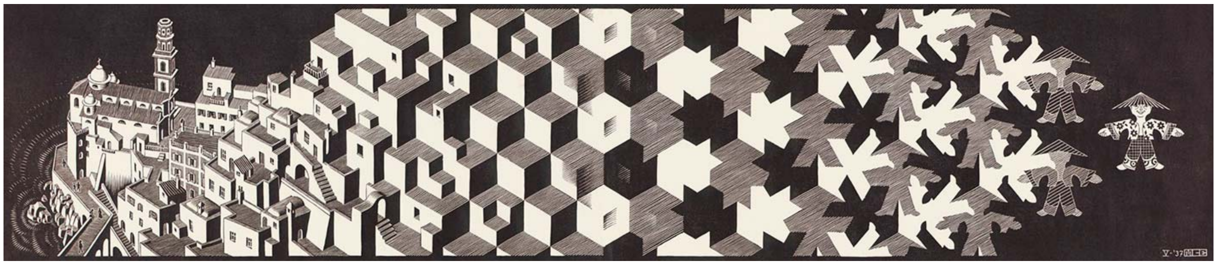

Elementary Functions and Calculus II

Math 132 Lecture notes

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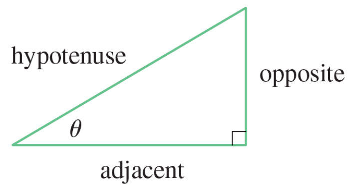
Contents

Chapter 1 | Differential Calculus of Trigonometric Functions

§1.1 Function Fundamentals

You most likely learned about trig functions in precalculus via right triangles. For an acute angle θ, the six trigonometric functions are defined as ratios of lengths of sides of a right triangle as follows:



(a) $\sin(\theta) = \dfrac{opp}{hyp}$

(b) $\cos(\theta) =$

(c) $\tan(\theta) = \dfrac{opp}{adj}$

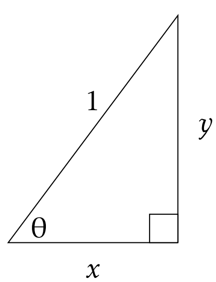
(d) $\csc(\theta) =$

(e) $\sec(\theta) = \dfrac{hyp}{adj}$

(f) $\cot(\theta) =$

Question 1.

Consider a right-angled triangle as below whose hypotenuse has length 1*. Use the definitions above to express x and y in terms of θ.*



How does the Pythagoras' identity relate x and y? Substitute both in terms of θ to establish the classical trigonometric identity.

The definition above doesn't apply to obtuse or negative angles, so we need to generalize it.

Definition 1.1.1

For a general angle θ in standard position, we let $P \equiv (x, y)$ be any point on the terminal side of θ and we let r be the distance $|OP|$ as in fig. 1.1. Then we can algebraically define

(a) $\sin(\theta) = \dfrac{y}{r}$ (b) $\cos(\theta) = \dfrac{x}{r}$ (c) $\tan(\theta) = \dfrac{y}{x}$

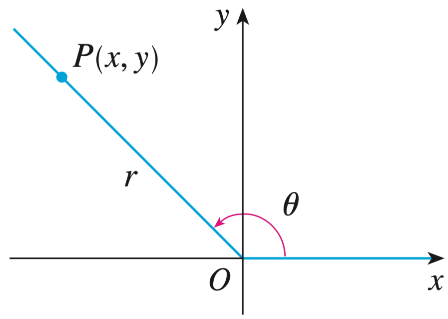


Figure 1.1: General Definition of Trigonometric Functions

Note: This definition automatically takes care of the sign. In particular,

(a) In the first quadrant, $\sin\theta$ is positive and $\cos\theta$ is positive.

(b) In the second quadrant, $\sin\theta$ is ________________ and $\cos\theta$ is ________________.

(c) In the third quadrant, $\sin\theta$ is ________________ and $\cos\theta$ is ________________.

(d) In the fourth quadrant, $\sin\theta$ is ________________ and $\cos\theta$ is ________________.

The picture below, derived using Euclidean geometry results, will help you recall the trig values of some common angles. **The circle in the picture has radius** 1**.**

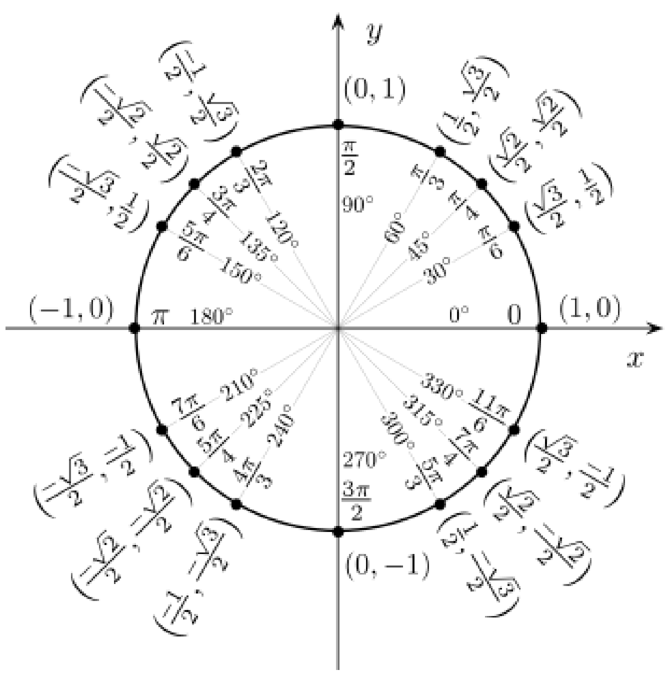


Figure 1.2: Common Trig Values

Example 1.1.2

Since the length of the radius is 1, we can use above picture to calculate

$$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2} \qquad \text{and} \qquad \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}.$$

Note: Recall that the equation of the circle is $x^2 + y^2 = 1$. Since $r = 1$, we get $\sin\theta = x$ and $\cos\theta = y$. So

$$\sin^2\theta + \cos^2\theta = 1$$

for all values of θ.

Question 2.

Try to use the values in figure 1.2 above, and your knowledge of the unit circle, to compute the trig values below.

(a) $\sin\left(\frac{-5\pi}{6}\right) =$

(b) $\cos\left(-\frac{3\pi}{4}\right) =$

(c) $\tan\left(\frac{14\pi}{3}\right) =$

(d) $\sec\left(\frac{7\pi}{6}\right) =$

(e) $\cot\left(\frac{7\pi}{6}\right) =$

In calculus, when we use the notation $f(x) = \sin(x)$ to mean the sine function, it is implicitly understood that $\sin x$ means the sine of the angle whose **radian** measure is x. We will discuss the domain and range of this function in the next section.

Question 3.

Use figure 1.2 to find values of x between 0 and 2π for which $\sin x = 0$?

What about values of x between 0 and 2π for which $\sin x = \frac{1}{2}$?

Can you find a real number x between 2π and 4π such that $\sin x = \frac{1}{2}$?

1.1.1 Domain and Range

Next, let's think about the domain and range of the trigonometric functions we defined above. As we saw in the last exercise, we can put any real number, positve or negative, and even those bigger than 2π as an angle on the unit circle. So the functions of sine and cosine are defined for any real number.

Hence, the domain of $\sin(x)$ and $\cos(x)$ is $\mathbb{R}$. What is the range of these two functions? ____________

Hint: They both have the same range. Write your answer using the interval notation.

Since the other four trigonometric functions are utilize division in their definitions we need to be careful about their domains.

■ **Question 4.** □

Find the domain of the $\tan x$ *function.* ***Hint:*** *Where is the function undefined?*

■ **Question 5.** □

Find the domain of the $\sec x$ *function.* ***Hint:*** *The answer is same as above. Why?*

1.1.2 Verifying Trigonometric Identities

With our knowledge of the interrelated definitions of the six standard trigonometric functions, we can verify a variety of trigonometric identities. The process involves utilizing our definitions of the trigonometric functions to manipulate one side of an identity to look like the other. Effectively, the act of going through this process is algebraic practice to familiarize ourselves with the definitions of the trigonometric functions.

■ **Question 6.** □

Show that $\cos t + \sin t \tan t = \sec t.$

■ **Question 7.** □

Show that $(1 - \cos^2 x)(1 + \cot^2 x) = 1.$

There are more examples in the textbook.

1.1.3 Graphs of the sine and cosine function

Before we look at the graphs of the two fundamental trigonometric functions, consider the following list of identities. Can you explain why these must be true using geometry on the unit circle?

For all $x \in \mathbb{R}$,

(a) $\sin(2\pi + x) = \sin(x)$ and $\cos(2\pi + x) = \cos(x)$

(b) $\sin\left(\frac{\pi}{2}-x\right)=\cos x$ and $\cos\left(\frac{\pi}{2}-x\right)=\sin x$

(c) $\sin\left(\frac{\pi}{2}+x\right)=\cos x$ and $\cos\left(\frac{\pi}{2}+x\right)=-\sin x$

(d) $\sin(\pi-x)=\sin x$ and $\cos(\pi-x)=-\cos x$

(e) $\sin(\pi+x)=-\sin x$ and $\cos(\pi+x)=-\cos x$

(f) $\sin\left(\frac{3\pi}{2}-x\right)=-\cos x$ and $\cos\left(\frac{3\pi}{2}-x\right)=-\sin x$

(g) $\sin\left(\frac{3\pi}{2}+x\right)=-\cos x$ and $\cos\left(\frac{3\pi}{2}+x\right)=\sin x$

(h) $\sin(2\pi-x)=\sin(-x)=-\sin x$ and $\cos(2\pi-x)=\cos(-x)=\cos x$

We will go over a couple of them during the tutorial in week 1. There are a lot more trig identity listed in the book. You do NOT need to memorize those. We will come back to them when we need them later.

Using these results to extend the graph of $\sin x$ and $\cos x$ beyond $[0, 2\pi]$, we get fig. 1.3.

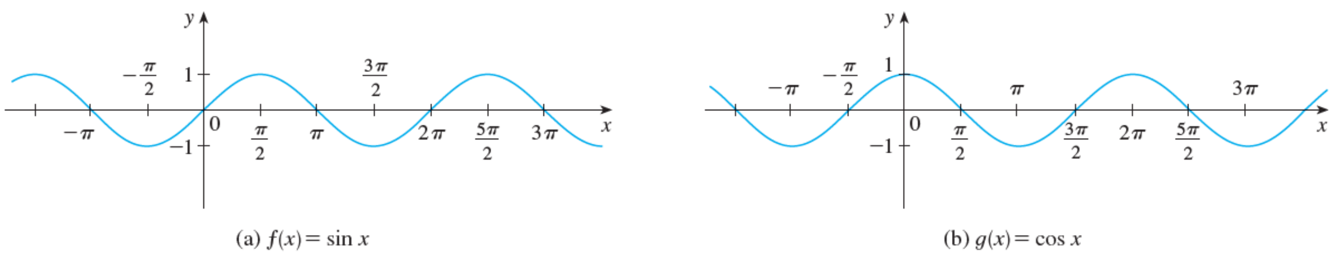


Figure 1.3: Graphs of $\sin x$ and $\cos x$

Question 8.

Use DESMOS to draw the graphs of $\tan x$, $\csc x$, *and* $\sec x$.

1.1.4 Period and Amplitude of Trigonometric Functions

Go to https://www.desmos.com/calculator/wyamilnmwe. Given a graph of $f(x)$, consider the graph of $g(x)=af(bx)$ and $h(x)=f(x+c)+d$. Explore the effect of the constants a, b, c and d on the graph using the sliders and discuss the questions with your group. Change the function $f(x)$ to experiment with other choices.

Question 9.

Match the function with the correct effect in the following table.

$f(ax)$	*Shift upwards by a*
$af(x)$	*Shrink horizontally by a factor of a*
$f(a+x)$	*Stretch vertically by a factor of a.*
$a+f(x)$	*Shift to the left by a.*

■ Question 10. ☐

Draw the graphs of $\sin(2x)$ *and* $\cos(2x)$ *in the space below without using DESMOS.*

Definition 1.1.3

A function f is called periodic if there is a positive number p such that

$$f(x+p) = f(x)$$

for all real numbers x in $\mathrm{Dom}(f)$. In this case, the smallest such p is called the 'period' of f.

■ Question 11. ☐

What is the period of $\sin x$*? Does it make sense from fig. 1.3?* ____________________

What is the period of $\sin(2x)$*?* ____________________

What is the period of $\sin(ax)$ *where* $a > 0$ *is a real number?* ____________________

What is the period of $\sin(ax)$ *if* $a < 0$*?* ____________________

Definition 1.1.4

If a periodic function f attains a maximum and a minimum, we define the amplitude of the function as **half** the vertical distance between the highest and the lowest point on the graph.

■ Question 12. ☐

What is the amplitude of $f(x) = \sin x$*?* ____________________

What is the amplitude of $f(x) = 2\sin(x)$*?* ____________________

What is the amplitude of $f(x) = 2 + \sin(x)$*?* ____________________

What is the period and amplitude of $f(x) = 5 + 4\sin(3x)$*?* ____________________

■ Question 13. ☐

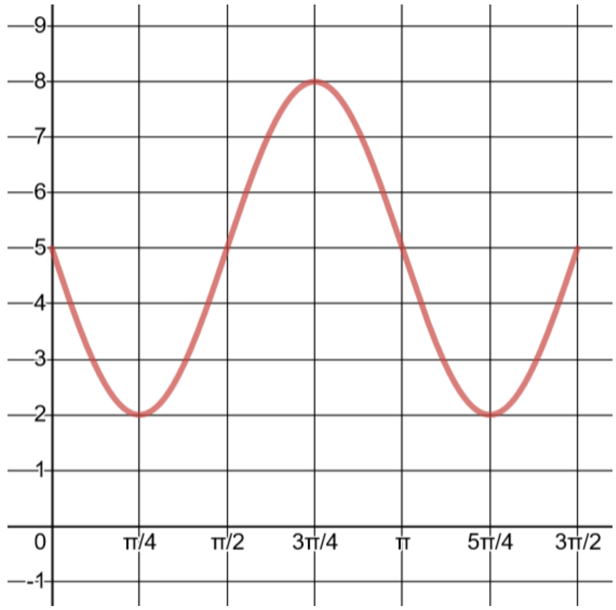
Find a periodic function with a period of 12 *and an amplitude of* 15*.*

■ Question 14. □

Find equation of a trigonometric function of the form

$$A + B\sin(Cx + D)$$

whose graph looks like below. You can use Desmos to check your answer.



*[**Hint**: Start with the graph of* $\sin(x)$ *in Desmos. Use what you learned in the last problem to shift/stretch the graph until you get to the figure above.]*

Then discuss the following questions:

(a) *How does the period depend on* C*?*

(b) *How does the amplitude depend on* B*?*

(c) *What does the constant* A *correspond to in the graph?*

(d) *What does the constant* D *correspond to in the graph?*

■ Question 15. □

The Bay of Fundy in Canada has the largest tides in the world. The difference between low and high water levels is 15 *meters. Assume the time between successive high tides is* 12 *hours. At a particular point, the depth of the water,* y *meters, is given as a function of time,* t*, in hours since midnight by*

$$y(t) = D + A\cos(B(t - C)).$$

(a) *What would be a possible physical interpretation of* D*?*

(b) *What is the value of* A*?*

(c) *What is the value of* B*?*

(d) *What would be a possible physical interpretation of* C*?*

1.1.5 Other Trigonometric Identities

We will not prove the following identities but they will be needed in the next section.

Angle Sum Identities

For any $\alpha+\beta$ in the natural domain of the functions below, the following relations hold.

$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta);\quad \sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\sin(\beta)\cos(\alpha);\quad \tan(\alpha+\beta)=\frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}.$$

The first two can be checked using geometry. The last one can be obtained by divding the first two.

Angle Difference Identities

For any α,β in the natural domain of the functions below, the following relations hold.

$$\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta);\quad \sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\sin(\beta)\cos(\alpha);\quad \tan(\alpha-\beta)=\frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}.$$

These can be obtained by reaplcing β with $-\beta$ in the sum identities.

Double Angle Identities

For any α in the natural domain of the functions below, the following relations hold.

$$\cos(2\alpha)=\cos^2(\alpha)-\sin^2(\alpha)=2\cos^2\alpha-1=1-2\sin^2\alpha;$$

$$\sin(2\alpha)=2\sin(\alpha)\cos(\alpha);\quad \tan(2\alpha)=\frac{2\tan(\alpha)}{1-\tan^2(\alpha)}.$$

These can be obtained by substituting β with α.

Converting Product to Sum

For any real numbers α and β, the following relations hold.

$$2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta)$$

$$2\cos\alpha\sin\beta=\sin(\alpha+\beta)-\sin(\alpha-\beta)$$

$$2\cos\alpha\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta)$$

$$2\sin\alpha\sin\beta=\cos(\alpha-\beta)-\cos(\alpha+\beta)$$

All of these can be obtained from the angle sum/difference identities by adding the appropriate two identities.

§1.2 Limits and Derivatives involving Trigonometric Functions

1.2.1 Continuity

Looking at the graphs of the sine and the cosine function, we can certainly guess that they are continuous. Let's see if we could prove that. Recall that $\sin\theta$ and $\cos\theta$ are coordinates of the point corresponding to angle θ on the unit circle. As this point approches $(1,0)$, the angle θ aproches 0. Thus

$$\lim_{\theta\to 0}\cos\theta = 1 \qquad \text{and} \qquad \lim_{\theta\to 0}\sin\theta = 0$$

Now we will need to use the sum identity for sine and cosine to show continuity of a general angle. Given any $c \in \mathbb{R}$, note that

$$\begin{aligned}
\lim_{x\to c}\sin x &= \lim_{(x-c)\to 0}\sin x\\
&= \lim_{\theta\to 0}\sin(\theta + c)\\
&= \lim_{\theta\to 0}(\sin\theta\cos c + \cos\theta\sin c)\\
&= 0\times\cos c + 1\times\sin c\\
&= \sin c
\end{aligned}$$

Hence the sine function is continuous at all $c \in \mathbb{R}$. Similarly we can show that cosine is continuous everywhere.

Question 16. □

Find the following limits.

(a) $\lim_{x\to\pi}(x^2\cos x + \sin(x/2))$.

(b) $\lim_{x\to\frac{\pi}{4}}\dfrac{\cos(2x)}{\cos x - \sin x}$.

1.2.2 Two Special Limits

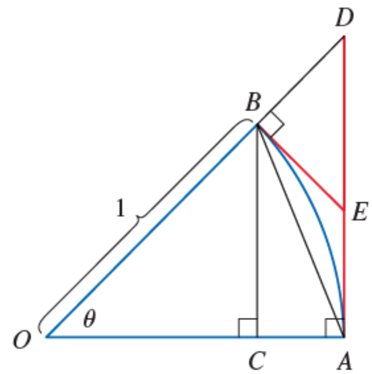
Before we can find the derivative of the sine function, we need to find a special limit involving trigonometric function. The proof particularly illuminating as it demostrates an interseting way to use the squeeze theorem.

Theorem 1.2.5

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \qquad \textit{and} \qquad \lim_{\theta\to 0}\frac{(\cos\theta - 1)}{\theta} = 1$$

We will work through the proof together. Let's do the first one.

Proof. Consider the following picture.



