y(a_2,b_2) \geq 0$

(c) $f_x(a_1,b_1) \geq f_x(a_2,b_2)$

(d) $f_{xx}(a_2,b_2) \geq 0$

(e) $f_{xy}(a_2,b_2) \geq 0$

12.5 Interpreting second partial derivatives

Question 12.94

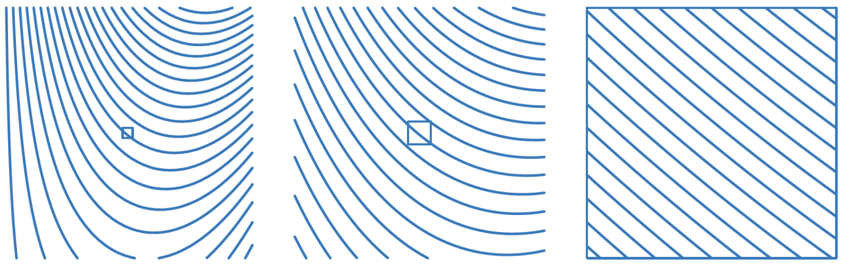
Go back to our example of a guitar string above. Let $f(x,t)$ be the vertical height of a guitar string at time t and position x units from one end of the guitar.

- What does f_{xx} represent?
- What does f_{xx} represent?
- What do f_{xt} and f_{tx} represent?

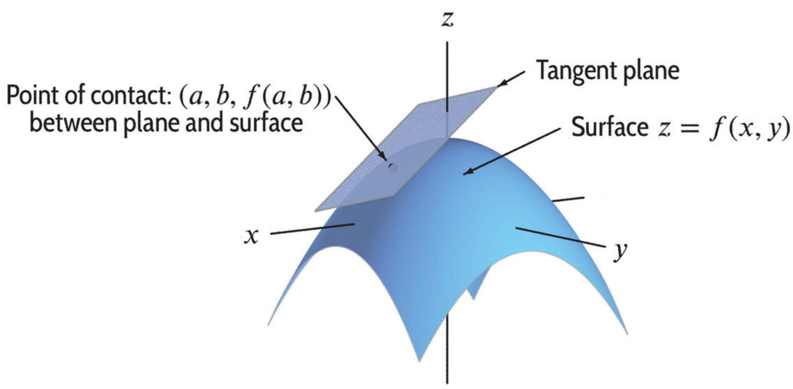
§13 | Tangent Plane and Linear Approximation

13.1 Tangent Plane

We have seen earlier that we can view the graph of a function $z = f(x,y)$ as a surface in 3D. If we zoom in on the graph, at most points it seems to flatten out and become planar. Similarly, if we zoom in at the contour plot of $f(x,y)$, the contours look more like equally spaced parallel lines, which are the contours of a linear function. (As we zoom in, we have to add more contours.)



Seeing a plane when we zoom in at a point tells us (provided the plane is not vertical) that $f(x,y)$ is closely approximated near that point by a *linear* function $L(x,y)$. Graphically, the graph of this linear function is a plane, which we call the *tangent plane* at that point.



Question 13.95

We are going to find the formula of the tangent plane at the point $(a, b, f(a, b))$ on the graph $z = f(x, y)$.

(a) The x grid curve (or $y = b$ trace) through this point is parametrized as $\vec{r}_1(x) = \langle x, b, f(x, b)\rangle$. Find the tangent vector to this curve at $x = a$. (This is a tangent vector to the surface at the point $(x, y, z) = (a, b, f(a, b))$.

(b) Find another tangent vector to the surface at the point $(x, y, z) = (a, b, f(a, b))$ using the y grid curve (i.e. $x = a$).

(c) Find the normal vector to the tangent plane using the two tangent vectors from parts (a) and (b).

(d) Now find the equation of the tangent plane. This is the plane through $(a, b, f(a, b))$ and perpendicular to the normal vector from part (c).

Question 13.96

Find the equation of the tangent plane to the elliptic paraboloid $z = x^2 + 2y^2$ at the point $(x, y, z) = (1, 1, 3)$.

13.2 Linear Approximation

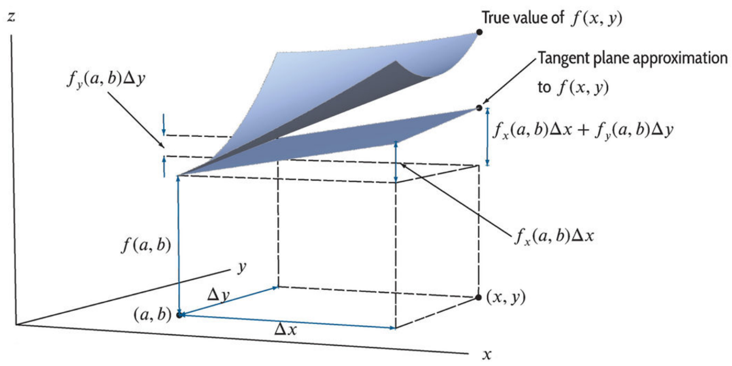
Since the tangent plane function matches f (and its partial derivatives) at (a, b) and is a linear function, it is called a *local linearization* of f near (a, b). We can use the local linearization to approximate a differentiable function at a point.

Theorem 13.1

The local linearization of a differentiable function $f(x, y)$ at a point (a, b) is given by

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Here is picture that explains above theorem graphically.



Question 13.97

- Approximate the values of $z = f(x,y) = x^2 + 2y^2$ at the points $(x,y) = (0.9, 1.1)$ and $(x,y) = (0.95, 0.95)$ using the linearization from the previous exercise.
- Calculate the actual values of $f(0.9, 1.1)$ and $f(0.95, 0.95)$. Compare them to your answer from last part. Could you have predicted whether your estimate was an over-estimate or an under-estimate before calculating the actual values?

Question 13.98

Approximate the value of $f(x,y) = ye^{xy}$ at the point $(x,y) = (0.90, 0.01)$. Which point should you chose to find the local linearization?

§14 | Directional Derivative and Gradient

The *directional derivative* of a function $f(x,y)$ in the direction of a unit vector $\vec{u}$ is the rate of change of the value of f in the direction of $\vec{u}$ and is denoted by $D_{\vec{u}} f$.

Suppose we start at (a,b) on the XY−plane and move in the direction of the unit vector $\vec{u}$ in the XY−plane. We want to compare the slope of the graph of f above points along the path. To do this, we consider different contractions h of the unit vector $\vec{u} = u_1\hat{i} + u_2\hat{j} = \langle u_1, u_2\rangle$ and find the function value at the end of such a contraction $f(a+hu_1, b+hu_2)$.

The average rate of change of $f(x,y)$ between the two points (a,b) and $(a+hu_1, b+hu_2)$ is given by the difference quotient

$$\frac{\Delta f}{displacement} = \frac{f(a+hu_1, b+hu_2) - f(a,b)}{\|h\vec{u}\|} = \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

and so to get the instantaneous rate of change we calculate the limit as $h \to 0$. Thus,

$$D_{\vec{u}} f(a,b) = \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

Now recall the differential dz or $df = f(x,y) - f(a,b)$ is in fact equal to $f_x(a,b)(x-a) + f_y(a,b)(y-b)$. This is because the instantaneous rate of change in f is the same as the instantaneous rate of change of the linearization $L(x,y)$ of f at the point P (why?). So

$$\begin{aligned} D_{\vec{u}} f(a,b) &= \lim_{h\to 0} \frac{f_x(a,b)hu_1 + f_y(a,b)hu_2}{h} \\ &= f_x(a,b)u_1 + f_y(a,b)u_2 \\ &= \left\langle f_x(a,b), f_y(a,b)\right\rangle \cdot \vec{u} \end{aligned}$$

The *two-dimensional* vector $\left\langle f_x(a,b), f_y(a,b)\right\rangle$ is special for its own reasons (as we will see next), and deserves its own name: we call it the *gradient* of f at (a,b). In general $\operatorname{grad} f$ is also denoted by ∇f and is equal to

$$\nabla f = \left\langle f_x, f_y\right\rangle$$

Theorem 14.1

The directional derivative of f at (a,b) in the direction of unit vector $\vec{u}$ is given by

$$D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}$$

Question 14.99

Find the directional derivative of the function $f(x,y) = x^2 + 2y^2$ in the direction of the vector $\langle 1,1 \rangle$.

14.1 Significance of the gradient

We observe that

$$D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u} = \|\nabla f(a,b)\| \cos(\theta)$$

where θ is the angle between ∇f and $\vec{u}$, since $\|\vec{u}\| = 1$. Hence starting at the point (a,b), the rate of change of $f(x,y)$ i.e. $D_{\vec{u}} f(a,b)$ is maximized when $\theta = 0$ i.e. when $\vec{u}$ and ∇f are parallel.

In other words,

Theorem 14.2

$D_{\vec{u}} f(a,b)$ is maximized in the direction of ∇f. Moreover, the maximum value of $D_{\vec{u}} f(a,b)$ is equal to $\|\nabla f\|$.

Question 14.100

Suppose we drop some water on the surface $z = x^3 y + y^2 x$ at the point $(1,1,2)$. Which way does the water flow?

Question 14.101

True or False: If we know the value of the directional derivative $D_{\vec{u}} f(a,b)$ in the direction of any two nonparllel different unit vectors, then we can determine $\nabla f(a,b)$.

14.2 Gradient and Level Curves

What happens to $f(x,y)$ along a level curve of f? If $\vec{u}$ is the direction of the tangent at the point $P = (a,b)$ to the level curve passing through P, then the rate of change of f in the direction of $\vec{u}$ is zero.

Hence for this choice of $\vec{u}$, we have

$$D_{\vec{u}} f(a,b) = 0 \implies \theta = \frac{\pi}{2}$$

where θ is the angle between $\vec{u}$ and $\nabla f(a,b)$. In other words,

Theorem 14.3

The gradient vector at a point is perpendicular to the level curve through that point.

Question 14.102

Let $f(x,y) = 3x^2 + 4y^2$.

(a) Find the parametric equation $\vec{r}(t)$ of the level curve of f that passes through the point $(1,1)$.

(b) Find the velocity $\vec{r}'(t)$ at $(1,1)$.

(c) Verify that $\vec{r}'(t)$ at $(1,1)$ is perpendicular to $\nabla f(1,1)$.

§15 | Three Dimensional Gradient and the Tangent Plane

The directional derivative formula and the gradient have the obvious extensions for functions of three-variables $f(x,y,z)$:

$$\begin{aligned} D_{\vec{u}} f(a,b,c) &= \nabla f(a,b,c) \cdot \vec{u} \\ &= \left\langle f_x(a,b,c), f_y(a,b,c), f_z(a,b,c) \right\rangle \cdot \vec{u} \\ &= \|\nabla f(a,b,c)\| \cos(\theta) \end{aligned}$$

so the gradient again points in the direction of greatest increase of f from (a,b,c) and has magnitude equal to that rate of change.

Question 15.103

A fly is flying around a room in which the temperature is given by $T(x,y,z) = x^2 + y^4 + 2z^2$. The fly is at the point $(1,1,1)$ and realizes that he's cold. In what direction should he fly to warm up most quickly?

Question 15.104

How should the gradient $\nabla f(a,b,c)$ be related to the level-surface of f through (a,b,c)?

Question 15.105

For each of the following functions $F(x,y,z)$, do the following

(a) Compute the gradient ∇F.

(b) Identify the level surface $F(x,y,z) = \textit{constant}$ through the point $(x,y,z) = (1,1,1)$.

(c) Find the tangent plane to the level surface from part (b) at $(1,1,1)$. Recall that the tangent plane to a surface $z = f(x,y)$ at the point (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

(d) Verify that the gradient vector ∇F is perpendicular to (the tangent plane of) the level surface at the point $(x,y,z) = (1,1,1)$.

$$\text{(I) } F(x,y,z) = 3x + 2y + z \qquad \text{(II) } F(x,y,z) = x^2 + y^2 - z^2$$

15.1 3D Gradient as the Normal vector

Since the gradient $\nabla f(a,b,c)$ is perpendicular to the level surface of f containing (a,b,c), it is the normal vector to the tangent plane of the level surface at (a,b,c). So even if the surface is not the graph of a function, we can use the three dimensional gradient to find tangent planes.

Question 15.106

Find the equation of tangent plane to the surface $xy + yz + zx = 5$ at the point $(1,1,2)$.

Question 15.107

Alice the "A" student is debating Chuck the "C" student. Alice says that the direction of greatest ascent on the graph $z = f(x,y)$ is in the direction ∇f. Chuck says that instead we should look at the level surface $F(x,y,z) = z - f(x,y) = 0$ and go in direction ∇F. Who is right? How would you explain this to the student that is wrong?

Question 15.108

You're hiking a mountain which is the graph of $f(x,y) = 15 - x^2 - 2xy - 3y^2$. You're standing at $(1,1,9)$. You wish to head in a direction which will maintain your elevation (so you want the instantaneous change in your elevation to be 0). How many possible directions are there for you to head? What are they?

Question 15.109

Show that every plane that is tangent to the cone $x^2 + y^2 = z^2$ passes through the origin.

Question 15.110

Where does the normal line to the paraboloid $z = x^2+y^2$ at the point $(1,1,2)$ intersect the paraboloid a second time?

Question 15.111

Find the angle of intersection between the curve given by its parametric equation $\vec{r}(t) = \langle t, 2t^2 \rangle$, and the parabola $y = x^2 + 4$.

Question 15.112

Find the tangent plane to the surface of the solid described by inequalities $0 \le x \le 6$ and $0 \le y^2 + z^2 \le 4$ at the point $(6,1,1)$.

Question 15.113

The ellipsoid $x^2+4y^2+9z^2 = 36$ and the surface $z = sin[\pi(x-y)]$ intersect in a curve, call it $\mathcal{C}$. Find the line tangent to $\mathcal{C}$ at the point $(6,0,0)$. (Please give a parametric vector equation for the line.)

Question 15.114

Consider the surface

$$\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}.$$

1. Find the equation of the tangent plane to this surface at a point (p,q,r).

2. Show that the sum of X-intercept, Y-intercept and Z-intercept of the above

tangent plane does not depend on p, q, and r.

§16 | Chain Rule

Recall from one-variable calculus the chain rule

$$\frac{d}{dt} f(g(t)) = f'(g(t)) \cdot g'(t)$$

which is stated as "the derivative of the outside function (evaluated at inside function), times the derivative of the inside function".

Now think of a particle is moving in a parameterized curve $\vec{r}(t) = \langle x(t), y(t) \rangle$ and we want to understand how some function $f(x,y)$ changes along the curve. For example, we might ask, what is the rate of change of height along a particular racetrack. For any composite function of the form $f(x(t),y(t)) = f(\vec{r}(t))$, the multivariable chain rule is

$$\frac{dz}{dt} = \frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle x'(t), y'(t) \rangle$$

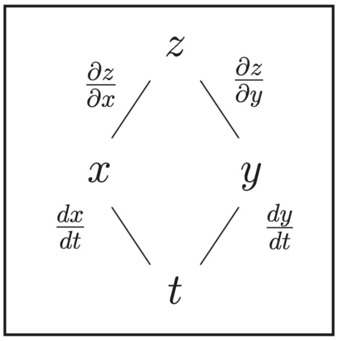
where the gradient ∇f acts like the derivative of the outside function f and the velocity $\vec{r}\,'(t)$ acts like the derivative of the inside vector-function $\vec{r}(t) = \langle x(t), y(t) \rangle$.

Notice that this just like the formula for the directional derivative, but now with $\vec{r}\,'(t)$ playing the role of the direction $\vec{u}$. Thus we can think of $\frac{df}{dt}$ as (a multiple of) the rate of change of f in the tangent direction of the parameterized motion.

16.1 Chain Rule diagram

A chain of dependence diagram is a convenient way to represent the chain rule for $z = f(x(t), y(t))$.

Then, the derivative $\dfrac{dz}{dt}$ is obtained by adding the products of the derivatives on each path of the diagram leading from z to t:

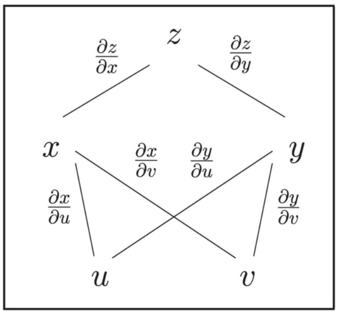


$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}x'(t) + \frac{\partial z}{\partial y}y'(t)\\ &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\end{aligned}$$

where the terms $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are both evaluated at $(x(t), y(t))$.

16.2 More complicated diagram

Curves in the xy-plane can also sometimes be described by even more parameters, like $x(u,v)$ and $y(u,v)$. Then the chain rule for differentiating $z = f(x(u,v), y(u,v))$ is even more complicated; but a dependence diagram can help.



Now the partial derivative $\frac{\partial z}{\partial u}$ is obtained by adding the products of the derivatives on each path from z to u in the diagram:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

and

the partial derivative $\frac{\partial z}{\partial v}$ is obtained by adding the products of the derivatives on each path from z to v in the diagram:

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Question 16.115

Let $w = x^2 e^y, x = 4u$, and $y = 3u^2 - 2v$. Compute $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ using the chain rule.

Question 16.116

Let $z = g(u,v)$ and $u = u(x,y,t), v = v(x,y,t)$ and $x = x(t)$ $y = y(t)$. Then how many terms are there in the chain rule expression for $\frac{dz}{dt}$?

Question 16.117

Let $p = g(u,v)$ be a differentiable function of two variables. Let $u = \frac{x}{y}$ and $v = \frac{y}{z}$. Show that

$$x\frac{\partial p}{\partial x} + y\frac{\partial p}{\partial y} + z\frac{\partial p}{\partial z} = 0$$

Question 16.118

1. Use the chain rule to compute $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ for $z = f(x(r,\theta), y(r,\theta))$ where

$$x(r,\theta) = r\cos(\theta) \quad y(r,\theta) = r\sin(\theta)$$

2. Show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

Question 16.119

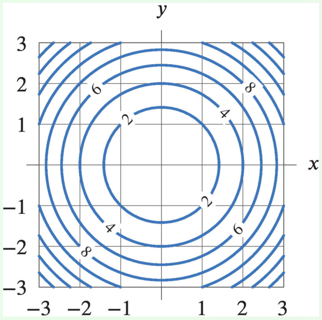
The figures below show the contour plot of $z = f(x,y)$ and the graphs of x and y, respectively, as functions of t. Decide if $\left.\dfrac{dz}{dt}\right|_{t=2}$ is

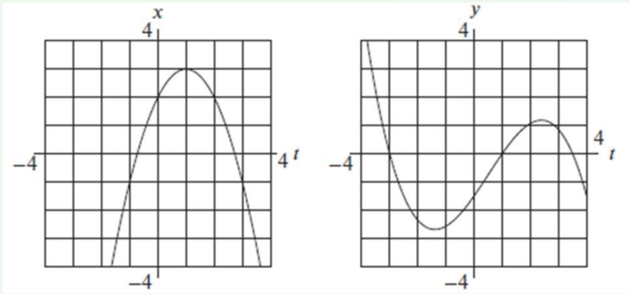
a) Positive

b) Negative

c) Approximately 0

d) Can't tell without further information





Question 16.120

Let $z = f(x,y) = x^2 + y^3$, and $x = x(s,t)$ and $y = y(s,t)$; i.e., x and y are functions of s and t. Suppose that when $(s,t) = (0,1)$, we have:

$$x(0,1) = -1, \quad x_s(0,1) = -4, \quad x_t(0,1) = -7, \quad y(0,1) = 2, \quad y_s(0,1) = 10, \quad y_t(0,1) = 5.$$

Compute $\dfrac{\partial z}{\partial t}$ at $(s,t) = (0,1)$.

