Calculus & Analytical Geometry II

Lecture 6 Worksheet

Spring 2021 **Subhadip Chowdhury** **Math 112**

§A. Some Trigonometric Identities

Before we look at integrals involving trigonometric functions, let us summarize some of the Pythagorean identities we should be aware of:

$$\sin^2 x + \cos^2 x = 1 \tag{1}$$
$$\tan^2 x + 1 = \sec^2 x \tag{2}$$
$$1 + \cot^2 x = \csc^2 x \tag{3}$$

Note that the second two identities are obtained from the first be means of division. In terms of integration, we really only concern ourselves with the first two. If you can integrate something with tangents and secants, then you can apply the same strategies to integrals involving cotangents and cosecants.

Next, we will list the angle sum identities:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

Question 1.

(a) Set $x = y$ in the above identities to derive a formula for

$\sin(2x) =$ ________________________ and $\cos(2x) =$ ________________________

(b) Use the identity for $\cos(2x)$ and the Pythagorean identity to show the power-reduction identity

$$\cos^2 x = \frac{1}{2}(1 + \cos(2x))$$

(c) Find a similar power-reduction identity for $\sin^2 x$.

§B. Strategy for integrating power of trigonometric functions

Odd Power

Example B.1

Suppose we want to find the integral $\int \sin^3 x\,\mathrm{d}x$. We can rewrite the integral as

$$\begin{aligned}\int \sin^3 x\,\mathrm{d}x &= \int \sin^2 x \sin x\,\mathrm{d}x \\ &= \int (1-\cos^2 x)\sin x\,\mathrm{d}x\end{aligned}$$

Now we can use a u-substitution. If we choose

$u =$ ______________________________, then $\mathrm{d}u =$ ______________________________ and consequently, we can rewrite above integral as

Now finish the integration.

Question 2.

Find $\int \cos^5 x\,\mathrm{d}x$.

Even Power

Example B.2

Suppose we want to find the integral $\int \cos^2 x \, dx$. We can rewrite the integral using a power-reduction identity as follows

$$\int \cos^2 x \, dx = \frac{1}{2} \int (1 + \cos(2x)) \, dx$$

Now finish the integration.

Question 3.

□

Find $\int \sin^4 x \, dx$.

§C. **Strategy for integrating product of powers trigonometric functions**

Consider an integral of the form

$$\int \sin^m x \cos^n x \, dx$$

where m and n are integers. Here is the general algorithm:

- If m is odd, rip off a $\sin x$ and use Pythagorean identity $\sin^2 x = 1 - \cos^2 x$ to turn everything else into cosine. Then use the u-substitution $u = \cos x$.
- If n is odd, rip off a $\cos x$ and use Pythagorean identity $\cos^2 x = 1 - \sin^2 x$ to turn everything else into sine. Then use the u-substitution $u = \sin x$.
- If both m and n are even, then use the power-reduction identities (possibly multiple times) to transform it into another integral where the exponents are odd. Then use one of the above two methods as appropriate.

Let's try some examples.

Question 4.

□

Find the following integrals.

(a) $\displaystyle\int \sin^8 x \cos^5 x \, dx$

(b) $\displaystyle\int \frac{\sin^5 x}{\cos^4 x} \, dx$

(c) $\displaystyle\int \tan^3 x \sec^3 x \, dx$

(d) $\displaystyle\int \cos^2 x \sin^2 x \, dx$

§D. **Strategy for integrating product of multiple angle trigonometric functions**

We can use the angle sum identities from the first page to derive the following formula:

$$\sin(ax)\sin(bx) = \frac{1}{2}[\cos((a-b)x) - \cos((a+b)x)]$$
$$\sin(ax)\cos(bx) = \frac{1}{2}[\sin((a-b)x) + \sin((a+b)x)]$$
$$\cos(ax)\cos(bx) = \frac{1}{2}[\cos((a-b)x) + \cos((a+b)x)]$$

Note: I will not ask you to memorize these formula. You can come back and refer to them whenever they are needed.

■ Question 5.

□

Find the following integrals.

(a) $\displaystyle\int \sin(9x)\sin(4x)\,\mathrm{d}x$

(b) $\displaystyle\int \sin(2x)\cos(3x)\,\mathrm{d}x$