Calculus & Analytical Geometry II

Lecture 18-19 Worksheet

Spring 2021 **Subhadip Chowdhury** **Math 112**

§A. Introduction to Infinity

What is Infinity? Wikipedia defines infinity as *"something that is boundless or endless, or else something that is larger than any real or natural number"*. In particular, note that infinity is not a number, it is merely the **opposite of finite.**

So in Mathematical terms, when we write $x \to \infty$, what we means is that x grows larger and larger without bound, but never really approaches anything. On the other hand, writing something like

$$\lim_{x\to\infty} f(x) = \mathrm{L}$$

makes perfect sense, because $f(x)$ can '*approach*' a finite value L even when x gets larger and larger. In other words, this is the same thing as saying the line $y = \mathrm{L}$ is a horizontal asymptote for the graph of $f(x)$.

For the rest of this semester, we will keep dealing with infinity a lot. Let's see an example on how they show up in applications of integrals.

Example A.1

A company with a large customer base has a call center that receives thousands of calls a day. After studying the data that represents how long callers wait for assistance, they find that the function $p(t) = 0.25e^{-0.25t}$ models the time customers wait in the following way: the **fraction** of customers who wait between $t = a$ and $t = b$ minutes is given by

$$\int_a^b p(t)\,\mathrm{d}t$$

Use this information to answer the following questions.

(a) Determine the fraction of callers who wait between 5 and 10 minutes. ____________________

(b) Determine the fraction of callers who wait between 10 and 20 minutes. ____________________

(c) Next, let's study the fraction who wait up to a certain number of minutes:

  (i) What is the fraction of callers who wait between 0 and 5 minutes? ____________________

  (ii) What is the fraction of callers who wait between 0 and 10 minutes? ____________________

(d) Let $F(b)$ represent the fraction of callers who wait between 0 and b minutes. Find a formula for $F(b)$ that involves a definite integral and evaluate it.

(e) What is the value of the limit $\lim_{b\to\infty} F(b)$? What is its meaning in the context of the problem?

Question 1.

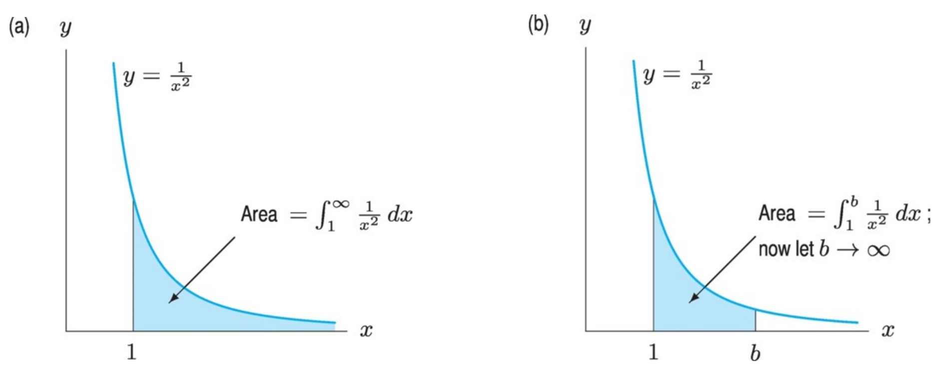
Consider the function $g(x) = \dfrac{1}{x^2}$.

(a) Let $b > 0$. What is the area under $g(x)$ on the interval $[1, b]$? Write down the corresponding definite integral and draw a diagram.

(b) Determine the area under $g(x)$ for $b = 10, 100,$ and 1000.

(c) As $b \to \infty$, what is happening to the values $\displaystyle\int_1^b g(x)\,\mathrm{d}x$? Formally answer this by computing the limit

$$\lim_{b\to\infty} \int_1^b g(x)\,\mathrm{d}x$$



§B. Improper Integrals

Integrals that involve infinity in some way are called Improper integrals. These extend the original definition of definite integrals by allowing ourselves to consider different kinds of intervals for integration.

