MATH 111 - Calculus and Analytic Geometry I

Lab 11 Worksheet

Fall 2020 **Subhadip Chowdhury** **Nov 17**

§A. Limits as Definite Integral

The definition of Definite Integral tells us that limits of Riemann sums can be written as an integral. For example, with right endpoint Riemann sum,

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(a+\frac{(b-a)i}{n}\right)\frac{b-a}{n} = \int_a^b f(x)\,dx$$

In the special case when $a=0$ and $b=1$, we can rewrite above result as

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n} = \int_0^1 f(x)\,dx$$

This tells us an algorithm to rewrite any limit of Riemann sum as a definite integral. Let's take a look at an example first.

Example A.1

Let's convert $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\sin^2\left(2\pi\frac{i}{n}\right)$ into a definite integral.

If we compare the summand $\frac{1}{n}\sin^2\left(2\pi\frac{i}{n}\right)$ to the term $f\left(\frac{i}{n}\right)\frac{1}{n}$, we find that $f\left(\frac{i}{n}\right)=\sin^2\left(2\pi\frac{i}{n}\right)$. That tells us

$$f(x)=\sin^2(2\pi x)$$

We conclude that the integral form of the limit is $\int_0^1 \sin^2(2\pi x)\,dx$.

Example A.2

Let's convert $\lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left(1+\frac{2i}{n}\right)$ into a definite integral.

First note that we can rewrite above limit as

$$\lim_{n\to\infty}\sum_{i=1}^{n} 2\left(1+\frac{2i}{n}\right)\frac{1}{n}$$

So the summand $2\left(1+\frac{2i}{n}\right)\frac{1}{n}$ must be the term $f\left(\frac{i}{n}\right)\frac{1}{n}$. Comparing the two terms, we get

$$f\left(\frac{i}{n}\right) = 2\left(1+\frac{2i}{n}\right)$$

That tells us

$$f(x) = 2(1+2x)$$

We conclude that the integral form of the limit is $\int_0^1 2(1+2x)\,dx$.

So we have the following algorithm.

Algorithm for rewriting limits of Riemann Sum as Definite Integrals

Step 1. Look carefully at the summand and isolate a $\frac{1}{n}$ from inside. This corresponds to Δx which will become dx.

Step 2. Write the remaining part as a function of $\frac{i}{n}$. Identify the function as f. Your function must not contain any i or n in it.

Step 3. Your integral will be $\int_0^1 f(x)\,dx$.

Let's try a more complicated example.

Question 1.

□

In the following problems, express the limits as definite integrals. **Do not evaluate the integrals.**

(a) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{9i+3n}{3in+2n^2}$

(b) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(5 + \frac{3i}{n}\right)^4 \left(\frac{2}{n}\right)$

(c) $\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{\sqrt{n^2 - i^2}}$

(d) $\lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(1 + 2\frac{i-1}{n}\right)$

§B. Problems using the Fundamental Theorem of Calculus

■ Question 2. ☐

Suppose $f''(x) = \sin(x)$, $f'(0) = 2$ and $f(0) = 3$. Find $f(x)$.

■ Question 3. ☐

Find $\dfrac{d}{dx}\displaystyle\int_1^{\ln x} \frac{1}{1+t^4}\,dt$.

■ Question 4. □

Use the graph of a function below to answer the given questions about the area function $\mathbf{F}(x) = \int_0^x f(t)\,dt$.

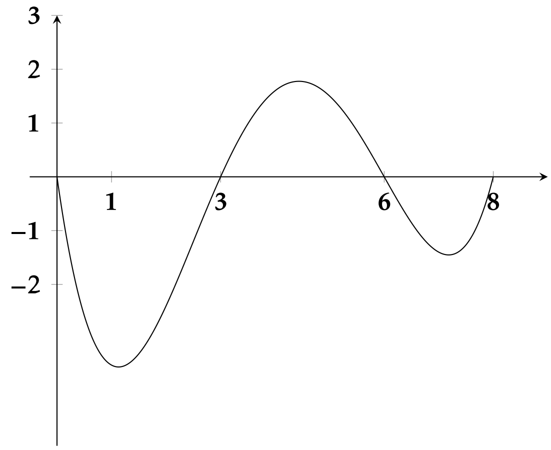


Figure 1: Graph of function $f(t)$ for problem #3

(a) Where is **F** increasing/decreasing?

