Calculus & Analytical Geometry II

Lecture 27-28 Worksheet

Spring 2021 **Subhadip Chowdhury** **Math 112**

§A. Alternating Series

The convergence tests that we have looked at so far apply only to series with positive terms. So how do we deal with series that has both positive and negative terms?

Definition A.1: Alternating Series

An **alternating series** is a series whose terms are alternately positive and negative. Here are two examples:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots=\sum_{i=1}^{\infty}(-1)^{i+1}\frac{1}{i}$$

$$-\frac{1}{2}+\frac{2}{3}-\frac{3}{4}+\frac{4}{5}-\frac{5}{6}+\frac{6}{7}-\cdots=\sum_{i=1}^{\infty}(-1)^{i}\frac{i}{i+1}$$

We see from these examples that the i-th term of an alternating series is of the form

$$a_i=(-1)^{i+1}b_i \quad \text{or} \quad a_i=(-1)^{i}b_i$$

where b_i is a positive number. (In fact, $b_i=|a_i|$.) The following test says that if the terms of an alternating series decrease toward 0 in absolute value, then the series converges.

Theorem A.2: Alternating Series Test

Let $\{b_n\}$ be a sequence of positive numbers. The alternating series

$$\sum_{i=1}^{\infty}(-1)^{i+1}b_i=b_1-b_2+b_3-b_4+b_5-b_6+\cdots$$

or

$$\sum_{i=1}^{\infty}(-1)^{i}b_i=-b_1+b_2-b_3+b_4-b_5+b_6-\cdots$$

converges if

(a) $b_{n+1}\leq b_n$ *for all n, i.e. the sequence $\{b_n\}$ is a decreasing sequence*

(b) $\lim\limits_{n\to\infty}b_n=0$, *i.e. the series passes the divergence test*

Before we ask why this theorem is true, let's look at some practice problems.

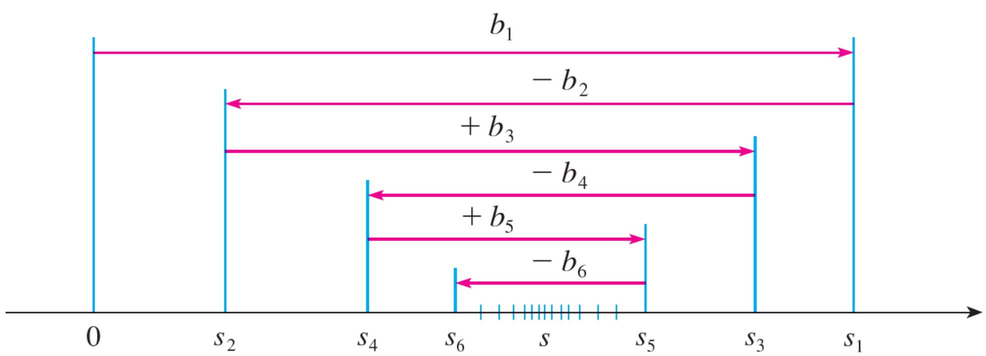
Question 1.

Test whether the following alternating series are convergent.

(a) $\sum_{i=1}^{\infty} \frac{(-1)^i}{i}$ (b) $\sum_{i=1}^{\infty} \frac{(-1)^i 3i}{4i-1}$

(c) $\sum_{i=1}^{\infty} (-1)^i \frac{i^2}{i^3+1}$ (d) $\sum_{i=1}^{\infty} (-1)^i \sin\left(\frac{\pi}{i}\right)$

The proof of the alternating series test relies on the fact that since $\{b_n\}$ is decreasing, the even partial sums are increasing and bounded, and the odd partial sums are decreasing and bounded. The following picture should give you an idea of what's happening.



Also try the following DESMOS link for an animation that plots the successive partial sums.

https://www.desmos.com/calculator/1m5pkonvgi

§B. Absolute and Conditional Convergence

Not every alternating series fits the conditions of the alt. series test. In particular, a series could be alternating with $\lim_{n\to\infty} b_n = 0$, but the general term might never be a decreasing sequence. For example,

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{2^3} - \frac{1}{3^3} + \cdots$$

In such cases, we still have one more test we can try.

Theorem B.3: Absolute Convergence Test

Given any sequence $\{a_n\}$, if $\sum |a_i|$ is convergent, then $\sum a_i$ is convergent.

How do we apply the test to the series above? Is the series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$$

convergent? ____ Let's try some more examples.

Question 2.

(a) $\displaystyle\sum_{i=1}^{\infty} \frac{(-1)^i}{i^3}$

(b) $\displaystyle\sum_{i=1}^{\infty} \frac{(-1)^i}{i(\ln i)^2}$

(c) $\displaystyle\sum_{i=1}^{\infty} (-1)^i \frac{\sin\left(\frac{i\pi}{4}\right)}{i^2}$

(d) $\displaystyle\sum_{i=1}^{\infty} \frac{\cos i}{i^2}$

Definition B.4

A series $\sum a_i$ is called **absolutely convergent** if the series of absolute values $\sum |a_i|$ is convergent.

A series $\sum a_i$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

With this definition, the absolute convergence test can be restated as

$$\text{Absolute Convergence} \implies \text{Convergence}$$

Question 3.

For the following series, determine whether they are absolutely convergent, conditionally convergent, or divergent.

(a) $\displaystyle\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i^3+1}$

(b) $\displaystyle\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{\sqrt{i}}$

(c) $\displaystyle\sum_{i=1}^{\infty} \frac{(-1)^i}{5i+1}$

(d) $\displaystyle\sum_{i=1}^{\infty} (-1)^i \ln\left(1+\frac{1}{i}\right)$