For example, suppose we have a function $f(x)$ that is continuous on the interval $[a, \infty)$. Then we can denote the area under the graph of $f(x)$ from a to infinity as $\int_a^\infty f(x)\,\mathrm{d}x$ and define it as

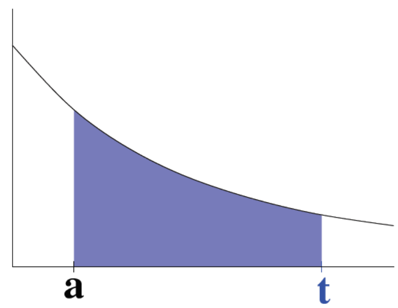
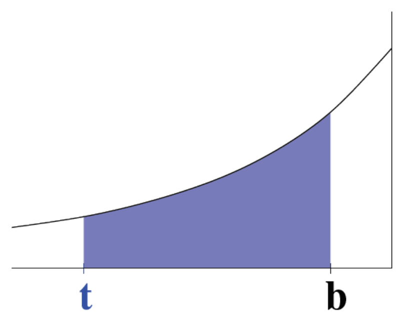
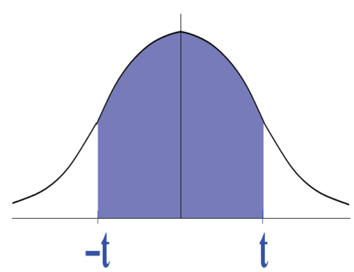
$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{t\to\infty}\int_a^t f(x)\,\mathrm{d}x$$

Warning: Suppose the antiderivative of $f(x)$ is $\mathrm{F}(x)$. Then it is tempting to write something like

$$\int_a^\infty f(x)\,\mathrm{d}x = \mathrm{F}(\infty) - \mathrm{F}(a)$$

But remember that ∞ is not a number, it is a different kind of object. So $\mathrm{F}(\infty)$ doesn't make mathematical sense. We should instead interpret it as $\lim_{t\to\infty}\mathrm{F}(t)$.

Depending on the direction of the limit, we can similarly define the following types of Improper integrals.

Improper Integral	**Definition**	**Picture**
$\int_a^\infty f(x)\,\mathrm{d}x$	$\lim_{t\to\infty}\int_a^t f(x)\,\mathrm{d}x$	
$\int_{-\infty}^b f(x)\,\mathrm{d}x$	$\lim_{t\to-\infty}\int_{-t}^b f(x)\,\mathrm{d}x$	
$\int_{-\infty}^\infty f(x)\,\mathrm{d}x$	$\lim_{t\to-\infty}\int_t^0 f(x)\,\mathrm{d}x + \lim_{t\to\infty}\int_0^t f(x)\,\mathrm{d}x$	

Since the improper integral is defined as a limit, it may or may not actually exist! In fact, we have the following definitions.

Definition B.2

We say that an improper integral **converges** if all of the defining limits exist and are finite. If any of the involved limits is infinite or does not exist, we say that the improper integral **diverges** .

Question 2.

(a) *Suppose $p > 0$ is some real number. Evaluate the integral* $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$.

(b) *Does your integral converge or diverge? How does the answer depend on p?*

Note: The result of above exercise will be of vital importance later on.

Question 3.

Evaluate $\displaystyle\int_{-\infty}^0 xe^x\,dx$.

The third entry in the above table is a bit unusual. When investigating an improper integral of the form $\displaystyle\int_{-\infty}^\infty f(x)\,dx$, we require that both $\displaystyle\int_{-\infty}^0 f(x)\,dx$ and $\displaystyle\int_0^\infty f(x)\,dx$ converge in order for the improper integral to be convergent. Should either of $\displaystyle\int_{-\infty}^0 f(x)\,dx$ or $\displaystyle\int_0^\infty f(x)\,dx$ diverge, then so too does $\displaystyle\int_{-\infty}^\infty f(x)\,dx$.

