MATH 111 - Calculus and Analytic Geometry I

Lecture 21 Worksheet

Fall 2020 | **Subhadip Chowdhury** | **Oct 5**

Title: Implicit Differentiation

Summary: We will learn to find slope of tangents to curves that are not graphs of functions.

§A. What is an Implicit Function

In all of our studies with derivatives so far, we have worked with functions whose formula is given explicitly in the form of $y = f(x)$. But not all planar curves are graphs of functions of the form $y = f(x)$. For example, take the case of a circle

$$x^2 + y^2 = 4.$$

Is this graph of a function? Since there are x−values that correspond to two different y−values, y is not a function of x on the whole circle. In fact, if we solve for y in terms of x, we quickly find that y is a function of x on the top half, and y is a different function of x on the bottom half.

Question 1.

Use DESMOS to draw the following curves on the plane and conclude that none of them can be written in the form $y = f(x)$.

a) $y^4 + xy = x^3 - x + 2$

b) $x^3 + y^3 = 3xy$

However, we can always draw **tangents** to a curve at any point! Thus, it makes sense to wonder if we can compute $\dfrac{dy}{dx}$ at any point on such curves, even though we cannot write y explicitly as a function of x.

In such cases, we say that the equation of the curve, $x^2 + y^2 = 4$ for example, defines y **implicitly** as a function of x. An implicitly defined curve can be broken into pieces where each piece can be defined by an explicit function of x.

§B. The Implicit Differentiation Process

As it should become clear by looking at the examples from question **1**, It is often rather difficult to solve expressions involving x and y to obtain the explicitly defined functions (the circle is a rare exception where the calculation is easy). However, by viewing y as an implicit function of x, we can still think of y as some function of x whose formula $f(x)$ is unknown, but which we **can** differentiate.

Just as y represents an unknown formula, so too its derivative with respect to x, given by $\dfrac{dy}{dx}$, will be (at least temporarily) unknown. Let's go back to the example of the circle

$$x^2 + y^2 = 4$$

Differentiating both sides of the equation with respect to x gives

$$\frac{d}{dx}\left[x^2 + y^2\right] = \frac{d}{dx}[4].$$

On the right, the derivative of the constant 4 is 0 , and on the left we can apply the sum rule, so it follows that

$$\frac{d}{dx}\left[x^2\right]+\frac{d}{dx}\left[y^2\right]=0$$

Note carefully the different roles being played by x and y. Because x is the independent variable, $\frac{d}{dx}\left[x^2\right]=2x$. But y is the dependent variable and y is an implicit function of x. Recall from last week where we computed $\frac{d}{dx}\left[f(x)^2\right]$. Computing $\frac{d}{dx}\left[y^2\right]$ is the same, and requires **the chain rule**, by which we find that $\frac{d}{dx}\left[y^2\right]=2y\frac{dy}{dx}$.

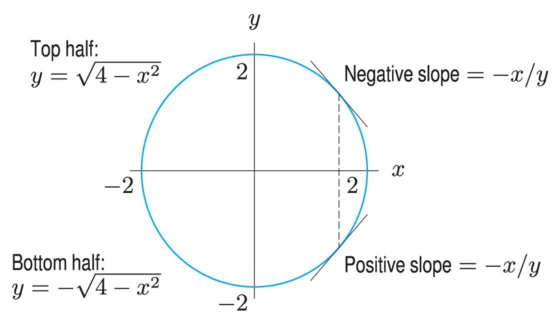
We now have that

$$2x+2y\frac{dy}{dx}=0$$

We solve this equation for $\frac{dy}{dx}$ by subtracting $2x$ from both sides and dividing by $2y$

$$\frac{dy}{dx}=-\frac{2x}{2y}=-\frac{x}{y}$$

The most important thing to observe here is that this expression for the derivative involves both x and y. This makes sense because there are two corresponding points on the circle for each value of x between -2 and 2, and the slope of the tangent line is different at each of these points.



