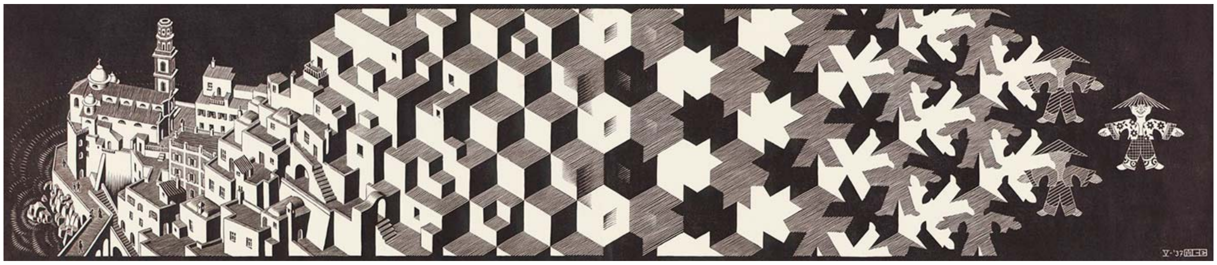

Theory of Integral Calculus

Math 125 Lecture notes

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Contents

1 Integration Techniques **1**
- 1.1 u-substitution . . . 1
- 1.2 Integration by parts . . . 2
- 1.3 Integral of Trigonometric Functions . . . 4
  - 1.3.1 Strategy for integrating power of trigonometric functions . . . 5
  - 1.3.2 Strategy for integrating product of powers trigonometric functions . . . 7
  - 1.3.3 Strategy for integrating product of multiple angle trigonometric functions . . . 8
- 1.4 Integrals via Trigonometric Substitution . . . 9

2 Definite Integrals **13**
- 2.1 Riemann Sum and Numerical Integration . . . 13
  - 2.1.1 The Area Problem . . . 13
- 2.2 Improper Integrals . . . 18
  - 2.2.1 Introduction to Infinity . . . 18
  - 2.2.2 Improper Integrals of Type I . . . 19
  - 2.2.3 Improper Integrals of Type II . . . 21
- 2.3 Comparison of Improper Integrals . . . 23

3 Introduction to Sequences and Series **26**
- 3.1 What is a Sequence? . . . 26
  - 3.1.1 Arithmetic and Geometric Progressions . . . 27
  - 3.1.2 Limit of a Sequence . . . 28
- 3.2 What is a Series? . . . 32
  - 3.2.1 Definitions . . . 33
  - 3.2.2 Geometric Series . . . 35
  - 3.2.3 Examples of Geometric Series . . . 36

4 Convergence and Divergence Tests for Series **38**
- 4.1 Divergence Test . . . 38
- 4.2 Integral Test for Positive Series . . . 39
- 4.3 Comparison Tests . . . 41
  - 4.3.1 Direct Comparison Test . . . 41
  - 4.3.2 Limit Comparison Test . . . 42
  - 4.3.3 A Bunch of Practice Problems . . . 43
  - 4.3.4 Some Useful Inequalities . . . 44
- 4.4 Alternating Series . . . 45
  - 4.4.1 Absolute and Conditional Convergence . . . 46
- 4.5 Two ways to test for Absolute Convergence . . . 48
  - 4.5.1 Ratio Test . . . 48
  - 4.5.2 Root Test . . . 49
- 4.6 Summary of the chapter . . . 51
  - 4.6.1 Problem Solving Strategy . . . 51
  - 4.6.2 Summary of Series Tests . . . 51

5 Approximating Functions as Series **55**

5.1 Power Series . . . 55
5.1.1 Radius and Interval of Convergence . . . 58
5.2 Representations of Functions as Power Series . . . 60
5.2.1 Using the Geometric Series . . . 60
5.2.2 Differentiation and Integration of Power Series . . . 61
5.3 More Practice Problems . . . 64
5.4 Taylor and Maclaurin Series . . . 65
5.4.1 How to find a Power Series representation? . . . 65
5.4.2 When is a function equal to its Taylor series? . . . 68

