# Calculus & Analytical Geometry II

## Lecture 23 Worksheet

**Spring 2021** **Subhadip Chowdhury** **Math 112**

### §A. **Introduction to Series**

What do we mean when we express a number as an infinite decimal? For instance, what does it mean to write

$$\pi = 3.14159265358979323846264338327950288\ldots?$$

The convention behind our decimal notation is that any number can be written as an infinite sum. Here it means that

$$\pi = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \frac{6}{10^7} + \frac{5}{10^8} + \cdots$$

where the three dots ($\cdots$) indicate that the sum continues forever, and the more terms we add, the closer we get to the actual value of $\pi$.

In general, if we try to add the terms of an infinite sequence $\{a_i\}_{i=1}^{\infty}$ we get an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_i + \cdots$$

which is called an infinite series (or just a series) and is denoted, for short, by the symbol

$$\sum_{i=1}^{\infty} a_i \quad \text{or} \quad \sum a_i$$

But does it always make sense to talk about the sum of infinitely many terms? What about

$$1 + 2 + 3 + 4 + 5 + \ldots$$

Clearly that some is infinite. In other words, the series $\sum_{i=1}^{\infty} i$ diverges. But how about the infinite sum

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

When we plot it in `DESMOS` (see here), we can see that the sum converges to 1.

**Warning:** Be sure to distinguish between the convergence of a sequence vs. the series. The two words do not mean the same thing in Calculus. For example, the sequence $\frac{1}{2^i} \to 0$, whereas the series $\sum \frac{1}{2^i} \to 1$.

**Question 1.**

*Check in* `DESMOS` *what happens when you change the series to*

*(i)* $\sum \frac{1}{i}$ *(ii)* $\sum \frac{(-1)^i}{i}$ *(iii)* $\sum (-1)^i$

**Question 2.** □

*Let* $s_n = \dfrac{n}{n+1}$. *Investigate the convergence of the sequence* $\{s_i\}_{i=1}^{\infty}$ *vs. the series* $\sum_{i=1}^{\infty} s_i$.

## §B. Definitions

First of all, notice that to define a series, we need to start with a sequence $\{a_n\}$. Now, consider a second sequence $\{s_n\}$ constructed from the original sequence as follows:

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ s_4 &= a_1 + a_2 + a_3 + a_4 \end{aligned}$$

and, in general,

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

This is called the sequence of partial sums.

**Definition B.1**

We define the **infinite series** $\sum_{i=1}^{\infty} a_i$ to be the limit of the sequence $\{s_n\}_{n=1}^{\infty}$ of **partial sums**, where

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

In other words,

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

The limit, if it exists, is called the **sum** of the series and we say that the series **converges**. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

**Example B.2**

Suppose we know that the sum of the first $n$ terms of the series $\sum_{i=1}^{\infty} a_i$ is

$$s_n = a_1 + a_2 + \cdots + a_n = \frac{2n}{3n+5}$$

Then the sum of the series is the limit of the sequence $\{s_n\}$ :

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2n}{3n+5} = \lim_{n\to\infty} \frac{2}{3+\frac{5}{n}} = \frac{2}{3}$$

## §C. Geometric Series

One of the most important examples of an infinite series is the geometric series that arises from a geometric sequence:

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + \cdots = \sum_{i=1}^{\infty} ar^{i-1}, \quad a \neq 0$$

### Question 3. ☐

(a) *Show that the partial sum sequence $\{s_n\}$ has the formula*

$$s_n = \frac{a(1-r^n)}{1-r}.$$

(b) *When does the limit $\lim_{n\to\infty} s_n$ exist?*

**Theorem C.3**

*The geometric series*

$$\sum_{i=1}^{\infty} ar^{i-1} = a + ar + ar^2 + \cdots$$

*is convergent if $|r| < 1$ and its sum is*

$$\sum_{n=1}^{\infty} ar^{n-1} = \underline{\hspace{3cm}} \quad \text{for } |r| < 1$$

*If $|r| \geqslant 1$, the geometric series is divergent.*

### Question 4. ☐

*Which of the following series are geometric? For those that are, identify $a$ and $r$. Which of the geometric series converge, and to what value?*

