MATH 111 - Calculus and Analytic Geometry I

Lecture 16 Worksheet

Fall 2020 **Subhadip Chowdhury** **Sep 23**

Title: The Chain Rule

Summary: We will learn about the fanciest and most used Differentiation rule.

§A. Motivation

Imagine we are moving straight upward in a hot air balloon. Let y be our distance from the ground. The air pressure, $\mathbf{P}$, is changing as a function of altitude, so $\mathbf{P} = f(y)$. How does our air pressure change with time?

Since air pressure is a function of height, $\mathbf{P} = f(y)$, and height is a function of time, $y = g(t)$, we can think of air pressure as a composite function of time, $\mathbf{H} = f(g(t))$, with f as the outside function and g as the inside function. The example suggests the following result, which turns out to be true:

Rate of change of composite function	=	Rate of change of outside function	×	Rate of change of inside function

§B. Chain Rule

Theorem B.1

If f and g are differentiable functions, then

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$$

⚠ The derivative of the outside function must be evaluated at the inside function.

This is called the **chain rule** because, if you have multiple compositions (i.e. several functions stuffed inside of each other) then you will end up with a "chain" of products in the derivative. The Leibniz notation is very suggestive and helpful for remembering how the chain rule works: for the function $y = f(u) = f(g(x))$, meaning $u = g(x)$, we have

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

⚠ $\frac{dy}{du}$ and $\frac{du}{dx}$ are NOT fractions.

Example B.1

Suppose $f(x) = x^3$ and $g(x) = \cos x$. Then $f(g(x)) = (\cos x)^3 = \cos^3(x)$. Then first note that

$$f'(x) = 3x^2 \implies f'(g(x)) = 3\cos^2 x.$$

Then using chain rule,

$$\frac{d}{dx} f(g(x)) = \underbrace{3\cos^2(x)}_{f'(g(x))} \cdot \underbrace{(-\sin x)}_{g'(x)} = -3\cos^2 x \sin x$$

■ Question 1.

Let $f(x)$ be a function with

$$f(1) = 1, \quad f(2) = 2, \quad f'(1) = 3, \quad f'(2) = 5.$$

If $g(x) = 2f(2x) + f(x)$, what is $g'(1)$?

■ Question 2.

Let f be a function which is differentiable on the entire real line. Find the derivative of $f(x^3) - (f(x))^3$.

■ Question 3.

Suppose $f(x)$ and $g(x)$ and their derivatives have the values given in the table.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	1	5	2	-5
1	3	-2	0	1
2	0	2	3	1
3	2	4	1	-6

Let $h(x) = f(g(x))$. Find the following.

a) $h'(0)$ b) $h'(1)$ c) $h'(2)$ d) $h'(3)$

§C. Chain Rule with more than two functions

Using the chain rule twice we can similarly write

$$f(g(h(x))) = f'(g(h(x)))g'(h(x))h'(x)$$

As an example, consider the function

$$P(x) = \frac{1}{\sin\left(\frac{x^2}{5}\right) - 1}$$

The order of operation here is as follows

$$x \longrightarrow \frac{x^2}{5} \longrightarrow \sin\left(\frac{x^2}{5}\right) - 1 \longrightarrow \frac{1}{\sin\left(\frac{x^2}{5}\right) - 1}$$

We broke our function into exact steps as above because we want to be able to take derivative at each step using the simpler rules we have learned so far. We would like to write $\mathbf{P}(x)$ as $f(g(h(x)))$. Note that h is applied first to x, and then g and then f. So in the above sequence of steps, we can identify f, g and h as follows:

$$x \xrightarrow{h} \underbrace{\frac{x^2}{5}}_{h(x)} \xrightarrow{g} \underbrace{\sin\left(\frac{x^2}{5}\right) - 1}_{g(h(x))} \xrightarrow{f} \underbrace{\frac{1}{\sin\left(\frac{x^2}{5}\right) - 1}}_{f(g(h(x)))}$$

where

$$h(x) = \frac{x^2}{5} \implies h'(x) = \frac{2x}{5}$$

$$g(x) = \sin x - 1 \implies g'(x) = \cos x \implies g'(h(x)) = \cos\left(\frac{x^2}{5}\right)$$

$$f(x) = \frac{1}{x} = x^{-1} \implies f'(x) = -x^{-2} \implies f'(g(h(x))) = -\left(\sin\left(\frac{x^2}{5}\right) - 1\right)^{-2}$$

So,

$$\begin{aligned} \mathbf{P}'(x) &= f'(g(h(x)))g'(h(x))h'(x) \\ &= -\left(\sin\left(\frac{x^2}{5}\right) - 1\right)^{-2} \cos\left(\frac{x^2}{5}\right)\frac{2x}{5} \end{aligned}$$

■ Question 4.

Find the derivative of

$$y = \frac{1}{(\tan(\sin(x)))^2}$$