# MATH 111 - Calculus and Analytic Geometry I

## Lecture 35 Worksheet

**Fall 2020** | **Subhadip Chowdhury** | **Nov 6**

**Title:** Riemann Sums

**Summary:** We will learn what a Riemann sum is and how to use them to estimate the area between a given curve and the horizontal axis over a particular interval.

**Related Reading**: Chapter 5.1 from the textbook.

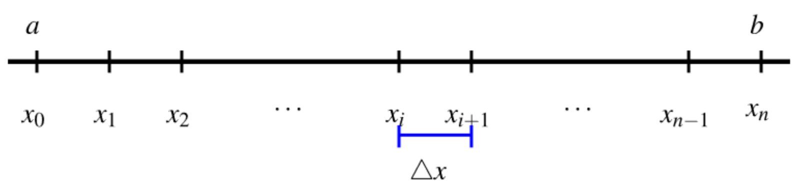
## §A. Sigma Notation for Total Area

Last time we learned, when a moving body has a positive velocity function $y = v(t)$ on a given interval $[a, b]$, the area under the curve over the interval gives the total distance the body travels on $[a, b]$. In general, we are also interested in finding the exact area bounded by $y = f(x)$ on an interval $[a, b]$, regardless of the meaning or context of the function $f$. For now, we continue to focus on finding an accurate estimate of this area by using a sum of the areas of rectangles. In this lecture, we will try to formalize the process.

First of all, let's divide the interval $[a, b]$ into $n$ subintervals as follows:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

Such a choice of subintervals is called a **Partition** of the interval $[a, b]$. We will say that the partition is **regular** if all the subintervals are of equal length.




For a regular partition, we will use $\Delta x$ to denote the length of each subinterval of the form $[x_i, x_{i+1}]$, where $0 \le i \le n - 1$. Check that

$$\Delta x = \frac{b-a}{n}$$

**Example A.1**

Consider again $y = f(x) = x^2$ on the interval $[2, 6]$. Let us write out a regular partition of the interval $[2, 6]$ by evenly spaced $n + 1$ points $x_), x_1, \ldots, x_n$.

We will have that $\Delta x = \dfrac{6-2}{n} = \dfrac{4}{n}$. Thus, our sub-intervals will be:

$$\left[2, 2+1\cdot\frac{4}{n}\right], \left[2+1\cdot\frac{4}{n}, 2+2\cdot\frac{4}{n}\right], \left[2+2\cdot\frac{4}{n}, 2+3\cdot\frac{4}{n}\right], \ldots, \left[2+(n-1)\cdot\frac{4}{n}, 2+n\cdot\frac{4}{n}\right].$$

Thus in general, the $i$-th subinterval $[x_i, x_{i+1}]$ can be written as $\left[a + i\frac{b-a}{n}, a + (i+1)\frac{b-a}{n}\right]$ for $i = 0, 1, \ldots, n-1$.

Now suppose we want to write down the formula for the left-endpoint approximation with $n$ rectangles. Geometrically, this means that the height of the $i$-th rectangle on the subinterval $[x_i, x_{i+1}]$ is the function value at the left end point, i.e. $f(x_i)$. So the area of this rectangle is

$$f(x_i)\Delta x = f\left(a + i\frac{b-a}{n}\right)\frac{b-a}{n}$$

And so the Left-endpoint approximation $\mathbf{S}_n$ is given by

$$\mathbf{S}_n = \sum_{i=0}^{n-1} f(x_i) \cdot \Delta x = \sum_{i=0}^{n-1} f\left(a + i\frac{b-a}{n}\right)\frac{b-a}{n}$$

**Example A.2**

In the case of $f(x) = x^2$ on the interval $[2, 6]$, we get

$$\begin{aligned}\mathbf{S}_n = \sum_{i=0}^{n-1} f(x_i) \cdot \Delta x &= \sum_{i=1}^{n} f\left(2 + \frac{4i}{n}\right) \cdot \frac{4}{n} \\ &= \sum_{i=0}^{n-1} \left(2 + \frac{4i}{n}\right)^2 \cdot \frac{4}{n}\end{aligned}$$

■ **Question 1.** □

Write down the formula for the right-endpoint approximation using the sigma notation.

## §B. Riemann Sums

So far the heights of the rectangles used for our approximation have been determined by evaluating the function at either the right or left endpoints of the subinterval $[x_i, x_{i+1}]$. In reality, there is no reason to restrict evaluation of the function to one of these two points only. We could evaluate the function at any random point $x_i^*$ in the subinterval $[x_i, x_{i+1}]$, and use $f(x_i^*)$ as the height of our rectangle. This gives us an estimate for the area of the form

$$\mathbf{A}_n = \sum_{i=0}^{n-1} f(x_i^*) \cdot \Delta x$$

A sum of this form is called a **Riemann sum**. Thus the left-endpoint approximation and the right-endpoint approximation are both examples of Riemann sums. There are other Riemann sums like Middle-point approximation, Upper sum, Lower sum etc depending on how we choose $x_i^*$. However, as $n \to \infty$, i.e. as the number of rectangles increase, all of the Riemann sums go towards the same limit, which is the actual area under the curve.

$$\boxed{\mathbf{A} = \text{Area under } f(x) \text{ on the interval } [a,b] = \lim_{n\to\infty} \mathbf{A}_n = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i^*)\cdot \Delta x}$$

Look at this applet for an interactive visualization of this concept:

https://webspace.ship.edu/msrenault/GeoGebraCalculus/integration_riemann_sum.html

## ■ Question 2. □

What are upper sums and lower sums?

Consider the function $f(x) = \sin(x)$ on the interval $[0,\pi]$. In this case, if we wanted to get the exact area under the curve, we could compute the following limit with right-endpoint Riemann sum:

$$\lim_{n\to\infty} \mathbf{A}_n = \lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\frac{\pi i}{n}\right)\frac{\pi}{n}$$

That does not look like a limit that we can actually compute, especially with the summation involved. We would need a formula for the sum of the sine function, and we do not readily have such a formula, nor does it really behoove us to figure one out.

However, for some simple polynomials of low degree, we can use the summation formulas from Lab 9 to actually compute the area under the curve using limits. We will only do this once.

## ■ Question 3. □

Consider again $f(x) = x^2$ on $[2,6]$. Compute the limit $\lim_{n\to\infty} \mathbf{A}_n$ using the sum formulas from Lab 9.

$$\lim_{n\to\infty} \mathbf{A}_n = \lim_{x\to\infty} \sum_{i=1}^{n} \left(2+\frac{4i}{n}\right)^2 \cdot \frac{4}{n}$$