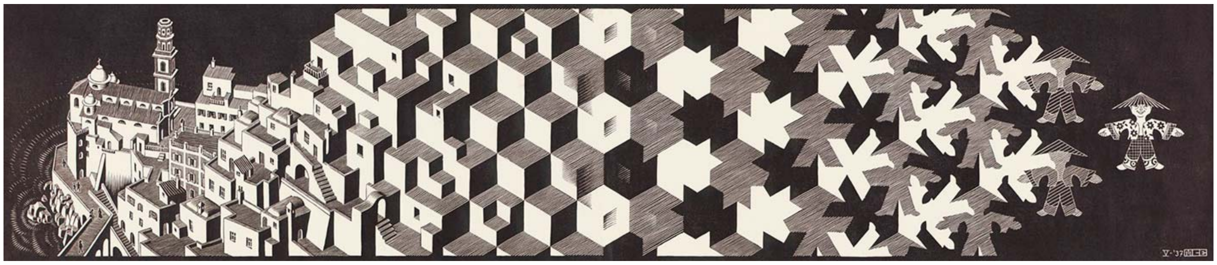

Elementary Functions and Calculus I

Math 131 Lecture notes

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Chapter 0 | Precalculus

§0.1 Number Sets

Definition 0.1.1

The most important sets of numbers are defined and denoted as follows:

- Natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$
- Integers, $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- Rational numbers, $\mathbb{Q} = \left\{ \frac{p}{q} \middle| p, q \in \mathbb{Z}, q \neq 0 \right\}$
- Real numbers, $\mathbb{R}$

The set of irrational numbers doesn't have a notation. It is instead denoted as $\mathbb{R} \setminus \mathbb{Q}$.

0.1.1 Density of Real numbers

Between any two different real numbers a and b, no matter how close together, there is another real number. Actually, we can say more. Between any two distinct real numbers, there are both a rational number and an irrational number. The fancy way to say this is as follows:

Theorem 0.1.2

The rational numbers are dense along the real line. Similarly, the irrational numbers are dense along the real line.

We will not prove this result here. But one of the most important consequences of the above result is the following, which may prove useful down the road:

> **Note:** The fact that rational numbers are dense in $\mathbb{R}$ means that given any real number $x \in \mathbb{R}$, we can find a rational number $q \in \mathbb{Q}$ as close to x as we want. Similarly, for irrationals.

Question 1. □

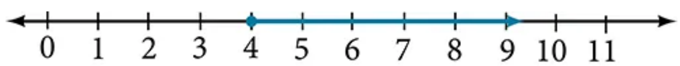
Find a rational number that is within 0.0000001 *distance of* π. *[Hint:* $\pi = 3.1415\ldots$*]*

§0.2 Inequalities and Absolute Values

I will assume at the start that you are familiar with solving equations such as $x^2 - 3x + 2 = 0$. If you are not sure what I mean by this, please let me know as soon as possible. In this lecture, we are going to talk about how to solve inequalities instead.

0.2.1 Interval Notations

Consider the inequality $x \geq 4$. We can describe the 'solution set' for this inequality as the set of all real numbers that are greater than or equal to 4. One way to describe this set is to use a number line as shown below.



The blue ray begins at $x = 4$ and, as indicated by the arrowhead, continues to infinity. Obviously, this is not a very compact way to describe the set, but I think a visual intuition will help a lot in the future.

The second method is to use the set-builder notation $\{x|x \geq 4\}$. But of course, this doesn't really illuminate much on the nature of the solutions. Nonetheless, the set notation also has its uses.

The third method is the **interval notation**, in which solution sets are indicated with parentheses or brackets. The solutions set in this case would be represented as $[4, \infty)$. We will focus on this method for this lecture.

Question 2.

Convert the following set-builder notations to interval notation. Some of them have been worked out for you.

(a) $\{x \mid a < x < b\} = (a, b)$

(b) $\{x \mid a < x < b\} = [a, b]$

(c) $\{x \mid a \leq x < b\} =$ ____________

(d) $\{x \mid a < x \leq b\} =$ ____________

(e) $\{x \mid x < a\} =$ ____________

(f) $\{x \mid x \geq b\} =$ ____________

(g) $\{x \mid x < a \text{ or } x > b\} =$ ____________

(h) $\{x \mid x \in \mathbb{R}\} =$ ____________

Question 3.

Express all real numbers less than −2 or greater than or equal to 3 in interval notation. ____________

0.2.2 Solving Inequalities

The following three properties of inequality will help you with transforming an inequality until the solution set becomes obvious.

Theorem 0.2.3

Addition Property: *If $a < b$, then $a + c < b + c$.*

Multiplication Property:

- *If $a < b$ and $c > 0$, then $ac < bc$*
- *If $a < b$ and $c < 0$, then $ac > bc$*

■ Question 4.

☐

Solve the following inequalities. Make sure you can explain which of the above properties you are using at each step.

(a) $3x - 2 > 1$

(b) $2 - 5x > 3$

Observation: A linear polynomial of the form $ax + b$ is zero when $x = \dfrac{-b}{a}$. It is negative on one side and positive on the other side of this value.

More complicated Inequalities in one variable

■ Question 5.

☐

Solve the following inequalities and write the solution sets using interval notation.

(a) $13 - 7x \geq 10x - 4$

(b) $x^2 - 5x + 6 \geq 0$

- *First factor the polynomial $x^2 - 5x + 6$.* ________________________________
- *Write down the values of x where the polynomial is equal to zero.* __________ *There are two values.*
- *Since the polynomial isn't zero otherwise, we have three intervals where it could be positive. Let's check each interval individually.*

  First interval: ______ *On this interval, the first factor is* ______ *and the second factor is* ______

  Second interval: ______ *On this interval, the first factor is* ______ *and the second factor is* ______

  Third interval: ______ *On this interval, the first factor is* ______ *and the second factor is* ______
- ***Final answer:*** ________________________ *Don't forget that the questions ask for '$\geq$'.*

(c) $\dfrac{x-1}{x+2} > 0$ *[**Hint:** Try something similar to the last example.]*

Compound Inequalities

A compound inequality includes two inequalities in one statement. A statement such as $6 \leq 4x - 3 < 9$ means $6 \leq 4x - 3$ and $4x - 3 < 9$. You can manipulate all three parts simultaneously, or you can deal with each part separately.

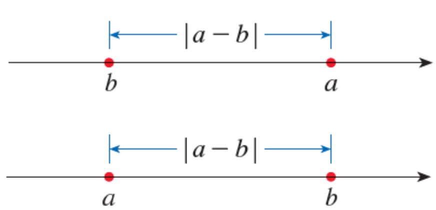
Question 6.

Solve the compound inequality: $3 + 2x > 7x - 2 > 5x - 10$.

What about the inequality: $3 + 2x < 7x - 2 < 5x - 10$? ______________________

0.2.3 Absolute Value Inequalites

I like to think of the absolute value of a number as its distance from the origin. A point located at $-x$ on the real number line has the same distance from the origin as x. Both have an absolute value of x, as it is x units away. This allows us to think of $|a - b|$ as the distance between two numbers a and b. Regardless of direction, positive or negative, the distance between the two points is represented as a positive number or zero.

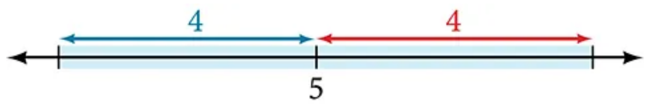


Example 0.2.4

- Consider the inequality $|x| > 4$. It is most likely that you may have seen how to solve this algebraically in the past. But I want to think of this as the set of numbers x, whose distance from 0 is more than 4. This allows us to break our solution set into two pieces, one to the left of the origin: ________________, and one to the right: ________________.

The final answer is the union of these two intervals. ________________

- Similarly, if we have $|x| < 7$, this is the set of points that are within 7 unit distance from the origin. In interval notation, this is ________________
- Finally, consider $|x-5| < 4$. This is the set of points whose distance from 5 is less than 4. Let's draw a Real number line.



What numbers are within 4 units of 5? Write it in interval notation: ________________

So we have established that

Theorem 0.2.5

- $|x| > c$ *is equivalent to* ________________
- $|x| < c$ *is equivalent to* ________________

Question 7.

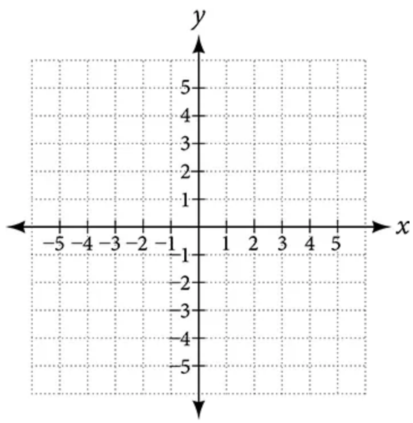
Solve the inequality $|2x+4| \leq 6$. □

Question 8.

Solve the inequality $-2|x-4| \leq -6$. □

§0.3 The Rectangular Coordinate System

The Cartesian coordinate system, also called the rectangular coordinate system, is based on a two-dimensional plane consisting of the x-axis and the y-axis. Perpendicular to each other, the axes divide the plane into four sections. Each section is called a **quadrant**. The center of the plane is the point at which the two axes cross. It is known as the **origin**, or point $(0,0)$.



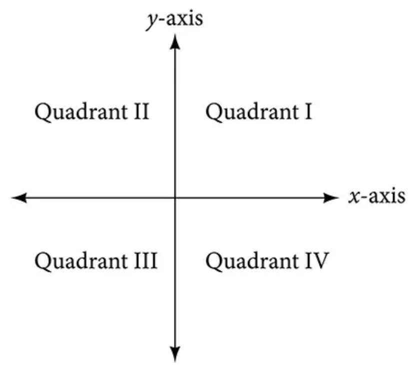


Figure 1: xy-plane

From the origin, each axis is further divided into equal units: increasing, positive numbers to the right on the x-axis and up the y-axis; decreasing, negative numbers to the left on the x-axis and down the y-axis. The axes extend to positive and negative infinity as shown by the arrowheads in fig. 1. Each point in the plane is identified by its x-coordinate, or horizontal displacement from the origin, and its y-coordinate, or vertical displacement from the origin. Together, we write them as an ordered pair indicating the combined distance from the origin in the form (x,y).

■ **Question 9.** □

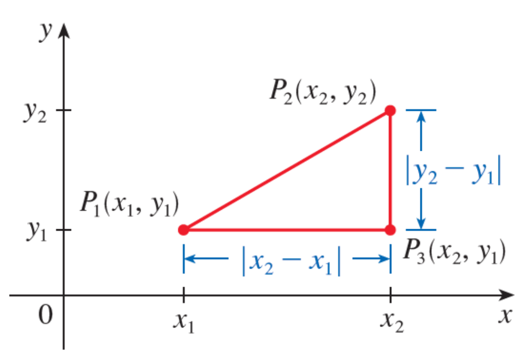
Plot the points $(-3,-2)$ *and* $(4,5)$ *on fig. 1.*

■ **Question 10.** □

Plot all the points (x,y) *where* $y=1$ *on fig. 1.*

0.3.1 The Distance Formula

Consider two points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ on the Cartesian plane. Complete the triangle as in the picture below:



■ Question 11.

Can you explain why the coordinates of P_3 is (x_2, y_1)?

We established in the last chapter that the absolute value function measures the distance between two numbers. Using this idea, we can measure the length of the base and the height of the right-angled triangle. Now use the Pythagorean theorem to complete the theorem below.

Theorem 0.3.6

The distance between two points (x_1, y_1) and (x_2, y_2) on the Cartesian plane is given by

____ .

■ Question 12.

Find the distance between the two points from fig. 1.

■ Question 13.

Find the point exactly halfway between the two points from fig. 1.

A common mathematical use of the xy-plane is to visually bring to life equations of two real-numbered variables. We will consider the equation of a circle and a straight line in the rest of this section.

0.3.2 The Equation of a Circle

Definition 0.3.7

A circle is a set of points that lie at a fixed distance (the radius) from a fixed point (the center).

Note: There are two important observations here:

(a) The interior of the circle is NOT part of the circle.

(b) The center of the circle is NOT part of the circle.

Example 0.3.8

onsider, for example, the circle of radius 3 with center at $(-1, 2)$. If (x, y) is a point on the circle then the distance of (x, y) from $(-1, 2)$ is 3. Using the distance formula from theorem 6, we get

$3 =$ ____ . Squaring both sides, we get

____ .

■ **Question 14.** □

Suppose the diameter of a circle has endpoints $(-1,-4)$ *and* $(5,-2)$.

(a) *Find the center and the radius of the circle.*

(b) *Find the equation of the circle.*

0.3.3 Equation of Straight Lines

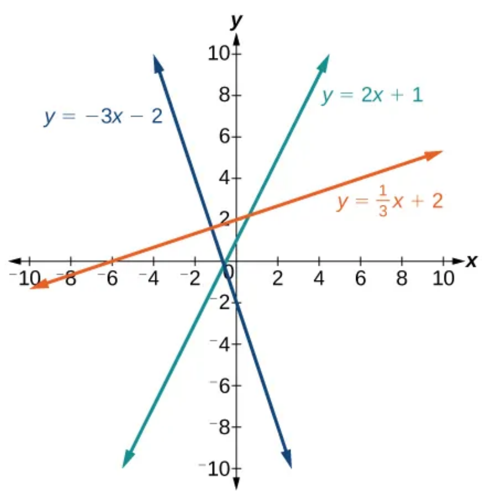
Besides circles, another set of simpler equations to start studying carefully are those for which both x and y are only of degree-one. Such an equation would be of the form $ax+by=c$, where a,b, and c are in $\mathbb{R}$.

- If b is zero, then such an equation has an equivalent form of $x=\frac{c}{a}$, which is satisfied by all points along a vertical line $\frac{c}{a}$ units from the origin.
- If b is not zero, then we can rewrite the equation into the following equivalent form:

  $y=$ ____ . $(\star)$

  This equation shows the linear dependence between x and y more clearly, since any change in x, say Δx, results in a proportional change in y, of $\Delta y=-\frac{a}{b}\Delta x$. Increase x by 1 and the corresponding y is $-\frac{a}{b}$; increase x by 2 and the corresponding y is $-\frac{2a}{b}$; and so on.

Regardless of the case, a degree-one equation of the form $ax+by=c$, where a,b, and c are in $\mathbb{R}$, graphically defines a line in the xy-plane.



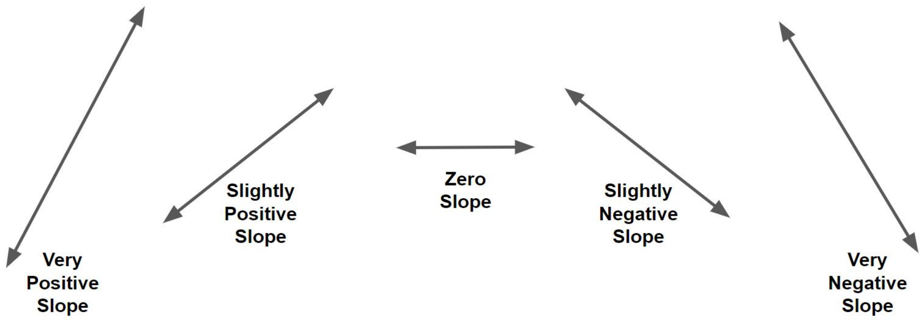
SLopes of Straight Lines

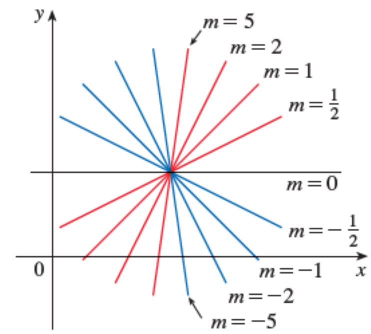
Definition 0.3.9

The slope of a *nonvertical* line refers to the ratio of the vertical change in y over the horizontal change in x between any two points on a line. It indicates the direction in which a line slants as well as its steepness.

We can quantify the slope of a straight line passing through two points (x_1, y_1) and (x_2, y_2) as

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$





Question 15.

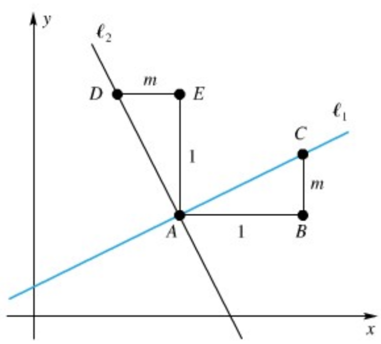
Find the slope of the straight line that passes through the points $(-2, 6)$ *and* $(1, 4)$.

Question 16.

How do the slopes of two nonvertical parallel lines related to each other?

Exploration Activity

Slopes of Perpendicular Lines: It turns out if m_1 and m_2 are the slopes of two perpendicular lines, then $m_1 m_2 = -1$. See this Khan Academy video (https://tinyurl.com/yx3d3v52) for an explanation that uses Euclidean geometry and congruent triangles.



From the equation (⋆) on the last page, we can rewrite a linear equation to find the ratio of Δy to Δx. This by definition then, is the slope of the line.

Question 17.

Find the slope of the straight line $3x + 4y = 5$.

0.3.4 Other Forms of Straight line equations

Point-Slope Form

Question 18. □

Answer YES or NO for each.

(a) *Given a line in the xy-plane can the value of its slope alone distinguish it from all other lines in the xy-plane?*

(b) *If we specify the slope of a straight line and a point it passes through, can we uniquely identify the equation of the line?*

So let's consider a straight line that passes through (x_1, y_1) and has slope m. Given a generic point (x, y) on this line, we can use the definition of slope to get

$$m = \frac{y - y_1}{x - x_1}, \qquad \text{or equivalently,} \qquad y - y_1 = m(x - x_1)$$

This is called the **point-slope form** of the equation of a straight line.

Question 19. □

Write the equation of the line with slope $m = -3$ *that passes through the point* $(4, 8)$.

Question 20. □

Write the equation of line parallel to $5x = 7 + y$ *and passing through the point* $(3, 5)$.

Question 21. □

Write the equation of line perpendicular to $5x - 3y + 4 = 0$ *and passing through the point* $(-1, 4)$.

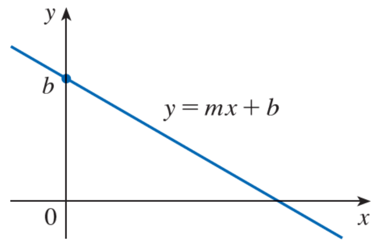
Slope-Intercept Form

Let's rewrite the Point-Slope form to isolate y. We get

$$y = mx - mx_1 + y_1 = mx + (y_1 - mx_1)$$

Where does this line intersect the x–axis?

This must be a point whose x-coordinate is zero. If it has to be on our straight line, this forces the y–cordinate of the intersection point to be $y_1 - mx_1$. Let's call it b, this is usually called the y–intercept.



The form $y = mx + b$ is accordingly called the **slope-intercept form** of the equation of a straight line.

Two-point Form

Question 22.

We want to find the equation of the line passing through the points $(3, 4)$ *and* $(0, -3)$.

(a) *First find the slope of the line.*

(b) *Next find its point-slope form (use either point). Can you explain why both points would give the same answer?*

(c) *Write the final equation in slope-intercept form.*

□

§0.4 Graphs of Equations and Curves in the Plane

The use of coordinates for points in the plane allows us to describe a curve (a geometric object) by an equation (an algebraic object). We saw how this was done for circles and lines in the previous section. Now we want to consider the reverse process: graphing an equation.

Definition 0.4.10

The **graph** of an equation in x and y consists of those points in the plane whose coordinates (x, y) satisfy the equation, that is, make it a true equality.

0.4.1 Using DESMOS to understand graphs

The most naive way of plotting a graph is to make a list of (x, y) pairs that satisfy an equation and then join the points by a smooth curve. We will do an example on the blackboard. However, the modern technology allows us to use a graphing calculator or a computer to do these jobs more efficiently. So while it is still important to have an understanding of why graphs of certain equations look a certain way, we don't need to plot graphs by hand!

Question 23. □

Start by opening `https://www.desmos.com/calculator` *on your computer. You can optionally sign up or log in to save and share your work.*

(a) *Type in the equation $y = mx + b$.*

- *Once this equation is entered DESMOS shows an option to add sliders for the values m and b. Click all.*
- *Once you click all, a line will appear on the grid as well as two sliders. One controls the m value and one controls the b value. The default settings allow you to change the value of each variable from -10 to 10. To change the range of numbers you can use click on either the upper or lower bound. You can now change the value of the bounds and the value of the step between each number on the slider. Change it range and step value.*
- *Explore what types of lines can be made when changing the values of each slider.*

(b) *Plot $y = x^2$. The graph is symmetric about the y–axis. Can you algebraically justify why this is the case?*

(c) *Type in $y = px^2 + qx + r$ and make sliders for $p, q,$ and r. To type powers of x, use the caret key(ˆ) on top of 6 in your keyboard. Observe that in general, any equation of the form*

"y equals degree-two polynomial in the variable x"

results in a parabola. How does the shape of the graph depend on p and r? What happened when p, the coefficient of x^2 is positive vs negative?

Answer: ______________________________

(d) *Type in any equation of your choice (e.g. $y = 2x^3 - 5x + 1$). Suppose we want to create a table of values from a graph. To see a table that reflects the values of this equation go to the gear button at the top left side of the screen. This is the edit list button.*

- *Two new options will appear. The first is to convert the equation to a chart. The second is to duplicate the equation. We will convert the equation to a chart.*

- *Desmos will fill five coordinates into the chart from x=-2 to x=2. If you want more values, you can fill in an x value into the chart, and the corresponding y value will be added.*

(e) *Start a new graph (or delete the plots so far). Try plotting $(x-h)^2+(y-k)^2=r^2$ and add sliders as usual.*

- *You should have got a circle. What do the number h and k represent?*
- *What does the quantity r represent?*

Answer: ____

(f) *Plot $x=y^2$. This should give you the picture of a sideways parabola.*

(g) *Plot $y=\sqrt{x}$. What's the difference between this graph and the graph of $x=y^2$?*

Answer: ____

(h) *Plot $y=x^3$. The graph is symmetric across the origin. Can you algebraically justify why this is the case?*

Answer: ____

(i) *Plot $x=y^3$. How does it compare to the graph of $y=x^3$?*

(j) *Plot $y=\sqrt[3]{x}=x^{1/3}$. How does it compare to the graph of $x=y^3$?*

Answer: ____

(k) *The last example graph we do today is that of $y=|x|$. This graph is particularly interesting in that while it can be sketched from left-to-right in one connected motion (without having to pick up one's writing utensil), its graph abruptly jumps in slope at the origin (which will be a useful feature to discuss more precisely in due time).*

0.4.2 Determining Intersection Points of Graphs

How can we use equations of curves to find their points of intersection? You have done some examples in the last tutorial. The process can be summarized as follows. We will follows along with an example.

Example 0.4.11

Suppose we want to find the intersection point between the straight line $2y-3x=2$ and the parabola $y=x^2-x-5$. Here are the steps.