Student Projects **71**
A Project 1: Fourier Coefficients . . . 72
B Project 2: Laplace Transform . . . 76
C Project 3: The Koch Snowflake . . . 79
D Project 4: Irrationality of e . . . 82

Chapter 1 | Integration Techniques

In this chapter, we will review some of the integration techniques you have learned in earlier courses and learn some more integration formula that you can add to your toolbox. Before we begin, here's a list of antiderivative formula you might need. All of these can be derived using the fundamental theorem of calculus. Here C is an arbitrary constant.

- $\int x^{\alpha}\,dx = \dfrac{x^{\alpha+1}}{\alpha+1} + C$, where $\alpha \neq -1$
- $\int \cos x\,dx = \sin x + C$
- $\int \sin x\,dx = -\cos x + C$
- $\int \sec^2 x\,dx = \tan x + C$
- $\int \csc^2 x\,dx = -\cot x + C$
- $\int \sec x \tan x\,dx = \sec x + C$
- $\int \csc x \cot x\,dx = -\csc x + C$
- $\int e^x\,dx = e^x + C$
- $\int \dfrac{1}{x}\,dx = \ln|x| + C$

§1.1 u-substitution

Algorithm for Solving u-substitution Problems

Step 1. Look carefully at the integrand and select an expression $g(x)$ within the integrand to set equal to u. Select $g(x)$ such that $g'(x)$ is also sitting somewhere else in your integrand.

Step 2. Substitute $g(x)$ by u and $g'(x)\,dx$ by du into the integral.

Step 3. We should now be able to evaluate the integral with respect to u. If the integral can't be evaluated we need to go back and select a different expression to use as u.

Step 4. Evaluate the integral in terms of u.

Step 5. Substitute the expression $g(x)$ back in place of u and write the final answer in terms of x.

Question 1. □

Find the following indefinite integrals:

(a) $\displaystyle\int \frac{x+1}{x^2+2x+19}\,dx$

(b) $\displaystyle\int \frac{\sin x}{\sqrt{\cos x + 4}}\,dx$

(c) $\displaystyle\int x\sqrt{x-1}\,dx$

(d) $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx$

Question 2. □

Find the following definite integrals:

(a) $\displaystyle\int_0^{\sqrt{\pi}} \theta\cos(\theta^2)\sin(\theta^2)\,d\theta$

(b) $\displaystyle\int_0^1 x\sqrt{1-x^2}\,dx$

(c) $\displaystyle\int_1^e \frac{1+\ln x}{x\ln x}\,dx$

(d) $\displaystyle\int_0^2 \frac{t}{\sqrt{5+t^2}}\,dt$

§1.2 Integration by parts

The integration by parts formula is given as follows:

Theorem 1.2.1

Given two differentiable functions $u(x)$ and $v(x)$,

$$\int uv'\,dx = uv - \int u'v\,dx$$

and

$$\int_a^b uv'\,dx = [u(x)v(x)]\Big|_a^b - \int_a^b u'(x)v(x)\,dx$$

Integration by parts is most useful when the integrand can be viewed as a product and when the integral on the right-hand side is simpler than that on the left. So here are some rules of thumb to choose u and v:

- Whatever you choose as v', you need to be able to find v.
- Our end goal is to make sure $u'v$ is easier to integrate than uv'. So it helps if u' is simpler than u (or at least no more complicated than u) and similarly, if v is simpler than v' (or at least no more complicated than v').

The following acronym can help in deciding what should be your u and what should be your v':

Preferred Choice for u ↓	**I**	- Inverse trigonometric	($\arcsin, \arccos, \arctan$ etc.)
	L	- Logarithmic	($\ln$)
	A	- Algebraic	(polynomials, rational functions etc.)
	T	- Trigonometric	($\sin, \cos, \tan$ etc.)
Preferred Choice for v' ↑	**E**	- Exponential	(e^x etc.)