In this figure, O is the origin, $\overline{OA} = \overline{OB} = 1$ and we have drawn an arc of the unit circle from A to B.

Since $\angle BOA = \theta$, we know that B has coordinates ____.

The point C is the foot of the perpendicular from B, so C has coordinates ________________________.

Finally, D is the point where a vertical line starting at A intersects OB. Since $\overline{OA} = 1$, using trigonometry we find that $\overline{AD} =$ ____.

Now, the area of $\triangle OAB = \frac{1}{2} \times OA \times CB =$ ____

The area of the pie-shaped region OAB can be found using the unitary method as ________________.

And the area of $\triangle OAD = \frac{1}{2} \times OA \times AD =$ ____

These three areas clearly follow a chain of inequality. Substituting the formulas, we get

$$\sin\theta < \theta < \tan\theta$$

From the first inequality, we get $\frac{\sin\theta}{\theta} < 1$. From the second inequality, we get

$$\theta < \frac{\sin\theta}{\cos\theta} \implies \underline{\qquad\qquad} < \frac{\sin\theta}{\theta}.$$

Now use the squeeze theorem. Both sides of the quantity $\frac{\sin\theta}{\theta}$ approches 1 as $\theta \to 0^+$. Hence,

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1$$

Finally, since $\frac{\sin\theta}{\theta}$ is an even function, we conclude that $\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1$, finishing the proof.

■

To calculate $\lim_{\theta\to 0}\frac{(\cos\theta - 1)}{\theta}$, we can use a sort of 'multiply by the conjugate' method.

$$\begin{aligned}
\lim_{\theta\to 0}\frac{(\cos\theta-1)}{\theta} &= \lim_{\theta\to 0}\left(\frac{\cos\theta-1}{\theta}\cdot\frac{\cos\theta+1}{\cos\theta+1}\right)\\
&= \lim_{\theta\to 0}\frac{\cos^2\theta-1}{\theta(\cos\theta+1)} = \lim_{\theta\to 0}\frac{-\sin^2\theta}{\theta(\cos\theta+1)}\\
&= -\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{\cos\theta+1}\right) = -\left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)\cdot\left(\lim_{\theta\to 0}\frac{\sin\theta}{\cos\theta+1}\right)\\
&= -1\cdot\left(\frac{\sin 0}{\cos 0+1}\right)\\
&= -1\cdot\left(\frac{0}{1+1}\right)\\
&= 0
\end{aligned}$$

The most important applications of both of these formulas will come in the next subsection when we evaluate the derivative of sine function. For now, let's see how we can use these limits in practice.

Example 1.2.6

Evaluate $\lim_{x\to 0}\frac{\sin 7x}{4x}$.

Solution: First of all, note that $7x$ and $4x$ are different values, so it's not quite in the $\frac{\sin\theta}{\theta}$ form yet. But we can rewrite it in the following way to make the angle same as the denominator:

$$\frac{\sin 7x}{4x} = \frac{7}{4}\left(\frac{\sin 7x}{7x}\right)$$

Now, if $\theta = 7x$, we get

$$\lim_{x\to 0}\frac{\sin 7x}{4x} = \frac{7}{4}\lim_{x\to 0}\frac{\sin 7x}{7x} = \frac{7}{4}\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \frac{7}{4}\cdot 1 = \frac{7}{4}$$

■ **Question 17.** □

Evaluate $\lim_{x\to 0}\frac{1-\cos 3x}{4x}$.

■ **Question 18.** □

Evaluate the following limits.

(a) $\lim_{x\to 0} x\cot x$.

(b) $\lim_{x\to 0}\frac{\cos x - 1}{\sin x}$.

(c) ***(Challenge Problem)*** $\lim_{x\to 0}\frac{1-\cos 4x}{1-\cos 2x}$.

1.2.3 Derivative of sine and cosine

First, let's look at the graph of $f(x) = \sin x$. We might ask, where is the derivative equal to zero? Then ask where the derivative is positive and where it is negative. In exploring these answers, we get something like the following graph.

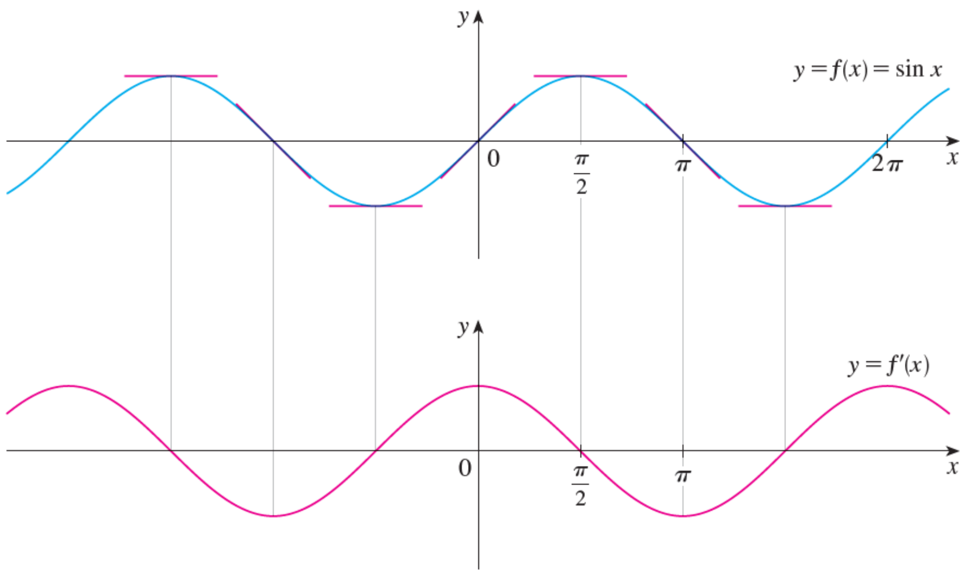


Figure 1.4: The sine function and its derivative

The graph of the derivative in Figure 1.4 looks suspiciously like the graph of the cosine function. This might lead us to conjecture, quite correctly, that the derivative of the sine is the cosine. For a more formal proof, we need to use angle-sum trigonometric identity:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Then we can write

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \sin x\left(\lim_{h\to 0}\frac{(\cos h - 1)}{h}\right) + \cos x\left(\lim_{h\to 0}\frac{\sin h}{h}\right) \\
&= \sin(x)\times 0 + \cos x \times 1 \\
&= \cos x
\end{aligned}$$

where we used the fact that $\lim\limits_{h\to 0}\dfrac{\sin h}{h} = 1$ and $\lim\limits_{h\to 0}\dfrac{(\cos h - 1)}{h} = 0$. Similarly, we can calculate the derivative of $\cos x$ using the identity for cosine angle-sum.

Theorem 1.2.7: Trig Derivatives

For x in radians,

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x \quad \textit{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$$

Question 19.

□

Use the quotient rule to find derivatives of the following trigonometric functions.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}[\tan x]$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}x}[\sec x]$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}[\cot x]$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}[\csc x]$

Question 20.

□

(a) *Find the derivative of* $f(x) = (\sin(3x))^2$.

(b) *Find the 27th derivative of* $\cos x$.

■ Question 21. □

Where does the gaph of the function $g(x) = \dfrac{\sec x}{1 + \tan x}$ *has a hrizontal tangent?*

■ Question 22. □

Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ *by implicit differntiation:*

$$\sin x + \cos y = \tan(xy)$$

1.2.4 Local and Global Extrema with Trigonometric Functions

Question 23.

Consider the function $f : [0, 2\pi] \to \mathbb{R}$ *defined by* $f(x) = \sin x - \sin^2 x$.

(a) *Find the critical points of* f.

(b) *Find the global maximum and minimum of* f.

Question 24.

Find and classify the critical points of the function $f(x) = \dfrac{\cos x}{2 + \sin x}$ *on* $(0, 2\pi)$.

Question 25.

☐

Consider the function $f : [0, 2\pi] \to \mathbb{R}$ *defined as* $f(x) = 2\cos x + \cos^2 x$.

(a) *Find the intervals of increase or decrease.*

(b) *Find the local maximum and minimum values.*

(c) *Find the intervals of concavity and the inflection points.*

(d) *Use the information from parts (a) – (c) to sketch the graph.*

1.2.5 Related Rates and Optimization Problems with Trigonometry

Question 26. □

A rocket is launched so that it rises vertically. A camera is positioned at 5000 *ft from the launch pad. When the rocket is* 1000 *ft above the launch pad, its velocity is* 600 *ft/sec. Find the necessary rate of change of the camera's angle as a function of time so that it stays focused on the rocket.*

Question 27. □

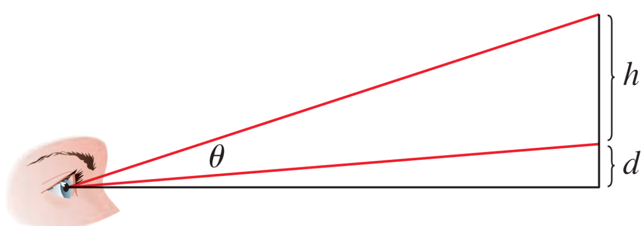
You are stationary on the ground and are watching a bird fly horizontally at a rate of 10 m/sec*. The bird is located* 40 m *above your head. How fast does the angle of elevation change when the horizontal distance between you and the bird is* 9 *m?*

Question 28. **Requires Law of Cosines**

The minute hand on a watch is 8 *mm long and the hour hand is* 4 *mm long. How fast is the distance between the tips of the hands changing at one o'clock?*

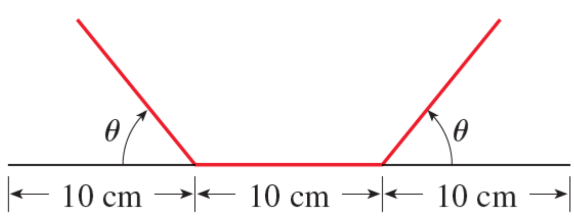
Question 29. □

A painting in an art gallery has height h and is hung so that its lower edge is a distance d above the eye of an observer (as in the figure). How far from the wall should the observer stand to get the best view? (In other words, where should the observer stand so as to maximize the angle θ *subtended at his eye by the painting?)*



Question 30. □

A rain gutter is to be constructed from a metal sheet of width 30 *cm by bending up one-third of the sheet on each side through an angle* θ*. How should* θ *be chosen so that the gutter will carry the maximum amount of water?*



Chapter 2 | Further Applications of Derivatives

§2.1 Curve Sketching

The following checklist is intended as a guide to sketching a curve $y = f(x)$ by hand. Not every item is relevant to every function. (For instance, a given curve might not have an asymptote or possess symmetry.) But the guidelines provide all the information you need to make a sketch that displays the **most important aspects** of the function.

A. Domain. It's often useful to start by determining the domain D of f, that is, the set of values of x for which $f(x)$ is defined.

B. Intercepts. The y-intercept is $f(0)$ and this tells us where the curve intersects the y-axis. To find the x-intercepts, we set $y = 0$ and solve for x. (You can omit this step if the equation is difficult to solve.)

C. Symmetry.

(i) If $f(-x) = f(x)$ for all x in D, that is, the equation of the curve is unchanged when x is replaced by $-x$, then f is an **even function** and the curve is symmetric about the y-axis. This means that our work is cut in half. If we know what the curve looks like for $x > 0$, then we need only reflect about the y-axis to obtain the complete curve. Here are some examples: $y = x^2, y = x^4, y = |x|$, and $y = \cos x$.

(ii) If $f(-x) = -f(x)$ for all x in D, then f is an **odd function** and the curve is symmetric about the origin. Again we can obtain the complete curve if we know what it looks like for $x > 0$. [Rotate 180° about the origin] Some simple examples of odd functions are $y = x, y = x^3, y = x^5$, and $y = \sin x$.

(iii) If $f(x+p) = f(x)$ for all x in D, where p is a positive constant, then f is called a periodic function and the smallest such number p is called the period. For instance, $y = \sin x$ has period 2π and $y = \tan x$ has period π. If we know what the graph looks like in an interval of length p, then we can use translation to sketch the entire graph.

D. Asymptotes.

(i) A line $y = L$ is a horizontal asymptote of a function $f(x)$ if $f(x) \to L$ as $x \to \infty$ or $x \to -\infty$. Note that a function may have different **horizontal asymptotes** as it goes towards $+\infty$ or $-\infty$. If it turns out that $f(x) \to \infty$ or $-\infty$ as $x \to \infty$, then we do not have an asymptote to the right, but this fact is still useful information for sketching the curve.

(ii) A line $x = K$ is a vertical asymptote of a function $f(x)$ if $f(x) \to \infty$ or $-\infty$ as $x \to K+$ and $x \to K-$. For rational functions, you can locate the **vertical asymptotes** by equating the denominator to 0 after canceling any common factors. But for other functions, this method does not apply.

E. Intervals of Increase or Decrease. Find the intervals on which $f'(x)$ is positive (f is increasing) and the intervals on which $f'(x)$ is negative (f is decreasing).

F. Local Maximum and Minimum Values. Follow the steps from last section.

G. Concavity and Points of Inflection. Compute $f''(x)$. The curve is concave upward where $f''(x) \geq 0$ and concave downward where $f''(x) \leq 0$. Inflection points occur where the direction of concavity changes.

H. Drawing the Graph. Using the information in items A-G, draw the graph. Sketch the asymptotes as dashed lines. Plot the intercepts, maximum and minimum points, and inflection points. Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes. If additional accuracy is desired near any point, you can compute the value of the derivative there. The tangent indicates the direction in which the curve proceeds.

■ Question 31. □

Analyze as suggested in the summary above and then sketch the graph.

(a) $f(x) = 8x^3 - 21x^2 + 18x + 2$

(b) $h(x) = \dfrac{\cos x}{2 + \sin x}$

§2.2 Mean Value Theorem

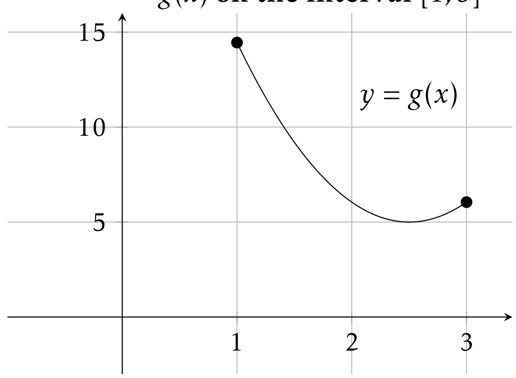
To motivate the topic, let's start with the following question: Suppose you leave your house and drive to your friend's house in a city 100 miles away, completing the trip in two hours. At any point during the trip do you necessarily have to be going 50 miles per hour?

In answering this question, it is clear that the average speed for the entire trip is 50 mph, but the question is whether or not your *instantaneous* speed is ever exactly 50 mph. In other words, does your speedometer ever read exactly 50 mph? The answer, as we will discover, under some very reasonable assumptions, is "yes."

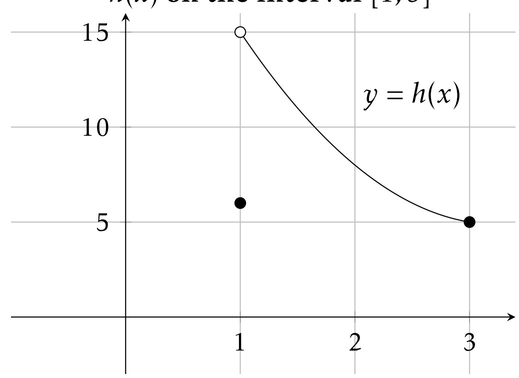
Question 32.

Draw the secant line between the endpoints $(a, f(a))$ and $(b, f(b))$ for the given interval $[a,b]$. Can you identify any point c, with $a < c < b$, such that the slope of the tangent line to the graph at $x = c$ is equal to the slope of the secant line between a and b?

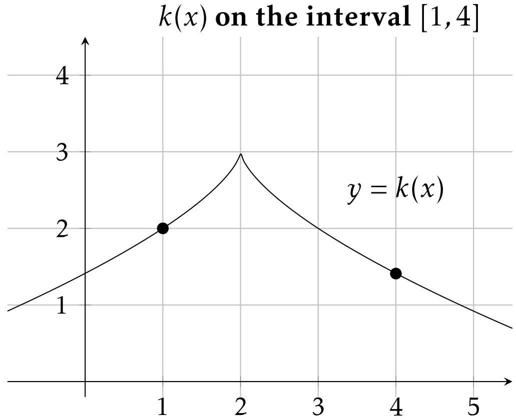
$g(x)$ **on the interval** $[1,3]$



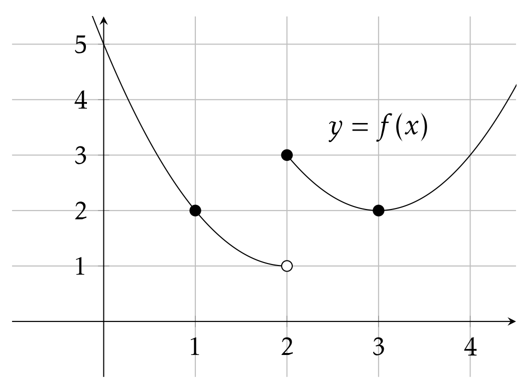
$h(x)$ **on the interval** $[1,3]$



$k(x)$ **on the interval** $[1,4]$



$f(x)$ **on the interval** $[1,3]$



Question 33.

Using your observations from these four cases, make a conjecture regarding when it is possible to find such a point c. In other words, what properties does the function need to have?

Theorem 2.2.8

If f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then there exists at least one number c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

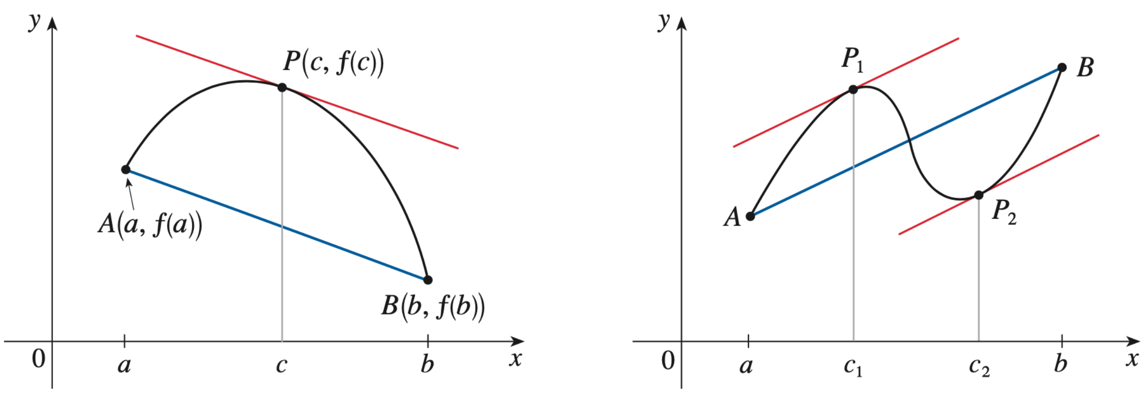
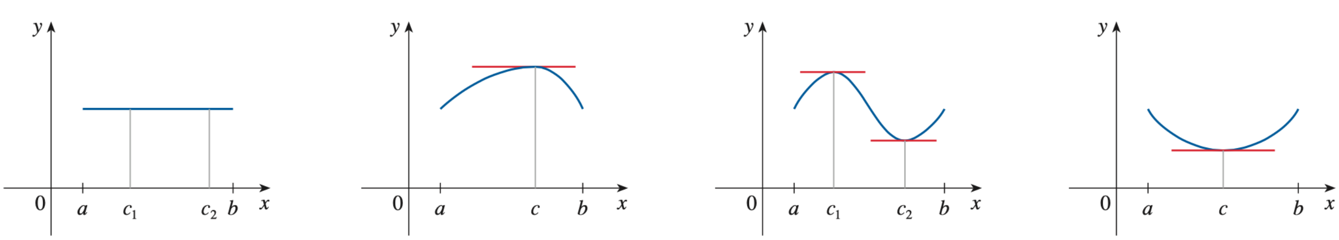


Figure 2.1: The function f attains the slope of the secant between a and b as the derivative at the point(s) $c \in (a,b)$.

