Calculus & Analytical Geometry II

Lecture 12 Worksheet

Spring 2021 | **Subhadip Chowdhury** | **Math 112**

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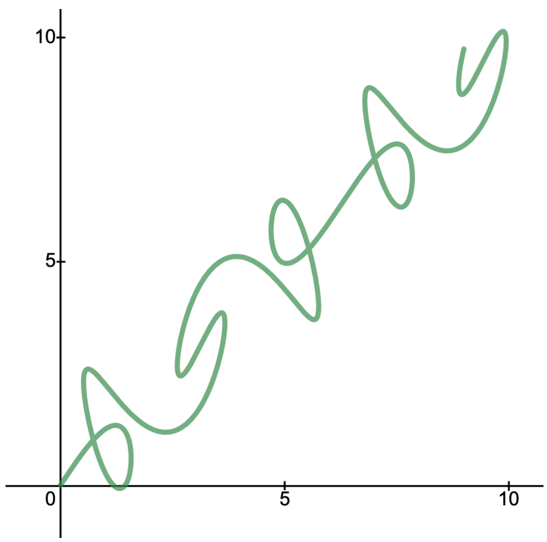
§A. Parametric Curves

Imagine that a particle moves along a complicated curve C as shown in the figure below. Clearly it is impossible to describe C by an equation of the form $y = f(x)$ or $x = g(y)$. Then how do calculate distance travelled by the particle in this case?



Suppose instead we are given that the $x-$ and $y-$coordinates of the particle are functions of time and so we can write $x = f(t)$ and $y = g(t)$. Such a pair of equations is often a convenient way of describing a curve and gives rise to the following definition.

Definition A.1: Parametric Curves

Suppose that x and y are both given as functions of a third variable t (called a **parameter**) by the equations

$$x = f(t) \qquad \text{and} \qquad y = g(t)$$

(called **parametric equations**). Each value of t (you can think of it as time) determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call a **parametric curve**.

Question 1.

Use DESMOS to sketch the curve defined by the parametric equations $x = t^2 - 2t$, and $y = t + 1$. Rewrite the equation of the curve using only x and y.

Note: This equation in x and y describes **where** the particle has been, but it doesn't tell us **when** the particle was at a particular point. The parametric equations have an advantage—they tell us **when** the particle was at a point. They also indicate which direction the particle was going!

Question 2.

☐

Identify the parametric curve represented by the equations

$$x(t) = \cos(t) \text{ and } y(t) = \sin(t) \quad \text{for } 0 \le t \le 2\pi$$

We can parametrize the same curve in different ways and interpret each parametrization as the motion of a particle with the parameter t being time.

Question 3.

☐

Explain why all of the parametrized curves below should look like a circle.

- $x(t) = \cos(2t)$ and $y(t) = \sin(2t)$ for $0 \le t \le 2\pi$
- $x(t) = \cos(t^2)$ and $y(t) = \sin(t^2)$ for $0 \le t \le 2\pi$
- $x(t) = \cos(t)$ and $y(t) = -\sin(t)$ for $0 \le t \le 2\pi$
- $x(t) = -\cos(t)$ and $y(t) = \sin(t)$ for $0 \le t \le 2\pi$
- $x(t) = -\cos(t)$ and $y(t) = -\sin(t)$ for $0 \le t \le 2\pi$

§B. Tangents of Parametric Curves

Recall that the slope of a tangent to a curve at any point is given by the derivative. In the case of a parametric curve given by the equations $x = f(t), y = g(t)$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{f'(t)}{g'(t)}$$

Question 4.

☐

Consider a parametric curve whose equation is given by $x = t^2$, $y = t^3 - 3t$. Use DESMOS to draw the curve.

(a) Find the point on the curve when the tangent is horizontal.

(b) Show that the curve has two tangents at the point $(3, 0)$ and find their equations.

§C. **Area under a Parametric Curve**

We know that the area under a curve $y = \varphi(x)$ from a to b is $\mathrm{A} = \int_a^b \varphi(x)dx$, where $\varphi(x) \geq 0$. If the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t), \alpha \leq t \leq \beta$, then we can calculate an area formula by using a u–Substitution as follows:

$$\mathrm{A} = \int_a^b y dx = \int_\alpha^\beta g(t) f'(t) \mathrm{d}t$$

Question 5.

□

The parametric curve with equations

$$x = t - \sin t, \quad y = 1 - \cos t$$

is called a **cycloid**. Draw it in `DESMOS`. Find the area under one arch of the cycloid.

§D. **Arc Length of a Parametric Curve**

We already know that the arc length L of a curve C given is given by

$$\mathrm{L} = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \mathrm{d}x = \int_a^b \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$$

So if the curve is parametrized as $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, we have

$$\mathrm{d}x = f'(t)\mathrm{d}t \text{ and } \mathrm{d}y = g'(t)\mathrm{d}t.$$

So the formula becomes

$$\mathrm{L} = \int_\alpha^\beta \sqrt{f'(t)^2 + g'(t)^2} \mathrm{d}t$$

Question 6.

□

Find the length of one arch of the cycloid.

§E. **Surface Area of Solids of Revolution**

In the same way as for arc length, we can adapt the formula for surface area. Suppose the curve C given by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, is rotated about the x-axis. *If* C *is traversed exactly once as* t *increases from* α *to* β, then the area of the resulting surface is given by

$$S = \int_{\alpha}^{\beta} 2\pi g(t)\sqrt{(f'(t))^2 + (g'(t))^2}\,\mathrm{d}t$$

■ Question 7. □

Find the surface area of a sphere of radius r. Recall that you found out the parametrization of a circle in question 2.

§F. **More Practice Problems**

■ Question 8. □

Graph the curve given by the parametric equations

$$x = e^t \cos t, \quad y = e^t \sin t, \quad 0 \leq t \leq \pi$$

and find its exact length.

■ Question 9. □

Find the area enclosed by the parametric curve $x = t^2 - 2t, y = \sqrt{t}$ and the y-axis.

■ Question 10. □

Find the arc length of the curve $y = x^2 - \dfrac{\ln x}{8}$ from $x = 1$ to $x = e$.

■ Question 11. □

Find the arc length of the curve $y^3 = x^2$ from $(1, 1)$ to $(8, 4)$.