Calculus & Analytical Geometry II

Lecture 14 Worksheet

Spring 2021 | **Subhadip Chowdhury** | **Math 112**

§A. Physics Application of Definite Integrals - Work

The last application of integration we will look at is the notion of Work - a concept from Physics, related to force, and intuitively defined as the energy required to 'push' or 'pull' an object along some distance.

Some Preliminary Facts

According to physics, when we have a constant force, **work** can be expressed as the product of force and distance.

$$\text{Work} = \text{Force} \times \text{Distance}$$

Weight is the force that gravity exerts on an object of a particular mass. In particular,

$$\text{Weight} = \text{Mass} \times \text{Acceleration due to Gravity}$$

Quantity	SI Units	USA Units
Mass	kilogram (kg)	slug
Force	Newton ($\text{kg} \cdot \frac{\text{m}}{\text{s}^2}$)	pound
Work	Joule ($\text{N} \cdot \text{m}$)	foot pound

The approximate value of acceleration due to gravity is 9.8 m/s^2 or 32 ft/s^2.

Example A.1

The weight of 1 kilogram of iron is

$$(1 \text{ kg}) \times (9.8 \text{ m/s}^2) = 9.8 \text{ N}$$

Example A.2

If a person drags a 20 kg mass 4 meters along the ground, the total work accomplished is

$$(20 \text{ kg}) \cdot (9.8 \text{ m/s}^2) \cdot (4 \text{ m}) = 784 \text{ N}$$

Unfortunately, the formula $\text{W} = \text{F} \cdot d$ only applies when the force is constant over the distance d. So what happens when the force varies? For example, the work done to compress (or elongate) a spring varies depending on how far the spring has already been compressed (or stretched).

Definition A.3

If a variable force $F(x)$ moves an object in a positive direction along the x-axis from point a to point b, the the **work** done on the object is

$$W = \int_a^b F(x)\,dx$$

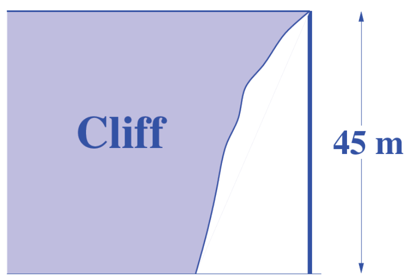
Note: When F is constant, the integral evaluates to $F \cdot (b-a) = F \cdot d$, which is the formula we stated in the last page.

We will consider two situations where a varying force accomplishes work.

§B. Work Done in Lifting

Suppose we are interested in the following problem:

A 45-meter heavy rope with a total mass of 30 kg is dangling over the edge of a cliff. Ignoring friction, how much work is needed to pull the rope up to the top of the cliff?



■ Question 1. □

(a) Explain what is **wrong** with the following solution.

Solution. First, note that the weight of the rope is

$$(30\text{ kg}) \times (9.8\text{ m/s}^2) = 294\text{ N}.$$

Hence the total work done is

$$\text{Work} = (\text{Force}) \times (\text{Distance}) = (294\text{ N}) \cdot (45\text{ m}) = 13230\text{ Joules}.$$

■

(b) Give a correct solution to this problem.

HINT: To account for the fact that different portions of the rope travel different distances to the top of the cliff, we imagine chopping the rope into tiny pieces of equal height Δy and then add up the total work necessary to move each slice to the top.

Question 2.

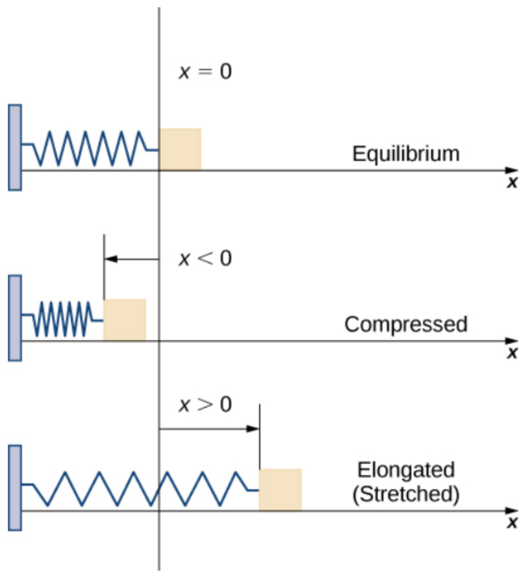
A leaky bucket is being hauled up from a 100 m deep well. When lifted from the water, the bucket and water together weigh 40 N. As the bucket is being hauled upward at a constant rate, the bucket leaks water at a constant rate so that it is losing weight at a rate of 0.1 N/m.

(a) Construct a function $B(h)$ that tells the weight of the bucket after the bucket has been lifted h m.

(b) What is the total amount of work accomplished in lifting the bucket to the top of the well?

§C. Work Required to Stretch or Compress a Spring

Consider a block attached to a horizontal spring. The block moves back and forth as the spring stretches and compresses.



According to Hooke's law, the force required to compress or stretch a spring from an equilibrium position is given by $F(x) = kx$ for some constant k. The constant k is called the spring constant and is always positive.

Question 3.

Suppose it takes a force of 10 N (in the negative direction) to compress a spring 0.2 m from the equilibrium position.

(a) Use the given fact that $F(-0.2) = -10$ to find k.

(b) Find the work done to *stretch* the spring 0.5 m beyond its natural length.

(c) How much work is required to *stretch* the spring from 1 m beyond its natural length to 1.5 m beyond its natural length?