MATH 111 - Calculus and Analytic Geometry I

Lecture 20 Worksheet

Fall 2020 **Subhadip Chowdhury** **Oct 2**

Title: Derivatives of Inverse Functions

Summary: We will find derivative of inverse trigonometric and inverse functions in general.

§A. Inverse Trignonometric Functions

First of all, we will have a brief review of inverse trigonometric functions. Trigonometric functions are periodic, so they fail to be one-to-one, and thus do not have inverse functions. However, we can restrict the domain of each trigonometric function so that it is one-to-one on that domain.

For instance, consider the sine function on the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Because no output of the sine function is repeated on this interval, the function is one-to-one and thus has an inverse. Thus, the function $f(x) = \sin x$ with domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and range $[-1, 1]$ has an inverse function f^{-1} such that

$$f^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

We call f^{-1} the **arcsine** function and write $f^{-1}(y) = \arcsin(y)$. It is especially important to remember that

$$y = \sin(x) \text{ and } x = \arcsin(y)$$

say the same thing. "The arcsine of y" means "the **angle** whose sine is y." For example, $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$ means that $\frac{\pi}{6}$ is the angle whose sine is $\frac{1}{2}$ which is equivalent to writing $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.

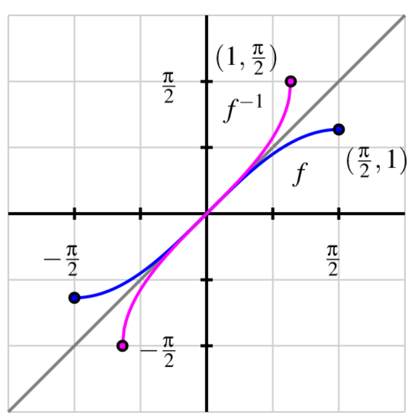


Figure 1: A graph of $f(x) = \sin x$ (in blue), restricted to the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, along with its inverse, $f^{-1}(x) = \arcsin(x)$ (in magenta).

§B. **Derivative of arcsin**

Next, we determine the derivative of the arcsine function. Letting $g(x) = \arcsin(x)$, our goal is to find $g'(x)$. since $g(x)$ is the angle whose sine is x it is equivalent to write

$$\sin(g(x)) = x$$

Differentiating both sides of the previous equation, we have

$$\frac{d}{dx}[\sin(g(x))] = \frac{d}{dx}[x]$$

The right hand side is simply 1 , and by applying the chain rule applied to the left side,

$$\cos(g(x))g'(x) = 1$$

Solving for $g'(x)$, it follows that

$$g'(x) = \frac{1}{\cos(g(x))}$$

Finally, we recall that $g(x) = \arcsin(x)$, so the denominator of $g'(x)$ is the function $\cos(\arcsin(x))$, or in other words, "the cosine of the angle whose sine is x." A bit of right triangle trigonometry allows us to simplify this expression considerably.

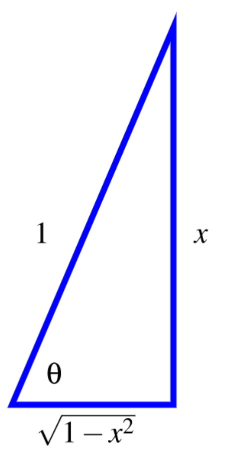


Figure 2: The right triangle that corresponds to the angle $\theta = \arcsin(x)$.

Let's say that $\theta = \arcsin(x)$, so that θ is the angle whose sine is x. We can picture θ as an angle in a right triangle with hypotenuse 1 and a vertical leg of length x, as shown in figure (2). The horizontal leg must be $\sqrt{1-x^2}$ by the Pythagorean Theorem.

Now, because $\theta = \arcsin(x)$, the expression for $\cos(\arcsin(x))$ is equivalent to $\cos(\theta)$. From the figure,

$$\cos(\arcsin(x)) = \cos(\theta) = \sqrt{1-x^2}.$$

Substituting this expression into our formula, $g'(x) = \dfrac{1}{\cos(\arcsin(x))}$, we have now shown that

$$g'(x) = \frac{1}{\sqrt{1-x^2}}$$

Theorem B.1

For all real numbers x such that $-1 < x < 1$

$$\boxed{\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}}$$

§C. Derivative of arctan

The derivative of $\arctan x$ is given by

$$\boxed{\frac{d}{dx}[\arctan(x)] = \frac{1}{1+x^2}}$$

Question 1.

