MATH 111 - Calculus and Analytic Geometry I

Lecture 19 Worksheet

Fall 2020 **Subhadip Chowdhury** **Sep 30**

Title: Derivatives of Exponential and Logarithm Functions

Summary: We will define e and learn how to calculate derivatives of exponential and logarithm function with general base.

§A. Exponential Function

Firstly, recall that exponential functions have the form $f(x) = b^x$, where $b > 0$ and $b \neq 1$. When $b > 1$, an exponential function models rapid **growth**, while if $0 < b < 1$, the function models rapid **decay**. See the graphs below in Figure 1.

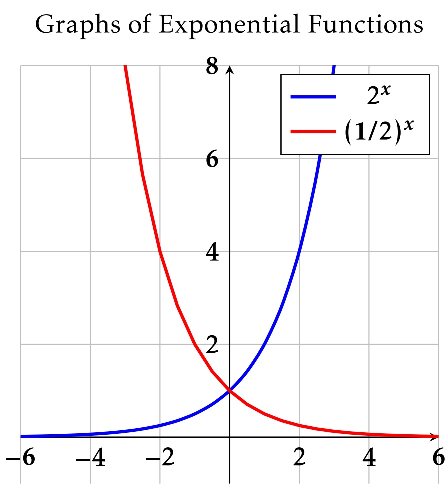

Figure 1: Graphs of $y = 2^x$ and $y = (\frac{1}{2})^x$

We can see from examples of graphs of exponential functions that they appear to be differentiable. If we want to determine the derivative, we will necessarily need to use the limit definition.

Note that the power rule does not apply to exponential functions!

We have,

$$\begin{aligned}\frac{d}{dx}[b^x] &= \lim_{h\to 0}\frac{b^{x+h} - b^x}{h}\\ &= b^x\left(\lim_{h\to 0}\frac{b^h - 1}{h}\right)\end{aligned}$$

Note that the quantity $\boxed{\lim_{h\to 0}\frac{b^h - 1}{h}}$ does not depend on x. Most likely, it is a constant that depends on b, let's call it $m(b)$. We will find what $m(b)$ is in a moment. However, we can still conclude that the function b^x is **proportional** to its own derivative! No function we have seen thus far has had this property!

Before we find the proportionality constant $m(b)$, let's consider the following question:

Does there exist a value b such that $\frac{d}{dx}[b^x] = b^x$?

If such a value of b can be found, then it must be true that $\lim_{h\to 0} \frac{b^h - 1}{h} = 1$. For very small values of h, we can rewrite this as

$$\frac{b^h - 1}{h} \approx 1 \iff b^h \approx 1 + h \iff b \approx (1+h)^{\frac{1}{h}}$$

In other words, b must be the limit of $(1+h)^{\frac{1}{h}}$ as h approaches 0.

Definition A.1

We define **the number e**, Euler's constant, as the value of the limit

$$e = \lim_{h\to 0}(1+h)^{\frac{1}{h}}$$

Equivalently, e is the unique number b for which $\lim_{h\to 0} \frac{b^h - 1}{h} = 1$.

Digression

Both of these can be used as **equivalent** definitions of e, meaning, you can assume one is true and use it to show that the other limit holds. There are actually several ways to define the number e. But in terms of Calculus, the above definition is the most natural one to give.

Summarizing, we have a derivative formula for the **very special** exponential function $f(x) = e^x$ as follows:

$$\boxed{\frac{d}{dx}[e^x] = e^x}. \tag{1}$$

So $f(x) = f'(x)$ for this function. This is one of the reasons why e^x is so special! In general, using chain rule, we can modify above statement to give the general rule:

$$\boxed{\frac{d}{dx}\left[e^{g(x)}\right] = e^{g(x)}g'(x)}. \tag{2}$$

To find the general formula for $m(b) = \lim_{h\to 0} \frac{b^h - 1}{h}$, we need to use the inverse function of e^x. Recall that the inverse function of $f(x) = e^x$ is called the **natural logarithm** and is written as $f^{-1}(x) = \ln x$. It follows that

$$\ln(e^b) = b = e^{\ln b}$$

for all real numbers $b > 0$. So we can replace the b in above limit and rewrite it as

$$\begin{aligned}
m(b) &= \lim_{h\to 0} \frac{b^h - 1}{h} \\
&= \lim_{h\to 0} \frac{\left(e^{\ln b}\right)^h - 1}{h} \\
&= \lim_{h\to 0} \frac{e^{h\ln b} - 1}{h} \\
&= \lim_{k\to 0} \frac{e^k - 1}{\left(\frac{k}{\ln b}\right)}, \quad \text{where } k = h\ln b \\
&= \ln b \left(\lim_{k\to 0} \frac{e^k - 1}{k}\right) \\
&= \ln b, \quad \text{because } \lim_{k\to 0} \frac{e^k - 1}{k} = \lim_{h\to 0} \frac{e^h - 1}{h} = 1 \text{ by definition}
\end{aligned}$$

Thus we have proved the general result that for any $b > 0$, we have

$$\boxed{\frac{d}{dx}[b^x] = b^x \ln b}. \tag{3}$$

Try your hand at using this derivative formula for the next several examples.

Question 1. **Exponential Derivative Practice**

Compute the derivative $\dfrac{dy}{dx}$.

a) $y = e^{x^2}$

b) $y = e^{\sin(x)}$

c) $y = x^2 e^{2x}$ (remember the product rule!)

d) $y = e^{e^x}$

e) $y = 3^x - 2^x$

f) $y = 2^{\sin(x)}$

§B. Logarithm Function

Next we would like to find the derivative of $f(x) = \log_b(x)$. Let's start with $\ln x = \log_e(x)$. Since $e^{\ln x} = x$, we can write

$$\frac{d}{dx} e^{\ln x} = \frac{d}{dx}[x] \implies e^{\ln x} \frac{d}{dx}[\ln x] = 1 \implies x \frac{d}{dx}[\ln x] = 1$$

Here in the second step, we used the chain rule and the formula from (2). So we conclude that,

$$\boxed{\frac{d}{dx}[\ln x] = \frac{1}{x}}. \tag{4}$$

In general, using chain rule, we can modify above statement to give the general rule:

$$\boxed{\frac{d}{dx}[\ln g(x)] = \frac{1}{g(x)} g'(x)}. \tag{5}$$

With this result, here's a second way to prove result (3). We will start with the identity $\ln(b^x) = x \ln b$. Then differentiating both sides with respect to x and using chain rule, we get

$$\frac{d}{dx} \ln b^x = \frac{d}{dx}[x \ln b] \implies \frac{1}{b^x} \frac{d}{dx} b^x = \ln b \implies \frac{d}{dx} b^x = b^x \ln b.$$

What about the derivative of the general function $f(x) = \log_b x$. Fortunately, this is easier to differentiate than the general exponential function, since $\log_b x = \dfrac{\ln x}{\ln b}$ using a property of logarithm. Hence we can conclude that

$$\boxed{\frac{d}{dx} \log_b x = \frac{1}{x \ln b}}. \tag{6}$$

■ Question 2. | Logarithm Derivative Practice

Compute the derivative $\dfrac{dy}{dx}$.

a) $y = \dfrac{x}{\ln(x)}$

b) $y = \tan(x) \cdot \ln(x)$

c) $y = \ln(9x^2 - 8)$

d) $y = \ln(\sec(x))$

e) $y = \ln(\ln(v))$

f) $y = \log_{10}(z^2 - 10z + 5)$

g) $y = \log_7(\sqrt{6^x + 1})$