- *Algebraically manipulate one of the equations to solve for one of the variables in terms of the other. Depending on your equation, it might be easier to solve for y interms of x or x in terms of y.*

  If we start with the parabola, we already have y interms of x. If we start with the line, we can rewrite it as $y=$ ____________________. We will continue the next steps using both methods.

- *Next, substitute the variable you solved for into the other equation.*

  If we start with the parabola, subsitute y into the equation of the straight line to get

$$2(x^2-x-5)-3x=2 \implies 2x^2-5x-10=2 \implies 2x^2-5x-12=0$$

If we start with the straight line, substitute what you got for y in the last step into the equation $x^2 - x - 5 =$ ______________________. Simplify, and you should get $2x^2 - 5x - 12 = 0$.

- *Once you have achieved an equation involving a single variable, solve for the variable.*

  Hence the solutions are $x =$ ______________________________________.

- *Find the corresponding values of the other coordinate by plugging your values from last step into either curve. Both give the same answers because that's what intersection means - points that are on both curves simultaneously.*

  The final points are: ______________________________________

Question 24.

Find and the points of intersection of the following graphs: $y = x - 1$ *and* $2x^2 + 3y^2 = 11$.

Note: While DESMOS can be used to also get the points of intersection easily, note that it will not give you an exact value unless your points of intersections are ''nice'' (e.g. integer or rationals). So you should know how to find such points algebraically as well.

§0.5 Functions, their Combinations, and their Graphs

0.5.1 Pattern Recognition and Mathematical Modeling

Mathematics is the art of making sense of patterns. One way that patterns arise is when two quantities are changing in tandem. In this setting, we may make sense of the situation by expressing the relationship between the changing quantities through words, through images, through data, or through a formula.

Example 0.5.12

In the late 1800s, the physicist Amos Dolbear was listening to crickets chirp and noticed a pattern: how frequently the crickets chirped seemed to be connected to the outside temperature. Dolbear's observations are summarized in the following table.

N (chirps per minute)	40	80	120	160
T (° Fahrenheit)	50°	60°	70°	80°

For a mathematical model, we often seek an algebraic formula that captures observed behavior accurately and can be used to predict behavior not yet observed. For the data in above table, we observe that each of the ordered pairs in the table make the equation

$$T = 40 + 0.25N$$

true. For instance, $70 = 40 + 0.25 \times 120$. Indeed, scientists who made many additional cricket chirp observations following Dolbear's initial counts found that the formula holds with remarkable accuracy for the snowy tree cricket in temperatures ranging from about 50°F to 85°F.

Using the model, if we hear snowy tree crickets chirping at a rate of 92 chirps per minute, what does Dolbear's model suggest should be the outside temperature? ____________________

0.5.2 Domain, Codomain, and Range

At its core, a function is a repeatable process that takes a collection of input values and generates a corresponding collection of output values with the property that **if we use a particular single input, the process always produces exactly the same single output.**

For instance, Dolbear's Law provides a process that takes a given number of chirps as inputs and reliably produces the corresponding temperature that corresponds to the number of chirps as output, and thus this equation generates a function. Using "D" to denote the "Dolbear" function, then we can represent the process as follows:

$$80 \xrightarrow{D} 60$$
$$120 \xrightarrow{D} 70$$
$$N \xrightarrow{D} 40 + 0.25N$$

Alternatively, we can also use the equivalent notation $D(N) = 40 + 0.25N$. Besides equation, tables and graphs are also particularly valuable ways to characterize and represent functions as we saw in last class using DESMOS.

If we name a given function f and call the collection of possible inputs to f the set A and the corresponding collection of potential outputs B, we say "f is a function from A to B".

Definition 0.5.13

Let f be a function from A to B. The set A of possible inputs to f is called the domain of f; the set B of potential outputs from f is called the codomain of f. We denote this as $f : A \to B$, or $A \xrightarrow{f} B$

The codomain of a function is the collection of possible outputs, which we distinguish from the collection of actual ouputs.

Definition 0.5.14

The range of f is the collection of all actual outputs of the function. That is, the range is the collection of all elements y in B for which it is possible to find an element x in A such that $f(x) = y$.

Question 25.

Find the domain of the following functions:

(a) $f(x) = \sqrt{x}$ ________________________

(b) $g(x) = \dfrac{1}{x-2}$ ________________________

Question 26.

Consider a spherical tank of radius 4 m that is filling with water. Let V be the volume of water in the tank (in cubic meters) at a given time, and h the depth of the water (in meters) at the same time. It can be shown using calculus that V is a function of h according to the rule

$$V = f(h) = \frac{\pi}{3}h^2(12-h)$$

(a) What values of h make sense as possible inputs in the context of this problem? ________________

This is the domain of f. Write your answer in interval notation.

(b) What values of V can you possibly get as outputs in the context of this problem? Be as broad as you want, no need to calculate any specific values yet. ________________________

This is the codomain of V. Write your answer in interval notation.

(c) Is there a value in your codomain that is not in the range? ____________________

(d) What is $f(4)$? You don't need to simplify. Can you write down an english sentence that gives a physcial interpretation (using appropriate units) of $f(4)$?

*(e) Consider the claim "**Since $f(9) = \frac{\pi}{3}9^2(12-9) \approx 254.47$, when the tank is 9 meters deep, there is about 254.47 cubic meters of water in the tank.**" Is this claim valid? Why or why not?*

Bubble Diagrams

The domain and codomain of a function don't have to be real numbers. You could have, for example, a function whose input is your name, and output is your CnetID! One useful representation of functions on finite sets is via **bubble diagrams**. To draw a bubble diagram for a function $f : A \to B$, draw one circle (i.e, a "bubble") for each of A and B and for each element of each set, put a dot in the corresponding set. Typically, we draw A on the left and B on the right. Next, draw an arrow from $x \in A$ to $y \in B$ if $f(x) = y$.

Example 0.5.15

Figure 2 depicts a bubble diagram for a function from domain $A = \{a, b, c, d\}$ to codomain $B = \{\diamondsuit, \clubsuit, \heartsuit, \spadesuit\}$. In this case, the range is equal to $\{\diamondsuit, \clubsuit, \spadesuit\}$.

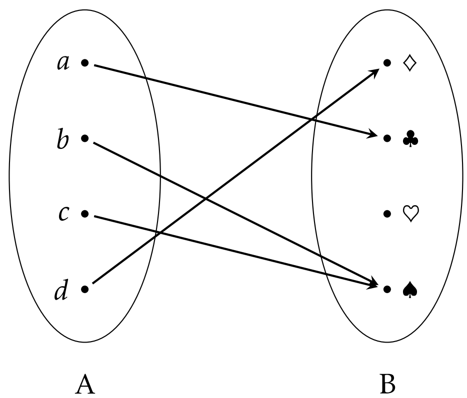


Figure 2: An example of a bubble diagram for a function.

Definition 0.5.16

When the rule for a function is given by an equation of the form $y = f(x)$, we call x the **independent variable** and y the **dependent variable**. Any value in the domain may be substituted for the independent variable. Once selected, this value of x completely determines the corresponding value of the dependent variable y.

0.5.3 Combinations of Real Functions

A function whose domain and codomain are both subsets of $\mathbb{R}$, is called a Real function. Such a function can be graphed on the Cartesian plane by plotting all the (input, output) pairs. Let's start with some examples of special families of Real functions.

Definition 0.5.17: Power Functions

A function of the form $f(x) = x^a$, where a is a constant, is called a power function. Here a does not necessarily have to an integer, it can be any positive or negative real number (or 0).

Essential Arithmetic Operations on Functions

Example 0.5.18

Cosider two function $f(x) = \dfrac{1}{x-2}$ and $g(x) = \sqrt{x}$. In the last page, you calculated the domain of both functions. Now let's say we create a new function calle $f + g$ whose value at x is the sum of $f(x)$ and

$g(x)$, i.e.

$$(f+g)(x) = f(x) + g(x)$$

The formula for $(f+g)(x)$ is ____

The domain of the new function $(f+g)$ is ________________________________

Similarly define $(f-g)(x)$ as $f(x) - g(x)$ and $(f \cdot g)(x)$ as $f(x) \cdot g(x)$. Write down the formulas.

$(f-g)(x) =$ __________________________ $(f \cdot g)(x) =$ __________________________

What can you say about the domain of $f-g$? _______________ and $f \cdot g$ _______________ ?

The function $\left(\dfrac{f}{g}\right)(x)$ defined as $\dfrac{f(x)}{g(x)}$ is a bit trickier, because we have an additonal restriction: we cannot divide by 0.

Writethe formaula. $\left(\dfrac{f}{g}\right)(x) =$ _______________. What is domain of this function? _______________

Question 27.

Suppose $F(x) = x^2 + 1$ *and* $G(x) = 2x + 1$. *What is the domain of* $\left(\dfrac{F}{G}\right)(x)$*?*

So, if we multiply a bunch of $f(x) = x$ functions together, we can create power functions with nonnegative integer exponents of x. If we then add multiple such functions with same or different exponents, we get Polynomial functions.

Definition 0.5.19: Polynomial Function

A function P is called a polynomial if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are constants called the coefficients of the polynomial.

The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$. If the leading coefficient $a_n \neq 0$, then the degree of the polynomial is n. For example, the function

$$P(x) = 2x^6 - x^4 + \frac{2}{5}x^3 + \sqrt{2}$$

is a polynomial of degree 6. If $n = 1$, we say that $P(x)$ is linear, if $n = 2$, it's a quadratic, and so on.

Finally, if we we take the quotient of two polynomial functions, we get the following:

Definition 0.5.20: Rational Function

A function of the form $\dfrac{P(x)}{Q(x)}$ where both P and Q are polynomial functions is called a **rational**

function.

Rational Function The following are examples of rational functions: $\frac{1}{x}$, $\frac{x^3+5x-5}{x^8-3}$.

Note: polynomials are rational functions with denominator 1. e.g. $x+1 = \frac{x+1}{1} = \frac{x+1}{x^0}$.

Question 28.

Find the domain of the rational function $f(x) = \frac{x^2-3}{x^2-2x-8}$

Composition of Functions

Because every function is a process, it makes sense to think that it may be possible to take two function processes and do one of the processes first, and then apply the second process to the result.

Example 0.5.21

uppose we know that y is a function of x according to the process defined by $y = f(x) = x^2 - 1$ and, in turn, x is a function of t via $x = g(t) = 3t - 4$. Is it possible to combine these processes to generate a new function so that y is a function of t?

When we have a situation like above where we use the output of one function as the input of another, we often say that we have "**composed two functions**". In addition, we use the notation $h(t) = f(g(t))$ to denote that a new function, h was the result of the composition.

Note: We also sometimes use the notation $h = f \circ g$ to denote that h is a composition of f and g. So we can write

$$(f \circ g)(x) = f(g(x))$$

Question 29.

Suppose $f(x) = \frac{1}{x-2}$ *and* $g(x) = \sqrt{x}$. *What is the function* $(f \circ g)(x)$? = ____________________

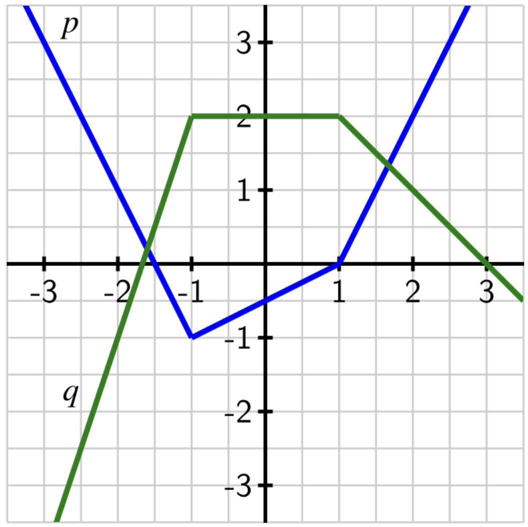
Find the domain of $f \circ g$.

What about $(g \circ f)(x)$? ____________________ *What is its domain?*

■ Question 30.

☐

Consider two function p and q whose braphs look like below:



Then $(p \circ q)(2) =$ ______________________________ .

Find an x such that $(q \circ p)(x) = 2$. ______________________________

Translation and Scaling of Functions

■ Question 31.

☐

Open a new Desmos graph and define the function $f(x) = |x|$. *Adjust the window so that the range is for* $-6 \le x \le 6$ *and* $-2 \le y \le 10$.

(a) *In Desmos, on line 2, enter* $g(x) = f(x) + a$. *You will get prompted to add a slider for a. Do so.*

(b) *Explore by moving the slider for a and write at least one sentence to describe the effect that changing the value of a has on the graph of g.*

(c) *Next, define the function* $h(x) = f(x - b)$ *add the slider for b. Move the slider for b and write at least one sentence to describe the effect that changing the value of b has on the graph of h.*

(d) *Now define the function* $p(x) = cf(x)$ *and add the slider for c. Move the slider for c and write at least one sentence to describe the effect that changing the value of c has on the graph of p. In particular, when* $c = -1$, *how is the graph of p related to the graph of f ?*

(e) *Finally, click on the icons next to g, h, and p to temporarily hide them, and go back to Line 1 and change your formula for f. You can make it whatever you'd like, but try something like* $f(x) = x^2 + 2x + 3$ *or* $f(x) = x^3 - 1$. *Then, investigate with the sliders a, b, and c to see the effects on g, h, and p (unhiding them appropriately). Write a couple of sentences to describe your observations of your explorations.*

0.5.4 Roots and Asymptotes

Definition 0.5.22

Roots are points on the x-axis where a function is zero. These can be found out by solving the equation $f(x) = 0$.

Thus for example, the roots of $f(x) = (x+2)(x-3)$ are ____________________

When sketching graphs of functions, sometimes those graphs appear to plateau in height as the graph extends far off to the right or the left. Other times, the function values spike, and the graph either ascending sharply toward or descending sharply toward . Such behavior is called **asymptotic**, and the horizontal lines to which a graph plateaus or the vertical lines to which a graph ascends or descends sharply are called **asymptotes** (horizontal and vertical, respectively).

Question 32.

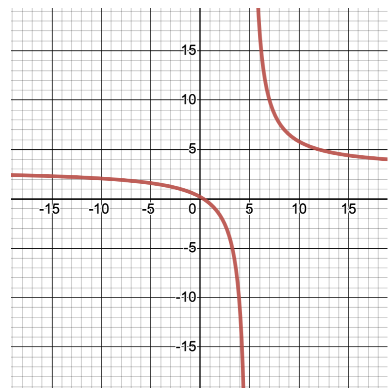
Consider the graph of the rational function $f(x) = \dfrac{3x-1}{x-5}$ *on the right.*

What happens to $f(x)$ *as* x *approches* 5 *from the right?* ____________

What happens to $f(x)$ *as* x *approches* 5 *from the left?* ____________

What happens to $f(x)$ *as* x *approches* $+\infty$ *or* $-\infty$*?* ____________

What are the vertical and horizontal asymptotes of this graph?



□

0.5.5 Symmetries of a Function

- If $f(-x) = f(x)$ for all x in the domain of f, that is, the equation of the curve is unchanged when x is replaced by $-x$, then f is called an **even function** and the curve is symmetric about the y-axis.

  This means when graphing such a function, our work is cut in half. If we know what the curve looks like for $x > 0$, then we need only reflect about the y-axis to obtain the complete curve. Here are some examples: $y = x^2, y = x^4$, and $y = |x|$.

- If $f(-x) = -f(x)$ for all x in the domain of f, then f is called an **odd function** and the curve is symmetric about the origin.

  Again we can obtain the complete curve if we know what it looks like for $x > 0$. [Rotate 180° about the origin] Some simple examples of odd functions are $y = x, y = x^3, y = x^5$, and $y = \sin x$.

Question 33.

Is the function $f(x) = \dfrac{x^3+3x}{x^4-3x^2+4}$ *odd, even, or neither?*

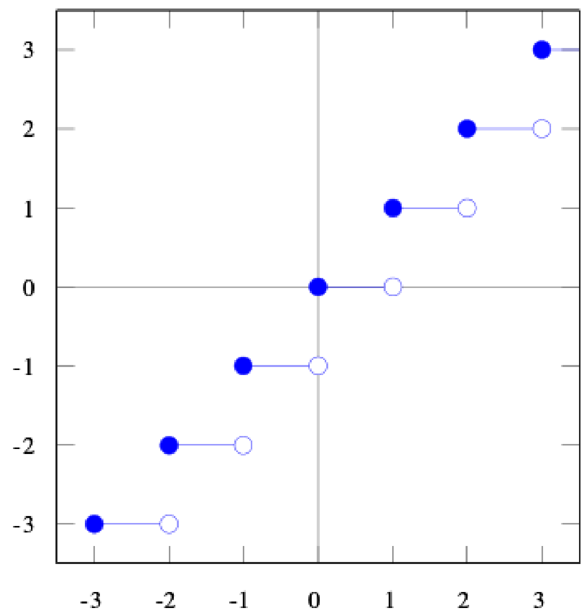
□

Exploration Activity

One of the interesting mathematical functions that often gets used as examples is the **greatest integer function** or **floor function** defined as

$$\lfloor x \rfloor = \text{ the greatest integer less than or equal to } x$$

The graph of $\lfloor x \rfloor$ is very interesting, it takes a 'jump' at each integer.



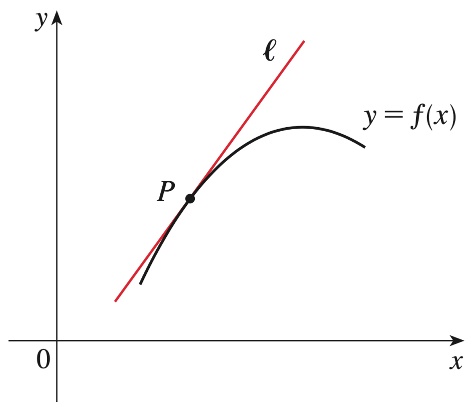
Chapter 1 | Limit and Continuity

§1.1 The Tangent Problem

Before we start the discussion on Limit, let's consider a geometric problem. Suppose we have the graph of a function $y = f(x)$ and we wish to find the equation of the tangent line ℓ to the graph at a point.

https://www.desmos.com/calculator/r1yukzurni

We quickly discover that one of the main obstacles behind figuring out the slope of ℓ is the fact that we need two points to compute the slope and we know only one point P where ℓ touches the curve.

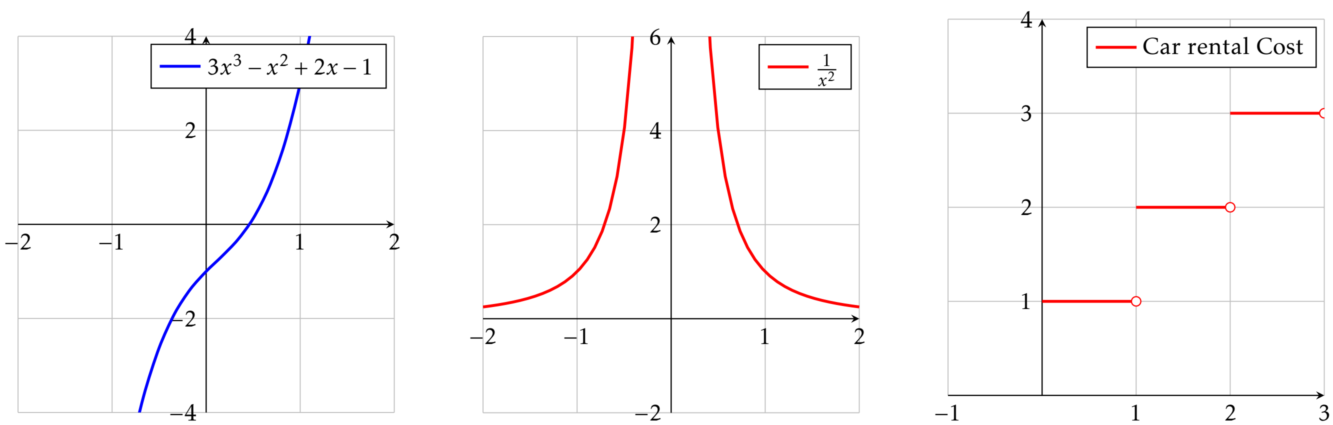


However, we find that if we start with the slope of a nearby secant PQ and compute its slope m_{PQ}, then the slope of the tangent can be numerically approximated by moving Q closer and closer to P and keeping track of what value m_{PQ} is approaching.

Today, we are going to talk about how we make this idea of "approaching" more mathematically precise by introducing some shorthand notations. However, before we go there, let's talk about another mathematical concept that also uses the idea of a function "approaching" a certain value.

§1.2 An Informal Introduction to Continuity

Consider graphs of the three functions as follows:



Informally, we say that a function is **continuous** if you can draw its graph without lifting the pencil from the paper. In the above three pictures, the first function is continuous everywhere; the second one is continuous on all intervals not containing 0, and the third function is continuous on intervals of the form $(n, n+1)$.

More specifically, A function $y = f(x)$ is called **continuous at a point** if nearby values of x give nearby values of y. In practical terms, this means small errors in the input lead to only small errors in the output. For the same reason, the idea of continuity is important in real life.

A bit more formally, we can say $f(x)$ is continuous at $x = a$ when the value of $f(x)$ "approaches" $f(a)$ as x approaches a (which we will denote in symbols as $x \to a$). So again we have a need to investigate what happens when a number x gets closer to a.

§1.3 What is a "Limit"?

We will write

$$\lim_{x\to a} f(x) = \mathrm{L}$$

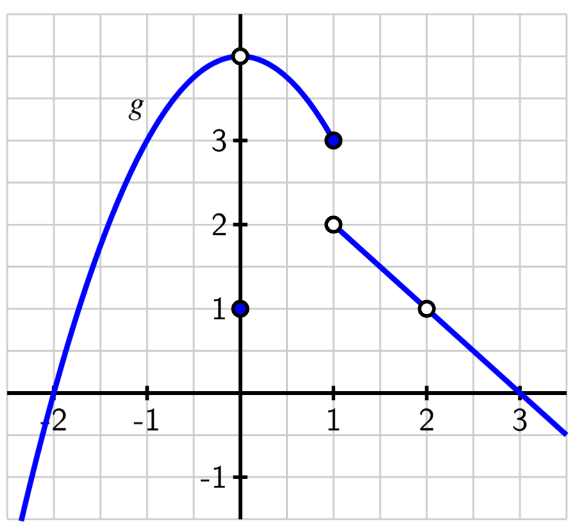
to mean that the values of $f(x)$ approaches L as x approaches a. We read it as:

"the limit of $f(x)$, as x approaches (or tends to) a, is equal to L".

In other words, the values of $f(x)$ tend to get closer and closer to the number L as x gets closer and closer to the number a (from either side of a).

Example 1.3.23

Consider the graph of $g(x)$ given below. Let us use this to show some examples of limits.



- For x-values like $x = -2, -1$, or $x = 3$, the values of $g(x)$ approach $g(-2)$, $g(-1)$, and $g(3)$, respectively. No strange behavior is happening right by these points. Hence, we could write:

$\lim_{x\to(-2)} g(x) =$ __________ $\lim_{x\to(-1)} g(x) =$ __________ $\lim_{x\to 3} g(x) =$ __________ .

- Next consider $x = 0$. Notice that $g(0)$ is defined and $g(0) =$ ________________________.

