# Calculus & Analytical Geometry II

## Lecture 9 Worksheet

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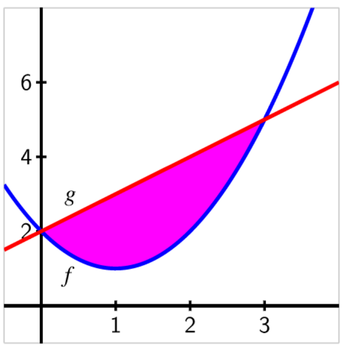
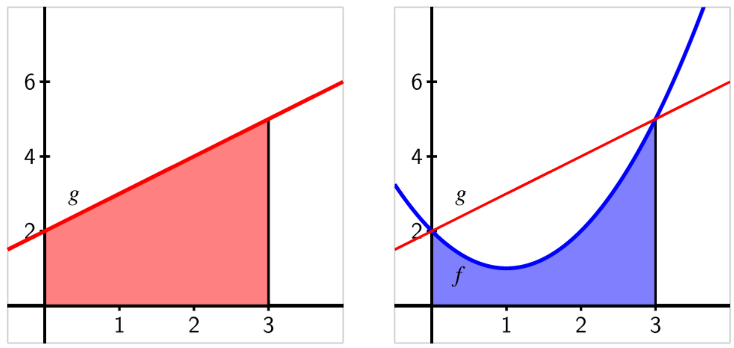
### §A. Area between two curves

In Calculus I, we learned that definite integral is equivalent to the area under a curve. In this lecture, we will discuss what happens when there are two curves in our graph and we want to find the region between them.

Consider the graphs of $f(x) = (x-1)^2 + 1$ and $g(x) = x + 2$. We wish to find the area bounded between the two graphs, the purple region in the picture below.




(a) Area between two graphs




(b) Area under each graph

We observe that the purple area is the same as the difference between the red and the blue area in the picture above. Mathematically, we subtract the area under the "bottom" curve from the area under the "top" curve, using the end points of the interval as our bounds.

We can find the red (and the blue) area by direct definite integration, but we need to first find out the bounds. Observe that the bounds are the points of intersection between the two graphs of $f$ and $g$. So setting $f(x) = g(x)$, we get

$$(x-1)^2 + 1 = x + 2 \implies x^2 - 2x + 1 + 1 - x - 2 = 0 \implies x^2 - 3x = 0 \implies x = 0 \text{ or } 3$$

Thus the red area (second picture) is equal to $\int_0^3 g(x)\,dx = \int_0^3 (x+2)\,dx =$ ________________________________

and the blue area (third picture) is equal to $\int_0^3 f(x)\,dx = \int_0^3 ((x-1)^2+1)\,dx =$ ____________________________

So finally, the purple area is equal to

$$\int_0^3 (g(x) - f(x))\,dx = \int_0^3 g(x)\,dx - \int_0^3 f(x)\,dx = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

## ■ Question 1. □

Find the area between the curves for the following problems. try to first sketch the functions and shade the area between the curves, and then find the area.

(a) $f(x) = 2 - x^2$, $g(x) = x^2$

(b) $f(x) = x^2, g(x) = 6 - x$

(c) $y = -x^2 + 3x + 1$, $y = -x + 1$

## §B. **Functions with more than two intersection points**

If the two functions cross each other more that two times, it is unclear which graph is the ''top'' one and which one is the ''bottom'' one. In fact, the top and the bottom functions switch at each intersection. In such cases, we will calculate the area as follows.

**Example B.1**

Consider the area of the region between the graphs $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$. Use `DESMOS` to find the values of $x$ where the two graphs intersect each other.

What do you notice? Which function is ''on top'' and which function is ''on bottom''?

On the interval $[-2, 0]$, we have _______ $\leq$ _______ . On the interval $[0, 2]$, we have _______ $\leq$ _______ .

Since we have a different ''top'' and ''bottom'' curve for each interval, we need to do a different integral for each interval, and then add them.

Set up and evaluate the two integrals. Remember, when finding the area between two curves, you take the integral of the ''top'' curve minus the ''bottom'' curve, using the end points of the interval as bounds.

Thus in general, we have the following theorem

**Theorem B.2**

*Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a,b]$. Let* R *denote the region between the graphs of $f(x)$ and $g(x)$, and be bounded on the left and right by the lines $x = a$ and $x = b$ respectively. Then, the area of* R *is given by*

$$\int_a^b |g(x) - f(x)|\, dx$$

## Question 2.

If R is the region between the graphs of the functions $f(x) = \sin x$ and $f(x) = \cos x$ over the interval $[0,1]$, find the area of region R.

## Question 3.

Sketch the region bounded by the functions $f(x) = \sqrt{3-x}$, $g(x) = x - 1$, and the $x$-axis, and find the area of the region.

# §C. **More Complex Regions**

**Example C.3**

How would you find the area of the region bounded by $f(x) = x^2$, $g(x) = 2 - x$, and the $x$–axis?

First we will draw the graphs and find the points of intersection. Afterwards, there are two ways to finish the problem.

- **Method 1.** We can divide the interval into two pieces. Then write each piece as an integrals with respect to $x$.

  $$\text{Area (R)} = \int_0^1 \underline{\hspace{4cm}}\,\mathrm{d}x + \int_1^2 \underline{\hspace{4cm}}\,\mathrm{d}x = \underline{\hspace{8cm}}.$$

- **Method 2.** We can treat the curves as functions of $y$ instead of as functions of $x$. Then we find the $y$-coordinates of the intersection points and calculate the area by integrating with respect to $y$.

Let's explain the second method a bit more. Consider, for example, the curves $x = y^2 - 1$ and $y = x - 1$. It is immediately clear that the parabola in the picture below is not the graph of a function of $x$ (why?), so our formula, area $= \int_a^b (g(x) - f(x))\,\mathrm{d}x$ doesn't work! In particular, the intersection points do not give correct bounds for $x$.

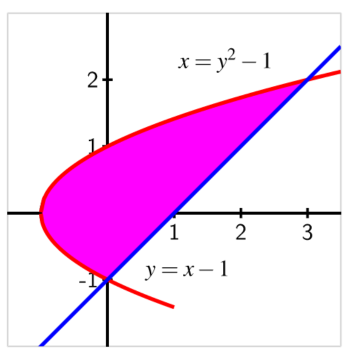



Figure 2: Area between two curves

So instead we treat the curves as

$$x = f(y) = y^2 - 1, \qquad x = g(y) = y + 1$$

and find the area between $y = -1$ and $y = 2$ as

$$\int_{-1}^{2} (g(y) - f(y))\,\mathrm{d}y = \underline{\hspace{12cm}}$$

## Question 4.

□

Find the area of the region bounded by $x + y^2 = 10$ and $x = (y-2)^2$.

## Question 5.

□

Find the area of the region bounded by $xy = 10$, $x = 0$, $y = 2$, and $y = 10$.