Calculus & Analytical Geometry II

Strategies for Applying Series Tests

Spring 2021 **Subhadip Chowdhury** **Math 112**

The following is adapted from your Openstax textbook chapter 5.6. The book has a really nice flowchart summarizing strategies for choosing the correct test, as well as a table with summaries of all tests. The book uses a_n instead of a_i. So to be consistent, we will use the notation $\sum a_n$ today.

§A. Problem Solving Strategy

Step 1. First thing to try: the **Divergence Test.** If $\lim_{n\to\infty} a_n \neq 0$, then we know that the series $\sum a_n$ diverges and we are done right away! Otherwise, $\lim_{n\to\infty} a_n = 0$ and we have to try an actual test.

Step 2. Next, try to see if the series looks familiar. Is it of a certain class that we know: a **Geometric Series**, a $p-$**series**, the Harmonic Series, etc. If so, check the ratio r or the power p to determine if the series converges.

Look for variations as well - if the series looks like a sum or difference of two geometric series, or a sum of a $p-$series and a geometric series, etc. Remember, the sum or difference of two converging series converges. (What happens if we add a diverging series with a converging series?)

Step 3. Identify if the series has all positive terms. If it does not, determine if it is an Alternating Series and try the **Alternating Series Test.**

Step 4. If the series has negative terms and is *not alternating*, you can try to determine if the series is absolutely convergent, since absolute convergence implies convergence. Remember, this means determining whether $\sum a_n$ converges. You can either use the tests below for positive series, or use the Ratio or Root Test, as these are tests for absolute convergence. Do the terms in the series contain a factorial or power? If the terms are powers such that $a_n = (b_n)^n$ try the root test first. Otherwise, try the ratio test first.

Step 5. Lastly for a non-positive series, make sure you have answered the question! Does it ask for absolute/conditional convergence, or simply asks whether the series converges or diverges? If the former, make sure you fully investigate the series by checking for absolute convergence, especially if it is an alternating series.

Step 6. Now, if your series has only positive terms (or you are examining $\sum |a_n|$), then we can apply the other three tests - the Direct & Limit Comparison Tests and the Integral Test. It's a good idea to try **Direct comparison** *first*. If that fails due to the inequality going the wrong way, then use the **Limit comparison** test.

Step 7. If all of the above has failed you, then we have the **Integral Test** as our backup.

§B. Summary of Series Tests

The following is copied from your textbook.

Series or Test	Conclusions	Comments
Divergence Test For any series $\sum_{n=1}^{\infty} a_n$, evaluate $\lim_{n\to\infty} a_n$.	If $\lim_{n\to\infty} a_n = 0$, the test is inconclusive.	This test cannot prove convergence of a series.
	If $\lim_{n\to\infty} a_n \neq 0$, the series diverges.	
Geometric Series $\sum_{n=1}^{\infty} ar^{n-1}$	If $\|r\| < 1$, the series converges to $a/(1-r)$.	Any geometric series can be reindexed to be written in the form $a + ar + ar^2 + \cdots$, where a is the initial term and r is the ratio.
	If $\|r\| \geq 1$, the series diverges.	
***p*-Series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$	If $p > 1$, the series converges.	For $p = 1$, we have the harmonic series $\sum_{n=1}^{\infty} 1/n$.
	If $p \leq 1$, the series diverges.	

Comparison Test For $\sum_{n=1}^{\infty} a_n$ with nonnegative terms, compare with a known series $\sum_{n=1}^{\infty} b_n$.	If $a_n \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.	Typically used for a series similar to a geometric or p-series. It can sometimes be difficult to find an appropriate series.
	If $a_n \geq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	
Limit Comparison Test For $\sum_{n=1}^{\infty} a_n$ with positive terms, compare with a series $\sum_{n=1}^{\infty} b_n$ by evaluating $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.	If L is a real number and $L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.	Typically used for a series similar to a geometric or p-series. Often easier to apply than the comparison test.
	If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.	
	If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	