(i) $\displaystyle\sum_{i=1}^{\infty} \frac{1}{2^i}$ (ii) $\displaystyle\sum_{i=1}^{\infty} \frac{2^{2i}}{3^{i-1}}$ (iii) $\displaystyle\sum_{i=1}^{\infty} 2(i-1)^3$ (iv) $\displaystyle\sum_{i=1}^{\infty} 3^{i+1}\frac{1}{4^i}$

### Question 5. ☐

*Write $0.9 + 0.09 + 0.009 + 0.0009 + \ldots$ as a geometric series and find the sum.*

### Question 6. ☐

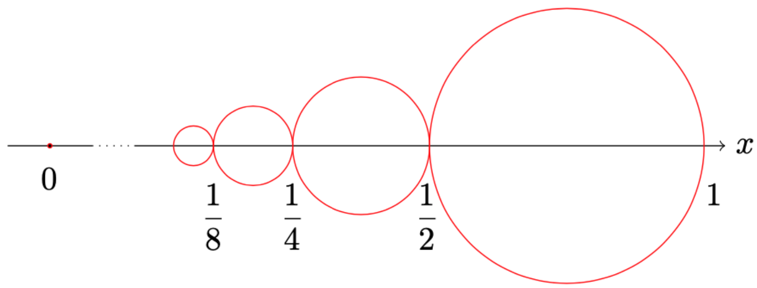
*Find a way to write the repeating decimal $0.\overline{123}$ as a geometric series and determine what rational number it is equal to.*

## §D. **Examples of Geometric Series**

Next we will explore several different geometric series by means of geometry, lending some credence to their name!

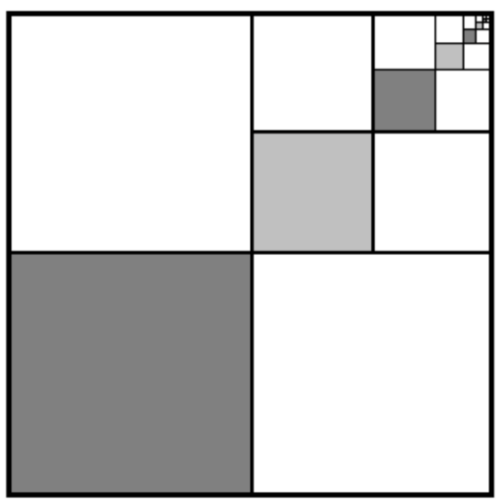
### Question 7.

*Find the total area of the infinitely many circles on the interval* $[0, 1]$ *pictured below by determining a geometric series that represents the area.*



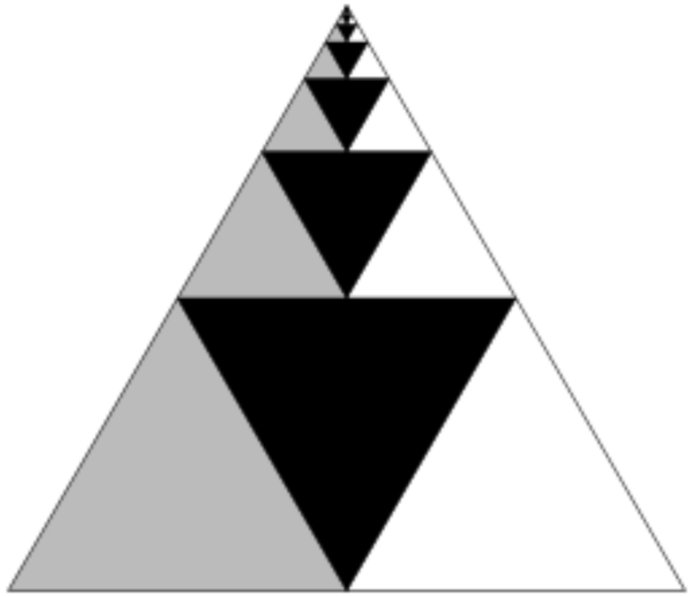

### Question 8.

*Find the total area of the shaded region inside the given unit square (i.e. the sides of the square are length* 1*).*



### Question 9.

*Suppose we take an equilateral triangle and divide it into four smaller triangles of equal area (suppose the area of the whole triangle is* 1*). If we continue this process for forever, we get something like what is sketched below. What is the total area outside of the black region?*



## ■ Question 10. **Optional**

*In the figure below there are infinitely many circles approaching the vertices of an equilateral triangle, each circle touching other circles and sides of the triangle. If the triangle has sides of length* 1*, find the total area occupied by the circles.*