MATH 111 - Calculus and Analytic Geometry I

Lecture 33 Worksheet

Fall 2020 | **Subhadip Chowdhury** | **Nov 2**

Title: Determining distance traveled from velocity

Summary: We will learn how the problem of finding distance traveled is related to finding the area under a certain curve, and look into some ways to approximate it.

Related Reading: Chapter 5.1 from the textbook.

§A. Motivating Question

If we know the velocity of a moving body at every point in a given interval, can we determine the distance the object has traveled on the time interval? What is the velocity is changing over time? Can we perhaps estimate it graphically?

Question 1. — Preview Example

Suppose that a person is taking a walk along a long straight path and walks at a constant rate of 3 miles per hour.

(a) On the left-hand axes provided below, sketch a labeled graph of the velocity function $v(t) = 3$.

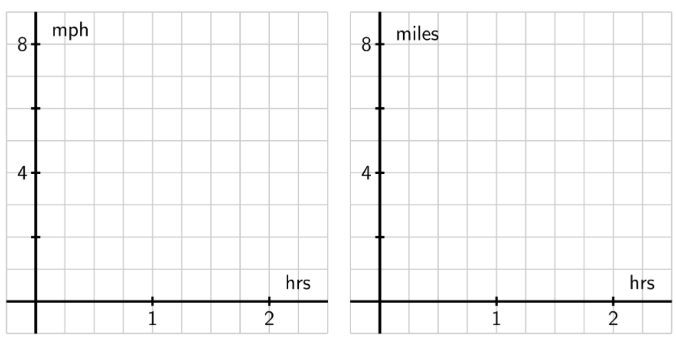


Figure 1: At left, axes for plotting $y = v(t)$; at right, for plotting $y = s(t)$.

Note that while the scale on the two sets of axes is the same, the units on the right-hand axes differ from those on the left. The right-hand axes will be used in question (d).

(b) How far did the person travel during the two hours? How is this distance related to the area of a certain region under the graph of $y = v(t)$?

(c) Find an algebraic formula, $s(t)$, for the position of the person at time t, assuming that $s(0) = 0$. Explain your thinking.

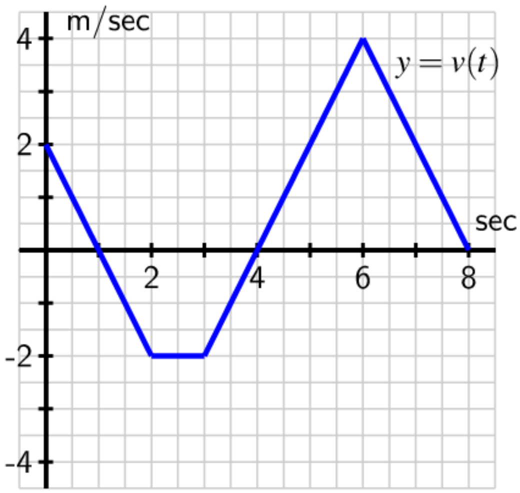
(d) On the right-hand axes provided in the picture above, sketch a labeled graph of the position function $y = s(t)$.

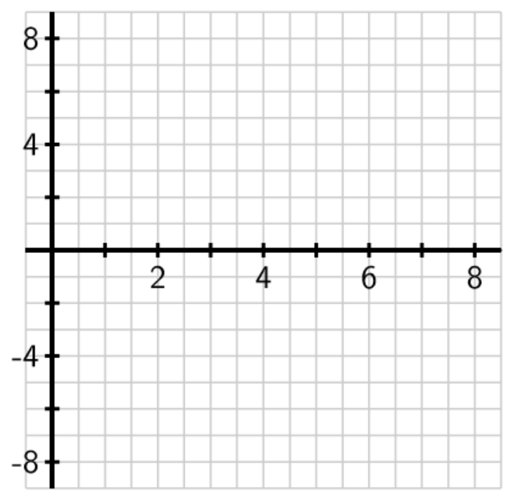
(e) For what values of t is the position function s increasing? Explain why this is the case using relevant information about the velocity function v.

§B. Area under the graph of the velocity function

In above example, we observed that when the velocity of a moving object is constant (and positive), the area under the velocity curve over an interval of time tells us the distance the object traveled. The situation is becomes gradually more complicated when the velocity function is not constant.

To begin with, Suppose that an object moving along a straight line path has its velocity v (in meters per second) at time t (in seconds) given by the piecewise linear function i.e. the function $v(t)$ is piece-wise defined and each part is a linear function. We view movement to the right as being in the positive direction (with positive velocity), while movement to the left is in the negative direction.





