Calculus & Analytical Geometry II

Lecture 10-11 Worksheet

Spring 2021 **Subhadip Chowdhury** **Math 112**

Just as we can use definite integrals to find area of specific regions, we can also use it to find the volume of three dimensional solids.

It is particularly straightforward when the cross sections of the solids have a consistent shape (e.g. a circle or square). So we will start with a type of solids known as **Solids of Revolution.**

§A. What is a Solid of Revolution

Start with a two dimensional region, e.g. the area under the graph of a function $y = f(x)$. Now revolve it around a straight line, e.g. the x–axis. The three dimensional solid you obtain this way is called a **solid of revolution**. The line around which you rotate is called the **axis of revolution.**

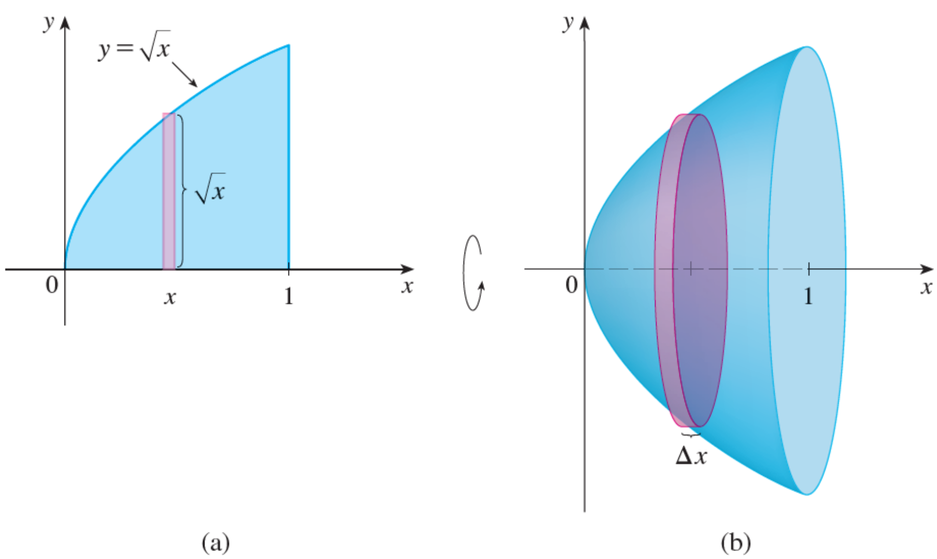
Example A.1

Suppose we wish to calculate the volume of the solid obtained by rotating the region under $y = \sqrt{x}$ about the x–axis for $0 \le x \le 1$.

- **Step 1. Visualize the Solid** Graph the function and the line it is to be rotated about. Label the curves. Make a 3-dimensional sketch of the solid.

  Go to this website to get a 3-D visual.

- **Step 2. Picture a slice** perpendicular to XY-plane. Each slice will produce a disk. Make a rough sketch of what a slice will look like.



- **Step 3.Express the volume of each slice** as volume of a really thin cylinder. So the volume of the slice is the area of the face ($\text{A}(x) = \pi f(x)^2$, because it's a circle!) times its thickness (Δx). But the radius of the circle depends on the function.

In this example, the volume of each slice is

$$A(x)\Delta x = \pi(\sqrt{x})^2 \Delta x$$

- **Step 4: Express the Volume of the Solid as an Integral and Solve.** The volume of the solid is the sum of the volumes of the individual disks. This is a **Riemann sum**, which becomes our integral! The limits of the integral are the boundary values of the variable of integration. Set up and evaluate the integral.

$$V = \int_a^b A(x)\,dx = \underline{\hspace{10cm}}$$

§B. The Disk and the Washer method

The process of calculating volumes of solids of revolutions as in the last example is called the **Disk Method**. Let's try some more examples to solidify (pun intended) the concept!

Question 1.

Calculate the volume of the solid obtained by rotating around the x-axis the region bounded by $y = x^3$, the x-axis, $x = 0$ and $x = 3$.

Question 2.