  But, which y-value does $g(x)$ approach as x gets closer and closer to (but not equal to) 0 from both sides? ________________________. Hence, $\lim_{x\to 0} g(x) =$ ________________________.

  Example 1.3.23

  Note: Limits tell us about **where a function is going**. It doesn't matter what $g(0)$ is; that has no bearing on the value of the limit, because the limit is trying to capture the behavior of g **around** $x = 0$ and not actually at $x = 0$.

- Consider $x = 2$. We can see on the graph that $g(2)$ is undefined, but everywhere around $x = 2$, the function $g(x)$ is defined and appears linear. Hence, we would say that $\lim_{x\to 2} g(x) =$ ____________.

- Now consider $x = 1$. Notice that the function g has a jump in its graph.

  For values $x < 1$, the function $g(x)$ is approaching the y-value of ________________________.

  But for values $x > 1$, $g(x)$ is approaching ____________________________________.

  Because $g(x)$ is trying to approach two different values from each side of $x = 1$, we say here that

  $$\lim_{x\to 1} g(x) \text{ does not exist.}$$

  You can also abbreviate "does not exist" with the letters DNE.

One of the most important takeaways from the above example is that even when $f(a)$ does not exist, the limit $\lim_{x\to a} f(x)$ might. Similarly, it is possible that $f(a)$ exists but $\lim_{x\to a} f(x)$ doesn't.

Question 34. ☐

Consider the function from the third picture above (cost of car rental). What is $f(1)$? What is $\lim_{x\to 1} f(x)$?

1.3.1 One-sided Limits

We actually have a notation to help us describe situations like the case of $x = 1$ in the last example 23, as well as the case of question 34.

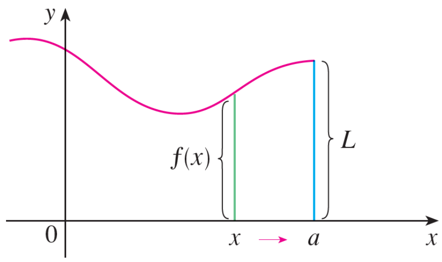
Definition 1.3.24: One Sided Limits

We write

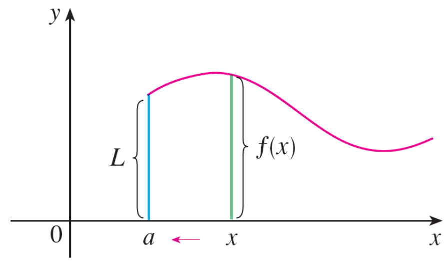
$$\lim_{x\to a^-} f(x) = L$$

and say the **left-hand limit** of $f(x)$ as x approaches a is equal to L if $f(x)$ approaches L as $x \to a$ with $x < a$ (i.e., **approaching from the left**)

Similarly, $\lim_{x\to a^+} f(x)$ denotes the **right-hand limit** where the notation $x \to a^+$ means that we consider only $x > a$ (i.e., **approaching from the right**).



(a) $\lim\limits_{x \to a^-} f(x) = L$



(b) $\lim\limits_{x \to a^+} f(x) = L$

So for the function $g(x)$ above in example 23, we could write:

$$\lim_{x \to 1^-} g(x) = 3 \quad \text{and} \quad \lim_{x \to 1^+} g(x) = 2.$$

Question 35.

The functions $f(x) = \frac{x^2-9}{x-3}$ and $g(x) = \frac{|x-3|}{x-3}$ are graphed below. Use the graphs to find the following limits.

(a) $\lim\limits_{x \to 3^+} \dfrac{x^2-9}{x-3} =$ ______________________

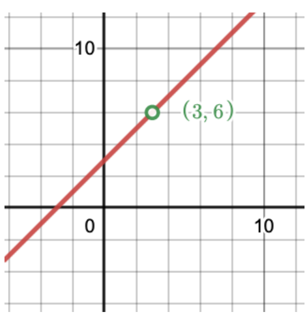
(d) $\lim\limits_{x \to 3^-} \dfrac{|x-3|}{x-3} =$ ______________________

(b) $\lim\limits_{x \to 3^-} \dfrac{x^2-9}{x-3} =$ ______________________

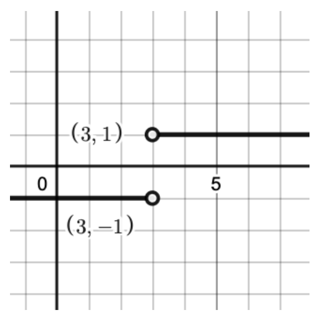
(e) $\lim\limits_{x \to 3^+} \dfrac{|x-3|}{x-3} =$ ______________________

(c) $\lim\limits_{x \to 3} \dfrac{x^2-9}{x-3} =$ ______________________

(f) $\lim\limits_{x \to 3} \dfrac{|x-3|}{x-3} =$ ______________________



(a) $f(x) = \frac{x^2-9}{x-3}$



(b) $g(x) = \frac{|x-3|}{x-3}$

Figure 1.1: Graphs for $f(x)$, $g(x)$, $h(x)$

Question 36.

(***Sketch a Graph I***) *Sketch a graph for a function $f(x)$ that satisfies the following:*

- $f(-2) = 2$ *and* $\lim\limits_{x \to (-2)} f(x) = 1$
- $f(-1) = 3$ *and* $\lim\limits_{x \to (-1)} f(x) = 3$
- $f(1)$ *is not defined and* $\lim\limits_{x \to 1} f(x) = 0$
- $f(2) = 1$ *and* $\lim\limits_{x \to 2} f(x)$ *does not exist.*

§1.4 A more formal definition of Limit

While the somewhat reasonable nature of our intuition of limits is satisfactory, (for a mathematician) a logically sound articulation is always preferred. As such, today we will work to develop a precise mechanism by which we can define the notion of limit of a function at a point.

1.4.1 The Closeness Game

For calculating limits, we wish to precisely address the question, "If x is close to a, what value is $f(x)$ close to?" We immediately run into an issue: how do you define "close" using math? We will get to that in a moment.

Let's work with a concrete example to showcase the process:

Example 1.4.25

Consider $\lim\limits_{x\to 3}(2x+1)$ which we expect converges to 7. To test whether our expectation is correct, we play a two-person game as follows:

- First, player 1 chooses a positive real number, we will call it ε, to quantify an allowable margin of error for how close $f(x)$ should be to 7 when x is close to 3.

  For example, say player 1 established an allowable error margin of 0.2. In other words, player 1 challenges player 2 to find values of x such that the output $f(x)$ is within the interval $(7-0.2, 7+0.2) = (6.8, 7.2)$.
- Next, to validate the first players expectations, and pass the test that player 1 has challenged player 2 with, player 2 must choose an allowable deviation in x around 3, which we will call δ, for which $f(x)$ will necessarily evaluate within the interval corresponding to the $(6.8, 7.2)$.
- So now player 2 asks themselves, "If we need $2x+1$ to evaluate to be between 6.8 and 7.2 , how near to 3 must our x be?"
- Player 2 decides to work backwards and considers when exactly does $2x+1$, our function, actually equal to 6.8 and when is it equal to 7.2?
- $$2x+1 = 6.8 \implies 2x = 5.8 \implies x = 1.9, \qquad 2x+1 = 7.2 \implies 2x = 6.2 \implies x = 3.1$$
- So player 2 announces that the allowable deviation from 3 in the input x can be at most 0.1.

Note: At this point, there are a couple items worth mentioning.

- Player 1 had a free choice of positive real number for the allowable error margin that they gave us. They happened to have chosen 0.2, but in principle they could have chosen any positive number (no matter how large or small).
- If they chose a smaller value for ε, we would have to respond with a smaller choice of δ in reply. But, if they chose a larger value for ε, the choice of δ we made would still work.

Question 37. □

Let's do another round of the closeness game. Can you find a δ that would work if $\varepsilon = 0.02$? What if $\varepsilon = 0.002$?

■ Question 38.

Have you found a full-proof strategy so that player 2 can always find a δ no matter what ε player 1 provides? □

If you said yes to the last question, Congratulations! You have "proved" that the limit of $2x+1$, as x approaches 3 is indeed 7. Now, if I ask you to write down what player 2's startegy should be - that's would be writing the "proof".

■ Question 39.

Let's do another example with a different limit. Suppose we wish to "prove" that □

$$\lim_{x\to 1}(3-2x) = 1.$$

(a) *Suppose player 1 provides $\varepsilon = 0.1$. Can you help player 2 find a δ? Use the same steps as the example.*

(b) *The goal is still to find a full-proof strategy so that player 2 can always find a δ no matter what ε player 1 provides. Can you come up with such a strategy?*

(c) *Is there a way to state δ in terms of ε?*

We will focus on how to write your argument formally in one coherent explanation next time. For now, let's end with the more precise mathematiocal definition of "closeness" as we promised before.

Exploration Activity

Here is an applet for you to experiment $\varepsilon - \delta$ proofs for a non-linear function:

https://www.geogebra.org/m/FQwxkVbK

1.4.2 Quantifying Closeness

Recall that the distance between two points a and b on a number line is given by $|a-b|$.

- The statement $|f(x)-L| < \varepsilon$ may be interpreted as: The distance between $f(x)$ and L is less than ε.
- The statement $0 < |x-a| < \delta$ may be interpreted as: $x \neq a$ and the distance between x and a is less than δ.

We also want to recall the following equivalent algebraic interpretation for absolute value:

- The statement $|f(x)-L| < \varepsilon$ is equivalent to the statement $L-\varepsilon < f(x) < L+\varepsilon$.
- The statement $0 < |x-a| < \delta$ is equivalent to the statement $a-\delta < x < a+\delta$ and $x \neq a$.

With these clarifications, we can state the formal epsilon-delta definition of the limit.

Definition 1.4.26

Suppose $f(x)$ is defined for all x over an open interval containing a, except possibly at a. Let L be a real number. Then the notation

$$\lim_{x\to a} f(x) = L$$

means:

For all $\varepsilon > 0$, there exists a $\delta > 0$, such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$.

In other words, we define the limit of the function $f(x)$ as x approaches a to be a number L such that $f(x)$ can be made to remain as close to L as we want by choosing x sufficiently close to a (but $x \neq a$).

This definition may seem rather complex from a mathematical point of view, but it becomes easier to understand if we break it down phrase by phrase. So let's translate each part to more intuitive ideas:

Definition	**Translation**
1. For every $\varepsilon > 0$,	1. For every positive distance ε from L,
2. there exists a $\delta > 0$,	2. There is a positive distance δ from a,
3. such that	3. such that
4. if $0 < \|x-a\| < \delta$,	4. if x is closer than δ to a and $x \neq a$,
5. then $\|f(x) - L\| < \varepsilon$.	5. then $f(x)$ is closer than ε to L.

Algorithm for Proving a Statement about Limits

Step 1. Let's begin the proof with the following statement: Let $\varepsilon > 0$.

Step 2. Next, we need to obtain a value for δ. After we have obtained this value, we make the following statement, filling in the blank with our choice of δ : Choose $\delta =$ ______________________________

Step 3. The next statement in the proof should be (at this point, we fill in our given value for a): Assume $0 < |x-a| < \delta$.

Step 4. Next, based on this assumption, we need to show that $|f(x) - L| < \varepsilon$, where $f(x)$ and L are our function $f(x)$ and our limit L. At some point, we need to use $0 < |x-a| < \delta$.

Step 5. We conclude our proof with the statement: Therefore, $\lim_{x\to a} f(x) = L$.

Example 1.4.27

Complete the proof that $\lim_{x\to -1}(4x+1) = -3$ by filling in the blanks. Do your scratchwork separately, it's not part of the proof.

Let ____.

Choose $\delta =$ ____.

Assume $0 < |x -$ ____________ $| < \delta$. Or equivalently, $0 < |x +$ ____________ $| < \delta$.

Then, $|$ ____________ $-$ ____________ $| =$ ____________ $=$ ____________ $< \varepsilon$.

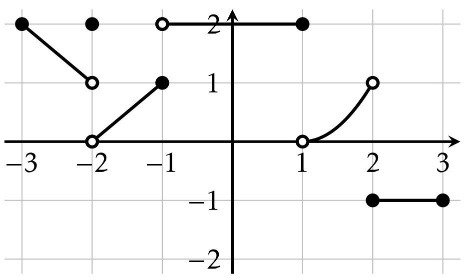
Therefore, ______________________ $=$ ______________________.

§1.5 Limit Laws

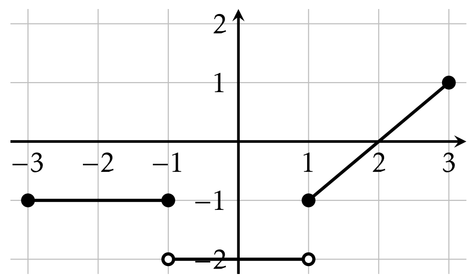
■ Question 40.

Oh no! These limits are wacky. Help me understand the solutions. All I have is the answers and not the reasons why the answers are what they are. Do this by providing the correct mathematical reasons/work explaining how one gets the correct answer.

Graph of f



Graph of g



(a) $\lim\limits_{x\to 0}(f(x)+g(x)) = 0$

(b) $\lim\limits_{x\to -2^-}\dfrac{f(x)}{g(x)} = -1$

(c) $\lim\limits_{x\to -2^+}\dfrac{f(x)}{g(x)} = 0$

(d) $\lim\limits_{x\to -2}\dfrac{f(x)}{g(x)} = DNE$

(e) $\lim\limits_{x\to -1}(f(x)+g(x)) = 0$

(f) $\lim\limits_{x\to -1}\dfrac{f(x)}{g(x)} = -1$

(g) $\lim_{x\to 2}(f(x)g(x)) = 0$

(h) $\lim_{x\to 1^+} f(g(x)) = 2$ *(and NOT 1)*

(i) $\lim_{x\to 2^-} \frac{f(x)}{g(x)} = -\infty$

(j) $\lim_{x\to 1^-} f(g(x)) = 2$

(k) $\lim_{x\to 2^+} \frac{f(x)}{g(x)} = -\infty$

(l) $\lim_{x\to 2} \frac{f(x)}{g(x)} = -\infty$

(m) $\lim_{x\to 3^-} f(g(x)) = 2$

(n) $\lim_{x\to -2^-} g(f(x)) = -1$

Note that this activity[*] is graded for participation only. You are not submitting anything. So you are responsible for your own understanding. The goal is for you to be able to explain why the answers are the way they are ***using mathematically correct language and notation****. Move on from a question only when you have fully understood what's going on there. Ask me or your teammate if you are not sure why the answer is correct. By the end of this activity, your goal is to understand how limits behaves for sum, product, ratio, and composition of functions.*

[*] From Active Calculus BOALA.

1.5.1 Fundamental Arithmetic of Limits

Theorem 1.5.28

If $\lim_{x\to c} f(x) = L$ *and* $\lim_{x\to c} g(x) = K$, *then*

(a) $\lim_{x\to c}[f(x) \pm g(x)] = L \pm K$

(b) $\lim_{x\to c}[f(x)g(x)] = LK$

(c) $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$ *if* $K \neq 0$

(d) $\lim_{x\to c} a = a$

(e) $\lim_{x\to c} x = c$

Consequences (Substitutions Rules)

- Use the product law repeatedly and multiply $f(x)$ to itself. What can you say about $\lim_{x\to c} f(x)^n$ where n is a positive integer? ______________________________

  Next apply quotient law to $\dfrac{1}{f(x)^n}$ where n is positive. What does it tell you about $\lim_{x\to c} f(x)^m$ where m is negative? ______________________________

- Replacing $f(x)$ with x, we get that $\lim_{x\to c} x^n =$ ______________ for positive integers n.

  Hence $\lim_{x\to c} ax^n =$ ______________________________.

  Then, by the addition law, $\lim_{x\to c} P(x) = P(c)$ for any polynomial function $P(x)$.

- Infact, extending the idea further, if we have a rational function of the form $f(x) = \dfrac{P(x)}{Q(x)}$, then by quotient law, $\lim_{x\to c} f(x) = f(c)$ *for any c in the domain of f.*

Note: The domain part is important as we don't want $Q(c) = 0$.

Thus we have found a collection of functions (polynomials and rational functions) where the limit can be calculated algebraically by direct substitution - these are precisely what we define to be **continuous** functions! Unfortunately, not all limits can be calculated by direct substitution (e.g. discontinuous functions).

1.5.2 Limits of Quotients

In calculus we often encounter limits of the form $\lim_{x\to c} \dfrac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are continuous. There are three types of behavior for this type of limit:

- When $g(c) \neq 0$, the limit can be evaluated by substitution (the quotient law).
- When $g(c) = 0$ but $f(c) \neq 0$, the limit is undefined.
- When $g(c) = 0$ and $f(c) = 0$, the limit may or may not exist and can take any value.

The third type is the most interesting and we will spend rest of this section learning different strategies to handle those limit.

Question 41.

Find the following limits.

(a) $\lim_{x\to 3} \dfrac{x^2-2x-3}{x-1}$ (b) $\lim_{x\to 2} \dfrac{x+1}{x-2}$

Limits of the Form $\frac{0}{0}$

Example 1.5.29: Factorize and Cancel

$$\lim_{x\to -2} \frac{x^2-4}{x+2} = \lim_{x\to -2} \frac{(x-2)(x+2)}{x+2} = \lim_{x\to -2}(x-2) = \lim_{x\to -2} x - \lim_{x\to -2} 2 = -2-2 = 4$$

Question 42.

Find the following limits.

(a) $\lim_{x\to 3} \dfrac{x^2-3x}{2x^2-5x-3}$. (b) $\lim_{x\to 0} \dfrac{(3+x)^2-9}{x}$ (c) $\lim_{x\to 1} \dfrac{x-1}{x^2-2x+1}$

Example 1.5.30: Multiply by Conjugate

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{x-4}{(x-4)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{4}$$

Question 43.

Find the following limits.

(a) $\lim_{x\to 5} \dfrac{\sqrt{x-1}-2}{x-5}$ (b) $\lim_{x\to -1} \dfrac{\sqrt{x+2}-1}{x+1}$ (c) ***[Optional]*** $\lim_{x\to 1} \dfrac{\sqrt[3]{x}-1}{x-1}$.

1.5.3 The Squeeze Theorem

The final topic of this section is the most aptly (unimaginatively?) named theorem with Limits, called the Squeeze Theorem (a.k.a. the Sandwich theorem, a.k.a the Pinching theorem, a.k.a. the Police theorem). The statement of the theorem essentially says that if a function has to "squeeze" in between two functions that are limiting to the same value then the first function has no choice other than to approach the same value.

Let's work out an example before we state the theorem formally.

Example 1.5.31

This example was moved to the Tuesday tutorial.

Consider the function $f(x) = x^2 \cos\left(\frac{\pi}{x}\right)$.

(a) Sketch a graph of the function $f(x)$ below. (You may want to use a graphing utility e.g. DESMOS)

(b) Also graph the functions $-x^2$ and x^2 along with $f(x)$.

(c) Based off your sketch, do you agree that $-x^2 \leq f(x) \leq x^2$ for all $x \neq 0$?

(d) What is $\lim_{x\to 0} -x^2$? What is $\lim_{x\to 0} x^2$? What can you conclude about $\lim_{x\to 0} x^2 \cos\left(\frac{\pi}{x}\right)$?

Theorem 1.5.32: The Squeeze Theorem

If we have

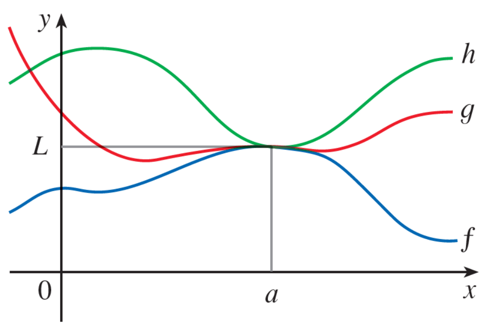
$$f(x) \leq g(x) \leq h(x)$$

for all x close to $x = a$ except possibly at $x = a$, and if

$$\lim_{x\to c} f(x) = \mathrm{L} = \lim_{x\to a} h(x),$$

then

$$\lim_{x\to a} g(x) = \mathrm{L}.$$



In time (first week of Winter Quarter), we will see a nice application of the Squeeze Theorem with an important limit of a trigonometric function (which will enable us access to ideas that will allow us to develop the ideas of calculus for trigonometric functions). Until Winter Quarter, trig functions are off the table (and we only need to focus on the idea of the theorem itself...which is pretty cool in its own right).

Question 44.

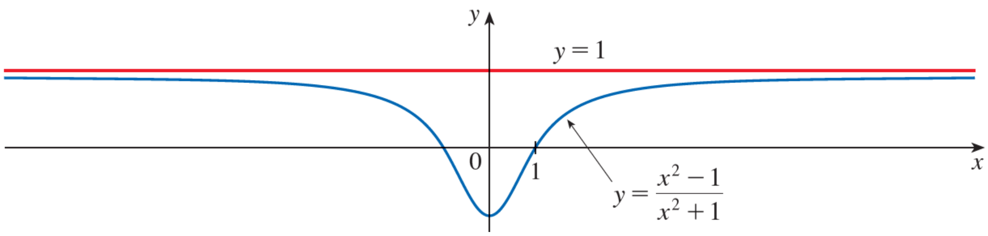
Use the Squeeze theorem to find the following limits:

(a) $\lim_{x\to 0} f(x)$ *where* $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ x^2 & \text{if } x \notin \mathbb{Q} \end{cases}$.

(b) $\lim_{x\to\infty} (3^x + 1)^{1/x}$.

§1.6 Horizontal Asymptotes

Let's begin by investigating the behavior of the function f defined by $f(x) = \dfrac{x^2-1}{x^2+1}$ as x becomes large.