Integral Test If there exists a positive, continuous, decreasing function f such that $a_n = f(n)$ for all $n \geq N$, evaluate $\int_N^\infty f(x)\,dx$.	$\int_N^\infty f(x)\,dx$ and $\sum_{n=1}^\infty a_n$ both converge or both diverge.	Limited to those series for which the corresponding function f can be easily integrated.
Alternating Series $\sum_{n=1}^\infty (-1)^{n+1} b_n$ or $\sum_{n=1}^\infty (-1)^n b_n$	If $b_{n+1} \leq b_n$ for all $n \geq 1$ and $b_n \to 0$, then the series converges.	Only applies to alternating series.
Ratio Test For any series $\sum_{n=1}^\infty a_n$ with nonzero terms, let $\rho = \lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert$.	If $0 \leq \rho < 1$, the series converges absolutely.	Often used for series involving factorials or exponentials.
	If $\rho > 1$ or $\rho = \infty$, the series diverges.	
	If $\rho = 1$, the test is inconclusive.	
Root Test For any series $\sum_{n=1}^\infty a_n$, let $\rho = \lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert}$.	If $0 \leq \rho < 1$, the series converges absolutely.	Often used for series where $\lvert a_n \rvert = b_n^n$.
	If $\rho > 1$ or $\rho = \infty$, the series diverges.	
	If $\rho = 1$, the test is inconclusive.	

Question 1.

Decide whether each of the following is a p–series, or a geometric series, or neither. If it is one of the two, indicate whether or not the series converges. If it's neither, you do not need to work further.

(a) $\displaystyle\sum_{n=1}^\infty \frac{1}{n\sqrt{n}}$ (b) $\displaystyle\sum_{n=1}^\infty \frac{3}{4^{n-1}}$ (c) $\displaystyle\sum_{n=1}^\infty \frac{n}{(3n)^5}$ (d) $\displaystyle\sum_{n=1}^\infty \frac{2^{2n+1}}{5^{n-1}}$

(e) $\displaystyle\sum_{n=1}^\infty \sqrt{\frac{2}{n}}$ (f) $\displaystyle\sum_{n=1}^\infty \frac{(-1)^n}{n}$ (g) $\displaystyle\sum_{n=1}^\infty \frac{(-2)^n}{e^{2n}}$

Solution. (a) Since $\displaystyle\sum_{n=1}^\infty \frac{1}{n\sqrt{n}} = \sum_{n=1}^\infty \frac{1}{n^{3/2}}$, we see that this is a p -series with $p = 3/2$. The series

(b) Since $\displaystyle\sum_{n=1}^\infty \frac{3}{4^{n-1}} = \sum_{n=1}^\infty 3\cdot\left(\frac{1}{4}\right)^{n-1}$, we see that this is a geometric series with $r = 1/4$. The series converges.

(c) Since $\displaystyle\sum_{n=1}^\infty \frac{n}{(3n)^5} = \frac{1}{3^5}\sum_{n=1}^\infty \frac{1}{n^4}$, we see that this is a constant multiple of a p-series with $p = 4$. The series converges.

(d) We have

$$\sum_{n=1}^\infty \frac{2^{2n+1}}{5^{n-1}} = \sum_{n=1}^\infty \frac{2^{2n}\cdot 2}{5^{n-1}} = \sum_{n=1}^\infty \frac{\left(2^2\right)^n\cdot 2}{5^{n-1}} = \sum_{n=1}^\infty 8\left(\frac{4}{5}\right)^{n-1}$$

so this series is geometric with $r = 4/5$. The series converges.

(e) Since $\displaystyle\sum_{n=1}^{\infty}\sqrt{\frac{2}{n}} = \sum_{n=1}^{\infty}\frac{\sqrt{2}}{\sqrt{n}} = \sqrt{2}\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$, we see that this series is a constant multiple of a p -series with $p = 1/2$. The series diverges.

(f) This series is not a p-series since its terms are alternating in sign, and it is not geometric since there is no common ratio among successive terms. Therefore, our answer is neither.

(g) We have

$$\sum_{n=1}^{\infty}\frac{(-2)^n}{e^{2n}} = \sum_{n=1}^{\infty}\frac{(-2)^n}{(e^2)^n} = \sum_{n=1}^{\infty}\left(\frac{-2}{e^2}\right)^n$$

so we see that this is a geometric series with $r = -2/e^2$. The series converges.

