MATH 111 - Calculus and Analytic Geometry I

Lecture 26 Worksheet

Fall 2020 | **Subhadip Chowdhury** | **Oct 16-19**

Title: Applied Optimization

Summary: In a setting where a situation is described for which optimal parameters are sought, we will describe how to develop a function that models the situation and use calculus to find the desired maximum or minimum.

Related Reading: Chapter 4.7 from the textbook.

§A. Preview Activity

Let us start by jumping right into an example.

Example A.1

A farmer wants to fence a rectangular grazing area along a straight river (no fence is needed along the river). There are 100 total feet of fencing available. What dimensions (length and width) will maximize the grazing area?

- We will start by drawing a picture. We are going label the known quantities and assign variables to the unknown ones. Let x denote the length of the side of the grazing area perpendicular to the river and let y denote the length of the side parallel to the river.
- The quantity we want to maximize is the Area, which is given by $\mathbf{A} = xy$.
- We know that the total amount of fencing used is $2x + y$, which is given to be 100. We will use the equation $2x + y = 100$ to rewrite the function $\mathbf{A}$ as function of one variable. Since $y = 100 - 2x$, we can write

$$\mathbf{A} = xy = x(100 - 2x) = 100x - 2x^2$$

- Note that $x > 0$ by definition. Also since $y > 0$, we have $100 - 2x > 0 \implies x < 50$. So we are trying to maximize $\mathbf{A}(x) = x(100 - 2x)$ on the interval $(0, 50)$.
- Let's find the global maximum of $\mathbf{A}(x)$ over the closed interval $[0, 50]$. If the maximum occurs in the interior, we have our answer. If it happens at the boundary, we will say that the function doesn't have a global max on the open interval $(0, 50)$. The steps are as follows:
  - First we find the critical points. We have $\mathbf{A}'(x) = 100 - 4x$. So the critical point is at $x = 25$.
  - We find that $\mathbf{A}(25) = 2500 - 1250 = 1250$. We also check that $\mathbf{A}(0) = \mathbf{A}(50) = 0$.
  - So the global maximum is when $x = 25$.
- In conclusion, the maximum area of the grazing field is 1250 sqft.

§B. The Problem Solving Strategy

The basic idea of the optimization problems that follow is the same. We have a particular quantity that we are interested in maximizing or minimizing. However, we also have some auxiliary condition that needs to be satisfied. The problem is that the function to optimize may not be explicitly provided. We may need to understand the problem by drawing a figure, introducing variables, and then by developing a formula for a function that models the quantity to be optimized. While there is no single algorithm that works in every situation where optimization is used, in most of the problems we consider, the following steps are helpful:

Algorithm for Solving Applied Optimization Problems

Step 1. **Draw a picture. Introduce and label variables.**

It is essential to first understand what quantities are allowed to vary in the problem and then to represent those values with variables. Constructing a figure with the variables labeled is almost always an essential first step. Sometimes drawing several diagrams can be especially helpful to get a sense of the situation.

Step 2. **Make sure that you know what quantity or function is to be optimized.**

Write down a formula for this quantity algebraically using the variables you introduced in the last step. This function is called the **Objective Function**.

Step 3. **Using information given in the problem, re-write your formula from Step 2 as a function of one variable.**

The information given in the problem regarding the relationship among the variables should aid you in making the necessary substitutions or eliminations in this step. The information given is usually in the form of other equations; we refer to this as a **constraint equations**. you have to eliminate all but one variable.

Step 4. **Decide the domain on which to optimize your Objective Function.**

Often the physical constraints of the problem will limit the possible values that the variables can take on. Thinking back to the diagram describing the overall situation and any relationships among variables in the problem often helps identify the smallest and largest values of the input variable.

Step 5. **Apply the techniques you know to identify the Max/Min(s).**

This always involves finding the critical numbers of the function first. Then evaluate the function at the endpoints and critical numbers to find the global max and/or min.