A special case of the Mean Value Theorem is called Rolle's Theorem.

Theorem 2.2.9: Rolle's Theorem

Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). If $f(a) = f(b)$, then there is at least one number c in (a,b) such that $f'(c) = 0$.



Sketch of Proof: The proof of Rolle's theorem follows from the Extreme Value Theorem which says that a continuous function on a closed interval must attain its extremum at some c. This fact along with the fact that local extrema are critical points gives us $f'(c) = 0$ at such points. ■

■ Question 34. □

Like other Math theorems, the MVT is a "If-Then" statement. There are some hypotheses and there is a conclusion. Can you identify which part is which?

Example 2.2.10

If the MVT applies to a given function, we can analytically determine the guaranteed value(s) c as described in the theorem. For example, suppose $f(x) = \sin(\pi x)$ and consider the interval $[0,2]$. We know that the sine function is continuous and differentiable everywhere, so $f(x)$ satisfies the hypotheses of the MVT. Hence, there should exist at least one number c in $(0,2)$ such that:

$$\frac{f(2) - f(0)}{2 - 0} = f'(c)$$

Since $f'(x) =$ ____________________________________, we can plug in f' and our function values to get an equation in terms of c:

$$\begin{aligned}\frac{\sin(2\pi) - \sin(0)}{2 - 0} &= \pi\cos(c\cdot\pi)\\ \implies 0 &= \pi\cos(c\cdot\pi)\\ \implies 0 &= \cos(c\cdot\pi)\end{aligned}$$

We know that $\cos(x) = 0$ when $x = \pm\frac{\pi}{2}, \pm\frac{5\pi}{2}, \ldots$ etc.. Thus, $c\pi =$ __________ $\implies c =$ __________.

The other possible solutions will yield a value of c outside of the given interval $[0,2]$. Look at a graph of $f(x)$ over the given interval to confirm this result yourself.

2.2.1 Applications

■ Question 35. □

Does the MVT apply to $g(x) = x^{1/3}$ on $[0,8]$? Why or why not? If so, find all values of c that satisfy the theorem.

■ Question 36. □

Explain why $h(x) = x^3 + 6x + 2$ satisfies the hypotheses of the MVT on the interval $[-1,3]$. Then find all values of c in $[-1,3]$ guaranteed by the theorem.

■ Question 37. □

Suppose f is a polynomial. Which of the following statements are correct? There may be more than one correct answer.

(a) Between any two consecutive roots of f, there must be at least one root of f'.

(b) Between any two consecutive roots of f', there must be at least one root of f.

(c) Between any two consecutive roots of f, there can be at most one root of f'.

(d) Between any two consecutive roots of f', there can be at most one root of f.

The Mean Value Theorem relates the function f and its derivative, f'. Since the derivative has many interpretations, e.g. instantaneous rate of change, slope of the tangent line, and velocity, it is no surprise that we can use the MVT in different contexts. The main significance of the Mean Value Theorem is that it enables us to obtain information about a function from information about its derivative.

■ Question 38. □

Consider a function $f(x)$ that is differentiable everywhere. If $f(5) = 2$ and $f'(x) \leq 3$ for all x, then what is the maximum possible value of $f(15)$?

■ Question 39. □

Let $g(x) = |x^2 - 1|$. Graph this function using Desmos and answer the questions below.

(a) Do the hypotheses of the MVT hold on $[0, 3]$? Does the conclusion hold? Explain.

(b) Do the hypotheses of the MVT hold on $[1, 3]$? Does the conclusion hold? Explain.

(c) Do the hypotheses of the MVT hold on $[-1, 3]$? Does the conclusion hold? Explain.

Exploration Activity

It is important to think about the chain of logic for a theorem. Let me use a Cats analogy. Consider the statement: If we have a cat, then we have a mammal. Note that the converse isn't true. Just because an animal is a mammal, it doesn't necessarily mean it's a cat. Indeed a dog is also a mammal. So the conclusion can be valid even when the hypothesis isn't. Relating to the problem above, in part (a), the conclusion is valid, even when the hypothesis isn't.

Similarly, when the hypothesis doesn't hold, we can't really say whether the conclusion holds or not. For example, if your animal is not a cat, we do not know if it is a mammal or not, it could be an octopus, or it could be a dog. Relating to the problem above, the hypothesis doesn't hold in both (a) and (c); but for one of them the conclusion holds, for the other it doesn't.

■ Question 40.

□

Show that $|\sin x - \sin y| \leq |x - y|$ *for all* x *and* y.

2.2.2 An Important Corollary

One interesting consequence of the MVT that will be useful later on is as follows:

Theorem 2.2.11

If $f'(x) = 0$ *for all* x *in an interval* I, *then* $f(x)$ *is constant on* I.

■ Question 41.

□

Suppose two different functions $f(x)$ *and* $g(x)$ *have the same derivative on some interval* I, *i.e.* $f'(x) = g'(x)$ *for all* $x \in \mathrm{I}$. *What can you say about the relationship between the two functions?*

We end with a wild MVT spotted on a bridge in Beijing!

§2.3 Antiderivatives

A recurring theme in our discussion of differential calculus has been the question: "Given information about the derivative of an unknown function f, how much information can we obtain about f itself?"

For example, over the last couple of classes, we endeavored to sketch a possible graph of f, given information about f' and f''. We investigated how the first derivative test enables us to determine where the original function is increasing and decreasing, as well as where f has relative extreme values. If we know a formula or graph of f', by computing $(f')' = f''$, we can find where the original function f is concave up and concave down. Thus, knowing f' and f'' enables us to understand the shape of the graph of f.

In what follows, we would like to go one step further. From the information about f', we'd like to not only determine a graph of f but also provide an accurate formula of f.

Definition 2.3.12

Let f be a function. We say a function F is **an** antiderivative of f if

$$F'(x) = f(x)$$

for all x in the domain of F.

Here are some examples:

F is antiderivative for...	the function f
$F(x) = x^2 + 1$	$f(x) = 2x$
$F(x) = x^2 + \pi$	$f(x) = 2x$
$F(x) = \dfrac{x^3}{3}$	$f(x) = x^2$
$F(x) = -\cos(x)$	$f(x) = \sin(x)$

Table 2.1: Examples of Antiderivatives F for some functions f

Question 42.

Verify that the function $F(x)$ *is an antiderivative for* $f(x)$ *by computing* F'.

(a) $F(x) = 5x^3 + 2x^2 + 3x + 11$ *and* $f(x) = 15x^2 + 4x + 3$.

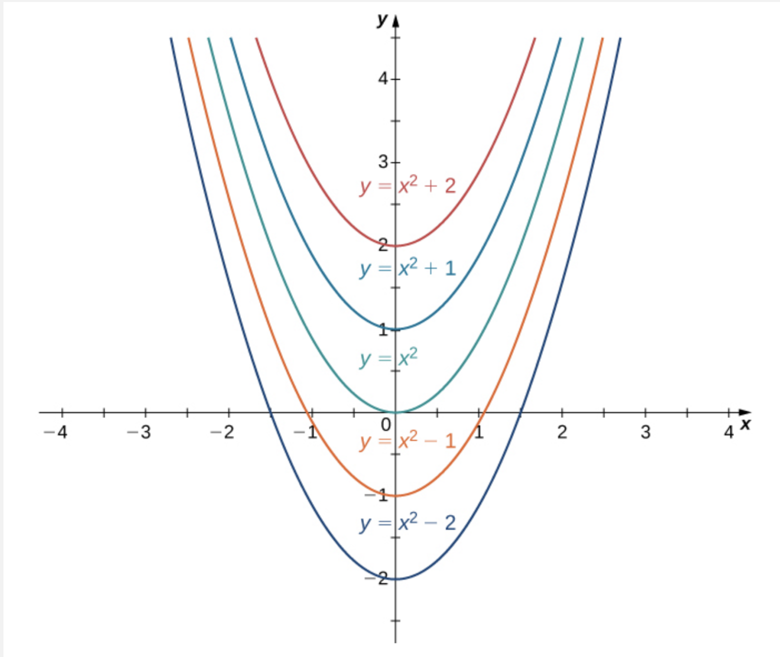
(b) $F(x) = \cos^2 x - \sin^2 x$ *and* $f(x) = -2\sin(2x)$.

Note: Note that we say "an" antiderivative of f rather than "the" antiderivative of f. That's because a function f can have multiple (in fact, infinitely many) antiderivatives!

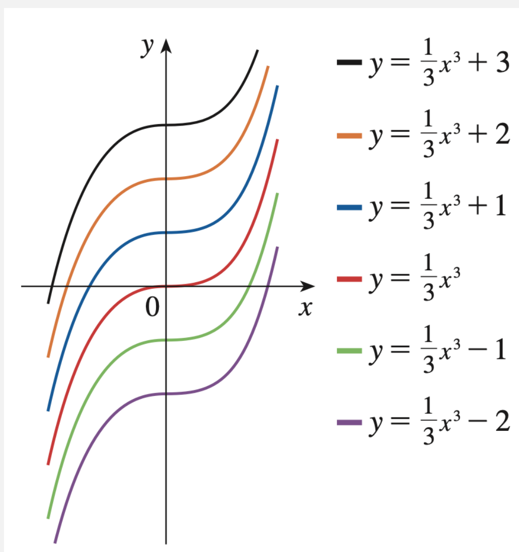
Consider the first example from table 2.1 above, $f(x) = 2x$. Notice that any of

$$x^2,\ x^2 + 1,\ x^2 - \sqrt{7},$$

etc. are antiderivatives for $f(x)$. Some of the the antiderivatives are graphed below which should give you a visual intuition of why this is true.



Here is another picture that shows the family of antiderivatives for x^2, which includes $\dfrac{x^3}{3} + C$ for variuous C values.



Question 43.

If $F(x)$ and $G(x)$ are two antiderivatives of the same function $f(x)$, then use the MVT to explain why $F(x) - G(x)$ must be a constant.

Question 44.

Find the most general antiderivative of each of the following functions.

(a) $f(x) = \dfrac{1}{x^2}$.

(b) $g(x) = \sin x$

A consequence of the last exercise is that now we can introduce a new notation to denote 'antiderivatives' and not have to be so verbose all the time.

Definition 2.3.13

Given a function f, the **indefinite integral** of f,

$$\int f(x)\,\mathrm{d}x,$$

denotes a general antiderivative of f. If $F(x)$ is a specific antiderivative of $f(x)$, then

$$\int f(x)\,\mathrm{d}x = F(x) + C.$$

for some arbitrary constant C. The expression $f(x)$ is called the *integrand* and the variable x is the *variable of integration*.

Warning: In practice, we use this integral notation $\int \ldots.\mathrm{d}x$ as a command or operator, saying "take the antiderivative." or "take the integral". But the answer that we write down is not one antiderivative, but a *family of antiderivatives*. That is why we have the *arbitrary constant* +C.

2.3.1 Properties of the Indefinite Integeral

We can take any derivative formula that you remember from before and make an antiderivative formula out of it. For example, as you may have noticed through the examples so far, the antiderivative version of the *power rule* goes as follows.

$$\boxed{\text{For } n \neq -1, \int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + C.}$$

We might refer to the above formula as the **reverse power rule** from time to time.

Note: Why do we have the condition $n \neq 1$? It's because the right hand side would be undefined if n were equal to 1. But that might raise the question, what is the antiderivative of $\dfrac{1}{x}$? We will anser that question in due time!

Example 2.3.14

If we use $n = \frac{3}{4}$ in the reverse power rule formula, we would get:

$$\int x^{\frac{3}{4}}\,\mathrm{d}x = \frac{x^{\frac{3}{4}+1}}{\frac{3}{4}+1} + C = \frac{x^{7/4}}{\frac{7}{4}} + C = \frac{4}{7}x^{\frac{7}{4}} + C,$$

where C is some arbitrary constant.

Since derivatives play nicely with sums and constant multiples, so do antiderivatives.

Theorem 2.3.15

Sums and differences:

$$\int (f(x) \pm g(x)) = \int f(x)\,\mathrm{d}x \pm \int g(x)\,\mathrm{d}x$$

Constant Multiples: *For any constant real number k,*

$$\int kf(x)\,\mathrm{d}x = k\int f(x)\,\mathrm{d}x$$

Question 45.

Use the sum, difference, and constant multiple properties above, along with the reverse power rule to compute some antiderivatives.

(a) $\int \left(3x^4 - 5x^{2/3}\right)\mathrm{d}x$

(b) $\int \left(\frac{5}{x^2} + 3x^{-10}\right)\mathrm{d}x$

Finally, considering our facts about trigonometric derivatives, we can produce the following six statements about trigonometric antiderivatives.

- $\int \cos(x)\,dx = \sin(x) + C;$
- $\int -\sin(x)\,dx = \cos(x) + C;$
- $\int \sec^2(x)\,dx = \tan(x) + C;$
- $\int \sec(x)\tan(x)\,dx = \sec(x) + C;$
- $\int -\csc(x)\cot(x)\,dx = -\csc(x) + C;$
- $\int -\csc^2(x)\,dx = \cot(x) + C.$

Question 46.

Evaluate $\int \left(5\sec^2 x + 2x^{4/5} + 3\right) dx$.

2.3.2 Initial Value Problems

We have established that $\int f(x)\,dx$ denotes the *general antiderivative* of the function f. But what if we wanted to know a particular or specific antiderivative? In this case, we must know a single value of the particular antiderivative F. This value that F takes on is called an *initial condition*. Determining the specific antiderivative F that satisfies a given initial condition is called an **initial-value problem**.

Example 2.3.16

Suppose we wanted the antiderivative $F(x) = \int 2x\,dx$ that satisfies $F(0) = 3$.

Analytically, we can solve this by writing down the general antiderivative first, then plugging in the initial condition to solve for the arbitrary constant C:

$$F(x) = \int 2x\,dx = x^2 + C$$

We want $F(0) = 3$, hence we want

$$F(0) = (0)^2 + C = 3,$$

and thus $C = 3$ for this particular function, meaning $F(x) = x^2 + 3$. ■

Here is a second, slightly more involved example.

Example 2.3.17

Find f if $f''(x) = 6x - 4$, $f(0) = 4$, and $f(1) = 1$.

The general antiderivative of $f''(x) = 6x - 4$ is

$$f'(x) = 6\frac{x^2}{2} - 4x + C = 3x^2 - 4x + C$$

Using the antidifferentiation rules once more, we find that

$$f(x) = 3\frac{x^3}{3} - 4\frac{x^2}{2} + Cx + D = x^3 - 2x^2 + Cx + D$$

To determine C and D we use the given conditions that $f(0) = 4$ and $f(1) = 1$. Since $f(0) = 0 + D = 4$, we have $D = 4$. Since

$$f(1) = 1 - 2 + C + 4 = 1$$

we have $C = -2$. Therefore the required function is

$$f(x) = x^3 - 2x^2 - 2x + 4$$

■ Question 47.

A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff, 432 ft above the ground.

(a) *Find its height above the ground t seconds later. You may use the fact that acceleration to the gravity is* 32 ft/s^2 *pointing downwards.*

(b) *When does it reach its maximum height?*

(c) *When does it hit the ground?*

□

Hints: $a(t) = -32$. $v(0) = 48$. $s(0) = 432$.

Answer to (a): $s(t) = -16t^2 + 48t + 432$.

Chapter 3 | Definite Integrals

Towards the end of the last chapter, we defined ‘‘Antiderivatives" and discussed how derivative rules can be ‘‘reversed’’ to produce Antiderivatives. However, there is a significant gap in our understanding of how the antiderivative of a function graphically relates to it. We know that derivatives can be interpreted as slope of the tangent to the graph, but what about antiderivatives? We will spend several of the next sections trying to answer this question before we come back to the more ‘‘algebraic’’ and mechanical process of finding antiderivative formulas.

To give an outline of this chapter, we will first start by focusing on the signed area between a function’s graph and the horizontal axis. When we say ‘‘signed area", we mean that we will account for area beneath the horizontal axis as negative and area above the horizontal axis as positive. Then we will show that the derivative of the area function is the original function, this is a result called the **Fundamental Theorem of Calculus**, which will prove that the area function is in fact the antiderivative.

§3.1 The Sigma Notation

Before we can describe a way to measure the area, we need to discuss some Mathmematical notations. In calculus, we do a lot of adding. One notation is used predominantly in mathematics to help write out long formulaic sums in a concise way. This notation uses the Greek letter $\sum$ and is called *Sigma Notation*.

Example 3.1.18

$$\sum_{i=1}^{5}(2i) = 2(1)+2(2)+2(3)+2(4)+2(5) = 30, \qquad \sum_{j=2}^{5} 2^j = 2^2+2^3+2^4+2^5 = 4+8+16+32 = 60$$

More generally,

$$\sum_{k=1}^{n} f(k) = f(1)+f(2)+\cdots+f(n)$$

■ **Question 48.** □

It’s your turn! Interpret the following sigma notations as sums and compute them.

(a) $\sum_{i=3}^{5}(i^2+1)$

(b) $\sum_{i=1}^{5} i$

(c) $\sum_{j=6}^{10}(j-5)$

(d) $\sum_{i=1}^{20} 2$

The following three formulas can be helpful when approximating area for some simple functions; they are also just cool formulas to know!

Theorem 3.1.19

The sum of n consecutive integers is given by:

$$\sum_{i=1}^{n} i = 1+2+\cdots+n = \frac{n(n+1)}{2}$$

The sum of consecutive integers square is given by:

$$\sum_{i=1}^{n} i^2 = 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

The sum of consecutive integers cubed is given by

$$\sum_{i=1}^{n} i^3 = 1^3+2^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4} = \left(\sum_{i=1}^{n} i\right)^2$$

Some properties involving sigma notation follow from our basic properties of arithmetic.

Theorem 3.1.20

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ represent two sequences of terms and let c be a constant. The following properties hold for all positive integers n and for integers m, with $1 \leq m \leq n$.

- $\sum_{i=1}^{n} c = nc$
- $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
- $\sum_{i=1}^{n} (a_i+b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
- $\sum_{i=1}^{n} (a_i-b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$
- $\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i$

Example 3.1.21

$$\begin{aligned}\sum_{i=1}^{6}(2i+3) &= 2\cdot\sum_{i=1}^{6} i + \sum_{i=1}^{6} 3\\ &= 2\cdot\frac{6\cdot 7}{2} + 3\cdot 6\\ &= 42+18=60\end{aligned}$$

Question 49.

