MATH 111 - Calculus and Analytic Geometry I

Lecture 8 Worksheet

Fall 2020 | **Subhadip Chowdhury** | **Sep 4**

Title: Evaluating Limits Analytically

Summary: We will learn various algebra techniques that will help us evaluate limits analytically. Then we will learn the second most important theorem of this chapter: The Squeeze Theorem.

Arithmetic of Limits

You might recall the following from our lab activity on Tuesday.

Theorem .1

If $\lim\limits_{x\to c} f(x) = L$ and $\lim\limits_{x\to c} g(x) = K$, then

(a) $\lim\limits_{x\to c}[f(x) \pm g(x)] = L \pm K$

(b) $\lim\limits_{x\to c}[f(x)g(x)] = LK$

(c) $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$ if $K \neq 0$

(d) $\lim\limits_{x\to c} f(x)^n = L^n$ for $n \in \mathbb{Z}$, $L \neq 0$ for $n < 0$

(e) $\lim\limits_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$ for all L if n is odd, and for $L > 0$ if n is even.

(f) $\lim\limits_{x\to c} a = a$

(g) $\lim\limits_{x\to c} x = c$

§A. Limits of Quotients

In calculus we often encounter limits of the form $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are continuous. There are three types of behavior for this type of limit:

- When $g(c) \neq 0$, the limit can be evaluated by substitution.
- When $g(c) = 0$ but $f(c) \neq 0$, the limit is undefined.
- When $g(c) = 0$ and $f(c) = 0$, the limit may or may not exist and can take any value.

The third type is the most interesting and we will spend our class today learning different strategies to handle those limit.

Question 1.

Find the following limits.

(a) $\lim\limits_{x\to 3} \dfrac{x^2+2x+1}{x-1}$

(b) $\lim\limits_{x\to 2} \dfrac{x+1}{x-2}$

Limits of the Form $\frac{0}{0}$

Example A.1: Factorize and Cancel

$$\lim_{x\to-2}\frac{x^2-4}{x+2}=\lim_{x\to-2}\frac{(x-2)(x+2)}{x+2}=\lim_{x\to-2}(x-2)=\lim_{x\to-2}x-\lim_{x\to-2}2=-2-2=4$$

Question 2.

Find the following limits.

(a) $\lim\limits_{x\to3}\dfrac{x^2-3x}{2x^2-5x-3}$. (b) $\lim\limits_{x\to0}\dfrac{(3+x)^2-9}{x}$ (c) $\lim\limits_{x\to1}\dfrac{x-1}{x^2-2x+1}$

Example A.2: Multiply by Conjugate

$$\lim_{x\to4}\frac{\sqrt{x}-2}{x-4}=\lim_{x\to4}\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)}=\lim_{x\to4}\frac{x-4}{(x-4)(\sqrt{x}+2)}=\lim_{x\to4}\frac{1}{\sqrt{x}+2}=\frac{1}{4}$$

Question 3.

Find the following limits.

(a) $\lim\limits_{x\to1}\dfrac{\sqrt[3]{x}-1}{x-1}$. (b) $\lim\limits_{x\to-1}\dfrac{\sqrt{x+2}-1}{x+1}$

Limits of the Form $\frac{\infty}{\infty}$

Example A.3: Divide by the highest power

$$\lim_{x\to\infty}\frac{3+4x^2}{x^2+3x+2}=\lim_{x\to\infty}\frac{\frac{3}{x^2}+4}{1+\frac{3}{x}+\frac{2}{x^2}}=\frac{0+4}{1+0+0}=4$$

Question 4.

Find the following limits.

(a) $\lim\limits_{x\to\infty}\dfrac{3+4x}{x^2+3x+2}$ (b) $\lim\limits_{x\to\infty}\dfrac{3+4x^2}{3x+2}$

§B. Limits of Combinations of Continuous Functions

The textbook uses the following **Composite Function Theorem** to show that the Trigonometric Functions are continuous on their respective domains, but we can also use it to evaluate some interesting limits.

