MATH 111 - Calculus and Analytic Geometry I

Lab 4 Worksheet: Applications of Derivatives

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§A. **Instructions**

Below are application problems form Section 3.4 of the textbook. These applications include problems on population modeling (biology/economics), marginal cost and revenue (economics), and motion of objects (physics). Every one of your team members should work on every single problem from this worksheet in order. You can check whether some of your answers are correct or not by using the 'Lab 4' edfinity assignment (it's not graded). On Friday, I will randomly select a member from your group to present their answer(s), so everyone should be participating in forming and recording solutions.

§B. **Change Formula**

The amount of change formula is less of an equation but an idea about how we can use the derivative of a function to **estimate** values of the function. Since the derivative gives us, by definition,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h},$$

this tells us that for very small values of $h \neq 0$, the value of $f'(a)$ is approximately equal to $\dfrac{f(a+h)-f(a)}{h}$. Thus, if we know a function value of $f(a)$, and we need to approximate a nearby value of $f(a+h)$, then we set the approximate value of $f(a+h)$ to be

$$\boxed{f(a+h) \approx f(a) + f'(a)h}.$$

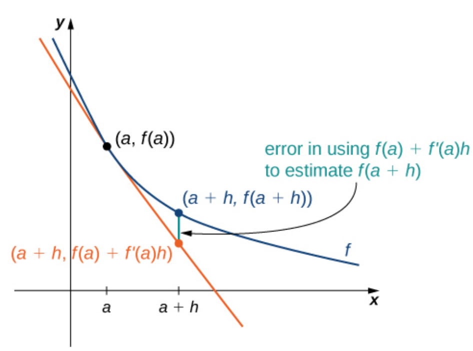

Figure 3.22 The new value of a changed quantity equals the original value plus the rate of change times the interval of change: $f(a+h) \approx f(a) + f'(a)h$.

Figure 1: From Page 267 of the textbook - picture showing that $f(a+h) \approx f(a) + f'(a)h$

See Figure 1 for a picture from the textbook showing this. The problems below about population and revenue/cost utilize this change formula. Here is the idea: suppose you have a current population or cost, say $f(a)$. And, you happen to know a growth rate, $f'(a)$. So, using just these two values, we can approximate $f(a+h)$, with good approximations for small values of h, and worse approximations for h larger.

Example B.1

Here is an example for above idea. Suppose $f(x) = x^2$. Then $f'(x) = 2x$. So we can write $f(1) = 1$ and $f'(1) = 2 \times 1 = 2$. So the approximate value of $(1.3)^2 = f(1.3)$ is

$$f(1) + f'(1) \times 0.3 = 1 + 2 \times 0.3 = 1.6.$$

Example B.2

Another example. Suppose $f(x) = \sqrt{x} = x^{1/2}$. Then $f'(x) = \frac{1}{2}x^{-1/2}$. So we can write $f(1) = 1$ and $f'(1) = \frac{1}{2}$. So the approximate value of $\sqrt{1.1} = f(1.1)$ is

$$f(1) + f'(1) \times 0.1 = 1 + 0.5 \times 0.1 = 1.05$$

§C. Population Change

To do problems related to population models, we need the following definition of **growth rate**:

Definition C.1

If $P(t)$ is the number of entities present in a population, then the **population growth rate** of $P(t)$ is defined to be $P'(t)$.

We can re-write the amount of change formula here as:

$$\boxed{P(a+h) \approx P(a) + hP'(a).}$$

Question 1.

Suppose the population of a country $P(t)$ is counted in millions where t is counted in years. What is the unit of the growth rate? Fill out Question 1 in 'Lab 4' Edfinity assignment.

Question 2. (Checkpoint 3.23)

The current population of a mosquito colony is $3,000$; that is, $P(0) = 3,000$. If $P'(0) = 100$, estimate the size of the mosquito population in 3 days, where t is measured in days.

Fill out Question 2 in 'Lab 4' Edfinity assignment.

In some problems, you might have to approximate the growth rate $P'(a) \approx \dfrac{P(a+h) - P(a)}{h}$, before you can approximate the population at a later time.

Question 3. [Like Example 3.37]

The population of a city is doubling every 6 years. If the current population is $12,000$, what will be the approximate population 2 years from now?

Fill out Question 3 in 'Lab 4' Edfinity assignment.

■ **Question 4. [Exercise 164]**

A small town in Ohio commissioned an actuarial firm to conduct a study that modeled the rate of change of the town's population. The study found that the town's population (measured in thousands of people) can be modeled by the function $P(t) = -\frac{1}{3}t^3 + 64t + 3000$, where t is measured in years.

(a) Find $P'(1), P'(2), P'(3),$ and $P'(4)$ and interpret what the results mean for the town.

