Calculus & Analytical Geometry II

Lecture 2-3 Worksheet

Spring 2021 | **Subhadip Chowdhury** | **Math 112**

§A. Integration by Substitution

Suppose we want to find $\int e^{5x}\,\mathrm{d}x$. What is the antiderivative $F(x)$ of the function $f(x) = e^{5x}$? We can do this in two different ways.

Guess and Check

$$\begin{aligned}
\text{Guess:}\quad & F(x) = e^{5x} + C \\
\text{Check:}\quad & F'(x) = 5e^{5x} \\
\text{How do we fix this?}\quad & \frac{1}{5}F'(x) = e^{5x} \\
\text{We divide by 5.}\quad & F(x) = \frac{1}{5}e^{5x} + C \\
\int e^{5x}\,\mathrm{d}x = \quad & \frac{1}{5}e^{5x} + C
\end{aligned}$$

Now let's do it in another way!

Substitution

We start by creating a new variable for our "inner function". Let's call it $\heartsuit$.

$$\begin{gathered}
\text{Let } \heartsuit = 5x \\
\text{Then } d\heartsuit = 5\,\mathrm{d}x \\
\implies \frac{1}{5}d\heartsuit = dx
\end{gathered}$$

Now we use our new code to convert our integral to hearts.

$$\begin{aligned}
\text{Substitute:}\quad & \int e^{5x}\,\mathrm{d}x = \int e^{\heartsuit} \cdot \frac{1}{5}d\heartsuit \\
\text{Pull the constant to the outside:}\quad & \frac{1}{5}\int e^{\heartsuit}d\heartsuit \\
\text{Integrate:}\quad & \frac{1}{5}e^{\heartsuit} + C \\
\text{Plug in the original variable:}\quad & \frac{1}{5}e^{5x} + C
\end{aligned}$$

Both methods give the same answer!

§B. Reversing the Chain Rule

Suppose $F'(x) = f(x)$. Recall that the Chain Rule states:

$$\frac{d}{dx}[F(g(x))] = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x)$$

So equivalently, restating this relationship in terms of an indefinite integral,

$$\int f(g(x)) \cdot g'(x)\, dx = F(g(x)) + C$$

How do we use this in practice?

Theorem B.1: u-substitution

Let $u = g(x)$, where $g'(x)$ is continuous over an interval I, and let $f(x)$ be continuous over $g(\mathrm{I})$. Let $F(x)$ be an antiderivative of $f(x)$. Then $du = g'(x)dx$, and we can write

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du = F(u) + C = F(g(x)) + C$$

Example B.2

Let's find $\displaystyle\int xe^{x^2}\, dx$.

The inside function is x^2, with derivative $2x$. The integrand has a factor of x, and since the only thing missing is a constant factor, we try $u = x^2$ to get

$$du = u'(x)dx = 2xdx \implies xdx = \frac{1}{2}du$$

Thus,

$$\int xe^{x^2}\, dx = \int e^u \cdot \frac{1}{2}\, du = \frac{1}{2}\int e^u\, du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C$$

Example B.3

Let's find $\displaystyle\int x^3\sqrt{x^4+5}\, dx$.

The inside function is $x^4 + 5$, with derivative $4x^3$. The integrand has a factor of x^3, and since the only thing missing is a constant factor, we try $u = x^4 + 5$ to get

$$du = u'(x)dx = 4x^3dx \implies x^3dx = \frac{1}{4}du$$

Thus,

$$\int x^3\sqrt{x^4+5}\, dx = \int \sqrt{u} \cdot \frac{1}{4}\, du = \frac{1}{4}\int u^{1/2}\, du = \frac{1}{4}\frac{u^{3/2}}{3/2} + C = \frac{1}{6}(x^4+5)^{3/2} + C$$

Algorithm for Solving u-substitution Problems

Step 1. Look carefully at the integrand and select an expression $g(x)$ within the integrand to set equal to u. Select $g(x)$ such that $g'(x)$ is also sitting somewhere else in your integrand.

Step 2. Substitute $g(x)$ by u and $g'(x)dx$ by du into the integral.

Step 3. We should now be able to evaluate the integral with respect to u. If the integral can't be evaluated we need to go back and select a different expression to use as u.

Step 4. Evaluate the integral in terms of u.

Step 5. Sustitute the expression $g(x)$ back in place of u and write the final answer in terms of x.

Note: It is often helpful to choose u to be the "inside" of some other function.

Question 1.

Find the following indefinite integrals.

(a) $\displaystyle\int \frac{x^3}{\sqrt{1+x^4}}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{\sin x}{\cos^3 x}\,\mathrm{d}x$

(c) $\displaystyle\int x\sin(x^2+5)\,\mathrm{d}x$

(d) $\displaystyle\int \frac{e^{1/x}}{x^2}\,\mathrm{d}x$

§C. **Evaluating definite integrals via u-substitution**

Let's solve the following integral using substitution: $\int_0^1 x^3(2x^4+1)^{10}\,\mathrm{d}x$

Let $\star = 2x^4+1$, find $d\star =$

Solve for $x^3 dx$:

Now use your "$\star$ code" to translate from x's to $\star$'s:

Now we must find the limits of integration in terms of $\star$'s, instead of x's.