Deriavtion of the formula

The following prompts will lead you to develop the derivative of the inverse tangent function yourself!

(a) Let $r(x) = \arctan(x)$. Use the relationship between the arctangent and tangent functions to rewrite this equation using only the tangent function.

(b) Differentiate both sides of the equation you found in (a). Solve the resulting equation for $r'(x)$, writing $r'(x)$ as simply as possible in terms of a trigonometric function evaluated at $r(x)$.

(c) Recall that $r(x) = \arctan(x)$. Update your expression for $r'(x)$ so that it only involves trigonometric functions and the independent variable x.

(d) Introduce a right triangle with angle θ so that $\theta = \arctan(x)$. What are the three sides of the triangle?

(e) In terms of only x and 1, what is the value of $\cos(\arctan(x))$?

(f) Use the results of your work above to find an expression involving only 1 and x for $r'(x)$.

Question 2.

Derivative Practice

Compute the derivative of the following functions.

(i) $f(x) = x^3 \arctan(x) + e^x \ln(x)$

(ii) $p(t) = 2^{t \arcsin(t)}$

(iii) $h(z) = (\arcsin(5z) + \arctan(4-z))^{27}$

(iv) $s(y) = \cot(\arctan(y))$

(v) $m(v) = \ln\left(\sin^2(v) + 1\right)$

(vi) $g(w) = \arctan\left(\frac{\ln(w)}{1+w^2}\right)$

§D. Derivative of the Inverse Function

Suppose f and g are differentiable functions that are inverses of each other, i.e. $y = f(x)$ if and only if $x = g(y)$. Then we can write $f(g(x)) = x$ for every x in the domain of f^{-1}. Differentiating both sides of this equation, we have

$$\frac{d}{dx}[f(g(x))] = \frac{d}{dx}[x]$$

and by the chain rule,

$$f'(g(x))g'(x) = 1$$

Solving for $g'(x)$, we have $g'(x) = \dfrac{1}{f'(g(x))}$. In other words,

Theorem D.1

Suppose that the domain of a function f is an open interval $\mathbf{I}$ and that f is differentiable and one-to-one on this interval. Then f^{-1} is differentiable at any point x in the range of f at which $f'(f^{1-}(x)) \neq 0$, and its derivative is

$$\boxed{\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}}$$

The formula for the derivatives of **arcsin** and **arctan** obtained above are just applications of this result.

Question 3.

Let g denote the inverse function of f. Suppose

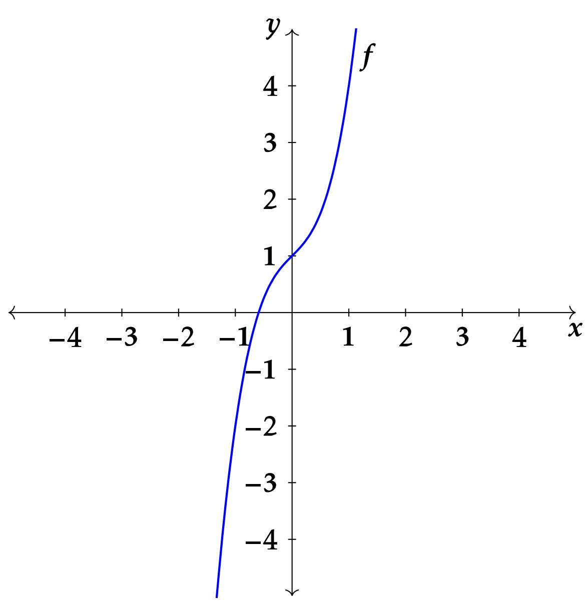
$$f(3) = -6, \quad f'(3) = 2/3, \quad f(-6) = 2, \quad f'(2) = 1,$$

$$f'(-6) = 3, \quad f'(-1) = -6, \quad f'(-6) = 5$$

What is $g'(-6)$?

Question 4.

Let $g(x)$ be the inverse function of $f(x) = 2x^3 + x + 1$. What is $g'(4)$?



■ Question 5.

Let $f(x) = 2x - \sin(x)$ (graphed below) and let $g(x)$ be the inverse function of $f(x)$. Then find $g'(2\pi)$.