Calculate the volume of the solid obtained by rotating around the x-axis the region bounded by $y = \dfrac{1}{x+1}$, $y = 0$, $x = 0$, and $x = 4$. Give an exact answer.

Question 3.

Calculate the volume of the solid obtained by rotating around the x-axis the region bounded by $y = e^{-x} + 1$, $y = 0$, $x = 0$, and $x = 4$. Give an exact answer.

Rotating About y-axis

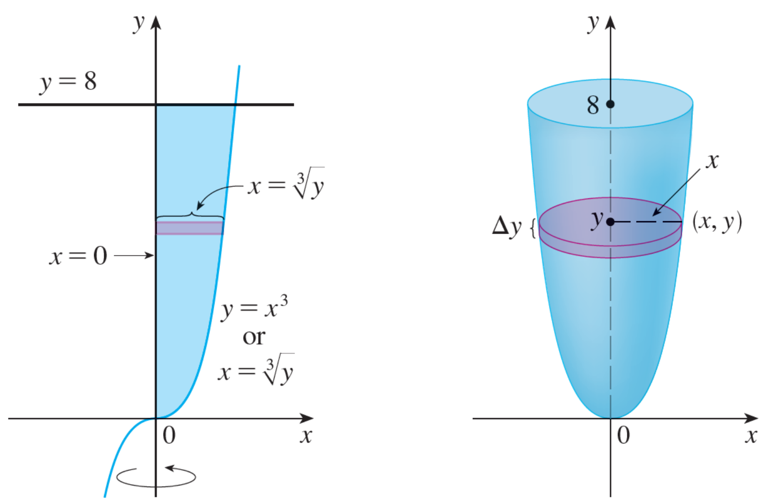
Each of the above problems had you computing the volume of a solid formed by rotating about the x-axis. But we can form a solid by rotating about the y-axis too! Let's work out an example first.

Example B.2

Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.

- **Step 1.** Visualize the solid using the geogebra applet.
- **Step 2.** Slice the solid into disks **perpendicular to the axis of revolution.**

  Note that the radius of the disk is given by the value of x, as a function of y, So we will need to solve for x from the equation $y = x^3$. Thus the radius of the slice pictured below is $x = g(y) = \sqrt[3]{y}$.



(a) (b)

- **Step 3.** The volume of the slice is given by $A(y)\Delta y = \pi g(y)^2 \Delta y = \pi(y^{1/3})^2 \Delta y$.
- **Step 4.** The total volume is the limit of the Riemann sum with bounds on y. Note that the upper and lower bound on the integral must be obtained from the picture using algebra. In this case, the lower bound of y is obtained when $x = 0$, i.e. when $y = 0^3 = 0$. The upper bound is given in the problem as $y = 8$. So the integral is

$$V = \int_0^8 \pi y^{2/3}\, dy = \underline{\hspace{10cm}}$$

Question 4.

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Calculate the volume of the solid obtained by rotating about the y-axis the region bounded by $y = \dfrac{1}{x+1}$, $y = \dfrac{1}{5}$, and $y = 1$.