Another interesting thing is to note that the formula doesn't work when $y=0$, which makes sense since the tangents are vertical there. In general, this process of implicit differentiation leads to a derivative whenever the expression for the derivative does not have a zero in the denominator.

Example B.1

Here is another example to help you see the chain rule. How would you find $\dfrac{d}{dx}\sin(y)$?

Just think of y as $g(x)$. Then with the chain rule, you would get:

$$\frac{d}{dx}\sin(g(x))=\cos(g(x))\cdot g'(x).$$

Now change $g(x)$ back to y and note that $g'(x)=y'=\dfrac{dy}{dx}$. So,

$$\frac{d}{dx}\sin(y)=\cos(y)\frac{dy}{dx}.$$

Question 2.

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Differentiate each expression as indicated. Assume that the variables $x, y,$ and t may mutually depend on each other.

(a) $\dfrac{d}{dx}[y^3]$

(b) $\dfrac{d}{dy}[y^3]$

(c) $\dfrac{d}{dt}[y^3]$

(d) $\dfrac{d}{dx}[\sin y]$

(e) $\dfrac{d}{dy}[\sin y]$

(f) $\dfrac{d}{dt}[\sin y]$

(g) $\dfrac{d}{dx}[x^2y^2]$

(h) $\dfrac{d}{dy}[x^2y^2]$

(i) $\dfrac{d}{dt}[x^2y^2]$

There is a big difference between writing $\dfrac{d}{dx}$ and $\dfrac{dy}{dx}$. For example,

$$\frac{d}{dx}\left[x^2+y^2\right]$$

gives an instruction to take the derivative with respect to x of the quantity x^2+y^2, presumably where y is a function of x. On the other hand,

$$\frac{dy}{dx}\left(x^2+y^2\right)$$

means the product of the derivative of y with respect to x with the quantity x^2+y^2. Make sure to use the correct notation for the correct purpose.

Question 3.

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For each of the following curves, first find a formula for $\dfrac{dy}{dx}$. Then compute $\dfrac{dy}{dx}$ at the specified (x, y) point. Then, write the equation of the tangent line at the given point.

(a) $\sqrt{x}-\sqrt{y}=-1$, $(1,4)$

(b) $x^3+y^2-2xy=2$, $(-1,1)$

(c) $xy^2+3x^3y-y=3$, $(1,1)$

(d) $\tan(y)+y^2=x^2$, (π,π)

(e) $\sin(x+y)+\cos(x-y)=1$, $(\pi/2,\pi/2)$

(f) $x\ln y+y^3=3\ln x+1$, $(1,1)$

§C. Horizontal and Vertical Tangents

It is natural to ask where the tangent line to a curve is vertical or horizontal. The slope of a horizontal tangent line must be zero, while the slope of a vertical tangent line is undefined. Often the formula for $\dfrac{dy}{dx}$ is expressed as a quotient of functions of x and y, say

$$\frac{dy}{dx}=\frac{p(x,y)}{q(x,y)}$$

The tangent line is horizontal precisely when the numerator is zero and the denominator is nonzero, making the slope of the tangent line zero. If we can solve the equation $p(x,y)=0$ for either x and y in terms of the other, we can substitute that expression into the original equation for the curve. This gives an equation in a single variable, and if we can solve that equation we can find the point(s) on the curve where $p(x,y)=0$. At those points, the tangent line is horizontal. Similarly, the tangent line is vertical whenever $q(x,y)=0$ and $p(x,y)\neq 0$, making the slope undefined.

■ Question 4. □

Find all points where the tangent line to the curve $x^3+y^3=3xy$ is either horizontal or vertical. Does your answer match the picture you drew in question **1**?

■ Question 5. □

Find all points where the tangent line to the curve $2y^3+y^2-y^5=x^4-2x^3+x^2$ is either horizontal or vertical. Be sure to use `DESMOS` to plot this implicit curve and to visually check the results of algebraic reasoning that you use to determine where the tangent lines are horizontal and vertical.