MATH 111 - Calculus and Analytic Geometry I

Lecture 15 Worksheet

Fall 2020 **Subhadip Chowdhury** **Sep 21**

Title: Quotient Rule

Summary: We will talk about an important differentiation rules that might not match your expectations.

§A. Trigonometric Functions

Before moving forward with other basic rules of differentiation, let's increase the size of our formula catalogue a bit. We are going to look at the function $f(x) = \sin(x)$ and try to find its derivative.

First, let's look at the graph of $f(x) = \sin x$.

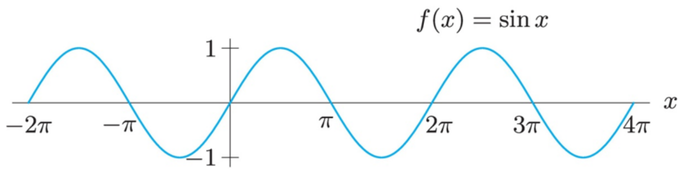

Figure 1: The sine function

We might ask, where is the derivative equal to zero? Then ask where the derivative is positive and where it is negative. In exploring these answers, we get something like the following graph.

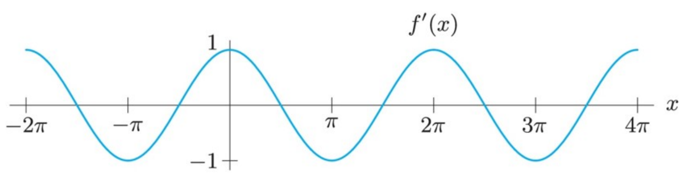

Figure 2: Derivative of the sine function

The graph of the derivative in Figure 2 looks suspiciously like the graph of the cosine function. This might lead us to conjecture, quite correctly, that the derivative of the sine is the cosine. For a more formal proof, we need the following trigonometric identity:

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

Then we can write

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin x(\cos h - 1) + \cos x\sin h}{h}\\
&= \sin x\lim_{h\to 0}\frac{(\cos h - 1)}{h} + \cos x\lim_{h\to 0}\frac{\sin h}{h}\\
&= \sin(x)\times 0 + \cos x\times 1\\
&= \cos x
\end{aligned}$$

where we used the fact that $\lim_{h\to 0} \frac{\sin h}{h} = 1$ and $\lim_{h\to 0} \frac{(\cos h - 1)}{h} = 0$. A proof of the second result is at the end of this worksheet. Similarly, we can calculate derivative of $\cos x$ using the identity

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

Theorem A.1: Trig Derivatives

For x in radians,

$$\frac{d}{dx}\sin x = \cos x \quad \text{and} \quad \frac{d}{dx}\cos x = -\sin x$$

We will learn derivatives of the other trigonometric functions after we learn Quotient Rule.

§B. Quotient Rule

As with the product rule, the derivative formula for the derivative of a quotient of two differentiable functions is anything but what we would expect.

Question 1.

Come up with an example of two functions $f(x)$ and $g(x)$ such that $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] \neq \frac{f'(x)}{g'(x)}$.

Here is the actual quotient rule formula:

Theorem B.1

For $f(x)$ and $g(x)$ differentiable functions,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Question 2.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-2	4	5	-1	2
1	3	-1	2	-2

Table 1: Table for Problem 8

Let's practice! Use Table 1 to compute the given derivative value.

(a) Find $h'(1)$ if $h(x) = \frac{f(x)}{g(x)}$.

(b) Find $k'(-2)$ if $k(x) = \frac{xg(x)}{f(x)}$.

(c) Find $\mathbf{L}'(\mathbf{1})$ if $\mathbf{L}(x) = \dfrac{x^3+4}{f(x)+g(x)}$.

An immediate and useful application of the quotient rule is extending the power rule to include negative exponents.

Theorem B.2: Power Rule Part II

For any non-zero integer n,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Question 3.

Use the quotient rule to prove that the power rule applies for negative integers.

Question 4.

Calculate $\mathbf{F}'(0)$ for the function $\mathbf{F}(x) = \dfrac{x^9+x^8+4x^5-7x}{x^4-3x^2+2x+1}$. This problem can be made easier by computing $f(0), g(0), f'(0)$, and $g'(0)$ separately, and then combining the numbers into the quotient rule formula. Once you have $\mathbf{F}'(0)$, write an equation for the tangent line at $x=0$.

Question 5.

Find all values of $x=a$ such that the tangent line to $f(x) = \dfrac{x-1}{x+8}$ at $x=a$ passes through the origin.

Question 6.

Use the quotient rule to find derivatives of the following trigonometric functions.

(a) $\dfrac{d}{dx}[\tan x]$

(b) $\dfrac{d}{dx}[\cot x]$

(c) $\dfrac{d}{dx}[\sec x]$

(d) $\dfrac{d}{dx}[\csc x]$

Question 7.

See if you can't derive the quotient rule formula using the help of Figure 3. Follow the steps below.

(a) Firstly, write out $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right]$ using the definition of the derivative. Then, combine your fractions in the numerator.

(b) Write out the rectangles **C** and **D** in terms of $f(x), g(x), f(x+h)$, and $g(x+h)$.

(c) In part (1), did you get the difference $f(x+h)g(x) - f(x)g(x+h)$? Identify this difference in terms of areas in Figure 3.

(d) Put everything together and you've just proven the quotient rule!!

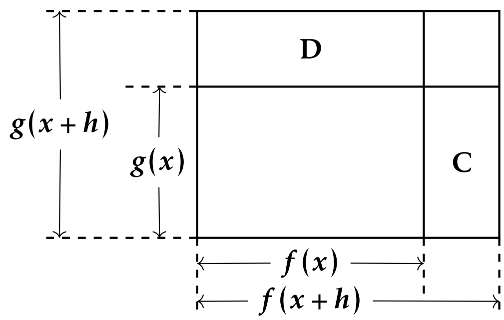

Figure 3: Rectangle Aid for Quotient Rule Formula

§C. Proof of a certain result

To calculate $\lim_{h\to 0} \frac{(\cos h - 1)}{h}$, we can use a sort of 'multiply by the conjugate' method.

$$\begin{aligned}
\lim_{h\to 0} \frac{(\cos h - 1)}{h} &= \lim_{h\to 0} \frac{(\cos h - 1)(\cos h + 1)}{h(\cos h + 1)} \\
&= \lim_{h\to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} \\
&= \lim_{h\to 0} \frac{-\sin^2 h}{h(\cos h + 1)} \\
&= -\left(\lim_{h\to 0} \frac{\sin h}{h}\right)\left(\lim_{h\to 0} \frac{\sin h}{\cos h + 1}\right) \\
&= -(1)\left(\frac{\sin 0}{\cos 0 + 1}\right) \\
&= -(1)\left(\frac{0}{1+1}\right) \\
&= 0
\end{aligned}$$