MATH 111 - Calculus and Analytic Geometry I

Lecture 37-39 Worksheet

Fall 2020 | **Subhadip Chowdhury** | **Nov 11,13,16**

Title: The Fundamental Theorem of Calculus

Summary: We will learn why differentiation and integration are considered opposite processes and evaluate some simple integrals.

Related Reading: Chapter 5.(3,4) from the textbook.

§A. Antiderivative and The Indefinite Integral

Definition A.1

If g and G are functions such that $G' = g$, we say that G is an **antiderivative** of g.

Example A.1

For example, if $g(x) = 3x^2+2x$, then $G(x) = x^3+x^2$ is an antiderivative of g, because $G'(x) = g(x)$. Note that we say "an" antiderivative of g rather than "the" antiderivative of g, because $H(x) = x^3 + x^2 + 5$ is also a function whose derivative is g, and thus H is another antiderivative of g.

Example A.2

Let $f(x) = 2x$. Notice that any of

$$x^2,\ x^2+1,\ x^2-\sqrt{7},$$

etc. are antiderivatives for $f(x)$. In fact, **all** antiderivatives for $f(x) = 2x$ are necessarily of the form $x^2 + C$, for any arbitrary constant C. See Figure 1 below.

■ Question 1.

□

(a) Write down a few antiderivatives for $f(x) = e^{2x}$. What form must all antiderivatives of $f(x)$ have?

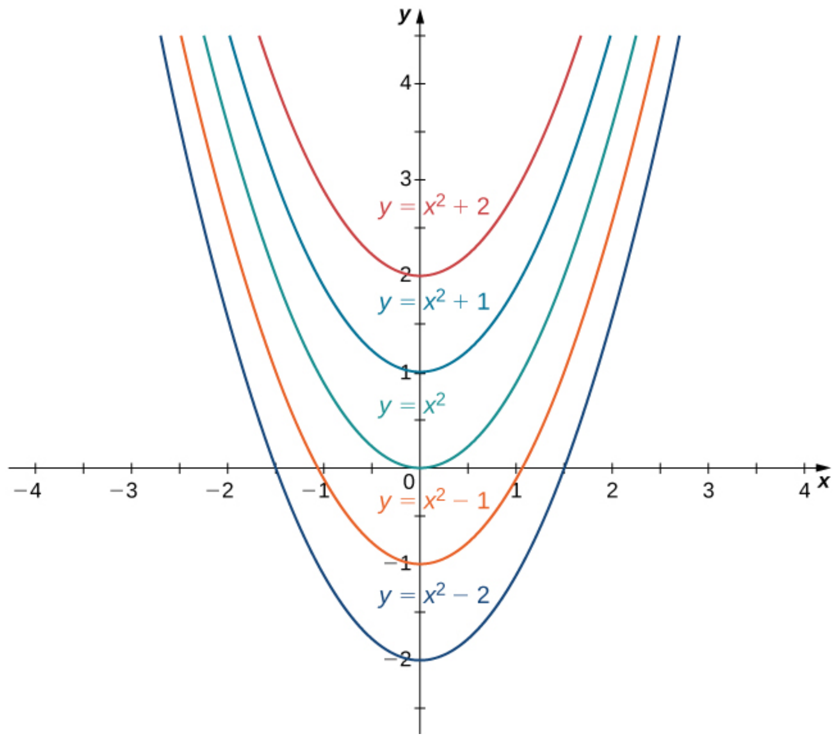

Figure 1: Family of antiderivatives $x^2 + \mathbf{C}$ for the function $f(x) = 2x$

(b) Using absolute value, the function $\mathbf{F}(x) = \mathbf{ln}\,|x|$ can be written $\mathbf{ln}\,|x| = \begin{cases} \mathbf{ln}(x) & \text{if } x > 0 \\ \mathbf{ln}(-x) & \text{if } x < 0 \end{cases}$ Verify that $\mathbf{ln}\,|x|$ is an antiderivative for $f(x) = \dfrac{1}{x}$. What form must all antiderivatives of $f(x) = \dfrac{1}{x}$ have?

Suppose that $\mathbf{F}(x)$ and $\mathbf{G}(x)$ are both antiderivatives for a function $f(x)$ on the interval $[a, b]$. Then $\mathbf{F}'(x) = f(x) = \mathbf{G}'(x)$ on $[a, b]$. The following consequence of the Mean Value Theorem from Chapter 4.4 tells us that $\mathbf{F}(x)$ and $\mathbf{G}(x)$ must differ by only a constant:

Theorem A.1

If $\mathbf{F}$ is any antiderivative of f on an interval $\mathbf{I}$, then for any constant $\mathbf{C}$, the function $\mathbf{G}(x) = \mathbf{F}(x) + \mathbf{C}$ is also an antiderivative of f on that interval. Moreover, **any** antiderivative of f on the interval $\mathbf{I}$ can be expressed in the form $\mathbf{F}(x) + \mathbf{C}$ by choosing the constant $\mathbf{C}$ appropriately.

With all of this in mind, we define the **indefinite integral** as follows.