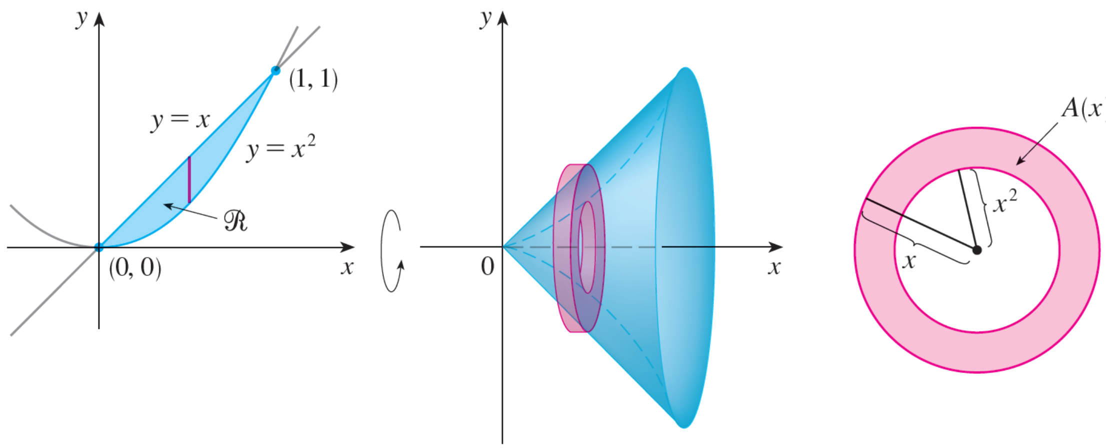
THE WASHER METHOD

The washer method is a modified version of the disk method for cases when we have a region bounded by two curves. Let's take a look at an example.

Example B.3

The region $\mathscr{R}$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

- **Step 1.** First Visualize the region $\mathscr{R}$ and the then solid. This time, the solid of revolution looks like a 'container'. It's outer surface is formed by revolving the curve $y = x$, and the inner surface is formed by revolving the curve $y = x^2$.



(a) (b) (c)

- **Step 2.** Picture the slices perpendicular to the axis of revolution, in this case the x–axis. The cross-sections no longer look like full disks. Instead they have the shape of a *washer* (an annular ring) with an inner radius x^2 and an outer radius x.

- **Step 3.** The volume of a thin slice is the area of the washer times the thickness. Evidently, the area of washer cross-section by subtracting the area of the inner circle from the area of the outer circle:

$$A(x) = \pi x^2 - \pi(x^2)^2 = \pi(x^2 - x^4)$$

- **Step 4.** The bounds on the integral are obtained from the point of intersection of the two curves $y = x$ and $y = x^2$. Therefore we have

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^2 - x^4)\,dx = \underline{\hspace{10cm}}$$

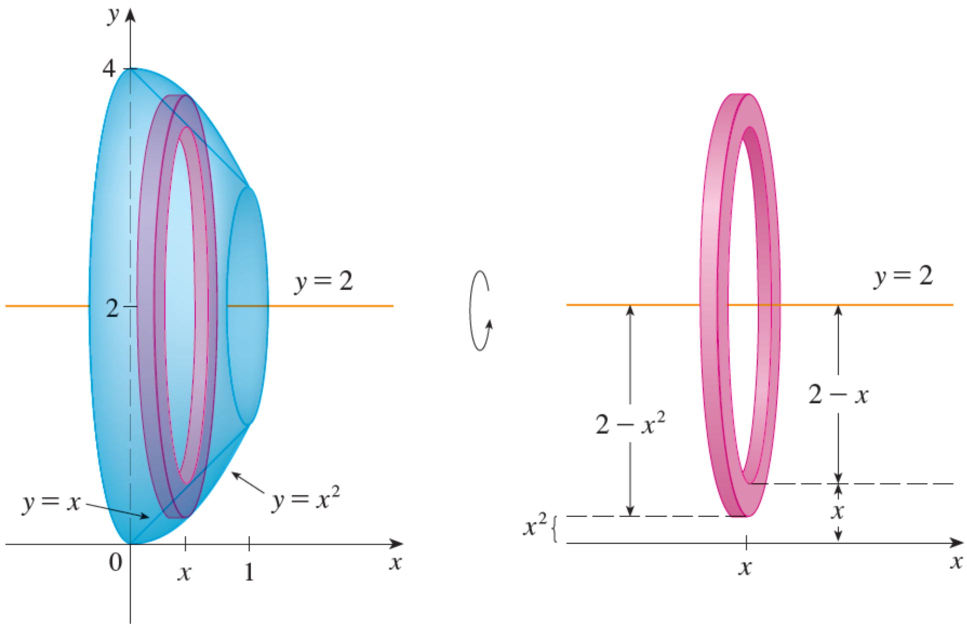
Rotating about other straight lines parallel to the axes

We don't have to necessarily rotate a region about the x-axis or y-axis to get a solid of revolution! Typically, this will result in a solid with a hole in it, thus utilizing the washer method. Give that a try in the next example, where we use the same region as the previous example, but rotate about a different line.

