Calculus & Analytical Geometry II

Lecture 38-40 Worksheet

Spring 2021 | **Subhadip Chowdhury** | **Math 112**

§A. Recap on Power Series

Here's a summary of some of the Power Series we derived in the last worksheet.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots \qquad \text{for } |x| < 1$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - \dots \qquad \text{for } |x| < 1$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \qquad \text{for } -1 < x \leq 1$$

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \qquad \text{for } -1 \leq x \leq 1$$

One important thing to recall is that the left-hand function and the right-hand power series are only equal on the interval of convergence. Here is a `DESMOS` page to showcase this fact:

https://www.desmos.com/calculator/cwt347lgkj

In this worksheet, we investigate the more general problems: Which functions have power series representations? How can we find such representations?

§B. Taylor Series

Our first main goal is as follows:

> **Main Goal:** Suppose a function $f(x)$ can be represented by a power series
>
> $$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots \qquad \text{on the interval } |x-a| < R$$
>
> Then find the values of c_n for all n.

Step 1. Suppose $F(x)$ denotes the power series

$$F(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$$

So we are given that the function $f(x)$ and $F(x)$ agree with each other on the interval $(a-R, a+R)$. In particular, this means $f(a) = F(a)$.

Substitute $x = a$ in the formula for $F(x)$. What does this tell you about c_0?

$c_0 =$ ____.

Step 2. Again let's start with the equation $f(x) = F(x)$. Take the derivative of both sides with respect to x.

$f'(x) = F'(x) =$ ______________________________ .

Now substitute $x = a$ in both sides of above equation. What does this tell you about c_1?

$c_1 =$ ______________________________ .

Step 3. Take the derivative of both sides with respect to x one more time.

$f''(x) = F''(x) =$ ______________________________ .

Again substitute $x = a$ in both sides of above equation. What does this tell you about c_2?

$c_2 =$ ______________________________ .

Step 4. If you repeat the process,

$c_3 =$ ______________________________ .

Step 5. Can you identify a pattern for the coefficients of the power series $F(x)$? Write down the general formula for c_n.

> **Notation:** The nth derivative of $f(x)$ is denoted as $f^{(n)}(x)$.

$c_n =$ ______________________________ .

The above formula remains valid even for $n = 0$ if we adopt the conventions that $0! = 1$ and $f^{(0)}(x) = f(x)$. Thus we have proved the following theorem.

Theorem B.1

If $f(x)$ has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad \text{for } |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Definition B.2: Taylor Series of f at a

The **Taylor series** of the function f at a (or about a or centered at a) is given by

$$\begin{aligned} P(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

The Nth partial sum (i.e. sum of the first N terms) of the series, $\mathrm{P}_{\mathrm{N}}(x)$, is called the Nth degree Taylor Polynomial of $f(x)$, that is,

$$\mathrm{P}_{\mathrm{N}}(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Warning: The Taylor series of a function $f(x)$ is not necessarily equal to $f(x)$ for all x. Theorem 1 tells us that if $f(x)$ has a power series expansion, then the expansion is the Taylor series. However, there do exist functions that are **not equal to** their Taylor series.

We will see a theorem later that tells us under what conditions the Taylor series of a function actually converges to the original function.

The case of when the Taylor series of a function is centered at $a = 0$ arises frequently enough that it has its own special name. the Taylor series for $f(x)$ centered at $x = 0$ is called the **Maclaurin Series** for $f(x)$ and has the form

$$\mathrm{P}(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Example B.3

Let's build the Taylor series for $f(x) = e^x$ centered at $x = 0$.

Step 1. $f(x) =$ ______________________ $\Longrightarrow f(0) =$ ______________________

Step 2. $f'(x) =$ ______________________ $\Longrightarrow f'(0) =$ ______________________

Step 3. $f''(x) =$ ______________________ $\Longrightarrow f''(0) =$ ______________________

Step 4. $f^{(n)}(x) =$ ______________________ $\Longrightarrow f^{(n)}(0) =$ ______________________

Therefor, the Taylor series of e^x centered at $x = 0$ is given by

$\mathrm{P}(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots =$ ______________________________

Example B.4

Let's build the Taylor series for $g(x) = \sin(x)$ centered at $x = 0$.

Step 1. $g(x) =$ ______________________ $\Longrightarrow g(0) =$ ______________________

Step 2. $g'(x) =$ ______________________ $\Longrightarrow g'(0) =$ ______________________

Step 3. $g''(x) =$ ______________________ $\Longrightarrow g''(0) =$ ______________________

Step 4. $g^{(n)}(x) =$ ______________________ $\Longrightarrow g^{(n)}(0) =$ ______________________

Therefor, the Taylor series of $g(x) = \sin(x)$ centered at $x = 0$ is given by

$$P(x) = g(0) + \frac{g'(0)}{1!}x + \frac{g''(0)}{2!}x^2 + \frac{g'''(0)}{3!}x^3 + \cdots = \underline{\hspace{10cm}}$$

Question 1.

☐

Determine the interval of convergence for the above two power series.

Question 2.

☐

Find the Taylor series of $h(x) = \cos x$, centered at $x = 0$ using theorem 1. Compare this to the derivative of the power series for $\sin(x)$.

