Calculus & Analytical Geometry II

Lecture 21-22 Worksheet

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§A. Sequence

A **sequence** can be defined as an infinite set of real numbers written in a definite order. There is a more precise definition using functions, we will come to that later.

We can write denote a sequence of real numbers as follows

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

Here a_1 is the first term in the sequence, a_2 is the second etc. and more generally, a_n, the term with **index** n, is called the nth term. We also use the notation

$$\{a_n\}_{n=1}^{\infty}$$

to denote above sequence.

Sequences can be described in a couple of different ways. We can provide an **explicit** ''closed-form'' formula for the nth term.

Example A.1

Some examples are

(a) $a_n = \dfrac{1}{n}$ for all n. This is the sequence $1, 1/2, 1/3, 1/4, \ldots$

(b) $a_n = \dfrac{n}{n+1}$ for all n. This is the sequence $1/2, 2/3, 3/4, 4/5, \ldots$

(c) $a_n = \cos\left(\dfrac{n\pi}{6}\right)$ for all n. This is the sequence $\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}, 0, -\dfrac{1}{2}, \ldots$

■ **Question 1.** □

Write down the first 6 terms of the sequence defined as $\{a_n\} = \left\{\dfrac{(-1)^n}{n}\right\}$.

Alternately, we can define a sequence **recursively**. This means, we define the nth term of the sequence using the term(s) before it.

Example A.2

Some examples are

(a) $a_n = \dfrac{a_{n-1}}{2}$ for all $n > 1$ and $a_1 = 1$. This is the sequence $1, 1/2, 1/4, 1/8, \ldots$

(b) $a_1 = 0$ and $a_n = 3a_{n-1} + 4$ for all $n > 1$. This is the sequence $0, 4, 16, 52, \ldots$

(c) The **Fibonacci Sequence** is one of the famous examples of integer sequences. It is defined as:

$$F_1 = F_2 = 1, \quad F_n = F_{n-1} + F_{n-2} \text{ for all } n \geq 3$$

This is the sequence $1, 1, 2, 3, 5, 8, 13, \ldots$.

Note: Observe that we need two pieces of information to define a sequence this way. We need the defining relation and we need the starting points.

Question 2.

Give the first 6 terms of the following sequences and then guess a closed form formula for the nth term. You don't have to prove the formula.

(a) $a_1 = 1, \quad a_2 = 3, \quad a_{n+1} = 2a_n - a_{n-1}$ *for* $n \geq 2$.

(b) $a_1 = 1, \quad a_2 = 3, \quad a_{n+1} = 3a_n - 2a_{n-1}$ *for* $n \geq 2$.

Finally, there are some sequences which don't have a simple defining equation.

Example A.3

Some examples are,

(a) $a_n =$ the nth prime number.

(b) $a_n =$ the nth digit of π.

§B. Arithmetic and Geometric Progressions

An **Arithmetic Progression** (AP) or arithmetic sequence is a sequence of numbers where the **difference** between the consecutive terms is constant. For example,

$$4, 9, 14, 19, 24, 29, \ldots$$

If we denote this sequence by $\{A_n\}$, we can define it recursively by $A_1 = 4$ and $A_n = A_{n-1} + 5$ for $n \geq 2$.

Question 3.

Find an explicit formula for the sequence.

In general, to define an AP, we need two information: the starting value, usually denoted $A_1 = a$; and we need the common difference, usually denoted d. Then the formula for the nth term is given by

$A_n =$ ______________________________

A sequence where the **ratio** of consecutive terms is constant, is called an **Geometric Progression** (GP) or geometric series. For example,

$$2, 4, 8, 16, 32, \ldots$$

$$\frac{-3}{7}, \frac{3}{4}, \frac{-21}{16}, \frac{147}{64}, \ldots$$

Question 4.

Identify the common ratio in the two examples above.

Similar to an AP, we need two pieces of information to define an GP $\{G_n\}$. We need the starting value $G_1 = a$; and we need the common ratio, call it r. Then we have the following formula:

$G_n =$ ____

§C. Limit of a Sequence

A sequence such as $a_n = \dfrac{n}{n+1}$, can be pictured by plotting its graph, as in fig. 1.