Step 6. **Finally, bring all your information together, and answer whatever questions were posed by the problem.**

Make sure that you have answered the correct question: does the question seek the absolute maximum of a quantity, or the values of the variables that produce the maximum? Also make sure to answer all asked questions! (Many problems have multiple parts!)

Familiarity with common geometric formulas is particularly essential in solving optimization problems. Sometimes those involve perimeter, area, volume, or surface area of geometric objects. At other times, the constraints of a problem introduce right triangles (where the Pythagorean Theorem or trigonometric relationships apply) or other shapes whose formulas provide relationships among the variables. So it is recommended to brush up on those result as we progress further.

§C. **Practice Problems**

Question 1.

□

A box with an open top of fixed volume $\mathbf{V} = \mathbf{4000}$ m^3 with a square base is to be constructed. Find the dimensions of the box that minimize the amount of material used in its construction.

Question 2.

□

According to U.S. postal regulations, the girth plus the length of a parcel sent by mail may not exceed **108** inches, where by ''girth'' we mean the perimeter of the smallest end. What is the largest possible volume of a rectangular parcel with a square end that can be sent by mail? What are the dimensions of the package of largest volume?

Question 3.

□

A metal can manufacturer needs to build cylindrical cans with volume **300** cubic centimeters. The material for the side of a can costs **0.03** cents per cm^2, and the material for the bottom and top of the can costs **0.06** cents per cm^2. What is the cost of the least expensive can that can be built?

Question 4.

□

Consider your typical piece of notebook paper. The dimensions are likely to be **8.5** by **11** inches. Suppose we remove a square of side length x from the corners of the paper, and we then fold up each newly formed flap to make an open-top box. How large must the removed squares be in order to achieve a box with the largest possible volume?

Question 5.

□

Consider the region in the $x - y$ plane that is bounded by the x axis and the function $f(x) = 25 - x^2$. Construct a rectangle whose base lies on the x-axis and is centered at the origin, and whose sides extend vertically until they intersect the curve $y = 25 - x^2$. Which such rectangle has the maximum possible area?

Question 6.

□

Two vertical poles of heights **60** ft and **80** ft stand on level ground, with their bases **100** ft apart. A cable that is stretched from the top of one pole to some point on the ground between the poles, and then to the top of the other pole. What is the minimum possible length of cable required? Justify your answer completely using calculus.

Question 7.

□

A **20** cm piece of wire is cut into two pieces. One piece is used to form a square and the other to form an equilateral triangle. How should the wire be cut to maximize the total area enclosed by the square and triangle? to minimize the area?

Question 8.

□

Consider an isosceles triangle that circumscribes a circle of radius **1**. What is the smallest possible area of the triangle?

■ Question 9. □

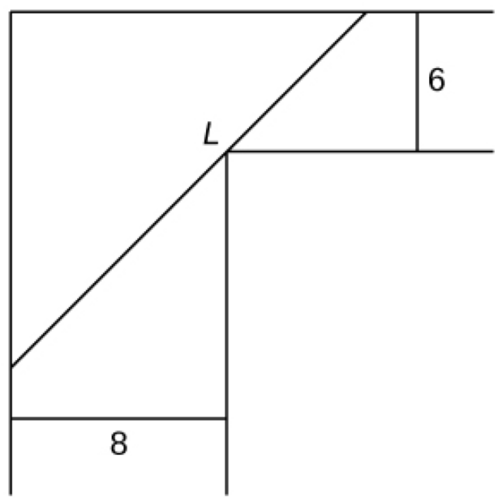
Find the volume of the largest right circular cylinder that fits in a sphere of radius **1**.

■ Question 10. □

Find the dimensions of a right circular cone with surface area $\mathbf{S = 4\pi}$ that has the largest volume.

■ Question 11. □

You are moving into a new apartment and notice there is a corner where the hallway narrows from **8** ft to **6** ft. What is the length of the longest item that can be carried horizontally around the corner?



■ Question 12. □

A window is composed of a semicircle placed on top of a rectangle. If you have **20** ft of window-framing materials for the outer frame (three sides of the rectangle and the half-circle), what is the maximum size (i.e. area) of the window you can create?