Calculus & Analytical Geometry II

Lecture 24 Worksheet

Spring 2021 | **Subhadip Chowdhury** | **Math 112**

§A. Introduction

First recall the Monotone Convergence Theorem that we discussed last time.

Theorem A.1: Monotone Convergence Theorem

If a sequence $\{a_n\}$ is monotone and bounded, then it converges.

So we can see that there are two ways a sequence can diverge:

- Either the sequence is unbounded. For example, $a_n = 2^n$.
- Or the sequence is bounded, but oscillating. For example, $a_n = (-1)^n$.

We can use similar ideas to show that a series is divergent.

Example A.2

Consider the series $\sum_{i=1}^{\infty} \frac{1}{\sqrt{i}}$. We can write

$$s_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \ldots + \frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}} = \sqrt{n}$$

So the sequence s_n is unbounded, and $\lim_{n\to\infty} s_n$ does not exist.

Example A.3

Consider the harmonic series $\sum_{i=1}^{\infty} \frac{1}{i}$. We are going to show that it is divergent. Observe that we can write

$$
\begin{aligned}
s_2 &= 1 + \frac{1}{2} \\
s_4 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} \\
s_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2} \\
s_{16} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \ldots + \frac{1}{8}\right) + \left(\frac{1}{9} + \ldots + \frac{1}{16}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right) \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{4}{2}
\end{aligned}
$$

So in general, $s_{2^n} > 1 + \frac{n}{2}$. This shows that $s_n \to \infty$, because it's unbounded! Therefore the harmonic series diverges.

§B. Convergence/Divergence Tests for Series

We have the Monotone Convergence Theorem that gives us a nice criterion for knowing when a sequence converges. Next we will investigate if there are similar tests to check when a series converges or diverges.

Let's start with the following claim.

Theorem B.4

If a series $\sum_{i=1}^{\infty} a_i$ *is convergent, then* $\lim_{a_i} = 0$.

Question 1.

□

Let's prove the theorem above. If $\sum_{i=1}^{\infty} a_i$ *is convergent, then* $\lim_{n\to\infty} s_n = L$ *exists. Now write*

$$a_n = s_n - s_{n-1}$$

Does that make sense? What can you say about $\lim_{n\to\infty} a_n$*?*

Warning: This theorem does not say that if $\lim_{a_i} = 0$, then $\sum_{i=1}^{\infty} a_i$ is convergent. Harmonic series is a clear counterexample.

For example,

If I'm not in California, then I'm not in Los Angeles. $\longleftarrow$ True

If I'm in California, then I'm in Los Angeles. $\longleftarrow$ False

So we have the first test for convergence.

Divergence Test

Theorem B.5: Divergence Test

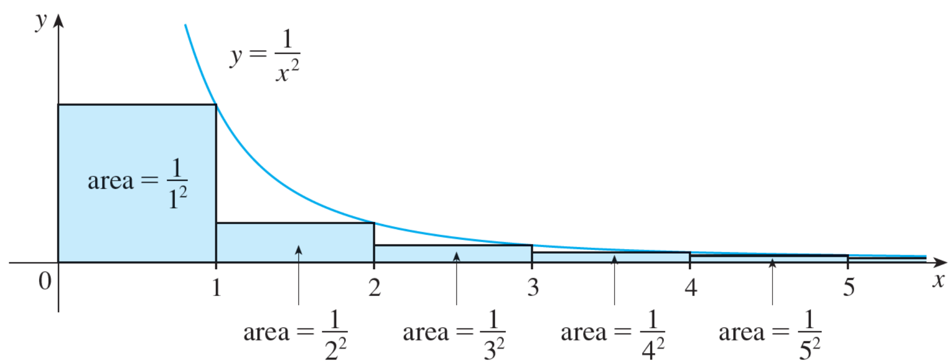
If $\lim_{n\to\infty} a_n \neq 0$, *then* $\sum_{i=1}^{\infty} a_i$ *is divergent.*

Question 2.

Show that the series $\sum_{i=1}^{\infty} \frac{i^2}{3i^2-6i+2}$ *is divergent.*

Integral Test for Positive Series

Consider $\sum_{i=1}^{\infty} \frac{1}{i^2}$. There is a relationship between this series and the improper integral $\int_1^{\infty} \frac{1}{x^2}\,\mathrm{d}x$.



$$s_n = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} < 1 + \int_1^{n+1} \frac{1}{x^2}\,\mathrm{d}x$$

Because we know $\lim_{n\to\infty} \int_1^{n+1} \frac{1}{x^2}\,\mathrm{d}x = \int_1^{\infty} \frac{1}{x^2}\,\mathrm{d}x < \infty$, we have that the sequence of partial sums for $\sum_{i=1}^{\infty} \frac{1}{i^2}$ is bounded. Hence, this series converges.

Observe that we can also use the upper Riemann sum to set up an upper bound for the integral as follows:

$$\int_1^{n+1} \frac{1}{x^2}\,\mathrm{d}x < 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2}$$

which is not very useful for comparison here, but may be useful in case we want to test for divergence.

Combining these observations, we get the following test for convergence.

Theorem B.6: Integral Test

Suppose f is a

- *continuous*
- *positive*
- *decreasing*

function on $[1,\infty)$ and let $a_i = f(i)$ for $i = 1,2,3,\ldots$.

Then the series $\sum_{i=1}^{\infty} a_i$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,\mathrm{d}x$ is convergent.

Question 3.

p**-series**

Describe the convergence and divergence of the p−series for different values of the constant p.

$$\sum_{i=1}^{\infty} \frac{1}{i^p}$$

Question 4.

□

First make sure that the integral test is applicable for the following series. Then use it to test the convergence of the following series:

(a) $\displaystyle\sum_{i=1}^{\infty} \frac{1}{2i+3}$

(b) $\displaystyle\sum_{i=1}^{\infty} \frac{e^{-i}}{1+e^{-2i}}$

(c) $\displaystyle\sum_{i=1}^{\infty} \frac{\ln(i)}{i^2}$