Calculus & Analytical Geometry II

Strategies for Applying Series Tests

Spring 2021 | **Subhadip Chowdhury** | **Math 112**

The following is adapted from your Openstax textbook chapter 5.6. The book has a really nice flowchart summarizing strategies for choosing the correct test, as well as a table with summaries of all tests. The book uses a_n instead of a_i. So to be consistent, we will use the notation $\sum a_n$ today.

§A. **Problem Solving Strategy**

Step 1. First thing to try: the **Divergence Test.** If $\lim\limits_{n\to\infty} a_n \neq 0$, then we know that the series $\sum a_n$ diverges and we are done right away! Otherwise, $\lim\limits_{n\to\infty} a_n = 0$ and we have to try an actual test.

Step 2. Next, try to see if the series looks familiar. Is it of a certain class that we know: a **Geometric Series**, a $p-$**series**, the Harmonic Series, etc. If so, check the ratio r or the power p to determine if the series converges.

Look for variations as well - if the series looks like a sum or difference of two geometric series, or a sum of a $p-$series and a geometric series, etc. Remember, the sum or difference of two converging series converges. (What happens if we add a diverging series with a converging series?)

Step 3. Identify if the series has all positive terms. If it does not, determine if it is an Alternating Series and try the **Alternating Series Test.**

Step 4. If the series has negative terms and is *not alternating*, you can try to determine if the series is absolutely convergent, since absolute convergence implies convergence. Remember, this means determining whether $\sum a_n$ converges. You can either use the tests below for positive series, or use the Ratio or Root Test, as these are tests for absolute convergence. Do the terms in the series contain a factorial or power? If the terms are powers such that $a_n = (b_n)^n$ try the root test first. Otherwise, try the ratio test first.

Step 5. Lastly for a non-positive series, make sure you have answered the question! Does it ask for absolute/conditional convergence, or simply asks whether the series converges or diverges? If the former, make sure you fully investigate the series by checking for absolute convergence, especially if it is an alternating series.

Step 6. Now, if your series has only positive terms (or you are examining $\sum |a_n|$), then we can apply the other three tests - the Direct & Limit Comparison Tests and the Integral Test. It's a good idea to try **Direct comparison** *first*. If that fails due to the inequality going the wrong way, then use the **Limit comparison** test.

Step 7. If all of the above has failed you, then we have the **Integral Test** as our backup.

§B. **Summary of Series Tests**

The following is copied from your textbook.

Series or Test	Conclusions	Comments
Divergence Test For any series $\sum_{n=1}^{\infty} a_n$, evaluate $\lim_{n\to\infty} a_n$.	If $\lim_{n\to\infty} a_n = 0$, the test is inconclusive.	This test cannot prove convergence of a series.
	If $\lim_{n\to\infty} a_n \neq 0$, the series diverges.	
Geometric Series $\sum_{n=1}^{\infty} ar^{n-1}$	If $\|r\| < 1$, the series converges to $a/(1-r)$.	Any geometric series can be reindexed to be written in the form $a + ar + ar^2 + \cdots$, where a is the initial term and r is the ratio.
	If $\|r\| \geq 1$, the series diverges.	
***p*-Series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$	If $p > 1$, the series converges.	For $p = 1$, we have the harmonic series $\sum_{n=1}^{\infty} 1/n$.
	If $p \leq 1$, the series diverges.	

Comparison Test For $\sum_{n=1}^{\infty} a_n$ with nonnegative terms, compare with a known series $\sum_{n=1}^{\infty} b_n$.	If $a_n \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.	Typically used for a series similar to a geometric or p-series. It can sometimes be difficult to find an appropriate series.
	If $a_n \geq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	
Limit Comparison Test For $\sum_{n=1}^{\infty} a_n$ with positive terms, compare with a series $\sum_{n=1}^{\infty} b_n$ by evaluating $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.	If L is a real number and $L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.	Typically used for a series similar to a geometric or p-series. Often easier to apply than the comparison test.
	If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.	
	If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	