Question 5.

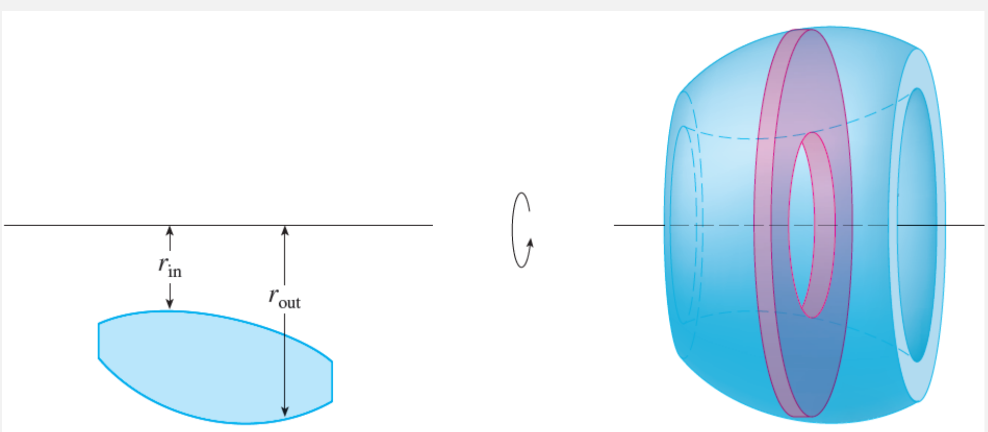
Find the volume of the solid obtained by rotating the region in example 3 about the line $y = 2$.

Here's a picture to help you get started.



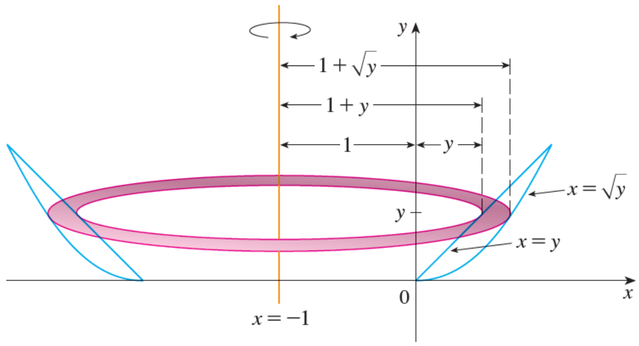
Note: In general if the cross-section is a washer, we find the inner radius r_{in} and outer radius r_{out} as the distance from the axis of revolution using a sketch and compute the area of the washer by subtracting the area of the inner disk from the area of the outer disk:

$$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2 = \pi\left(r_{out}^2 - r_{in}^2\right)$$



Question 6.

Find the volume of the solid obtained by rotating the region in example 3 about the line $x = -1$.



Question 7.

Set up the integrals needed to find the volume of the solid generated by revolving the region bounded by $y = 2x^2$, $y = 0$, and $x = 2$ about the given line. After you have set up all four, then you can evaluate (or leave the evaluating until after class, as setting these up is generally the trickiest part!).

(i) the y–axis

(ii) the x–axis

(iii) the line $y = 8$

(iv) the line $x = 2$

Question 8.

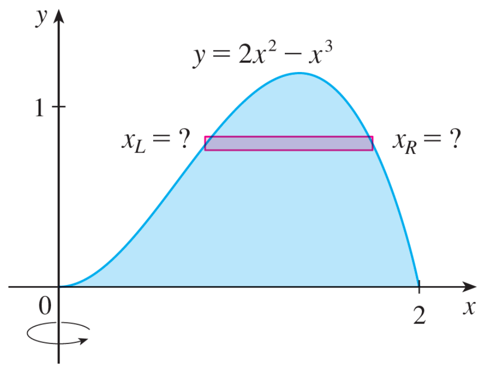
Set up the integrals needed to find the volume of the solid generated by revolving the region bounded by $y = e^x$, $y = 0$, $x = -1$, and $x = 1$ about the given line. Leave evaluating until after class.

