Calculus & Analytical Geometry II

Lecture 26 Worksheet

Spring 2021 **Subhadip Chowdhury** **Math 112**

Thus far, we have seen two "types" of series for which we can easily determine their convergence or divergence:

(a) Geometric Series (b) p-Series

For Geometric series, we are in fact able to compute the sum of the series. For p-Series, we generally can only determine convergence or divergence.

Additionally, we have the following two tests so far:

(a) Divergence Test (b) Integral Test

Each test has its limitations. The Divergence Test actually doesn't tell when a given series converges, it's a test of when we can conclude that it diverges. The Integral Test is only applicable to continuous positive decreasing functions. So we can't use it for a series that has terms involving $(-1)^n$ or $\sin(n)$ or $n!$.

We add two more tests to our repertoire in this section: the Direct and Limit Comparison Tests.

§A. Direct Comparison Test

In the comparison tests the idea is to compare a given series with a series that is known to be convergent or divergent. Let's consider an example.

Example A.1

The series

$$\sum_{i=1}^{\infty} \frac{1}{2^i+1} \tag{1}$$

looks like the series $\sum_{i=1}^{\infty} \frac{1}{2^i}$, which is a geometric series with

$a =$ ____________ and $r =$ ____________ and is therefore ____________.

We can then conclude that $s_n = \sum_{i=1}^{n} \frac{1}{2^i+1}$ is bounded above by the sum of the geometric series,

which is equal to ________________________.

Additionally, because each term $\frac{1}{2^i+1}$ is positive, the partial sum sequence $\{s_n\}$ is increasing.

So, by ____________ ____________ theorem, the series (1) ____________.

Thus similar to the comparison test for improper integrals, we can write a comparison test for series, *which also has the restriction that all the terms of both the series must be positive.*

Theorem A.2: Direct Comparison Test

Suppose that $\sum a_i$ *and* $\sum b_i$ *are series with positive terms.*

- *If* $\sum b_i$ *is convergent and* $a_i \leq b_i$ *for all* n*, then* $\sum a_i$ *is also convergent.*
- *If* $\sum b_i$ *is divergent and* $a_i \geq b_i$ *for all* n*, then* $\sum a_i$ *is also divergent.*

Note:

- The purpose of the Comparison Test is to use a "nice" series $\sum b_n$ (whose convergence or divergence behavior we already know) to determine whether or not a second series $\sum a_n$ converges.
- We often choose $\sum b_n$ to be either a p-series or a geometric series that is somehow similar to $\sum a_n$.

Question 1.

Test each of the following series for convergence.

(a) $\displaystyle\sum_{i=1}^{\infty} \frac{5}{3^i+2}$

(b) $\displaystyle\sum_{i=1}^{\infty} \frac{4}{\sqrt{3i^2-i-2}}$

(c) $\displaystyle\sum_{i=1}^{\infty} \frac{i+\sin^2 i}{i^3+1}$

(d) $\displaystyle\sum_{i=1}^{\infty} \frac{4}{i\sqrt{i}+2^i}$

(e) $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i!}$

(f) $\displaystyle\sum_{i=1}^{\infty} \frac{2^i}{\ln(i)5^{i+1}}$

§B. Limit Comparison Test

Consider the series $\displaystyle\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} \frac{i^2}{i^4-i-1}$. Try a direct comparison. What goes wrong?

In a situation such as this, where the inequality you come up with *goes the wrong direction*, then you can try to apply the following test instead.

Theorem B.3: Limit Comparison Test

Let $\sum a_i$ *and* $\sum b_i$ *be series with positive terms. If*

$$\lim_{i\to\infty} \frac{b_i}{a_i} = c$$

for some finite constant c *and* $c > 0$*, then either both series converge or both diverge.*

Note: The Limit Comparison Test shows that if we have a series $\sum \frac{p(i)}{q(i)}$ of rational functions where $p(i)$ is a polynomial of degree m and $q(i)$ a polynomial of degree l, then the series $\sum \frac{p(i)}{q(i)}$ will behave like the series

$$\sum \frac{i^m}{i^l} = \sum \frac{1}{i^{l-m}}$$

So this test allows us to determine the convergence or divergence of series whose terms are rational functions.

Question 2.

Use the Limit Comparison Test to determine the convergence or divergence of the following series.

(a) $\sum_{i=1}^{\infty} \frac{2i^2+3i}{\sqrt{5+i^5}}$

(b) $\sum_{i=1}^{\infty} \frac{1}{2^i-1}$

(c) $\sum_{i=1}^{\infty} \frac{1}{\sqrt{i}+1}$

(d) $\sum_{i=1}^{\infty} \frac{i^3}{\sqrt{i^8+2i^3+1}}$

(e) $\sum_{i=1}^{\infty} \frac{2^i+1}{5^i+4}$

(f) $\sum_{i=1}^{\infty} \frac{e^i+1}{ie^i+1}$

§C. A Bunch of Practice Problems

Question 3.

For the problems below, first try to find a series to do a direct comparison with to determine convergence or divergence. If your inequality goes the wrong way, try the limit comparison test! If that doesn't seem to work, then try to change the series you are comparing to!

(a) $\sum_{i=1}^{\infty} \frac{\sqrt{i}}{i-3}$

(b) $\sum_{i=1}^{\infty} \frac{\sqrt{i}}{5i^2+2}$

(c) $\sum_{i=1}^{\infty} \frac{\sqrt{i+1}+\sqrt{i}}{i}$

(d) $\sum_{i=1}^{\infty} \frac{1}{\ln(i)}$

(e) $\sum_{i=1}^{\infty} \frac{i!}{(i+2)!}$ *(Simplify first)*

(f) $\sum_{i=1}^{\infty} \frac{1}{i^2-i\sin i}$

(g) $\sum_{i=1}^{\infty} \frac{\sqrt[3]{i}}{\sqrt{i^3+4k+3}}$

(h) $\sum_{i=1}^{\infty} \frac{1+\cos i}{e^i}$

(i) $\sum_{i=1}^{\infty} \frac{(2i-1)(i^2-1)}{(i+1)(i^2+4)^2}$

(j) $\sum_{i=1}^{\infty} \frac{i+3^i}{i+2^i}$

Some Useful Inequalities

Plot the following functions in DESMOS to confirm yourself.

- $\sin x \le x$ for all $x \ge 0$.
- $\ln x \le x$ for all $x \ge 0$.
- $\ln x \le x^p$ for sufficiently large x, for any $p > 0$. Try plotting for example, $x^{1/3}$.
- $x^p \le e^x$ for sufficiently large x, for any $p > 0$. Try plotting for example, x^3.

Question 4.

More Challenging Problems

(a) *Consider* $\displaystyle\sum_{i=1}^{\infty}\left(1-\cos\frac{1}{i}\right)$*. Test its convergence by limit comparing with* $\displaystyle\sum_{i=1}^{\infty}\frac{1}{i^2}$.

(b) *If* $\sum a_i$ *is a convergent series with positive terms, is it true that* $\sum \sin(a_i)$ *is also convergent? Try limit comparison test.*

(c) *If* $\sum a_i$ *and* $\sum b_i$ *are convergent series with positive terms, is it true that* $\sum a_i b_i$ *is also convergent? Try direct comparison test.*