# MATH 111 - Calculus and Analytic Geometry I

## Lecture 13 Worksheet

**Fall 2020** | **Subhadip Chowdhury** | **Sep 16**

**Title:** Basic Differentiation Rules

**Summary:** In these notes, we discuss and question the why's and how's of the derivative rules developed in Section 3.3 of our textbook. We not only want to be able to use these differentiation rules to compute derivatives, but we want to understand why the formulas are what they are.

## §A. Notation

Before moving forward, first a few words about notations. There are several different ways we can denote the derivative of $f$.

- If $f(x)$ is a function of $x$, then $f'(x)$ means the derivative of $f(x)$ with respect to $x$.
- We can forego the variable to denote the derivative of $f$ by $f'$.
- If $y = f(x)$, we also write $f'(x) = \dfrac{dy}{dx}$ or $\dfrac{df(x)}{dx}$ or $\dfrac{d}{dx}f(x)$.

> The significance of the third notation is to make you realise that $\dfrac{dy}{dx}$ is NOT a fraction. $\dfrac{d}{dx}$ is an operation we perform on $f(x)$ to produce $f'(x)$.

- For example if $f(x) = \sin x$, we can write $\dfrac{d}{dx}\sin x$ to mean the derivative of the function $\sin x$.

## §B. Basic Arithmetic

**Theorem B.1**

- If $c$ is a constant, then $\dfrac{d}{dx}[c] = 0$
- If $c$ is a constant and $f(x)$ is differentiable, then $\dfrac{d}{dx}[cf(x)] = cf'(x)$
- If $f(x)$ and $g(x)$ are differentiable, then $\dfrac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

Most of these results can be immediately understood by drawing a picture. For example, the fact that derivative of a constant is zero is equivalent to saying that the slope of a horizontal straight line is zero. For a proof of the next fact, look at figure (1).

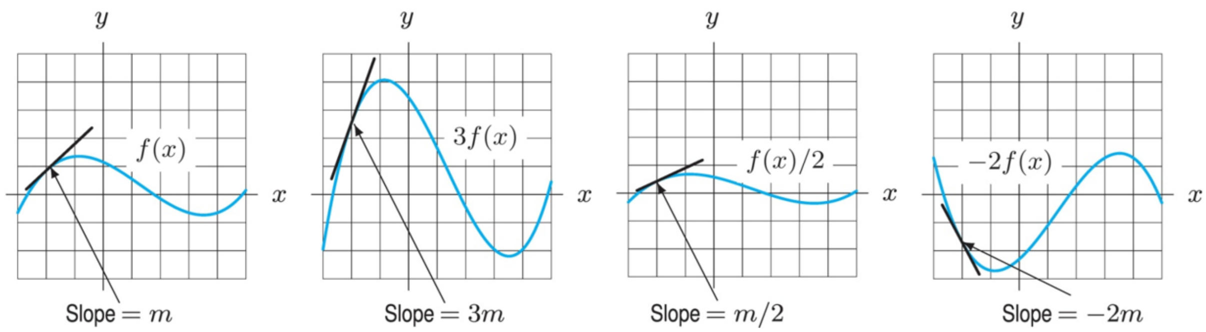


Figure 1: Derivative of multiple is multiple of derivative

They can be also proved easily by writing down the limit definition. For example,

$$\begin{aligned}\frac{d}{dx}[f(x)+g(x)] &= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= f'(x)+g'(x)\end{aligned}$$

## Question 1.

Here is a question to practice just these basic formulas. Using Table 1, compute the following:

(a) Find $h'(1)$ if $h(x) = 5 - f(x)$.

(b) Find $k'(-2)$ if $k(x) = -\frac{1}{2}g(x)$.

(c) Find $p'(-2)$ if $p(x) = 2f(x) + 3g(x)$.

| $x$ | $f(x)$ | $f'(x)$ | $g(x)$ | $g'(x)$ |
|---|---|---|---|---|
| $-2$ | $-6$ | $9$ | $-10$ | $16$ |
| $1$ | $5$ | $-3$ | $3$ | $-2$ |

Table 1: Table for Question 1

## §C. Powers and Polynomials

The most common rule that any student who has taken some Calculus will inevitably remember is the power rule.

**Theorem C.1: Power Rule Part I**

For any positive integer $n$,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

## Question 2.

We will prove the Power rule ourselves by using a factorization formula.

(a) First, confirm the following factorization identity:

$$y^n - x^n = (y-x)(y^{n-1} + y^{n-2}x + \cdots + yx^{n-2} + x^{n-1})$$

(b) Next recall that

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}$$

So when $f(x) = x^n$ for some postive integer $n$, we get

$$f'(x) = \lim_{y \to x} \frac{y^n - x^n}{y - x}$$

Now use part (a) to calculate the limit yourself.

## ■ Question 3.

Let $f(x) = x^5 - 2x^4 + 3x + 12$. Use the power rule and the basic derivative formulas to compute $f'(x)$.

## ■ Question 4.

The graph of $y = x^3 - 9x^2 - 16x + 1$ has a slope of $5$ at two points. Find the coordinates of the points.

## ■ Question 5.

Find the equation of the line tangent to the graph of $f(x) = x^2 - 4x + 6$ at $x = 1$.

**Definition C.1**

The derivative of a function is itself a function, so we can find the derivative of a derivative, called a second derivative. For example, derivative of displacement is velocity, and derivative of velocity is acceleration! Similarly, we can calculate a third derivative, fourth derivative and so on. We use the notations

$$f''(x) = \frac{d}{dx}\frac{d}{dx}f(x) = \frac{d^2}{dx^2}f(x) = \frac{d^2y}{dx^2}$$

$$f'''(x) = \frac{d}{dx}\frac{d}{dx}\frac{d}{dx}f(x) = \frac{d^3}{dx^3}f(x) = \frac{d^3y}{dx^3}$$

$$f^{(n)}(x) = \underbrace{\frac{d}{dx}\frac{d}{dx}\cdots\frac{d}{dx}}_{n \text{ times}}f(x) = \frac{d^n}{dx^n}f(x) = \frac{d^ny}{dx^n}$$

## ■ Question 6.

For the polynomial $f(x) = x^5 + x^3 + x + 1$, find

(a) $f''(x)$, i.e. the second derivative of $f$.

(b) $f^{(5)}(x)$, i.e. the fifth derivative of $f$.

(c) $f^{(6)}(x)$, i.e. the sixth derivative of $f$.

## ■ Question 7.

If the position of a body, in meters, is given as a function of time $t$, in seconds, by

$$s(t) = -4.9t^2 + 5t + 6,$$

find the velocity and acceleration of the body at time $t = 3$ seconds. Use proper units.