Try to use above formulas, along with simple properties of sums, to compute larger sums with ease:

(a) $\displaystyle\sum_{i=1}^{100} i^2$

(b) $\displaystyle\sum_{k=0}^{20}(k^2-5k+1)$

(c) $\displaystyle\sum_{j=11}^{20}(j^2-10j)$

(d) $\displaystyle\sum_{i=0}^{5} \frac{i}{n^2}$

(e) $\displaystyle\sum_{i=1}^{n} \frac{i}{n^2}$

(f) $\displaystyle\frac{1}{n}\sum_{i=1}^{n} \frac{i}{n}$

(g) $\displaystyle\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{i}{n}$

§3.2 Why area?

It is very natural to wonder: why would we try to compute area under a curve? Let's consider a motivating question. If we know the velocity of a moving body at every point in a given interval, can we determine the distance the object has traveled on the time interval? What is the velocity is changing over time? Can we perhaps estimate it graphically?

Example 3.2.22

Suppose that a person is taking a walk along a long straight path and walks at a constant rate of 3 miles per hour.

(a) On the left-hand axes provided below, sketch a labeled graph of the velocity function $v(t) = 3$.

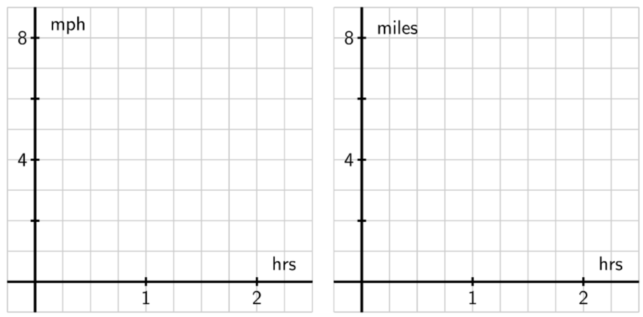


Fig. At left, axes for plotting $y = v(t)$; at right, for plotting $y = s(t)$.

Note that while the scale on the two sets of axes is the same, the units on the right-hand axes differ from those on the left. The right-hand axes will be used in question (d).

(b) How far did the person travel during the two hours? How is this distance related to the area of a certain region under the graph of $y = v(t)$?

(c) Find an algebraic formula, $s(t)$, for the position of the person at time t, assuming that $s(0) = 0$. Explain your thinking.

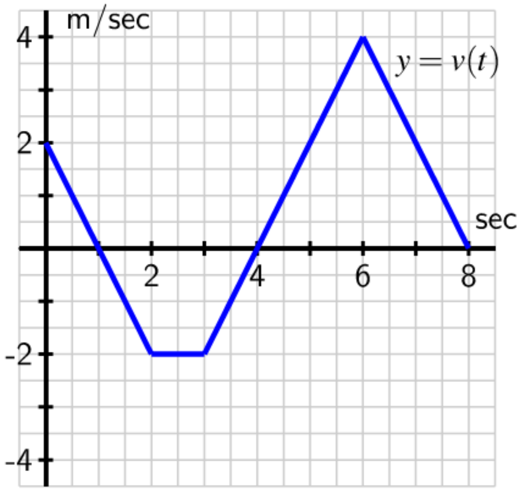
(d) On the right-hand axes provided in the picture above, sketch a labeled graph of the position function $y = s(t)$.

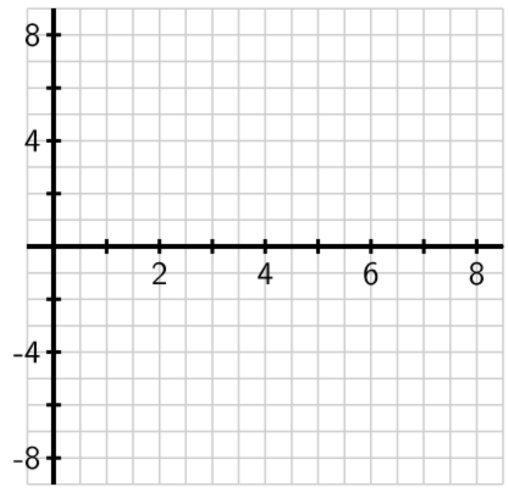
(e) For what values of t is the position function s increasing? Explain why this is the case using relevant information about the velocity function v.

3.2.1 Area under the graph of the velocity function

In above example, we observed that when the velocity of a moving object is constant (and positive), the area under the velocity curve over an interval of time tells us the distance the object traveled. The situation is becomes gradually more complicated when the velocity function is not constant.

To begin with, Suppose that an object moving along a straight line path has its velocity v (in meters per second) at time t (in seconds) given by the piecewise linear function i.e. the function $v(t)$ is piece-wise defined and each part is a linear function. We view movement to the right as being in the positive direction (with positive velocity), while movement to the left is in the negative direction.





Question 50.

Suppose further that the object's initial position at time $t = 0$ is $s(0) = 1$.

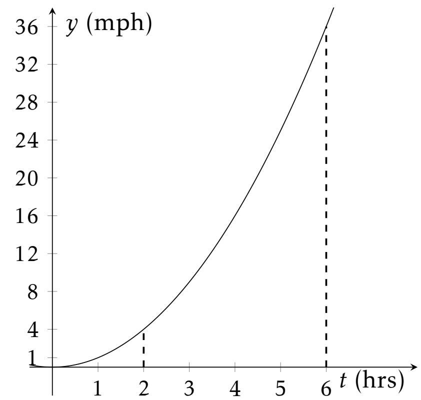
(a) *Determine the total distance traveled and the total change in position on the time interval $0 \le t \le 2$. What is the object's position at $t = 2$?*

(b) *On what time intervals is the moving object's position function increasing? Why? On what intervals is the object's position decreasing? Why?*

(c) *What is the object's position at $t = 8$? How many total meters has it traveled to get to this point (including distance in both directions)? Is this different from the object's total change in position on $t = 0$ to $t = 8$?*

(d) *Find the exact position of the object at $t = 1, 2, 3, \ldots, 8$ and use this data to sketch an accurate graph of $y = s(t)$ on the axes provided at right in the figure above. How can you use the provided information about $y = v(t)$ to determine the concavity of s on each relevant interval?*

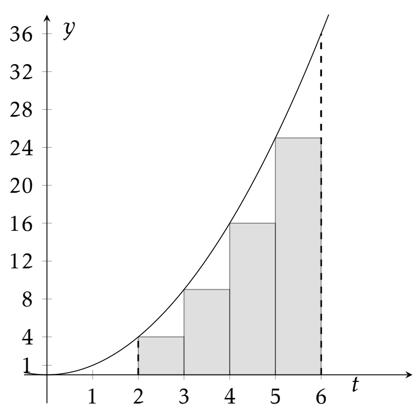
Next consider the case of a general velocity function such as $v(t) = t^2$ given below. Let's try to approximate the area under the curve $y = t^2$ between $t = 2$ and $t = 6$, as seen in the graph to the left.



We can estimate the area under the curve using rectangles. Let's use four. Since we have four rectangles, the base of each rectangle is 1 unit long, because our interval (from $x = 2$ to $x = 6$) is 4 units long.

So, we have four rectangles with bases in the intervals $[2,3]$, $[3,4]$, $[4,5]$, $[5,6]$. But, where do we get the height of our rectangle from? There are two simple ways to do this.

Left-Endpoint Approximation



When using left-endpoints for our rectangles, first evaluate the function at each *left* end point of the intervals.

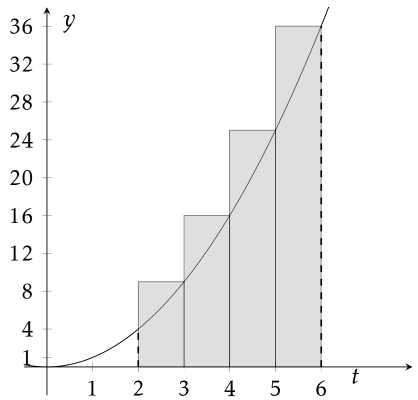
In our case we have: $v(2) = 4$, $v(3) = 9$, $v(4) = 16$, and $v(5) = 25$.

Then, draw a rectangle in each interval with a height equal to that of the value of the function at the left-endpoints, as seen in the graph.

■ Question 51. □

Using the four rectangles, what is the approximate area? Do you think this is an over estimate or an under estimate? Explain.

Right-Endpoint Approximation



When using right-endpoints for our rectangles, first evaluate the function at each *right* end point of the intervals.

In our case we have: $v(3) = 9$, $v(4) = 16$, $v(5) = 25$, and $v(6) = 36$.

Then, draw a rectangle in each interval with a height equal to that of the value of the function at the right-endpoints, as seen in the graph.

Question 52.

Using the four rectangles, what is the approximate area? Do you think this is an over estimate or an under estimate? Explain.

Question 53.

Write down the area under $y = x^2$ on $[2,6]$ with 8 rectangles in expanded form first and then using the sigma notation.

(a) ***Left-Endpoint Approximation:***

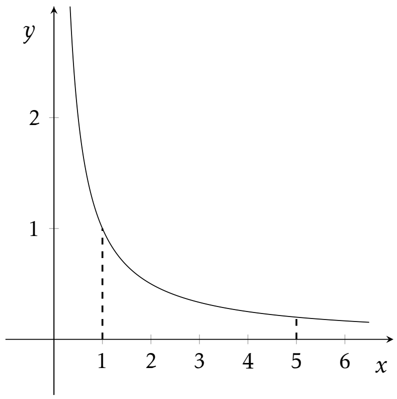
(b) ***Right-Endpoint Approximation:***

Question 54.

What would happen if we had 100 rectangles? 1,000 rectangles? An infinite number of rectangles? Would our approximation get better or worse?

■ Question 55.

Approximate the area under the curve $y = \dfrac{1}{x}$ *from* $x = 1$ *to* $x = 5$ *with* 4 *rectangles, using each method. The graph is given below.*



§3.3 Riemann Sum and Numerical Integration

Fianlly, let's consider the more general problem. We wish to find the are under the function $y = f(x)$ from $x = a$ to $x = b$ over the X-axis. This is the region S is fig. 3.1.

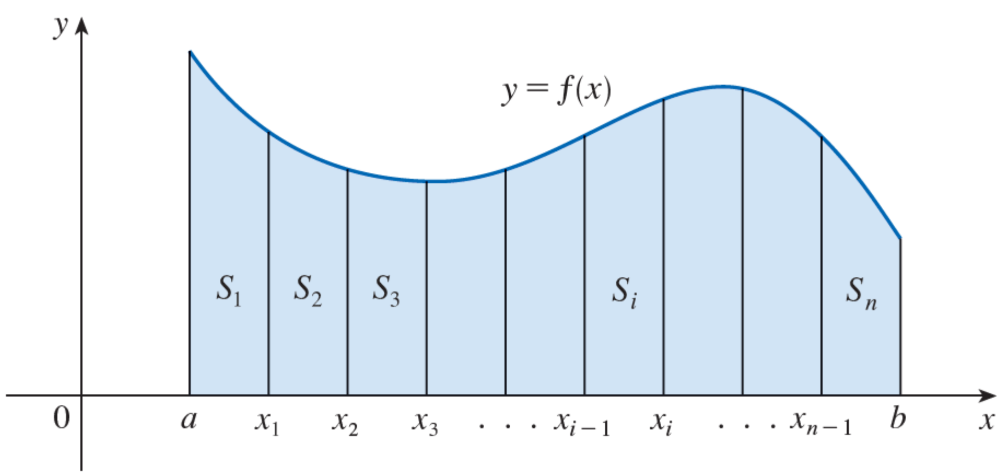


Figure 3.1

Partition and Sample Points

We start by subdividing the region S into n strips $S_1, S_2, S_3, \ldots, S_n$ of equal width as in fig. 3.1. Since the total width of the interval is $b-a$, we find that the width of each of the n strips is $\Delta x =$ ________________.

These strips divide the interval $[a,b]$ into n subintervals $[x_0,x_1],[x_1,x_2],[x_2,x_3],\ldots,[x_{n-1},x_n]$ where $x_0 = a$ and $x_n = b$. Check that the strip S_i corresponds to the interval $[x_{i-1},x_i]$, for $i = 1,2,3,\ldots,n$.

Now we are going to try to approximate the are of each strip S_i by a rectangle of width Δx. Intuitively, we should choose the height of the rectangle to be the "average" value of $f(x)$ over the interval $[x_{i-1},x_i]$.

But recall that we might not know the exact formula of the function f. So, the best we can hope for, is an arbitrary choice of x_i^* in the interval $[x_{i-1},x_i]$ and take the value of f at that point to be the height of the rectangle (see fig. 3.2).

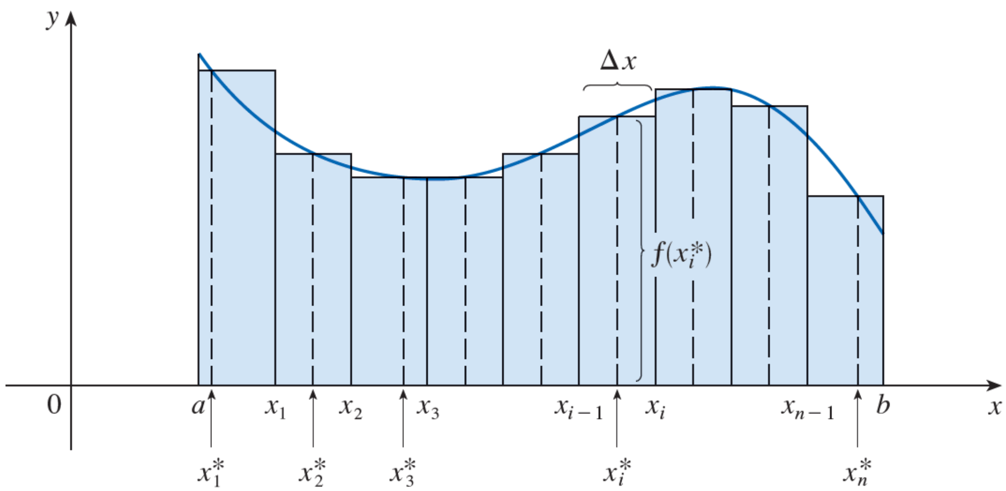


Figure 3.2

Limit of Total Area

What we think of intuitively as the "area" of the region S is thus approximated by the sum of the areas of the rectangles in fig. 3.2, which is

$$A_n^* = f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + \ldots + f(x_n^*)\Delta x$$

Definition 3.3.23

The sum that appears in the above formula, also abbreviated as $\sum_{i=1}^{n} f(x_i^*)\Delta x$, is called a **Riemann sum** after the German mathematician Bernhard Riemann.

Now, as n increases, the subintervals get thinner, and the value of $f(x_i^*)$ becomes closer to the actual average of f over the interval $[x_{i-1},x_i]$. So it stands to reason that the limiting value of A_n as $n \to \infty$ gives the area A of the region S. In other words,

Definition 3.3.24

If $f(x) \geq 0$ on $[a,b]$, we *define* the **area under the graph of f over** $[a,b]$ to be

$$A = \lim_{n\to\infty} A_n^*$$

Note: The choice of x_i^* here is arbitrary, as long as it is in the interval $[x_{i-1}, x_i]$. Here are some choices that are used in practice:

- If $x_i^* = x_{i-1}$, the left end-point, we denote the Riemann sum as L_n.
- If $x_i^* = x_i$, the right end-point, we denote the Riemann sum as R_n.
- If $x_i^* = \dfrac{x_{i-1}+x_i}{2}$, the mid-point, we denote the Riemann sum as M_n.
- If x_i^* is the point where $f(x)$ achieves its maximum in the interval $[x_{i-1}, x_i]$, we call that Riemann sum the "upper sum", denoted $\mathcal{U}_n$.
- If x_i^* is the point where $f(x)$ achieves its minimum in the interval $[x_{i-1}, x_i]$, we call that Riemann sum the "lower sum", denoted $\mathcal{L}_n$.

Look at this applet for an interactive visualization of different Riemann sums.

https://webspace.ship.edu/msrenault/GeoGebraCalculus/integration_riemann_sum.html

Exploration Activity

Another approximation, called the Trapezoidal Rule, results from averaging the left-endpoint and the right-endpoint Riemann sums:

$$\begin{aligned} A_n^* &= \frac{1}{2}\left[\sum_{i=1}^{n} f(x_{i-1}^*)\Delta x + \sum_{i=1}^{n} f(x_i^*)\Delta x\right] \\ &= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right] \end{aligned}$$

This essentially replaces the rectangles with trapezoids in each strip to evaluate the area.

3.3.1 Sum to Integral

Now that we have formally defined what "area" means, we are in a position to finally talk about how integration comes into picture.

Definition 3.3.25: Definition of a Definite Integral

If f is a function defined for $a \leq x \leq b$, we divide the interval $[a,b]$ into n subintervals of equal width $\Delta x = (b-a)/n$. We let $x_0(=a), x_1, x_2, \ldots, x_n(=b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in these subintervals, so x_i^* lies in the i th subinterval $[x_{i-1}, x_i]$.

Then the **definite integral** of f from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is integrable on $[a,b]$.

Notice the caveat about "all possible choice of sample points". That means when f is integrable,

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \mathrm{L}_n = \lim_{n\to\infty} \mathrm{R}_n = \lim_{n\to\infty} \mathrm{M}_n = \lim_{n\to\infty} \mathcal{U}_n = \lim_{n\to\infty} \mathcal{L}_n$$

Each of the different sampling choices gives a different approximation for the actual integral, depending on how large n is. But as n increases, all choices go towards the same limit.

So the integral is actually a limit! And from differential calculus, you know that limits sometimes don't exist. Fortunately, we have the following theorem:

Theorem 3.3.26

If f is continuous on $[a,b]$ or has only a finite number of jump discontinuities, then f is integrable (i.e. the limit exists) on $[a,b]$.

Exploration Activity

On the other hand, it is possible to define functions $f(x)$ such that the limit doesn't exist. For example, the Dirichlet function. This is different from saying that there is no antiderivative formula. Rather we are saying that it is impossible to define $\int f(x)\,\mathrm{d}x$ in any meaningful way and even a computer cannot approximate a value for it.

3.3.2 Limits as Definite Integral

The definition of Definite Integral tells us that limits of Riemann sums can be written as an integral. For example, with right endpoint Riemann sum,

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + \frac{(b-a)i}{n}\right)\frac{b-a}{n} = \int_a^b f(x)dx$$

In the special case when $a = 0$ and $b = 1$, we can rewrite above result as

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n} = \int_0^1 f(x)dx$$

This tells us an algorithm to rewrite any limit of Riemann sum as a definite integral. Let's take a look at an example first.

Example 3.3.27

Let's convert $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\sin^2\left(2\pi\frac{i}{n}\right)$ into a definite integral.

If we compare the summand $\frac{1}{n}\sin^2\left(2\pi\frac{i}{n}\right)$ to the term $f\left(\frac{i}{n}\right)\frac{1}{n}$, we find that $f\left(\frac{i}{n}\right)=\sin^2\left(2\pi\frac{i}{n}\right)$. That tells us

$$f(x)=\sin^2(2\pi x)$$

We conclude that the integral form of the limit is $\int_0^1 \sin^2(2\pi x)dx$.