Note: This acronym is not an end-all-be-all rule on how to perform integration by parts. Sometimes **L-I-A-T-E** works better. In general, if you are stuck with your choice of u and v, switch the two and see if your integration becomes simpler.

■ **Question 3.** □

Find the following integrals

(a) $\int (x^2+1)\sin x \, dx$

(b) $\int x \arcsin x \, dx$

(c) $\int_0^1 \arctan x \, dx$

(d) $\int x^3 \sin(x^2) \, dx$

(e) $\int_0^{\pi/4} x \sin(2x) \, dx$

(f) $\int x^5 e^{x^3} \, dx$

(g) $\int_0^3 \ln(x^2+1) \, dx$ *[Hint: This one might be tricky, we will revisit it next week.]*

§1.3 Integral of Trigonometric Functions

You already know how to integrate some of the simpler trigonometric integrands.

Question 4.

Use u-substitution to find

(a) $\int \sin x \cos^2 x \, dx$

(b) $\int \tan x \, dx$

Before learning about the more complicated trig integrals, let us summarize some of the Pythagorean identities we should be aware of:

$$\sin^2 x + \cos^2 x = 1 \tag{1.1}$$

$$\tan^2 x + 1 = \sec^2 x \tag{1.2}$$

$$1 + \cot^2 x = \csc^2 x \tag{1.3}$$

Note: The second two identities are obtained from the first be means of division. In terms of integration, we really only concern ourselves with the first two. If you can integrate something with tangents and secants, then you can apply the same strategies to integrals involving cotangents and cosecants.

Next, we will list the angle sum identities:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y \tag{1.4}$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y \tag{1.5}$$

■ Question 5. □

(a) *Set $x = y$ in the above identities to derive a formula for*

$\sin(2x) =$ .... *and*

$\cos(2x) =$ ....

(b) *Use the identity for* $\cos(2x)$ *and the Pythagorean identity to derive the power-reduction identities*

$$\cos^2 x = \frac{1}{2}(1 + \cos(2x)) \qquad \textit{and} \qquad \sin^2 x = \frac{1}{2}(1 - \cos(2x))$$

1.3.1 Strategy for integrating power of trigonometric functions

Odd Power

Example 1.3.2

Suppose we want to find the integral $\int \sin^3 x \, dx$. We can rewrite the integral as

$$\begin{aligned} \int \sin^3 x \, dx &= \int \sin^2 x \sin x \, dx \\ &= \int (1 - \cos^2 x) \sin x \, dx \end{aligned}$$

Now we can use a u-substitution. If we choose

$u =$ ...., then $du =$ .... and consequently,

we can rewrite above integral as

Now finish the integration.

■ Question 6. □

Find $\int \cos^5 x \, \mathrm{d}x$.

Even Power

Example 1.3.3

Suppose we want to find the integral $\int \cos^2 x \, \mathrm{d}x$. We can rewrite the integral using a power-reduction identity as follows

$$\int \cos^2 x \, \mathrm{d}x = \frac{1}{2} \int (1 + \cos(2x)) \, \mathrm{d}x$$

Now finish the integration.

Question 7.

Find $\int \sin^4 x \, \mathrm{d}x$.

1.3.2 Strategy for integrating product of powers trigonometric functions

Consider an integral of the form

$$\int \sin^m x \cos^n x \,\mathrm{d}x$$

where m and n are integers. Here is the general algorithm:

Step 1. If m is odd, isolate a $\sin x$ and use Pythagorean identity $\sin^2 x = 1 - \cos^2 x$ to turn everything else into cosine. Then use the u-substitution $u = \cos x$.

Step 2. If n is odd, isolate a $\cos x$ and use Pythagorean identity $\cos^2 x = 1 - \sin^2 x$ to turn everything else into sine. Then use the u-substitution $u = \sin x$.

Step 3. If both m and n are even, then use the power-reduction identities (possibly multiple times) to transform it into another integral where the exponents are odd. Then use one of the above two methods as appropriate.

