Calculus & Analytical Geometry II

Lecture 4-5 Worksheet

Spring 2021 **Subhadip Chowdhury** **Math 112**

§A. Reversing the Product Rule

The method of u-substitution reverses the chain rule. Now we introduce integration by parts, which is based on the product rule.

First, let us recall what product rule says. Given two differentiable functions $u(x)$ and $v(x)$, the derivative of the product $u(x)v(x)$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}[u(x)v(x)] = u'(x)v(x) + u(x)v'(x)$$

We can rewrite this as

$$uv' = \frac{\mathrm{d}}{\mathrm{d}x}(uv) - u'v$$

and integrate both sides to get

$$\int uv'\,\mathrm{d}x = \int \frac{\mathrm{d}}{\mathrm{d}x}(uv)\,\mathrm{d}x - \int u'v\,\mathrm{d}x$$

Since the antiderivative of the derivative of uv is just uv, we get the integration by parts formula

Theorem A.1

$$\int uv'\,\mathrm{d}x = uv - \int u'v\,\mathrm{d}x$$

You may be familiar with some other ways of writing the same thing:

$$\int u\,dv = uv - \int v\,du$$

$$\int uv\,\mathrm{d}x = u\left(\int v\,\mathrm{d}x\right) - \int u'\left(\int v\,\mathrm{d}x\right)dx$$

■ **Question 1.** □

Explain why the last two are equivalent formulation of the integration by parts formula.

§B. Integration by Parts techniques

Integration by parts is most useful when the integrand can be viewed as a product and when the integral on the right-hand side is simpler than that on the left. Let's try some examples.

Example B.2

Suppose we want to find $\int x \ln x \, dx$. First convince yourself that you cannot find the integral by simple u-substitution.

Our first goal is to rewrite the integral as

$$\int x \ln x \, dx = \int u v' \, dx$$

Fortunately, the integrand here is a product of two functions, x and $\ln x$. So one of them should be $u(x)$ and the other one is $v'(x)$.

However note that the formula requires us to find $v(x)$ on the right hand side. So we should choose $v'(x)$ to be the factor that is easier to integrate!

Which one is easier to integrate? Is it $\int x \, dx$ or is it $\int \ln x \, dx$? ____________________

So we choose $v'(x) =$ ____________ and consequently, $v(x) = \int v'(x) \, dx =$ ____________

And we have $u(x) =$ ________________ and consequently, $u'(x) =$ ________________

Now fill in the right hand side of the formula:

$\int u v' \, dx = uv - \int u' v \, dx =$ ____

Now finish the integration.

Some Rules of Thumb for Choosing u and v'.

- Whatever you let v' be, you need to be able to find v.
- Our end goal is to make sure $u'v$ is easier to integrate than uv'. So it helps if u' is simpler than u (or at least no more complicated than u) and similarly, if v is simpler than v' (or at least no more complicated than v').

Question 2.

Try it yourself. Find $\int x \cos x \, dx$.

■ Question 3.

Using Integration By Parts more than once

Find $\displaystyle\int x^2 e^{-x}\,\mathrm{d}x$.

There are some examples which don't look like good candidates for integration by parts because they don't appear to involve products, but for which the method works well.

■ Question 4.

□

Find $\displaystyle\int \ln x\,\mathrm{d}x$.

Sometimes you must simplify your integral expression using a substitution first before you can use integration by parts.

Example B.3

Let's find $\int x^3 \sin(x^2)\,\mathrm{d}x$. It is not immediately obvious what we should choose as $v'(x)$.

- On one hand, if $v'(x) = x^3$, then $v(x)$ is more complicated than $v'(x)$. Also u' is actually worse than $u(x)$.
- On the other hand, if $v'(x) = \sin(x^2)$, we don't actually know how to find $v(x)$.

So let's check if we can simplify the integral a bit using substitution first. Let's choose z to be the "inside" function (we are using a different letter so as not to confuse with u and v above).

So $z =$ ____________________ and $dz =$ ____________________

So we can rewrite the integral as $\int x^3 \sin x^2\,\mathrm{d}x =$ ____________________

Now finish the integration.

Question 5.

Find $\int x^5 e^{x^3}\,\mathrm{d}x$.

What to do when the original integral reappears on the right-hand side of integration by parts?

Example B.4

Let's find $\int e^x \sin x \, dx$.

Neither e^x nor $\sin x$ becomes simpler when differentiated, but we try choosing $u(x) = e^x$ and $v'(x) = \sin x$ anyway. Then $u'(x) = e^x$ and $v(x) = -\cos x$, so integration by parts gives

$\int e^x \sin x dx =$ ____

The integral that we obtain on the right hand side is no simpler than the original one, but at least it's no more difficult. Let's try by parts once more.

At first glance, it appears as if we have accomplished nothing because we have arrived back at the original integral $\int e^x \sin x \, dx$, which is where we started. How does this help you solve the integral?

§C. One more rule of thumb for choosing u and v

The following acronym can help in deciding what should be your u and what should be your v':

Preferred Choice for $u \downarrow$	**I**	-Inverse trigonometric
	L	-Logarithmic
	A	-Algebraic
	T	-Trigonometric
Preferred Choice for $v' \uparrow$	**E**	-Exponential

Note: This acronym is not an end-all-be-all rule on how to perform integration by parts. Sometimes **L-I-A-T-E** works better. In general, if you are stuck with your choice of u and v, switch the two and see if your integration becomes simpler.

§D. Practice Problems

Question 6.

Find the following integrals.

(a) $\displaystyle\int (x^2+1)\sin x \, dx$

(b) $\displaystyle\int x \arcsin x \, dx$

(c) $\displaystyle\int \sec^3 x \, \mathrm{d}x$ [HINT: Let $u = \sec x$ and $v' = \sec^2 x$.]

(d) $\displaystyle\int \cos(\ln x) \, \mathrm{d}x$

§E. Evaluating Definite Integrals Using Integration by Parts

If we combine the formula for integration by parts with the Fundamental Theorem of Calculus, we can evaluate definite integrals by parts.

Theorem E.5

$$\int_a^b uv'\,\mathrm{d}x = [u(x)v(x)]\Big|_a^b - \int_a^b u'(x)v(x)\,\mathrm{d}x$$

Question 7.

Find the following integrals.

(a) $\displaystyle\int_0^{\pi/4} x\sin(2x)\,\mathrm{d}x$

(b) $\displaystyle\int_0^1 \arctan x\,\mathrm{d}x$

(c) $\displaystyle\int_1^{\sqrt{3}} \arctan(1/x)\,\mathrm{d}x$

(d) $\displaystyle\int_0^{3} \ln(x^2+1)\,\mathrm{d}x$ [HINT: This one might be tricky, we will revisit it at the end of the week.]