Question 4.

Evaluate $\displaystyle\int_{-\infty}^\infty \frac{e^x}{1+e^x}\,dx$.

Question 5.

Evaluate $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$.

Question 6.

To see why improper integrals over $(-\infty,\infty)$ are defined the way they are, evaluate the following

$$\lim_{t\to\infty}\int_{-t}^{t} \frac{1}{1+x^2}\,dx.$$

How does your answer compare to that in the previous question?

■ Question 7. ☐

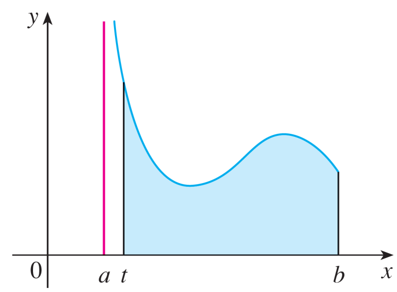
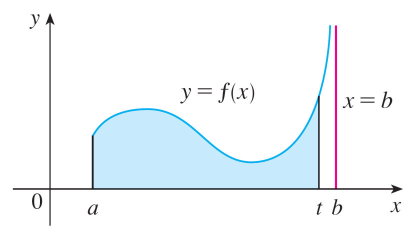
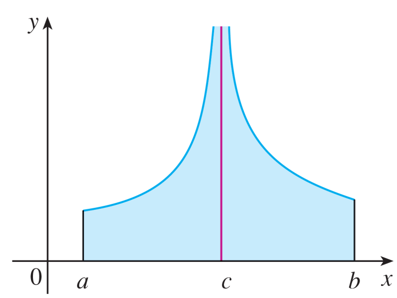
Here is another example to see why improper integrals over $(-\infty, \infty)$ are defined as they are.

(a) *Evaluate* $\displaystyle\lim_{t\to\infty} \int_{-t}^{t} x\,\mathrm{d}x.$

(b) *Compare the previous answer with the definition for* $\displaystyle\int_{-\infty}^{\infty} x\,\mathrm{d}x.$

§C. Improper Integrals where the integrand becomes unbounded

Suppose a function $f(x)$ has an infinite discontinuity at a point. There are three situations we can examine where a continuous function might be integrated over an interval containing such a discontinuity.

Improper Integral	Definition	Picture
	When the integrand becomes infinite	
$\displaystyle\int_a^b f(x)\,\mathrm{d}x$	$\displaystyle\lim_{t\to a^+} \int_t^b f(x)\,\mathrm{d}x$	
$\displaystyle\int_a^b f(x)\,\mathrm{d}x$	$\displaystyle\lim_{t\to b^-} \int_a^t f(x)\,\mathrm{d}x$	
$\displaystyle\int_a^b f(x)\,\mathrm{d}x$	$\displaystyle\lim_{t\to c^-} \int_a^t f(x)\,\mathrm{d}x + \lim_{t\to c^+} \int_t^b f(x)\,\mathrm{d}x$	

■ Question 8. ☐

For each of the following, explain why the integral is improper. Then determine whether it converges or diverges. If it does converge, find its value.

(a) $\displaystyle\int_0^1 x \ln x \, \mathrm{d}x$

(b) $\displaystyle\int_0^1 \ln x \, \mathrm{d}x$

(c) $\displaystyle\int_{-1}^1 \frac{1}{\sqrt[3]{x^4}} \, \mathrm{d}x$

(d) $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}} \, \mathrm{d}x$

(e) $\displaystyle\int_0^3 \frac{1}{x-1} \, \mathrm{d}x$ *[WARNING: The answer is not* $\ln(2)$*.]*

L'Hôpital's Rule. You will need to use L'Hôpital's Rule to calculate some of the limits above. Here is a quick recap in case you forgot about it.

Suppose you wish to calculate $\lim\limits_{x\to c} \frac{f(x)}{g(x)}$, where the limit has an indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then we can compute the limit using a ratio of derivatives:

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$