Calculus & Analytical Geometry II

Lecture 29 Worksheet

Spring 2021 **Subhadip Chowdhury** **Math 112**

Last time we learned that *absolute convergence* of a series implies *convergence*. This allows us to test the convergence of a series $\sum a_i$ with positive and negative terms by instead looking at the convergence of a series $\sum |a_i|$ with only positive terms. However, we still need to test the convergence of the second series by other means.

Since the second series has positive terms only, we can now apply integral, direct comparison, or limit comparison tests. Consider the following example. What would you apply here?

Example .1

Consider $\displaystyle\sum_{i=1}^{\infty} \frac{(-1)^i \ln(i)}{i!}$. Does this series converge absolutely?

We could try a comparison for $\displaystyle\sum_{i=1}^{\infty} \frac{\ln(i)}{i!}$. Did you run into any difficulty?

§A. Ratio Test

The following test is very useful in determining whether a given series is absolutely convergent.

Theorem A.2: Ratio Test

Let $\sum a_i$ be a series with nonzero terms and suppose the following limit exists:

$$\mathrm{L} = \lim_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right|$$

(a) If $0 \le \mathrm{L} < 1$, then $\sum a_i$ converges absolutely.

(b) If $\mathrm{L} > 1$, then $\sum a_i$ diverges.

(c) If $\mathrm{L} = 1$, then the Ratio Test is inconclusive.

The ratio test is ideal for any series involving factorials or exponential expressions. We won't concern ourselves with the details of the proof, you can find it in your textbook and it's a little too technical. For the curious, note that it involves geometric series and the direct comparison test.

Note: Part (c) of the Ratio Test says that if $\lim_{i\to\infty} |a_{i+1}/a_i| = 1$, the test gives no information. For instance, for the convergent series $\sum 1/n^2$ we have

$$\left|\frac{a_{i+1}}{a_i}\right| = \frac{\frac{1}{(i+1)^2}}{\frac{1}{i^2}} = \frac{i^2}{(i+1)^2} = \frac{1}{\left(1+\frac{1}{i}\right)^2} \to 1 \quad \text{as } n \to \infty$$

whereas for the divergent series $\sum 1/n$ we have

$$\left|\frac{a_{i+1}}{a_i}\right| = \frac{\frac{1}{i+1}}{\frac{1}{i}} = \frac{i}{i+1} = \frac{1}{1+\frac{1}{i}} \to 1 \quad \text{as } i \to \infty$$

Therefore, if $\lim_{i\to\infty} |a_{i+1}/a_i| = 1$, the series $\sum a_i$ might converge or it might diverge. In this case the Ratio Test fails and we must use some other test.

Question 1.

Use the Ratio Test to test each of the following series for convergence.

(a) $\sum \frac{(-1)^i \ln(i)}{i!}$.

(b) $\sum \frac{(-1)^i i^2}{i!}$

(c) $\sum \frac{3^i}{i!}$

(d) $\sum \frac{i^i}{i!}$

(e) $\sum \frac{(-1)^i (i!)^2}{(2i)!}$

(f) $\sum \frac{(-1)^i i^3}{3^i}$

§B. Root Test

Another test similar to the ratio test to check absolute convergence is the following. It is almost identical to the ratio test in its conclusion, but you compute a different limit, as per its name.

Theorem B.3: Root Test

Let $\sum a_i$ be a series with nonzero terms and suppose the following limit exists:

$$L = \lim_{i\to\infty} \sqrt[i]{|a_i|} = \lim_{i\to\infty} |a_i|^{1/i}$$

(a) *If $0 \le L < 1$, then $\sum a_i$ converges absolutely.*

(b) *If $L > 1$, then $\sum a_i$ diverges.*

(c) *If $L = 1$, then the Ratio Test is inconclusive.*

You will want to use the root test in only certain circumstances when its application is easier than the ratio test. This will be when a series has general term involving exponentials. In particular, if individual terms a_i look like $(b_i)^i$, then applying the root test allows you to only have to evaluate the (likely much simpler) limit $\lim_{i\to\infty} b_i$.

> **Note:** If $L = 1$ in the Ratio Test, don't try the Root Test because L will again be 1. And if $L = 1$ in the Root Test, don't try the Ratio Test because it will fail too.

■ Question 2.

Use the Ratio Test to test each of the following series for convergence.

(a) $\sum \left(\frac{i^2+1}{2i^2+3}\right)^i$

(b) $\sum \frac{2^{i^2}}{i!}$ *[Hint: $2^{i^2} = (2^i)^i$ and $i! < i^i$.]*

(c) $\sum \frac{(-1)^{i-1}}{(\ln i)^i}$

(d) $\sum (\arctan i)^i$