Question 16.121

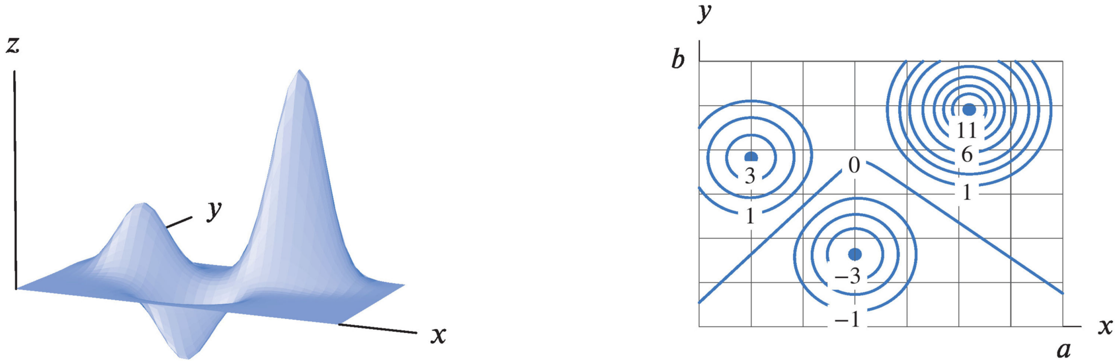
The length of a side of a triangle is increasing at a rate of 3 in/s, the length of another side is decreasing at a rate of 2 in/s, and the contained angle θ is increasing at a rate of 0.05 radian/s. How fast is the area of the triangle changing when $x = 40$ in, $y = 50$ in, and $\theta = \pi/6$?

§17 | Local Optimization (Mar 30, Book chapter 15.1)

- f has a local maximum at the point P_0 if $f(P_0) \geq f(P)$ for all points P near P_0.
- f has a local minimum at the point P_0 if $f(P_0) \leq f(P)$ for all points P near P_0.

17.1 How to detect a local extremum?

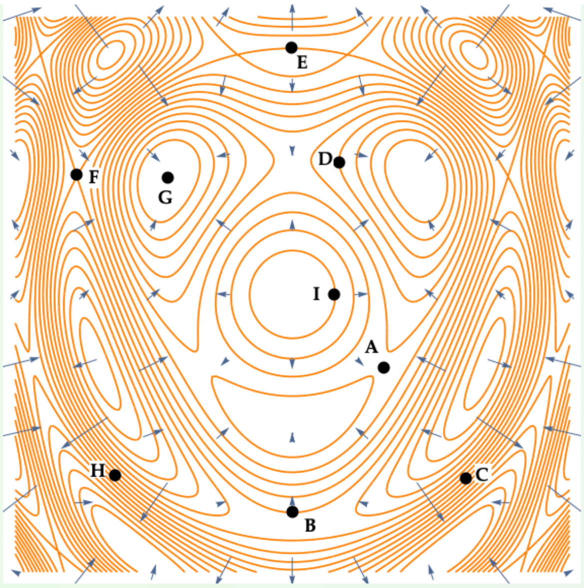
Consider the graph and contour plot of a function $z = f(x, y)$ as follows:



What does the contour plot look like around a local extremum? What can you say about the gradient at the local extremum? Use similar logic to answer the following question.

Question 17.122

A function $f(x, y)$ of two variables has level curves as shown in the picture. Some of the gradient vectors are also given for your convenience. Fill in the table below with all appropriate choices of points.



Enter all possible choices from A-I	is/are point(s) where
	$f_x(x,y) = 0$ and $f_y(x,y) \neq 0$
	$f_y(x,y) = 0$ and $f_x(x,y) \neq 0$
	$f_y(x,y) = 0$ and $f_x(x,y) = 0$
	$f(x,y)$ has either a local maxima or a local minima.

17.2 Stationary Points

A point (a,b) is called a *stationary point* for the function $z = f(x,y)$ iff $f_x(a,b) = 0$ and $f_y(a,b) = 0$. That is, critical points are those points where $\nabla f = \left\langle f_x, f_y \right\rangle = 0$.

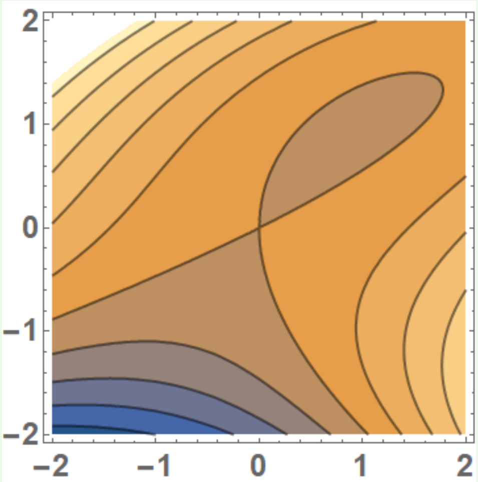
From the table above, it should be clear that a local maxima or local minima is a stationary point, but not all stationary points are local extremum.

Note that the "critical points" of f include the stationary points as well as the points where the gradient is undefined. If f is differentiable everywhere, both are the same things.

Question 17.123

Find the critical points of the function $f(x,y) = 8y^3 + 12x^2 - 24xy$. Are all of them local extrema?

Here is a contour plot to help you.



A critical point which is neither a local maximum nor a local minimum is called a saddle point.

17.3 Classifying Stationary Points - The Second-Derivative test

The *Hessian* of a function $f(x,y)$ is defined to be the matrix $\mathrm{H} = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$. Suppose (a,b) is a point where $\nabla f(a,b) = \vec{0}$ and let

$$\mathrm{D} = \det(\mathrm{H}(a,b)) = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2$$

The second-derivative test says

- If $\mathrm{D} > 0$ and $f_{xx}(a,b) > 0$, then f has a local minimum at (a,b).
- If $\mathrm{D} > 0$ and $f_{xx}(a,b) < 0$, then f has a local maximum at (a,b).
- If $\mathrm{D} < 0$, then f has a saddle point at (a,b).
- If $\mathrm{D} = 0$, anything can happen: f can have a local maximum, or a local minimum, or a saddle point, or none of these, at (a,b).

Question 17.124

How does the contour plot look like near a saddle point? Identify the saddle points in the plots above.

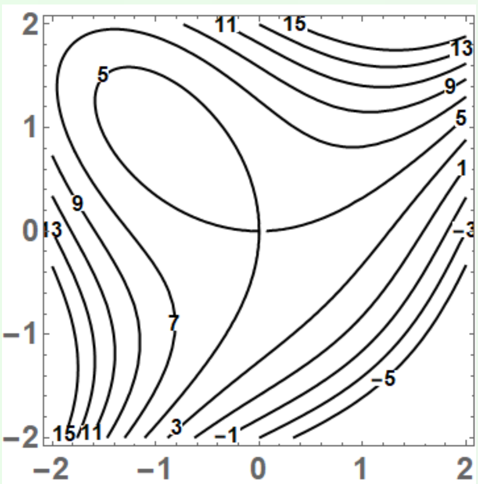
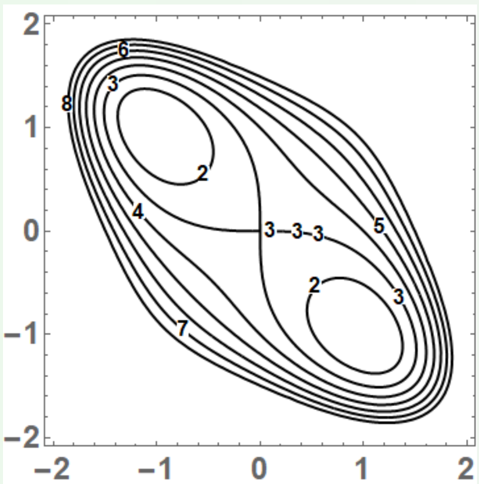
Question 17.125

Find and classify all stationary points of the following functions:

1. $f(x,y) = x^2 + 2xy + 2y^2 - 8y + 12$ **Ans:** $(-4,4)$, minimum
2. $f(x,y) = x^2 - 2y^2 + xy - 4$ **Ans:** $(0,0)$, saddle
3. $f(x,y) = 8 - x^2 - xy - y^2$ **Ans:** $(0,0)$, maximum
4. $f(x,y) = x^4 + y^4 + 4xy + 3$ **Ans:** $(-1,1)$, minimum; $(0,0)$, saddle; $(1,-1)$, minimum
5. $f(x,y) = y^2 + 2xy - x^2 + 2x - 2y + 36$ **Ans:** $(1,0)$, saddle
6. $f(x,y) = 5 - x^3 + y^3 + 3xy$ **Ans:** $(-1,1)$, minimum; $(0,0)$, saddle

Question 17.126

Two of the above functions are shown as contour plots below. For each plot, predict the location of the stationary points and classify them as maxima, minima, or saddle points. Can you identify which functions are plotted here?

Project 4 | Stationary Points with Mathematica (Due April 12)

A stationary point for a function of two variables is a point where both first partial derivatives equal zero. In this project you will investigate the stationary points for the following functions:

$$
\begin{aligned}
f(x,y) &= x^3 + 3xy + y^3 \\
g(x,y) &= x^2 + 6xy + y^2 + 14x + 10y \\
h(x,y) &= 16x^2 + 8xy + y^2
\end{aligned}
$$

17.1 Computing Stationary Points

Not only will *Mathematica* calculate the partial derivatives for you, but it also has a built-in `NSolve` command that you can use to find the points where the derivatives are both zero.

- Define the first function by executing the command

```
f[x_,y_] := x^3 + 3*x*y + y^3
```

  Similarly define $g(x,y)$ and $h(x,y)$ next.

- Solve for the stationary points by executing the command

```
Solve[Grad[f[x,y],{x,y}]=={0,0}, {x, y}, Reals]
```

- We can define a routine `StatPts` in Mathematica that will take f, g or h as input and produce the list of stationary points directly as follows. Type and execute

```
StatPts[func_] := Solve[Grad[func[x,y], {x, y}] == {0, 0}, {x, y}, Reals]
```

Check that `StatPts[f]` produces the same list of points as part (2). The advantages of doing this step are as follows.

- First, to find the stationary points of `g` and `h`, we can skip writing a long command as in part (2). Instead, we can directly get a list by using `StatPts[g]` and `StatPts[h]`.
- Secondly, we have given the list of stationary points a name, that we can refer to later.

► Find and record these stationary points for future reference in the table below. *Note that the third function $h(x,y)$ has an entire line of stationary points, and you should choose any* ***one*** *of these for your investigation.*

Question 17.127

Draw the 3D plots and the contour plots of the functions f, g and h and try to visually classify your critical points as local maximum, local minimum, or saddle point. *Choose a big enough domain so that it contains all the stationary points you are looking at.* Recall that the command for 3D Plot looks like

```
Plot3D[f[x,y],{x,a,b}, {y,c,d}]
```

and the command for Contour Plot looks like

```
ContourPlot[f[x,y],{x,a,b}, {y,c,d}]
```

Note that you will need to replace a, b, c and d with appropriate numbers to see the full pictures.

Question 17.128

Fill out the following table with information you obtained from above plots.

Function:	$f(x,y)$		$g(x,y)$	$h(x,y)$
Stationary points: $x =$				
$y =$				
Sign of $\frac{\partial^2}{\partial x^2}$:				
Sign of $\frac{\partial^2}{\partial y^2}$:				
Classification:				

17.2 Using Second Derivative test

You may have found that it was hard to visually classify the stationary points of g. We can use the second derivative test to give a definite answer in such cases.

- Recall that the Hessian is the matrix $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$. We can think of this as the gradient of the gradient of f. Type and execute

  ```
  H[func_]:=Grad[Grad[func[x,y],{x,y}],{x,y}]
  ```

  to define a routine `H`. Check that `H[f,{x,y}]` produces the Hessian matrix for f.

- We can define the determinant of the Hessian `d` as

  ```
  d[func_]:=Det[H[func]]
  ```

  Execute the command above.

- Find `d[f]`. Next find the determinant for other two functions by changing f to g and h.

- We can evaluate the determinant at each of the stationary point above as follows. Type and execute

  ```
  d[f]/.StatPts[f]
  ```

  to evaluate $d(f,x,y)$ at the stationary points of f. Recall that the '`/.`' operator was used in last lab, it's called `ReplaceAll`.

- You can get the top left entry in the Hessian matrix by typing `H[f][[1,1]]`. Find the value of f_{xx} at the stationary points by executing

  ```
  H[f][[1,1]]/.StatPts[f]
  ```

  Write down the values of f_{xx}, g_{xx}, and h_{xx} at corresponding stationary points from the table. You can copy and paste above lines of codes and change f to g or h everywhere to investigate stationary points of g and h.

Question 17.129

Classify the stationary points as local maxima, local minima or saddle point using the values you got in the last step. Check that your classification is consistent with the table above.

Question 17.130

Is the following "second derivative test" a valid method?

Consider a stationary point (a,b) for any function $f(x,y)$.

If $f_{xx}(a,b) > 0$ and $f_{yy}(a,b) > 0$, then (a,b) is a local minima.

If $f_{xx}(a,b) < 0$ and $f_{yy}(a,b) < 0$, then (a,b) is a local maxima.

We can combine all of the above steps into one routine as follows:

```
ClassifyStationaryPoints[f_, {x_, y_}] :=
Module[{X, P, H, g, d, S},
X = {x, y};
P = Solve[Grad[f, X] == 0, X, Reals];
H = Grad[Grad[f, X], X];
g = H[[1, 1]];
d = Det[H];
S[d_, g_] := If[d < 0, "saddle", If[g > 0, "minimum", "maximum"]];

TableForm[{x, y, d, g, S[d, g], f} /. P,
TableHeadings -> {None, {x, y, "D", "f_xx", "Type", "f"}}]]
```

Note in particular the `If` command in there that does the second derivative test and the `TableForm` command that writes the results nicely in a table. Read the documentation for the commands at home.

Execute above routine.

Question 17.131

(a) Type and execute

```
ClassifyStationaryPoints[4 x y - x^3 y - x y^3, {x, y}]
```

to find and classify all the stationary points of $u(x,y) = 4xy - x^3y - xy^3$.

(b) Draw a contour plot to confirm your answer.

Question 17.132

(a) Use the routine to find and classify all stationary points of $v(x,y) = 5(y^2 - x^2)e^{-x^2-y^2}$.

(b) Draw a contour plot to confirm your answer.

Project 5 | (Optional) Taylor Approximation and Second Derivative Test

17.1 Why does the Second Derivative Test Work?

It combines 2 ideas:

(a) It's not too hard to classify critical points for *quadratic* polynomials in two variables, i.e. polynomials where no term is bigger than x^2, y^2, or xy.

(b) We can use Taylor series to approximate any (smooth) function by a quadratic polynomial.

Let's begin with the first point.

17.2 Understanding quadratics in two variables

► Give a simple explanation why $f(x,y) = x^2 + y^2$ has a minimum at $(0,0)$.

Hint: what is the minimum possible value this function could have?

► For the simple function $f(x) = ax^2$, can you can tell whether it's concave up or down at $x = 0$ just by the sign of a? When a is positive, is it concave up or down? When a is negative?

Even though the example $x^2 + y^2$ is easy, let's think about it in a different way. A function $z = f(x,y)$ has a local minimum at a point if it's "concave up in every direction" at that point. To be more precise, when we take any vertical slice through this point, the resulting curve should be concave up.

Now let's convince ourselves that $f(x,y) = x^2 + y^2$ is concave up on every vertical slice going through $(0,0,0)$.

► Show it's concave up for the two standard cross-sections $x = 0$ and $y = 0$.

► Think about intersecting $z = x^2 + y^2$ with the vertical plane $y = x$ (you could call this the "45-degree vertical plane" I suppose if you looked down on it from above). Note that this plane does go through the origin. Justify that the slice of the surface lying on this plane is concave up. (Just plug $y = x$ into the equation!)

- Now do the same thing with slice from the vertical plane $y = 2x$. And $y = -3x$. And $y = 0.02x$. It should be concave up on each of these slices.
- Every vertical slice is found by intersecting the surface with the plane $y = mx$ for some m. (Or $x = 0$ if you want to think about the slice where m goes to infinity.) Plug $y = mx$ into $z = x^2 + y^2$ and get an equation of the form $z = ax^2$. What is a in terms of m? Argue that a is always positive.

This last part shows that the surface is concave up in every direction, thus we really do have a local minimum at the origin!

- Repeat the same analysis as above for the quadratic function $z = g(x, y) = x^2 - y^2$, at the critical point $(0, 0)$ and show that it is concave up in some directions and concave down in others. This explains why g has a saddle point at the origin.
- Repeat the same analysis as above for the quadratic function $z = h(x, y) = x^2 + 3xy + y^2$ at the critical point $(0, 0)$. Is it concave up in every direction, or are there some directions where it is concave down? Then say whether this critical point is a local max or min or saddle point.
- One more specific example: try $z = f(x, y) = 2x^2 - 4xy + 3y^2$. When you plug in $y = mx$, you get $z = p(m) \cdot x^2$ where $p(m)$ is a quadratic in m. How can you tell if $p(m)$ takes on both positive and negative values?

  Hint: quadratic equation, discriminant. Justify that the critical point at $(0, 0)$ is a local min.

17.3 The General Quadratic

- Show that the quadratic $z = q(x, y) = Ax^2 + Bxy + Cy^2$ has a critical point at $(0, 0)$.
- Intersect the surface $z = q(x, y)$ with the vertical plane $y = mx$. As before, we get $z = p(m)x^2$ where $p(m)$ is a quadratic in m. Write down $p(m)$.
- Convince yourself that if $p(m)$ takes on only positive values (for any choice of m), then the critical point is a local minimum. And that if $p(m)$ takes on only negative values, then the critical point is a local max. And that if $p(m)$ can be either positive or negative depending on the choice of m, then the critical point is a saddle point.
- Show that the sign of the expression $B^2 - 4AC$ determines whether $p(m)$ takes on only positive (or only negative) values or both positive and negative values.

17.4 More general Quadratic?

The most general formula of a quadratic function is

$$Q(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + G$$

We can rewrite this by completing some squares as follows:

$$Q(x,y) = A\left(x+\frac{D}{2A}\right)^2 + C\left(y+\frac{E}{2C}\right)^2 + Bxy + \underbrace{\left(G-\frac{D^2}{4A}-\frac{E^2}{4C}\right)}_{\text{constant}}$$

For the following three questions, You can try experimenting in *Matheamtica*.

- What is the effect on the graph of Q if we change G?
- What is the effect on the graph of Q if we change E?
- What is the effect on the graph of Q if we change D?

We should be able to conclude that the *shape* of Q is unaffected by different choices of D, E and G. So we can assume $D = E = G = 0$. Hence the shape of a general quadratic can be fully understood by analyzing $q(x,y)$ from above.

17.5 The Taylor Quadratic

Recall that the linear approximation of the function $f(x,y)$ at $(0,0)$corresponds to the tangent plane at $(0,0)$:

$$T(x,y) = f_x(0,0)x + f_y(0,0)y + f(0,0)$$

Note that a linear approximation essentially finds a linear function such that the slope of the function matches with the linear function in $x-$ and $y-$ direction. But clearly it can't distinguish between local max/min/saddle point.

So we want to find a better approximation of f by matching more derivatives. For that we'll need something more complicated than linear functions, a quadratic function.

Suppose $Q(x,y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + G$ such that

$$\begin{aligned}
Q(0,0) &= f(0,0)\\
Q_x(0,0) &= f_x(0,0)\\
Q_y(0,0) &= f_y(0,0)\\
Q_{xx}(0,0) &= f_{xx}(0,0)\\
Q_{yy}(0,0) &= f_{yy}(0,0)\\
Q_{xy}(0,0) &= f_{xy}(0,0)\\
Q_{yx}(0,0) &= f_{yx}(0,0)
\end{aligned}$$

- Find A, B, C, D, E, G.