Question 2.

Suppose further that the object's initial position at time $t = 0$ is $s(0) = 1$.

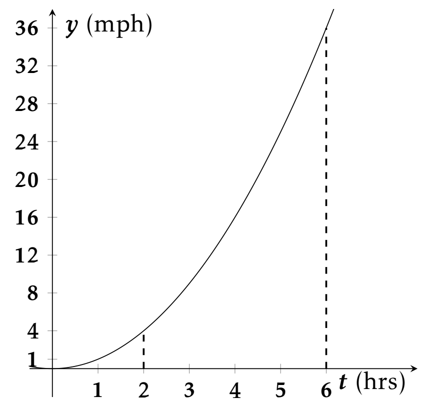
(a) Determine the total distance traveled and the total change in position on the time interval $0 \leq t \leq 2$. What is the object's position at $t = 2$?

(b) On what time intervals is the moving object's position function increasing? Why? On what intervals is the object's position decreasing? Why?

(c) What is the object's position at $t = 8$? How many total meters has it traveled to get to this point (including distance in both directions)? Is this different from the object's total change in position on $t = 0$ to $t = 8$?

(d) Find the exact position of the object at $t = 1, 2, 3, \ldots, 8$ and use this data to sketch an accurate graph of $y = s(t)$ on the axes provided at right in the figure above. How can you use the provided information about $y = v(t)$ to determine the concavity of s on each relevant interval?

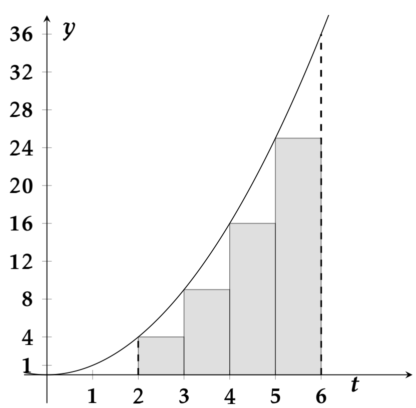
Next consider the case of a general velocity function such as $v(t) = t^2$ given below. Let's try to approximate the area under the curve $y = t^2$ between $t = 2$ and $t = 6$, as seen in the graph to the left.



We can estimate the area under the curve using rectangles. Let's use four. Since we have four rectangles, the base of each rectangle is 1 unit long, because our interval (from $x = 2$ to $x = 6$) is 4 units long.

So, we have four rectangles with bases in the intervals $[2, 3]$, $[3, 4]$, $[4, 5]$, $[5, 6]$. But, where do we get the height of our rectangle from? There are two simple ways to do this.

Left-Endpoint Approximation



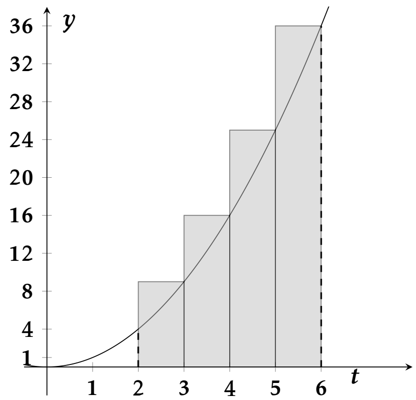
When using left-endpoints for our rectangles, first evaluate the function at each *left* end point of the intervals.

In our case we have: $v(2) = 4$, $v(3) = 9$, $v(4) = 16$, and $v(5) = 25$.

Then, draw a rectangle in each interval with a height equal to that of the value of the function at the left-endpoints, as seen in the graph.

Using the four rectangles, what is the approximate area? Do you think this is an over estimate or an under estimate? Explain.

Right-Endpoint Approximation



When using right-endpoints for our rectangles, first evaluate the function at each *right* end point of the intervals.

In our case we have: $v(3) = 9$, $v(4) = 16$, $v(5) = 25$, and $v(6) = 36$.

Then, draw a rectangle in each interval with a height equal to that of the value of the function at the right-endpoints, as seen in the graph.

Using the four rectangles, what is the approximate area? Do you think this is an over estimate or an under estimate? Explain.

■ Question 3.

□

Approximate the area under $y = x^2$ on $[2, 6]$ with 8 rectangles, using left-endpoints and then right-endpoints.

(a) Left-Endpoint Approximation

(b) Right-Endpoint Approximation

What would happen if we had 100 rectangles? 1,000 rectangles? An infinite number of rectangles?

■ Question 4.

Now, you try!

Approximate the area under the curve $y = \frac{1}{x}$ from $x = 1$ to $x = 5$ with 4 rectangles, using each method. The graph is given below.