Definition A.2

Given a function f, the **indefinite integral** of f,

$$\int f(x)\,dx,$$

denotes the most general antiderivative of f. If $\mathbf{F}(x)$ is an antiderivative of $f(x)$, then we write

$$\int f(x)\,dx = \mathbf{F}(x) + \mathbf{C},$$

where $\mathbf{C}$ is an arbitrary constant.

Example A.3

Just as the **power rule** was very useful when taking derivatives, it's corresponding antiderivative formula is also very useful:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + \mathbf{C} \text{ for } n \neq -1.$$

This can be proved simply by checking that the derivative of $\dfrac{x^{n+1}}{n+1}$ is indeed x^n. Any derivative formula from Chapter 3 can be reverse-engineered to yield an antiderivative formula.

Question 2.

☐

Find the indefinite integrals of $f(x) = x^{-1}$ and $f(x) = e^x$.

Question 3.

☐

Find the indefinite integrals of $f(x) = \sin x$ and $f(x) = \cos x$.

§B. **Integration as Anti-differentiation**

Suppose we know the position function $s(t)$ and the velocity function $v(t)$ of an object moving in a straight line, and for the moment let us assume that $v(t)$ is positive on $[a,b]$. Then, as shown in Figure 2, we know two different ways to compute the distance, $\mathbf{D}$, the object travels: one is that $\mathbf{D} = s(b) - s(a)$, the object's change in position. The other is the area under the velocity curve, which is given by the definite integral, so $\mathbf{D} = \int_a^b v(t)dt$.

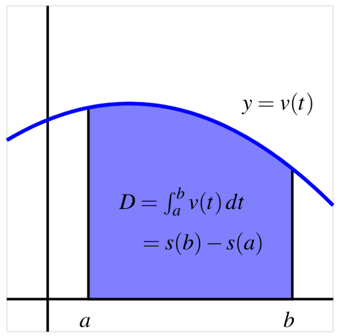

Figure 2: Finding distance traveled in two different ways

Since both expressions tell us the distance travelled, it follows that they must be equal. So

$$s(b) - s(a) = \int_a^b v(t)\,dt$$

The following theorem summarizes above observation.

Theorem B.1: The Fundamental theorem of Calculus Part I

Let f be continuous on $[a,b]$. Then for any antiderivative $\mathbf{F}$ of f on $[a,b]$,

$$\int_a^b f(x)\,dx = \mathbf{F}(b) - \mathbf{F}(a).$$

Theorem B.2: Corollary of FTC: Net Change Theorem

Let $\mathbf{F}(t)$ be some quantity with a continuous rate of change $\mathbf{F}'(t)$. Then

$$\int_a^b \mathbf{F}'(t)\,dt = \mathbf{F}(b) - \mathbf{F}(a)$$

In other words, the integral of a rate of change gives total change.

The FTC is one of the most important theorems we will learn in Calculus as it allows us to easily compute definite integrals. All we need to do is determine an antiderivative for the function $f(x)$. Since all antiderivatives differ by a constant, you can simply pick the choice that is easiest to use.

Example B.1

For example, suppose you are computing $\int_0^1 x^2\,dx$. We can write the general antiderivative here as $\int x^2\,dx = \frac{x^3}{3} + C$, where C is any constant. For the purposes of evaluating the definite integral, we can just use the $\frac{x^3}{3}$ part, like so:

$$\begin{aligned}\int_0^1 x^2\,dx &= \left.\frac{x^3}{3}\right|_0^1 \\ &= \frac{1}{3}\end{aligned}$$

Note the following shorthand notation:

$$F(x)\Big|_a^b = F(b) - F(a).$$

We will often write this as part of our work when evaluating definite integrals, as it allows us to write down the antiderivative first before trying to plug values into it (and likely confusing ourselves in the process).

■ Question 4. □

Determine the exact distance traveled on $[1,5]$ by an object with velocity function $v(t) = 3t^2 + 40$ m/sec.

■ Question 5. □

Evaluate the following definite integrals.

(a) $\displaystyle\int_1^2 \left(6x^2 - 3x\right)\, dx$

(b) $\displaystyle\int_{-1}^1 (5y^4 + y^2 - y)\, dy$

(c) $\displaystyle\int_{-8}^8 x^{1/3}\, dx$

(d) $\displaystyle\int_{-2}^{-1} \frac{1}{x^3}\, dx$

(e) $\displaystyle\int_0^\pi \sin(x)\, dx$

(f) $\displaystyle\int_0^{\pi/2} \cos(\theta)\, d\theta$

(g) $\displaystyle\int_0^{3} |x^2 - 1|\, dx$

(h) $\displaystyle\int_0^{4} |x^2 - 4x + 3|\, dx$

(i) $\displaystyle\int_0^{\pi} 3|\cos x|\, dx$

■ Question 6. □

Using your answers from Problem 1, write out each indefinite integral. Then, evaluate the given definite integral.

(a) Determine $\int e^{2x}\,dx$. Then, compute $\int_0^1 e^{2x}\,dx$.

(b) Determine $\int \frac{1}{x}\,dx$. Then, compute $\int_1^e \frac{1}{x}\,dx$.

■ Question 7. □

Use what you remember about derivatives to determine the general antiderivative. Then, evaluate the given definite integral.