■

■ Question 2. □

For each of the following series, find either a p -series or a geometric series that would be an appropriate candidate for comparison. You need not actually perform the comparison test.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{5n^2}{2n^3-1}$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{3n}{\sqrt{n^5+n^4+2}}$ (c) $\displaystyle\sum_{n=1}^{\infty}\frac{3^n+1}{2^n-1}$ (d) $\displaystyle\sum_{n=1}^{\infty}\frac{4}{n(n+3)}$

(e) $\displaystyle\sum_{n=1}^{\infty}\frac{3^n}{5^n+n}$ (f) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{n^2\sqrt{6n-1}}$ (g) $\displaystyle\sum_{n=1}^{\infty}\sqrt{\frac{4^n}{3^{2n}+100}}$ (h) $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{6^n-n}}{4^{2n}+n\sqrt{n}}$

Solution. (a) Since

$$\frac{5n^2}{2n^3-1} \geq \frac{5n^2}{2n^3} = \frac{5}{2}\cdot\frac{1}{n}$$

the series $\displaystyle\frac{5}{3}\sum_{n=1}^{\infty}\frac{1}{n}$ would be a good candidate for comparison.

(b) Since

$$\frac{3n}{\sqrt{n^5+n^4+2}} \leq \frac{3n}{\sqrt{n^5}} = 3\cdot\frac{1}{n^{3/2}}$$

the series $\displaystyle 3\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ would be a good candidate for comparison.

(c) Since

$$\frac{3^n+1}{2^n-1} \geq \frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n$$

the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n$ would be a good candidate for comparison.

(d) Since

$$\frac{4}{n(n+3)} \leq \frac{4}{n^2} = 4\cdot\frac{1}{n^2}$$

the series $\displaystyle 4\sum_{n=1}^{\infty}\frac{1}{n^2}$ would be a good candidate for comparison.

(e) Since

$$\frac{3^n}{5^n+n} \leq \frac{3^n}{5^n} = \left(\frac{3}{5}\right)^n$$

the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n$ would be a good candidate for comparison.

(f) Since

$$\frac{n^2}{n^2\sqrt{6n-1}} \geq \frac{n^2}{n^2\sqrt{6n}} = \frac{1}{\sqrt{6}}\cdot\frac{1}{n^{1/2}}$$

the series $\displaystyle\frac{1}{\sqrt{6}}\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ would be a good candidate for comparison.

(g) Since

$$\sqrt{\frac{4^n}{3^{2n}+100}} \leq \sqrt{\frac{2^{2n}}{3^{2n}}} = \left(\frac{2}{3}\right)^n$$

the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ would be a good candidate for comparison.

(h) Since

$$\frac{\sqrt{6^n-n}}{4^{2n}+n\sqrt{n}} \leq \frac{\sqrt{6^n}}{4^{2n}} = \frac{(\sqrt{6})^n}{16^n} = \left(\frac{\sqrt{6}}{16}\right)^n$$

the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{\sqrt{6}}{16}\right)^n$ would be a good candidate for comparison.

■

■ Question 3.

□

Determine which of the following series converge. Justify your conclusions with the appropriate explanations.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ (b) $\displaystyle\sum_{n=1}^{\infty}2^{-n}$ (c) $\displaystyle\sum_{n=1}^{\infty}\frac{n+5}{5^n}$ (d) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\sqrt{n(n-1)}}$

(e) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n n^3}{n^2}$ (f) $\displaystyle\sum_{n=1}^{\infty}\frac{2n}{8n-5}$ (g) $\displaystyle\sum_{n=1}^{\infty}\frac{3^n}{(2n)!}$ (h) $\displaystyle\sum_{n=1}^{\infty}2^n$

(i) $\displaystyle\sum_{n=1}^{\infty}\frac{n-1}{n^2\sqrt{n^3+1}}$ (j) $\displaystyle\sum_{n=1}^{\infty}\left(-\frac{1}{3}\right)^n$ (k) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ (l) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\ln n}{\sqrt{n}}$

Solution. (a) Let $a_n = 1/\sqrt{n}$. Then we have

$$a_{n+1} = \frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}} = a_n \quad \text{for all } n$$

and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}(1/\sqrt{n}) = 0$. Therefore, since $a_{n+1} \leq a_n$ for all n, and since $\lim\limits_{n\to\infty} a_n = 0$, the given series converges by the Alternating Series Test.