Example 3.3.28

Let's convert $\lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left(1+\frac{2i}{n}\right)$ into a definite integral.

First note that we can rewrite above limit as

$$\lim_{n\to\infty}\sum_{i=1}^{n}2\left(1+\frac{2i}{n}\right)\frac{1}{n}$$

So the summand $2\left(1+\frac{2i}{n}\right)\frac{1}{n}$ must be the term $f\left(\frac{i}{n}\right)\frac{1}{n}$. Comparing the two terms, we get

$$f\left(\frac{i}{n}\right)=2\left(1+\frac{2i}{n}\right)$$

That tells us

$$f(x)=2(1+2x)$$

We conclude that the integral form of the limit is $\int_0^1 2(1+2x)dx$.

So we have the following algorithm.

Algorithm for rewriting limits of Riemann Sum as Definite Integrals

Step 1. Look carefully at the summand and isolate a $\frac{1}{n}$ from inside. This corresponds to Δx which will become dx.

Step 2. Write the remaining part as a function of $\frac{i}{n}$. Identify the function as f. Your function must not contain any i or n in it.

Step 3. Your integral will be $\int_0^1 f(x)dx$.

■ Question 56.

□

Express each of the following limits as a definite integral:

(a) $\lim_{n\to\infty} \sum_{i=1}^{n} (4x_i^* - (x_i^*)^2)\,\Delta x$ *over* $[4,7]$

(b) $\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^*)^2 \sin(x_i^*)\,\Delta x$ *over* $[0,\pi]$

(c) $\lim_{n\to\infty} \sum_{i=1}^{n} \dfrac{i}{n^2}$

(d) $\lim_{n\to\infty} \sum_{i=1}^{n} \dfrac{9i+3n}{3in+2n^2}$

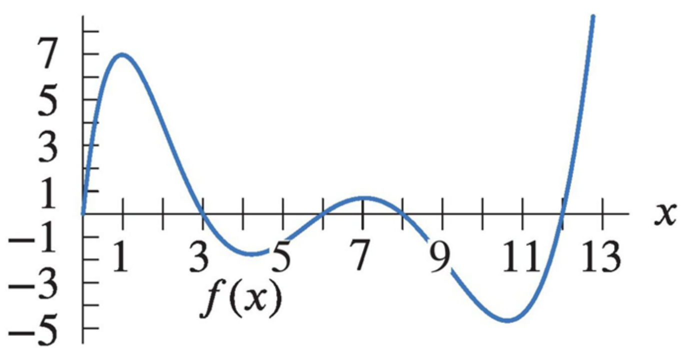
3.3.3 Properties of Definite Integral

Because a definite integral is simply a limit of sums, and both sums and limits behave nicely with addition and constant multiplication, we have familiar **linearity rules** for definite integrals. We will list some of these properties below.

- $\displaystyle\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$
- $\displaystyle\int_a^a f(x)\,\mathrm{d}x = 0$
- $\displaystyle\int_a^b c\,\mathrm{d}x = c(b-a)$
- $\displaystyle\int_a^b [f(x)+g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$
- $\displaystyle\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$
- $\displaystyle\int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$
- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x \geq \int_a^b g(x)\,\mathrm{d}x$
- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $\displaystyle m(b-a) \leq \int_a^b f(x)\,\mathrm{d}x \leq M(b-a)$.

Question 57. □

Using the figure below, find whether each of the following expressions is positive, negative, or zero. You may assume that the picture is up to scale.



(a) $\displaystyle\int_3^8 f(x)\,\mathrm{d}x$

(b) $\displaystyle\int_6^0 f(x)\,\mathrm{d}x$

(c) $\displaystyle\int_8^{12} (f(x)+6)\,\mathrm{d}x$

(d) $\displaystyle\int_0^3 f(x+3)\,\mathrm{d}x$

Question 58.

□

If f is an even function and $\displaystyle\int_1^2 f(t)\,\mathrm{d}t = 4$ *and* $\displaystyle\int_1^5 f(t)\,\mathrm{d}t = 6$, *then find* $\displaystyle\int_{-5}^{-4} f(t)\,\mathrm{d}t$.

Dummy Variable.

We end this section with the observation that while the symbol $\displaystyle\int f(x)\,\mathrm{d}x$ denotes an antiderivative **function** of $f(x)$, the symbol $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ is not a function, it's a number. As such, the x can be replaced by any other letter (provided, of course, that it is replaced in each place where it occurs). Thus

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(t)\,\mathrm{d}t = \int_a^b f(u)\,\mathrm{d}u = \int_a^b f(y)\,\mathrm{d}y = \cdots = \int_a^b f$$

§3.4 The Fundamental Theorem of Calculus, Part I

The Fundamental Theorem of Calculus is appropriately named because it establishes a connection between the two branches of calculus: differential calculus and integral calculus. Differential calculus arose from the tangent problem, whereas integral calculus arose from a seemingly unrelated problem, the area problem. In what follows, we will try to establish that differentiation and integration are inverse processes, and the Fundamental Theorem of Calculus will provide the precise inverse relationship.

Let's start by defining something called an Accumulation Function. As you read the definition and the example, compare them to the ideas of how we evaluated position from the velocity graph.

Definition 3.4.29

Let f be a continuous function on $[a,b]$, and define a function F by

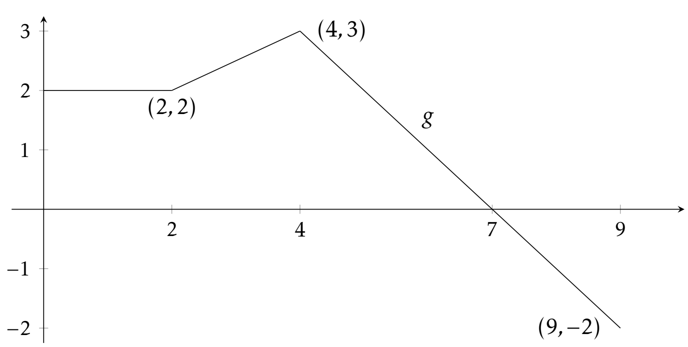
$$\mathrm{F}(x) = \int_a^x f(t)\,\mathrm{d}t$$

In other words, the value of $\mathrm{F}(x)$ is precisely the signed area between the graph of the function $f(x)$ and the horizontal axis on the interval $[a,x]$. A function of the form $\mathrm{F}(x)$ is called an *accumulation function.*

Let's see an example in practice.

Example 3.4.30

Let g be the function graphed below and let $\mathrm{G}(x) = \int_0^x g(t)\,\mathrm{d}t$.



(a) Find the largest open interval on which G is increasing. ________________

(b) Find the largest open interval on which G is decreasing. ________________

(c) What is the minimum of G on $[0,9]$? ________________

(d) What is the maximum of G on $[0,9]$? ________________

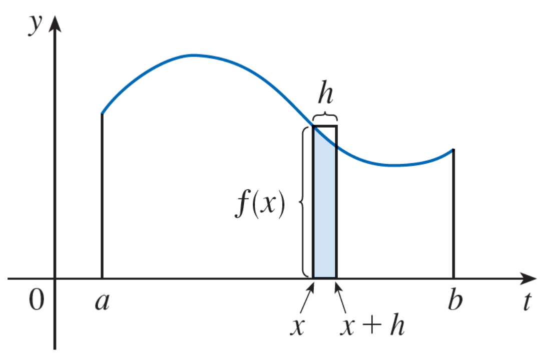
We had theorized earlier that in general, the area under the velocity graph should give us the position function. In the same vein, we would like to claim that the accumulation function of a function f is an Antiderivative of f. Let's see if we can prove that.

■ Question 59.

□

Let $\mathrm{F}(x)$ *be an accumulation function as above. Recall that* $\dfrac{\mathrm{d}}{\mathrm{d}x}[\mathrm{F}(x)] = \lim\limits_{h\to 0} \dfrac{\mathrm{F}(x+h)-\mathrm{F}(x)}{h}$. *Show that the quantity* $\mathrm{F}(x+h)-\mathrm{F}(x)$ *is equal to* $\displaystyle\int_x^{x+h} f(t)\,\mathrm{d}t$.

Now, the definite integral $\displaystyle\int_x^{x+h} f(t)\,\mathrm{d}t$, i.e. the quantity $\mathrm{F}(x+h)-\mathrm{F}(x)$, corresponds to the area of the shaded region in the picture below.



■ Question 60.

□

What is the area of the rectangle in above picture ?

While the shaded area is not quite equal to the area of the rectangle, our intuition about Riemann sums earlier tells us that as $h \to 0$, the shaded area will approach the area of the rectangle. Then,

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\,\mathrm{d}t = \lim_{h\to 0} \frac{h\cdot f(x)}{h} = f(x)$$

This result can be proved more rigorously with the Squeeze theorem, but I am not going to include it here. Check the textbook if you are interested.

So we have established the following.

Theorem 3.4.31: The Fundamental Theorem of Calculus Part I

Let f be a continuous function on $[a,b]$ and define an accumulation function $F(x)$ by:

$$F(x) = \int_a^x f(t)\,dt.$$

Then $F(x)$ is continuous on $[a,b]$, differentiable on (a,b), and $F'(x) = f(x)$. In other words,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

■ Question 61.

☐

When did we show that F is continuous?

■ Question 62.

☐

Apply the FTC part I to find the following derivatives.

(a) $\displaystyle\frac{d}{dx}\int_0^x \sqrt{1+t^2}\,dt$

(b) $\displaystyle\frac{d}{dx}\int_{-2}^x \tan\left(\frac{1}{u^2+1}\right)du.$

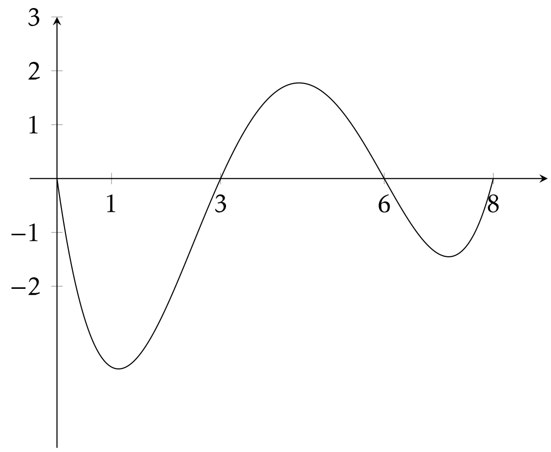
(c) $\displaystyle\frac{d}{dx}\int_1^{x^4} \sec t\,dt$ **Hint:** *You will need the Chain Rule.*

(d) $\displaystyle\frac{d}{dx}\int_{\sqrt{x}}^1 \tan(y)\,dy.$

■ Question 63.

☐

Use the graph of a function below to answer the given questions about the area function $F(x) = \int_0^x f(t)\,dt$.



(a) *Where is* F *increasing/decreasing?*

(b) *Find all critical points of* F *and determine which are local extrema.*

(c) *Since* F *is continuous on* $[0,8]$, *it must have absolute extrema. Where do the absolute extrema for* F *occur?*

(d) *Where is* F *concave up/concave down?*

(e) *Does* F *have any points of inflection?*

(f) *Sketch a possible graph of* F.

§3.5 The Fundamental Theorem of Calculus, Part II

So far, the only method we know for computing definite integrals is using the definition as a limit of Riemann sums and we saw that this procedure is sometimes long and difficult. The second part of the Fundamental Theorem of Calculus, which follows easily from the first part, provides us with a much simpler method for the evaluation of integrals.

Before stating the theorem, we will make some observations that will gradually lead us to the main result.

Suppose f is a continuous function on $[a,b]$. Define an accumulation function G as $G(x) = \int_a^x f(t)\,dt$.

Then using the first FTC, $G'(x) =$ ______________________________.

Suppose F is an antiderivative of f. What is the relation between F and G? ______________

Using the definition, $G(a) =$ ______________ $\implies F(a) =$ ______________

Hence $F(b) - F(a) =$ ______________________________.

Theorem 3.5.32: The Fundamental theorem of Calculus Part I

Let f be a continuous function on $[a,b]$. Then for any antiderivative F of f on $[a,b]$,

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Theorem 3.5.33: Corollary of FTC: Net Change Theorem

Let $F(t)$ be some quantity with a continuous rate of change $F'(t)$. Then

$$\int_a^b F'(t)\,dt = F(b) - F(a)$$

In other words, the integral of a rate of change gives total change.

3.5.1 Why is this theorem the best thing since sliced bread?

We no longer need Riemann sums to find definite integrals if we know an antiderivative! Huzzah!

Find an antiderivative F of $\sin x$. ______________________________

Hence $\int_0^\pi \sin x\,dx = F(\pi) - F(0) =$ ______________________________.

Example 3.5.34

One more example. Suppose you are computing $\int_0^1 x^2\,dx$. We can write the general antiderivative here as $\int x^2\,dx = \frac{x^3}{3} + C$, where C is any constant. To evaluate the definite integral, we can just use the $\frac{x^3}{3}$ part, like so:

$$\begin{aligned}\int_0^1 x^2\,dx &= \frac{x^3}{3}\bigg|_0^1 \\ &= \frac{1}{3}\end{aligned}$$

Note the following shorthand notation:

$$F(x)\Big|_a^b = F(b) - F(a).$$

We will often write this as part of our work when evaluating definite integrals, as it allows us to write down the antiderivative first before trying to plug values into it (and likely confusing ourselves in the process).

Question 64.

Evaluate the following definite integrals. The first one has been done for you as another example.

(a) $\int_1^2 \left(6x^2 - 3x\right) = \left(\frac{6x^3}{3} - \frac{3x^2}{2}\right)\Bigg|_1^2 = \left(2(2)^3 - \frac{3(2)^2}{2}\right) - \left(2 - \frac{3}{2}\right) = (16-6) - \frac{1}{2} = \frac{19}{2}.$

(b) $\int_{-1}^1 (5y^3 + y^2 - y)\,dy$

(c) $\int_{-8}^8 x^{1/3}\,dx$

(d) $\displaystyle\int_{0}^{\pi/2} \cos(\theta)\, d\theta$

(e) $\displaystyle\int_{0}^{3} |x^2 - 1|\, dx$

(f) $\displaystyle\int_{0}^{4} |x^2 - 4x + 3|\, dx$

■ Question 65.

□

What is wrong with the following calculation?

$$\int_{-1}^{3} \frac{1}{x^2}\, dx = \int_{-1}^{3} x^{-2}\, dx = \left.\frac{x^{-1}}{-1}\right|_{-1}^{3} = \left.\frac{-1}{x}\right|_{-1}^{3} = \frac{-1}{3} - \frac{-1}{-1} = -\frac{4}{3}$$

§3.6 The Method of Substitution

Suppose we want to find $\int \sin(5x)\,\mathrm{d}x$. What is the antiderivative $F(x)$ of the function $f(x) = \sin(5x)$? We can do this in two different ways.

Guess and Check

$$\begin{aligned}
\text{Guess:} &\quad F(x) = -\cos(5x) + C \\
\text{Check:} &\quad F'(x) = 5\sin(5x) \\
\text{How do we fix this?} &\quad \frac{1}{5}F'(x) = \sin(5x) \\
\text{We divide by 5.New guess:} &\quad F(x) = -\frac{1}{5}\cos(5x) + C \\
\int \sin(5x)\,\mathrm{d}x = &\quad -\frac{1}{5}\cos(5x) + C
\end{aligned}$$

Now let's do it in another way!

Substitution

We start by creating a new variable for our "inner function". Let's call it $\heartsuit$.

$$\begin{aligned}
\text{Let } \heartsuit &= 5x \\
\text{Then } \frac{\mathrm{d}\heartsuit}{\mathrm{d}x} &= 5 \\
\text{Let's rewrite it as } \mathrm{d}\heartsuit &= 5\,\mathrm{d}x \\
\implies \frac{1}{5}\mathrm{d}\heartsuit &= \mathrm{d}x
\end{aligned}$$

Now we use our new code to convert our integral to hearts.

$$\begin{aligned}
\text{Substitute:} &\quad \int \sin(5x)\,\mathrm{d}x = \int \sin\heartsuit \cdot \frac{1}{5}\,\mathrm{d}\heartsuit \\
\text{Pull the constant to the outside:} &\quad \frac{1}{5}\int \sin\heartsuit\,\mathrm{d}\heartsuit \\
\text{Integrate:} &\quad -\frac{1}{5}\cos\heartsuit + C \\
\text{Plug in the original variable:} &\quad -\frac{1}{5}\cos(5x) + C
\end{aligned}$$

Both methods give the same answer!

3.6.1 Reversing the Chain Rule

Suppose $F'(x) = f(x)$. Recall that the Chain Rule states:

$$\frac{\mathrm{d}}{\mathrm{d}x}[F(g(x))] = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x)$$

So equivalently, restating this relationship in terms of an indefinite integral,

$$\int f(g(x)) \cdot g'(x)\,\mathrm{d}x = \mathrm{F}(g(x)) + \mathrm{C}$$

How do we use this in practice?

Theorem 3.6.35: u-substitution

Let $u = g(x)$, where $g'(x)$ is continuous over an interval I*, and let $f(x)$ be continuous over $g(\mathrm{I})$. Let $\mathrm{F}(x)$ be an antiderivative of $f(x)$. Then $du = g'(x)dx$, and we can write*

$$\int f(g(x)) \cdot g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u = \mathrm{F}(u) + \mathrm{C} = \mathrm{F}(g(x)) + \mathrm{C}$$

Example 3.6.36

Let's find $\displaystyle\int x\cos(x^2)\,\mathrm{d}x$.

The inside function is x^2, with derivative $2x$. The integrand has a factor of x, and since the only thing missing is a constant factor, we try $u = x^2$ to get

$$\mathrm{d}u = u'(x)\,\mathrm{d}x = 2x\,\mathrm{d}x \implies x\,\mathrm{d}x = \frac{1}{2}\mathrm{d}u$$

Thus,

$$\int x\cos(x^2)\,\mathrm{d}x = \int \cos u \cdot \frac{1}{2}\mathrm{d}u = \frac{1}{2}\int \cos u\,\mathrm{d}u = \frac{1}{2}\sin u + \mathrm{C} = \frac{1}{2}\sin(x^2) + \mathrm{C}$$

Example 3.6.37

Let's find $\displaystyle\int x^3\sqrt{x^4+5}\,\mathrm{d}x$.

The inside function is $x^4 + 5$, with derivative ____________________. The integrand has a factor of x^3, and since the only thing missing is a constant factor, we try $u =$ ____________________________ to get

$\mathrm{d}u = u'(x)\,\mathrm{d}x =$ _______________________________ $\mathrm{d}x \implies x^3\,\mathrm{d}x =$ _______________________________ $\mathrm{d}u$

Thus,

$\displaystyle\int x^3\sqrt{x^4+5}\,\mathrm{d}x = \int$ _____________ $\cdot$ _____________ $\mathrm{d}u$ [convert everything in terms of u]

$=$ ____________________________________ [pull the constant out]

$=$ ____________________________________ [Do the integration]

$=$ ____________________________________ [Substitute u back]

Algorithm for Solving u-substitution Problems

Step 1. Look carefully at the integrand and select an expression $g(x)$ within the integrand to set equal to u. Select $g(x)$ such that $g'(x)$ is also sitting somewhere else in your integrand.

