Calculus & Analytical Geometry II

Lecture 17 Worksheet

Spring 2021 **Subhadip Chowdhury** **Math 112**

Differential Equations show up everywhere in the world around us whenever a quantity changes over time according to some rule. Unfortunately, the rules may be complex enough that the differential equation is really complicated and solving it is not always feasible. But for a certain type of differential equation, we can have a simple algorithm to get exact analytical solutions of differential equations.

§A. Analytical Solution of Differential Equations

Example A.1

Suppose an ant is moving along a curve in the XY-plane such that for every point (x, y) on the curve, the tangent line to the curve is perpendicular to the line joining (x, y) to the origin $(0, 0)$. What does the path of the ant look like?

The slope of the tangent to the curve at point (x, y) is given by _____

The slope of the line joining (x, y) to the origin $(0, 0)$ is given by _____

If the two lines are perpendicular, what can we say about the product of the slopes? _____

You have found a differential equation by translating the word problem to Math. Now solve the differential equation.

Definition A.2

A differential equation is called separable if it can be written in the form

$$\frac{dy}{dx} = f(x)g(y).$$

■ Question 1. □

Which one of the following DEs are separable?

(i) $y' = x\sec(y)$ *(ii)* $y' = x + \sin(y)$ *(iii)* $y' = 1 + y^2$

(iv) $y' = xy + 3x - 2y - 6$ *(v)* $y' = x + y$ *(vi)* $x^2y' = e^y$

§B. Algorithm for solving a Separable DE

Step 1. Factor the right hand side of the ODE as a product $f(x)g(y)$.

Step 2. The values of y for which $g(y) = 0$ are called equilibrium solutions. Make a list of these.

Step 3. Separate the 'x's and the 'y's to two different sides and rewrite the ODE as an equality of differentials. Then integrate both sides. Your answer may be an implicit curve in x and y.

Step 4. If an initial condition is given, use it to find the constant of integration.

■ Question 2. □

Solve the separable DEs from Question 1. Check your answers using a computer.

■ Question 3. □

Solve the differential equation

$$\frac{dy}{dx} = ky$$

and check that the only non-constant solution is the exponential function $y(x) = y_0 e^{kx}$ for some arbitrary constant y_0 (which is the initial value).

§C. More Practice Problems

■ Question 4. □

Find the general solution to the differential equation $x^2y' + y = 0$.

■ Question 5. □

Find the specific solution to the initial value problem $y' = e^{x-y}$, $y(0) = 0$.

■ Question 6. □

Find the general solution to the differential equation $e^x \dfrac{dx}{dt} = \dfrac{3t^2}{1+x}$.

■ Question 7. □

Suppose you are having a dinner party for a large group of people, and you decide to make 2 gallons of chili. The recipe calls for 2 teaspoons of hot sauce per gallon, but you misread the instructions and put in 2 tablespoons of hot sauce per gallon. (Since each tablespoon is 3 teaspoons, you have put in 6 teaspoons per gallon, which is a total of 12 teaspoons of hot sauce in the chili.) You don't want to throw the chili out because there isn't much else to eat (and some people like hot chili), so you serve the chili anyway. However, as each person takes some chili, you fill up the pot with beans and tomatoes without hot sauce until the concentration of hot sauce agrees with the recipe. Suppose the guests take 1 cup of chili per minute from the pot (there are 16 cups in a gallon), how long will it take to get the chili back to the recipe's concentration of hot sauce? How many cups of chili will have been taken from the pot?