You can see that as x grows larger and larger, the values of $f(x)$ get closer and closer to 1. (The graph of f approaches the horizontal line $y = 1$ as we look to the right.) In fact, it seems that we can make the values of $f(x)$ as close as we like to 1 by taking x sufficiently large. This situation is expressed symbolically by writing

$$\lim_{x\to\infty} \frac{x^2-1}{x^2+1} = 1$$

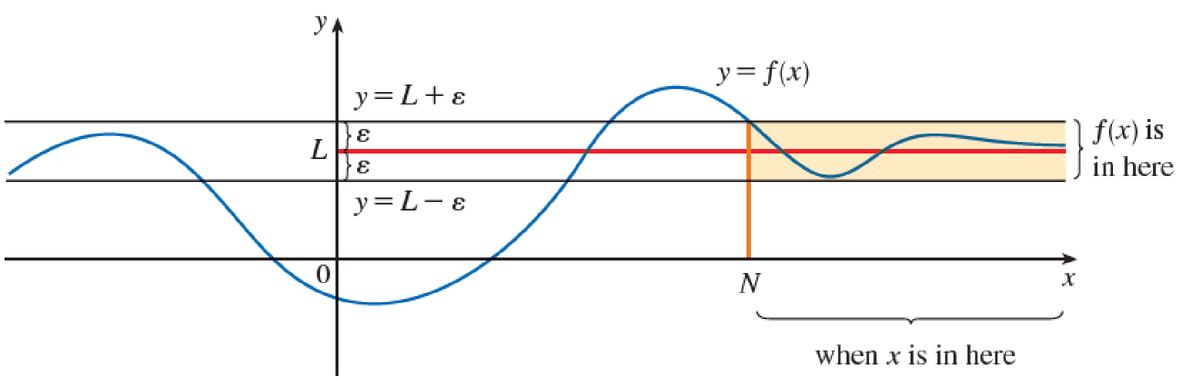
Definition 1.6.33: Horizontal Asymptotes and Limits

Let f be a function defined on the interval $[c,\infty)$ for som real number c. We say $\lim\limits_{x\to\infty} f(x) = \mathrm{L}$ if the value of $f(x)$ can be be made to stay as close as L as we want, by making x sufficiently large. More precisely,

$$\lim_{x\to\infty} f(x) = \mathrm{L}$$

means that, for every positive number ε there is a positive number N such that

$$\text{if } x > \mathrm{N} \text{ then } |f(x) - \mathrm{L}| < \varepsilon.$$



Warning: The symbol ∞ is not a number. It is used to denote the adjective "not finite". The notation $\lim\limits_{x\to\infty} f(x) = \mathrm{L}$ is read as

"the limit of $f(x)$, as x approaches infinity, is L" but what it really means is that

"the limit of $f(x)$, as x increases without bound, is L"

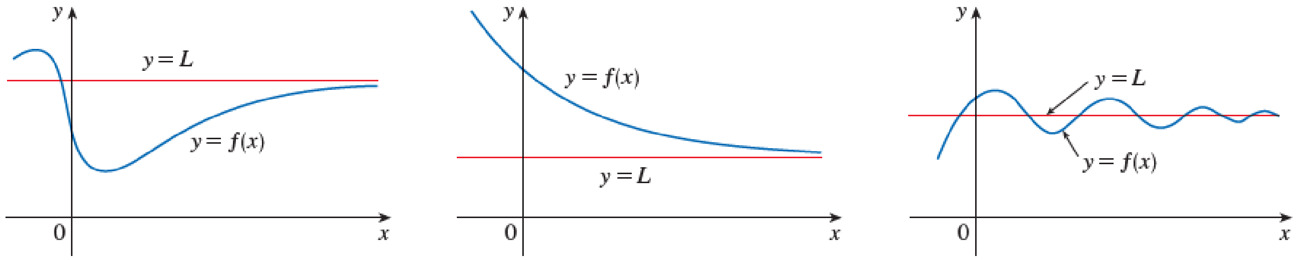


Figure 1.2: Example cases illustrating $\lim_{x\to\infty} f(x) = \mathrm{L}$

Question 45.

Can you similarly define $\lim_{x\to-\infty} f(x) = \mathrm{L}$*?*

Exploration Activity

Use DESMOS to plot the function $f(x) = \left(1+\frac{1}{x}\right)^x$. Does it have a horizontal asymptote? The limit $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ is an irrational number which does not have a close-form algebraic formula. It's called Euler's constant, and is denoted by e.

Definition 1.6.34

If either $\lim_{x\to\infty} f(x) = \mathrm{L}$ or $\lim_{x\to-\infty} f(x) = \mathrm{L}$, we say that the line $y = \mathrm{L}$ is an **horizontal asymptote** to the graph of the function $f(x)$.

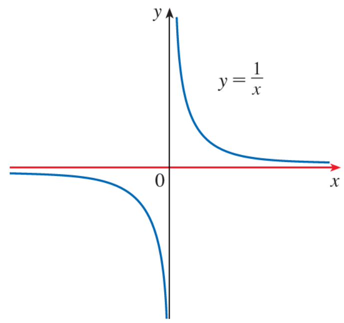
Example 1.6.35

Let's find $\lim_{x\to\infty}\frac{1}{x}$. Observe that when x is large, $\frac{1}{x}$ is small. For instance,

If $x = 100$, we have $\frac{1}{x} =$ ____________________. If $x = 100000$, we have $\frac{1}{x} =$ ____________________.

In fact, by taking x large enough, we can make $\frac{1}{x}$ as close to 0 as we want. Can you see why?

Then, by definiton $\lim_{x\to\infty}\frac{1}{x} = 0$.



Question 46.

Consider the piecewise defined function $g(x) = \begin{cases} 3-2x & x \leq 0 \\ 1+x & x > 0 \end{cases}$. *Find* $\lim\limits_{x\to\infty} g\left(\frac{1}{x}\right)$ *and* $\lim\limits_{x\to-\infty} g\left(\frac{1}{x}\right)$.

Evaluating Limits at Infinity

Before we begin, let's use the limit laws from last section on the example from last page. Suppose $r > 0$ is a positive real number. Then

$\lim\limits_{x\to\infty} \frac{1}{x^r} =$ ______________________________

Question 47.

What happens to the limit is $r < 0$? What if $r = 0$?

Example 1.6.36: $\frac{\infty}{\infty}$ form

Consider the limit $\lim\limits_{x\to\infty} \frac{3+4x^2}{x^2+3x+2}$. We are going to algebraically evalaute it as follows.

$$\lim_{x\to\infty} \frac{3+4x^2}{x^2+3x+2} = \lim_{x\to\infty} \frac{\frac{3+4x^2}{x^2}}{\frac{x^2+3x+2}{x^2}} = \lim_{x\to\infty} \frac{\frac{3}{x^2}+4}{1+\frac{3}{x}+\frac{2}{x^2}} = \frac{\lim\limits_{x\to\infty}\left(\frac{3}{x^2}+4\right)}{\lim\limits_{x\to\infty}\left(1+\frac{3}{x}+\frac{2}{x^2}\right)} = \frac{0+4}{1+0+0} = 4$$

Question 48.

Find the following limits.

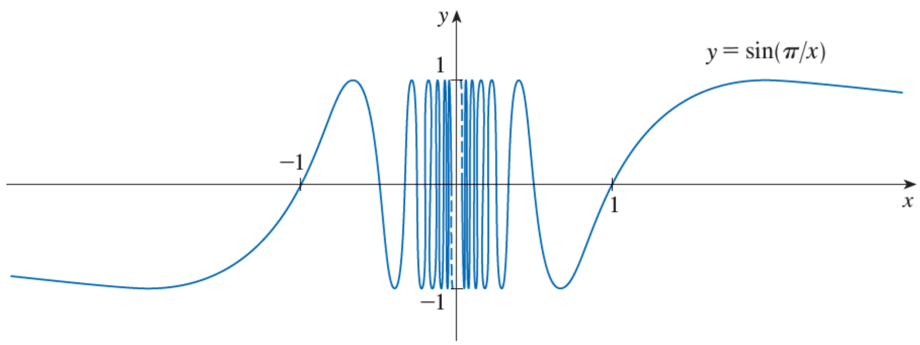
(a) $\lim\limits_{x\to\infty} \frac{3x^2-x-2}{5x^2+4x+1}$

(b) $\lim\limits_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5}$

§1.7 Vertical Asymptotes and Infinite Limits

How Can a Limit Fail to Exist? Two ways we have seen so far are:

- when the left- and right-hand limits are not equal.
- when $f(x)$ is so chaotic that even the one-handed limits do not exist. Here's a picture to illustrate the case of $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$.



- A third possible way the limit of $f(x)$ at a number a can fail to exist is when $f(x)$ grows arbitrarily large (in absolute value) as x approaches a. This is our next topic of discussion.

Definition 1.7.37

Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large postive value by taking x sufficiently close to a, but not equal to a.

Question 49.

Can you give a precise definition of the above idea?

□

Warning: While ∞ is not anumber, there is a distinction between writing $+\infty$ and $-\infty$ for a limit.

Writing $\lim_{x\to a} f(x) = \infty$ means $f(x)$ **increases** without bound as x approaches a from **both sides.**

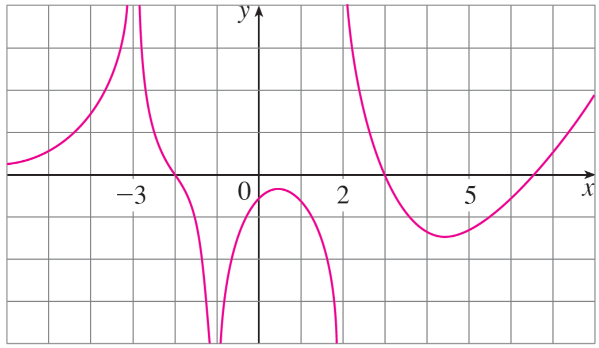
Similarly, $\lim_{x\to a} f(x) = -\infty$ means $f(x)$ **decreases** without bound as x approaches a.

If $\lim_{x\to a^+} f(x) = \infty$ and $\lim_{x\to a^-} f(x) = -\infty$, then we will say $\lim_{x\to a} f(x)$ does not exist and avoid using the notation of infinity.

Question 50.

For the following graph, evaluate all the limits below.

□



$\lim_{x\to -3^-} f(x) =$ ____________ $\lim_{x\to -1^-} f(x) =$ ____________ $\lim_{x\to 2^-} f(x) =$ ____________

$\lim_{x\to -3^+} f(x) =$ ____________ $\lim_{x\to -1^+} f(x) =$ ____________ $\lim_{x\to 2^+} f(x) =$ ____________

$\lim_{x\to -3} f(x) =$ ____________ $\lim_{x\to -1} f(x) =$ ____________ $\lim_{x\to 2} f(x) =$ ____________

Definition 1.7.38

If one or both of the one-sided limits of a function $f(x)$ at a point a approach infinity or negative infinity, we say that the graph of $f(x)$ has a **vertical asymptote** at $x = a$.

Question 51.

Each of the following limit is infinite. Can you see why? Intuitively, it's because as approaches the corresponding, the denominators become smaller and smaller, closer to zero; but the numerator approaches a nonzero number. So the ratio becomes larger and larger (either in the positive or negative direction) and goes towards infinity. Try to figure out using a number line whether the function is positive or negative on both sides of the asymptote. Use that to intuitively evaluate the limit as $+\infty$ or $-\infty$.

(a) $\lim_{x\to 5^-} \frac{2x}{x-5}$.

(b) $\lim_{x\to 3^+} \frac{\sqrt{x}}{(x-3)^5}$.

(c) $\lim_{x\to -2^-} \frac{x^2+2x}{x^2+4x+4}$.

Question 52.

(***Sketch a Graph II***) *Sketch a graph for a function $f(x)$ that satisfies the following:*

- $\lim_{x\to(-2)^-} f(x) = +\infty$
- $\lim_{x\to(-2)^+} f(x) = -\infty$
- $f(0) = 2$ *and* $\lim_{x\to 0} f(x) = 1$
- $\lim_{x\to 3} f(x)$ *does not exist but* $\lim_{x\to 3^+} f(x) = -4$

§1.8 Continuity

We noticed in the previous sections that the limit of a function as x approaches a can often be found simply by calculating the value of the function at a. Functions having this property are called **continuous** at a. We will see that the mathematical definition of continuity corresponds closely with the meaning of the word continuity in everyday language.

Definition 1.8.39: Continuity at a Point

The function $f(x)$ is said to be **continuous** at a point $x = a$ if

- f is defined at $x = a$, and
- if $\lim_{x \to a} f(x)$ exists, and
- if $\lim_{x \to a} f(x) = f(a)$

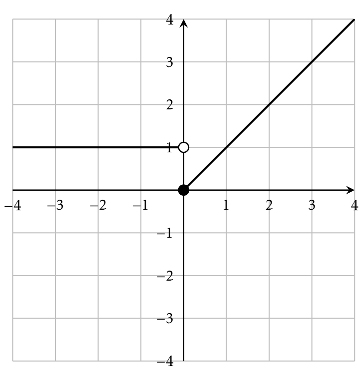
In other words, $f(x)$ can be made to remain as close as we want to $f(c)$ provided x is chosen close enough to c.

Warning: An important difference between limits and continuity: a limit is only concerned with what happens **near** a point, but continuity depends on what happens **near** a point and **at** that point.

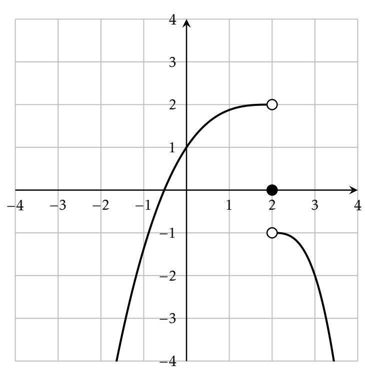
Question 53.

In the following pictures, if you believe the function is discontinuous at a point, discuss ***why*** *you think it's discontinuous at that point. Which of the three parts in the definition does it fail to satisfy (if discontinuous)?*

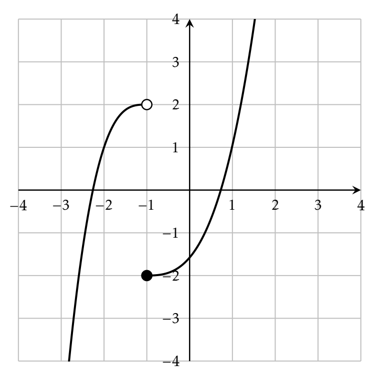
(a)



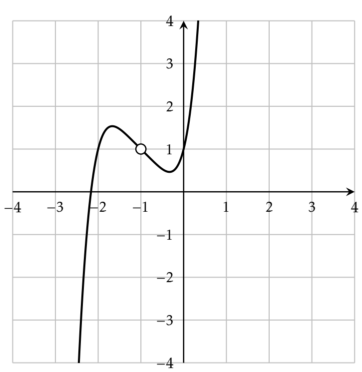
(b)



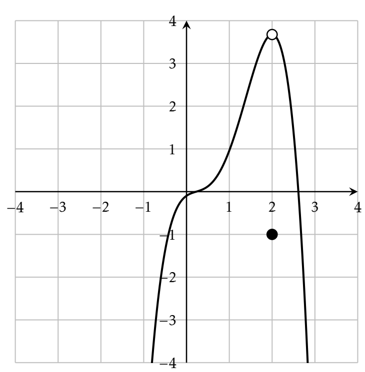
(c)



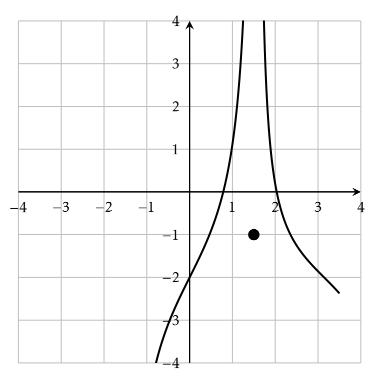
(d)

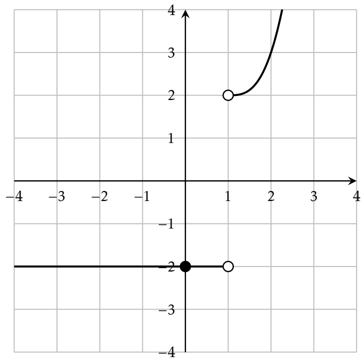
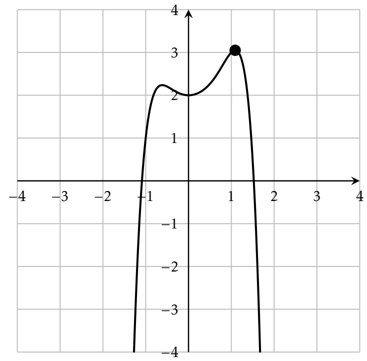


(e)



(f)



(g)  (h) 

Definition 1.8.40: Continuity on an Interval

A function f is said to be **continuous on an open interval** (a,b) if it is continuous at every point in the interval.

A function f is said to be **continuous from the right at** c if $\lim\limits_{x\to c^+} f(x) = f(c)$.

A function f is said to be **continuous from the left at** c if $\lim\limits_{x\to c^-} f(x) = f(c)$.

A function f is said to be continuous on a closed interval $[a,b]$ if it is continuous on (a,b), and continuous from the right at a and continuous from the left at b.

Most of the functions you have studied prior to Calculus are continuous functions. Or at least, functions that are continuous on their domains. Here is a summary:

- Polynomials are continuous for all real numbers.
- Even fractional powers like $y = \sqrt{x}$ or $y = x^{1/4}$ are continuous on their domains $[0,\infty)$.
- Rational functions are continuous on their domain, e.g. $y = \frac{x^2+1}{x-3}$ is continuous on the intervals $(-\infty,3)$ and $(3,\infty)$.
- Exponential and logarithm functions are continuous on their domains.

Theorem 1.8.41: Continuity of Sums, Products, and Quotients of Functions

Suppose that f and g are continuous on an interval and that k is a constant. Then, on that same interval,

(a) $kf(x)$ *is continuous.*

(b) $f(x)+g(x)$ *is continuous.*

(c) $f(x)g(x)$ *is continuous.*

(d) $f(x)/g(x)$ *is continuous, provided* $g(x) \neq 0$ *on the interval.*

Theorem 1.8.42: Composite Limit Theorem

If $f(x)$ is continuous at L, *and* $\lim\limits_{x\to a} g(x) = \mathrm{L}$, *then*

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(\mathrm{L}).$$

As a consequence, if g is continuous at a, and f is continuous at $g(a)$, then the composite $f \circ g$ is continuous at a.

1.8.1 Using Continuity to evaluate a Limit

The last condition of continuity, that $\lim_{x\to a} f(x) = f(a)$, tells us how to compute limits for continuous functions.

Example 1.8.43

(a) The function $p(x) = 2x^3 - 6x^2 + 5$ is a polynomial, which is continuous for any real number. Hence, if we wanted to know, say, $\lim_{x\to 2} p(x)$, we simply have to **evaluate $p(x)$ at $x = 2$**:

$\lim_{x\to 2}\left(2x^3 - 6x^2 + 5\right) =$ ____________________________________

(b) Even a somewhat scary-looking limit is easy to evaluate if the function is continuous at the point in question.

$\lim_{x\to 25}\left(x^{\sqrt{x}+1} - (x-20)^2\right) =$ ____________________________________

(c) We can use what we know about continuity to help us determine if a piecewise function is continuous without even needing to graph it (although it's still fun to graph just about anything). For

$$f(x) = \begin{cases} 5x+2, & \text{if } 0 \le x < 2 \\ 6, & \text{if } x = 2 \\ x^2 + x + 6, & \text{if } x > 2 \end{cases}$$

we can evaluate left and right-hand limits at $x = 2$:

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 5x + 2 =$ ____________________________________

and

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2 + x + 6 =$ ____________________________________ .

From what we have learned previously, we know that $\lim_{x\to 2} f(x) = 12$ because the left and right hand limits are the same. However, $f(2) = 6 \neq 12$, so $f(x)$ is **not continuous** at $x = 2$.

In what way could we adjust the function f so that it is continuous at $x = 2$? ____________

■ **Question 54.** □

Determine a value of c that will make the function g continuous for all real numbers.

$$g(x) = \begin{cases} cx & \text{if } 0 \le x < 2 \\ 3x^2 & \text{if } 2 \le x \end{cases}$$

1.8.2 Intermediate Value Theorem

Let's work through a thought exercise.

- What was your height at birth? ______________________________

  The average baby born measures about 20 inches at birth.

- What is your height today? ______________________________

- Sketch a graph of how your height has changed over the course of your lifetime.

Question 55. □

Was there a day in your life when you measured exactly 40 inches?

Question 56. □

(a) *A continuous function $y = f(x)$ is known to be negative at $x = 0$ and positive at $x = 1$. Explain why the equation $f(x) = 0$ must have at least one solution between $x = 0$ and $x = 1$? Illustrate with a sketch.*

(b) *How would your observation generalize if instead of $x = 0$ and $x = 1$, you had $x = a$ and $x = b$? Also what if we didn't require the function to be positive and negative at the two end points? Can we still conclude the existence of a root? If not a root, what can we conclude?*

Theorem 1.8.44

Suppose f is continuous on a closed interval $[a,b]$. If k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a,b]$ such that $f(c) = k$.

Question 57. □

(a) *Draw a picture to explain the Intermediate Value Theorem (IVT) in your own words.*

(b) *Why do we need continuity for the Intermediate Value Theorem?*

(c) *Prove that the function $f(x) = x^3 + x + 1$ has at least one root between -1 and 0.*

Chapter 2 | The Derivative

§2.1 Introduction to Derivatives

2.1.1 The Velocity Problem

Suppose you drop a ball from the top of a tower: how long does it take it to reach the ground? How fast is the ball moving at any given point in time?

Questions like the above are known as "the velocity problem", and are central to the study of differential calculus. Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. (This model for free fall neglects air resistance.) If the height of the tower is H (in meters), and the current height of the ball after t seconds is denoted by $s(t)$ (measured in meters), then Galileo's observation is expressed by the equation

$$s(t) = \mathrm{H} - 4.9t^2$$

The graph of this quadratic function is clearly a parabola. You can see a general figure below in 2.1.

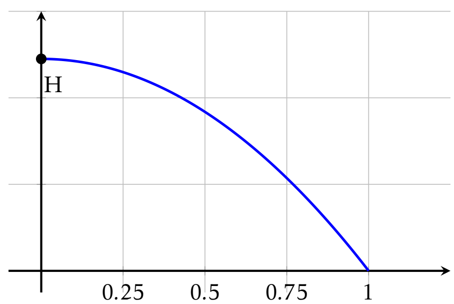


Figure 2.1: Position model of a ball falling from a height H

Example 2.1.45

In the above scenario, suppose we wish to find the velocity of the ball after 0.5 seconds. You may have seen before that velocity as displacement over time. However, the difficulty in finding the velocity at 0.5 seconds is that we are dealing with a single instant of time $t = 0.5$, so no time interval is involved.

However, we can approximate the desired quantity by computing the average velocity over the brief time interval of a hundredth of a second from $t = 0.5$ to $t = 0.51$.

$$\begin{aligned}
\text{Average Velocity} = v_{avg} &= \frac{\text{change in position}}{\text{time elapsed}} \\
&= \frac{s(0.51) - s(0.5)}{0.01} \\
&= \frac{[\mathrm{H} - 4.9(0.51)^2] - [\mathrm{H} - 4.9(0.5)^2]}{0.01} \\
&\approx -4.949
\end{aligned}$$

The negative denotes the fact that the ball is moving downward.

Now, if we were to reduce the time interval even further, we would probably get a better approximation. Go to

https://www.desmos.com/calculator/wbxi71j3av

and try it yourself.

In general,

Definition 2.1.46

For an object moving in a straight line with position function $s(t)$, the **average velocity** of the object on the interval from $t = a$ to $t = b$, denoted by v_{avg}, is given by

$$v_{avg} = \frac{s(b) - s(a)}{b - a} \quad \text{on interval } [a, b]$$

Question 58.

What is v_{avg} on the time interval $[0.5, 0.501]$? What about $[0.5, 0.50001]$? Can you guess what value v_{avg} is approaching?

Let's introduce some terminology to formally describe exact situations like this.