(i) the line $x = 1$

(ii) the line $y = -5$

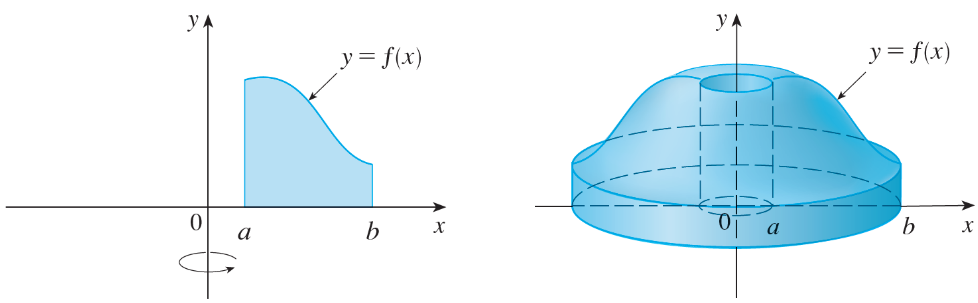
§C. The Shell Method

Some volume problems are very difficult to handle by the disk or washer method. For instance, let's consider the problem of finding the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

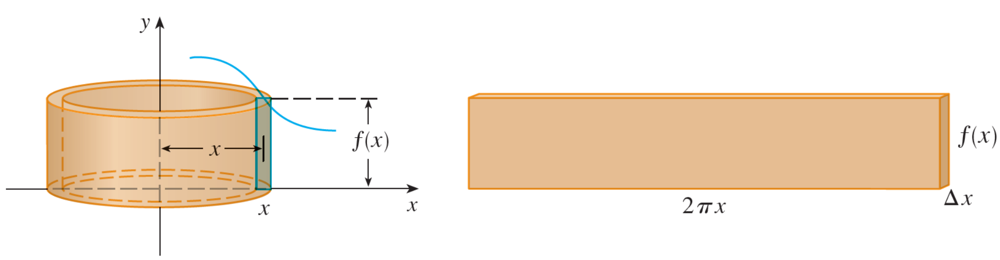


If we slice perpendicular to the y-axis, we get a washer. But to compute the inner radius and the outer radius of the washer, we'd have to solve the cubic equation $y = 2x^2 - x^3$ for x in terms of y; that's not easy. Fortunately, there is a method, called the method of cylindrical shells, that is easier to use in such a case. We will mention it briefly for the sake of completion.

Suppose we wish to find the volume of a solid S obtained by rotating about the y-axis the region bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$, where $b \ge a > 0$.

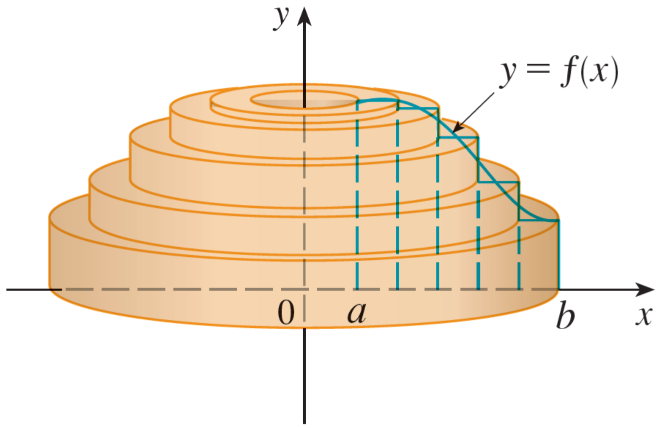


We are going to calculate this by dividing the region into thin rectangles (thickness=Δx) with height $f(x)$ and finding the volume of the cylinder formed by revolving it.



The volume of the cylinder is given by

$$\underbrace{2\pi x}_{\text{circumference}} \cdot \underbrace{f(x)}_{\text{height}} \cdot \underbrace{dx}_{\text{thickness}}$$



Then we sum up all this volumes, take a limit of the Riemann sum and voila! We have an integral formula

$$V = \int_a^b 2\pi x f(x)\,\mathrm{d}x$$

Question 9.

Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

Question 10.

Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the y-axis.