(b) Since $\sum 2^{-n} = \sum \left(\frac{1}{2}\right)^n$, we see that this series is a geometric series with $r = 1/2$. Therefore, the series converges. (Note: The Ratio Test would also work on this series.)

(c) Because the series contains a term with 5^n in it, we will apply the Ratio Test. We have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{n+6}{5^{n+1}} \cdot \frac{5^n}{n+5} \right| = \frac{1}{5} \cdot \lim_{n\to\infty} \frac{n+6}{n+5} = \frac{1}{5}$$

Therefore, since $\lim_{n\to\infty} |a_{n+1}|/|a_n| = 1/5 < 1$, the given series converges by the Ratio Test.

(d) First, note that

$$\frac{1}{\sqrt{n(n-1)}} \geq \frac{1}{\sqrt{n^2}} = \frac{1}{n}$$

so we choose to compare the given series with the series $\sum \frac{1}{n}$. Since $\frac{1}{\sqrt{n(n-1)}} \geq \frac{1}{n} \geq 0$ for all $n \geq 2$, and since $\sum \frac{1}{n}$ is a divergent p -series ($p = 1$), the given series diverges by the Comparison Test.

(e) This series is alternating with $a_n = n^3/n^2 = n$, but the Alternating Series Test does not give any information since $\lim_{n\to\infty} a_n \neq 0$. However, note that

$$\left\{(-1)^n \frac{n^3}{n^2}\right\}_{n=1}^{\infty} = \{(-1)^n n\}_{n=1}^{\infty} = -1, 2, -3, 4, -5, 6, \ldots$$

Thus, $\lim_{n\to\infty} ((-1)^n n)$ does not exist since the above sequence does not approach any one number, and it therefore follows that the given series diverges by the Test for Divergence.

(f) Since

$$\lim_{n\to\infty} \frac{2n}{8n-5} = \lim_{n\to\infty} \frac{2}{8} = \frac{1}{4} \neq 0$$

the given series diverges by the Test for Divergence.

(g) Since this series has a factorial in it, we will apply the ratio test. We have

$$\lim_{n\to\infty} \left| \frac{3^{n+1}}{(2(n+1))!} \cdot \frac{(2n)!}{3^n} \right| = \lim_{n\to\infty} \frac{3}{(2n+2)(2n+1)} = 0$$

Therefore, since $\lim_{n\to\infty} |a_{n+1}|/|a_n| = 0 < 1$, the given series converges by the Ratio Test.

(h) This is a geometric series with $r = 2$. Therefore, since $|r| > 1$, the series diverges by the Geometric Series Test. (Note: The Test for Divergence and the Ratio Test would also work for this series.)

(i) First, note that

$$\frac{n-1}{n^2\sqrt{n^3+1}} \leq \frac{n}{n^2\sqrt{n^3}} = \frac{n}{n^{7/2}} = \frac{1}{n^{5/2}}$$

so we choose to compare the given series with the series $\sum \frac{1}{n^{5/2}}$. Since $0 \leq \frac{n-1}{n^2\sqrt{n^3+1}} \leq \frac{1}{n^{5/2}}$ for all n, and since $\sum \frac{1}{n^{5/2}}$ is a convergent p -series ($p = 5/2$), the given series converges by the Comparison Test.

(j) Since this series is geometric with $r = -1/3$, it converges by the Geometric Series Test. (Note: The Ratio Test would also work on this series.)