These choices lead us to the *Taylor Quadratic* for $f(x,y)$ at $(0,0)$.

$$\mathrm{TQ}(x,y) = \frac{f_{xx}(0,0)}{2}x^2 + f_{xy}(0,0)xy + \frac{f_{yy}(0,0)}{2}y^2 + f_x(0,0)x + f_y(0,0)y + f(0,0)$$

Or more generally the Taylor Quadratic approximation at a point (a,b) is given by,

$$\mathrm{TQ}(x,y) = \frac{f_{xx}(a,b)}{2}(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{f_{yy}(a,b)}{2}(y-b)^2$$
$$+ f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

Using the same argument as last section, we can then conclude that the shape of this quadratic is entirely determined by

$$4\mathrm{AC} - \mathrm{B}^2 = 4\frac{f_{xx}(a,b)}{2}\frac{f_{yy}(a,b)}{2} - (f_{xy}(a,b))^2 = \det[\text{Hessian}]$$

- Apply the $4\mathrm{AC} - \mathrm{B}^2$ rule to determine the graph shapes of the Taylor quadratics:
  a) $-2xy - 4y^2 - 3x$
  b) $-3(x+1)^2 - 2(x+1)(y-1) - 4(y-1)^2 - 2(x+1) - 6(y-1)$
- We can also rewrite $q(x,y)$ as

$$q(x,y) = \mathrm{A}x^2 + \mathrm{B}xy + \mathrm{C}y^2 = \mathrm{A}\left(x + \frac{\mathrm{B}}{2\mathrm{A}}y\right)^2 + \left(\mathrm{C} - \frac{\mathrm{B}^2}{4\mathrm{A}}\right)y^2$$

  So when $\mathrm{B}^2 - 4\mathrm{AC} = 0$, the y^2 term vanishes. What is the shape of the surface

$$z = \mathrm{A}\left(x + \frac{\mathrm{B}}{2\mathrm{A}}y\right)^2?$$

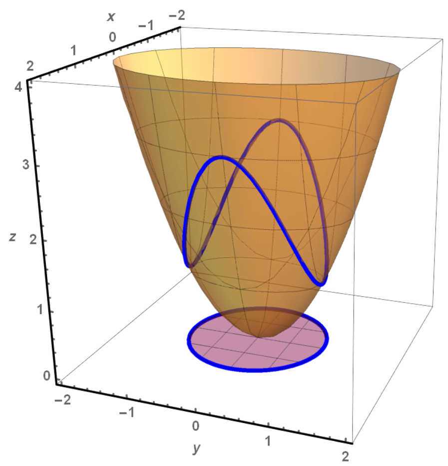
- What can you conclude when $\mathrm{B}^2 - 4\mathrm{AC} = 0$? Is it a local max/min/saddle point? Try experimenting in *Matheamtica*.

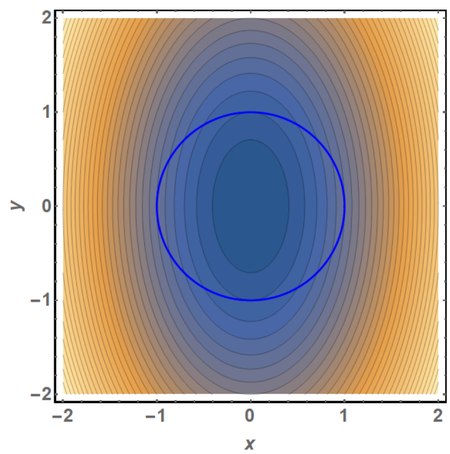
§18 | Constrained Optimization (Apr 1, Book chapter 15.3)

18.1 Motivation

A racehorse lives in a valley which happens to be the graph of $f(x,y) = 3x^2 + y^2$. He is doomed to wander his racetrack, which is the set of points in the valley where $x^2 + y^2 = 1$. The racehorse secretly wishes to be a mountain climber, and his fantasy is to escape from his racetrack and take the steepest path up the mountain. At what point should he make his escape, and in what direction should he run?

Below we have drawn 3-D depiction of the valley and the racetrack in the first picture, and the contour plot of the function $f(x,y)$ along with the projection of the racetrack in the second picture.



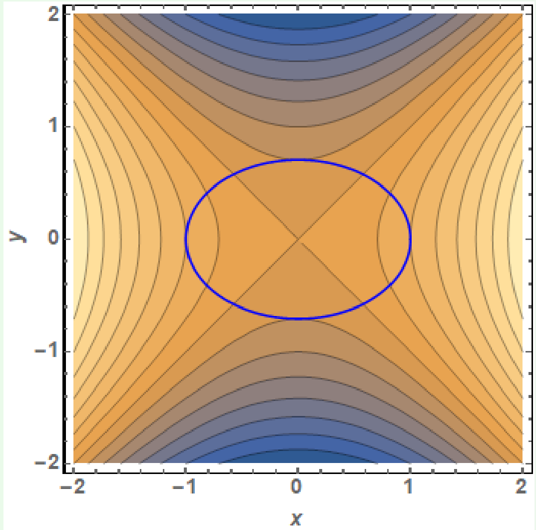


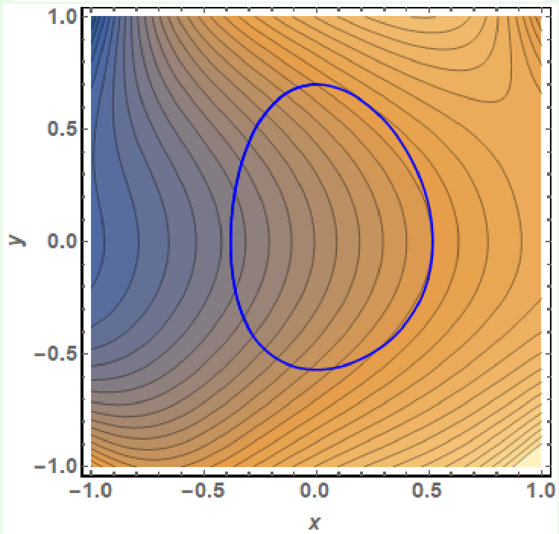
It makes sense that the horse will make his escape when $f(x,y)$ is largest along the racetrack. Can you see how to locate those points on the contour plot?

18.2 Graphical Reasoning

Question 18.133

Consider in general, the contours of a differentiable function $f(x,y)$ overlaid with a curve $g(x,y)=c$.





(a) For each plot, locate the local and global maximizer(s) and minimizer(s) of $f(x,y)$ on the curve $g(x,y)=c$.

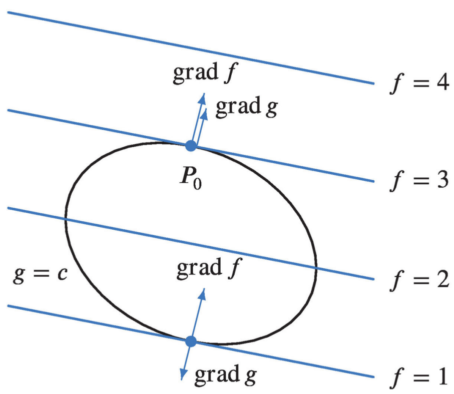
(b) How are the tangents to the level curve of f and the curve $g(x,y)=c$ related at the extreme points?

(c) The curve $g(x,y)=c$ is evidently a level curve for the function $g(x,y)$. What does this imply about the relationship between ∇f and ∇g at the extreme points of f on the curve $g(x,y)=c$?

[Hint: How is the gradient of any function related to the level curves of that function?]

(d) Find the other points on the curve $g(x,y) = c$ which satisfy the gradient relationship. What is the behavior of f relative to the curve $g(x,y) = c$ at these points?

18.3 Algebraic Solution



From the last exercise, we observe that ∇f must be parallel to ∇g at the extreme points (a,b) of f on the curve $g(x,y) = c$. So we should be able to find some scalar λ such that

$$\nabla f(a,b) = \lambda \nabla g(a,b)$$

This is called the *Lagrange multiplier rule* in honor of Joseph-Louis Lagrange, and the particular scalar λ that lines up the two vectors exactly is called the *Lagrange multiplier*.

In the picture above ∇f and ∇g lie on the same line at the extreme points.

18.3.1 Algorithm

Step 1. To find critical points of a function $f(x,y)$ on a constraint curve $g(x,y) = c$, first solve the following system of simultaneous equations for x, y, and λ:

$$\begin{aligned} f_x(x,y) &= \lambda g_x(x,y) \\ f_y(x,y) &= \lambda g_y(x,y) \\ g(x,y) &= c \end{aligned}$$

Step 2. Once you have found all the critical points (a,b), evaluate $f(a,b)$ at each of them. The critical points where f is greatest are maxima and the critical points where f is smallest are minima.

In case of functions of three variables, Lagrange Multiplier problems with one constraint equation means solving a system of four simultaneous equations.

18.3.2 Tips for Solving the System of Equations

Solving the system of equations can be hard! Here are some possible approaches for solving these systems of equations:

- Solve for x, y, z in terms of λ. Then you can plug these back into the constraint, find the value of λ, which will give you x, y, z.
- Or, eliminate λ to solve for one variable in terms of the others. Substitute them in to $g(x, y) = c$ to solve for x and y.
- Remember that whenever you take a square root, you must consider both the positive and the negative square roots.
- Remember that whenever you divide an equation by an expression, you must be sure that the expression is not 0. It may help to split the problem into two cases: first solve the equations assuming that a variable is 0, and then solve the equations assuming that it is not 0.

18.3.3 Interpretation of λ as a rate of change

The value of λ can be interpreted as the rate of change of the optimum value of f as c increases (where $g(x, y) = c$). Suppose the optimum value of f is obtained at the point (a, b), where both a and b depend on the choice of c, and suppose the optimum value is given by

$$f_{min}(c) = f(a(c), b(c))$$

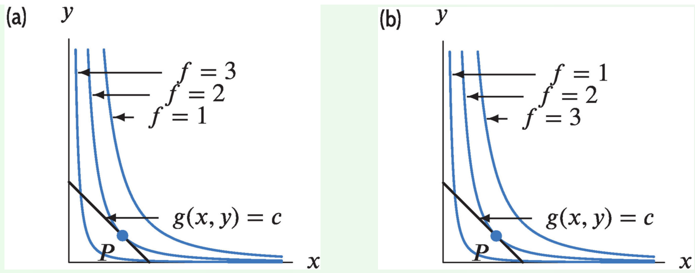
as a function of c. Then

$$\lambda = \frac{d}{dc} f_{min}(c)$$

18.4 Practice Problems

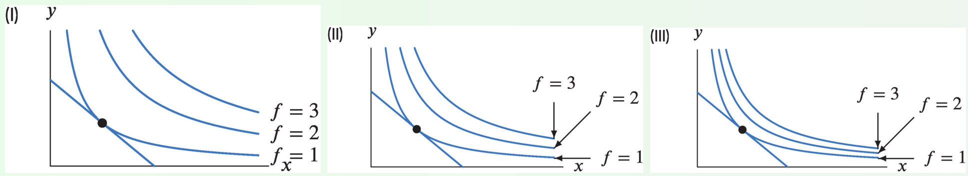
Question 18.134

The image below shows the contours of a function f. For the graphs (a) and (b), Is P a maximizer or minimizer of f subject to $g(x, y) = x + y = c$ with $x, y \geq 0$? What is the sign of λ? Where could the other extreme be located?



Question 18.135

The images below show the optimal point (marked with a dot) in three optimization problems with the same constraint. Arrange the corresponding values of λ in increasing order. (Assume λ is positive.)



Question 18.136

For each of the following problems find the maxima and minima of f on the constraint curve (or the surface) $g = c$.

(a) $f(x,y) = xy, \quad g(x,y) = 3x^2 + y^2 = 6$

(b) $f(x,y,z) = x^2 - y^2, \quad g(x,y,z) = x^2 + 2y^2 + 3z^2 = 1$

Question 18.137

Let $T(x,y,z) = x - 2y + 5z + 64$ give the temperature, in degrees Celsius, at the point (x,y,z), where x, y, and are measured in meters. Find the maximum and minimum temperature on the sphere $x^2 + y^2 + z^2 = 120$.

Project 6 | Rocket Science (Due April 19)

Many rockets, such as the *Pegasus XL* currently used to launch satellites and the *Saturn V* that first put men on the moon, are designed to use three stages (engines) in their ascent into space. A large first stage initially propels the rocket until its fuel is consumed, at which point the stage is jettisoned to reduce the mass of the rocket. The smaller second and third stages function similarly in order to place the rocket's payload into orbit about the earth. (With this design, at least two stages are required in order to reach the necessary velocities, and using three stages has proven to be a good compromise between cost and performance.) Our goal here is to determine the individual masses of the three stages, which are to be designed to minimize the total mass of the rocket while enabling it to reach a desired velocity.

For a single-stage rocket consuming fuel at a constant rate, the change in velocity resulting from the acceleration of the rocket vehicle has been modeled by

$$\Delta v = -c \ln\left(1 - \frac{(1-S)M_r}{P + M_r}\right)$$

where

- M_r is the mass of the rocket engine including initial fuel,
- P is the mass of the payload,
- S is a structural factor determined by the design of the rocket (specifically, it is the ratio of the mass of the rocket vehicle without fuel to the total mass of the rocket with payload),
- and c is the *(constant)* speed of exhaust relative to the rocket.

Now consider a rocket with *three* stages and a payload of mass A. Assume that outside forces are negligible and that c and S remain constant for each stage. If M_i is the mass of the ith stage, we can initially consider the rocket engine (i.e. the first stage) to have mass M_1 and its payload to have mass $M_2 + M_3 + A$; the second and third stages can be handled similarly.

Question 18.138

Show that the velocity attained after all three stages have been jettisoned is given by

$$v = c\left[\ln\left(\frac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A}\right) + \ln\left(\frac{M_2 + M_3 + A}{SM_2 + M_3 + A}\right) + \ln\left(\frac{M_3 + A}{SM_3 + A}\right)\right]$$

We wish to minimize the total mass $M = M_1 + M_2 + M_3$ of the rocket engine subject to the constraint that the final velocity v from Problem 1 is equal to some desired velocity v_f. The method of Lagrange multipliers is appropriate here, but difficult to implement using the current expressions.

To simplify, we define variables N_i so that the constraint equation may be expressed as

$$v_f = c(\ln N_1 + \ln N_2 + \ln N_3).$$

Thus for example $N_1 = \frac{M_1+M_2+M_3+A}{SM_1+M_2+M_3+A}$.

Question 18.139

Show that

$$\begin{aligned} \frac{M_1 + M_2 + M_3 + A}{M_2 + M_3 + A} &= \frac{(1-S)N_1}{1-SN_1} \\ \frac{M_2 + M_3 + A}{M_3 + A} &= \frac{(1-S)N_2}{1-SN_2} \\ \frac{M_3 + A}{A} &= \frac{(1-S)N_3}{1-SN_3} \end{aligned}$$

and conclude that

$$\frac{M + A}{A} = \frac{(1-S)^3 N_1 N_2 N_3}{(1-SN_1)(1-SN_2)(1-SN_3)} \qquad (\star)$$

But now M is difficult to express in terms of the N_i's. So instead of minimizing M, we are going to find N_i's that minimize $\ln \frac{M+A}{A}$.

Question 18.140

To see why this works, show that $\ln \frac{M+A}{A}$ is an increasing function of M. Explain why this implies that $\ln \frac{M+A}{A}$ is minimized at the same place where M is minimized.

Question 18.141

Now write $\ln \frac{M+A}{A}$ as a function of N_1, N_2, and N_3 by using the result of equation ($\star$). Then use Lagrange multipliers to minimize $\ln \frac{M+A}{A}$ subject to the constraint $c(\ln N_1 + \ln N_2 + \ln N_3) = v_f$. Find expressions for the values of N_i in terms of v_f where the minimum occurs.

[Hint: Use properties of logarithms to help simplify the expressions. You should be getting $N_1 = N_2 = N_3$.]

Question 18.142

Show that the minimum value of M as a function of v_f is

$$M = \frac{A(1-S)^3 e^{v_f/c}}{\left[1 - Se^{v_f/(3c)}\right]^3} - A$$

Question 18.143

If we want to put a three-stage rocket into orbit 100 miles above the earth's surface, a final velocity of approximately 17,500 mph is required. Suppose that each stage is built with a structural factor $S = 0.2$ and an exhaust speed of $c = 6000$ mph.

(a) Find the minimum total mass M of the rocket engines as a function of A.

(b) Find the mass of each individual stage as a function of A. (They are not equally sized!)

§19 | Global Optimization (Apr 3, Book chapter 15.2)

19.1 Definitions

- f has a global maximum on a two-dimensional region R at the point P_0 if $f(P_0) \geq f(P)$ for all points $P \in R$.
- f has a global minimum on a two-dimensional region R at the point P_0 if $f(P_0) \leq f(P)$ for all points $P \in R$.

If the region R is not stated explicitly, we take it to be the whole xy-plane unless the context of the problem suggests otherwise.

19.2 How do we know whether a function has a global maximum or minimum?

- A *closed* region is one which contains its boundary.
- A *bounded* region is one which does not stretch to infinity in any direction.

Theorem 19.1: Extreme Value Theorem

If f is a continuous function on a closed and bounded region R, then f achieves its global maximum and minimum at some points in R.

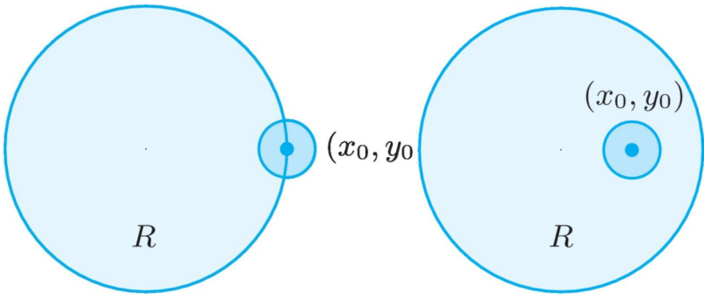
The global extrema could be on the boundary or in the interior of R.

Question 19.144

For the function $f(x,y) = (x-2)^2 + y^2$, find a region R such that

(a) f attains a global maximum value of 4 and a global minimum value of 0 over R.

(b) f attains a global maximum value of 3 and a global minimum value of 1 over R.



(a) A point on the boundary (b) A point in the interior

(c) f attains a global maximum value of 9 but has no global minimum over R.

19.3 Algorithm for Global Optimization

Question 19.145

Consider the function

$$F(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 1$$

defined on the disk D of radius $\sqrt{2}$ centered at the origin i.e.

$$D = \{(x, y) \mid x^2 + y^2 \leq 2\}.$$

Follow the steps below to find the global maximum and minimum of f on D.

(a) Find all the stationary critical points of F. [Hint: There are 4 such points.]

(b) In the list of critical points from part (a), identify the ones lying *inside* D (excluding the boundary).

(c) Evaluate F at the critical point(s) from part (b).

(d) Use Lagrange multiplier to find the maximum and minimum of $F(x, y)$ subject to the constraint $x^2 + y^2 = 2$. Note that this gives the extreme values of F on the boundary circle of D.

(e) Evaluate F at the critical point(s) from part (d).

(f) Compare the extreme values of F from part (c), and the extreme values of F from part (e), to find the global maximum and minimum of $F(x, y)$ on D.

19.4 Practice Problems

Question 19.146

For which of the regions D described below is it true that every continuous function $f(x, y)$ must attain an global maximum value and global minimum value on D? (There may be more than one.)

(i) D is the set of points (x, y) such that $|x| \leq 4$ and $|y| < 2$

(ii) D is the set of points (x, y) such that $|x + y| \leq 1$

(iii) D is the set of points (x, y) such that $x^2 + 4y^2 \leq 1$

(iv) D is the set of points (x, y) such that $x^2 + 4y \leq 1$

(v) D is the set of points (x, y) such that $-x \leq y \leq x$

Then do the following:

(a) For one of the regions D that you picked, find the global minimum and global maximum value of $f(x, y) = x^2 - 4x + y^2$ on the region.

(b) For one region you didn't pick, find a function $f(x, y)$ which has either no maximum or no minimum on the region D.

Question 19.147

Find all global extrema of the function

$$f(x, y) = 2x^3 + 2y^3 - 3x^2 - 3y^2 + 6$$

on the disc $D = \{(x, y) \mid x^2 + y^2 \leq 4\}$.

Question 19.148

Find the point (or points) on the graph of the function $f(x, y) = 2x^2 - y^2 + 1$ that is closest to the origin.

Project 7 | (Optional) Ordinary Linear Regression

Suppose that a scientist has reason to believe that two quantities x and y are related linearly, that is, $y = mx + b$, at least approximately, for some values of m and b. The scientist performs an experiment and collects data in the form of points $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$, and then plots these points. The points don't lie exactly on a straight line, so the scientist wants to find constants m and b so that the line $y = mx + b$ "fits" the points as well as possible (see figure 19.1).

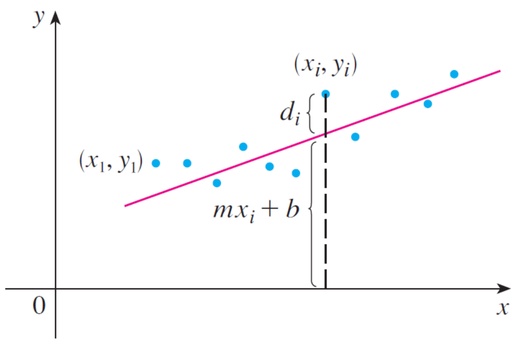


Figure 19.1

For each data point (x_i, y_i), observe that the corresponding point directly above or below it on the best fit line has y-coordinate $b + mx_i$. Let d_i be the vertical distance between each data point and the corresponding point on the straight line found above (see figure 19.1). Then for each data point (x_i, y_i), we get

$$d_i = y_i - (b + mx_i)$$

We can think of these d_is as error measurement of each data point. The method of *Ordinary Linear Regression* tries to minimize the **S**um of the **S**quares of the **E**rrors. In Statistics, this is known as SSE or RSS (Residual Sum of Squares). It is given by the function $f(b, m)$ as follows:

$$f(b, m) = \sum_{i=1}^{n} (y_i - (b + mx_i))^2$$

Our goal is to find b and m that minimizes $f(b, m)$.