Definition 2.1.47

For an object moving in a straight line with position function $s(t)$, the **instantaneous velocity** of the object at $t = a$, denoted by $v_{instant}$, is given by

$$v_{instant} = \lim_{b \to a} \frac{s(b) - s(a)}{b - a}$$

Example 2.1.48

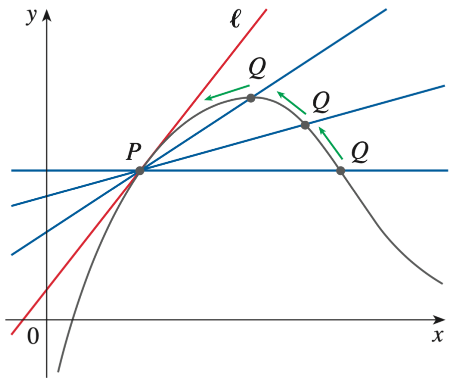
Going back to the situation with the ball, at $t = 0.5$,

$$v_{instant} = \lim_{b \to 0.5} \frac{[\mathrm{H} - 4.9b^2] - [\mathrm{H} - 4.9(0.5)^2]}{b - 0.5} = -4.9 \lim_{b \to 0.5} \frac{b^2 - (0.5)^2}{b - 0.5} = -4.9 \lim_{b \to 0.5} (b + 0.5) = -4.9$$

Note: It may not be always easy to calculate the limit for a different function $s(t)$. But that's a question for another chapter, more specifically, when we study differentiation formulas.

2.1.2 The Tangent Problem

Another way to interpret the average velocity is to observe that the difference quotient in the example above is also the slope of the secant joining the point $\mathrm{P} = (a, s(a))$ and the point $\mathrm{Q} = (b, s(b))$. Then the idea of shortening the interval gradually is the same as moving Q closer to P. However, we observe from the picture below that as Q moves closer to P, the secant lines are approaching towards the tangent line at P.



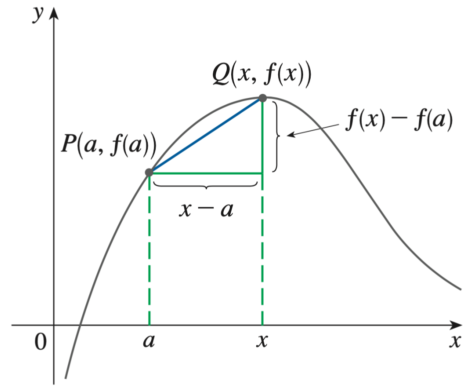


Figure 2.2

In other words, the instantaneous velocity must correspond to the slope of the tangent!

Replacing $s(t)$ with a general function $f(x)$, we arrive at the following definition:

Definition 2.1.49

The tangent line to the curve $y = f(x)$ at the point $\mathrm{P}(a, f(a))$ is the line through P with slope

$$m = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

provided that this limit exists.

■ Question 59.

Find the equation of the tangent line to the curve $y = \dfrac{3}{x}$ *at the point* $\mathrm{P}(3, 1)$. □

2.1.3 Derivative and Rate of Change

So far, we have the same type of limit in both the velocity problem and the tangent problem. It turns out that a similar limit arises whenever we calculate a *rate of change* in any of the STEM fields, such as a rate of reaction in chemistry or a marginal cost in economics.

Suppose y is a quantity that depends on another quantity x, i.e. $y = f(x)$ is a function of x. If x changes from a to b, then the change in x is $\Delta x = b - a$ and the corresponding change in y is $\Delta y = f(b) - f(a)$. The difference quotient we have seen multiple times so far, $\frac{\Delta y}{\Delta x}$ is called the **average rate of change** of y with respect to x over the interval $[a, b]$ and the limit of the quotient (as $b \to a$) is defined as the **instantaneous rate of change** of y with respect to x at $x = a$.

Definition 2.1.50

The **derivative** of a function $f(x)$ at a point $x = a$, denoted $f'(a)$, is

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

if this limit exists.

In other words, the derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

Question 60.

A manufacturer produces bolts of a fabric with a fixed width. The cost of producing x yards of this fabric is $C = f(x)$ dollars.

(a) *What is the meaning of the derivative $f'(x)$? What are its units?*

(b) *In practical terms, what does it mean to say that $f'(1000) = 9$?*

Now if we let a vary over all possible real numbers, we could ask the same question at every possible point: what is the derivative of f at $x = a$? Changing our point of view and replacing a by a variable x, we obtain

$$f'(x) = \lim_{b \to x} \frac{f(b) - f(x)}{b - x}$$

which we can think of as the **derivative function**.

Alternate Formulation: If we substitute b with $x + h$, the condition $b \to x$ can be replaced by $h \to 0$. Then the definition can be rewritten as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example 2.1.51

Open the DESMOS link below to find the graph of a function $f(x)$ is given below. Can we use it to give a rough sketch of the graph of the derivative f'?

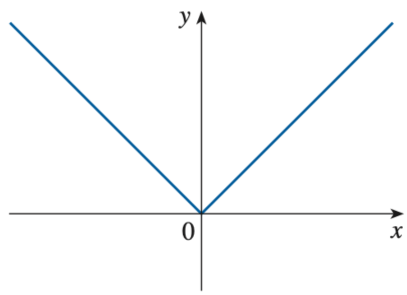
https://www.desmos.com/calculator/mwatkatvza

§2.2 The Derivative Function and Differentiability

The process of finding the derivative of a function is called **differentiation**. A function $f(x)$ is said to be **differentiable** at a if its derivative exists at $x = a$. By now, we have seen lots of examples when a limit doesn't exist or is undefined. If the limit in the definition of the derivative doesn't exist at $x = a$, we say that the derivative $f'(a)$ doesn't exist. So in that case, we say that the function $f(x)$ is not differentiable at $x = a$. A function is said to be differentiable on an open interval (a, b) if its derivative exists at every x in the interval.

Example 2.2.52

Consider the function $f(x) = |x|$ graphed below.



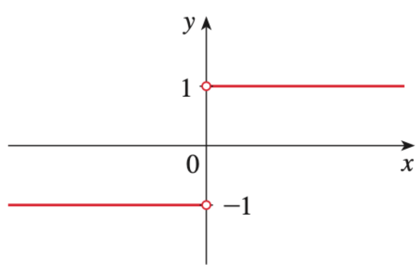
Let's try to evaluate the derivative at $x = 0$ using the limit definition from above. The function is defined and continuous at $x = 0$. In fact $f(0) = |0| = 0$. Now what can we say about the quantity

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{|0+h|-|0|}{h} = \underline{\hspace{8cm}}$$

Similarly,

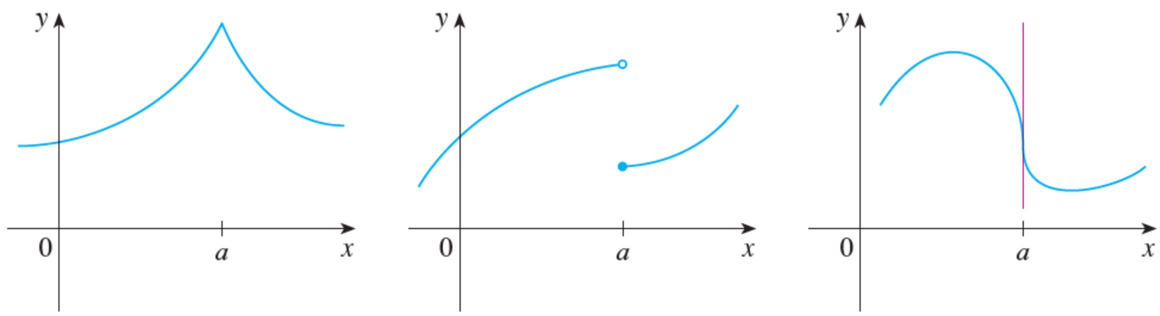
$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{|0+h|-|0|}{h} = \underline{\hspace{8cm}}$$

Another way to confirm these two values is to calculate the slopes of tangents to the graph from the left and right as x approaches 0.



Either way, we find that the left hand limit and the right hand limit does not match! Hence the overall limit doesn't exist. We conclude that $f(x) = |x|$ is NOT differentiable at $x = 0$.

In general, if the graph of a function f has a "corner" or "cusp" in it, then the graph of f has no tangent at this point and f is not differentiable there. In trying to compute $f'(x)$, we find that the left and right limits are different. This is the case in the first picture below. Here are two other scenarios.



(a) A corner (b) A discontinuity (c) A vertical tangent

Figure 2.3: Three ways for f not to be differentiable at a

Warning: If P is the point $(a, f(a))$ on the graph of $f(x)$, sometime we will say differentiable at P (or the derivative at P) interchangeably with differentiability at a (or derivative at a). Both $f'(\mathrm{P})$ and $f'(a)$ will mean the **instantaneous rate of change** of the function $f(x)$ at a, and this is the same as the slope of the tangent line to the curve $y = f(x)$ at the point P.

Question 61.

The function $y = p(x)$ is pictured in Figure 2.4 below. Each portion of the graph is a straight line.

(a) *For what values of a, does $\lim_{x\to a} p(x)$ not exist?*

(b) *For which x-values is $f(x)$ **not** continuous?*

(c) *For which x-values is $f(x)$ **not** differentiable?*

(d) *Sketch an accurate graph of $p'(x)$ on the same picture.*

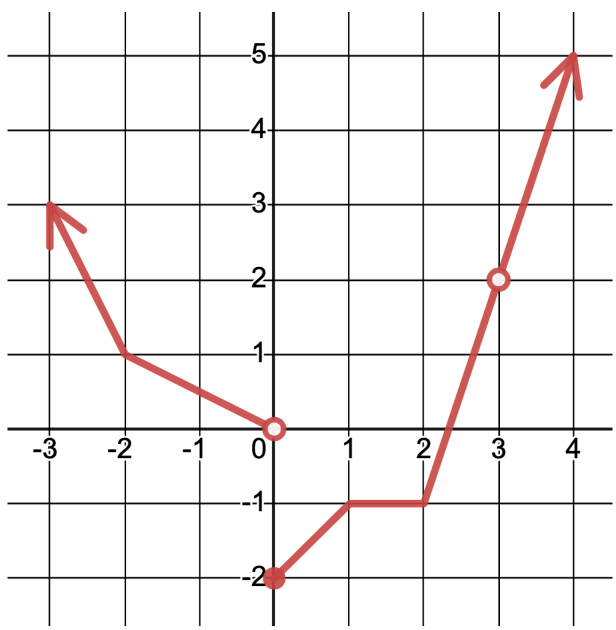
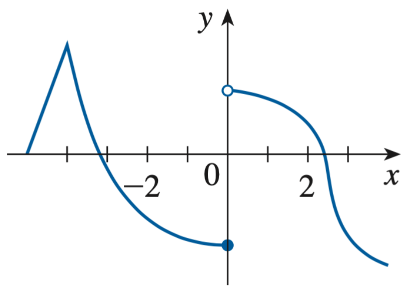


Figure 2.4: Piecewise linear function, $y = p(x)$

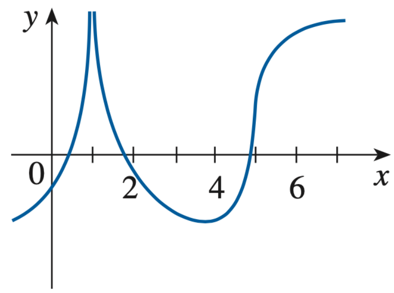
> **Note:** From example 52 and fig. 2.3, we also observe the following relation between continuity and differentiability. A continuous function may not be always differentiable. A discontinuous function is definitely not differentiable. However, a differentiable function has to be continuous. We will not prove these results in this course.

Question 62. □

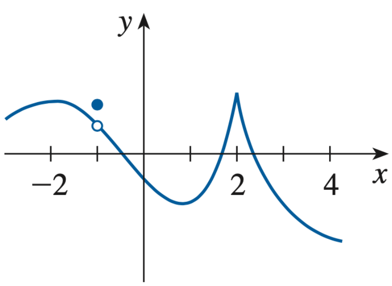
For each of the functions $f(x)$ given below, determine, with reasons, the numbers at which $f(x)$ is not differentiable.



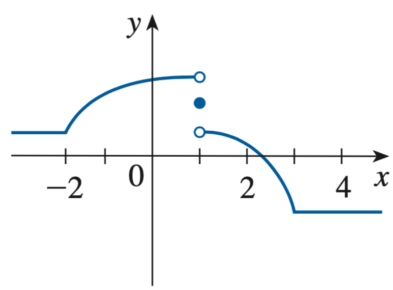
(a)



(c)



(b)



(d)

2.2.1 Some Key Notations

Before we introduce our derivative formulas in the next section, here is a quick word on derivative notation. We have been using f' to denote the derivative of f, but there are other ways to do this that you will see in your text or elsewhere online.

If we write $y = f(x)$, then we can write the expressions

$$\frac{\mathrm{d}y}{\mathrm{d}x} \quad \text{or} \quad \frac{\mathrm{d}f}{\mathrm{d}x}$$

instead of f'. We read these expressions as "dee-y dee-x" or "dee-f dee-x." This notation is referred to as **Leibnitz notation**.

The notation $\frac{\mathrm{d}y}{\mathrm{d}x}$ comes from the notation $\frac{\Delta y}{\Delta x}$ that is often used to denote the slope of a line. Although we read $\frac{\Delta y}{\Delta x}$ as "change in y over change in x," we view $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a single symbol, not as a quotient of two quantities.

Since we can use $\frac{\mathrm{d}y}{\mathrm{d}x}$ in place of f', to write out an expression like $f'(2)$ with Leibnitz notation, we would write something like $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=2}$, which says "evaluate $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = 2$."

Lastly, we use a variation of the Leibnitz notation as a command for taking the derivative. That is, the following

$$\frac{\mathrm{d}}{\mathrm{d}x}[\square]$$

says to "take the derivative of $\square$ with respect to x." For example, $\frac{\mathrm{d}}{\mathrm{d}x}[7x+1] = 7$.

§2.3 Essential Derivative Formulas

2.3.1 Constant, Multiple and Sum of Functions

Let's start with the simplest of all functions, the constant function $f(x) = c$. Its graph is a horizontal line with slope zero at every point. Thus, its derivative should be zero everywhere. We summarize this with the following rule.

Theorem 2.3.53

For any real number c, if $f(x) = c$, then $\dfrac{\mathrm{d}}{\mathrm{d}x}[f(x)] = 0$.

Question 63.

Check this using the limit definition of derivative.

Next, we have two rules regarding sum and constant multiples.

Theorem 2.3.54

- *If c is a constant and $f(x)$ is differentiable, then* $\dfrac{\mathrm{d}}{\mathrm{d}x}[cf(x)] = cf'(x)$.
- *If $f(x)$ and $g(x)$ are differentiable, then* $\dfrac{\mathrm{d}}{\mathrm{d}x}[f(x) \pm g(x)] = f'(x) \pm g'(x)$.

A visual justification for the first fact can be observed from the picture below.

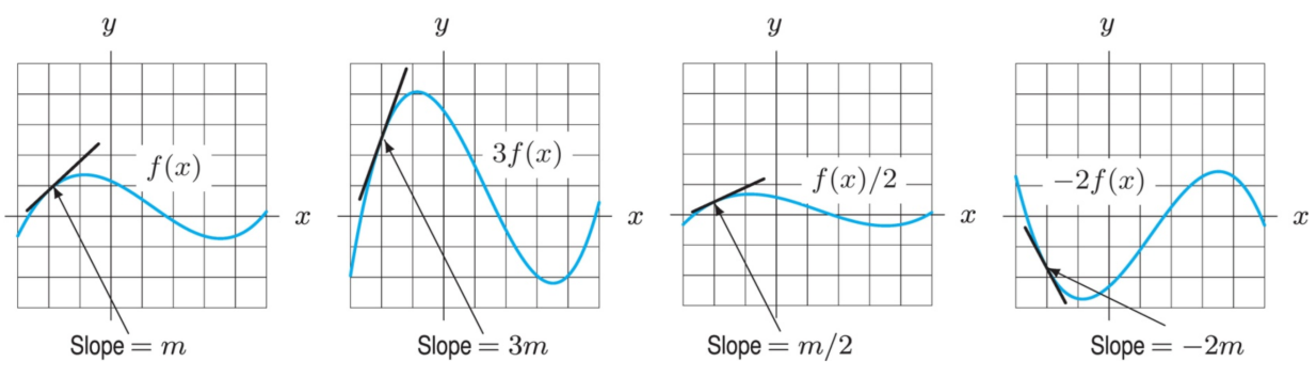


Figure 2.5: Derivative of multiple is multiple of derivative

These facts can be also checked using the limit definition.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[f(x)+g(x)] &= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= f'(x)+g'(x)
\end{aligned}$$

■ Question 64.

Using Table 2.1, compute the following:

(a) *Find $h'(1)$ if $h(x) = 5 - f(x)$.*

(b) *Find $k'(-2)$ if $k(x) = -\frac{1}{2}g(x)$.*

(c) *Find $p'(-2)$ if $p(x) = 2f(x) + 3g(x)$.*

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-2	-6	9	-10	16
1	5	-3	3	-2

Table 2.1

2.3.2 Powers and Polynomials

We are going to try to use the limit definition of derivatives to determine the derivative of $f(x) = x^n$, where n is some positive integer. Recall that,

$$\text{If } f(x) = x^n, \text{ then } f'(x) = \lim_{y\to x}\frac{f(y)-f(x)}{y-x} = \lim_{y\to x}\frac{y^n - x^n}{y-x}.$$

We will leave it as an exercise to show that the following factorization is true.

■ Question 65.

Show that

$$y^n - x^n = (y-x)\left(y^{n-1} + y^{n-2}x + y^{n-3}x^2 + \ldots + y^2x^{n-3} + yx^{n-2} + x^{n-1}\right) \quad \textit{for any positive integern.}$$

Then $f'(x) = \lim_{y\to x}\dfrac{y^n - x^n}{y-x}$

$= \lim_{y\to x}\left(y^{n-1} + y^{n-2}x + y^{n-3}x^2 + \ldots + y^2x^{n-3} + yx^{n-2} + x^{n-1}\right)$

$=$ ____

This is a total of n terms, each of which simplifies to ____________________.

Hence $f'(x) =$ ____________________________________

We conclude that

Theorem 2.3.55: Power Rule

If n is any real number then

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^n] = nx^{n-1}$$

Once we know the derivative of any power of x, we can use the constant multiple and the sum/difference rules to find derivatives of any polynomials.

■ Question 66.

☐

Let $f(x) = x^5 - 2x^4 + 3x + 12$. Use the power rule and the basic derivative formulas to compute $f'(x)$.

■ Question 67.

☐

Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

■ Question 68.

☐

(a) *Find the slope of the tangent line to the graph of $f(z) = \sqrt{z} + \dfrac{1}{z}$ at the point where $z = 4$.*

(b) *Find the equation of the tangent line to the graph of $f(z) = \sqrt{z} + \dfrac{1}{z}$ at the point where $z = 4$.*

■ Question 69. **Homework**

The graph of $y = x^3 - 9x^2 - 16x + 1$ *has a slope of* 5 *at two points. Find the coordinates of the points.*

■ Question 70. **Homework**

Using the Sum and Constant Multiple rules and the rules for Power, Constant, and Exponential functions, compute the derivative for each function with respect to the given independent variable. Make sure to write your answer using proper derivative notation.

(a) $f(x) = x^{5/3} - x^4 + 2^x$

(b) $h(z) = \sqrt{z} + \dfrac{1}{z^4}$

(c) $s(y) = (y^2 + 1)(y^2 - 1)$ *(distribute first)*

(d) $p(a) = 3a^4 - 2a^3 + 7a^2 - a + 12$

(e) $q(x) = \dfrac{x^3 - x + 2}{x}$ *(break it into three fractions first)*

§2.4 Product and Quotient Rule

By definition, the derivative is a limit of a quotient, which can be annoyingly cumbersome to work with. Last time, we observed trends that enabled us to compute the derivative of any polynomial function rather swiftly. Today, we will observe more trends in the derivative so that we can compute the derivative of any rational function with relative efficiency as well.

2.4.1 Product Rule

Question 71. ☐

Let $f(x) = x^5$ and $g(x) = x^2 - 1$.

(a) $f'(x) =$ ______________ *and* $g'(x) =$ ______________.

(b) $f'(x)g'(x) =$ ______________.

(c) *Simplify the product* $h(x) = f(x)g(x) =$ ______________. *Then* $h'(x) =$ ______________.

(d) *Is it the case that* $f'(x)g'(x) = h'(x)$? *Yes/No?* ______________

Here is the actual product rule formula:

Theorem 2.4.56: Product Rule

If $f(x)$ and $g(x)$ are two differentiable functions, then

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

In words: *The derivative of a product is the derivative of the first times the second plus the first times the derivative of the second.For $f(x)$ and $g(x)$ differentiable functions,*

Intuitively, it makes some sense that the rate of change of a product needs to take into account how fast f and g are changing, but also needs to account for how large f and g are at the particular value of x.

Derivation of Product Rule

Why does the product rule look this way? If we start trying to write out $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)]$ using the definition, we get the following limit:

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

To help us see how we can go from this limit to the product rule, we will use some geometry to help us conceptualize.

Question 72. ☐

Use Figure 2.6 to aid in this problem. Write your answers in terms of $f(x), g(x), f(x+h)$, and $g(x+h)$.

(a) *What is the area of the rectangle denoted by* A? ______________

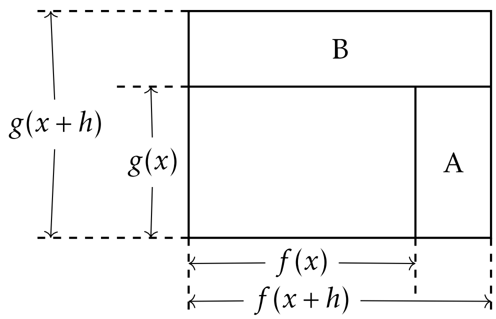


Figure 2.6: Rectangle Aid for Product Rule Formula

(b) *What is the area of the rectangle denoted by* B?

____

(c) *What area in the figure does $f(x+h)g(x+h)-f(x)g(x)$ correspond to? Write this area in terms of* A *and* B.

Using the formulas from the previous problem, we can now see where the product rule comes from. The numerator $f(x+h)g(x+h)-f(x)g(x)$ can be replaced in our expression for $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)]$ as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\ &= \lim_{h\to 0}\frac{(f(x+h)-f(x))g(x)+(g(x+h)-g(x))f(x+h)}{h}\end{aligned}$$

We can now break this fraction into pieces and use the fact that $f(x)$ and $g(x)$ are both differentiable to get our product formula. But note, we also use the fact that $\lim_{h\to 0} f(x+h)=f(x)$, which is true because f is necessarily continuous since it is differentiable.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] &= \lim_{h\to 0}\frac{(f(x+h)-f(x))\,g(x)+(g(x+h)-g(x))\,f(x+h)}{h}\\ &= \lim_{h\to 0}\frac{(f(x+h)-f(x))\,g(x)}{h}+\lim_{h\to 0}\frac{(g(x+h)-g(x))\,f(x+h)}{h}\\ &= \left[\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right]\left[\lim_{h\to 0}g(x)\right]+\left[\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right]\left[\lim_{h\to 0}f(x+h)\right]\\ &= f'(x)g(x)+g'(x)f(x)\end{aligned}$$

Example 2.4.57

Suppose we wan to use the product rule to differentiate function $h(x)=\sqrt{x}(a+bx)$.

