Theory of Differential Calculus

Application/Extension Problem 1

Fall 2021 **Subhadip Chowdhury** **Math 115**

§A. What this AEP is about

In mathematics and computing, a **root-finding algorithm** is an algorithm for finding roots of continuous functions. A **root** of a function $f(x)$ is a number x such that $f(x) = 0$. In general, if the function $f(x)$ is not overly simple, the roots of $f(x)$ cannot be computed exactly, nor expressed in closed form. So we might ask, how does your calculator (or a computer) find roots of functions or solutions of equations in general?

Below we will describe two such algorithms your calculator might use to provide **approximate answer** when it solves for a root. Note that most root-finding algorithms apriori assumes that a root exists. The algorithm itself does not guarantee that it will find any or all the roots. In particular, if such an algorithm does not find any root, that does not necessarily mean that a root does not exist!

§B. Prerequisites and tech requirements

Before starting this AEP, you should

- know what a continuous function looks like,
- know what it means to have a root of a function,
- know the conclusion of Intermediate Value Theorem and how it is applied,
- understand the interpretation of derivative as slope of tangent,
- know how to find equation of the tangent line to a given curve $y = f(x)$ at a point $(a, f(a))$.
- be comfortable using `DESMOS` to evaluate a function repeatedly at different values.

Please come talk to me if you are not comfortable with any of these topics.

§C. Submission Instructions and Grading criteria

To submit this AEP:

- Create a handwritten or typed document with your solution. Convert the document/picture of the document to JPG or PDF format using an app or software. Please do not submit MS Word documents.
- Upload the file to the appropriate AEP assignment link on Moodle.

This AEP will be graded based on the EMPX rubric. You can check your Moodle gradebook to see your grade and and view feedback left by the professor. These appear as text annotations on your PDF submission or as general comments next to the grade. Grades of **E** or **M** may not have much feedback. Grades of **P** or **X** always have feedback, so please look carefully for this.

In order to earn an **E** or **M**, your submission must:

- show all of your work neatly and in a ordered manner.
- back up any claim you make with sufficient proof.

- explain your reasoning in a way that could be understood by a classmate who understands the mathematical concepts but has no familiarity with the particular problem being solved.

In short, readers of your work should not have to fill in any details or guess your thought process.

> **Important Note.** There will be one chance to revise and resubmit this AEP within a week from when it is first graded and returned to you.

§D. Learning Targets covered

Besides regular PE credits, getting an **E** or **M** in this AEP will also earn you an 'S' in learning target L3 (IVT).

§E. AEP Task

Consider the polynomial $P(x) = x^3 + 3x - 1$. Your goal is to find a zero of this function: i.e., a number a so that $P(a) = 0$. Although there is an algebraic technique for finding a zero of a cubic polynomial, we are going to approximate a zero.

GOAL: We want to manually find an approximate value for a within 10^{-2} of the actual value of a.

Feel free to use DESMOS or a calculator throughout this AEP to calculate the value of $P(x)$ as needed. Note however that this AEP is describing how a computer or calculator finds a root of polynomial. So if you just copy the root from DESMOS, that defeats the purpose of the exercise.

First Algorithm: Bisection Method

Question 1.

To begin, show that the equation $P(x) = 0$ has at least one solution in the interval $[-1, 1]$. You must give a good justification that such a solution exists.

Our goal is find the location of the root. One way to approximate the root is to **bisect** the interval $[-1, 1]$ (i.e. break it into two equal halves). Then we can aks which one of those halves is the root in.

Question 2.

(a) *Determine whether $P(x) = 0$ has a solution in $[-1, 0]$ or $[0, 1]$ (as you did in the previous step), and*

(b) *then repeat the process with the new interval containing a solution (i.e. bisect it again).*

Note that each time you bisect the interval, you get an interval half the length of the previous one.

Question 3.

How many times do you need to repeat the bisection process to have a sufficiently accurate (see the goal) answer? Don't just give a number here; write down and show the steps.

What's your final approximation? Your answer should look like a fraction of the form $\frac{a}{b}$, and not like a decimal expansion.

Second Algorithm: Using Derivative

Calculus gives us another way to perform the search. We are going to use the interpretation of derivative as the slope of tangent at a point.

Question 4.

(a) *Let $y = f(x)$ be a function of x. What is the slope of of the line L tangent to the graph of f at $(x_1, f(x_1))$?*

(b) *What is the equation of the line L?*

(c) *Suppose the tangent line L intersects the x–axis at x_2. Find x_2 in terms of x_1 and f. See fig. 1 for a picture.*

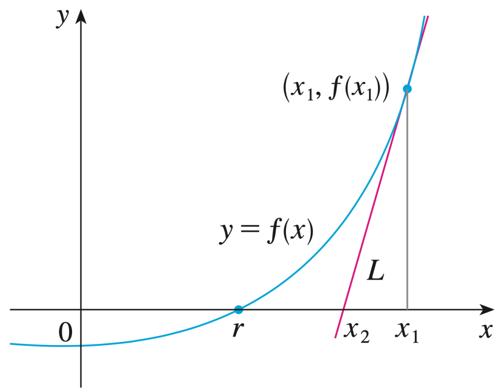

Figure 1

The main idea behind the second algorithm is that this new x-intercept will typically be a better approximation to the function's root than the original guess, and the method can be iterated.

Let's return to the original problem of $P(x) = x^3 + 3x - 1$. We will begin with one of the end points of the original interval; this is your original guess a.

Let $x_1 = 1$.

Question 5.

(a) *Apply the operation in the previous step, obtaining the x-intercept of the tangent line at $(a, P(a))$ as a new guess, which hopefully is a better approximation to a solution of the equation $x^3 + 3x - 1 = 0$ than the end point you started with.*

What is x_2?

(b) *Is this answer within the desired margin of error (i.e. within 10^{-2}) from the answer you obtained using the bisection method?*

(c) *When you get an answer within the margin of error you may stop. Otherwise, repeat the operation, this time beginning with your latest guess x_2 as the new a and find $x_3, x_4, \ldots$ so on.*

Comparison

Question 6.

Compare the two techniques for finding a solution.

(a) *Which is easier to understand in your opinion? Why?*

(b) *Which is faster; that is, which leads to an answer within the desired degree of accuracy in the fewest number of iterations?*