Integral Test If there exists a positive, continuous, decreasing function f such that $a_n = f(n)$ for all $n \geq N$, evaluate $\int_N^\infty f(x)\,dx$.	$\int_N^\infty f(x)\,dx$ and $\sum_{n=1}^\infty a_n$ both converge or both diverge.	Limited to those series for which the corresponding function f can be easily integrated.
Alternating Series $\sum_{n=1}^\infty (-1)^{n+1} b_n$ or $\sum_{n=1}^\infty (-1)^n b_n$	If $b_{n+1} \leq b_n$ for all $n \geq 1$ and $b_n \to 0$, then the series converges.	Only applies to alternating series.
Ratio Test For any series $\sum_{n=1}^\infty a_n$ with nonzero terms, let $\rho = \lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert$.	If $0 \leq \rho < 1$, the series converges absolutely.	Often used for series involving factorials or exponentials.
	If $\rho > 1$ or $\rho = \infty$, the series diverges.	
	If $\rho = 1$, the test is inconclusive.	
Root Test For any series $\sum_{n=1}^\infty a_n$, let $\rho = \lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert}$.	If $0 \leq \rho < 1$, the series converges absolutely.	Often used for series where $\lvert a_n \rvert = b_n^n$.
	If $\rho > 1$ or $\rho = \infty$, the series diverges.	
	If $\rho = 1$, the test is inconclusive.	

Question 1.

Decide whether each of the following is a p–series, or a geometric series, or neither. If it is one of the two, indicate whether or not the series converges. If it's neither, you do not need to work further.

(a) $\displaystyle\sum_{n=1}^\infty \frac{1}{n\sqrt{n}}$ (b) $\displaystyle\sum_{n=1}^\infty \frac{3}{4^{n-1}}$ (c) $\displaystyle\sum_{n=1}^\infty \frac{n}{(3n)^5}$ (d) $\displaystyle\sum_{n=1}^\infty \frac{2^{2n+1}}{5^{n-1}}$

(e) $\displaystyle\sum_{n=1}^\infty \sqrt{\frac{2}{n}}$ (f) $\displaystyle\sum_{n=1}^\infty \frac{(-1)^n}{n}$ (g) $\displaystyle\sum_{n=1}^\infty \frac{(-2)^n}{e^{2n}}$

Question 2.

For each of the following series, find either a p -series or a geometric series that would be an appropriate candidate for comparison. You need not actually perform the comparison test.

(a) $\displaystyle\sum_{n=1}^\infty \frac{5n^2}{2n^3-1}$ (b) $\displaystyle\sum_{n=1}^\infty \frac{3n}{\sqrt{n^5+n^4+2}}$ (c) $\displaystyle\sum_{n=1}^\infty \frac{3^n+1}{2^n-1}$ (d) $\displaystyle\sum_{n=1}^\infty \frac{4}{n(n+3)}$

(e) $\displaystyle\sum_{n=1}^\infty \frac{3^n}{5^n+n}$ (f) $\displaystyle\sum_{n=1}^\infty \frac{n^2}{n^2\sqrt{6n-1}}$ (g) $\displaystyle\sum_{n=1}^\infty \sqrt{\frac{4^n}{3^{2n}+100}}$ (h) $\displaystyle\sum_{n=1}^\infty \frac{\sqrt{6^n-n}}{4^{2n}+n\sqrt{n}}$

Question 3.

Determine which of the following series converge. Justify your conclusions with the appropriate explanations.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ (b) $\sum_{n=1}^{\infty} 2^{-n}$ (c) $\sum_{n=1}^{\infty} \frac{n+5}{5^n}$ (d) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n(n-1)}}$

(e) $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{n^2}$ (f) $\sum_{n=1}^{\infty} \frac{2n}{8n-5}$ (g) $\sum_{n=1}^{\infty} \frac{3^n}{(2n)!}$ (h) $\sum_{n=1}^{\infty} 2^n$

(i) $\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n^3+1}}$ (j) $\sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^n$ (k) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ (l) $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{\sqrt{n}}$

Question 4.

Determine if the given series is converging or diverging. If the series has negative terms, determine if the series is absolutely convergent, conditionally convergent, or divergent.

(a) $\sum_{n=1}^{\infty} \frac{n^2+2n}{n^3+3n^2+1}$ (b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(3n+1)}{n!}$ (c) $\sum_{n=1}^{\infty} \frac{e^n}{n^4}$ (d) $\sum_{n=1}^{\infty} \frac{3^n}{(n+1)^n}$

(e) $\sum_{n=1}^{\infty} \frac{2^n-5^n}{7^n}$ (f) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ (g) $\sum_{n=1}^{\infty} \frac{1+(-1)^n}{n}$ (h) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

(i) $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}-(\ln(n))^4}$ (j) $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ (k) $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^{2/3}}$ (l) $\sum_{n=1}^{\infty} \frac{n^5}{5^n}$

(m) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}(\ln(n))^2}$ (n) $\frac{1}{2}-\frac{1}{5}+\frac{1}{4}-\frac{1}{25}+\frac{1}{8}-\frac{1}{125}+\dots$