Here's a `DESMOS` page with the three series:

https://www.desmos.com/calculator/oxdwh8kcds

Question 3.

☐

Find the Taylor series for $f(x) = \ln x$ centered at $a = 1$.

Question 4.

☐

Find the Maclaurin series of $f(x) = (1+x)^k$, where k is some real number.

Note: This series is called the **binomial series**. If k is a non-negative integer, then the terms are eventually 0 and so the series becomes finite.

§C. When is a function equal to the sum of its Taylor series?

Let's start with an example. First build the Taylor series for $f(x) = \dfrac{1}{1-x}$ centered at $x = 0$.

Step 1. $f(x) =$ ______________________ $\Longrightarrow f(0) =$ ______________________

Step 2. $f'(x) =$ ______________________ $\Longrightarrow f'(0) =$ ______________________

Step 3. $f''(x) =$ ______________________ $\Longrightarrow f''(0) =$ ______________________

Step 4. $f^{(n)}(x) =$ ______________________ $\Longrightarrow f^{(n)}(0) =$ ______________________

We immediately notice that the Taylor series is in fact the geometric series expansion we have seen before. We also know that this series only converges to $\dfrac{1}{1-x}$ when $|x| < 1$.

In general, as with any convergent series, The Taylor series $P(x)$ of a function $f(x)$ will converge to $f(x)$ if and only the sequence of partial sums, i.e. the "Taylor Polynomials" converge to $f(x)$. In other words, if $f(x) = P(x)$, then

$$\lim_{n\to\infty} P_N(x) = f(x)$$

Definition C.5

The Nth **remainder** of the Taylor series $R_N(x)$ is defined as

$$R_N(x) = f(x) - P_N(x)$$

In other words, $R_N(x)$ is the sum of the Taylor series starting at the $(n+1)$th term and it tells us how well the partial sums $P_N(x)$ approximate the function $f(x)$.

Using this new notation,

$$\lim_{n\to\infty} P_n(x) = f(x) \text{ is equivalent to } \lim_{n\to\infty} R_n(x) = 0.$$

So we get the following theorem.

Theorem C.6: Taylor's Theorem

Suppose $f(x)$ has derivatives of all orders on an interval $I = (a-R, a+R)$, for $R > 0$. Then the Taylor series

$$P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

converges to $f(x)$ for all x in I if and only if

$$\lim_{N\to\infty} R_N(x) = 0$$

for all x in I.

So the proof of when a Taylor series converges to the function depend on calculating the limit of $R_n(x)$. We will not focus on that particular problem in this worksheet. However, we will use the idea of remainder to explain we can approximate a function using its Taylor polynomials.

Example C.7

Consider the following problem. We wish to approximate $\sin(1)$ numerically using the Taylor series for $\sin(x)$ centered at $x = 0$. Use the Taylor series from example 4.

Step 1. $P_1(x) =$ ______________________ $\implies P_1(1) =$ ______________________

Step 2. $P_3(x) =$ ______________________ $\implies P_3(1) =$ ______________________

Step 3. $P_5(x) =$ ______________________ $\implies P_5(1) =$ ______________________

There is definitely still some error in this approximation. In fact, the error when you you approximate $\sin(1)$ using $P_N(x)$ is exactly equal to the remainder $R_N(x)$. So if we can somehow give a numerical upper bound for $R_N(x)$, we will have a good idea of how good our approximation is.

Fortunately, we have a theorem that does the job for us. We will not prove it here, but you can check it out in your textbook. It actually follows from the Mean value Theorem that there exists a number z between a and x such that

$$R_N(x) = \frac{f^{(N+1)}(z)}{(N+1)!}(x-a)^{N+1}.$$

Going back to example 7, we can now find the maximum possible error for using P_5 to approximate $\sin(1)$. We look at

$$R_5(1) = \left|\frac{f^{(6)}(z)(1-0)^6}{6!}\right| = \left|\frac{\cos(z)(1)^6}{6!}\right|$$

where z is some number between 0 and 1. But we know that $|\cos z| \leq$ ______________ . Hence we have,

$$R_5(1) = \left|\frac{\cos(z)(1)^6}{6!}\right| \leq \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

In general, we get the following theorem.

Theorem C.8: Taylor's Inequality

If there exists a real number $M > 0$ such that $\left|f^{(N+1)}(x)\right| \leq M$ for $|x-a| \leq d$, then remainder $R_N(x)$ of the Taylor series satisfies the inequality

$$|R_N(x)| \leq \frac{M}{(N+1)!}|x-c|^{N+1} \quad \textit{for } |x-a| \leq d$$

■ Question 5. ☐

(a) *Find the fourth order Taylor polynomial for* $\ln(x)$ *(see question 3).*

(b) *Use it to approximate* $\ln(1.5)$.

■ Question 6. ☐

(a) *Approximate the function* $f(x) = \sqrt[3]{x}$ *by a Taylor polynomial of degree two at* $a = 8$.

(b) *How accurate is the approximation for* $7 \leq x \leq 9$?

■ Question 7. □

Use the third nonzero Taylor polynomial of e^{-x^2} *to approximate the value of* $\int_0^1 e^{-x^2}\,\mathrm{d}x$*. What is the maximum possible error in your approximation?*