Question 19.149

Show that the partial derivatives $\frac{\partial f}{\partial b}$ and $\frac{\partial f}{\partial m}$ are given by

$$\frac{\partial f}{\partial b} = -2\sum_{i=1}^{n}(y_i - (b + mx_i))$$

and

$$\frac{\partial f}{\partial m} = -2\sum_{i=1}^{n}(y_i - (b + mx_i)) \cdot x_i$$

Question 19.150

Show that the critical point equations $\frac{\partial f}{\partial b} = 0$ and $\frac{\partial f}{\partial m} = 0$ lead to a pair of simultaneous linear equations in b and m:

$$\begin{aligned} nb + \left(\sum_{i=1}^{n} x_i\right) m &= \sum_{i=1}^{n} y_i \\ \left(\sum_{i=1}^{n} x_i\right) b + \left(\sum_{i=1}^{n} x_i^2\right) m &= \sum_{i=1}^{n} x_i y_i \end{aligned}$$

Question 19.151

Solve the equations above for b and m, and show that

$$b = \frac{\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \qquad m = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

Question 19.152

Use the second derivative test to make sure that the stationary point we obtained above is in fact a local minimum.

Question 19.153

Either using the formula or from the construction of $f(b, m)$, find the line of best fit for the following data points: $(1, 1), (2, 1)$, and $(3, 3)$.

Go to this link: `https://mathlets.org/mathlets/linear-regression/` to experiment with best fit lines yourself.

§20 | Definite Integration in two variables (Apr 6, Book Chapter 16.1)

20.1 Introduction

In one-variable calculus, we learned that the definite integral $\int_a^b f(x)\mathrm{d}x$ gave the area under the graph of $f(x)$ over the interval $[a,b]$ on the x -axis. For two-variable functions, the input space is the xy-plane, so we compute volume integrals over two-dimensional regions R of the plane:

$$\int_{\mathrm{R}} f(x,y)\,d\mathrm{A}$$

which measure the amount of volume under the graph of $f(x,y)$ and over the region R. Note that R is NOT necessarily a rectangle. The dA indicates that the integration is taken over a region R with (A)rea in the xy-plane.

20.2 Construction

Recall that we approximate definite integrals in one-variable calculus By chopping the interval $[a,b]$ into subintervals and summing the areas of rectangles (usually) based on each subinterval.

For volume integrals, we can chop R into subregions and sum the volumes of rectangular (usually) prisms based on each subregion. As with one-variable integrals, using smaller and smaller subregions of R leads to better and better approximations.

Digression

The Riemann Sum definition: If R is a rectangle, we can express this approximation procedure more formally using Riemann sums and limits as follows.

Suppose the function $f(x,y)$ is continuous on R, the rectangle $a \le x \le b, c \le y \le d$. Consider the partitions

$$\mathrm{P} : \{a = x_0 < x_1 < \cdots < x_n = b\}$$

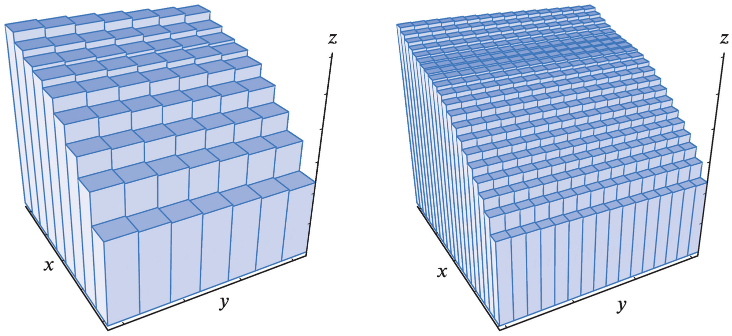
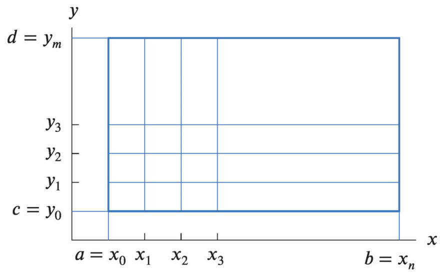


Figure 20.1: Approximating volume under a graph with finer and finer Riemann sums

and

$$Q : \{c = y_0 < y_1 < \cdots < y_m = d\}$$



For a choice of points $\left(x^*_{ij}, y^*_{ij}\right)$ in the (i, j)th subrectangle $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$,we define the Riemann Sum of f over R as

$$S^*_{(P,Q)} = \sum_{i=1}^{n}\sum_{j=1}^{m} f\left(x^*_{ij}, y^*_{ij}\right)\Delta x_i \Delta y_j = \sum_{i=1}^{n}\sum_{j=1}^{m} f\left(x^*_{ij}, y^*_{ij}\right)(x_i - x_{i-1})\left(y_j - y_{j-1}\right)$$

Also let $\|(P, Q)\|$ denote the maximum area of a subrectangle i.e.

$$\|(P, Q)\| = \max_{i,j}(x_i - x_{i-1})\left(y_j - y_{j-1}\right)$$

Then the definite integral of f over R is given by

$$\int_R f\, dA = \lim_{\|(P,Q)\| \to 0} S^*_{(P,Q)}$$

In the case the partitions are uniform, above limit can be rewritten as

$$\int_R f\,dA = \lim_{n,m\to\infty} S^*_{(P,Q)}$$

Question 20.154

If R is any region in the plane, what does the double integral $\iint_R 1\,dA$ represent? Why?

Question 20.155

Suppose the shape of a flat plate is described as a region R in the plane, and $f(x,y)$ gives the density (mass of unit area) of the plate at the point (x,y) in kilograms per square meter. What does the double integral $\iint_R f(x,y)\,dA$ represent? Why?

Question 20.156

What does the double integral $\dfrac{\iint_R f(x,y)\,dA}{\iint_R 1\,dA}$ represent? Why?

Question 20.157

Let R be the rectangle $-1 \le x \le 1, -1 \le y \le 1$, and T is the top half $-1 \le x \le 1, 0 \le y \le 1$, and L is the left half $-1 \le x \le 0$ $-1 \le y \le 1$.

Without evaluating any of the integrals, decide which of them are positive, which are negative, and which are zero.

a) $\int\limits_R x\,dA$ b) $\int\limits_R y\,dA$ c) $\int\limits_T y\,dA$ d) $\int\limits_R \left(x - x^2\right) dA$

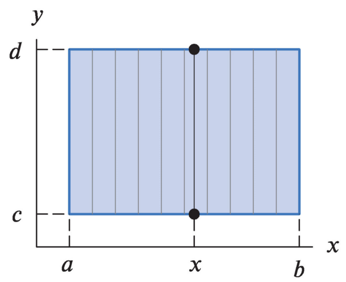
e) $\int\limits_T \left(y - y^2\right) dA$ f) $\int\limits_L \left(x^2 - x\right) dA$ g) $\int\limits_L \left(y + y^3\right) dA$ h) $\int\limits_R (2x + 3y)\,dA$

Question 20.158

For the same set of integrals above, arrange $(a), (c), (e)$ and (f) in increasing order.

§21 | Iterated Integrals (Apr 8, Book Chapter 16.2)

Consider the volume integral of a function f over a rectangular region R. Instead of dicing R into a grid to approximate the integral, we can cut it in "vertical strips" (in the y-direction) in the xy-plane. Then, we build solid slices whose heights are determined by the value of $f(x,y)$ along one edges of each strip.



Volume of such a solid slice is given by

$$\underbrace{\left(\int_c^d f(x_{i-1},y)dy\right)}_{\text{Area of a vertical cross-section parallel to YZ-plane}} \times \underbrace{(x_i - x_{i-1})}_{\text{width in X-direction}}$$

Now if we sum up the volumes of these solids and take a limit, we should get back the volume integral

$$\int_R f\,dA = \lim_{\Delta x_i \to 0} \sum_{i=1}^n \left(\int_c^d f(x_{i-1},y)dy\right)(x_i - x_{i-1}) = \int_a^b \int_c^d f(x,y)dydx$$

This is called an *iterated integral*.

21.1 Type I and Type II regions

A region R is called *Type I* if it can be written in the following way:

$$R = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

We can then compute a double integral as

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

A region R is called *Type II* if it can be written in the following way:

$$R = \{(x,y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$$

We can then compute a double integral as

$$\iint_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

For a quick reference, the rules on the limits of an iterated integral are as follows:

- The limits on the outer integral must be constants.
- The limits on the inner integral can involve only the variable in the outer integral. For example, if the inner integral is with respect to x, its limits can be functions of y.

Question 21.159

Match the integral with the appropriate region of integration.

(i) R_1 : The triangle with vertices $(0,0),(2,0),(0,1)$

(ii) R_2 : The triangle with vertices $(0,0),(0,2),(1,0)$

(iii) R_3 : The triangle with vertices $(0,0),(2,0),(2,1)$

(iv) R_4 : The triangle with vertices $(0,0),(1,0),(1,2)$

a) $\int_0^1 \int_0^{2-2x} f(x,y)dydx$

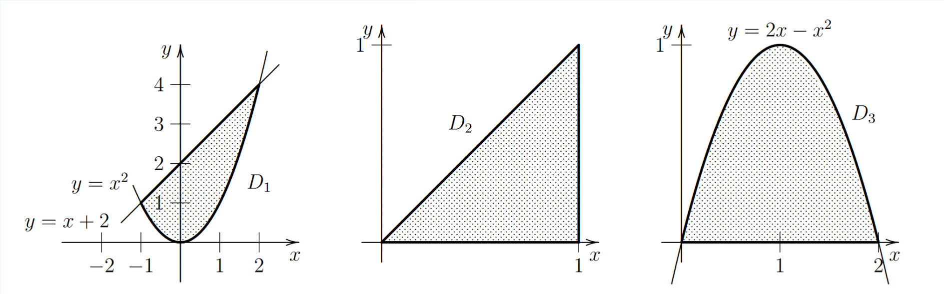
b) $\int_0^1 \int_0^{2-2y} f(x,y)dxdy$

c) $\int_0^1 \int_0^{2x} f(x,y)dydx$

d) $\int_0^1 \int_{2y}^{2} f(x,y)dxdy$

Question 21.160

Here are some type I regions. For each of them set up the given double integral as an iterated integral and evaluate it.

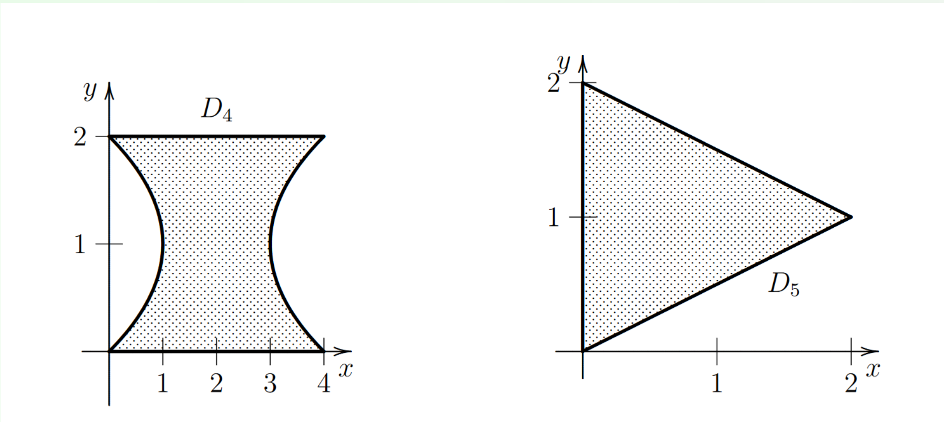


a) $\iint_{D_1} xy\,dA$ Ans: 45/8 b) $\iint_{D_2} e^{x^2}\,dA$ Ans: $(e-1)/2$

c) $\iint_{D_3} (x-1)y^2\,dA$ Ans: 0 d) $\iint_{D_1} 1\,dA$ Ans: 4.5

Question 21.161

Here are some type II regions. For each of them set up the given double integral as an iterated integral and evaluate it. The curves in D_4 are $x = 2y - y^2$ and $x = y^2 - 2y + 4$.



a) $\iint_{D_2} (1-y)^3\,dA$ Ans: 1/5 b) $\iint_{D_4} (y-1)x^2\,dA$ Ans: 0

c) $\iint\limits_{D_2} cos(x^2)dA$ Ans: $\frac{\sin(1)}{2}$ d) $\iint\limits_{D_5} (x-1)dA$ Ans: $-2/3$

21.2 Switching the Order of Integration

If a region is both type I and type II, you may find that one order of integration will be simpler to deal with than the other. Sometimes, when converting a double integral to an iterated integral, we decide the order of integration based on the integrand, rather than the shape of the region - some integrands are easy to integrat ewith respect to one variable and much harder (or even impossible) to integrate with respect to the other.

Question 21.162

For each of the following integrals, draw the region in question, write down an integral with the reverse order of integration, then finally integrate.

a) $\int_0^2 \int_x^2 (x+y)dydx$ Ans: 4 b) $\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1}dydx$ Ans: 26

c) $\int_0^1 \int_y^1 e^{x^2}dxdy$ Ans: $\frac{e-1}{2}$ d) $\int_0^{\pi/2} \int_y^{\pi/2} \frac{\sin(x)}{x}dxdy$ Ans: 1

e) $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} (3x-x^3)^{10}dxdy$ Ans: $\frac{2^{12}}{33}$ f) $\int_1^{e^3} \int_{\ln(y)}^3 (e^x-x)^5dxdy$ Ans: $\frac{e^3-4}{6}$

§22 | Double Integral in Polar Coordinates (Apr 10, Book Chapter 16.4)

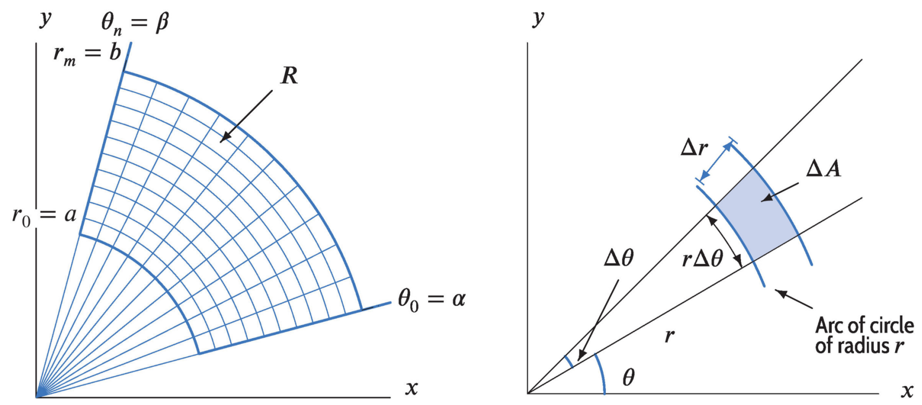
22.1 When to Use it

We will use Polar coordinates when the region of integration is easier to express in polar coordinates. This might happen mostly two cases:

- The region is interior of a polar curve of the form $r = f(\theta)$.
- The region is circularly symmetric and either the bounds or the integrand has terms involving $x^2 + y^2$.

22.2 What is dA in Polar Coordinates?

Consider the way a Riemann Sum is constructed on a region with polar coordinates. Subdividing a region of the form $\mathrm{R} = \{(r,\theta) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta\}$ into smaller partitions gives us the following pictures:



Accordingly, $\Delta\mathrm{A}$ is approximated as $r\Delta r\Delta\theta$ and after taking a limit of the Riemann sum we get the following identity

$$\iint_{\mathrm{R}} f(x,y)\,d\mathrm{A} = \iint_{\mathrm{R}} f(r\cos(\theta), r\sin(\theta))\,r\,dr\,d\theta$$

Note that extra r in the second integral. This shows up due to the fact that the area infinitesimal dA does not look like a small square in polar coordinates. A more detailed explanation will be given after we learn about Jacobians and change of variables.

Digression

When doing integrals in polar coordinates, you often need to integrate trigonometric functions. The double-angle formulas are very useful for this.

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta$$

These two identities make it easy to integrate $\sin^2\theta$ and $\cos^2\theta$.

Question 22.163

For the following problems

(a) Sketch the region of integration.

(b) Try to describe the region in polar coordinates and decide whether you should use polar coordinates or cartesian coordinates.

(c) Evaluate the integral.

(i) $\iint_R \sqrt{x^2+y^2}\,dA$, where R is the region $x^2+y^2 \le 1$. Ans: $2\pi/3$

(ii) $\iint_R x\,dA$, where R is the region $x^2+y^2 \le 1, x \ge 0$. Ans: $2/3$

(iii) $\iint_R (x+y)^2\,dA$, where R is the region $1 \le x^2+y^2 \le 9, x \ge 0, y \ge 0$. Ans: $10\pi + 20$

(iv) $\iint_R \sqrt{x^2+y^2}\,dA$, where R is the region $0 \le x \le 1, 0 \le y \le 1$ Ans: $2/3$

(v) $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} e^{x^2+y^2}\,dy\,dx$ Ans: $\pi\frac{e^4-1}{2}$

(vi) $\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sqrt{1-x^2-y^2}\, dx dy$ Ans: $2\pi/3$

(vii) $\int_{\pi/4}^{\pi/2} \int_{1/\sin\theta}^{4/\sin\theta} r dr d\theta$ Ans: 7.5

Question 22.164

1. Find the area of the region R lying between the curves $r = 2 + sin(3\theta)$ and $r = 4 - \cos(3\theta)$.
2. The region R lying inside the curve $r = 2 + \sin 3\theta$ and outside the curve $r = 3 - \sin(3\theta)$ consists of three pieces. Find the area of one of those pieces.
3. Find the area of the region which lies inside the circle $x^2 + (y-1)^2 = 1$ but outside the circle $x^2 + y^2 = 1$.

§23 | Triple Integrals (Apr 13 and 15, Book Chapter 16.3)

Note: We will learn this topic using a Mathematica notebook. I have only compiled the main results and some exercises here.

23.1 Volume and Mass as an Iterated Integral

The volume of a three-dimensional solid W is given by $\iiint\limits_{W} dV$. If the density at point (x, y, z) is given by $f(x, y, z)$, then it's mass is given by $\iiint\limits_{W} f\,dV$.

For suitable three dimensional regions W, the triple integral can be represented as an iterated integral of the form:

$$\int\limits_{W} f\,dV = \int\limits_{a}^{b} \left(\int\limits_{\psi_1(z)}^{\psi_2(z)} \left(\int\limits_{\phi_1(y,z)}^{\phi_2(y,z)} f(x, y, z)\,dx \right) dy \right) dz$$

where y and z are treated as constants in the innermost (dx) integral, and z is treated as a constant in the middle (dy) integral.

Other orders of integration are possible and sometimes necessary to make the integration feasible. In general, the rules on the limits on a triple integral are as follows:

- The limits for the outer integral are constants.
- The limits for the middle integral can involve only one variable (that in the outer integral).
- The limits for the inner integral can involve two variables (those on the two outer integrals).

23.2 Concept Test

Question 23.165

What does the integral $\int_0^1\int_0^1\int_0^1 2dzdydx$ represent?

(a) Twice the volume of a cube of side 1.

(b) The volume of a cube of side length 1.

(c) Twice the volume of a sphere of radius 1.

(d) The volume under the plane $z = 2$ and over a square of side length 1 in the xy-plane.