Step 2. Substitute $g(x)$ by u and $g'(x)\,\mathrm{d}x$ by $\mathrm{d}u$ into the integral.

Step 3. We should now be able to evaluate the integral with respect to u. If the integral can't be evaluated we need to go back and select a different expression to use as u.

Step 4. Evaluate the integral in terms of u.

Step 5. Substitute the expression $g(x)$ back in place of u and write the final answer in terms of x.

Note: It is often helpful to choose u to be the ''inside'' of some other function.

Question 66.

☐

Find the following indefinite integrals.

(a) $\displaystyle\int \frac{x^3}{\sqrt{1+x^4}}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{\sin x}{\cos^3 x}\,\mathrm{d}x$

(c) $\displaystyle\int x\sin(x^2+5)\,\mathrm{d}x$

(d) $\displaystyle\int \frac{x+1}{(x^2+2x+19)^2}\,\mathrm{d}x$

3.6.2 Evaluating definite integrals via u-substitution

Let's solve the following integral using substitution: $\displaystyle\int_0^1 x^3(2x^4+1)^{10}\,dx$

Let $\star = 2x^4+1$, then $d\star =$ ____________________________________

Solve for $x^3\,dx$: ____________________________ . Now use your "$\star$ code" to translate the integrand (just the inside) from x's to $\star$'s: ____________________________

Now we must find the limits of integration in terms of $\star$'s, instead of x's.

While using x's, the limits of integration were $x=0$ and $x=1$: $\displaystyle\int_{x=0}^{x=1} x^3(2x^4+1)^{10}\,dx$

Find the new lower limit. Let $x=0$ and solve for $\star$: ____________________________

Find the new upper limit. Let $x=1$ and solve for $\star$: ____________________________

Plug the limits of integration into our new integral: ____________________________

Evaluate the integral:

Alternately, we can evaluate the corresponding indefinite integral, as we did on the previous worksheet. Then, once we have x's back in our answer, we can evaluate the integral using the given limits of integration on x. That would proceed as follows:

$$\begin{aligned}\int x^3(2x^4+1)^{10}\,dx &= \frac{1}{8}\int u^{10}\,du\\ &= \frac{1}{8}\frac{u^{11}}{11}+C = \frac{1}{88}(2x^4+1)^{11}+C\end{aligned}$$

And then,

$$\int_0^1 x^3(2x^4+1)^{10}\,dx = \frac{1}{88}(2x^4+1)^{11}\Big|_0^1 = \frac{1}{88}(3^{11}-1)$$

Question 67.

□

Evaluate the following definite integrals using substitution:

(a) $$\int_0^1 x\sqrt{1-x^2}\,dx$$

(b) $$\int_0^1 x^2 \cos\left(\frac{\pi}{2}x^3\right)dx$$

(c) $$\int_0^{\sqrt{\pi}} \theta\cos(\theta^2)\sin(\theta^2)\,d\theta$$

§3.7 Further Properties of Definite Integrals

In section 3.3.3, we gave a list of properties of Definite Integrals. In this section, we will look at some more properties that can be derives using periodicity, symmetry, etc. and discuss a way to find the "average" value of an integral.

■ Question 68. □

Recall that if f and g are two integrable functions such that $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,\mathrm{d}x \geq \int_a^b g(x)\,\mathrm{d}x$.

Using this, or otherwise, explain why

$$\int_a^b |f(x)|\,\mathrm{d}x \geq \left|\int_a^b f(x)\,\mathrm{d}x\right|$$

for any integrable function f.

3.7.1 Use of Symmetry and Periodicity

Recall that an even function is one satisfying $f(-x) = f(x)$, whereas an odd function satisfies $f(-x) = -f(x)$. The graph of the former is symmetric with respect to the y-axis; the graph of the latter is symmetric with respect to the origin.

■ Question 69. □

Draw the graph of an even function f of your choice on an interval $[-a, a]$. Using the symmetry in the picture explain why the following must be true.

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x$$

■ Question 70. □

Draw the graph of an odd function g of your choice on an interval $[-a,a]$. Using the symmetry in the picture explain why the following must be true.

$$\int_{-a}^{a} g(x)\,\mathrm{d}x = 0$$

■ Question 71. □

Draw the graph of an arbitrary function h of your choice on an interval $[a,b]$.

(a) In the same picture, can you draw the graph of $h(a+b-x)$ on the interval $[a,b]$?

(b) Use symmetry arguments to explain why

$$\int_{a}^{b} h(x)\,\mathrm{d}x = \int_{a}^{b} h(a+b-x)\,\mathrm{d}x.$$

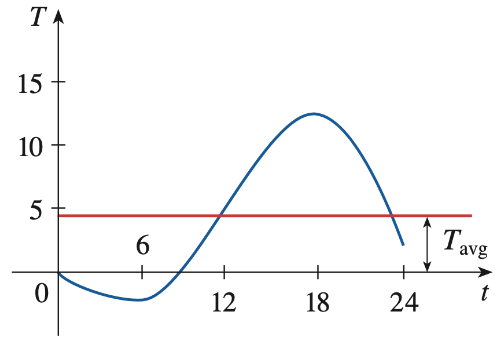
■ Question 72. □

Draw the graph of a periodic function f of your choice on an interval $[a,b]$. Suppose the period is p. Using the picture explain why the following must be true.

$$\int_{a}^{b} f(x)\,\mathrm{d}x = \int_{a+p}^{b+p} f(x)\,\mathrm{d}x$$

3.7.2 Average Value and MVT for Integrals

We knowhow to calculate average of finitely many quantities, but how do we, for example, compute the average temperature during a day if infinitely many temperature readings are possible? The figure below shows the graph of a temperature function $T(t)$, where t is measured in hours and T in °C, and a guess at the average temperature, T_{avg}.



So we propose the following definition (that has justification using Riemann sums).

Definition 3.7.38: Average Value of a Function

If f is integrable on the interval $[a,b]$, then the average value of f on $[a,b]$ is

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

Geometrically, for a positive function, we can think of the average value of a function being the $\frac{\text{area}}{\text{width}}$.

Question 73.

Assume that the temperature function above has the formula $T(t) = 5 - 8\sin\left(\frac{\pi t}{12}\right)$ *where t is measured in hours after the midnight. Find the average* T_{avg} *over a day.*

□

Question 74.

Find the average of the function $g(x) = 2\sin x - \sin(2x)$ *on* $[0,\pi]$. *(will be in tutorial tomorrow).*

□

From the last picture, you can clearly see that there was a moment of the day (in fact two different moments) when the temperature measured exacty T_{avg}. This is an existence statement reminiscent of the MVT we learned earlier. In fact, this statement is true in general, and is called the Mean Value Theorem for Integrals.

Theorem 3.7.39: Mean Value Theorem for Integrals

If f is continuous on $[a,b]$, then there exists a number $c \in [a,b]$ such that

$$f(c) = f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x.$$

The Mean Value Theorem for Integrals is a consequence of the Mean Value Theorem for derivatives and the Fundamental Theorem of Calculus. We will not prove it here.

■ Question 75. □

Point out all the differences between this theorem and the MVT for derivatives you learned earlier.

■ Question 76. □

Consider the function $f(x) = \dfrac{1}{x^2}$ defined on the interval $[1,3]$.

(a) *Find the average value of f on $[1,3]$.*

(b) *Find a number c in the interval $[1,3]$ such that $f(c) = f_{avg}$.*

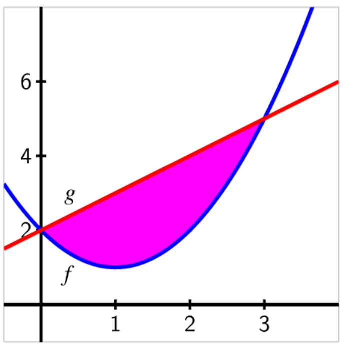
(c) *Sketch the graph of f and a rectangle whose base is $[1,3]$ and whose area is the same as the area under the graph of f.*

Chapter 4 | Applications of the Integral

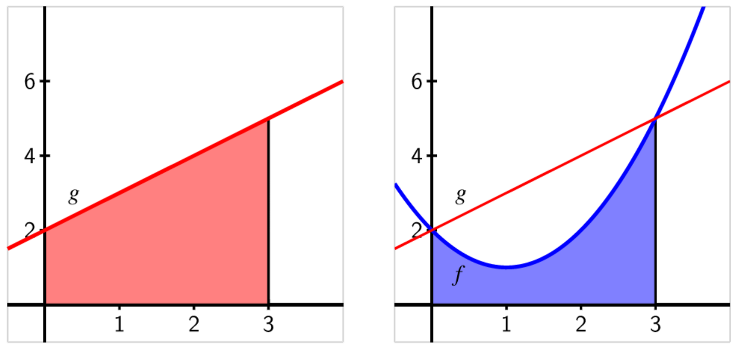
§4.1 Area between two curves

last week we learned that definite integral is equivalent to the area under a curve. In this section, we will discuss what happens when there are two curves in our graph and we want to find the region between them.

Consider the graphs of $f(x) = (x-1)^2 + 1$ and $g(x) = x + 2$. We wish to find the area bounded between the two graphs, the region in the first picture below.



(a) Area between two graphs



(b) Area under each graph

We observe that the area of the first region is the same as the difference between the two areas pictured to the right. Mathematically, we subtract the area under the "bottom" curve from the area under the "top" curve, using the end points of the interval as our bounds.

We can find the later two areas by direct definite integration, but we need to first find out the bounds. Observe that the bounds are the points of intersection between the two graphs of f and g.

Question 77. □

Show that the solutions to $f(x) = g(x)$ are $x = 0$ and 3.

Thus the area in the second picture is equal to $\displaystyle\int_0^3 g(x)\,dx = \int_0^3 (x+2)\,dx =$ ______________________

and the area in the third picture is equal to $\displaystyle\int_0^3 f(x)\,\mathrm{d}x = \int_0^3 ((x-1)^2+1)\,\mathrm{d}x =$ ________________________

So finally, the area between the two curves is equal to

$$\int_0^3 (g(x)-f(x))\,\mathrm{d}x = \int_0^3 g(x)\,\mathrm{d}x - \int_0^3 f(x)\,\mathrm{d}x = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

■ Question 78.

Find the area between the curves for the following problems. try to first sketch the functions and shade the area between the curves, and then find the area.

(a) $f(x) = 2-x^2$, $g(x) = x^2$

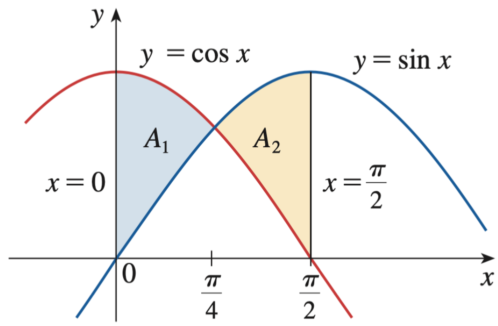
(b) $f(x) = x^2, g(x) = 6-x$

4.1.1 Functions with more than two intersection points

If the two functions cross each other more that two times, it is unclear which graph is the ''top'' one and which one is the ''bottom'' one. In fact, the top and the bottom functions switch at each intersection. In such cases, we will calculate the area as follows.

Example 4.1.40

Consider the area of the region bounded by the curves $f(x) = \sin x$ and $g(x) = \cos x$ between $x = 0$ and $x = \pi$. A point of intersection occurs when $\sin x = \cos x$, that is, when $x = \pi/4$ (in this interval).



What do you notice? Which function is ''on top'' and which function is ''on bottom''?

On the interval $[0, \pi/4]$, we have ____________________ $\leq$ ____________________.

On the interval $[\pi/4, \pi/2]$, we have ____________________ $\leq$ ____________________.

Since we have a different ''top" and ''bottom" curve for each interval, we need to do a different integral for each interval, and then add them.

Set up and evaluate the two integrals. Remember, when finding the area between two curves, you take the integral of the ''top" curve minus the ''bottom" curve, using the end points of the interval as bounds.

Thus in general, we have the following theorem

Theorem 4.1.41

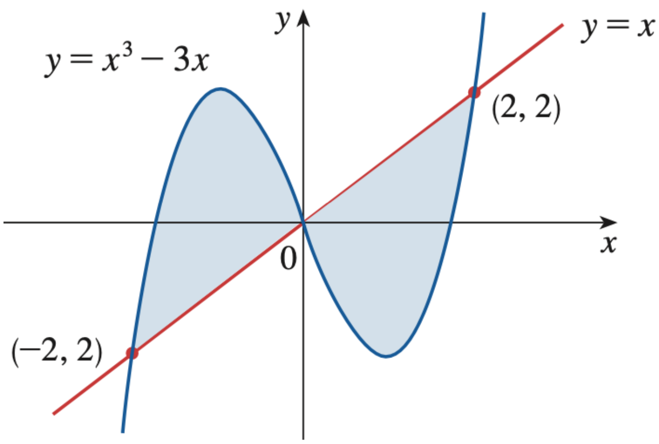
Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a,b]$. Let R denote the region between the graphs of $f(x)$ and $g(x)$, and be bounded on the left and right by the lines $x = a$ and $x = b$ respectively. Then, the area of R is given by

$$\int_a^b |g(x) - f(x)|\,dx$$

Question 79.

☐

Set up integrals that calculate the area of the following region. You do not need to evalaute the integrals.



Question 80.

☐

Sketch the region bounded by the functions $f(x) = \sqrt{3-x}$, $g(x) = x - 1$, and the x-axis, and set up integrals that calculate the area of the region.

4.1.2 More Complex Regions

Example 4.1.42

How would you find the area of the region bounded by $f(x) = x^2$, $g(x) = 2 - x$, and the x–axis?

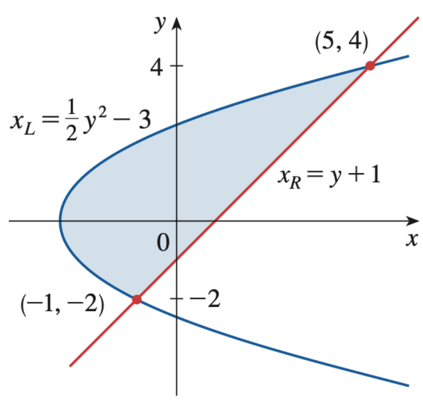
First we will draw the graphs and find the points of intersection. Afterwards, there are two ways to finish the problem.

- **Method 1.** We can divide the interval into two pieces. Then write each piece as an integrals with respect to x.

  $$\text{Area (R)} = \int_0^1 \underline{\hspace{4cm}}\,dx + \int_1^2 \underline{\hspace{4cm}}\,dx = \underline{\hspace{8cm}}.$$

- **Method 2.** We can treat the curves as functions of y instead of as functions of x. Then we find the y-coordinates of the intersection points and calculate the area by integrating with respect to y.

Let's explain the second method a bit more. Consider, for example, the curves $y^2 = 2x + 6$ and $y = x - 1$. It is immediately clear that the parabola in the picture below is not the graph of a function of x (why?), so our formula, area $= \int_a^b (g(x) - f(x))\,dx$ doesn't work! In particular, the intersection points do not give correct bounds for x.



So instead we treat the curves as

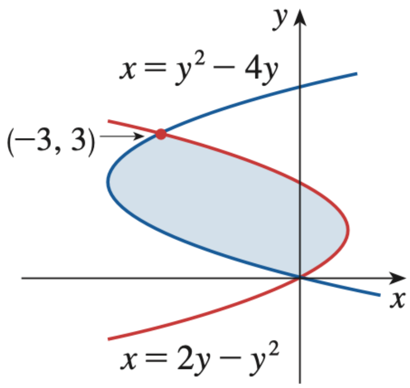
$$x = f(y) = \frac{1}{2}y^2 - 3, \qquad x = g(y) = y + 1$$

and find the area between $y = -2$ and $y = 4$ as

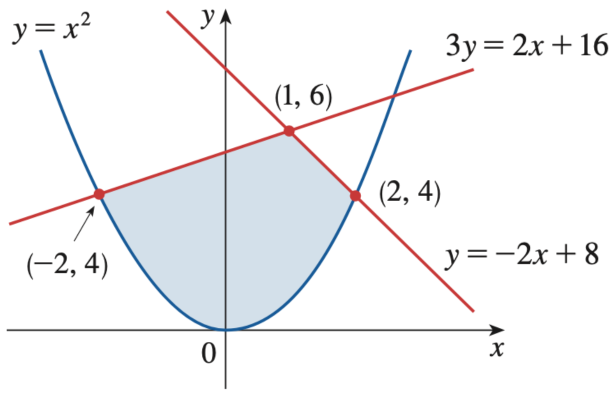
$$\int_{-2}^{4} (g(y) - f(y))\,dy = \underline{\hspace{12cm}}$$

■ Question 81.

Set up integrals that calculate the area of the following regions. You do not need to evalaute the integrals.



(a)



(b)

Chapter 5 | Integration Techniques

§5.1 Integration by Parts

Every differentiation rule has a corresponding integration rule. For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The integration rule that corresponds to the Product Rule for differentiation is called **integration by parts**.

First, let us recall what the product rule says. Given two differentiable functions $u(x)$ and $v(x)$, the derivative of the product $u(x)v(x)$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}[u(x)v(x)] = u'(x)v(x) + u(x)v'(x)$$

For simplicity of notation, let's rewrite this as

$$uv' = \frac{\mathrm{d}}{\mathrm{d}x}(uv) - u'v$$

and integrate both sides to get

$$\int uv'\,\mathrm{d}x = \int \frac{\mathrm{d}}{\mathrm{d}x}(uv)\,\mathrm{d}x - \int u'v\,\mathrm{d}x$$

Since the antiderivative of the derivative of uv is just uv, we get the integration by parts formula

Theorem 5.1.43

$$\int uv'\,\mathrm{d}x = uv - \int u'v\,\mathrm{d}x$$

Note: You may be familiar with some other ways of writing the same thing:

$$\int u\,dv = uv - \int v\,du$$

$$\int uv\,\mathrm{d}x = u\left(\int v\,\mathrm{d}x\right) - \int u'\left(\int v\,\mathrm{d}x\right)dx$$

Question 82. □

Explain why the last two are equivalent formulations of the integration by parts formula.

5.1.1 When do we use Integration by Parts

Integration by parts is most useful when the integrand can be viewed as a product of two different types of functions inside the integrated (algebraic, trigonometric, exponential, etc.). Let's try some examples.

Example 5.1.44

Suppose we want to find $\int x \sin x \, dx$. First, convince yourself that you cannot find the integral by simple u-substitution.

Our first goal is to rewrite the integral as

$$\int x \sin x \, dx = \int u v' \, dx$$

Fortunately, the integrand here is a product of two functions, x and $\ln x$. So one of them should be $u(x)$ and the other one is $v'(x)$.

However, note that the formula requires us to find $v(x)$ on the right-hand side and then integrate $u'v$. So we should choose u and v in a way so that integrating $u'v$ becomes easier than the original.

So we have $u(x) =$ ____________________ and consequently, $u'(x) =$ ____________________

And we choose $v'(x) =$ ______________ and consequently, $v(x) = \int v'(x) \, dx =$ ______________

Now fill in the right hand side of the formula:

$\int u v' \, dx = uv - \int u' v \, dx =$ ____

and finish the integration.

Some Rules of Thumb for Choosing u and v'.