This is a product of the functions $f(x)=\sqrt{x}$ and $g(x)=a+bx$. Hence, the derivative $h'(x)$ will have the derivative of the first function times the second function, plus the first function times the derivative of the second:

$f'(x) =$ ______________________________ $g'(x) =$ ______________________________

$h'(x) = f'(x)g(x) + f(x)g'(x) =$ ____

Note: There is some algebraic simplification that we **could do** to this derivative, but it generally is not really worth it to bother simplifying the derivative that results from the product rule unless the problem asks you to do so. The only times it can be advantageous to simplify a derivative would be if you need to set the derivative equal to something and solve for x, or if you want to make taking another derivative easier. Usually, we will want to know the derivative at a point, and in that case we can just evaluate the derivative there using substitution first, and then simplifying.

Question 73.

☐

Use table 2.2 to compute the given derivative value.

(a) *Find* $h'(1)$ *if* $h(x) = f(x)g(x)$.

(b) *Find* $k'(-2)$ *if* $k(x) = xf(x) - 2g(x)$.

(c) *Find* $p'(-2)$ *if* $p(x) = \dfrac{f(x)g(x)}{2} + x^2 f(x)$.

(d) *Find* $q'(1)$ *if* $q(x) = f(x)^2$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-2	-6	9	-10	16
1	5	-3	3	-2

Table 2.2

2.4.2 Quotient Rule

As with the product rule, the derivative formula for the derivative of a quotient of two differentiable functions is anything but what we would expect.

Question 74. **Homework**

Come up with an example of two functions $f(x)$ and $g(x)$ such that $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] \neq \dfrac{f'(x)}{g'(x)}$.

Here is the actual quotient rule formula:

Theorem 2.4.58: Quotient Rule

For $f(x)$ and $g(x)$ differentiable functions,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Example 2.4.59

Consider a simple rational function like $R(x) = \dfrac{2x+1}{3x^2+x+1}$. Since $R(x)$ is naturally defined as a quotient, we must use the quotient rule to determine its derivative. Sometimes, it can be helpful to identify the **top** and the **bottom** of the quotient, write out their derivatives separately, and then combine everything into the quotient rule formula.

Taking our own advice, let's identify our rational function as $R(x) = \dfrac{f(x)}{g(x)}$, so that $f(x) = 2x+1$ and $g(x) = 3x^2+x+1$. Then $f'(x) =$ ____________________ and $g'(x) =$ ____________________.

Substituting into the quotient rule formula, we have:

$R'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} =$ ____

For the purposes of computation, this is a **perfectly acceptable answer**! If we needed to take another derivative or determine something else specific related to $R(x)$'s rate of change, we might need to do some algebra to simplify $R'(x)$. However, for determining, say, the slope of the tangent line to $R(x)$ at the point $(1, R(1))$, we can just use the formula we have and a calculator to determine the slope of the tangent line:

$$R'(1) = \frac{(2)(5) - (3)(7)}{5^2} = \frac{-11}{25}$$

Question 75. □

Find the derivative of $F(x) = \dfrac{x^2-2x-8}{x^2-9}$ *by recognizing it as a quotient and applying the Quotient Rule for derivatives. Then determine $F'(0)$ using your derivative equation.*

More space in next page.

■ Question 76.

□

Use table 2.3 to compute the given derivative value.

(a) *Find* $h'(1)$ *if* $h(x) = \dfrac{f(x)}{g(x)}$.

(b) *Find* $k'(-2)$ *if* $k(x) = \dfrac{xg(x)}{f(x)}$.

(c) *Find* $L'(1)$ *if* $L(x) = \dfrac{x^3+4}{f(x)+g(x)}$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-2	4	5	-1	2
1	3	-1	2	-2

Table 2.3

■ Question 77. **Homework**

See if you can derive the quotient rule formula using the help of Figure 2.7. Follow the steps below.

(a) *Firstly, write out* $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{f(x)}{g(x)}\right]$ *using the definition of the derivative. Then, combine your fractions in the numerator.*

(b) *Write out the rectangles* C *and* D *in terms of* $f(x), g(x), f(x+h)$*, and* $g(x+h)$.

(c) *In part (1), did you get the difference* $f(x+h)g(x) - f(x)g(x+h)$*? Identify this difference in terms of areas in Figure 2.7.*

(d) *Put everything together and you've just proven the quotient rule!!*

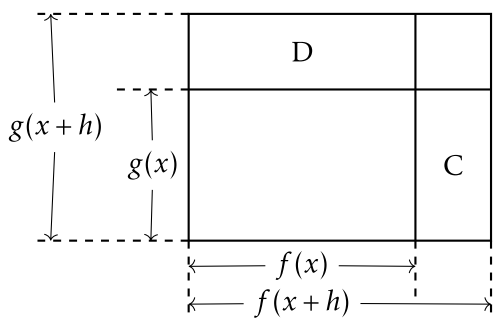


Figure 2.7: Rectangle Aid for Quotient Rule Formula

■ Question 78. **Practice Problems**

Use the graphs of functions $f(x)$ *(in blue) and* $g(x)$ *(in red) in fig. 2.8 to determine the given derivative values (**if they exist!**)*

(a) *Let* $h(x) = 3f(x) - g(x)$.
- (i) $h'(1)$
- (ii) $h'(3)$
- (iii) $h'(4)$

(b) *Let* $h(x) = \frac{1}{2}f(x)g(x)$.
- (i) $h'(1)$
- (ii) $h'(3)$
- (iii) $h'(4)$

(c) *Let* $h(x) = \dfrac{g(x)}{f(x)}$
- (i) $h'(1)$
- (ii) $h'(3)$
- (iii) $h'(4)$

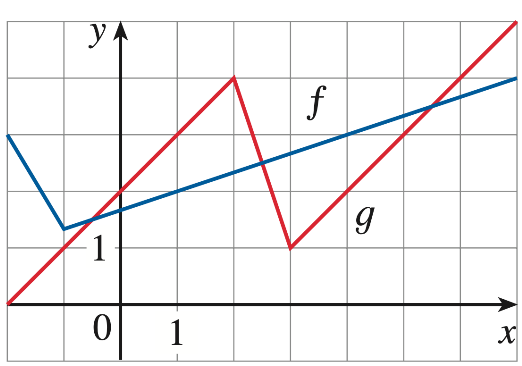


Figure 2.8

§2.5 Chain Rule

Imagine we are moving straight upward in a hot air balloon. Let y be our distance from the ground. The air pressure, P, is changing as a function of altitude, so $P = f(y)$. How does our air pressure change with time?

Since air pressure is a function of height, $P = f(y)$, and height is a function of time, $y = g(t)$, we can think of air pressure as a composite function of time, $H = f(g(t))$, with f as the outside function and g as the inside function. The example suggests the following result, which turns out to be true:

$$\text{Rate of change of composite function} = \text{Rate of change of outside function} \times \text{Rate of change of inside function}$$

Theorem 2.5.60

If f and g are differentiable functions, then

$$\frac{\mathrm{d}}{\mathrm{d}x}f(g(x)) = f'(g(x)) \cdot g'(x)$$

Warning: The derivative of the outside function must be evaluated at the inside function.

This is called the **chain rule** because, if you have multiple compositions (i.e. several functions stuffed inside of each other) then you will end up with a "chain" of products in the derivative. The Leibniz notation is very suggestive and helpful for remembering how the chain rule works: for the function $y = f(u) = f(g(x))$, meaning $u = g(x)$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

Example 2.5.61

Consider the functions $f(x) = \sqrt{x}$ and $g(x) = 3x-4$. The composite function here is $f(g(x)) = \sqrt{3x-4} = (3x-4)^{1/2}$. Notice the structure of this function:

$$\underbrace{(3x-4)^{1/2}}_{\text{outside function is the 1/2 power}} \qquad \sqrt{\underbrace{3x-4}_{\text{inside function is the polynomial}}}$$

We will always try to identify the **inside function** $g(x)$ and the **outside function** $f(x)$ before starting the chain rule process. Learning to spot insides and outsides correctly is the key to using chain rule.

The derivative of the **outside function** is $f'(x) = \frac{1}{2}x^{-1/2}$. Keeping $g(x)$ inside of $f'(x)$ looks like $f'(g(x)) = \frac{1}{2}(3x-4)^{-1/2}$.

The derivative of the **inside function** is $g'(x) = 3$.

Putting all these pieces of the chain rule formula together, we get:

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = \frac{1}{2}(3x-4)^{-1/2} \cdot 3.$$

So to compute the derivative $\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))]$, we take the derivative of the outside, put the inside function back inside the derivative of the outside, and then multiple by the derivative of the inside.

Note: Here is an informal rule that may help you in identifying the inside vs. outside function.

The last step in evaluation is the first function for differentiation.

For example, the **last** step in evaluting $(2x+1)^3$ at some point is to cube the value of$2x+1$. So if you wanted to apply Chain rule to do derivative, we need to apply it to the cubing function first. The cube here is the "outside" function.

Similarly, the last step in calculating $\frac{x^2+1}{x^2-1}$ is to take the quotient, after you evalute the numerator and denominator. So when taking derivative, we start with the quotient rule first.

■ Question 79.

Differentiate $h(x) = (2x^3 + 2x + 1)^4$ *using the chain rule.*

□

■ Question 80.

Let $f(x)$ *be a function with*

$$f(1) = 1, \quad f(2) = 2, \quad f'(1) = 3, \quad f'(2) = 5.$$

If $g(x) = 2f(2x) + f(x)$, *what is* $g'(1)$*?*

□

■ Question 81.

Suppose $f(x)$ *and* $g(x)$ *and their derivatives have the values given in the table.*

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	1	5	2	-5
1	3	-2	0	1
2	0	2	3	1
3	2	4	1	-6

Let $h(x) = f(g(x))$. *Find the following.*

(a) $h'(0)$ *(b)* $h'(1)$ *(c)* $h'(2)$ *(d)* $h'(3)$

□

Chain Rule with more than two functions

Using the chain rule twice we can similarly write

$$f(g(h(x))) = f'(g(h(x)))g'(h(x))h'(x)$$

As an example, consider the function

$$\mathrm{P}(x) = \sqrt{(2x+1)^3+1}$$

The order of operation here is as follows

$$x \longrightarrow 2x+1 \longrightarrow (2x+1)^3+1 \longrightarrow \sqrt{(2x+1)^3+1}$$

We broke our function into exact steps as above because we want to be able to take derivative at each step using the simpler rules we have learned so far. We would like to write $\mathrm{P}(x)$ as $f(g(h(x)))$. Note that h is applied first to x, and then g and then f. So in the above sequence of steps, we can identify f, g and h as follows:

$$x \xrightarrow{h} \underbrace{2x+1}_{h(x)} \xrightarrow{g} \underbrace{(2x+1)^3+1}_{g(h(x))} \xrightarrow{f} \underbrace{\sqrt{(2x+1)^3+1}}_{f(g(h(x)))}$$

where

$$\begin{aligned}
&h(x) = 2x+1 && \implies h'(x) = 2 && \\
&g(x) = x^3+1 && \implies g'(x) = 3x^2 && \implies g'(h(x)) = 3(2x+1)^2 \\
&f(x) = \sqrt{x} = x^{1/2} && \implies f'(x) = -\frac{1}{2}x^{-1/2} && \implies f'(g(h(x))) = -\frac{1}{2}\left((2x+1)^3+1\right)^{-1/2}
\end{aligned}$$

So,

$$\begin{aligned}
\mathrm{P}'(x) &= f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x) \\
&= -\frac{1}{2}\left((2x+1)^3+1\right)^{-1/2} \cdot 3(2x+1)^2 \cdot 2 \\
&= \frac{3(2x+1)^2}{\sqrt{(2x+1)^3+1}}
\end{aligned}$$

2.5.1 Practice Problems for all Differentiation Rules Combined

Question 82.

□

Sometimes, we might have to differentiate using a combination of the chain rule and the product rule, or the chain rule and the quotient rule. Try that in the two problems below.

(a) *Find* $f'(x)$ *where* $f(x) = (2x^4+3x^2+1)^5 \cdot (5x^3-x)^8$.

(b) *Find* $g'(x)$ *where* $g(x) = \left(\dfrac{x^4+3}{x^2-1}\right)^5$.

■ Question 83.

☐

Use table 2.4 to evaluate the following:

(a) *If* $p(x) = f(g(x))$, *compute* $p'(1)$.

(b) *If* $q(x) = g(f(x))$, *compute* $q'(-2)$.

(c) *If* $\mathrm{H}(x) = \dfrac{1}{(x^4 + g(x))^2}$, *compute* $\mathrm{H}'(1)$.

(d) *If* $\mathrm{R}(x) = \left(\dfrac{f(x)}{g(x)}\right)^2$, *compute* $\mathrm{R}'(2)$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-2	4	5	-1	2
1	-2	-1	2	-2
2	1	-2	-3	1

Table 2.4: Table for Problem 83

§2.6 Higher Order Derivatives and Physical Applications

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the second derivative of f because it is the derivative of the derivative of f.

For example, derivative of displacement is velocity, and derivative of velocity is acceleration!

Definition 2.6.62

Using diffent notations, we write the higher order derivatives of $y = f(x)$ as

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}f(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x) = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

$$f'''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}f(x) = \frac{\mathrm{d}^3}{\mathrm{d}x^3}f(x) = \frac{\mathrm{d}^3y}{\mathrm{d}x^3}$$

$$f^{(n)}(x) = \underbrace{\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\cdots\frac{\mathrm{d}}{\mathrm{d}x}}_{n \text{ times}}f(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}f(x) = \frac{\mathrm{d}^ny}{\mathrm{d}x^n}$$

■ **Question 84.** □

For the polynomial $f(x) = x^5 + x^3 + x + 1$, find

(a) $f''(x)$, i.e. the second derivative of f.

(b) $f^{(5)}(x)$, i.e. the fifth derivative of f.

(c) $f^{(6)}(x)$, i.e. the sixth derivative of f.

§2.7 Application in Physics - Motion along a line

As we have seen previously, we can use the derivative to think about the velocity of an object in motion. We summarize these facts below:

Definition 2.7.63

Let $s(t)$ denote the position of an object at time t.

- The velocity function $v(t)$ of the object at time T is given by $v(t) = s'(t)$.
- The speed of the object at time t is $|v(t)|$, the absolute value of $v(t)$.
- The acceleration function $a(t)$ of the object at time t is given by $a(t) = v'(t) = s''(t)$.

■ Question 85. □

The position function $s(t) = t^2 - 2t - 4$ represents the position of the back of a car coming out of a garage, and then driving in a straight line, with s in feet and t in seconds. In this case, $t = 0$ represents the time at which the car starts.

(a) *Find $s(0)$ and $s(1)$. What do these values tell you about how the car starts to move?*

(b) *Find the velocity function at time t. Find the acceleration at time t.*

(c) *When does the car stop moving for a moment? What is the acceleration at that moment?*

(d) *When is the car moving forward (that is, in the positive direction)?*

(e) *What can you say about the velocity at the moment when $s(t) = 4$?*

(f) *Find the total distance traveled by the car during the first four seconds.*

■ Question 86. □

A potato is launched vertically upward from a potato gun, with an initial velocity of 100 ft/s, and from atop an 96-foot-tall building. The height of the potato t seconds after being fired is given by $s(t) = -16t^2 + 80t + 96$ feet.

(a) *When does the potato reach it's maximum height?*

(b) *How long is the potato in the air?*

(c) *What is the speed of the potato when it hits the ground?*

We can tell if an object is **slowing down** or **speeding up** by examining the signs of the velocity and acceleration functions:

Slowing down: $a(t)$ has opposite sign from $v(t)$.

Speeding Up: $a(t)$ and $v(t)$ have the same sign.

Alternately, we can also think about the speed, $|v(t)|$. If $|v(t)|$ is increasing, then we are speeding up, and if $|v(t)|$ is decreasing, then we are slowing down.

Question 87.

The function $s(t) = 2t^3 - 3t^2 - 12t + 8$ *gives the position of a particle moving along a horizontal line. When is the particle slowing down or speeding up?*

□

§2.8 Implicit Differentiation

In all of our studies with derivatives so far, we have worked with functions whose formula is given explicitly in the form of $y = f(x)$. But not all planar curves are graphs of functions of the form $y = f(x)$. For example, take the case of a circle

$$x^2 + y^2 = 4.$$

Is this graph of a function? Since there are x–values that correspond to two different y–values, y is not a function of x on the whole circle. In fact, if we solve for y in terms of x, we quickly find that y is a function of x on the top half, and y is a different function of x on the bottom half.

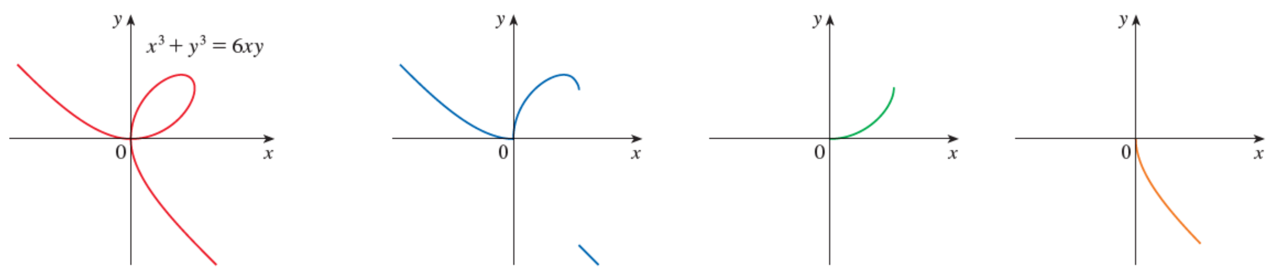
Question 88.

Use DESMOS *to draw the following curves on the plane and conclude that none of them can be written in the form* $y = f(x)$.

(a) $y^4 + xy = x^3 - x + 2$ *(b)* $x^3 + y^3 = 6xy$

However, we can always draw **tangents** to a curve at any point! Thus, it makes sense to wonder if we can compute $\dfrac{dy}{dx}$ at any point on such curves, even though we cannot write y explicitly as a function of x.

In such cases, we say that the equation of the curve, $x^2 + y^2 = 4$ for example, defines y **implicitly** as a function of x. **An implicitly defined curve can be broken into pieces where each piece can be defined by an explicit function of x.**



2.8.1 The Implicit Differentiation Process

As it should become clear by looking at the examples from question question 88, It is often rather difficult to solve expressions involving x and y to obtain the explicitly defined functions (the circle is a rare exception where the calculation is easy). However, by viewing y as an implicit function of x, we can still think of y as some function of x whose formula $f(x)$ is unknown, but which we **can** differentiate.

Finding $\dfrac{dy}{dx}$ in such scenario involves the method of **implicit differentiation**. This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for $\dfrac{dy}{dx}$.

Example 2.8.64

Let's showcase with the earlier example of the circle

$$x^2 + y^2 = 4$$

Differentiating both sides of the equation with respect to x gives

$$\frac{d}{dx}\left[x^2+y^2\right]=\frac{d}{dx}[4].$$

On the right, the derivative of the constant 4 is 0 , and on the left we can apply the sum rule, so it follows that

$$\frac{d}{dx}\left[x^2\right]+\frac{d}{dx}\left[y^2\right]=0$$

Note carefully the different roles being played by x and y. Because x is the independent variable, $\frac{d}{dx}\left[x^2\right]=2x$. But y is the dependent variable and y is an implicit function of x. Recall from last week where we computed $\frac{d}{dx}\left[f(x)^2\right]$. Computing $\frac{d}{dx}\left[y^2\right]$ is the same, and requires **the chain rule**, by which we find that $\frac{d}{dx}\left[y^2\right]=2y\frac{dy}{dx}$.

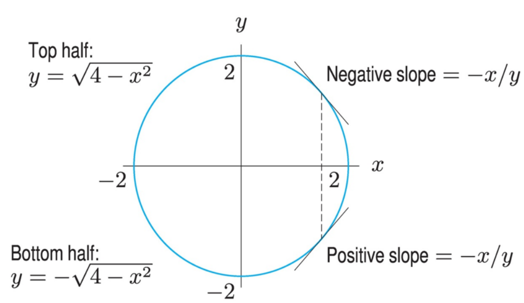
We now have that

$$2x+2y\frac{dy}{dx}=0$$

We solve this equation for $\frac{dy}{dx}$ by subtracting $2x$ from both sides and dividing by $2y$

$$\frac{dy}{dx}=-\frac{2x}{2y}=-\frac{x}{y}$$

The most important thing to observe here is that this expression for the derivative involves both x and y. This makes sense because there are two corresponding points on the circle for each value of x between -2 and 2, and the slope of the tangent line is different at each of these points.



Note: Another interesting observation is that the formula doesn't work when $y=0$, which makes sense since the tangents are vertical there. In general, this process of implicit differentiation leads to a derivative whenever the expression for the derivative does not have a zero in the denominator.

Example 2.8.65

Here is another example to help you see the chain rule. How would you find $\frac{d}{dx}f(y)$?

Just think of y as $g(x)$. Then with the chain rule, you would get:

$$\frac{d}{dx}f(g(x))=f'(g(x))\cdot g'(x).$$

Now change $g(x)$ back to y and note that $g'(x) = y' = \dfrac{\mathrm{d}y}{\mathrm{d}x}$. So,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(y) = f'(y)\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Question 89.

Differentiate each expression as indicated. Assume that the variables x, y, and t may mutually depend on each other.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}[y^3]$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}y}[y^3]$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}t}[y^3]$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}[x^2y^2]$

(e) $\dfrac{\mathrm{d}}{\mathrm{d}y}[x^2y^2]$

(f) $\dfrac{\mathrm{d}}{\mathrm{d}t}[x^2y^2]$

Warning: There is a big difference between writing $\dfrac{\mathrm{d}}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. For example,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2+y^2\right]$$

gives an instruction to take the derivative with respect to x of the quantity x^2+y^2, presumably where y is a function of x. On the other hand,

$$\frac{\mathrm{d}y}{\mathrm{d}x}\left(x^2+y^2\right)$$

means the product of the derivative of y with respect to x with the quantity x^2+y^2. Make sure to use the correct notation for the correct purpose.

Question 90.

For each of the following curves,

- *first find a formula for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.*
- *Then evaluate $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at the specified (a,b) point.*

- *Then, write the equation of the tangent line at the given point.*

(a) $\sqrt{x} - \sqrt{y} = -1$, $(1, 4)$

(b) $x^3 + y^2 - 2xy = 2$, $(-1, 1)$

(c) $xy^2 + 3x^3y - y = 3$, $(1, 1)$

2.8.2 Horizontal and Vertical Tangents

It is natural to ask where the tangent line to a curve is vertical or horizontal. The slope of a horizontal tangent line must be zero, while the slope of a vertical tangent line is undefined. Often the formula for $\dfrac{dy}{dx}$ is expressed as a quotient of functions of x and y, say

$$\frac{dy}{dx} = \frac{p(x,y)}{q(x,y)}$$

The tangent line is horizontal precisely when the numerator is zero and the denominator is nonzero, making the slope of the tangent line zero. If we can solve the equation $p(x,y) = 0$ for either x and y in terms of the other, we can substitute that expression into the original equation for the curve. This gives an equation in a single variable, and if we can solve that equation we can find the point(s) on the curve where $p(x,y) = 0$. At those points, the tangent line is horizontal. Similarly, the tangent line is vertical whenever $q(x,y) = 0$ and $p(x,y) \neq 0$, making the slope undefined.