(b) Find all critical points of **F** and determine which are local extrema.

(c) Since **F** is continuous on $[0, 8]$, it must have absolute extrema. Where do the absolute extrema for **F** occur?

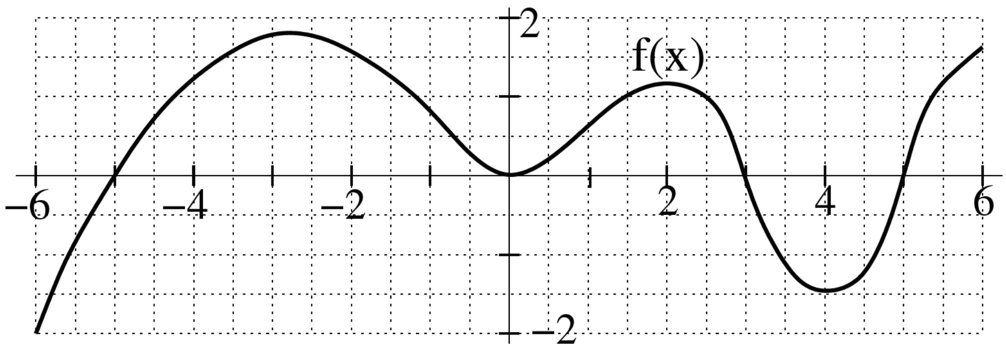
(d) Where is **F** concave up/concave down?

(e) Does **F** have any points of inflection?

(f) Sketch a possible graph of **F**.

■ Question 5. ☐

Let $\mathbf{F}(x) = \int_0^x f(t)\,dt$ where $f(x)$ is the function whose graph is given below.



(a) What are the critical points of $\mathbf{F}(x)$**?**

(b) Where is $\mathbf{F}(x)$ increasing? decreasing?

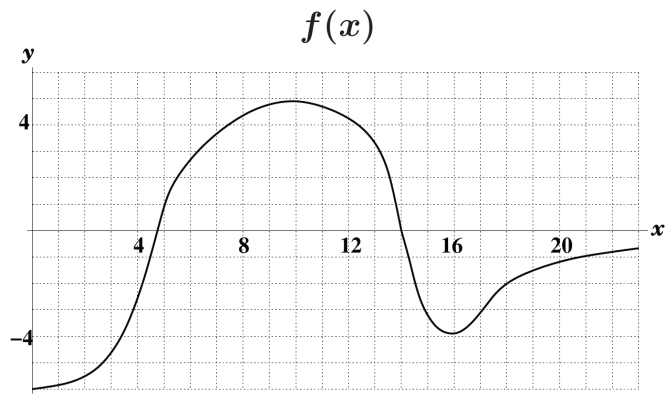
(c) Locate all places where $\mathbf{F}(x)$ has a local maximum or a local minimum, and make it clear which are which.

(d) Where is $\mathbf{F}(x)$ concave up? concave down?

(e) Where does $\mathbf{F}$ attain its maximum value?

■ Question 6. □

Let f be the function whose graph is given below, and define a new function $\mathbf{F}$ by the equation $\mathbf{F}(x) = \int_0^x f(t)dt$.



Given below are several lists of numbers. Rank each list in order from smallest to largest.

(a) $0, f'(3), f'(4), f'(9), f'(14), f'(15)$

(b) $0, \mathbf{F}(3), \mathbf{F}(4), \mathbf{F}(6), \mathbf{F}(13), \mathbf{F}(14)$

(c) $0, \mathbf{F}'(0), \mathbf{F}'(4), \mathbf{F}'(7), \mathbf{F}'(10), \mathbf{F}'(15), \mathbf{F}'(18), \mathbf{F}'(23)$