Question 23.166

For each integral (a) - (d) that makes sense, match it with its region of integration, I or II.

a) $\displaystyle\int_1^3\int_{y-1}^2\int_0^y f(x,y,z)dzdxdy$

b) $\displaystyle\int_1^3\int_0^y\int_2^{y-1} f(x,y,z)dxdydz$

c) $\displaystyle\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} f(x,y,z)dzdydx$

d) $\displaystyle\int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} f(x,y,z)dzdxdy$

I. The region below the plane $z = y$ and above the triangle with vertices $(0,1),(2,1),(2,3)$ in the xy-plane.

II. The region between the upper hemisphere of $x^2+y^2+z^2 = 1$ and the xy-plane.

Question 23.167

Find a solid W such that the integral

$$I = \iiint_W (x^2 - x)dV$$

a) is positive.

b) is negative

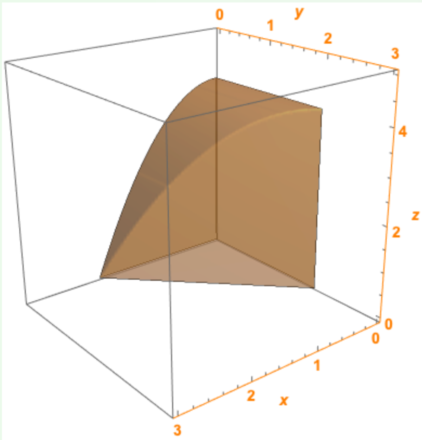
c) is zero.

23.3 Practice Problems

Question 23.168

Do the same for the region T in the first octant bounded by the plane $x + y = 2$ and the parabolic cylinder $z = 4 - x^2$.

a) $dz\,dy\,dx$ b) $dy\,dz\,dx$ c) $dy\,dx\,dz$

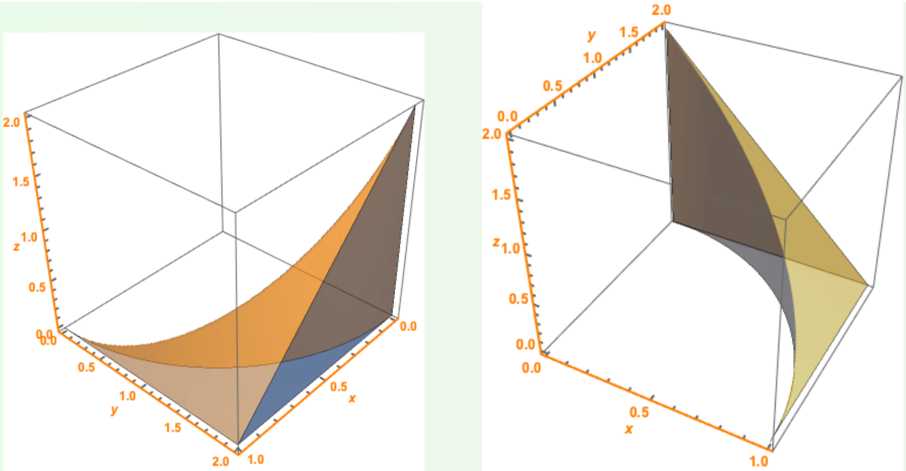


Question 23.169

Consider the region T in the first octant bounded by the elliptical cylinder $4x^2 + y^2 = 4$, the plane $2x + z = 2$ and the plane $y = 2$. For the given order of integration, write an iterated integral equivalent to the triple integral $\iiint_T 1\,dV$.

a) $dx\,dy\,dz$ b) $dy\,dx\,dz$ c) $dz\,dx\,dy$

Here are two pictures of the region T from two different viewpoints.

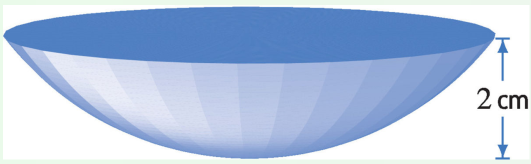


23.4 Practice Problems with Polar Coordinates

The following exercises require you to set up the base of integration in polar coordinates. The coordinate system (r, θ, z) is called a cylindrical coordinate system. Don't forget the extra r when setting up polar integrals.

Question 23.170

A spherical ball of radius 5 cm is sliced off by a plane that leaves the following solid. Write an iterated triple integral in $dzdrd\theta$ which represents the volume of this region.

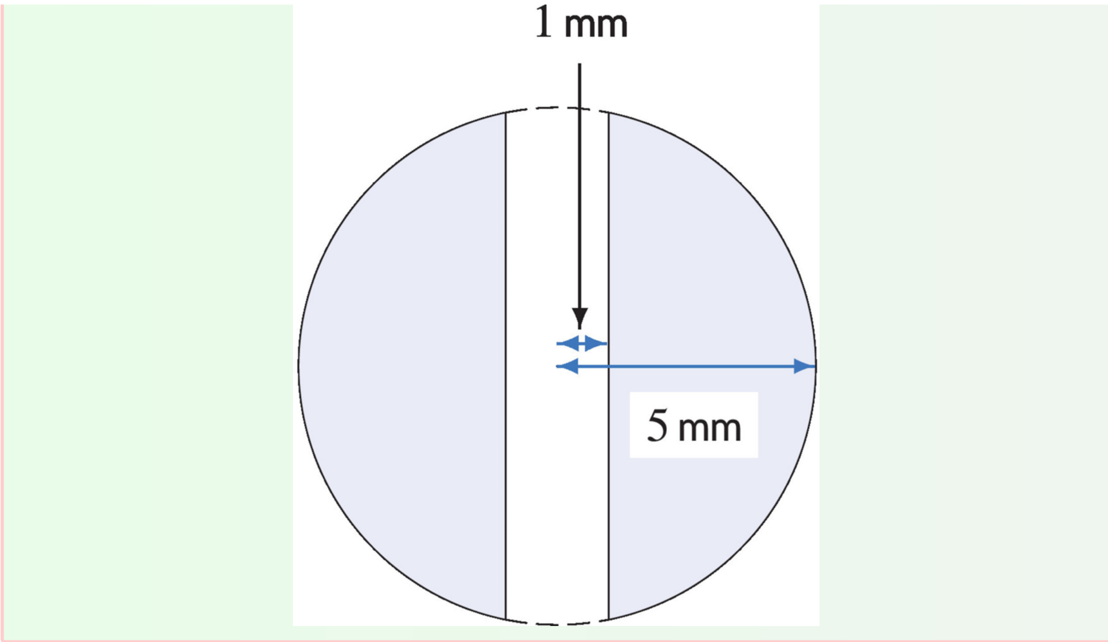


Question 23.171

A bead is made by drilling a cylindrical hole of radius 1 mm through a sphere of radius 5 mm.

(a) Set up a triple integral in $dzdrd\theta$ that represents the volume of the bead.

(b) Evaluate the integral.


1 mm
5 mm

Project 8 | Normal Probability Distribution (Due April 26)

In statistical applications it is important to know the exact value of the area under the bell–shaped curve $y = e^{-t^2/2}$, i.e., we need to evaluate the improper integral

$$k = \int_{-\infty}^{\infty} e^{-t^2/2} dt$$

Using integral k, the standard normal density function is given by

$$f(t) = \frac{1}{k} e^{-t^2/2}.$$

In this project, as a byproduct of our ability to evaluate double integrals using polar coordinates, we will obtain an exact value for k.

Notice that, by symmetry, we have

$$k = \int_{-\infty}^{\infty} e^{-t^2/2} dt = 2 \int_{0}^{\infty} e^{-t^2/2} dt$$

In addition, by definition of the improper integral from 0 to ∞, we also have

$$\int_{0}^{\infty} e^{-t^2/2} dt = \lim_{a \to \infty} \int_{0}^{a} e^{-t^2/2} dt$$

Question 23.172

(a) Let a be positive, and let D_a be the square domain $[0, a] \times [0, a]$. Show that

$$\iint\limits_{D_a} e^{-\left(x^2+y^2\right)/2}\, dA = \left(\int_0^a e^{-t^2/2}\, dt \right)^2$$

(b) Use part (a) to show that

$$\lim_{a\to\infty} \iint\limits_{D_a} e^{-\left(x^2+y^2\right)/2}\, dA = \frac{1}{4}k^2$$

(c) Now designate by S_a the quarter-disk of radius a consisting of points with polar coordinates satisfying $0 \leq \theta \leq \pi/2$ and $0 \leq r \leq a$, and designate by T_a the quarter-disk of radius $\sqrt{2}a$ consisting of points with polar coordinates satisfying $0 \leq \theta \leq \pi/2$ and $0 \leq r \leq \sqrt{2}a$. Explain geometrically why

$$\iint\limits_{S_a} e^{-\left(x^2+y^2\right)/2}\, dA \leq \iint\limits_{D_a} e^{-\left(x^2+y^2\right)/2}\, dA \leq \iint\limits_{T_a} e^{-\left(x^2+y^2\right)/2}\, dA$$

(d) Transform

$$\iint\limits_{S_a} e^{-\left(x^2+y^2\right)/2}\, dA$$

into an iterated integral in polar coordinates, and evaluate this integral exactly.

(e) Transform

$$\iint\limits_{T_a} e^{-\left(x^2+y^2\right)/2}\, dA$$

into an iterated integral in polar coordinates, and evaluate this integral exactly.

(f) Show that the integrals in parts (d) and (e) both approach the same limiting value as a approaches infinity.

(g) Use the results of parts (a) through (f) to find the value of k.

§24 | Vector Fields (April 17, Book Chapter 17.3)

24.1 Motivation

Given a function $f(x,y)$ of two variables, consider what the gradient function $\vec{\nabla} f(x,y)$ represents. To each point in the plane it associates a vector that represents the magnitude and direction greatest increase of $f(x,y)$ at the at point. We have learned earlier that vector $\vec{\nabla} f(x,y)$ is always perpendicular to the contour plot of f through the point (x,y). Using this knowledge we can draw all these vectors in XY-plane as in the following picture:

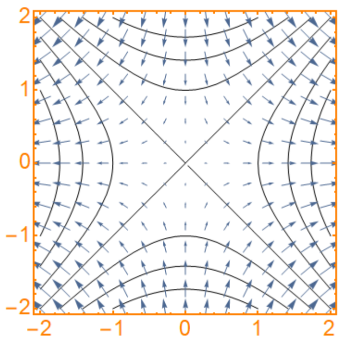


Figure 24.1: Contour Plot and Gradients of $f(x,y) = x^2 - y^2$

Thus $\vec{\nabla} f$ is a function that takes vector inputs (position vector of a point) and gives a vector output. In general, functions of this kind are given a special name: they are called *Vector Fields*.

Definition 4. A vector field in 2-space is a function

$$\vec{\mathbf{F}}(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$$

whose value at a point (x,y) is a 2-dimensional vector. Similarly, a vector field in 3-space is a function $\vec{\mathbf{F}}(x,y,z)$ whose values are 3-dimensional vectors.

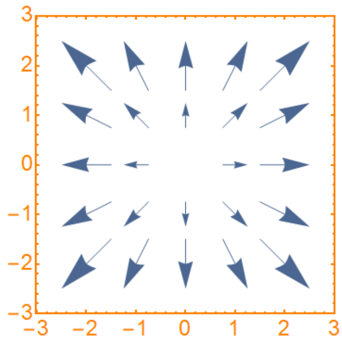
There are many other examples of vector fields that we come across in real life frequently e.g. Ocean currents, wind flow in your weather app, electromagnetic force field, gravitational force etc. It is not necessary that a vector field $\vec{\mathbf{F}}$ comes from gradient of a function.

24.2 Basic Examples

Example 5. Consider the vector field $\vec{\mathbf{F}}(x,y) = x\hat{\imath} + y\hat{\jmath}$. We are interested in what it looks like.

- Along the x-axis, we have $y = 0$, so the arrows are purely horizontal.
- Along the y-axis, we have $x = 0$, so the arrows are purely vertical.
- What about the arrow at other points? According to the formula, the vector at the point (x,y) is equal to its position vector. So the arrow points radially outward and has length equal to the distance of the point from the origin.

With these observations, our vector field looks like:



Question 24.173

Plot the following vector fields:

(a) $\vec{\mathbf{F}}(x,y) = y\hat{\imath}$

(b) $\vec{\mathbf{F}}(x,y) = (x+y)\hat{\imath} + (x-y)\hat{\jmath}$

Question 24.174

Give examples of vector fields where:

(a) all output vectors are parallel to the x-axis, and have the same magnitude if they are located on the same vertical line.

(b) all output vectors are unit length, and perpendicular to the position vectors $x\hat{\imath} + y\hat{\jmath}$.

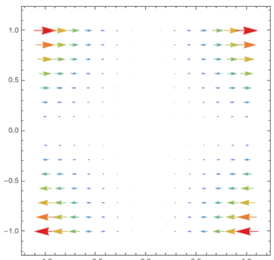
Question 24.175

Match the vector fields.

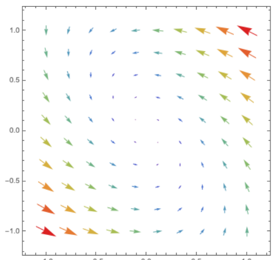
a) $\langle y, 1\rangle$ b) $\langle 0, 2y\rangle$ c) $\langle -x, -2y\rangle$ d) $\langle -2y, 3x\rangle$

e) $\langle 0, x^2y\rangle$ f) $\langle -2y, -x\rangle$ g) $\langle x^2y, 0\rangle$ h) $\langle -x, 0\rangle$

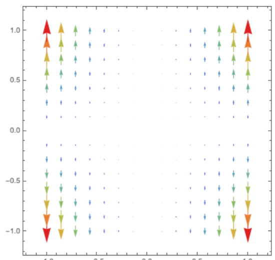
i) $\langle -2y, 1\rangle$ j) $\langle -y - x, x\rangle$ k) $\langle -y, x\rangle$ l) $\langle x^2, y^2\rangle$



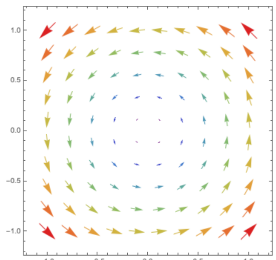
(1)



(2)



(3)



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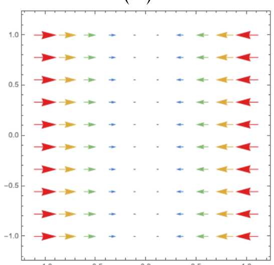

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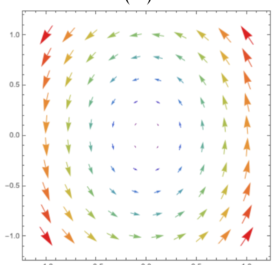

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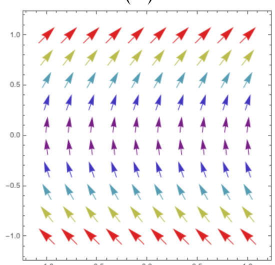

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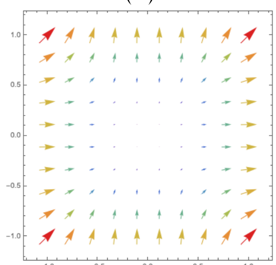

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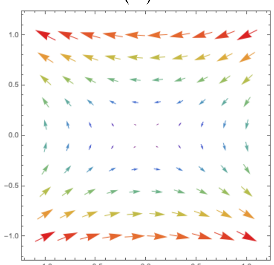

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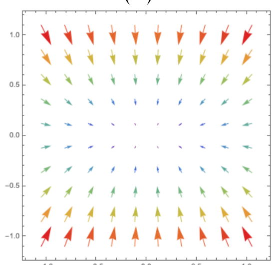

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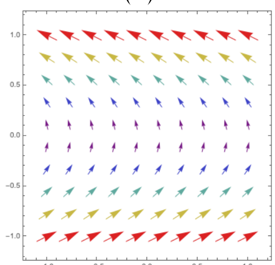

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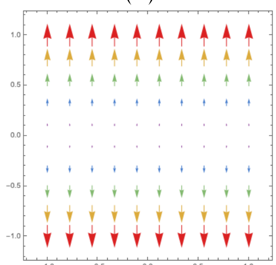


(12)

24.3 Gradient Vector Field

A vector field $\vec{\mathbf{F}}(x,y)$ is called a gradient vector field if it can be written as the gradient of some underlying function $f(x,y)$ i.e.

$$\vec{\mathbf{F}} = \vec{\nabla} f = \left\langle f_x, f_y \right\rangle$$

In that case, the function f is called the potential function for the field $\vec{\mathbf{F}}$ (the terminology comes from Physics and has to do with potential energy of objects).

Question 24.176

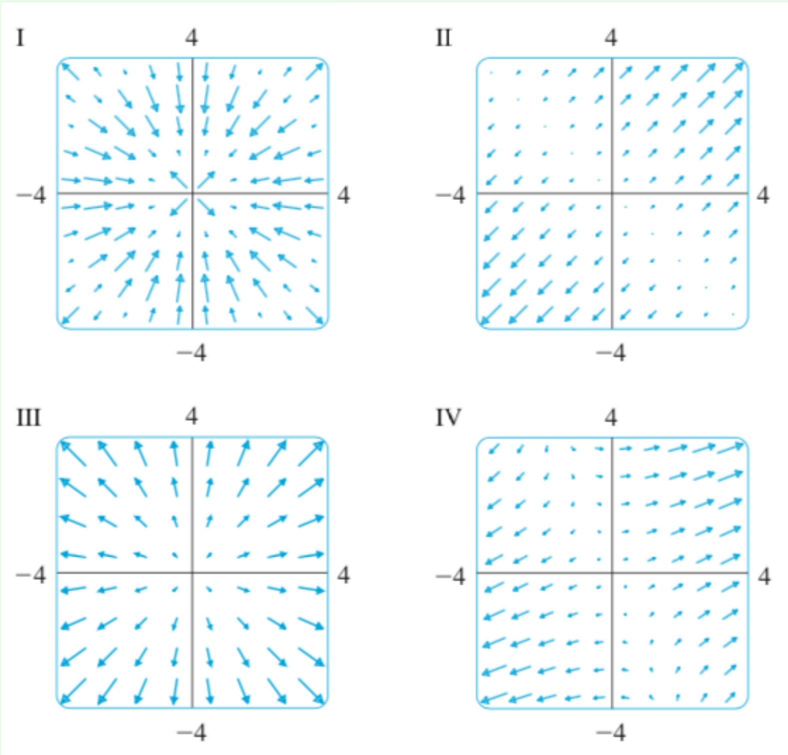
Match the following functions with their gradient vector fields.

a) $x^2 + y^2$

b) $x(x+y)$

c) $(x+y)^2$

d) $\sin\sqrt{x^2+y^2}$



Gradient vector fields have some very nice properties (which we'll explore soon), so it will be important for us to identify when a given vector field is a gradient vector field. Recall that Clairaut's theorem says $f_{xy} = f_{yx}$ for a smooth function. That means for a gradient vector field $\vec{F} = \vec{\nabla} f$, we must have $P_y = Q_x$. As a contrapositive, we have the following test.

Theorem 24.1

If $\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$, then $\vec{F}$ is not a gradient vector field.

The natural follow-up question to ask is whether the equality of the two partials then guarantees a gradient vector field. Unfortunately, we cannot yet answer this question with the material covered so far. We will show that the answer is affirmative in a couple of chapters.

24.3.1 Finding the potential function

Assume that you are told that a given vector field $\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$ is in fact a gradient vector field. How do we then find the exact formula of a function $f(x,y)$ such that $\vec{F} = \vec{\nabla} f$?

Here is an algorithm. Note that we want to find f such that

$$f_x = P \text{ and } f_y = Q$$

Integrating the first equation with respect to x gives us

$$f(x,y) = \int P(x,y)dx + g(y)$$

The extra $g(y)$ serves as a replacement for the arbitrary constant term '$+c$' because $\dfrac{\partial g(y)}{\partial x} = 0$. With this presumptive model for $f(x,y)$ we must have

$$\frac{\partial}{\partial y}\left(\int P(x,y)dx + g(y)\right) = Q$$

From here you can solve for g. We can also start from Q if that seems easier.

It's best to learn the algorithm by doing examples:

Example 6. Suppose $\vec{F}(x,y) = (x+y)\hat{\imath} + (x-y)\hat{\jmath} = \vec{\nabla} f(x,y)$. Then $f_x = x+y$ and $f_y = x-y$. The first equation tells us

$$f(x,y) = \frac{x^2}{2} + xy + g(y)$$

Then

$$x - y = \frac{\partial}{\partial y}\left(\frac{x^2}{2} + xy + g(y)\right) = 0 + x + g'(y) \implies g'(y) = -y \implies g(y) = -\frac{y^2}{2} + c$$

So overall, $f(x,y) = \frac{x^2}{2} + xy - \frac{y^2}{2} + c$.

Question 24.177

For each of the following vector fields, find a potential function or explain why one doesn't exists.