- Whatever you let v' be, you need to be able to find v.
- Our end goal is to make sure $u'v$ is easier to integrate than uv'. So it helps if u' is simpler than u (or at least no more complicated than u) and similarly, if v is simpler than v' (or at least no more complicated than v').

Question 83.

Try it yourself. Find $\int x^2 \cos x \, dx$. *You will need to use Integration By Parts more than once.*

Sometimes you must simplify your integral expression using a substitution first before you can use integration by parts.

Example 5.1.45

Let's find $\int x^3 \sin(x^2)\, dx$. It is not immediately obvious what we should choose as $v'(x)$.

- On one hand, if $v'(x) = x^3$, then $v(x)$ is more complicated than $v'(x)$. Also u' is actually worse than $u(x)$.
- On the other hand, if $v'(x) = \sin(x^2)$, we don't actually know how to find $v(x)$.

So let's check if we can simplify the integral a bit using substitution first. Let's choose z to be the "inside" function (we are using a different letter so as not to confuse with u and v above).

So $z =$ ____________________ and $dz =$ ____________________

So we can rewrite the integral as $\int x^3 \sin x^2 \, dx =$ ____________________

Now finish the integration.

Note: Before we do more examples, we are going to mention two specific results that we can use as formulas from now on. Both of these can be found using u−substitution.

$$\int \sin(ax)\,\mathrm{d}x = -\frac{\cos(ax)}{a} + C, \qquad \int \cos(ax)\,\mathrm{d}x = \frac{\sin(ax)}{a} + C$$

5.1.2 Evaluating Definite Integrals Using Integration by Parts

If we combine the formula for integration by parts with the Fundamental Theorem of Calculus, we can evaluate definite integrals by parts.

Theorem 5.1.46

$$\int_a^b uv'\,\mathrm{d}x = [u(x)v(x)]\Big|_a^b - \int_a^b u'(x)v(x)\,\mathrm{d}x$$

Question 84. □

Find $\displaystyle\int_0^{\pi/4} (x+1)\cos(2x)\,\mathrm{d}x$.

Chapter 6 | Expanding Our Library of Functions

§6.1 The Natural Logarithm Function

Let's start with a peculiar omission we purposefully ignored bach when we derived the power rule for integrals. The rule says

$$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + C$$

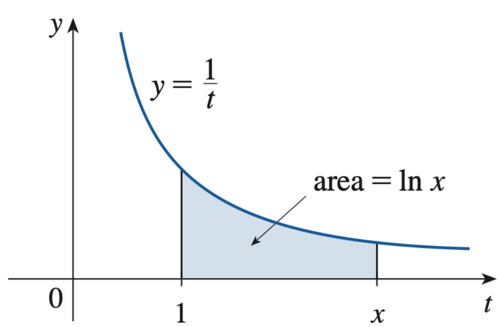
What happens if $n = -1$?

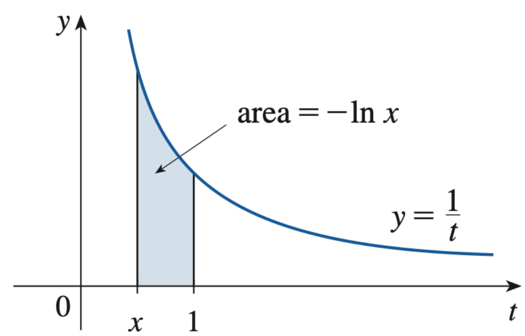
Clearly, the answer above wouldn't make any sense. On the other hand, $f(x) = \dfrac{1}{x}$ is a continuous function on $(0, \infty)$, and hence integrable. This gives us an idea to construct an antiderivative of $\dfrac{1}{x}$ using the accumulation funciton! We are going to avoid starting at 0 since the function is n't continuous there. Let's start at 1 instead.

Definition 6.1.47

Construct the accumulation function $F(x) = \displaystyle\int_1^x \frac{1}{t}\,\mathrm{d}t$ for $x > 0$. By the first FTC, $F'(x) = \dfrac{1}{x}$ for all $x > 0$.

The function $F(x)$ is called the natural logarithm function and is denoted by $\ln(x)$.





Note: By definition, if $x > 1$, the integral is positive, and hence $\ln(x) > 0$ for $x > 1$. Similalrly, $\ln(x) < 0$ for $0 < x < 1$.

Question 85.

Use a Riemann sum approximation to show that $\dfrac{1}{2} < \ln(2) < 1$.

6.1.1 Properties of the natural Logarithm function

Question 86. ☐

Suppose x and y and positive real numbers. Use the fact that $\frac{d}{dx}[\ln x] = \frac{1}{x}$ to show that

$$\ln(xy) = \ln x + \ln y.$$

Solution. Fix an arbitrary value of y and suppose $f(x) = \ln(xy)$. Then $f'(x) =$ ______________________.

Since $f(x)$ and $\ln(x)$ both have the same derivative, they must differ by a constant! Hence $f(x) = \ln x + C$.

At $x = 1$, we have $f(1) = \ln(y)$. On the other hand, $\ln(1) =$ ________. Hence $c =$ ________. ■

We can similarly show the following results that are being left as exercises for you to check at home. If x and y are positive numbers and r is a rational number, then

- $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$.
- $\ln(x^r) = r\ln(x)$.

As $r \to \infty$, we can clearly see that the right hand side increases without bound. Hence we have,

- $\lim_{x\to\infty} \ln(x) = \infty$.

Finally,

Suppose $t = \frac{1}{x}$, and let $t \to \infty$. Then $x \to 0^+$. Fill in the blanks:

$\lim_{x\to 0^+} \ln x = \lim_{t\to\infty} \ln\left(\frac{1}{t}\right) =$ ____.

Next, since $\frac{d}{dx}[\ln x] = \frac{1}{x}$ for $x > 0$, we can find

- $\frac{d^2}{dx^2}[\ln x] =$ ____ < 0.

So we conclude that $\ln(x)$ is increasing and concave ______________________.

Question 87. ☐

Putting all this information together, draw a graph of the function $\ln x$ on $(0, \infty)$.

6.1.2 Integrations involving $\ln(x)$

Finally, let's go back to our original question. We started this section by trying to find an antiderivative of $\frac{1}{x}$. We have found one for $x > 0$. But what about $x < 0$? We will use the Chain Rule in an interseting way to answer that question.

Suppose $x < 0$ is a real number. Then, $-x$ is a postive real number and $\ln(-x)$ is defined. Now by the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln(-x)] = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$$

So we have actually found another antiderivative for $\frac{1}{x}$, but this time the domain of the antiderivative is all negative numbers. Our final conclusion is

Theorem 6.1.48

$$\int \frac{1}{x}\,\mathrm{d}x = \begin{cases} \ln(x) + C, & \textit{for } x > 0 \\ \ln(-x) + C, & \textit{for } x < 0 \end{cases}$$

Or equivalently,

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C \textit{ for } x \neq 0$$

Question 88. □

Evaluate the following integrals.

(a) $\displaystyle\int \frac{1}{2x+1}\,\mathrm{d}x.$

(b) $\displaystyle\int \tan x\,\mathrm{d}x.$

(c) $\displaystyle\int \frac{1}{x\ln x}\,\mathrm{d}x$

(d) $\displaystyle\int \ln x\,\mathrm{d}x$ *[**Hint:** Use IBP with $u = \ln x$, $\mathrm{d}v = \mathrm{d}x$.]*

6.1.3 Integrating Rational Functions

Example 6.1.49

Suppose we want to find $\displaystyle\int \frac{x^2}{x-1}\,\mathrm{d}x$. When the integrand is a rational function and the numerator is of equal or greater degree than the denominator, we will start with a long division first.

Using a long division, $x^2 = (x-1)\cdot x + (x-1)\cdot 1 + 1$. Hence

$$\int \frac{x^2}{x-1}\,\mathrm{d}x = \int \left[(x+1) + \frac{1}{x-1}\right]\mathrm{d}x$$

Now, complete the integration below.

§6.2 Inverse Functions and their Derivatives

Consider an experiment in which a biologist started a bacteria culture with 100 bacteria in a limited nutrient medium; the size of the bacteria population was recorded at hourly intervals. The number of bacteria N is a function of the time t, i.e. $N = f(t)$.

Suppose, however, that the biologist changes her point of view and becomes interested in the time required for the population to reach various levels. In other words, she is thinking of t as a function of N. This function is called the inverse function of f, denoted by f^{-1}, and read "f inverse." Here $t = f^{-1}(N)$ is the time required for the population level to reach N.

The values of f^{-1} can be found by reading Table 1 from right to left or by consulting Table 2. For instance, $f^{-1}(550) = 6$ because $f(6) = 550$.

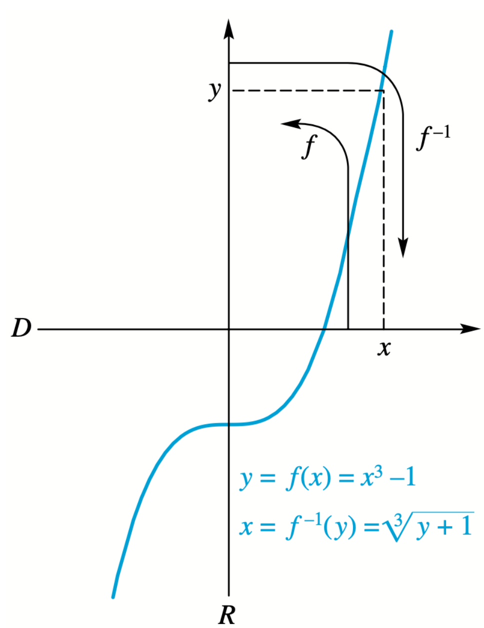
Table 1 ***N*** **as a function of** ***t***

t (hours)	$N = f(t)$ = population at time t
0	100
1	168
2	259
3	358
4	445
5	509
6	550
7	573
8	586

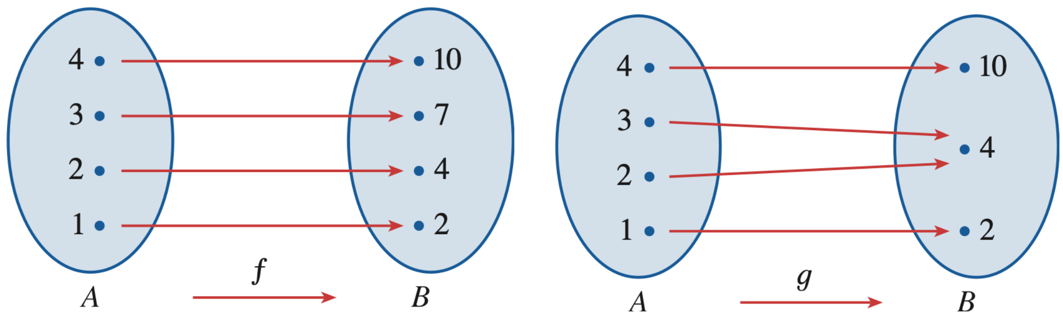
Table 2 ***t*** **as a function of** ***N***

N	$t = f^{-1}(N)$ = time to reach N bacteria
100	0
168	1
259	2
358	3
445	4
509	5
550	6
573	7
586	8

In general, assume a function f takes a number x from its domain D and assigns to it a single value y from its range R. Then in some cases, we can reverse f; that is, for any given y in the range R, we can **unambiguously** go back and find the x from which it came. This new function that takes y and assigns x to it is f^{-1}.



Not all functions possess inverses. Compare, for example, the functions f and g whose bubble diagrams are shown in the figure below. Note that f never takes on the same value twice (any two inputs in A have different outputs), whereas g does take on the same value twice (both 2 and 3 have the same output, 4).



Question 89.

Check that f^{-1} is well-defined but g^{-1} isn't.

Definition 6.2.50

A function f is called a one-to-one function if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \iff x_1 \neq x_2$$

In other words, distinct values of x must always lead to distinct values of $y = f(x)$.

If a horizontal line intersects the graph of f in more than one point, then there are **distinct** numbers x_1 and x_2 such that $f(x_1) = f(x_2)$. This means that f is not one-to-one. Therefore we have the following geometric method for determining whether a function is one-to-one.

Horizontal Line Test. A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Question 90.

*Sketch a graph of a function that is **not** one-to-one.*

Question 91.

Is the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ one-to-one? ____________________________

What about $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$? ____________________________

What about $h : (0,\infty) \to (0,\infty)$ defined by $h(x) = x^2$? ____________________________

6.2.1 Procedure for finding f^{-1}

Note that if $y = f(x)$, then $x = f^{-1}(y)$ (assuming f is one-to-one). So, to find the inverse of a function, we will use the following three-step procedure:

Algorithm for finding a formula for f^{-1}

Step 1. Write $y = f(x)$ and solve for x in terms of y (if this is possible).

Step 2. Use the solution for x (in terms of y) to write $f^{-1}(y)$.

Step 3. Interchange the roles of x and y to get the formula for $f^{-1}(x)$.

Question 92.

Find the inverse function of $f(x) = x^3 + 2$.

There are also cases where a function has an inverse, but it is not practical to find a formula for it.

Question 93.

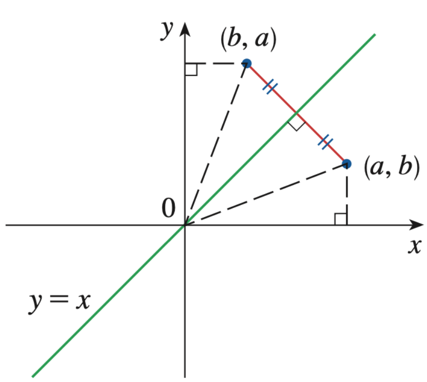
Show that the function $f(x) = x^5 + 2x + 1$ is increasing on $\mathbb{R}$. Use this to justify why f must be one-to-one (and hence has an inverse).

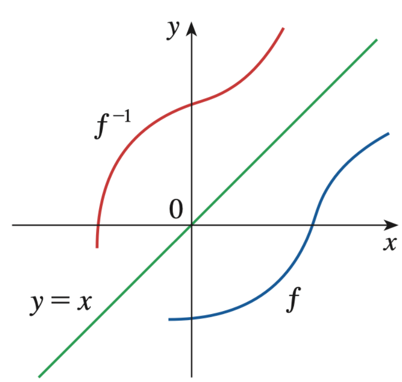
6.2.2 Graph of f^{-1}

The principle of interchanging x and y to find the inverse function also gives us the method for obtaining the graph of f^{-1} from the graph of f. Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point (a,b) is on the graph of f if and only if the point (b,a) is on the graph of f^{-1}. But we get the point (b,a) from (a,b) by reflecting about the line $y = x$.

Theorem 6.2.51

The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.





6.2.3 Derivtive of f^{-1}

If the inverse of f is f^{-1}, then the inverse of f^{-1} is f. One function kind of undoes the other.

$$f^{-1}(f(x)) = x \qquad \text{and} \qquad f(f^{-1}(x)) = x$$

Assume f is differentiable and let's apply the chain rule to the second equation.

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(f^{-1}(x))] = \frac{\mathrm{d}}{\mathrm{d}x}[x] \qquad \implies f'(f^{-1}(x)) \cdot \frac{\mathrm{d}}{\mathrm{d}x}[f^{-1}(x)] = 1 \qquad \implies \frac{\mathrm{d}}{\mathrm{d}x}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$$

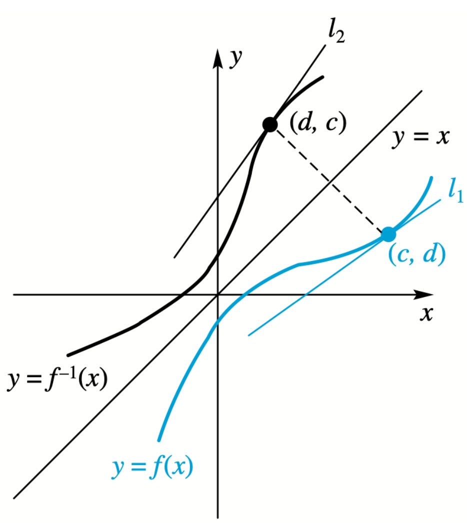
■ **Question 94.** □

Let $f(x) = x^3 + 1$. Find $f^{-1}(9)$.

The above result is sometimes called the **Inverse Function Theorem**. If we choose $d = f(c)$, so that $c = f^{-1}(d)$, then the result says

$$\left(f^{-1}\right)'(d) = \frac{1}{f'(f^{-1}(d)} = \frac{1}{f'(c)}$$

In other words, the derivative of f at c, i.e. the slope of the tangent at (c,d) to the graph of f, and the derivative of f^{-1} at d, i.e. the slope of the tangent at (d,c) to the graph of f^{-1}, are reciprocal of each other. This fact should also make sense using the reflection symmetry in the picture below.



§6.3 The Natural Exponential Function

The inverse of the natural logarithm function we introduced last week is called the natural exponential function, and is denoted by $\exp(x)$. Thus

$$y = \exp(x) \iff x = \ln(y)$$

and

$$\exp(\ln(x)) = x \qquad \text{and} \qquad \ln(\exp(x)) = x$$

Question 95. ☐

Explain why $\ln x$ *must be an invertible function.*

Using the fact that $\ln(x)$ and $\exp(x)$ are inverses of each other, we get that

The domain of $\exp(x)$ is ____________________, since the range of $\ln(x)$ is ____________________.

The range of $\exp(x)$ is ____________________, since the domain of $\ln(x)$ is ____________________.

Question 96. ☐

Draw the graph of the natural logarithm and the natural exponential function in the same picture. Point out the intercept values.

6.3.1 Why is it called "exponential"?

Recall that $\ln(x)$ is a continuous function that takes on arbitrarily large values. Then the ____________ says that there must be a number where $\ln(x)$ takes the value 1. Using the fact that ln is one-to-one, we can also conclude that this number must be unique. The number is denoted e after the mathematician Leonhard Euler.

Definition 6.3.52

e is the unique positive real number such that $\ln(e) = 1$.

Now, for any rational number r, we have $\ln(e^r) =$ ____________________________________.

Then, by definition of inverse $\exp($____________$) = e^r$ for all rational r. This gives us a way to define e^x in general for irrational x values by setting it equal to $\exp(x)$.

> **Note:** This may seem circuitous or even unnecessary at first sight. But ask yourself, how would you define $e^{\sqrt{2}}$? When r is rational, e.g. $3/4$, we can define $e^{3/4} = \sqrt[4]{e^3}$. So we can say that it's e multiplied to itself three times, and then we find a number whose fourth power is e^3. But can you describe $e^{\sqrt{2}}$ in a similar way? What does it mean, using elementary mathematical language, to raise a number to $\sqrt{2}^{\text{th}}$ power? This is why we need this roundabout way of defining e^x.

So, $\exp(x) = e^x$ satisfies all the familiar laws of exponents.

We have, $e^a e^b = \exp(\ln(e^a e^b)) = \exp(\underline{\hspace{6cm}}) = \exp(\underline{\hspace{6cm}}) = e^{a+b}$

We can similarly show that $e^{a-b} = \dfrac{e^a}{e^b}$.

■ Question 97. □

Use the derivative of the inverse formula from the last section to find the derivative of $\exp(x)$.

■ Question 98. □

Find $\displaystyle\int e^x \, \mathrm{d}x$.