(k) For this series, applying the Ratio Test yields $L = 1$ and is therefore inconclusive, and all attempts to compare it to a related p -series or geometric series fail. Therefore, since $f(x) = 1/\left(x(\ln x)^2\right)$ is continuous, positive, and decreasing on $[2, \infty)$, we will apply the Integral Test. We have

$$\begin{aligned}\int_2^\infty \frac{1}{x(\ln x)^2}dx = \lim_{t\to\infty}\int_2^t \frac{1}{x(\ln x)^2}dx &= \lim_{t\to\infty}\left[\frac{(\ln x)^{-1}}{-1}\right]_2^t \qquad \text{(substituting } w = \ln x, dw = (1/x)dx)\\ &= \lim_{t\to\infty}\left[-\frac{1}{\ln t}+\frac{1}{\ln 2}\right]\\ &= \frac{1}{\ln 2}\end{aligned}$$

indicating that the above integral converges to $1/(\ln 2)$. Therefore, since the integral $\int_2^\infty \frac{1}{x(\ln x)^2}dx$ converges, the given series also converges by the Integral Test.

(l) Since this is an alternating series, we will let $a_n = (\ln n)/\sqrt{n}$ and try the Alternating Series Test. To do this, we must first show that $\lim_{n\to\infty} a_n = 0$, which we can do by showing that the function $f(x) = (\ln x)/\sqrt{x}$ approaches 0 as x approaches infinity. Since

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty}\frac{1/x}{1/(2\sqrt{x})} = \lim_{x\to\infty}\frac{2\sqrt{x}}{x} = \lim_{x\to\infty}\frac{2}{\sqrt{x}} = 0$$

we therefore conclude that $\lim_{n\to\infty} a_n = 0$, so one of the two conditions of the Alternating Series Test is satisfied. We also need to show that $a_{n+1} \le a_n$, which is equivalent to showing that a_n decreases in value as n increases. We can do this by showing that the function $f(x) = (\ln x)/\sqrt{x}$ is decreasing, which we will show by demonstrating that the derivative of $f(x)$ is negative. Using the Quotient Rule and simplifying, we have

$$f'(x) = \frac{\sqrt{x}\cdot\frac{1}{x} - (\ln x)\cdot\frac{1}{2\sqrt{x}}}{x} = \frac{2-\ln x}{2x\sqrt{x}}$$

Therefore, we see that $f'(x) < 0$ if $\ln x > 2$, which occurs when $x > e^2 \approx 7.39$. Thus, we conclude that $a_{n+1} \le a_n$ for all $n \ge 8$.

To summarize, since we have shown above that $\lim_{n\to\infty} a_n = 0$, and that $a_{n+1} \le a_n$ for all $n \ge 8$, it follows that $\sum_{n=8}^\infty \frac{(-1)^n \ln n}{\sqrt{n}}$ converges by the Alternating Series Test. Finally, since the given series $\sum_{n=1}^\infty \frac{(-1)^n \ln n}{\sqrt{n}}$ is obtained by simply adding 7 finite numbers to the front of $\sum_{n=8}^\infty \frac{(-1)^n \ln n}{\sqrt{n}}$, it follows that $\sum_{n=1}^\infty \frac{(-1)^n \ln n}{\sqrt{n}}$ also converges.

■

■ Question 4.

□

Determine if the given series is converging or diverging. If the series has negative terms, determine if the series is absolutely convergent, conditionally convergent, or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2n}{n^3+3n^2+1}$ *(b)* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(3n+1)}{n!}$ *(c)* $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^4}$ *(d)* $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{(n+1)^n}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n-5^n}{7^n}$ *(f)* $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ *(g)* $\displaystyle\sum_{n=1}^{\infty} \frac{1+(-1)^n}{n}$ *(h)* $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}-(\ln(n))^4}$ *(j)* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ *(k)* $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^{2/3}}$ *(l)* $\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{5^n}$

(m) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}(\ln(n))^2}$ *(n)* $\frac{1}{2}-\frac{1}{5}+\frac{1}{4}-\frac{1}{25}+\frac{1}{8}-\frac{1}{125}+\dots$

Solution. (a) Diverges.

(b) Converges. Does it converge conditionally?

(c) Diverges.

(d) Converges.

(e) Converges.

(f) Converges. Does it converge absolutely or conditionally?

(g) Diverges.

(h) Diverges. What about the series $\sum \frac{n!}{n^n}$?

(i) Converges.

(j) Diverges.

(k) Converges.

(l) Converges.

(m) Converges.

(n) Converges. Does it converge absolutely?

■