MATH 111 - Calculus and Analytic Geometry I

Lecture 5 Worksheet

Fall 2020 **Subhadip Chowdhury** **Aug 28**

Title: Finding Limits numerically and graphically

Summary: We will have a more formal introduction to limits and see how to calculate them using a table of values or the graph of a function.

§A. A numerical approach

Continuing our discussion from last class, let's jump directly into an example and see if we can recall the ideas of limits.

Example A.1

Consider the graph of $f(x) = \dfrac{\sin x}{x}$. Use Desmos to draw its graph. Fill out the following table of values from the graph.

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
f(x)							

Make a conjecture about the value of the limit $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$.

Can you always trust tables?

Example A.2

Consider the function $f(x) = \sin\left(\dfrac{\pi}{x}\right)$. Use the following tables to make a guess as to the value of $\lim\limits_{x\to 0} f(x)$.

TABLE 1. Fill out the table by hand.

x	$f(x)$	x	$f(x)$
0.1		-0.1	
0.01		-0.01	
0.001		-0.001	

What is your guess for $\lim\limits_{x\to 0} \sin\left(\dfrac{\pi}{x}\right)$ using TABLE 1?

TABLE 2. Fill out the table by hand.

x	$f(x)$	x	$f(x)$
$\frac{2}{5}$		$-\frac{2}{5}$	
$\frac{2}{83}$		$-\frac{2}{83}$	
$\frac{2}{1601}$		$-\frac{2}{1601}$	

What is your guess for $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$ using TABLE 2?

Follow-Up Question:

If you have extra time, come back and try the same values in the tables, but for the function $g(x) = x\sin\left(\frac{\pi}{x}\right)$.

§B. A more formal definition of Limit

Definition B.1

Assume that a function f is defined on an interval around c, except perhaps at the point $x = c$. We define the limit of the function $f(x)$ as x approaches c to be a number $\mathbf{L}$ (if one exists) such that $f(x)$ can be made to remain as close to $\mathbf{L}$ as we want by choosing x sufficiently close to c (but $x \neq c$). If $\mathbf{L}$ exists, we write

$$\lim_{x\to c} f(x) = \mathbf{L}.$$

Digression

Here's a more succinct and abstract way to say the same thing:

For all $\epsilon > 0$, there exists a $\delta > 0$, such that if $0 < |x - c| \leq \delta$, then $|f(x) - \mathbf{L}| < \epsilon$.

You do not need to memorize this definition, but this is the most precise definition of limit. We will not use it. If you want to read more, see section 2.5 of the textbook.

■ Question 1.

Using this definition, what can you say about $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$?

§C. Extending the idea of Limit

Recall from our last class that the functional value at a point $f(c)$ and the limiting value at the same point $\lim_{x\to c} f(x)$ may not be the same. In fact, one or both of these may or may not exist. In light of all these possibilities, we want to introduce two notations which will allow us to describe the behavior of a function near a point c in a bit more detail.

Definition C.1: One Sided Limits

When we write $\lim_{x \to c} f(x)$, we mean the number that $f(x)$ approaches as x approaches c from both sides. It may or may not exist.

If we want x to approach c only through values greater than c (i.e. from the right hand side on the number line), we write $\lim_{x \to c^+} f(x)$ for the number that $f(x)$ approaches (assuming such a number exists), and we call it the **right-hand limit**.

Similarly, $\lim_{x \to c^-} f(x)$ denotes the number (if it exists) obtained by letting x approach c through values less than c, and we call it the **left-hand limit**.

For $\lim_{x \to c} f(x)$ to exist, both $\lim_{x \to c^+} f(x)$ and $\lim_{x \to c^-} f(x)$ must exist, and they must be equal to each other.

Question 2.

Recall the car rental price graph from last class. Let the function be denoted as $f(x)$.

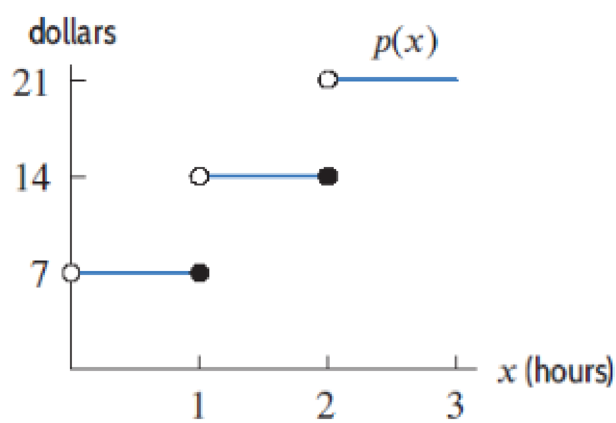

Figure 1

Use the picture to find

$\lim_{x \to 1^-} f(x) =$ $\qquad$ $\lim_{x \to 1^+} f(x) =$ $\qquad$ $\lim_{x \to 1} f(x) =$ $\qquad$ $f(1) =$

Question 3.

Use Desmos to draw the graph of the function $f(x) = \dfrac{|x-1|}{x-1}$. Use it to determine the given limits. Can you find these values without drawing a picture?

$\lim_{x \to 1^-} \dfrac{|x-1|}{x-1} =$ $\qquad$ $\lim_{x \to 1^+} \dfrac{|x-1|}{x-1} =$ $\qquad$ $\lim_{x \to 1} \dfrac{|x-1|}{x-1} =$

Definition C.2: Horizontal Asymptotes and Limits

We say $\lim_{x\to\infty} f(x) = \mathbf{L}$ if the value of $f(x)$ can be be made to stay as close as $\mathbf{L}$ as we want by making x sufficiently large. In that case we say the line $y = \mathbf{L}$ is an **horizontal asymptote** to the graph of the function $f(x)$. We can similarly define $\lim_{x\to-\infty} f(x) = \mathbf{L}$.

Example C.1

Recall from Monday we defined

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

Question 4.

Find $\lim_{x\to\infty} \arctan x$ and $\lim_{x\to-\infty} \arctan x$.

Definition C.3: Vertical Asymptotes and Limits

If one or both of the one-sided limit of a function $f(x)$ at a point c approach infinity or negative infinity, we say that the graph of $f(x)$ has a **vertical asymptote** at $x = c$.

Question 5.

For the following two graphs, find

$\lim_{x\to0^-} f(x) =$ $\qquad$ $\lim_{x\to0^+} f(x) =$ $\qquad$ $\lim_{x\to0} f(x) =$

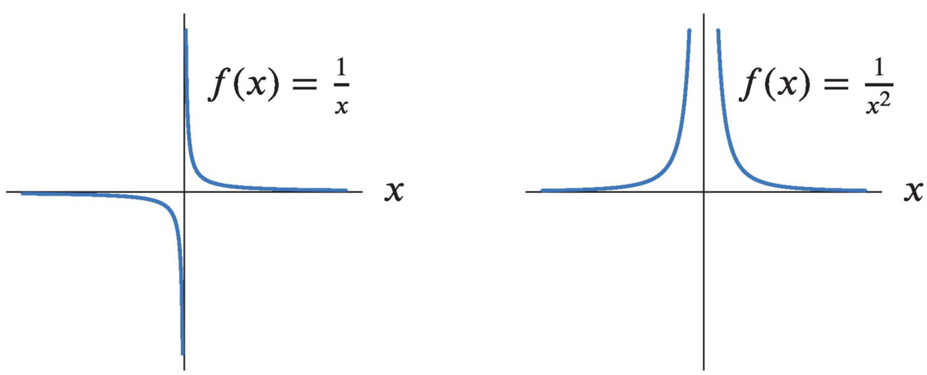

Figure 2