Theorem B.1: Composite Function Theorem

If $f(x)$ is continuous at $\mathbf{L}$, and $\lim\limits_{x\to a} g(x) = \mathbf{L}$, then

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(\mathbf{L}).$$

For the following problems, evaluate the given limit using the above theorem. Feel free to graph the function to confirm your answer.

1. $\lim\limits_{x\to \frac{2\pi}{3}} \sin\left(x - \frac{\pi}{3}\right)$.

2. $\lim\limits_{x\to \frac{2\pi}{3}} \sin\left(1 + \cos\left(x + \frac{\pi}{3}\right)\right)$.

3. $\lim\limits_{x\to \pi^-} \ln(\sin(x))$. (Note: recall that $\lim\limits_{x\to 0^+} \ln(x) = -\infty$.)

4. $\lim\limits_{x\to 1^+} \ln\left(\dfrac{(x-1)(x^2+2x-3)}{x^2-1}\right)$.

§C. The Squeeze Theorem

For the problems in this section, consider the function $f(x) = x^2 \cos\left(\dfrac{\pi}{x}\right)$.

1. Sketch a graph of the function $f(x)$. (You may want to use a graphing utility e.g. DESMOS)

2. Also graph the functions $-x^2$ and x^2 along with $f(x)$.

3. Based off your sketch, do you agree that $-x^2 \le f(x) \le x^2$ for all $x \ne 0$?

4. What is $\lim\limits_{x\to 0} -x^2$? What is $\lim\limits_{x\to 0} x^2$? What can you conclude about $\lim\limits_{x\to 0} x^2 \cos\left(\dfrac{\pi}{x}\right)$?

Theorem C.1: The Squeeze Theorem

If we have

$$b(x) \le f(x) \le a(x)$$

for all x close to $x = c$ except possibly at $x = c$, and if

$$\lim_{x\to c} b(x) = L = \lim_{x\to c} a(x),$$

then

$$\lim_{x\to c} f(x) = L.$$

The book uses the squeeze theorem to show that $\lim\limits_{x\to 0} \dfrac{\sin(x)}{x} = 1$, by using a geometric argument. I suggest you read it, but you do not need to memorize the exact proof. The key to applying the squeeze theorem is determining the inequality, which can come from geometry or an algebra identity. We are going to try to show the same limit as above for the function $f(x) = x^2 \cos\left(\dfrac{\pi}{x}\right)$, but this time without using graphs to justify our reasoning, by following the steps below.

5. What is the range of the cosine function $y = \cos(\theta)$?

6. Use your previous answer to find a constant k such that $-k \le \cos(\theta) \le k$ for every θ.

7. Replace θ with $\dfrac{\pi}{x}$. For what x-values is the inequality still valid?

8. Conclude that you have the same inequality as in (3) above, and apply the squeeze theorem to obtain the limit.

9. Try the same reasoning as above to show that $\lim\limits_{x\to 0^+} x \sin\left(\dfrac{\pi}{x}\right) = 0$.

10. Can you follow the same steps to show that $\lim\limits_{x\to 0^-} x \sin\left(\dfrac{\pi}{x}\right) = 0$?

Example C.1: Two Interesting Examples

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad\qquad \lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

Question 5.

Find $\lim\limits_{x\to 0} \dfrac{\sin 3x}{2x}$.

§D. **Bonus Problems**

Question 6.

Find $\lim\limits_{x \to 0} \dfrac{1 - \cos 4x}{1 - \cos 2x}$.

[**Hint**: You will need to use the double angle formula for cosine. Google it if you have forgotten.]

Question 7.

Find $\lim\limits_{x \to 0} \dfrac{\sin^2 x}{1 - \cos x}$.

[**Hint**: You can multiply both numerator and denominator by an expression to simplify.]