(a) Determine $\int \frac{1}{1+x^2}\,dx$. Then, compute $\int_{-1}^1 \frac{1}{1+x^2}\,dx$.

(b) Determine $\int \sec(x)\tan(x)\,dx$. Then, compute $\int_0^{\pi/4} \sec(x)\tan(x)\,dx$.

■ Question 8. □

For this problem, you are given the general antiderivative. Verify this, then use it to compute the given definite integral.

(a) Verify that $\int x\cos(x)\,dx = x\sin(x) + \cos(x) + C$. Then compute $\int_0^{\pi/2} x\cos(x)\,dx$.

(b) Verify that $\int \ln(x)\,dx = x\ln(x) - x + C$. Then compute $\int_1^{e} \ln(x)\,dx$.

§C. Differentiation as Anti-integration

Theorem C.1: The Fundamental theorem of Calculus Part II

Let f be a continuous function on $[a,b]$ and define an area function $F(x)$ by:

$$F(x) = \int_a^x f(t)\,dt.$$

Then $F(x)$ is continuous on $[a,b]$, differentiable on (a,b), and $F'(x) = f(x)$. In other words,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Because of this theorem, we know that any continuous function $f(x)$ necessarily has an antiderivative. That is, on the interval $[a, b]$ we know that $\mathrm{F}(x) = \int_a^x f(t)\,dt$ is an antiderivative by FTC II (theorem C.1). We can easily check that it satisfies FTC I (theorem B.1):

$$\mathrm{F}(b) - \mathrm{F}(a) = \int_a^b f(t)\,dt - \int_a^a f(t)\,dt = \int_a^b f(t)\,dt - 0 = \int_a^b f(t)\,dt.$$

■ Question 9. □

Make sure you can apply this theorem to the following problems:

(a) In the theorem above, what is $\mathrm{F}(a)$?

(b) Calculate $\dfrac{d}{dx}\displaystyle\int_{-2}^{x} \tan\left(\frac{1}{u^2+1}\right) du$.

(c) Calculate $\dfrac{d}{dx}\displaystyle\int_{\sqrt{x}}^{1} \tan(y)\,dy$.

(d) Calculate $\dfrac{d}{dx}\displaystyle\int_{\sqrt{x}}^{x^2} \tan(y)\,dy$. [Hint: Write the integral as a difference]

■ Question 10.

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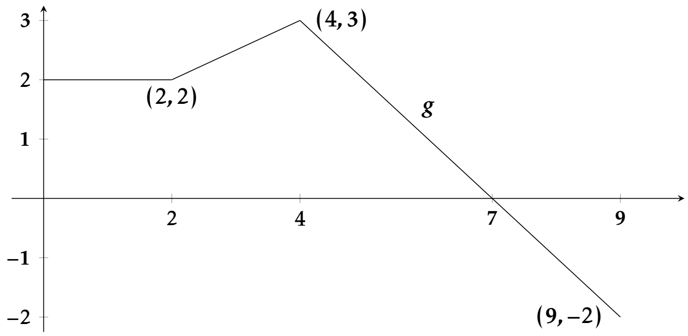
Let g be the function graphed below and let $G(x) = \int_0^x g(t)\,dt$, so $G'(x) = g(x)$.



(a) Evaluate each of the following:

(i) $G(0)$

(ii) $G(2)$

(iii) $G(4)$

(iv) $G(9)$

(v) $G'(4)$

(vi) $G'(7)$

(b) Find the largest open interval on which G is increasing.

(c) Find the largest open interval on which G is decreasing.

(d) What is the minimum of G on $[0, 9]$? Justify.

(e) What is the maximum of G on $[0, 9]$? Justify.

■ Question 11.

□

Use the graph of a function below to answer the given questions about the area function $\mathbf{F}(x) = \int_0^x f(t)\,dt$.

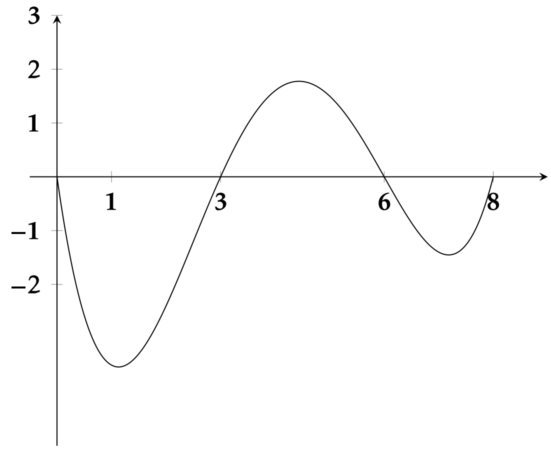


Figure 3: Graph of function $f(t)$ for problem #3

(a) Where is **F** increasing/decreasing?

(b) Find all critical points of **F** and determine which are local extrema.

(c) Since **F** is continuous on $[0, 8]$, it must have absolute extrema. Where do the absolute extrema for **F** occur?

(d) Where is **F** concave up/concave down?

(e) Does **F** have any points of inflection?

(f) Sketch a possible graph of **F**.