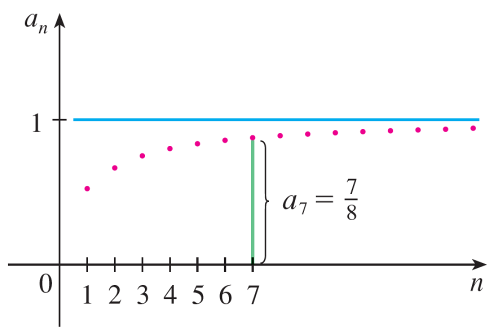

Figure 1

Here we are treating a sequence a_n as a function $f(n)$ whose domain is the set of positive integers, range is all real numbers, and its graph consists of isolated points with coordinates

$$(1, a_1) \qquad (2, a_2) \qquad (3, a_3) \qquad \ldots \qquad (n, a_n) \qquad \ldots$$

From fig. 1, it appears that the terms of the sequence $a_n = \dfrac{n}{n+1}$ are approaching 1 as n becomes large. In fact, the difference

$$1 - \frac{n}{n+1} = \frac{1}{n+1}$$

can be made as small as we like by taking n sufficiently large. We indicate this by writing

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

Thus the idea of limit of a sequence is the same exact idea as discussed in Calculus 1 regarding the limit of a function. For a function $y = f(x)$, we said that $\lim_{x\to c} f(x) = L$ if the function values $f(x)$ gets arbitrarily close to L as x approaches c. We can say the same thing for sequences:

Definition C.4

A sequence $\{a_n\}$ has the limit L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is convergent). Otherwise, we say the sequence **diverges** (or is divergent).

Question 5.

For the four examples given below (available HERE in DESMOS) determine if the sequence converges or diverges using the graphs.

(i) $a_n = \dfrac{(-1)^n n}{n+1}$

(ii) $b_n = \dfrac{3^n}{3n+1}$

(iii) $c_n = \dfrac{(-1)^n 2\ln(n)}{n}$

(iv) $d_n = \left(\dfrac{2}{3}\right)^n + \left(\dfrac{1}{4}\right)^n$

Of course, the graphs are not enough as proof of convergence. We should be able to find the limits of these sequences algebraically or analytically to confirm our guesses. Fortunately, with the idea that a sequence is just a function defined on the set of Natural numbers, the usual limit theorems all apply.

Example C.5

uppose $a_n = \dfrac{n^2+3n+1}{3n^2+2}$. Then the sequence is defined using the function $f(x) = \dfrac{x^2+3x+1}{3x^2+2}$, restricted to only natural numbers. We can use L'Hôpital's rule to easily find that $\lim_{x\to\infty} f(x) = \dfrac{1}{3}$. Then we can also conclude that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = \frac{1}{3}.$$

In fact, we have the following theorem;

Theorem C.6

Consider a sequence $\{a_n\}$ such that $a_n = f(n)$ for all $n \geq 1$. If there exists a real number L *such that*

$$\lim_{x\to\infty} f(x) = \mathrm{L}$$

then $\{a_n\}$ converges and

$$\lim_{n\to\infty} a_n = \mathrm{L}.$$

Question 6.

For each of the following sequences, decide whether they converge or diverge.

(i) $a_n = \dfrac{1-e^{-n}}{1+e^{-n}}$

(ii) $a_n = \dfrac{\ln(n)}{\ln(2n)}$

(iii) $a_n = \sin(n\pi)$

(iv) $a_n = \sin(\pi/n)$

(v) $a_n = n^{1/n}$

(vi) $a_n = \dfrac{5n^2+1}{e^n}$

Example C.7

We can use our knowledge of exponential functions to determine the convergence of some geometric sequences. Let $a_n = r^n$ where r is a real number. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } 0 \le r < 1 \\ 1 & \text{if } r = 1 \\ \text{undefined} & \text{if } r > 1 \end{cases}$$

Question 7.

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Consider the sequence from question 5.(iv) above. Can you use limit laws to determine the limit?

We can also modify the Squeeze Theorem for sequences as follows.

Theorem C.8: The Squeeze Theorem

If $a_n \le b_n \le c_n$ *for* $n \geqslant n_0$ *and* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, *then* $\lim_{n\to\infty} b_n = L$.

Question 8.

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Use the squeeze theorem to find the limit of the sequence from question 5.(iii) above.

Question 9.

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Use the squeeze theorem to show that if $\lim_{n\to\infty} |a_n| = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

We end this discussion with a couple of definitions.

Definition C.9

A sequence $\{a_n\}_{n=1}^{\infty}$ is called

increasing	if $a_{n+1} \ge a_n$
decreasing	if $a_{n+1} \ge a_n$
monotone	if it is either increasing or decreasing throughout.
bounded above	if there is a number M such that $a_n \le M$ for all $n \ge 1$
bounded below	if there is a number m such that $a_n \ge m$ for all $n \ge 1$

We will mention a final theorem without proof for the sake of completion. It is pretty straightforward to verify the statement by drawing a graph.

Theorem C.10

A monotone and bounded sequence is convergent.