Question 8. □

Find the following integrals.

(a) $\displaystyle\int \sin^8 x \cos^5 x \,\mathrm{d}x$

(b) $\displaystyle\int \frac{\sin^5 x}{\cos^4 x} \,\mathrm{d}x$

1.3.3 Strategy for integrating product of multiple angle trigonometric functions

We can use the angle sum identities from the first page to derive the following formula:

$$\begin{aligned}
\sin(ax)\sin(bx) &= \frac{1}{2}[\cos((a-b)x) - \cos((a+b)x)] \\
\sin(ax)\cos(bx) &= \frac{1}{2}[\sin((a-b)x) + \sin((a+b)x)] \\
\cos(ax)\cos(bx) &= \frac{1}{2}[\cos((a-b)x) + \cos((a+b)x)]
\end{aligned}$$

Note: I will not ask you to memorize these formula. You can come back and refer to them whenever they are needed.

Question 9. □

Find the following integrals.

(a) $\displaystyle\int \sin(9x)\sin(4x)\,\mathrm{d}x$

(b) $\displaystyle\int \sin(2x)\cos(3x)\,\mathrm{d}x$

§1.4 Integrals via Trigonometric Substitution

Recall the following two results from differential calculus:

Theorem 1.4.4

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2}, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

So we can immediately derive the following two integral formula

$$\int \frac{1}{1+x^2}\,\mathrm{d}x = \arctan x + C, \qquad \int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arcsin x + C$$

Now let's say we would like to integrate $\int \frac{1}{\sqrt{9-x^2}}\,\mathrm{d}x$. It looks almost like the above integral but not quite. Here's a strategy:

Example 1.4.5

Draw a right-angled triangle ABC with $\angle ABC = \frac{\pi}{2}$.

Let $AC = 3$ and $AB = x$. Let $\angle ACB = \theta$.

Then $\sin\theta =$

So, written as functions of θ, we can write $x =$ .... and $\mathrm{d}x =$ ....

Now use substitution to replace all of your x's in the original integral in terms of θ. Don't forget about $\mathrm{d}x$. We get,

$$\int \frac{1}{\sqrt{9-x^2}}\,\mathrm{d}x = \int$$

Note: The method employed above is called a trigonometric substitution and is a specific instance of something called *backwards substitution* or *reverse substitution*. The idea is, instead of letting $u = g(x)$ like for u-substitution, we let $x = g(u)$ and $\mathrm{d}x = g'(u)\,\mathrm{d}u$. This makes an integral initially look more complicated:

$$\int f(x)\,\mathrm{d}x = \int f(g(u))g'(u)\,\mathrm{d}u$$

but in particular cases, actually makes the integral simpler due to trigonometric identities!

The method of trigonometric substitution is particularly useful when the integrand contains an expression of the form

$$\sqrt{a^2-x^2}, \quad \sqrt{a^2+x^2}, \quad \text{or } \sqrt{x^2-a^2}$$

Expression	Substitution	Simplification
$\sqrt{a^2-x^2}$ or $\dfrac{1}{\sqrt{a^2-x^2}}$	$x = a\sin\theta$	$\sqrt{a^2-x^2} = a\cos\theta$ $\mathrm{d}x = a\cos\theta\,\mathrm{d}\theta$
$\sqrt{a^2+x^2}$ or $\dfrac{1}{a^2+x^2}$	$x = a\tan\theta$	$\sqrt{a^2+x^2} = a\sec\theta$ $\mathrm{d}x = a\sec^2\theta\,\mathrm{d}\theta$
$\sqrt{x^2-a^2}$ or $\dfrac{1}{\sqrt{x^2-a^2}}$	$x = a\sec\theta$	$\sqrt{x^2-a^2} = a\tan\theta$ $\mathrm{d}x = a\tan\theta\sec\theta\,\mathrm{d}\theta$

Digression

If your integrand has a term of the form $\frac{1}{x^2-a^2}$ or $\frac{1}{a^2-x^2}$, i.e. without the square root as above, it's probably an example of an integration technique called partial fractions. We will not cover it in this course.

