MATH 111 - Calculus and Analytic Geometry I

Lecture 36 Worksheet

Fall 2020 **Subhadip Chowdhury** **Nov 9**

Title: The Definite Integral

Summary: We will learn what the definite integral measures exactly, and some of the key properties of the definite integral.

Related Reading: Chapter 5.2 from the textbook.

§A. Definite Integral

If we want to find the area under a the graph of a function $f(x)$ on an interval $[a,b]$ using an infinite number of rectangles with the base length of the rectangles getting closer and closer to zero, we use Riemann sums. Last lecture we saw that as the number of rectangles got larger and larger, the values of the Left-endpoint Approximation $\mathbf{L}_n$, the right-endpoint Approximation $\mathbf{R}_n$, the midpoint approximation $\mathbf{M}_n$, and in general Riemann sums $\mathbf{S}_n$ all grew closer and closer to the same value.

$$\lim_{n\to\infty} \mathbf{L}_n = \lim_{n\to\infty} \mathbf{R}_n = \lim_{n\to\infty} \mathbf{M}_n = \lim_{n\to\infty} \mathbf{S}_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

In particular, the limit is the exact area under the curve $f(x)$ over the interval $[a,b]$. Today we are going to rephrase this quantity using the language of integration.

Definition A.1

The definite integral of a continuous function f on the interval $[a,b]$ is the real number given by

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

where $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$ (for $i = 0,\ldots,n$), and x_i^* satisfies $x_{i-1} \le x_i^* \le x_i$ (for $i = 1,\ldots,n$).

Since the definite integral is defined as a limit, it may not always be defined for a function $f(x)$. If the above limit exists, we say that the function $f(x)$ is **integrable**. In all of the examples we have looked at so far, $f(x)$ has been a continuous function. It is a nontrivial theorem to state that **continuous functions are integrable**, and we will use this fact without proof. If you are curious, you should try to come up with a function that is **not** integrable!

The first way we are going to "evaluate such integrals" is to look for common geometric shapes that we already know how to determine the area of (circles, triangles, rectangles, or a combination of those). Try some! To do so, sketch the given curve on its interval, then a find the area between the graph and the x-axis using geometry formulas.

Question 1.

Example Problems

1. $\displaystyle\int_{-4}^{6} 6\,dx$

2. $\displaystyle\int_{0}^{8} \frac{x}{4}\,dx$

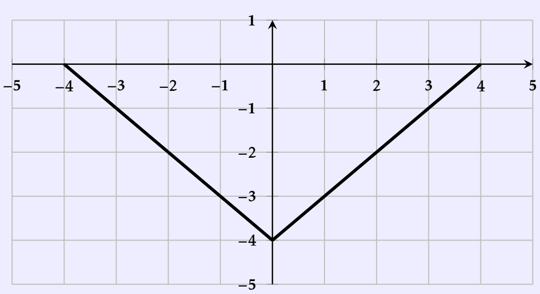
3. $\displaystyle\int_{-4}^{4} (4-|x|)\,dx$

4. $\displaystyle\int_{-3}^{3} \sqrt{9-x^2}\,dx$

Our definition of the definite integral allows for a few oddities. Consider the following example:

Example A.1

Consider $f(x) = |x| - 4$ on the interval $[-4, 4]$.



Try sketching the right-endpoint Riemann sum $\mathbf{R_8}$ on the graph of $f(x)$.

If we use our definition verbatim, we get the following calculation for $\mathbf{R}_8$:

$$\mathbf{R}_8 = (-1) + (-2) + (-3) + (-4) + (-3) + (-2) + (-1) + 0 = -16.$$

Based off this calculation of $\mathbf{R}_8$, what would you say is $\int_{-4}^{4} (|x| - 4)\, dx$?

Question 2.

Keeping in mind the last example, try to calculate the following integrals by thinking about the area formed below the x-axis and above the x-axis.

1. $\int_{-3}^{3} 2x\, dx$

2. $\int_{0}^{6} (x - 2)\, dx$

3. $\int_{-2}^{8} (3 - |x - 3|)\, dx$

4. $\int_{-1}^{1} x^3\, dx$

5. $\int_{0}^{2\pi} \sin(x)\, dx$

§B. **Properties of Definite Integrals**

Because a definite integral is simply a limit of sums, and both sums and limits behave nicely with addition and constant multiplication, we have familiar **linearity rules** for definite integrals. For example, we could write something like the following:

$$\int_a^b [f(x) + 2g(x)]\,dx = \int_a^b f(x)\,dx + 2\int_a^b g(x)\,dx$$

Question 3. □

Using what you know about area under a curve, try these! Sketches may help you "see" what is going on in each problem.

1. Given $\int_2^4 x^3\,dx = 60$ $\quad \int_2^4 x\,dx = 6$ $\quad \int_2^4 1\,dx = 2$ evaluate:

   (a) $\int_2^4 (10 + x)\,dx$

   (b) $\int_2^4 \left(10 + 4x - 3x^3\right)\,dx$

   (c) $\int_2^2 x^3\,dx$

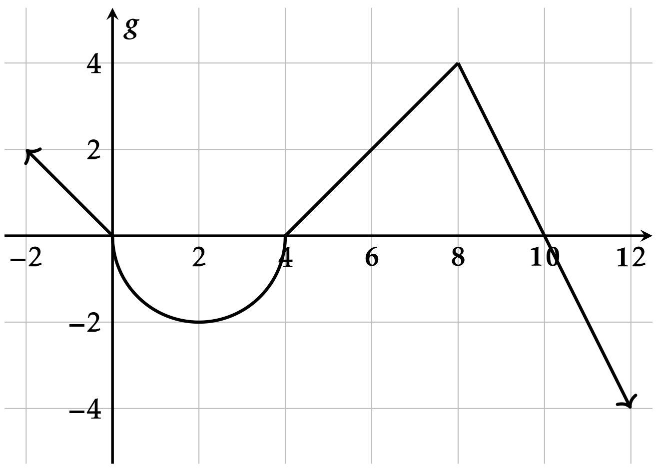
2. Given $\int_0^3 f(x)\,dx = 4$ and $\int_3^6 f(x)\,dx = -1$ evaluate:

   (a) $\int_0^6 f(x)\,dx$

(b) $\int_3^6 -5f(x)\,dx$

(c) $\int_6^3 f(x)\,dx$ (**Think about how we form Riemann sums**)

3. Below is a graph of the function $y = g(t)$ which is a piecewise function composed of line segments and a semi-circle.



(a) $\mathbf{G}(x) = \int_4^x g(t)\,dt$

$\mathbf{G}(4) =$

$\mathbf{G}(12) =$

$\mathbf{G}(10) =$

$\mathbf{G}(0) =$

■ Question 4.

□

Using what you know about area under a curve, explain why

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

■ Question 5.

Book problem 5.2.124

Suppose that $\mathbf{A} = \int_0^{\pi/2} \sin^2 t\,dt$ and $\mathbf{B} = \int_0^{\pi/2} \cos^2 t\,dt$. Show that $\mathbf{A} + \mathbf{B} = \frac{\pi}{2}$ and $\mathbf{A} = \mathbf{B}$. What is the value of $\mathbf{A}$?