(a) $x\hat{i}$

(b) $y\hat{i}$

(c) $(x^2 - y^2)\hat{i} - 2xy\hat{j}$

(d) $(2xy + 5)\hat{i} + (x^2 + 8y^3)\hat{j}$

(e) $2xe^{x^2}\sin y\hat{i} + e^{x^2}\cos y\hat{j}$

(f) $\left(yze^{xyz} + z^2\cos\left(xz^2\right)\right)\hat{i} + xze^{xyz}\hat{j} + \left(xye^{xyz} + 2xz\cos\left(xz^2\right)\right)\hat{k}$

(g) $\dfrac{-y\hat{i} + x\hat{j}}{x^2 + y^2}$

(h) $\dfrac{\vec{r}}{\|\vec{r}\|^3} = \dfrac{x}{(x^2+y^2+z^2)^{3/2}}\hat{i} + \dfrac{y}{(x^2+y^2+z^2)^{3/2}}\hat{j} + \dfrac{z}{(x^2+y^2+z^2)^{3/2}}\hat{k}$

§25 | Flow Lines of a Vector Fields (April 20, Book Chapter 17.4)

25.1 Motivation

In the last lecture we defined vector fields and learned how to graphically represent them on coordinate plane. For a practical application, consider a ocean current map that depicts the water velocity vector field and suppose we are interested in predicting the path of an iceberg that just showed up. The iceberg will obviously try to "go with the flow" and move along a curve such that it follows the arrow directions. In this chapter, we are interested in finding equations of those curves given the vector field. It will be helpful to follow along in the `lec25_flow_line.nb` file.

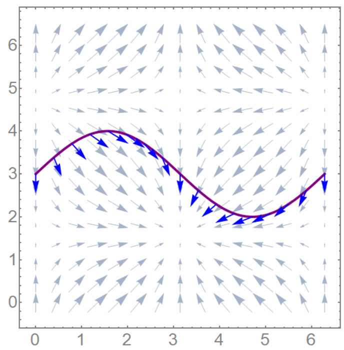
25.2 Vector Field along a Curve

Instead of looking at all of the vectors $\vec{\mathbf{F}}(x,y)$ in the plane, suppose we focus only on those along a given parameterized curve $\vec{r}(t) = \langle x(t), y(t)\rangle$. To express this symbolically, we write $\vec{\mathbf{F}}(\vec{r}(t))$ and interpret the input $\vec{r}(t)$ to the vector field as the position vector of the point $(x(t), y(t))$ coming from the parameterization.

Example 7. For the vector field $\vec{\mathbf{F}}(x,y) = \sin(x)\vec{i} + \cos(y)\vec{j}$ and the parameterized curve $\vec{r}(t) = \langle t, \sin(t)+3\rangle$, we get the (blue) vectors

$$\vec{\mathbf{F}}(\vec{r}(t)) = \vec{\mathbf{F}}(x(t), y(t)) = \sin(x(t))\vec{i} + \cos(y(t))\vec{j} = \sin(t)\vec{i} + (\cos(\sin(t)+3))\vec{j}$$

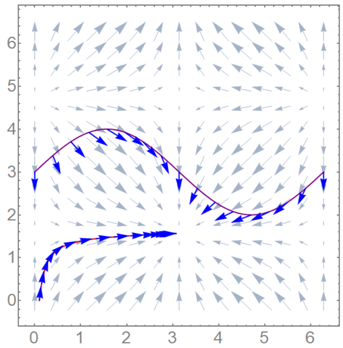
along the (purple) parameterized curve.



See Curve through vector field in `lec25_flow_line.nb`

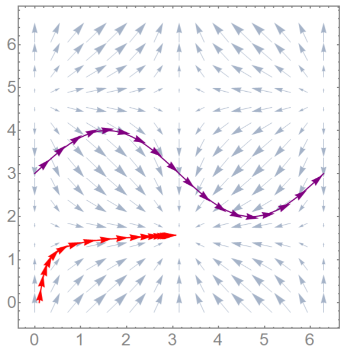
25.3 Flow Lines

There are special curves which fit the vector field perfectly, so that the arrows from the vector field are exactly the same as the velocity vectors associated with the parameterized curve. These curves are called flow lines because they “go with the flow” of the vector field (they aren’t always lines, but that’s what we call them).



See Flow line through vector field in `lec25_flow_line.nb`

In the image above, the shorter red curve “goes with the flow” of the vector field. To study flow lines carefully, we need to express this idea via an equation. We recall that the velocity vector $\vec{r}\,'(t)$ represents the tangent direction of the parameterized curve at any t. The image below shows the velocity vectors along our two different curves through the vector field.



See Velocity vectors along curves in `lec25_flow_line.nb`

Definition 8. A *flow line* of a vector field $\vec{\mathbf{F}}(x,y)$ is a path whose velocity vector equals the vector field along the path. Thus,

$$\vec{r}\,'(t) = \vec{\mathbf{F}}(\vec{r}(t))$$

The *flow* of a vector field is the family of all of its flow lines.

Obviously, not all parameterized curves $\vec{r}(t)$ satisfy this flow line equation (e.g., the purple curve in the above image), but we would like to identify the ones that do. This is relatively easy to do visually, however coming up with the formulas $\vec{r}(t)$ for the flow lines is not always so easy.

25.4 Identifying Flow Line formula

To identify the formula for the parameterized curve, we need to solve the flow line equation. If $\vec{\mathbf{F}}(x,y) = \mathrm{P}(x,y)\hat{\imath} + \mathrm{Q}(x,y)\hat{\jmath}$ then the equation can be rewritten as a system

$$\frac{dx}{dt} = \mathrm{P}(x,y)$$
$$\frac{dy}{dt} = \mathrm{Q}(x,y)$$

This is called a system of Differential Equations. There are no general methods of solving such a system and the solutions can be quite complicated. However, given a solution curve, it is easy to check that the curve indeed satisfies the differential equations.

Example 9. Let's take a look at some easy differential equations for which we can guess a solution.

Suppose $\vec{\mathbf{F}}(x,y) = 3\hat{\imath} + 2\hat{\jmath}$. The corresponding differential equations are

$$x'(t) = 3, \qquad y'(t) = 2$$

So the flow line curves are $\vec{r}(t) = \langle 3t + c_1, 2t + c_2 \rangle$. Specific values of the arbitrary constants c_1 and c_2 in the formula determines which specific flow line we are looking at.

Question 25.178

Determine the flow lines of the following vector fields.

a) $2\hat{\imath} + x\hat{\jmath}$

b) $-y\hat{\imath} + x\hat{\jmath}$

The differential equation

$$x'(t) = \sin(x)$$

is not the same as

$$x'(t) = \sin(t)$$

The later has the solution $x(t) = c - \cos(t)$ whereas the former has solution $x(t) = 2\,\mathrm{arccot}\left(e^{-t-c}\right)$.

25.5 Practice Problems

Question 25.179

Consider a vector field $\vec{\mathbf{F}} = -h_y\hat{\imath} + h_x\hat{\jmath}$ where $h(x,y)$ is a smooth function. Explain why

(a) $\vec{\mathbf{F}}$ is perpendicular to $\vec{\nabla}h$.

(b) the flow lines of $\vec{\mathbf{F}}$ are along the level curves of h.

Question 25.180

Show that every flow line of the vector field $\vec{\mathbf{F}}(x,y) = y\hat{\imath} + x\hat{\jmath}$ lies on a level curve for the function $f(x,y) = x^2 - y^2$.

Question 25.181

(a) Show that $h(t) = e^{-2at}\left(x^2+y^2\right)$ is constant along any flow line of $\vec{\mathbf{F}} = (ax-y)\hat{\imath} + (x+ay)\hat{\jmath}$.

(b) Show that points moving with the flow that are on the unit circle centered at the origin at time 0 are on the circle of radius e^{at} centered at the origin at time t.

25.6 Conceptual Problems

Question 25.182

Fill the boxes with 'certainly', 'possibly', or 'certainly not'.

(a) The plot of the vector field $\vec{\mathbf{G}}(x,y) = \vec{\mathbf{F}}(2x,2y)$ is $\boxed{}$ drawn by doubling the length of all the arrows in the plot of $\vec{\mathbf{F}}(x,y)$.

(b) If one papremetrization of a curve is a flow line for a vector field, then all of its paramterizations are $\boxed{}$ flow lines for the vector field.

(c) If the flow lines for the vector field $\vec{\mathbf{F}}(x,y)$ are all concentric circles centered at the origin, then the dot-product $\vec{\mathbf{F}}(x,y)\cdot(x\hat{\imath}+y\hat{\jmath})$is $\boxed{}$ equal to zero.

(d) If the flow lines for the vector field $\vec{\mathbf{F}}(x,y)$ are all straight lines parallel to the constant vector $\vec{v} = 3\hat{i} + 5\hat{j}$, then $\vec{\mathbf{F}}(x, y)$ is $\boxed{}$ equal to $\vec{v}$.

(e) The flow lines of the vector field $\vec{\mathbf{F}}(x, y) = e^x\hat{i} + y\hat{j}$ $\boxed{}$ cross the X-axis.

Question 25.183

Are the following true or false? If $\vec{r}(t)$ is a flow line for a vector field $\vec{\mathbf{F}}$, then

(a) $\vec{r}_1(t) = \vec{r}(t-5)$ is a flow line for the same vector field $\vec{\mathbf{F}}$.

(b) $\vec{r}_2(t) = \vec{r}(2t)$ is a flow line for the vector field $2\vec{\mathbf{F}}$.

(c) $\vec{r}_3(t) = 2\vec{r}(t)$ is a flow line for the vector field $2\vec{\mathbf{F}}$.

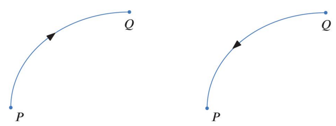
§26 | Line Integrals on Parameterized Curves (April 22,24, Book Chapter 18.1, 18.2)

26.1 Motivation

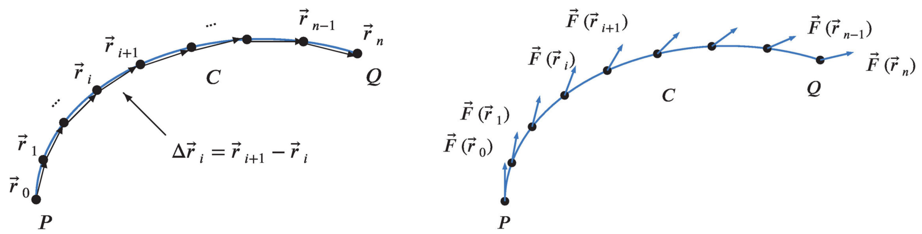
Now that we know about parameterized curves and vector fields, we are ready to introduce a new kind of integral which uses both of these concepts. The line integral measures the extent to which a curve in a vector field is, overall, going with the vector field or against it. For instance, a flow line goes completely with the flow of its vector field, so it should have a comparatively high line integral value.

26.2 Definition

Suppose we are given a vector field $\vec{\mathbf{F}}(x,y)$ and a parametrized curve $C : \vec{\boldsymbol{r}}(t), a \leq t \leq b$. First we will need to fix an *orientation* of C so that we know the direction of travel along the curve.



We will assume that the orientation is as in the first picture. Next, recall that we can use dot product to measure the extent to which two vectors point in the same or opposing directions. So we will break our parameterized curve C from $\vec{\boldsymbol{r}}(a) = P$ to $\vec{\boldsymbol{r}}(b) = Q$ into a linked trail of “secant” vectors $\Delta\vec{\boldsymbol{r}}_i$, connecting points along the curve, and then compare each of these vectors to the corresponding vector from the vector field vector at the same points.



We then add the dot products and create a Riemann sum whose limit gives us the Line Integral.

$$\int_C \vec{\mathbf{F}} \cdot d\vec{r} = \lim_{\|\Delta \vec{r}_i\| \to 0} \sum_{i=0}^{n-1} \vec{\mathbf{F}}(\vec{r}_i) \cdot \Delta \vec{r}_i$$

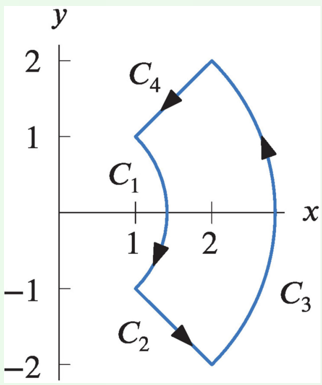
Note that the value of the line integral also defines the work done by the force $\vec{\mathbf{F}}$ as displacement happens along the curve C, which is an important quantity in physics.

Question 26.184

For each curve C_i in the picture and for each of the following vector fields, determine if possible whether $\int_C \vec{\mathbf{F}} \cdot d\vec{r}$ would be positive, negative, or zero.

a) $x\hat{i}$

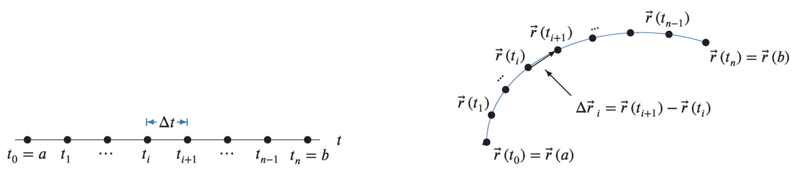
b) $y\hat{i} - x\hat{j}$



26.3 Evaluating a line integral

The key to evaluating line integration is to relate the secant vectors $\Delta \vec{r}_i$ to the velocity vectors $\vec{r}'(t_i)$ as

$$\Delta \vec{r}_i \approx \vec{r}'(t_i)\Delta t$$



As $\Delta t \to 0$, we then find the limit of the Riemann sum as

$$\int_C \vec{\mathbf{F}} \cdot d\vec{r} = \lim_{\Delta t \to 0} \sum_{a \le t \le b} \vec{\mathbf{F}}(\vec{r}(t)) \cdot \vec{r}'(t)\Delta t = \int_a^b \vec{\mathbf{F}}(\vec{r}(t)) \cdot \vec{r}'(t)dt$$

Observe that this is essentially a $u-$substitution, where we are changing the variable of integration to t.

Example 10. Let's compute the line integral

$$\int_C \langle x-y, x\rangle \cdot d\vec{r}$$

along the upper-half C of the unit circle traversed counterclockwise. First we need to parameterize C and find $\vec{r}'(t)$.

$$\vec{r}(t) = \cos t\hat{i} + \sin t\hat{j}, \qquad 0 \le t \le \pi$$

So $\vec{r}'(t) = \langle -\sin t, \cos t\rangle$. Then using the formula for line integrals,

$$\begin{aligned}
\int_C \langle x-y, x\rangle \cdot d\vec{r} &= \int_a^b \vec{\mathbf{F}}(\vec{r}(t)) \cdot \vec{r}'(t)dt \\
&= \int_0^\pi \underbrace{\langle \cos(t)-\sin(t), \cos(t)\rangle}_{\vec{\mathbf{F}}(\vec{r}(t))} \cdot \underbrace{\langle -\sin(t), \cos(t)\rangle}_{\vec{r}'(t)} dt \\
&= \int_0^\pi \Big(-\cos(t)\sin(t) + \sin^2(t) + \cos^2(t)\Big) dt \\
&= \int_0^\pi -\cos(t)\sin(t)\mathrm{d}t + \int_0^\pi 1 dt \\
&= \left(\frac{\cos^2(t)}{2} + t\right)\Bigg]_0^\pi = \pi
\end{aligned}$$

Question 26.185

Compute the line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{r}$ for the following pairs of vector fields $\vec{\mathbf{F}}$ and curves C.

(a) $\vec{\mathbf{F}} = \langle y, x\rangle$ and C is the quarter-circle centered at the origin starting at $(2,0)$ and proceeding counterclockwise to $(0,2)$.

(b) $\vec{\mathbf{F}} = \langle y, x\rangle$ and C is the line segment starting at $(2,0)$ and proceeding counterclockwise to $(0,2)$

(c) $\vec{\mathbf{F}} = \langle x, y, -2z \rangle$ and C is the "twisted cubic" $\vec{r}(t) = \left\langle t, t^2, t^3 \right\rangle$ from $t = 0$ to $t = 1$

(d) $\vec{\mathbf{F}} = \langle x, y, -2z \rangle$ and C is the line segment from $(0, 0, 0)$ to $(1, 1, 1)$

Question 26.186

True or False? The line integral of a vector field along a path depends on the parameterization of the curve.

§27 | The Fundamental theorem of Line Integrals (April 27, Book Chapter 18.3)

27.1 Introduction

Recall the Fundamental Theorem of (single-variable) Calculus:

$$\int_a^b f'(x)dx = f(b) - f(a)$$

This essentially says that integration is the opposite operation of differentiation (that's the reason we call the indefinite integral the anti-derivative). In this chapter we'll see a multivariable version of the Fundamental Theorem of Calculus, which uses the line integral of a (continuous) gradient $\vec{\mathbf{F}} = \nabla f$. Among other things, this result will allow us to compute some line integrals quickly and without resorting the approach from last time involving parameterization.

27.2 The Fundamental theorem of Line Integrals

27.2.1 Formulation

The single-variable FTC says that, to evaluate the integral of a function f' over an interval $[a, b]$, we find its antiderivative f, plug in the endpoints and subtract. By analogy, think of the potential function $f(x, y)$ as being an "antiderivative" for the gradient field $\vec{\mathbf{F}} = \nabla f$. And let C be a piece-wise smooth oriented path with starting point A and endpoint B. Then we have the following theorem

Theorem 27.1: The Fundamental Theorem for Line Integral

$$\int_{\mathrm{C}} \vec{\nabla} f \cdot d\vec{r} = f(\mathrm{B}) - f(\mathrm{A})$$

Before we prove this to be true, note how it simplifies the computation of line integrals for gradient fields! Instead of parametrizing C, we just have to identify its endpoints, find the potential function, and plug in the endpoints and subtract.

Proof 27.1

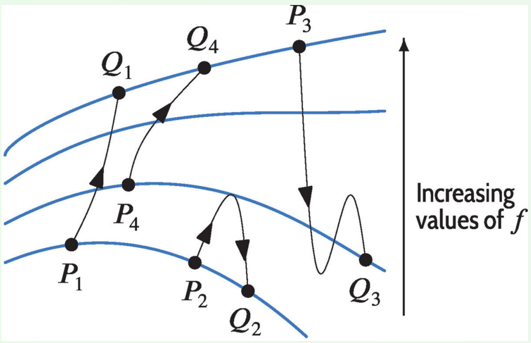
Suppose C is parametrized by $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, $a \le t \le b$. Recall that from last chapter, we have

$$\begin{aligned}
\int_C \vec{\nabla} f \cdot d\vec{r} &= \int_a^b \vec{\nabla} f \cdot \vec{r}'(t)\, dt \\
&= \int_a^b \left\langle f_x, f_y \right\rangle \cdot \langle x'(t), y'(t) \rangle\, dt \\
&= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \right) dt \\
&= \int_a^b \frac{df}{dt}\, dt \\
&= \int_A^B df = f(B) - f(A)
\end{aligned}$$

$\square$

Question 27.187

Consider the line integrals, $\int_{C_i} \vec{F} \cdot d\vec{r}$ for $i = 1, 2, 3, 4$, where $\vec{F}(x, y) = \vec{\nabla} f$ and C_i is the path from P_i to Q_i shown below. Some level curves of f are also shown in the figure.



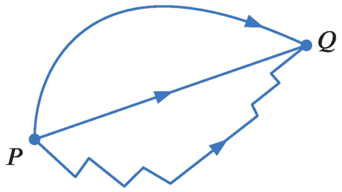
(a) Which of the line integral(s) is (are) zero?

(b) Arrange the four line integrals in ascending order (from least to greatest).

(c) Two of the nonzero line integrals have equal and opposite values. Which are they? Which is negative and which is positive?

27.3 Path Independence

Definition 11. A vector field is called *path-independent* or *conservative* if the value of its line integral is the same along any paths having the same endpoints.



In other words, the line integrals depend only on the starting and ending points and NOT on the particular path between those two points. Here 'path' means a piecewise smooth curves C that avoid any locations where $\vec{\mathbf{F}}$ is undefined.

How can we tell whether a given vector field is path-independent or path-dependent, without computing a bunch of line integrals to check? Note that with this new nomenclature, the FTLI can be stated as

$$\boxed{\text{Gradient Vector Field} \implies \text{Path independent}}$$

In fact, the converse is true as well. We are going to include the proof below as a digression, but it is not within the scope of the current discussion.