While using x's, the limits of integration were $x=0$ and $x=1$: $\int_{x=0}^{x=1} x^3(2x^4+1)^{10}\,\mathrm{d}x$

Find the new lower limit. Let $x=0$ and solve for $\star$:

Find the new upper limit. Let $x=1$ and solve for $\star$:

Plug the limits of integration into our new integral:

Evaluate the integral:

Alternately, we can evaluate the corresponding indefinite integral, as we did on the previous worksheet. Then, once we have x's back in our answer, we can evaluate the integral using the given limits of integration on x. That would proceed as follows:

$$\begin{aligned}\int x^3(2x^4+1)^{10}\,\mathrm{d}x &= \frac{1}{8}\int u^{10}\,\mathrm{d}u \\ &= \frac{1}{8}\frac{u^{11}}{11}+C = \frac{1}{88}(2x^4+1)^{11}+C\end{aligned}$$

And then,

$$\int_0^1 x^3(2x^4+1)^{10}\,\mathrm{d}x = \frac{1}{88}(2x^4+1)^{11}\Big|_0^1 = \ldots$$

Question 2.

Evaluate the following definite integrals using substitution:

(a) $\displaystyle\int_0^1 x\sqrt{1-x^2}\,\mathrm{d}x$

(b) $\displaystyle\int_0^1 x^2\cos\left(\frac{\pi}{2}x^3\right)\mathrm{d}x$

(c) $\displaystyle\int_0^{\sqrt{2}} xe^{-\left(\frac{x^2}{2}\right)}\,\mathrm{d}x$

§D. **Antiderivatives with Natural Log**

Recall the antiderivative for $y = \frac{1}{x}$:

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C.$$

Why is there an absolute value? Because the domain of $y = \frac{1}{x}$ is $(-\infty, 0) \cup (0, \infty)$, whereas the domain of $y = \ln x$ is only $(0, \infty)$. Although we never explicitly said this, antiderivatives must be defined on the same interval as the function (upto losing endpoints).

Question 3.

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The next few problems all involve substitution and this new antiderivative formula. Your goal is to do a substitution and get an integral that looks like $\int \frac{1}{u}\,\mathrm{d}u$.

(a) $\displaystyle\int \frac{x^2}{x^3+2}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{1}{x\ln x}\,\mathrm{d}x$

(c) $\displaystyle\int \frac{x+1}{x^2+2x+19}\,\mathrm{d}x$

(d) $\displaystyle\int \frac{e^x}{5+e^x}\,\mathrm{d}x$

(e) $\displaystyle\int \tan x\,\mathrm{d}x$

(Hint: write $\tan x = \dfrac{\sin x}{\cos x}$)

§E. **Manipulation Substitution**

Sometimes we need to manipulate an integral in ways that are more complicated than just multiplying or dividing by a constant. We need to eliminate all the expressions within the integrand that are in terms of x. When we are done, the new variable u should be the only variable in the integrand. In some cases, this means solving for x in terms of u.

Example E.4

Let's consider the example $\int x\sqrt{3x-4}\,\mathrm{d}x$. Let $u = 3x-4$. Then $du = 3\,\mathrm{d}x$ and notice that $x = \dfrac{u+4}{3}$.
Now substitute:

$$\int x\sqrt{3x-4}\,\mathrm{d}x = \int \left(\frac{u+4}{3}\right)\sqrt{u}\,\mathrm{d}u = \frac{1}{3}\int (u^{3/2} + 4\sqrt{u})\,\mathrm{d}u$$

Finish the integration.

Question 4.

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Find the following indefinite integrals. Remember that you can manipulate your substitution equations!

(a) $\displaystyle\int (x+1)\sqrt{2-x}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{1}{\sqrt{x}+1}\,\mathrm{d}x$

§F. Concept Check

Question 5.

Explain why the two antiderivatives are really, despite their apparent dissimilarity, different expressions of the same problem. You do not need to evaluate the integrals.

(a) $\displaystyle\int \frac{e^x\,dx}{1+e^{2x}}$ and $\displaystyle\int \frac{\cos x\,dx}{1+\sin^2 x}$

(b) $\displaystyle\int \frac{\ln x}{x}\,dx$ and $\displaystyle\int x\,dx$

(c) $\displaystyle\int e^{\sin x}\cos x\,dx$ and $\displaystyle\int \frac{e^{\arcsin x}}{\sqrt{1-x^2}}\,dx$

(d) $\displaystyle\int (\sin x)^3\cos x\,dx$ and $\displaystyle\int \left(x^3+1\right)^3 x^2\,dx$

(e) $\displaystyle\int \sqrt{x+1}\,dx$ and $\displaystyle\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$

Question 6.

Simplify the following to an equivalent expression which contains g but no integral sign.

(a) $\displaystyle\int g'(x)(g(x))^4\,\mathrm{d}x$

(b) $\displaystyle\int g'(x)e^{g(x)}\,\mathrm{d}x$

(c) $\displaystyle\int g'(x)\sin(g(x))\,\mathrm{d}x$

(d) $\displaystyle\int g'(x)\sqrt{1+g(x)}\,\mathrm{d}x$

Question 7.

More practice problems.

(a) $\displaystyle\int_1^e \frac{1+\ln x}{x\ln x}\,\mathrm{d}x$

(b) $\displaystyle\int \ln x \frac{\sqrt{1-(\ln x)^2}}{x}\,\mathrm{d}x$

(c) $\displaystyle\int \frac{x^3+x^2+2x}{x^2+1}\,\mathrm{d}x$