Question 91. □

Find all points where the tangent line to the curve $x^3 + y^3 = 3xy$ is either horizontal or vertical. Does your answer match the picture you drew in question 1?

Question 92. □

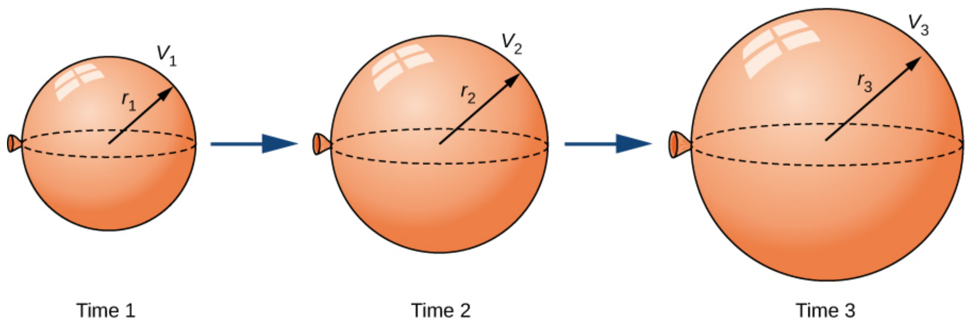
Find all points where the tangent line to the curve $2y^3 + y^2 - y^5 = x^4 - 2x^3 + x^2$ is either horizontal or vertical. Be sure to use `DESMOS` *to plot this implicit curve and to visually check the results of algebraic reasoning that you use to determine where the tangent lines are horizontal and vertical.*

§2.9 Related Rates

In most of our applications of the derivative so far, we have been interested in the instantaneous rate at which one variable, say y, changes with respect to another, say x, leading us to compute and interpret $\frac{dy}{dx}$. We next consider situations where several variable quantities are related, but where each quantity is implicitly a function of time, which will be represented by the variable t. Through knowing how the quantities are related, we will be interested in determining how their respective rates of change with respect to time are related. We call these **related rates problems**.

Example 2.9.66

Consider the following scenario. A spherical balloon is being inflated at a constant rate of 20 cubic inches per second. We want to knwo How fast the radius of the balloon is changing at the instant the balloon's diameter is 12 inches. Is the radius changing more rapidly when $d = 12$ or when $d = 16$? Why?



(a) Recall that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$. Note that in the setting of this problem, both V and r are changing as time t changes, and thus both V and r may be viewed as implicit functions of t, with respective derivatives $\frac{dV}{dt}$ and $\frac{dr}{dt}$.

Differentiate both sides of the equation $V = \frac{4}{3}\pi r^3$ with respect to t (using the chain rule on the right) to find a formula for $\frac{dV}{dt}$ that depends on both r and $\frac{dr}{dt}$.

(b) At this point in the problem, by differentiating we have **related the rates** of change of V and r. Recall that we are given in the problem that the balloon is being inflated at a constant rate of 20 cubic inches per second. Is this rate the value of $\frac{dr}{dt}$ or $\frac{dV}{dt}$? Why?

(c) From part (b), we know the value of $\frac{dV}{dt}$ at every value of t. Next, observe that when the diameter of the balloon is 12, we know the value of the radius. In the equation from part (a), substitute these values for the relevant quantities and solve for the remaining unknown quantity, which is $\frac{dr}{dt}$. How fast is the radius changing at the instant $d = 12$?

(d) How is the situation different when $d = 16$? When is the radius changing more rapidly, when $d = 12$ or when $d = 16$?

2.9.1 Practice Problems

Algorithm for Solving Related Rates Problems

Step 1. Identify the quantities in the problem that are changing and choose clearly defined variable names for them. Draw one or more figures that clearly represent the situation.

Step 2. State, in terms of the variables, all rates of change that are known or given and identify the rate(s) of change to be found.

Step 3. Find an equation that relates the variables whose rates of change are known to those variables whose rates of change are to be found.

Step 4. Using implicit differentiation, differentiate both sides of the equation found in step 3 with respect to t to relate the rates of change of the involved quantities.

Step 5. Substitute all known values into the equation from step 4, then solve for the unknown rate of change.

Warning: When solving a related-rates problem, it is crucial not to substitute known values too soon. For example, if the value for a changing quantity is substituted into an equation before both sides of the equation are differentiated, then that quantity will behave as a constant and its derivative will not appear in the new equation found in step 4.

Question 93. □

Suppose $z^2 = x^2 + y^2$. *Find* $\frac{\mathrm{d}z}{\mathrm{d}t}$ *at* $(x, y) = (1, 3)$ *if* $\frac{\mathrm{d}x}{\mathrm{d}t} = 4$ *and* $\frac{\mathrm{d}y}{\mathrm{d}t} = 3$.

Question 94. □

You and a friend are riding your bikes to a restaurant that you think is East; your friend thinks the restaurant is North. You both leave from the same point, with you riding at 16 km/hr *east and your friend riding* 12 km/hr *north. After you traveled* 4 km, *at what rate is the distance between you changing?*

Question 95. □

Imagine a rectangle with whose length x is increasing at a rate of 0.2 m/s *and whose width y is decreasing at a rate of* 0.1 m/s. *How fast is the area of rectangle changing at the moment when $x = 3$ m and $y = 2$ m.*

Question 96. □

The radius of a circle increases at a rate of 2 m/sec. *Find the rate at which the area of the circle increases when the radius is* 5 m.

Question 97. □

A cylindrical tank is leaking water at a constant rate. The cylinder has a height of 2 m and a radius of 2 m. We notice that the rate at which water level is decreasing is 10 cm/min *when the water level is* 1 m. *Find the rate at which the water is leaking out.*

Question 98. □

Gravel is being unloaded from a truck and falls into a pile shaped like a cone at a rate of 10 ft^3/min. *The radius of the cone base is three times the height of the cone. Find the rate at which the height of the gravel changes when the pile has a height of 5 ft.*

■ Question 99. □

A 6 ft tall person walks away from a 10 ft lamppost at a constant rate of 3 ft/sec. What is the rate that the tip of the shadow moves away from the pole when the person is 10 ft away from the pole?

■ Question 100. □

A baseball diamond is a square with side 90 ft. *A batter hits the ball and runs toward first base with a speed of* 24 ft/s.

(a) *At what rate is his distance from second base decreasing when he is halfway to first base?*

(b) *At what rate is his distance from third base increasing at the same moment?*

■ Question 101. □

A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?

■ Question 102. □

If the minute hand of a clock has length r (in centimeters), find the rate at which it sweeps out area as a function of r.

■ Question 103. Requires Trig.

A rocket is launched so that it rises vertically. A camera is positioned 5000 ft from the launch pad. When the rocket is 1000 ft above the launch pad, its velocity is 600 ft/sec. Find the necessary rate of change of the camera's angle as a function of time so that it stays focused on the rocket.

■ Question 104. Requires Trig.

You are stationary on the ground and are watching a bird fly horizontally at a rate of 10 m/sec. *The bird is located* 40 m *above your head. How fast does the angle of elevation change when the horizontal distance between you and the bird is 9 m?*

■ Question 105. Requires Trig.

The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?

Chapter 3 | Application of Derivatives - Shape of a Graph

§3.1 Maximum and Minimum Values

In many different settings, we are interested in knowing where a function achieves its least and greatest values. These can be important in applications — say to identify a point at which maximum profit or minimum cost occurs — or in theory to characterize the behavior of a function or a family of related functions.

In the rest of this course we are going to learn how to use Derivatives to find such maximum or minimum values. Let's start by giving a precise definition and look at some example cases.

Definition 3.1.67

Given a function f, we say that $f(c)$ is a **absolute** or **global maximum** of f on an interval I if $f(c) \geq f(x)$ for all x in I.

Similarly we call $f(c)$ a **absolute** or **global minimum** of f on an interval I whenever $f(c) \leq f(x)$ for all x in I.

Note: If I is not specified, we take I to be $(-\infty, \infty)$, the set of all real numbers.

Question 106. □

For each of the following functions, use the graph to figure out whether it has a global maximum or minimum on the specified interval. If yes, find the maximum and minimum values.

(a) $f(x) = x^2$ *on* $\mathbb{R}$.

(b) $g(x) = x^2$ *on* $[-2, 2]$.

(c) $h(x) = |x|$ *on* $[-3, 4]$.

(d) $p(x) = \dfrac{1}{x}$ *on* $[1, \infty)$.

(e) $q(x) = x + 1$ *on* $(-2, 2)$.

From the examples above, we observe functions seem to have extreme values when the domain is restricted to a closed interval. In fact, this is due to following theorem (we will not prove it) that gives conditions under which a function is guaranteed to possess extreme values.

Theorem 3.1.68: Extreme Value Theorem

*If f is **continuous** on a **closed** interval $[a, b]$, then f **attains** a global maximum value $f(c)$ and a global minimum value $f(d)$ at some numbers c and d in $[a, b]$.*

Warning: Note the usage of the word 'attain' here. It specifically says that not only the global max and min exist, but also they are attainable! In other words, we can find some numbers c and d in $[a,b]$ such that $f(c)$ is the global min and $f(d)$ is the global max.

The theorem does not tell us where these extreme values occur, but rather only that they must exist. To figure out where the extreme values are, we need to make a series of observation.

- Suppose we have a function f that is has a global maximum $f(c)$ somewhehre in the interior of an interval I.

  So we are excluding the cases where the maximum is at one of the boundary points.

- Also suppose that f is differentiable at c (i.e. no cusp or discontinuity etc.) This means f looks relatively nice and 'smooth' near $x = c$, and $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists.

- Since $f(x) \leq f(c)$ for all $x \in \mathrm{I}$, what can we say about the sign of $\lim_{x\to c^+} \frac{f(x)-f(c)}{x-c}$? Is it non-positive or non-negative? ______________________________________

- Similarly, what can we say about the sign of $\lim_{x\to c^-} \frac{f(x)-f(c)}{x-c}$? Is it non-positive or non-negative? _____

- If $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists, then the left-hand and the right-hand limits must be equal. What can you conclude using your observations from the last two parts? ____________________

We summarize above observations using the following theorem.

Theorem 3.1.69: Critical Point Theorem

Let f be defined on an interval I *containing the point c. If $f(c)$ is a global extremum, then c must be a* ***critical point****; i.e. c is one of the following three posiiblities:*

(i) an end point of I;

(ii) a ***stationary*** *point of f, i.e. a point where $f'(c) = 0$;*

(iii) a singular point of f, i.e. a point where $f'(c)$ does not exist.

Finding the global maximum and minimum of a continuous function f on the interval $[a,b]$

Step 1. Calculate $f'(x)$ and find where it is 0 or undefined inside the interval (a,b).

Step 2. Find the value of the function $f(p)$ for every critical point p from first step.

Step 3. Find the values of $f(x)$ at the endpoints of the interval, i.e. find $f(a)$ and $f(b)$.

Step 4. The largest of the values from Steps 2 and 3 is the global maximum value; the smallest of these values is the global minimum value.

3.1.1 Practice Problems

■ Question 107.
☐

Identify all the critical points and find the global maximum and minimum of the following functions on the specified interval.

(a) $f(x) = -2x^3 + 3x^2$ *on* $\left[-\frac{1}{2}, 2\right]$

(b) $f(x) = x^{2/3}$ *on* $[-1, 8]$

■ Question 108.
☐

For each question, sketch the graph of a function f that has the given properties.

(a) *f is continuous, but not necessarily differentiable, has domain $[0,6]$, reaches a maximum of 6 (attained when $x = 5$), and a minimum of 2 (attained when $x = 3$). Additionally, $x = 1$ and $x = 5$ are the only stationary points.*

(b) *f is continuous but not necessarily differentiable, has domain $[0,6]$, reaches a maximum of 6 (attained when $x = 0$) and a minimum of 0 (attained when $x = 6$). Additionally, f has two stationary points and two singular points in $(0,6)$.*

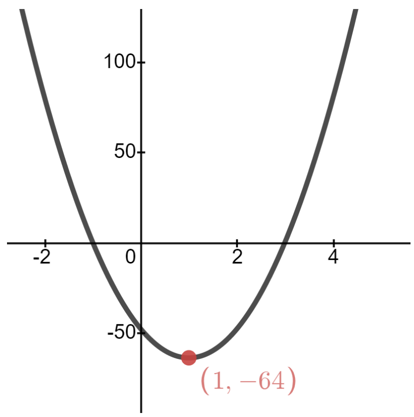
§3.2 Monotonicity and Concavity

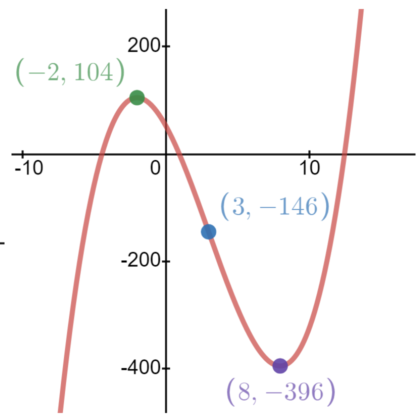
3.2.1 The First Derivative and Increasing/Decreasing Intervals

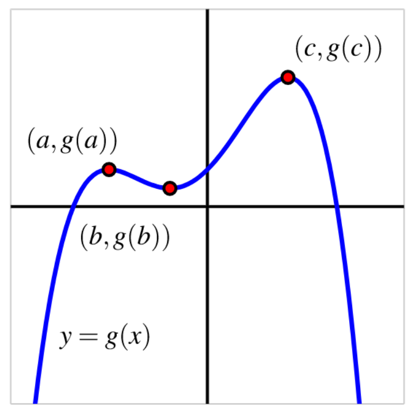
A function f is said to be **increasing** on an interval (a,b) if $f(x) < f(y)$ whenever $x < y$ on the given interval. Similarly, a function f is **decreasing** on an interval (a,b) if $f(x) > f(y)$ whenever $x < y$ on the given interval. In other words: a function is **increasing** if it the y-values are always getting smaller on an interval. A function is **decreasing** if the y-values are always getting bigger on an interval.

■ Question 109. □

For each of the functions, find the values of x for which the function is increasing and decreasing. What is the sign of the derivative on the corresponding intervals?







Intuitively, if a function is such that moving right implies that function values go up, this would coincide with our notion of a positive first derivative (i.e., a function whose graph is upward sloping). This can be formally proved using the limit definition of derivative.

Theorem 3.2.70

If $f'(x) > 0$ for all x in an interval (a,b), then f is increasing on (a,b).

Similarly, if $f'(x) < 0$ for all x in an interval (a,b), then f is decreasing on (a,b).

The relation between increasing or decreasing intervals and extremas of a function will be explored in the next section.

3.2.2 Concavity of a graph

Definition 3.2.71

Let f be a differentiable function on an open interval I. Then f is said to be **concave up** on I if and only if f' is increasing on I, and f is said to be **concave down** on I if and only if f' is decreasing on I,

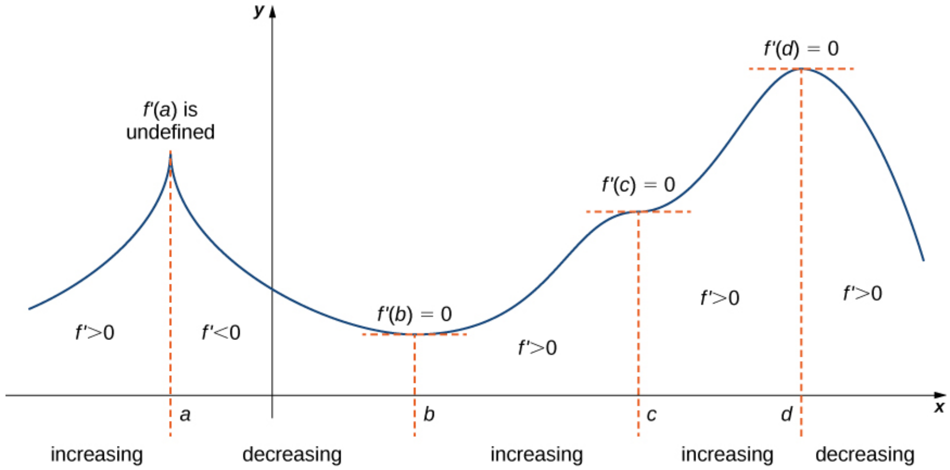
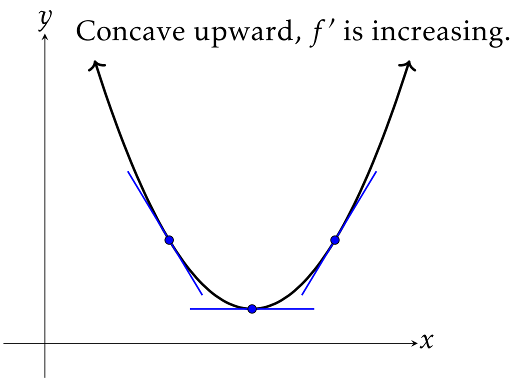
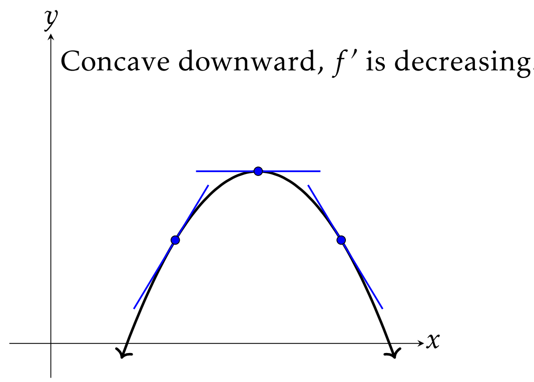


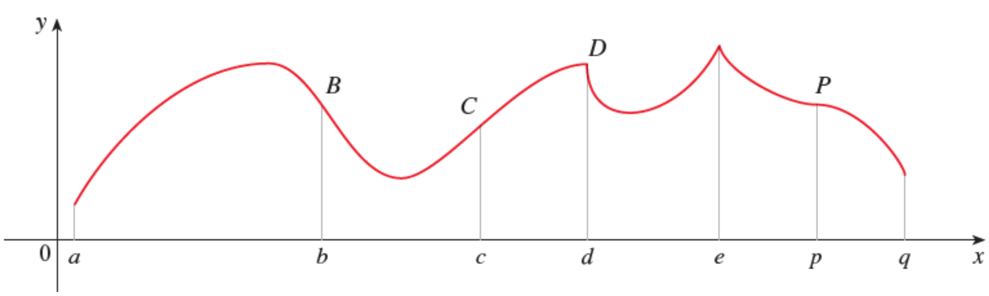
Figure 3.2: Example of increasing and decreasing intervals





■ Question 110.

Identify the intervals on which f is concave up or concave down.



When $f'(x)$ is increasing, its derivative $f''(x)$ is positive. Similarly, when f' decreases, f'' is negative. So we get the following test;

Theorem 3.2.72: Concavity Test

Let f be twice differentiable on the open interval I.

(a) *If $f''(x) > 0$ on an interval* I, *then the graph of f is concave upward on* I.

(b) *If $f''(x) < 0$ on an interval* I, *then the graph of f is concave downward on* I.

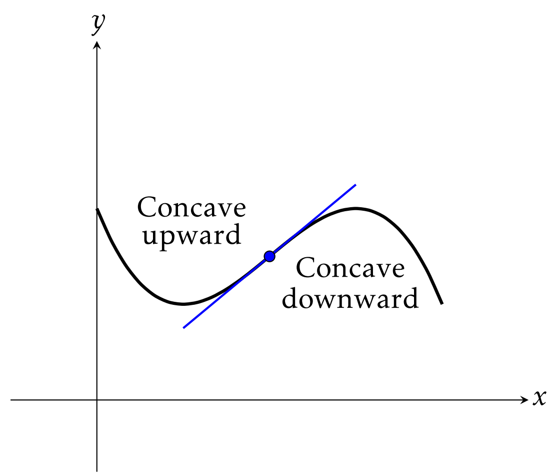
■ Question 111.

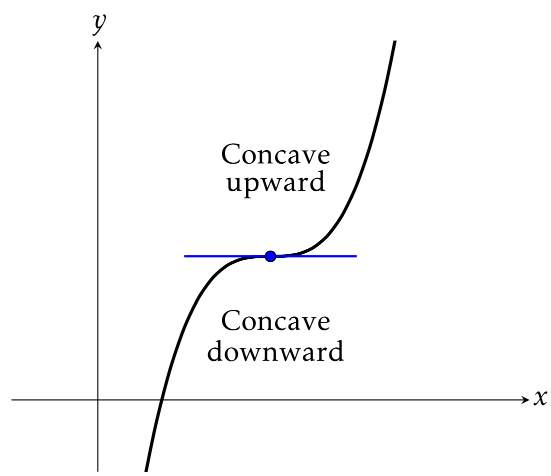
☐

Where is $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$ *increasing, decreasing, concave up, and concave down?*

Definition 3.2.73: Inflection Point

A point p, at which the graph of a continuous function, f, changes concavity is called an **inflection point** of f.





Similar to critical points, these inflection points **may** occur when $f''(x) = 0$ or when $f''(x)$ is undefined. To test whether p is an inflection point, we need to check whether f'' changes sign at p.

■ Question 112.

☐

Consider $f(x) = x^3 - 3x^2 - 9x - 1$. *Determine the intervals where* $f(x)$ *is concave up and concave down, and list any points of inflection.*

■ Question 113.

☐

Sketch a possible graph of a function f *that satisfies the following conditions:*

(a) $f'(x) > 0$ *on* $(-\infty, 1)$, $f'(x) < 0$ *on* $(1, \infty)$

(b) $f''(x) > 0$ *on* $(-\infty, -2)$ *and* $(2, \infty)$, $f''(x) < 0$ *on* $(-2, 2)$

(c) $\lim_{x \to -\infty} f(x) = -2$, $\lim_{x \to \infty} f(x) = 0$

§3.3 Summary of Curve Sketching

The following checklist is intended as a guide to sketching a curve $y = f(x)$ by hand. Not every item is relevant to every function. (For instance, a given curve might not have an asymptote or possess symmetry.) But the guidelines provide all of the information you need to make a sketch that displays the **most important aspects** of the function.

A. **Domain.** It's often useful to start by determining the domain D of f , that is, the set of values of x for which $f(x)$ is defined.

B. **Intercepts.** The y-intercept is $f(0)$ and this tells us where the curve intersects the y-axis. To find the x-intercepts, we set $y = 0$ and solve for x. (You can omit this step if the equation is difficult to solve.)

C. **Symmetry.**

(i) If $f(-x) = f(x)$ for all x in D, that is, the equation of the curve is unchanged when x is replaced by $-x$, then f is an **even function** and the curve is symmetric about the y-axis. This means that our work is cut in half. If we know what the curve looks like for $x > 0$, then we need only reflect about the y-axis to obtain the complete curve. Here are some examples: $y = x^2, y = x^4, y = |x|$, and $y = \cos x$.

