Calculus & Analytical Geometry II

Lecture 7 Worksheet

Spring 2021 **Subhadip Chowdhury** **Math 112**

§A. Trigonometric Substitution

Recall the following two results from Calc I:

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}, \qquad \frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

So we can immediately derive the following two integral formula

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C, \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

Now let's say we would like to integrate $\int \frac{1}{\sqrt{9-x^2}}\,dx$. It looks almost like the above integral but not quite. Here's a strategy:

Draw a right-angled triangle ABC with $\angle ABC = \frac{\pi}{2}$.

Let $AC = 3$ and $AB = x$. Let $\angle ACB = \theta$.

Then $\sin\theta =$

So, $x =$ ____________________ and $dx =$ ____________________

Now use substitution to replace all of your x's in the original integral in terms of θ. Don't forget about dx. We get,

$$\int \frac{1}{\sqrt{9-x^2}}\,dx = \int$$

The method employed above is called a trigonometric substitution and is a specific instance of something called *backwards substitution* or *reverse substitution*. The idea is, instead of letting $u = g(x)$ like for u-substitution, we let $x = g(u)$ and $\mathrm{d}x = g'(u)\,\mathrm{d}u$. This makes an integral initially look more complicated:

$$\int f(x)\,\mathrm{d}x = \int f(g(u))g'(u)\,\mathrm{d}u$$

but in particular cases, actually makes the integral simpler due to trigonometric identities! It is particularly useful when the integrand contains an expression of the form

$$\sqrt{a^2-x^2}, \quad \sqrt{a^2+x^2}, \quad \text{or } \sqrt{x^2-a^2}$$

Expression	Substitution	Simplification
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$\sqrt{a^2-x^2} = a\cos\theta$ $\mathrm{d}x = a\cos\theta\,\mathrm{d}\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$\sqrt{a^2+x^2} = a\sec\theta$ $\mathrm{d}x = a\sec^2\theta\,\mathrm{d}\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\sqrt{x^2-a^2} = a\tan\theta$ $\mathrm{d}x = a\tan\theta\sec\theta\,\mathrm{d}\theta$

■ Question 1. □

Let's try to find the following integrals using above substitutions.

(a) $\displaystyle\int \frac{2}{x\sqrt{x^2-25}}\,\mathrm{d}x$

(b) $\int \sqrt{9-x^2}\,\mathrm{d}x$ [Hint: You might need the trigonometric identity $\sin(2\theta) = 2\sin\theta\cos\theta$.]

(c) $\int \frac{(1-x^2)^{3/2}}{x^6}\,\mathrm{d}x$

(d) $\displaystyle\int \frac{1}{1+16x^2}\,\mathrm{d}x$

(e) $\displaystyle\int \frac{1}{x^2+4x+5}\,\mathrm{d}x$

(f) You will need know the integral $\int \sec x \, dx$ to do the next problem.

Here's a not very intuitive first step. Rewrite the integral as

$$\int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \, dx$$

Then try a u-substitution!

(g) $\int \sqrt{1 + x^2} \, dx$