MATH 111 - Calculus and Analytic Geometry I

Lab 10 Worksheet

Fall 2020 **Subhadip Chowdhury** **Nov 10**

§A. Properties of Definite Integrals

Question 1. Limits as Definite Integral

In the following problems, express the limits as definite integrals. **Do not evaluate the integrals.**

(a) $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{1}{n} \sin^2\left(2\pi \frac{i}{n}\right)$

(b) $\lim\limits_{n\to\infty} \frac{2}{n} \sum\limits_{i=1}^{n} \left(1 + \frac{2i}{n}\right)$

(c) $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \left(5 + \frac{3i}{n}\right)^4 \left(\frac{2}{n}\right)$

(d) $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{9i + 3n}{3in + 2n^2}$

(e) **(Optional)** $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{1 - i + 2n}{1 - i + n} \left(\frac{1}{n}\right)$

We learned the following properties of Definite Integrals in class yesterday.

Theorem A.1: Properties about the Limit of Integration

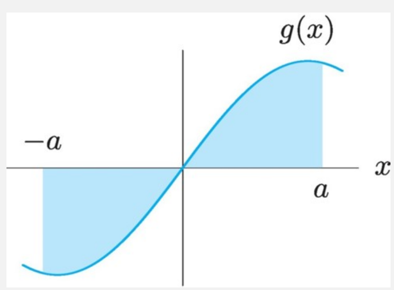
- $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$
- $\int_a^a f(x)dx = 0$
- $\int_b^a f(x)dx = -\int_a^b f(x)dx$

Theorem A.2: Properties of Sums and Constant multiples of the Integrand

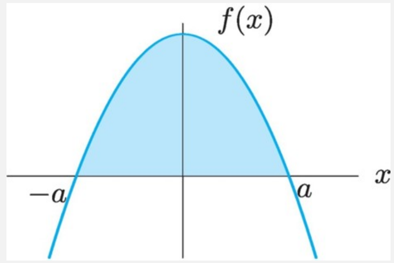
- $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
- $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ (if c is constant)

Theorem A.3: Using Symmetry to Evaluate Definite Integrals

- If f is an odd function, then $\int_{-a}^a g(x)\,dx = 0$



- If f is an even function, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$



■ Question 2.

Using what you know about area under a curve, explain why

$$\int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx$$

■ Question 3.

Book problem 5.2.124

Suppose that $\mathbf{A} = \displaystyle\int_{0}^{\pi/2} \sin^2 t\,dt$ and $\mathbf{B} = \displaystyle\int_{0}^{\pi/2} \cos^2 t\,dt$. Show that $\mathbf{A} + \mathbf{B} = \dfrac{\pi}{2}$ and $\mathbf{A} = \mathbf{B}$. What is the value of $\mathbf{A}$?

Question 4.

Book Problems 5.2.141

Draw the graph of $f(x) = \sin(2\pi x)$ using Desmos, or look it up in the textbook.

(a) Explain why $\int_0^1 \sin(2\pi t)\,dt = 0$

(b) Explain why, in general $\int_a^{a+1} \sin(2\pi t)\,dt = 0$ for any real number a.

Question 5.

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If a function $f(x)$ is 1-periodic (i.e. $f(t+1) = f(t)$), odd, and integrable over $[0,1]$, is it always true that $\int_0^1 f(t)\,dt = 0$?

Question 6.

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If a function $f(x)$ is T-periodic (i.e. $f(x+T) = f(x)$) and $\int_0^T f(x)\,dx = A$, is it necessarily true that $\int_a^{a+T} f(x)\,dx = A$ for all A?

Definition A.1: Definite Integrals as an Average

Let $f(x)$ be continuous over the interval $[a, b]$. Then, the average value of the function $f(x)$ (or $_{ave}$) on $[a, b]$ is given by

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x)\, dx$$

■ Question 7.

Book Problem 5.2.112

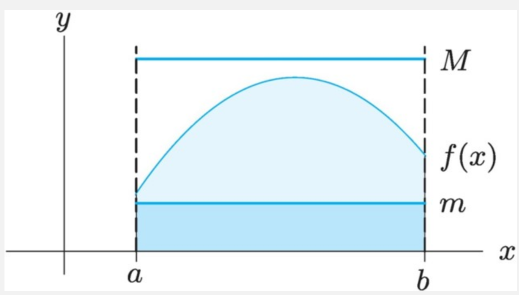
Find the average value f_{ave} for the function $f(x) = \sqrt{4 - x^2}$ on the interval $[0, 2]$.

Theorem A.4: Comparison of Definite Integrals

Let f and g be continuous functions.

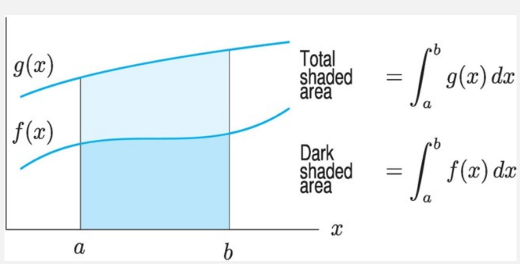
- If $m \leq f(x) \leq \mathbf{M}$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\, dx \leq \mathbf{M}(b-a)$$



- If $f(x) \leq g(x)$ for $a \leq x \leq b$, then

$$\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$$



■ Question 8.

Book Problems 5.2.(106,108,109)

Use the comparison theorems to answer the following questions:

(a) Show that $\displaystyle\int_0^1 \sqrt{1+x^3}\,dx \leq \int_0^1 \sqrt{1+x^2}\,dx$.

(b) Suppose you are given that $\sin t \geq \dfrac{2t}{\pi}$ for all $0 \leq t \leq \dfrac{\pi}{2}$. Show that $\displaystyle\int_0^{\pi/2} \sin t\,dt \geq \frac{\pi}{4}$.

(c) Show that $\displaystyle\int_{-\pi/4}^{\pi/4} \cos t\,dt \geq \frac{\pi\sqrt{2}}{4}$.