(b) Find $P''(1), P''(2), P''(3),$ and $P''(4)$ and interpret what the results mean for the town.

§D. Changes in Cost and Revenue

Firstly, we need to define what **marginal** means in economics concerning price, revenue, and profit (it simply means derivative!):

Definition D.1

- If $C(x)$ is the cost of producing x items, then the **marginal cost** $MC(x)$ is given by $MC(x) = C'(x)$.
- If $R(x)$ is the revenue obtained from selling x items, then the **marginal revenue** $MR(x)$ is given by $MR(x) = R'(x)$.
- If $P(x) = R(x) - C(x)$ is the profit obtained from selling x items, then the **marginal profit** $MP(x)$ is given by $MP(x) = P'(x) = R'(x) - C'(x)$.

In this context, we won't ever be producing or selling fractions of items, hence, **the values of x here will always be positive integers**. So when we talk about the amount of change formula in this context, $C(x+h) \approx C(x) + C'(x)h$, the smallest non-zero value for h is $h = 1$:

$$C(x+1) \approx C(x) + C'(x).$$

In other words, we have

$$C(x+1) \approx C(x) + MC(x) \quad \text{or equivalently,} \quad MC(x) \approx C(x+1) - C(x)$$

Thus, we can interpret marginal cost of producing x units as the change associated with producing one additional item. We have similar statements for marginal revenue and marginal profit.

■ **Question 5.**

Suppose the cost of producing x gallons of gas is denoted as $C(x)$ dollars. What is the unit of the marginal cost?

Fill out Question 4 in 'Lab 4' Edfinity assignment.

■ **Question 6. [Exercise 160]**

The cost function, in dollars, of a company that manufactures food processors is given by

$$C(x) = 200 + \frac{7}{x} + \frac{x^2}{7},$$

where x is the number of food processors manufactured.

(a) Find the marginal cost.

(b) Find the marginal cost of manufacturing 12 food processors.

(c) Approximately how much does it cost to produce one more food processor after the 12?

(d) Find the actual cost of producing the 13th food processor and compare how far off the approximation is from the actual cost.

Fill out Question 5 in 'Lab 4' Edfinity assignment.

■ Question 7. [Exercise 162]

A profit is earned when revenue exceeds cost. Suppose the profit function for a skateboard manufacturer is given by $P(x) = 30x - 0.3x^2 - 250$, where x is the number of skateboards sold. Estimate the profit gained from the sale of the 30th skateboard using marginal profit. How far off is this from the exact profit of the 30th skateboard?

■ Question 8. [Checkpoint 3.24]

Suppose that the profit obtained form the sale of x fish dinners is given by $P(x) = -0.03x^2 + 8x - 50$. Estimate the profit from the sale of the 101st fish dinner using the profit and the marginal profit at the hundredth dinner.

§E. Motion along a line

As we have seen previously, we can use the derivative to think about the velocity of an object in motion. We summarize these facts below:

Definition E.1

Let $s(t)$ denote the position of an object at time t.

- The velocity function $v(t)$ of the object at time $\mathbf{T}$ is given by $v(t) = s'(t)$.
- The speed of the object at time t is $|v(t)|$, the absolute value of $v(t)$.
- The acceleration function $a(t)$ of the object at time t is given by $a(t) = v'(t) = s''(t)$.

■ Question 9. [Exercise 155]

The position function $s(t) = t^2 - 2t - 4$ represents the position of the back of a car backing out of a driveway, and then driving in a straight line, with s in feet and t in seconds. In this case, $s(t) = 0$ represents the time at which the back of the car is at the garage door, so $s(0) = -4$ is the starting position of the car, 4 feet inside the garage.

What can you say about the velocity when $s(t) = 4$?

Fill out Question 6 in 'Lab 4' Edfinity assignment.

■ Question 10. [Exercise 157]

A potato is launched vertically upward from a potato gun, with an initial velocity of 100 ft/s, and from atop an 85-foot-tall building. The distance in feet that the potato travels is given by $s(t) = -16t^2 + 100t + 85$.

(a) When does the potato reach it's maximum height?

(b) How long is the potato in the air?

(c) What is the speed of the potato when it hits the ground?

We can tell if an object is **slowing down** or **speeding up** by examining the signs of the velocity and acceleration functions:

Slowing down: $a(t)$ has opposite sign from $v(t)$

Speeding Up: $a(t)$ and $v(t)$ have the same sign.

We can also think about the speed, $|v(t)|$. If $|v(t)|$ is increasing, then we are speeding up, and if $|v(t)|$ is decreasing, then we are slowing down.

Question 11. [Exercise 150]

The function $s(t) = 2t^3 - 3t^2 - 12t + 8$ gives the position of a particle moving along a horizontal line. When is the particle slowing down or speeding up?