Calculus & Analytical Geometry II

Lecture 12 Worksheet

Spring 2021 | **Subhadip Chowdhury** | **Math 112**

In the last worksheet, we calculated volumes of solids of revolution using definite integrals. The method we used can be summarized as follows:

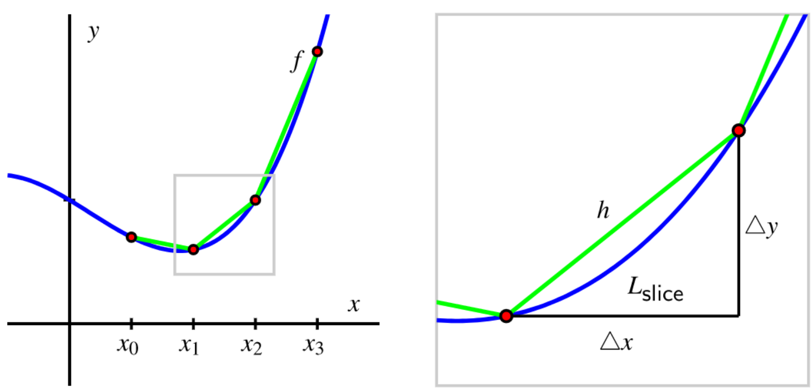
- Divide the solid into small pieces whose volume we can easily approximate;
- Add the contributions of all the pieces, obtaining a Riemann sum that approximates the total volume;
- Take the limit as the number of terms in the sum tends to infinity, giving a definite integral for the total volume;
- Evaluate the definite integral, either exactly or approximately.

The same method can be used to calculate total length of curves and surface areas of solids of revolution, as we will see in this lecture.

§A. Arc Length

Consider the curve $y = f(x)$. We wish to find its length from $x = a$ to $x = b$. The question might arise, for example, when an ant (a point object) is moving along the curve, and we wish to find the total distance covered (not the displacement).

We first start by taking a partition of the interval $[a, b]$, i.e. 'chop it up' into some small pieces. Then on each sub-interval of width Δx, we estimate the length of the curve as a line segment.



This way, the entire curve is approximated as a polygonal path and the total approximate length is the Riemann sum given as

$$\mathrm{L} = \sum_{i=1}^{n} \mathrm{L}_{\text{slice}} \approx \sum_{i=1}^{n} h = \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \cdot \Delta x$$

As the number of subintervals increase, and the length of each subinterval Δx tends to zero, the Riemann sum becomes the definite integral.

$$\lim_{n\to\infty}\sum_{i=1}^{n} \mathrm{L}_{\text{slice}} \to \int_a^b \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\cdot \mathrm{d}x$$

Rewriting $\mathrm{d}y/\mathrm{d}x$ as $f'(x)$, we have the following theorem.

Theorem A.1

Given a differentiable function f on an interval $[a,b]$, the total arc length, L, along the curve $y=f(x)$ from $x=a$ to $x=b$ is given by

$$\mathrm{L}=\int_a^b \sqrt{1+f'(x)^2}\,\mathrm{d}x$$

Note: If a curve has the equation $x=g(y)$, $c \le y \le d$, and $g'(y)$ is continuous, then by interchanging the roles of x and y in above formula, we can write its length as

$$\mathrm{L}=\int_c^d \sqrt{1+g'(y)^2}\,\mathrm{d}y.$$

Question 1.

□

Find the length of the curve $y=2x^{3/2}+2$ on the interval $[1,4]$.

Question 2.

□

Find the length of the curve $y^2=x^3$ between the points $(1,1)$ and $(4,8)$.

Question 3.

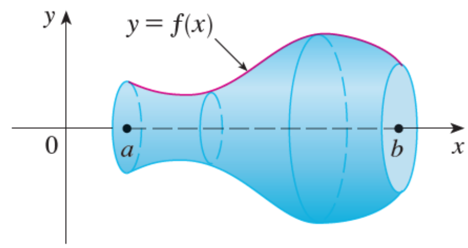
□

Set up, but do not evaluate, an integral for the length of the arc of the hyperbola $xy=1$ from the point $(1,1)$ to the point $(2,\frac{1}{2})$.

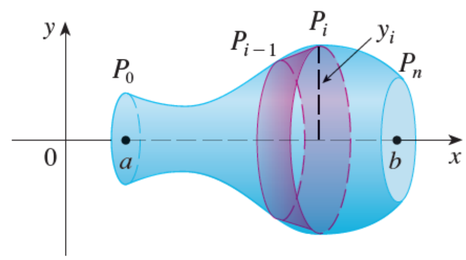
§B. Surface of Revolution

If we take a curve $y = f(x)$ on the interval $[a, b]$ and rotate around an axis, we will form a **surface of revolution.** In the previous lectures, we took a region and rotated it to form a solid. Here, by taking just a curve, we are taking just part of the boundary of a region. So, we are forming the surface of those same solids you examined in the last lectures.

We will tart off the same way as always, by breaking the interval $[a, b]$ into small subintervals, approximating the surface area of the 'band' formed over each subinterval, adding them to form a Riemann sum, and finally taking the limit to obtain a definite integral.



(a) Surface of revolution



(b) Approximating band

If Δx is small enough, then we can approximate the band as a cylinder whose *height* is equal to the arc length, and the *radius* is equal to $f(x)$. Thus the approximate surface area of each band is given by

$$2\pi f(x) \cdot \sqrt{1 + f'(x)^2} \cdot \Delta x$$

and the total surface area is given by the following formula.

Theorem B.2

Suppose $y = f(x)$ is differentiable and positive on the interval $[a, b]$. Then the total surface area of the surface of revolution obtained by revolving $y = f(x)$, $a \leq x \leq b$ around the x–axis, is given by

$$\mathrm{SA} = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\,\mathrm{d}x$$

If $f(x)$ is not necessarily positive, we replace $f(x)$ with $|f(x)|$ in the above formula.

Note: If the curve is given by the equation $x = g(y)$ on the interval $[c, d]$, rotated about the y-axis, the formula becomes:

$$\text{SA} = \int_c^d 2\pi g(y)\sqrt{1 + g'(y)^2}\,\mathrm{d}y$$

Question 4. ☐

Find the area of the surface generated by revolving $y = \frac{x}{2}$ about the x-axis.

Question 5. ☐

The curve $y = \sqrt{4 - x^2}$, $-1 \le x \le 1$, is an arc of the circle $x^2 + y^2 = 4$. Find the area of the surface obtained by rotating this arc about the x-axis. (The surface is a portion of a sphere of radius 2.)

Question 6. ☐

The arc of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$ is rotated about the y-axis. Find the area of the resulting surface.

Question 7. ☐

Find the **exact** area of the surface generated by rotating the curve $y = e^x$, $0 \le x \le 1$, about the x-axis.