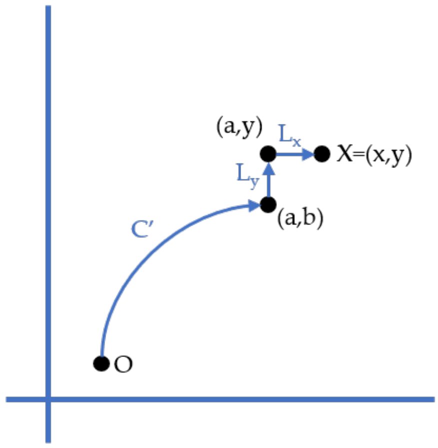
Digression

Suppose $\vec{\mathbf{F}} = P\hat{\imath} + Q\hat{\jmath}$ is a path-independent vector field. Consider the function f defined as follows:

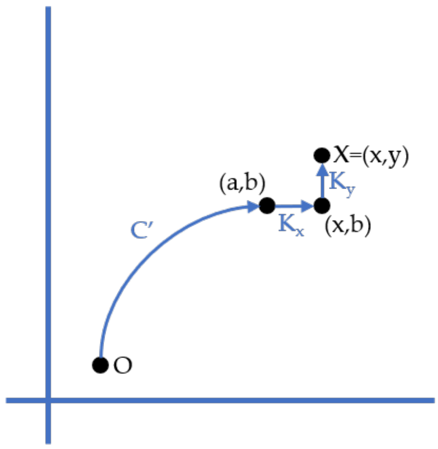
$$f(x,y) = \int_C \vec{\mathbf{F}} \cdot d\vec{r}$$

where C is a path from a fixed starting point O to a variable point $X = (x,y)$. This integral has the same value for any path from O to X because $\vec{\mathbf{F}}$ is path-independent.

The two figures below then explain why $\vec{\nabla} f = \vec{\mathbf{F}}$.



The path $C' + L_y + L_x$ is used to show $f_x = P$



The path $C' + K_x + K_y$ is used to show $f_y = Q$

So we can conclude that

$$\boxed{\text{Gradient Vector Field} \iff \text{Path independent}}$$

So it is a natural question to ask if a given vector field is a gradient vector field or not since that would make our life considerably easier! And if it is a gradient vector field, how can find f so that $\vec{\mathbf{F}} = \vec{\nabla} f$? Fortunately, we have already answered these questions in a previous chapter.

Note that we still haven't proved that a vector field is gradient v.f. if $Q_x = P_y$, although it is true in *most* cases.

Question 27.188

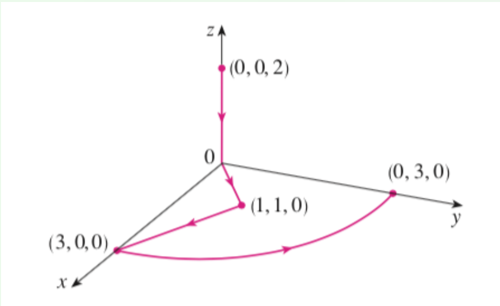
Evaluate the following line integrals. Not all of them can be done using FTLI. Identify the ones that are gradient vector fields.

(a) $\int \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y) = xy\,\vec{i} + x^2\,\vec{j}$ and C is given by $\vec{r}(t) = \sin t\,\vec{i} + (1+t)\,\vec{j}$, $0 \le t \le \pi$

(b) $\int \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y) = \left(4x^3y^2 - 2xy^3\right)\vec{i} + \left(2x^4y - 3x^2y^2 + 4y^3\right)\vec{j}$ and C is given by $\vec{r}(t) = (t + \sin \pi t)\,\vec{i} + (2t + \cos \pi t)\,\vec{j}$, $0 \le t \le 1$

(c) $\int \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y,z) = \sin y\,\vec{i} + x\cos y\,\vec{j} - \sin z\,\vec{k}$, and C is the helix $x = 3\cos t, y = t, z = 3\sin t$ from $(3,0,0)$ to $(0,\pi/2,3)$

(d) $\int \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y,z) = \left(3x^2yz - 3y\right)\vec{i} + \left(x^3z - 3x\right)\vec{j} + \left(x^3y + 2z\right)\vec{k}$ and C is the curve shown below.



(e) $\int \vec{F} \cdot d\vec{r}$, where

$$\vec{F}(x,y,z) = \left\langle 4xe^{2x^2+3y^2+4z^2}, 6ye^{2x^2+3y^2+4z^2}, 8ze^{2x^2+3y^2+4z^2} \right\rangle$$

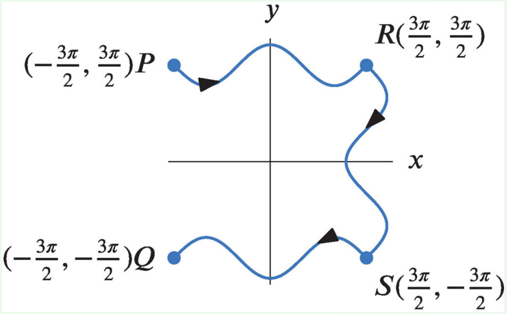
and C is

$$\vec{r}(t) = \langle (2+\cos(7t))\cos(t), (2+\cos(7t))\sin(t), \sin(7t) \rangle$$

parameterized by $0 \le t \le \pi$ starting at $t = 0$ and ending at $t = \pi$.

(f) $\int \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y) = xy^2\,\vec{i} + x^2y\,\vec{j}$, and C is $\vec{r}(t) = \cos t\,\vec{i} + 2\sin t\,\vec{j}$, $\quad 0 \le t \le \pi/2$

(g) $\int \vec{\mathbf{F}} \cdot d\vec{r}$, where $\vec{\mathbf{F}}(x,y) = \left(\sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)\right)\hat{\imath} - \left(\cos\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right)\right)\hat{\jmath}$, and C is as in the picture below:



27.4 Circulation Freedom

Definition 12. A curve C is called *closed* (or a closed loop) if it starts and ends at the same point.

Line integral on a closed loop is sometimes explicitly denoted by the symbol $\oint$, although it is not necessary.

Definition 13. A vector field $\vec{\mathbf{F}}$ is called *circulation-free* if $\oint\limits_{C} \vec{\mathbf{F}} \cdot d\vec{r} = 0$ for *every* closed path C.

Again, here 'path' means a piecewise smooth curve C that avoids any locations where $\vec{\mathbf{F}}$ is undefined.

One of the immediate observations we could make from the definition is that for a vector field

$$\boxed{\text{Path independent} \implies \text{Circulation-free}}$$

The converse is non-trivial but still easy to prove.

Question 27.189

Explain why circulation-free vector fields are path-independent.

27.5 Conclusion

Overall, the results of this chapter show the following

$$\boxed{\text{Gradient v.f.} \iff \text{Path independent} \iff \text{Circulation-free}}$$

§28 | Irrotational Vector Fields (April 29, Book Chapter 18.4)

28.1 Curl and Rotation

Definition 14. Given a vector field $\vec{\mathbf{F}} = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$, the curl of the vector field is defined as

$$\operatorname{curl}(\vec{\mathbf{F}}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Of course, this is only defined at points where P and Q are differentiable with respect to y and x respectively.

Definition 15. A vector field is called *irrotational* if its curl is zero at every point where curl is defined.

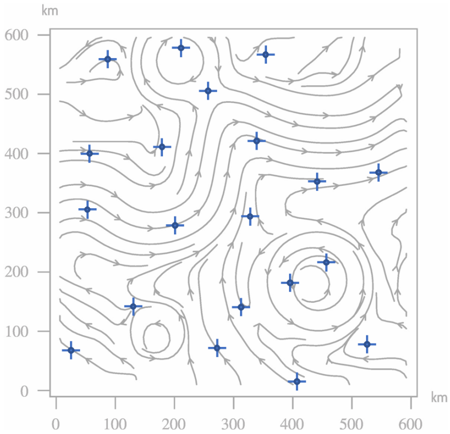
You might recall seeing the terms in $\operatorname{curl}(\vec{\mathbf{F}})$ before. In fact, we have proved the following in an earlier chapter using Clairaut's theorem.

$$\boxed{\text{Gradient vector field} \implies \text{Irrotational}}$$

28.1.1 Understanding Curl

The nomenclature suggests that $\operatorname{curl}(\vec{\mathbf{F}})$ has something to do with rotation in a vector field. There is a way to visually interpret the algebraic quantity that shows how curl corresponds to rotation.

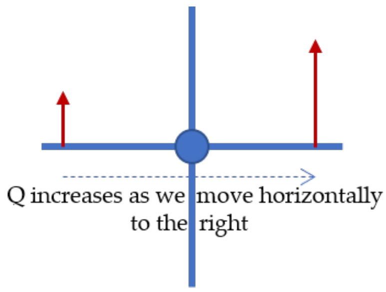
Imagine an infinitesimally small paddle-wheel lying in a vector field $\vec{\mathbf{F}}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$. We will show that as long as $\vec{\mathbf{F}}$ is smooth, the curl measures how much (and which way) it turns at any point.



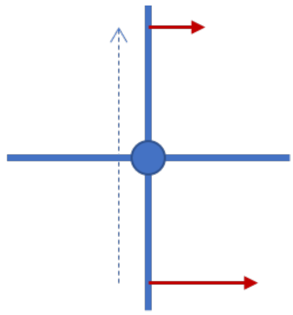
A positive curl means the paddle-wheel will rotate counterclockwise, and a negative curl means the paddle-wheel rotates clockwise. An irrotational vector field has no spinning paddle-wheels anywhere.

In general, the first term $\frac{\partial Q}{\partial x}$ in the curl function measures the effect of the vector field $\vec{\mathbf{F}}$ on the horizontal part of the paddle wheel, and a positive value corresponds to a counter-clockwise contribution to the rotation of the paddle-wheel. This follows because Q is the j-component of the vector field, and its partial derivative with respect to x measures its change when moving in horizontally, keeping y fixed.

The picture below shows a situation where $\frac{\partial Q}{\partial x}$ is *positive* since the (red) vectors representing the j-component $Q(x, y)$ of the vector field increases as we move horizontally. In this case, the paddle-wheel rotates counterclockwise.



Likewise, the second term $\frac{\partial P}{\partial y}$ in the curl function measures the effect of the vector field on the vertical part of the paddle wheel, but now a negative value corresponds to a counterclockwise contribution to the rotation of the paddle-wheel (that's why this term appears in the curl with a negative sign).

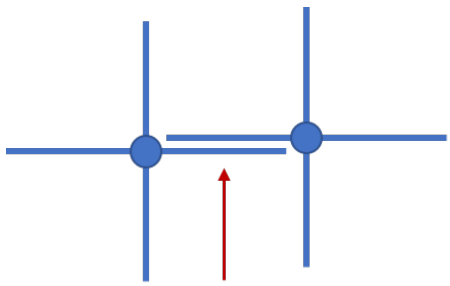


P decreases as we move vertically up

28.2 Green's Idea: Volume Integrate the Curl

The fact that gradient vector fields are both circulation free and irrotational lead a famous mathematical physicist named George Green to guess that there might be a relationship between circulation integrals to volume integrals of the curl.

To approximate the volume integral of the curl function over a closed bounded region R, we can first break R into smaller rectangular sub-rectangles and then add up the volumes of the rectangular solids with heights equal to the curl at one of the corners of the sub-rectangles.



The key thing to notice here is that the vector field spins "adjacent" paddle-wheels in opposite directions: The above red vector gives counterclockwise spin to the left paddle-wheel and the same amount of clockwise spin to the right paddle-wheel. This means that the curls associated with interior points of the region R will essentially be cancelled out by each other. So, when we take the limit of the volume approximation, all that is left is the spin due to the (counterclockwise) push of the vector field along the boundary of the region. This is measured by the line integral of the vector field around the (counterclockwise oriented) boundary.

This is of course, very much of a heuristic reasoning. An exact algebraic proof can be found in the book if you are interested.

28.2.1 Careful Formulation of Green's Theorem

Green's Theorem is a statement about only the 2D plane, not 3D space. Here's the setup: Let C be a piecewise smooth simple closed curve that is the boundary of a simply-connected region R in the plane. Let $\vec{\mathbf{F}} = P\,\vec{\boldsymbol{i}} + Q\,\vec{\boldsymbol{j}}$ be a smooth vector field defined on all of R and C.

Definition 16. To say a curve is *simple* means that it doesn't intersect itself, and to say a curve is *closed* means that it is a closed loop, that it starts and ends at the same point.

Definition 17. A region is called *simply-connected* if it is just one piece (connected) and doesn't have any holes in it.

Definition 18. A vector field is said to be *smooth* if its first partials are continuous functions.

Since C is the boundary of R, we will denote it as $C = \partial R$ and stop referring to C explicitly. When we consider the boundary curve of a simply-connected region, we always orient the curve so that the region is on the left as we follow the curve.

Theorem 28.1: Green's Theorem

Green's theorem relates the line integral of $\vec{\mathbf{F}}$ over ∂R to the volume integral of curl(F) over R.

$$\oint_{\partial R} \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{r}} = \iint_{D} \operatorname{curl}(F)\, dA = \iint_{R} (Q_x - P_y)\, dA.$$

You may also see this written in the Leibniz notation for partial derivatives.

$$\oint_{\partial R} P\,dx + Q\,dy = \iint_{R} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Question 28.190

Use Green's theorem to explain why smooth irrotational vector fields are guaranteed to be circulation free, and hence gradient vector fields.

28.3 The Venn Diagram

Example 19. Consider the following particularly illuminating example. We start with the vector field

$$\vec{\mathbf{F}}(x,y) = \frac{y}{x^2+y^2}\hat{\imath} - \frac{x}{x^2+y^2}\hat{\jmath}$$

(a) Check that the vector field is irrotational.

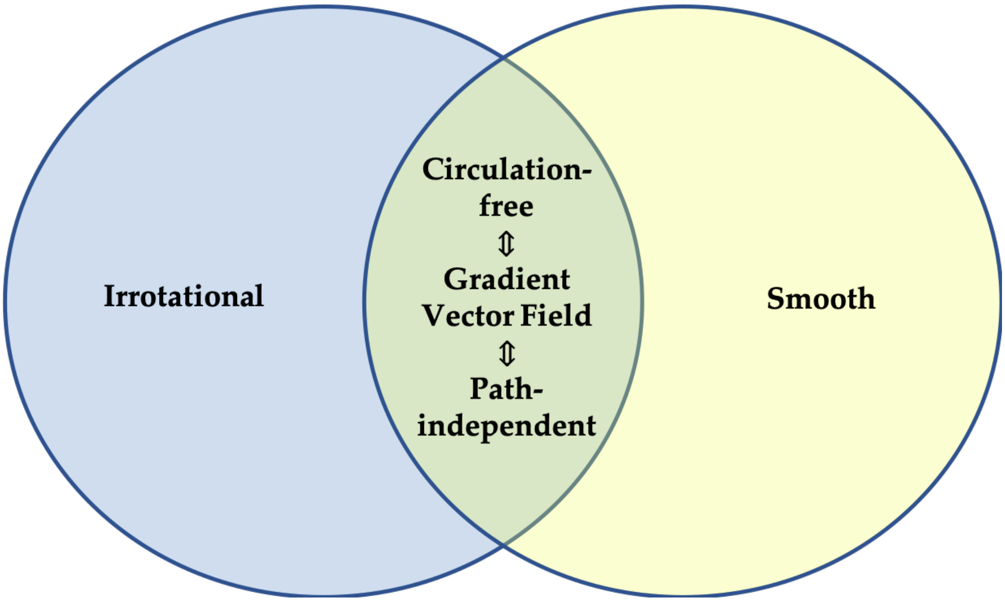
(b) Use a parameterization to compute $\oint_C \vec{\mathbf{F}} \cdot d\vec{r}$ where C is the unit circle traversed once in a counterclockwise direction.

(c) Explain how you can conclude that $\vec{\mathbf{F}}$ is not a gradient vector field.

(d) Let $f(x,y) = \arctan\frac{y}{x}$. Calculate $\vec{\nabla} f$.

(e) How do you explain the apparent contradiction?

This example highlights the requirement of smoothness in Green's theorem. In particular it shows that not all irrotational vector fields are gradient vector fields. Overall, the venn diagram of vector fields looks as follows:



§29 | Green's Theorem - Applications and Generalizations (April 31,...)

29.1 Applications of Green's Theorem

29.1.1 Evaluating Difficult Line Integrals

Example 20. If you're asked to find the line integral of an ugly vector field over a closed curve, you should look to see if Q_x and P_y are drastically more simple. If so, use Green's Theorem!

Question 29.191

Evaulate

$$\oint_C (2y + \sqrt{9 + x^3})\, dx + (5x + e^{\tan^{-1} y})\, dy,$$

where C is the circle $x^2 + y^2 = 4$ in the plane oriented counterclockwise.

Example 21. If you're asked to find the line integral over a closed curve that is clearly the boundary of a nice region, it's often a good idea to use Green's Theorem to switch to the double integral over the interior.

Question 29.192

Integrate $\vec{\mathbf{F}} = xy\,\vec{i} + e^x\,\vec{j}$ over the boundary of the rectangle determined by $0 \le x \le 2$, $0 \le y \le 3$, oriented clockwise around the boundary.

Question 29.193

Find the line integral of $\vec{\mathbf{F}} = 3xy\,\vec{i} + 2x^2\,\vec{j}$ over the curve C defined as follows: follow the curve $y = x^2 - 2x$ from $(0,0)$ to $(3,3)$, then follow the line $y = x$ from $(3,3)$

back to $(0,0)$.

Question 29.194

Evaluate

$$\oint_C \left(3y - e^{\sin x}\right) dx + \left(7x + \sqrt{y^4+1}\right) dy$$

where C is the circle $x^2 + y^2 = 9$, oriented clockwise.

Example 22. Consider the following problem.

Find the line integral of

$$\vec{\mathbf{G}}(x,y) = \left(x+y\right)\vec{\boldsymbol{i}} + \left(2x + y\ln\left(\csc\sqrt{1-y^5}\right)\right)\vec{\boldsymbol{j}}$$

over C_1*, the upper half of the unit circle from* $(1,0)$ *to* $(-1,0)$.

The problem is that the vector field is ugly, so parametrizing C_1 is just going to lead to an impossible integral. So we would like to use Green's Theorem, but this isn't a closed curve!

Here's how to fix that issue. Let C_2 be the straight line segment from $(-1,0)$ to $(1,0)$. Now $C_1 + C_2$ is a closed loop.

(a) Let R be the region enclosed by C_1+C_2. Use Green's Theorem to compute $\displaystyle\oint_{C_1+C_2} \vec{\mathbf{G}}\cdot d\vec{\boldsymbol{r}}$.

(b) Parametrize C_2 and directly calculate $\displaystyle\int_{C_2} \vec{\mathbf{G}}\cdot d\vec{\boldsymbol{r}}$. (Note that $y = 0$ everywhere on C_2, which is helpful.)

(c) Write $\displaystyle\oint_{C_1+C_2} \vec{\mathbf{G}}\cdot d\vec{\boldsymbol{r}} = \int_{C_1} \vec{\mathbf{G}}\cdot d\vec{\boldsymbol{r}} + \int_{C_2} \vec{\mathbf{G}}\cdot d\vec{\boldsymbol{r}}$ and use your answers to part (a) and (b) to finish off the problem and find the line integral of $\vec{\mathbf{G}}$ along C_1.

Question 29.195

Evaluate the integral.

Question 29.196

Compute the line integral of the vector field

$$\vec{\mathbf{F}}(x,y,z) = \left\langle \cos(x), 2+\cos(y), e^z + x\left(y^2+z^2\right)\right\rangle$$

along the curve

$$\vec{r}(t) = \langle t, \cos(t), \sin(t)\rangle \text{ with } 0 \le t \le 3\pi.$$

[Hint: Write $\vec{\mathbf{F}}$ as $\vec{\mathbf{G}} + \vec{\mathbf{H}}$ where $\vec{\mathbf{G}}$ is a gradient vector field. Then do the two integrals separately.]