■ Question 99. □

Find $\dfrac{\mathrm{d}}{\mathrm{d}x}[e^{\sqrt{x}}]$.

■ Question 100. □

Find $\displaystyle\int e^{-7x} \, \mathrm{d}x$.

■ Question 101.

□

Evaluate the following integrals.

(a) $\displaystyle\int xe^{1-x^2}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{e^x}{e^x-1}\,\mathrm{d}x$

(c) $\displaystyle\int e^x \sin x\,\mathrm{d}x$

§6.4 The General Exponential and Logarithmic Functions

6.4.1 Defining General Exponential

Last section, we defined e^x as $\exp(x)$ which in turn is defined as the inverse of the $\ln(x)$ function. This gives us a way to define irrational powers of e. Now we want to extend the definition in a way so that we can also define other numbers of the form b^x where b and x are any real numbers.

Definition 6.4.53

For any positive real number b, we define b^x as $e^{x\ln b}$.

Where did this weird thing come from? Let's work a bit backward. Using a property of logarithm, $x\ln b$ is the same as $\ln(b^x)$, even if we don't know what b^x is. On the other hand, since exp and ln are inverses of each other,

$$\exp(\ln(b^x)) = b^x \implies b^x = e^{\ln(b^x)} = e^{x\ln b}$$

Warning: Since the definition requires $\ln(b)$ to be defined, b has to be positive (why?). So, this definition only works for $b > 0$. That doesn't mean b^x doesn't exist when $b < 0$. For example, $(-2)^2$ makes sense. It only means that b^x doesn't always make sense if $b < 0$. For example, $(-2)^{1/2}$ does not exist.

Question 102. □

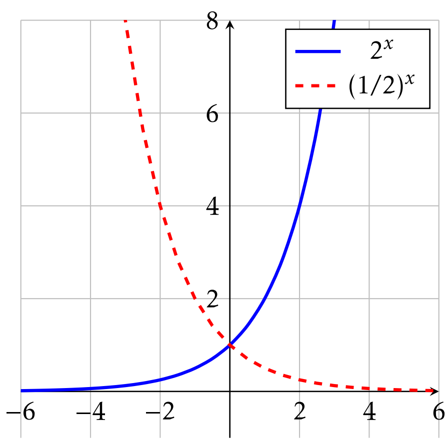
Suppose $b > 0$. Find $\dfrac{\mathrm{d}}{\mathrm{d}x}[b^x]$ using the Chain Rule.

Question 103. □

What happens when $b = 1$?

Functions of the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$ are called General Exponential Functions. When $b > 1$, an exponential function models rapid **growth**, while if $0 < b < 1$, the function models rapid **decay**. The graphs of $f(x) = 2^x$ and $f(x) = (0.5)^x$ are shown in the figure below. In both cases the domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

Graphs of Exponential Functions



Question 104. □

What is $\int b^x \, \mathrm{d}x$?

Question 105. □

Find $\int 3^{x^3} x^2 \, \mathrm{d}x$.

6.4.2 Inverse of the General Exponential

Question 106. □

Explain why $f(x) = b^x$ *is invertible for any* $b > 0$.

Definition 6.4.54

The inverse of the general exponential function $f(x) = b^x$ is known as the general (or common) logarithm function and is denoted as $f^{-1}(x) = \log_b(x)$.

Question 107. □

Show that $\log_b(x) = \dfrac{\ln x}{\ln b}$.

Question 108. □

Find the derivative of $\log_b(x)$ *using the above exercise.*

§6.5 Exponential Growth and Decay Models

So far in Differential and Integral Calculus, we have seen that a function's derivative and integral relates to information about its change. For example, the derivative tells us the rate at which the function is changing. On the other hand, the integral, as a consequence of the Fundamental Theorem of Calculus, helps us determine the total change of a function over an interval from the function's rate of change. This might lead you to believe that many real-world phenomena that change over time, can be modeled by mathematical equations. In fact, as long as a system changes according to a fixed rule, we should be able to describe the process using functions, and predict future behavior using calculus!

This is the study of Differential Equations and Mathematical Modeling, which we will briefly touch upon in this section.

6.5.1 Growth and Decay Models

Let's start with a population model. Suppose we have a Petri dish full of bacteria and we wish to examine the population of these cells as a function of time, say $P(t)$. If we have an initial population of 100 cells and every 1 hour each cell divides into 3 more cells, then we can determine an expression for $P(t)$ as follows:

$$\begin{aligned} &P(0) = 100 \\ &P(1) = 3 \cdot 100 = 300 \\ &P(2) = 3 \cdot 300 = 3^2 \cdot 100 = 900 \end{aligned}$$

Assuming nothing inhibits the growth of the cells and the pattern continues, we can see that the values of $P(t)$ can be written as $P(t) =$ ________________________________, an exponential function.

Several real-life phenomena, such as population growth, radioactive decay, compounding interest, etc., can be modeled via exponential functions.

Definition 6.5.55

Systems that exhibit exponential growth or decay can be modeled according to the mathematical equation

$$y(t) = y_0 e^{kt}$$

where y_0 represents the initial state of the system (i.e. the value $y(0)$) and k is a constant, called the growth constant or decay constant, depending on whether it is positive or negative.

Usually, the initial value of y at $t = 0$, i.e. $y_0 = y(0)$ is provided to us in the model and we have to determine k experimentally or algebraically. Let's take a look at an example.

Example 6.5.56

Let $y(t)$ represent the population of human beings on Earth (in billions) since 2000 (i.e. so t is in years, with $t = 0$ corresponding to the year 2000). We can model the population growth using an exponential model, the reasons are explained in the next subsection.

The population was recorded to be approximately 6 billion in 2000. Hence, we could write $y(0) = y_0 = 6$ and model the population via the function

$$y(t) = y_0 e^{kt} = 6e^{kt}$$

(a) Given that the world population was around 7.4 billion in 2015, determine the growth constant k for our population model. Do not simplify your answer.

(b) Once you have k, use your completed model to calculate $y(20)$ (i.e., the world population in 2020). How close is this to the recorded population in 2020? Use the internet to check!

(c) Predict the world population for 2050 using the model. How close is this to other predictions? The UN has some world population prediction numbers, so you could try doing an internet search to see how things compare!

■ Question 109.

Global temperatures have been rising, on average, for more than a century, sparking concern that the polar ice will melt and sea levels will rise. With t in years since 1880, we can give an approximate model for the average global temperature in Celsius as a function of t as follows

$$h(t) = 13.63e^{kt}$$

(a) The average global temperature in 2020 is 13.86° C. Find the value of k using DESMOS.

(b) Florida will face chronic floods when the average global temperature becomes 14° C. Assuming the current trend of global warming continues, estimate the year when this will happen.

6.5.2 A Differential Equation

Consider the exponential growth function $y(t)$ given by $y(t) = y_0 e^{kt}$. Let's examine the derivative of y:

$$\begin{aligned} y'(t) &= k \cdot y_0 e^{kt} \\ &= k \cdot y(t) \end{aligned}$$

So a feature of the exponential functions is that the derivative is proportional to the original function. It turns out (see a proof in Canvas) that this is in fact *unique* to exponential functions and serves as a defining characteristic. For now, note that this explains why we usually model population growth using an exponential model, as one can argue the rate of change of population over time should be proportional to the current population at the time.

We conclude that a function modeling exponential growth or decay is a solution to an equation of the form:

$$y' = ky, \qquad \text{where } k \text{ is a constant.}$$

Equations of the above form are called "Differential Equations". We will not go into further details at this point, there are further Math courses that will cover more examples of DEs.

6.5.3 Doubling Time and Half-life

Question 110. □

Suppose we have an exponential process that starts at time $t = 0$ with y_0 and changes over time according to $y(t) = y_0 e^{kt}$. Assuming $k > 0$, i.e. $y(t)$ is increasing, how long will it take for $y(t)$ to be double the initial value?

Solution. Let T be the amount of time it takes for the quantity $y(t)$ to double. That is,

$$y(\mathrm{T}) = 2y(0)$$

Since $y(\mathrm{T}) = y(0)e^{k\mathrm{T}}$, we can rewrite above equation as ____.

Then, we solve for T. (Finish the work)

■

Question 111. □

Check that with this value of T, we have $y(t + \mathrm{T}) = 2y(t)$ for any t. That means the quantity y doubles after every T time interval.

The value of T you found above is called the **doubling time** of the quantity $y(t)$. Note that the doubling time is related to the growth constant k. So given the doubling time for a specific quantity, you can determine the growth constant k. Try to do that in the problem below.

■ Question 112. □

Suppose a colony of cells doubles in size in 3 *hours. How long will it take the colony to reach* 10 *times its initial population? (Note: just use* y_0 *for the initial population, as you don't need to know what it is exactly to solve this problem).*

The same expression you found for doubling time is known as the **half-life** for a quantity if it is modeled by exponential *decay*. In other words, if a quantity decays exponentially, the half-life, denoted $T_{1/2}$ is the amount of time it takes the quantity to be reduced by half.

■ Question 113. □

Show that $T_{1/2} = \dfrac{-\ln 2}{k}$.

6.5.4 Another Mathematical Model - Newton's Law of Cooling

Newton's law of cooling is a differential equation that predicts the cooling of a warm object when placed in a cold environment. We all are aware that hot things will cool down to room temperature, and if you have hot soup in a ceramic bowl versus a glass bowl, your soup might cool down faster in one over the other.

Theorem 6.5.57: Newton's Law of Cooling

The rate at which an object cools is proportional to the temperature difference between the object and its surroundings.

Let's see if we can translate the verbal description to a differential equation. Consider, for example, you take a cake out of an oven and place it in a room where the temperature is 20° C. Let $T(t)$ represent the temperature of the object, in ° C, after t minutes.

Then

$$\begin{aligned} \frac{dT}{dt} &= \text{rate of change of temperature (° C per minute)} \\ 20 - T &= \text{temperature difference between room and cooling object} \end{aligned}$$

and the modeling Differential Equation is

$$\frac{dT}{dt} = k(20 - T)$$

where $k > 0$. Note that usually $20 - T$ would be negative, which agrees with the fact that T decreases over time.

Question 114. □

Check that all functions of the form

$$T(t) = 20 + (T_0 - 20)e^{-kt}$$

satisfy the differential equation.

Question 115. □

Our room temperature is about 30° C. Experiments have shown that a ceramic coffee mug has a cooling constant of $k = 0.12\,\frac{1}{min}$.

(a) *If my coffee begins class the class at about 80° C, then how cold is it after 50 minutes?*

(b) *Suppose coffee is only enjoyable for temperatures greater than 38° C. How long do I have to drink this mug of coffee before it is no longer enjoyable?*

§6.6 Inverse Trignonometric Functions

For the last thing this quarter, we will introduce the last class of transcendental functions we care about - the inverse trigonometric functions.

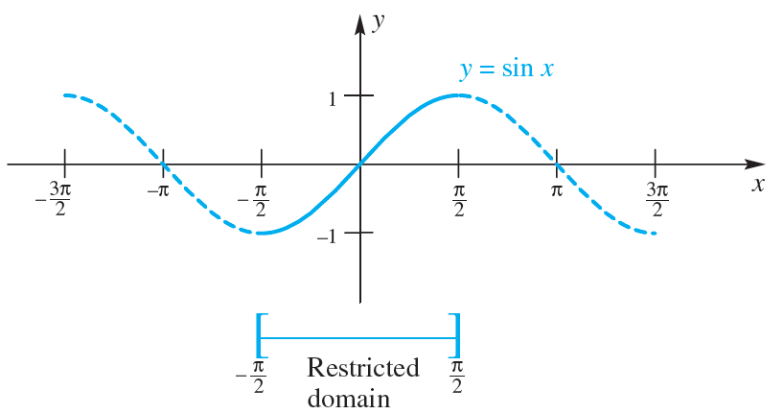
Question 116.

Explain why all six trigonometric functions are terrible candidates for having an inverse.

Nonetheless, we are going to introduce a notion of "inverse" for them by using a procedure called restricting the domain, which was discussed briefly in a previous section.

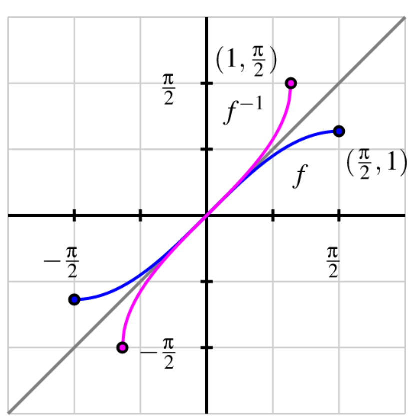
6.6.1 Construction of arcsin, arccos, and arctan

Let's start with the sine function. We are going to restrict the function to the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



Because no output of the sine function is repeated on this interval, the function is one-to-one and thus has an inverse. Thus, the function $f(x) = \sin x$ with domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and range $[-1, 1]$ has an inverse function f^{-1} such that

$$f^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



We call f^{-1} the **arcsine** function and write $f^{-1}(y) = \arcsin(y)$. Thus

Theorem 6.6.58

$$\arcsin(\sin(x)) = x \textit{ for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

and

$$\sin(\arcsin(x)) = x \textit{ for } -1 \le x \le 1$$

Example 6.6.59

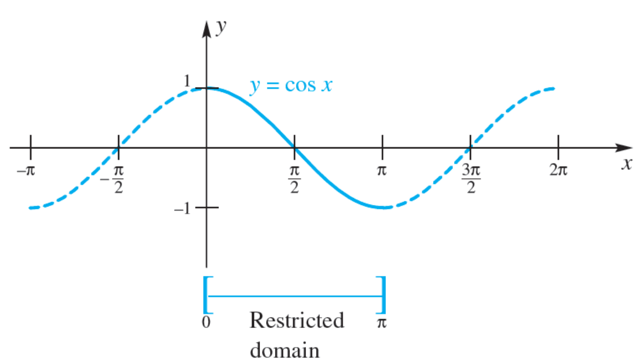
Remember that "The arcsine of y" means "the **angle** whose sine is y." but the angle must be chosen between $[-\pi/2, \pi/2]$. For example, we know that

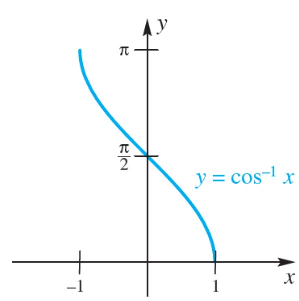
$$\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$$

However, $\arcsin\left(\frac{1}{2}\right)$ is not equal to $\frac{5\pi}{6}$. Instead, $\arcsin\left(\frac{1}{2}\right) =$ ______________________________________.

Warning: The textbook uses the notation $\sin^{-1}$ to denote arcsin. This is a common notation, but I am not fond of it. Since $\sin(x)$ is not invertible with its default domain, arcsine is not exactly the inverse of sine, but rather the inverse of restricted sine. So I think it deserves a different name/notation! The notation arcsin is pretty common itself and many authors do use it.

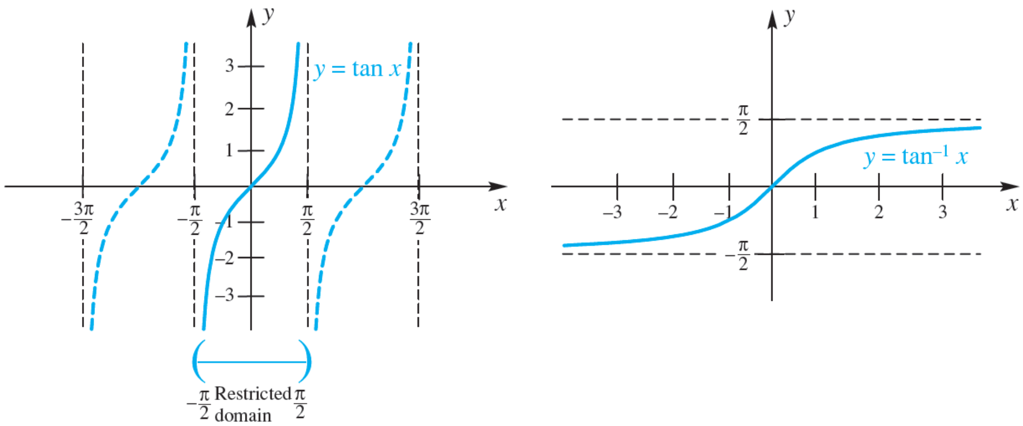
The functions arccos and arctan are constructed similarly.





Question 117.

The domain of $\arccos(x)$ *is* ______________________________ *and the range is* ______________________________.



Question 118.

The domain of arctan(x) *is* ______________ *and the range is* ______________.

6.6.2 Derivative of arcsin

Next, we are going to find the derivative of the arcsine function. Using the derivative of inverse formula we get

$$\frac{\mathrm{d}}{\mathrm{d}x}[\arcsin(x)] = \frac{1}{\qquad\qquad\qquad} \qquad \textit{[fill in the denomminator]}$$

Now suppose arcsin(x) is the angle θ. Then by definition, $\sin(\theta) = x$. On the other hand, the denominator of $g'(x)$ is equivalent to $\cos(\theta)$.

Using the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, we can solve for $\cos\theta$ in terms of x.

$\cos(\theta) =$ ______________________________

Now substitute this expression into our formula of $g'(x)$ to find the following result.

Theorem 6.6.60

For all real numbers x such that $-1 < x < 1$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}}$$

Note: An interesting fact to note here is that the theorem restricts x to the interior of the domain of arcsin. Can you see why?

6.6.3 Derivative of arctan

The derivative of arctan x is given by

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}[\arctan(x)] = \frac{1}{1+x^2}}$$

Question 119.

The following prompts will lead you to develop the derivative of the inverse tangent function yourself!

(a) *Let* $r(x) = \arctan(x)$. *Use the relationship between the arctangent and tangent functions to rewrite this equation using only the tangent function.*

(b) *Differentiate both sides of the equation you found in (a). Solve the resulting equation for* $r'(x)$, *writing* $r'(x)$ *as simply as possible in terms of a trigonometric function evaluated at* $r(x)$.

(c) *Recall that* $r(x) = \arctan(x)$. *Update your expression for* $r'(x)$ *so that it only involves trigonometric functions and the independent variable* x.

(d) *Introduce an angle* θ *so that* $\theta = \arctan(x)$. *What is* $\tan(\theta)$. *Use the trigonometric identity* $\sec^2\theta - \tan^2\theta = 1$ *to solve for* $\sec^2\theta$.

(e) *Use the results of your work above to find an expression involving only 1 and* x *for* $r'(x)$.

6.6.4 Practice Problems

Let's end with two more formulas for the sake of completion.

$\boxed{\dfrac{d}{dx}[\arccos(x)] = \dfrac{-1}{1+x^2}}$ for $-1 < x < 1$, and $\boxed{\dfrac{d}{dx}[\text{arcsec}(x)] = \dfrac{1}{|x|\sqrt{1-x^2}}}$ for $|x| > 1$

Question 120.

Compute the derivative of the following functions.

(a) $f(x) = x^3 \arctan(x) + e^x \ln(x)$

(b) $p(t) = 2^{t \arcsin(t)}$

(c) $h(z) = (\arcsin(5z) + \arctan(4-z))^{27}$

(d) $g(w) = \arctan\left(\dfrac{\ln(w)}{1+w^2}\right)$