# MATH 111 - Calculus and Analytic Geometry I

## Lecture 10 Worksheet

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**Title:** The Derivative Function

**Summary:** We will define the derivative function and learn how to calculate it from the definition. We will find that knowing the derivative of the function at every point would produce some valuable information about the behavior of the function.

### §A. Average Versus Instantaneous Rate of Change

Instantaneous Rate of Change is a generalization of the notion of instantaneous velocity and measures how fast a particular function is changing at a given point. If the original function represents the position of a moving object, this instantaneous rate of change is precisely the instantaneous velocity of the object. We saw yesterday that the instantaneous rate of change can be interpreted geometrically on the function's graph, and this connection is fundamental to many of the main ideas in calculus.

**Definition A.1**

For a function $f(x)$, we define the average rate of change of $f$ over the interval $a \leq x \leq b$ to be

$$\mathbf{AV}_{[a,b]} = \frac{f(b)-f(a)}{b-a}$$

Equivalently, if we want to consider the average rate of change of $f(x)$ on an interval of the form $[a, a+h]$, we can write

$$\mathbf{AV}_{[a,a+h]} = \frac{f(a+h)-f(a)}{h}$$

**Definition A.2**

The **derivative** of $f$ at $a$, written as $f'(a)$, is defined as the instantaneous rate of change of $f$ at $a$. In other words,

$$f'(a) := \lim_{b\to a} \frac{f(b)-f(a)}{b-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

The process of finding the derivative of a function is called **differentiation**. A function $f(x)$ is said to be **differentiable** at $a$ if its derivative exists at $x = a$. By now, we have seen lots of examples when a limit doesn't exist or is undefined. If the limit in the definition of the derivative doesn't exist at $x = a$, we say that the derivative $f'(a)$ doesn't exist. So in that case, we say that the function $f(x)$ is not differentiable at $x = a$. A function is said to be differentiable on an open interval $(a, b)$ if its derivative exists at every $x$ in the interval.

To emphasize that $f'(a)$ is the rate of change of $f(x)$ as the variable $x$ changes, we will say more precisely that, $f'(a)$ is the derivative of $f$ with respect to $x$ at $x = a$.

**Example A.1**

We want use the limit definition to find the slope of the tangent line to the graph of $f(x) = x^2 - 8x$ at $(2,-4)$. This is the value of $f'(2)$.

$$\begin{aligned} f'(2) &= \lim_{b\to 2} \frac{f(b) - f(2)}{b-2} \\ &= \lim_{b\to 2} \frac{b^2 - 8b - 2^2 + 8 \times 2}{b-2} \\ &= \lim_{b\to 2} \frac{(b^2 - 2^2) - 8(b-2)}{b-2} \\ &= \lim_{b\to 2} \frac{(b-2)(b+2) - 8(b-2)}{b-2} \\ &= \lim_{b\to 2} b + 2 - 8 = -4. \end{aligned}$$

■ **Question 1.**

Now find the slope of the tangent line to the graph of $f(x) = x^2 - 8x$ at $(c, f(c))$. This gives you a formula for $f'(c)$.

Does the value of $f'(c)$ match with your answer from last question when $c = 2$?

## §B. The Derivative Function

Generalizing the idea of derivative at a point, we can define the **Derivative Function** as follows:

**Definition B.1**

For any function $f(x)$ the derivative function $f'(x)$ is defined by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

■ **Question 2.**

Use either of the limit definition to find the derivative of the following functions. Also find the point(s) at which the function is not differentiable.

(a) $f(x) = b + mx$.

(b) $f(x) = \sqrt{x}$.

(c) $f(x) = \frac{1}{x}$.

### Graphical Interpretation

■ **Question 3.**

Sketch the derivative of the function $f(x)$ graphed below.

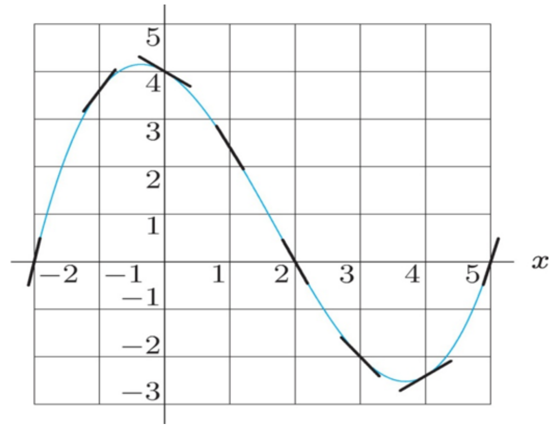


Figure 1: Estimating the derivative graphically as the slope of the tangent line

From the graphs and from our lab yesterday, we make the following observations:

**Theorem B.1**

If $f' > 0$ on an interval, then the slope of the tangent to the graph of $f$ is positive and consequently, $f$ increasing over that interval.

Similarly, if $f' < 0$ on an interval, then $f$ is decreasing over that interval. Finally, if $f' = 0$ on an interval, then $f$ is constant over that interval.

## §C. **Differentiability and Continuity**

**Theorem C.1**

If $f$ is differentiable at $a$, then $f$ is continuous at $a$.

In other words, $\boxed{\text{Differentiability} \implies \text{Continuity.}}$ The converse is **not** true. Below are some examples where $f(x)$ is continuous at $0$, but isn't differentiable at $0$.

### ■ Question 4.

For each of the following functions, draw its graph in Desmos. Explain why $f$ is continuous at $x = 0$ but $f'(0)$ is not defined.

(a) $f(x) = |x|$.

(b) $f(x) = \sqrt[3]{x}$.

(c) $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.