29.1.2 Calculating Area

Consider the following vector fields:

$$\vec{\mathbf{F}}_1 = x\,\vec{j}, \qquad \vec{\mathbf{F}}_2 = -y\,\vec{i}, \qquad \vec{\mathbf{F}}_3 = -\tfrac{1}{2}y\,\vec{i} + \tfrac{1}{2}x\,\vec{j}.$$

What is $Q_x - P_y$ for each of these fields? Applying Green's Theorem to a region D, we get that

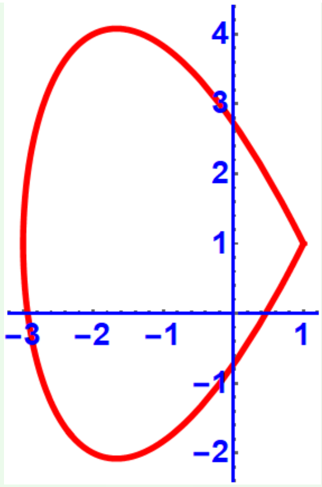
$$\oint_{\partial D} \vec{\mathbf{F}}_1 \cdot d\vec{r} \;=\; \oint_{\partial D} \vec{\mathbf{F}}_2 \cdot d\vec{r} \;=\; \oint_{\partial D} \vec{\mathbf{F}}_3 \cdot d\vec{r} \;=\; \iint_D 1\,dA \;=\; \text{Area of D. (!)}$$

Question 29.197

An ellipse with semi-major axis a and semi-minor axis b is parametrized by $x = a\cos t,\ y = b\sin t$ for $0 \le t \le 2\pi$. Use $\vec{\mathbf{F}}_3$ to find the area inside this ellipse.

Question 29.198

Let C be the curve parametrized by $\vec{r}(t) = (t^2-3)\,\vec{i} + (t^3-4t+1)\,\vec{j}$, $-2 \le t \le 2$. This is a closed loop. Use Green's Theorem and $\vec{\mathbf{F}}_1$ to find the area inside this loop.



29.2 Extended Versions of Green's Theorem

29.2.1 Finite union of simple regions

Although we have proved Green's Theorem only for the case where D is simple, we can now extend it to the case where D is a *finite union of simple regions*. For example, if D is the region shown in Figure 29.1, then we can write $D = D_1 \cup D_2$, where D_1 and D_2 are both simple. The boundary of D_1 is $C_1 + C_3$ and the boundary of D_2 is $C_2 + (-C_3)$. so, applying Green's Theorem to D_1 and D_2 separately, we get

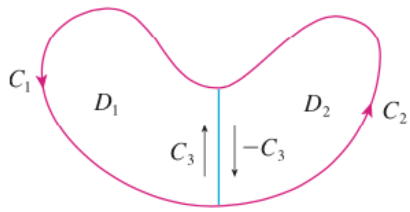


Figure 29.1

$$\oint_{C_1+C_3} \vec{\mathbf{F}} \cdot d\vec{r} = \iint_{D_1} \text{curl}(F)\, dA$$

$$\oint_{C_2+(-C_3)} \vec{\mathbf{F}} \cdot d\vec{r} = \iint_{D_2} \text{curl}(F)\, dA$$

Adding the two integrals above, we get,

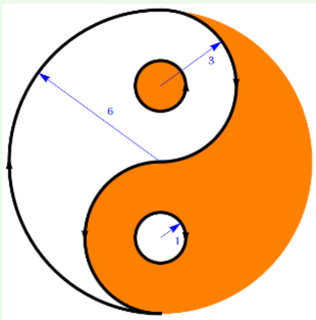
$$\oint_{C_1+C_2} \vec{\mathbf{F}} \cdot d\vec{r} = \iint_{D} \operatorname{curl}(F)\, dA$$

Question 29.199

Let C be the boundary curve of the white Yang part of the Ying-Yang symbol in the disc of radius 6. You can see in the picture that the curve C has three parts, and that the orientation of each part is given. Find the line integral of the vector field

$$\vec{\mathbf{F}}(x, y) = \langle -y + \sin(e^x), x \rangle$$

around C. Notice that the Ying and the Yang have the same area.



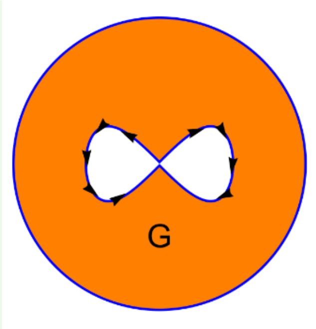
Question 29.200

Look at the shaded region G bounded by a circle of radius 2 and an inner *figure eight lemniscate* with parametric equation

$$\vec{r}(t) = \langle \sin(t), \sin(t)\cos(t) \rangle$$

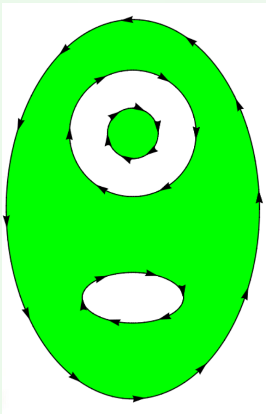
with $0 \le t \le 2\pi$. The picture shows the curve and the arrows indicate some of the velocity vectors of the curve. Find the area of this region G.

[Hint: Use Green's theorem and the vector field $x\hat{j}$.]



Question 29.201

Compute the line integral of $\vec{\mathbf{F}}(x,y) = \left\langle 5y + 3y^2, 6xy + y^5 \right\rangle$ along the boundary of the green *Cyclops region* given in the figure below. There are four boundary curves, *oriented as shown in the picture*: a large ellipse of area 16, two circles of area 2 (the eyeball) and 1 (the iris) as well as a small ellipse (the mouth) of area 3.



29.2.2 Finite intersection of simple regions

We can similarly extend the theorem to regions that are finite *intersections* of simple regions.

Question 29.202

Evaluate

$$\oint_C y^2 dx + 3xy dy$$

where C is the boundary of the semiannular region D in the upper half plane between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

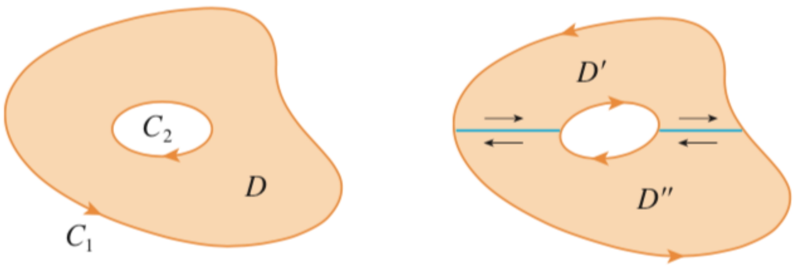


Figure 29.2

29.2.3 Not simply-connected regions

Green's Theorem can be also extended to apply to regions with holes, that is, regions that are not simply-connected. Observe that the boundary C of the region D in Figure 29.2 consists of two simple closed curves C_1 and C_2. We assume that these boundary curves are oriented so that the region D is always on the left as the curve C is traversed. *Thus the positive direction is counterclockwise for the outer curve C_1 but clockwise for the inner curve C_2.* If we divide D into two regions D′ and D″ by means of the lines shown in Figure 29.2 and then apply Green's Theorem to each of D′ and D″, we get

$$\begin{aligned}\iint_D \operatorname{curl}(F)\, dA &= \iint_{D'} \operatorname{curl}(F)\, dA + \iint_{D''} \operatorname{curl}(F)\, dA \\ &= \oint_{\partial D'} \vec{\mathbf{F}} \cdot d\vec{r} + \oint_{\partial D''} \vec{\mathbf{F}} \cdot d\vec{r}\end{aligned}$$

Since the line integrals along the common boundary lines are in opposite directions, they cancel each other out and we get

$$\iint_D \operatorname{curl}(F)\, dA = \oint_{C_1} \vec{\mathbf{F}} \cdot d\vec{r} + \oint_{C_2} \vec{\mathbf{F}} \cdot d\vec{r} = \oint_{C_1+C_2} \vec{\mathbf{F}} \cdot d\vec{r}$$

Question 29.203

If $\vec{\mathbf{F}}(x,y) = \dfrac{-y\,\vec{\boldsymbol{i}} + x\,\vec{\boldsymbol{j}}}{x^2+y^2}$, show that $\displaystyle\oint_C \vec{\mathbf{F}}\cdot d\vec{\boldsymbol{r}} = 2\pi$ for every positively oriented closed path C that encloses the origin.

Question 29.204

Evaluate $\displaystyle\oint_C \vec{\mathbf{F}}\cdot d\vec{\boldsymbol{r}}$ where

$$\vec{\mathbf{F}}(x,y) = \frac{2xy\,\vec{\boldsymbol{i}} + \left(y^2 - x^2\right)\vec{\boldsymbol{j}}}{\left(x^2+y^2\right)^2}$$

and C is a positively oriented closed path that encloses the origin.

29.3 More Practice Problems

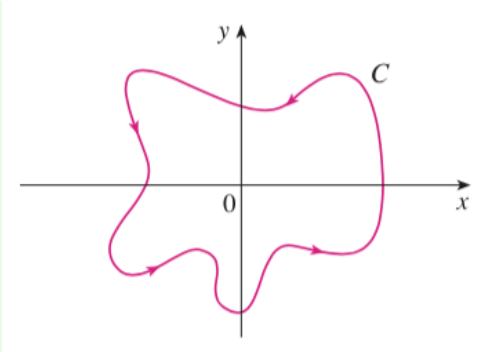
Question 29.205

Evaluate the following line integrals.

(a) $\displaystyle\oint_C y\,dx + \left(x + y^2\right)dy$ where C is the ellipse $4x^2 + 9y^2 = 36$ with counterclockwise orientation.

(b) $\displaystyle\oint_C \sqrt{1+x^3}\,dx + 2xy\,dy$ where C is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,3)$

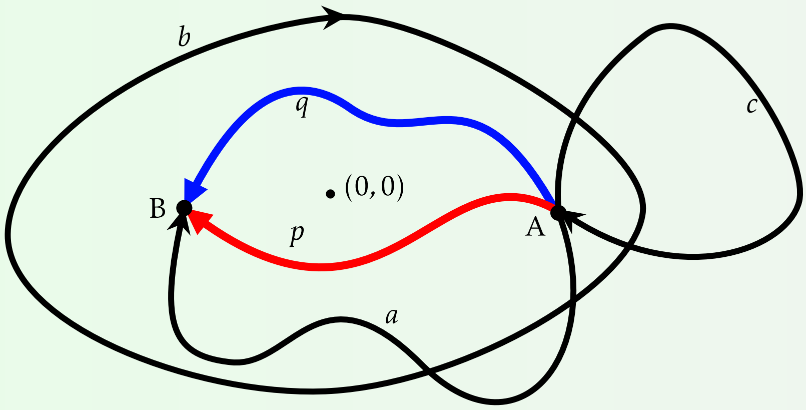
(c) $\displaystyle\oint_C \vec{\mathbf{F}}\cdot d\vec{\boldsymbol{r}}$, where $\vec{\mathbf{F}}(x,y) = \frac{\left(2x^3+2xy^2-2y\right)\vec{\boldsymbol{i}} + \left(2y^3+2x^2y+2x\right)\vec{\boldsymbol{j}}}{x^2+y^2}$ and C is the curve shown below.



(d) $\oint_C \vec{\mathbf{F}} \cdot d\vec{r}$, where $\vec{\mathbf{F}}(x,y,z) = \langle y\cos x - xy\sin x, xy + x\cos x\rangle$, and C is the triangle from $(0,0)$ to $(0,4)$ to $(2,0)$ to $(0,0)$.

Question 29.206

Suppose $\vec{\mathbf{F}}$ is an irrotational vector field in the plane (that is, its curl is everywhere zero) that is not defined at the origin $O = (0,0)$. Suppose the line integral of $\vec{\mathbf{F}}$ along the path p from A to B is 5 and the line integral of $\vec{\mathbf{F}}$ along the path q from A to B is -4. Find the line integral of F along the paths a, b and c.



Project 9 | (Optional) 3D Curl and Divergence

29.1 3D Curl

Consider the mathematical abstract 'construct' $\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$. This is called a "differential operator". It's an operation you can apply to a function of a Field. We will use it to give a new and more general definition of curl.

Definition 23. Given a vector field $\vec{\mathbf{F}} = \mathrm{P}\,\vec{i} + \mathrm{Q}\,\vec{j} + \mathrm{R}\,\vec{k}$, we define the *curl* of $\vec{\mathbf{F}}$ to be the *vector field*

$$\operatorname{curl}\vec{\mathbf{F}} = \vec{\nabla}\times\vec{\mathbf{F}}.$$

You should check that in the case of 2-dimensional vector fields, above definition gives the correct familiar formula for grad $\vec{\mathbf{F}}$ and curl $\vec{\mathbf{F}}$.

Since we have defined the differential operator $\vec{\nabla}$ which looks like a vector, we might naturally ask what is the dot product of the operator and the vector field. Note that the dot product $\vec{\nabla}\cdot\vec{\mathbf{F}}$ is NOT the gradient because gradient of a vector field doesn't make sense. The gradient of a function $g(x,y,z)$ can be written as $\vec{\nabla}g$ without any 'dot' (since it's not a dot product). We will see how to interpret the dot product in the next section.

Question 29.207

Write out the formula for curl $\vec{\mathbf{F}}$ for a 3D vector field using $\hat{\imath},\hat{\jmath},\hat{k}$ notation.

Question 29.208

Now suppose $\vec{\mathbf{F}}$ is a gradient vector field. I.e.,

$$\vec{\mathbf{F}} = \mathrm{P}\,\hat{\imath} + \mathrm{Q}\,\hat{\jmath} + \mathrm{R}\,\hat{k} = g_x\,\hat{\imath} + g_y\,\hat{\jmath} + g_z\,\hat{k}$$

for some function $g(x,y,z)$. Use Clairaut's theorem to prove that curl $\vec{\mathbf{F}} = 0$.

This statement is sometimes written as

$$\vec{\nabla} \times (\vec{\nabla} g) = 0.$$

Question 29.209

(a) Show that

$$\vec{\mathbf{F}}(x, y, z) = \left\langle y^2 z^3, 2xyz^3, 3xy^2 z^2 \right\rangle$$

is an irrotational vector field.

(b) Find a function f such that $\mathrm{F} = \vec{\nabla} f$ and conclude that $\vec{\mathbf{F}}$ is a conservative vector field.

Question 29.210

Show that Green's Theorem (in 2D) can be rewritten as

$$\oint_{\partial R} \vec{\mathbf{F}} \cdot d\vec{r} = \iint_{R} \left((\text{curl}\, \vec{\mathbf{F}}) \cdot \hat{k} \right) dA.$$

This is called the **vector form** of Green's Theorem. It generalizes to 3D situations in the form of *Stokes' Theorem*!

29.2 Divergence

If the curl can be interpreted of as a *vector derivative* of the vector field, we define the *scalar derivative* of the vector field as follows.

Definition 24. If $\frac{\partial P}{\partial x}$, $\frac{\partial Q}{\partial y}$, and $\frac{\partial R}{\partial z}$ exist, then the *divergence* of $\vec{\mathbf{F}}$ is defined to be the *scalar* quantity

$$\text{div}\, \vec{\mathbf{F}} = \vec{\nabla} \cdot \vec{\mathbf{F}} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Question 29.211

Show that If $\vec{\mathbf{F}} = \mathrm{P}\,\hat{\imath} + \mathrm{Q}\,\hat{\jmath} + \mathrm{R}\,\hat{k}$ is a vector field defined on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\operatorname{div}\operatorname{curl}\vec{\mathbf{F}} = 0.$$

This is also sometimes written as

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{\mathbf{F}}) = 0.$$

Question 29.212

Show that the vector field

$$\vec{\mathbf{F}}(x,y,z) = xz\hat{\imath} + xyz\hat{\jmath} - y^2\hat{k}$$

cannot be written as the curl of another vector field, that is, $\vec{\mathbf{F}} \neq \operatorname{curl}\vec{\mathbf{G}}$.

Question 29.213

Show that Green's Theorem can also be written in (yet another vector form) as

$$\oint\limits_{\partial \mathrm{R}} \vec{\mathbf{F}}\cdot\vec{\boldsymbol{n}}\,ds = \iint\limits_{\mathrm{R}} \operatorname{div}\vec{\mathbf{F}}\,d\mathrm{A}$$

where $\vec{\boldsymbol{n}}$ is the outward unit normal vector to $\partial\mathrm{R}$.

Above result generalizes to 3D situations in the form of *Divergence Theorem*!

29.2.1 Understanding Divergence

The reason for the name divergence can be understood in the context of fluid flow. If $\vec{\mathbf{F}} = \mathrm{P}\,\hat{\imath} + \mathrm{Q}\,\hat{\jmath} + \mathrm{R}\,\hat{k}$ the velocity of a fluid (or gas), then $\operatorname{div}\vec{\mathbf{F}}$ represents the net rate of change (with respect to time) of the mass of fluid (or gas) flowing from the point (x,y,z) per unit volume. In other words, $\operatorname{div}\vec{\mathbf{F}}$ measures the tendency of the fluid to diverge from the point (x,y,z). if $\operatorname{div}\vec{\mathbf{F}} = 0$ then the vector field $\vec{\mathbf{F}}$ is said to be *incompressible*.

29.3 Laplace Operator

For the sake of completion we also mention another differential operator that occurs when we compute the divergence of a gradient vector field.

$$\mathrm{div}(\vec{\boldsymbol{\nabla}} f) = \vec{\boldsymbol{\nabla}} \cdot \left(\vec{\boldsymbol{\nabla}} f\right)$$

is abbreviated as $\nabla^2 f$, and the operator ∇^2 is called the *Laplace operator*. $\nabla^2 f$ is also denoted as Δf.

Question 29.214

Check that

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

§30 | A 2-page Summary of Line Integrals

30.1 The Theorems

We learned the following theorems over the last couple of lectures.

Theorem 30.1: Line Integral on Parameterized Curves

If the curve C can be parametrized as $\vec{r}(t)$, $a \le t \le b$, then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{r} = \int_a^b \vec{\mathbf{F}}(\vec{r}(t)) \cdot \vec{r}'(t)dt$$

Theorem 30.2: Fundamental Theorem of Line Integrals

If the vector field $\vec{\mathbf{F}}$ is a gradient vector field i.e. $\vec{\mathbf{F}} = \nabla f$, and the curve C starts at P and ends at Q, then

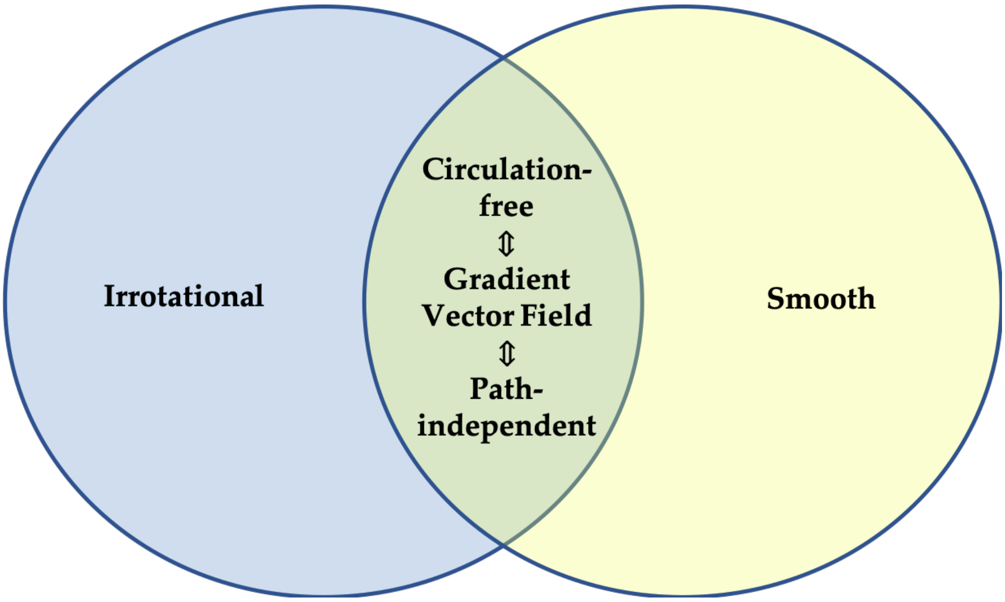
$$\int_C \vec{\mathbf{F}} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\mathrm{Q}) - f(\mathrm{P})$$

Theorem 30.3: Green's Theorem

If C is a *simple, closed, oriented* curve and the vector field $\vec{\mathbf{F}}$ is *smooth* over the simply-connected region R enclosed by C (oriented so that R is always to the left of C), then

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{r} = \iint_R \operatorname{curl} \vec{\mathbf{F}}\, dA$$

30.2 The Venn Diagram



30.3 The Flowchart