(ii) If $f(-x) = -f(x)$ for all x in D, then f is an **odd function** and the curve is symmetric about the origin. Again we can obtain the complete curve if we know what it looks like for $x > 0$. [Rotate 180° about the origin] Some simple examples of odd functions are $y = x, y = x^3, y = x^5$, and $y = \sin x$.

(iii) If $f(x+p) = f(x)$ for all x in D, where p is a positive constant, then f is called a periodic function and the smallest such number p is called the period. For instance, $y = \sin x$ has period 2π and $y = \tan x$ has period π. If we know what the graph looks like in an interval of length p, then we can use translation to sketch the entire graph.

D. **Asymptotes.**

(i) A line $y = L$ is a horizontal asymptote of a function $f(x)$ if $f(x) \to L$ as $x \to \infty$ or $x \to -\infty$. Note that a function may have different **horizontal asymptotes** as it goes towards $+\infty$ or $-\infty$. If it turns out that $f(x) \to \infty$ or $-\infty$ as $x \to \infty$, then we do not have an asymptote to the right, but this fact is still useful information for sketching the curve.

(ii) A line $x = K$ is a vertical asymptote of a function $f(x)$ if $f(x) \to \infty$ or $-\infty$ as $x \to K+$ and $x \to K-$. For rational functions you can locate the **vertical asymptotes** by equating the denominator to 0 after canceling any common factors. But for other functions this method does not apply.

E. **Intervals of Increase or Decrease.** Find the intervals on which $f'(x)$ is positive (f is increasing) and the intervals on which $f'(x)$ is negative (f is decreasing).

F. **Local Maximum and Minimum Values.** Follow the steps from last section.

G. **Concavity and Points of Inflection.** Compute $f''(x)$. The curve is concave upward where $f''(x) \geq 0$ and concave downward where $f''(x) \leq 0$. Inflection points occur where the direction of concavity changes.

H. **Drawing the Graph.** Using the information in items A-G, draw the graph. Sketch the asymptotes as dashed lines. Plot the intercepts, maximum and minimum points, and inflection points. Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes. If additional accuracy is desired near any point, you can compute the value of the derivative there. The tangent indicates the direction in which the curve proceeds.

■ Question 114.

Sketch the graph of the function $f(x) = x^{2/3}(6-x)^{1/3}$.

□

■ Question 115.

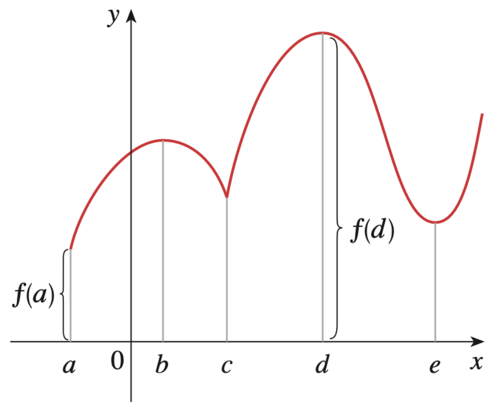
Sketch the graph of a single function that satisfies ***all*** *of the following conditions.*

- $f'(0) = f'(4) = 0, \quad f'(x) = 1$ *if* $x < -1$
- $f'(x) > 0$ *if* $0 < x < 2$
- $f'(x) < 0$ *if* $-1 < x < 0$ *or* $2 < x < 4$ *or* $x > 4$
- $\lim\limits_{x\to 2^-} f'(x) = \infty, \lim\limits_{x\to 2^+} f'(x) = -\infty$
- $f''(x) > 0$ *if* $-1 < x < 2$ *or* $2 < x < 4$
- $f''(x) < 0$ *if* $x > 4$

□

§3.4 Testsing for Local Extrema

Consider the graph of a function $f(x)$ as below.



Clearly, the function f has an global maximum at d and global minimum at a. Note that $(d, f(d))$ is the highest point on the graph and $(a, f(a))$ is the lowest point. However, if we consider only values of x near b [for instance, if we restrict our attention to the interval (a, c)], then $f(b)$ is the largest of those values of $f(x)$ and is called a **local maximum** value of f. Likewise, $f(c)$ is called a **local minimum** value of f because $f(c) \leq f(x)$ for x "near" c [in the interval (b, d) for instance]. The function f also has a local minimum at e. In general, we have the following definition.

Definition 3.4.74: Local Extrema

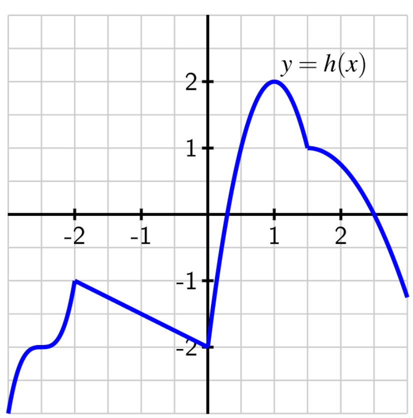
The number $f(c)$ is a

- local maximum value of f if $f(c) \geq f(x)$ when x is near c.
- local minimum value of f if $f(c) \leq f(x)$ when x is near c.

Note: If we say that something is true near c, we mean that it is true on some open interval containing c. Thus a local maximum or minimum can't occur at an endpoint.

Question 116. ☐

Consider the function h given by the graph in the figure below.



(a) *Identify all of the values of c in $(-3,3)$ for which $h(c)$ is a local maximum of h*

(b) *Identify all of the values of c in $(-3,3)$ for which $h(c)$ is a local minimum of h.*

(c) *Does h have a global maximum on the interval $[-3,3]$? If so, what is the value of this global maximum?*

(d) *Does h have a global minimum on the interval $[-3,3]$? If so, what is its value?*

(e) *Identify all values of c for which $h'(c) = 0$.*

(f) *Identify all values of c for which $h'(c)$ does not exist.*

(g) *True or false: every local maximum and minimum of h occurs at a point where $h'(c)$ is either zero or does not exist.*

(h) *True or false: at every point where $h'(c)$ is zero or does not exist, h has a local maximum or minimum.*

3.4.1 The First Derivative Test

Notice that the global maximum and minimum values that are not at the booundary always occur at local maximum or minimum values, so if we are working on an optimization problem, we start by looking for local extreme values.

Theorem 3.4.75: Fermat's theorem

If f has a local extremum at $x = c$, then c is a critical point.

Warning: The converse of Fermat's theorem is not true. Not every critical point is a local extremum. Consider for example, a function $f(x)$ from question 110. The function f has four critical points: a, b, c, and d. The function has local maxima at a and d, and a local minimum at b, The function does not have a local extremum at c.

Perhaps, the most interesting observation to make here is that the sign of f' changes at every local extrema.

Theorem 3.4.76: First Derivative Test

If c is a critical point of a continuous function f that is differentiable near c (except possibly at $x = c$), then

(i) *f has a relative maximum at c if and only if f' changes sign from positive to negative at c, and*

(ii) *f has a relative minimum at c if and only if f' changes sign from negative to positive at c.*

Question 117. □

Find the local extreme values of $f(x) = \dfrac{1}{3}x^3 - x^3 - 3x + 4$ *on* $\mathbb{R}$.

■ Question 118.

Suppose that $g(x)$ is a function continuous for every value of $x \neq 2$ whose first derivative is

$$g'(x) = \frac{(x+4)(x-1)^2}{x-2}.$$

Further, assume that it is known that g has a vertical asymptote at $x = 2$.

(a) *Determine all critical points of g.*

(b) *By developing a carefully labeled first derivative sign chart, decide whether g has as a local maximum, local minimum, or neither at each critical point.*

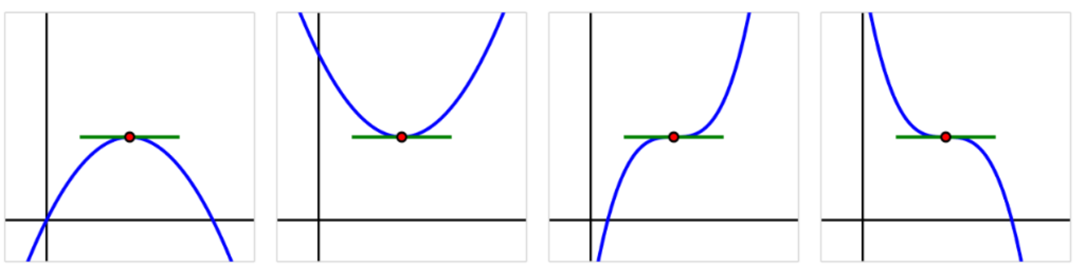
(c) *Does g have a global maximum? global minimum? Justify your claims.*

(d) *What is the value of $\lim_{x\to\infty} g'(x)$? What does the value of this limit tell you about the long-term behavior of g?*

(e) *Sketch a possible graph of $y = g(x)$.*

3.4.2 The Second Derivative Test

Sometimes (but not always), it is more eficient to use the second derivative to verify if a critical point is a local extrema instead of the first derivative test. In the last section, we saw that there are four possibilities for the graph of a function f with a horizontal tangent line at a critical point.



From the pictures, we can conclude the following.

Theorem 3.4.77: Second Derivative Test

If p is a critical point of a continuous function f such that $f'(p) = 0$ and $f''(p) \neq 0$, then f has a local maximum at p if and only if $f''(p) < 0$, and f has a local minimum at p if and only if $f''(p) > 0$.

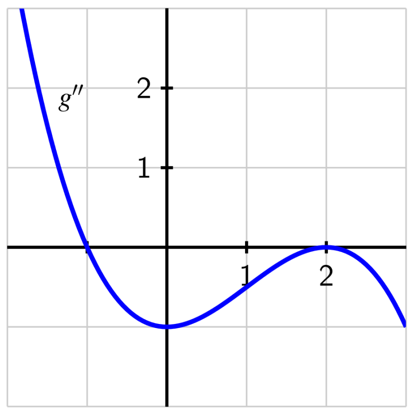
Warning: In the event that $f''(p) = 0$, the second derivative test is inconclusive. That is, the test doesn't provide us any information. This is because if $f''(p) = 0$, it is possible that f has a local minimum, local maximum, or neither.

■ Question 119. □

Use the second derivative test to find and classify the local extrema of the function $f(x) = x^3 - 3x^2 - 9x$.

■ Question 120. □

Consider a function $g(x)$ *whose second derivative* g'' *is given by the following graph.*



(a) *Find the* x *-coordinates of all points of inflection of* g.

(b) *Fully describe the concavity of* g *by making an appropriate sign chart.*

(c) *Suppose you are given that* $g'(-1.6) = 0$. *Is there a local maximum, local minimum, or neither (for the function* g*) at this critical point of* g, *or is it impossible to say? Why?*

(d) *Assuming that* $g''(x)$ *is a polynomial (and that all important behavior of* g'' *is seen in the graph above), what degree polynomial do you think* $g(x)$ *is? Why?*

■ Question 121. □

Find and classify the local extrema of the function $f(x) = 4x^6 - 6x^4$.

§3.5 Applied Optimization

The goal of the last section in our course is to directly apply what we have learned about finding extrema of functions to real-life applications. We want to see how the techniques of Calculus can be used to take function models and find precise optimization solutions.

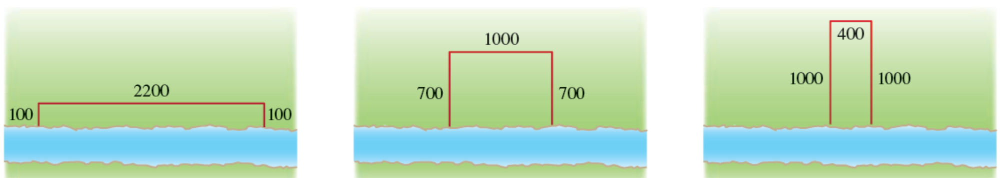
In solving such practical problems the greatest challenge is often to convert the word problem into a mathematical optimization problem by setting up the function that is to be maximized or minimized. Let's work through an example to understand the steps required:

Example 3.5.78

A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Step 1. **Draw a picture.** *We will always try to start by drawing a picture to have a visual representation of our problem. Having a picture let's us experiment with some specific cases to get a feeling for what is happening.*

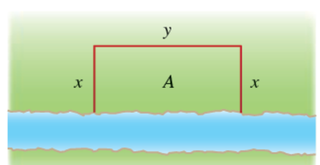
The figure below (not to scale) shows three possible ways of laying out the 2400 ft of fencing.



Area = 100 · 2200 = 220,000 ft^2 Area = 700 · 1000 = 700,000 ft^2 Area = 1000 · 400 = 400,000 ft^2

We see that when we try shallow, wide fields or deep, narrow fields, we get relatively small areas. It seems plausible that there is some intermediate configuration that produces the largest area.

Step 2. **Introduce and label variables.** *Next step is to understand what quantities are allowed to vary in the problem and then to represent those values with variables. So we will construct a general figure with the variables labeled.*



Let x and y be the depth and width of the rectangle (in feet) in the general case.

Step 3. **Make sure that you know what quantity or function is to be optimized.**

Write down a formula for this quantity algebraically using the variables you introduced in the last step. This function is called the ***Objective Function****.*

We wish to maximize the area A of the rectangle. So next we express A in terms of x and y.

$$A = \text{Area} = V = \text{length} \times \text{width} = xy$$

In order to do apply the techniques we have learned in this course so far, we need a function of **one** variable. So we go to the next step.

Step 4. **Using information given in the problem, re-write your formula from Step 3 as a function of ONE variable.** *The information given in the problem regarding the relationship among the variables should aid you in making the necessary substitutions or eliminations in this step. The information given is usually in the form of other equations; we refer to this as a* ***constraint equations****. Remember, you have to eliminate all but one variable.*

We want to express A as a function of just one variable, so we eliminate y by expressing it in terms of x. To do this we use the given information that the total length of the fencing is 2400 ft. Thus

$$2x + y = 2400$$

From this equation we have $y = 2400 - 2x$, which gives

$$A = xy = x(2400 - 2x) = 2400x - 2x^2$$

Step 5. **Decide the domain on which to optimize your Objective Function.** *Often the physical constraints of the problem will limit the possible values that the variables can take on. Thinking back to the diagram describing the overall situation and any relationships among variables in the problem often helps identify the smallest and largest values of the input variable.*

Note that the largest x can be is 1200 (this uses all the fence for the depth and none for the width) and x can't be negative, so the function that we wish to maximize is

$$A(x) = 2400x - 2x^2, \qquad 0 \le x \le 1200$$

We know that any continuous function on a closed interval will always have an absolute maximum and an absolute minimum. These are exactly what we wish to find.

Step 6. **Apply the techniques you know to identify the Max/Min(s).** *This always involves finding the critical numbers of the function first. Then evaluate the function at the endpoints and critical numbers to find the global max and/or min.*

The derivative is $A'(x) = 2400 - 4x$, so to find the critical numbers we solve the equation

$$2400 - 4x = 0$$

which gives $x = 600$.

The maximum value of A must occur either at this critical number or at an endpoint of the interval. Since $A(0) = 0$, $A(1200) = 0$ and $A(600) = 720000$, the maximum value of A must be $A(600) = 720000$.

[We could have also done above step using the second derivative test instead. How would that work?]

Step 7. **Finally, bring all your information together, and answer whatever questions were posed by the problem.** *Make sure that you have answered the correct question: does the question seek the absolute maximum of a quantity, or the values of the variables that produce the maximum? Also make sure to answer all asked questions! (Many problems have multiple parts!)*

The example asks for the dimensions of the field that gives the maximum. So we still need to find out y. The corresponding y-value is $y = 2400 - 2x = 2400 - 1200 = 1200$. So the rectangular field should be 600 ft deep and 1200 ft wide.

Here's a flowchart to summarize the process and help you with examples below:

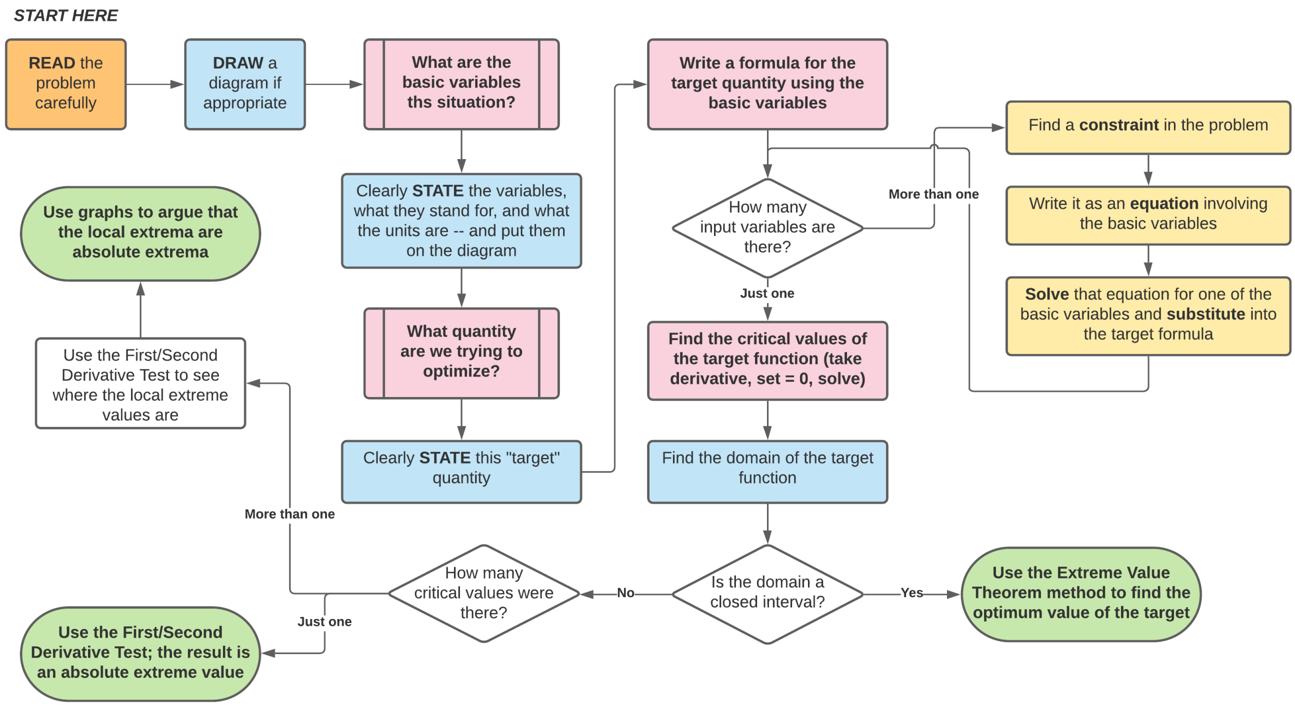


Figure 3.3: Flowchart for Optimization Problems (Picture Courtesy: Robert Talbert)

Warning: Familiarity with common geometric formulas is particularly essential in solving optimization problems. Sometimes those involve perimeter, area, volume, or surface area of geometric objects. At other times, the constraints of a problem introduce right triangles (where the Pythagorean Theorem apply) or other shapes whose formulas provide relationships among the variables. So it is highly recommended that you brush up on those result as we progress further.

Question 122.

You want to build a box for your cat to sit in. The box will have no top, a square base, and rectangular sides. Looking at the size of your cat, you want the box to have a volume of 150 in^3. What should be the dimensions of the box in order to ***minimize the surface area*** *(uses the least amount of material) of the box?* *

(a) *Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.*

(b) *Write an expression for the total area.*

(c) *Use the given information to write an equation that relates the variables.*

(d) *Use part (c) to write the total area as a function of one variable.*

(e) *Finish solving the problem using Calculus.*

**Answer:* For x the length of the square base and y the height of the side, the minimal surface area occurs when $x \approx 6.69$in and $y \approx 3.35$in.

■ Question 123.

☐

A retailer has been selling 1200 tablet computers a week at \$350 each. The marketing department estimates that if the price is lowered by \$10, an additional 80 tablets will sell each week.[†]

(a) *What should the price be set at in order to maximize revenue?*

(b) *If the retailer's weekly cost function is $C(x) = 35000 + 120x$ what price should it choose in order to maximize its profit?*

■ Question 124.

☐

Suppose we have a piece of cardboard that is 10 inches by 15 inches. We need to remove squares of side length x from the four corners of the cardboard, and then fold up each newly formed flap to make an open-top box. Additionally, the box must be at least 1 inch deep, but no more than 3 inches deep. What is the maximum possible volume of a box that we can make? What is the minimum volume that we can make?

■ Question 125.

☐

Consider a wire of length 4 feet. Suppose the wire is cut into two pieces and denote one of the pieces with length x. One piece is formed into a square, and the other is formed into a circle. What value of x will maximize the sum of the areas of the square and the circle? [‡]

■ Question 126.

☐

A nurse takes a patient's pulse three times and measures 70 bpm, 80 bpm, and 120 bpm (bpm stands for "beats per minute").

(a) *To identify a more accurate reading of the patient's pulse, the nurse wants to minimize the function*

$$P(x) = (x-70)^2 + (x-80)^2 + (x-120)^2.$$

What value of x will be a minimum for P?

> ***Note:*** *This kind of function is similar to how one finds a "line of best fit" or regression line in statistics*

(b) *Suppose the nurse noticed the patient was nervous for the third reading. They decide to adjust $P(x)$ to give less weight to the third reading like so:*

$$P(x) = (x-70)^2 + (x-80)^2 + \frac{1}{2}(x-120)^2.$$

Now what value of x minimizes the function?[§]

■ Question 127.

☐

Find the area of the largest rectangle that can be inscribed in a semicircle of radius 10.

[†] **Answer:** (a) \$250, (b)\$310

[‡] **Answer:** The maximum occurs when you just make a circle.

[§] **Answer:** For the first function, the minimum is $x = 90$bpm. For the weighted function, the minimum is $x = 74$bpm.

■ Question 128. □

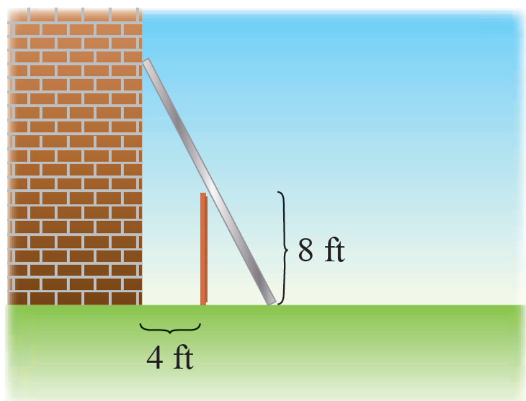
An oil refinery is located on the north bank of a straight river that is 2 km wide. A pipeline is to be constructed from the refinery to storage tanks located on the south bank of the river 10 km downstream of the refinery. The cost of laying pipe is \$10,000/km over land to a point P along the north bank and then \$40,000/km under the river to the tanks. To minimize the cost of pipeline, where should P be located?¶

■ Question 129. □

Two posts, on 12 feet high and the other 28 feet high stand 30 feet apart. They are to be stayed by two wires, attached to a single stake, running from ground level to the top of each post. Where should the stake be placed to use the least amount of wire?‖

■ Question 130. Challenge Problem!

A fence 8 ft tall runs parallel to a tall building at a distance of 4 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?



¶**Answer:** P should be located $10 - \frac{2}{\sqrt{15}}$ km downriver from the refinery.

‖**Answer:** 9 ft from the 12 ft post.