Using this technique, we can derive the following integration formula that you may use without proof from now on:

Theorem 1.4.6

- $\displaystyle\int \frac{\mathrm{d}u}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C$
- $\displaystyle\int \frac{\mathrm{d}u}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$

Question 10. □

Derive the above two formula yourself.

Question 11.

☐

Find the following integrals using above substitutions.

(a) $\displaystyle\int \frac{1}{9+25x^2}\,\mathrm{d}x$ *[Hint: Can you rewrite $25x^2$ as u^2?]*

(b) $\displaystyle\int \frac{e^x}{\sqrt{64-e^{2x}}}\,\mathrm{d}x$

(c) $\displaystyle\int \frac{2}{x\sqrt{x^2-25}}\,\mathrm{d}x$

We will work out the steps for this one as an example. Looking at the integrand, it has an expression of the form $\sqrt{x^2-a^2}$. Here the value of $a =$ ....

So, we will substitute $x = a\sec\theta =$ .... in the integrand. Check from the above table that this changes the expression $\sqrt{x^2-a^2}$ to $a\tan\theta =$ .... Don't forget to substitute the extra x outside the square root and the $\mathrm{d}x$ in the numerator. That changes the integrand to:

Now complete the integration:

(d) $\int \sqrt{9-x^2}\,\mathrm{d}x$ *[Hint: You might need the trigonometric identity* $\sin(2\theta) = 2\sin\theta\cos\theta$*.]*

(e) $\int \frac{(1-x^2)^{3/2}}{x^6}\,\mathrm{d}x$

(f) $\int \frac{1}{x^2+4x+5}\,\mathrm{d}x$ *[Hint: Complete the square in the denominator.]*

(g) $\int_0^1 \frac{\arctan x}{1+x^2}\,\mathrm{d}x$

Chapter 2 | Definite Integrals

§2.1 Riemann Sum and Numerical Integration

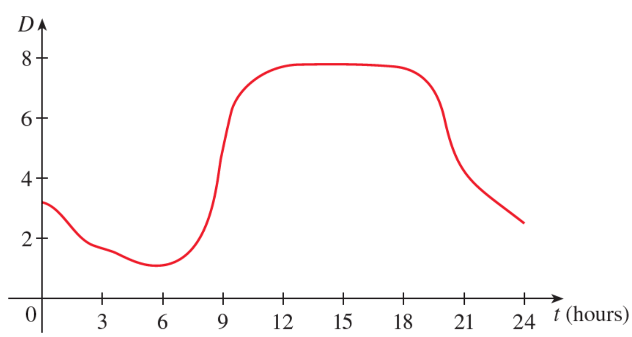
By now, you have learned about the Fundamental Theorem of Calculus and how it relates differentiation to integration. We used that idea to derive a lengthy list of antiderivative formulas and also various integration techniques for you to use. Although it was necessary to go through this process to establish a solid foundation, and hone your algebraic and analytic skills, you will very rarely find these exact formula ever used directly in real-life applications.

Of course, when talking about applications of integrals, we are specifically looking at definite integrals, which can be physically interpreted as the area 'under' a curve. However there are two situations in which it is impossible to find the exact value of a definite integral.

- Firstly, note that in order to evaluate $\int_a^b f(x)\,\mathrm{d}x$ using the Fundamental Theorem of Calculus we need to know an antiderivative of f. But sometimes, it is difficult, or even impossible, to find an antiderivative. For example, it is impossible to evaluate the following integrals exactly:

$$\int_0^1 e^{x^2}\,\mathrm{d}x \qquad \int_{-1}^1 \sqrt{1+x^3}\,\mathrm{d}x \qquad \int_0^\pi \sin(x^2)\,\mathrm{d}x$$

- The second scenario is when the graph of the function is determined from a scientific experiment through instrument readings or collected data. There may be no formula for the function in such a case. For example, suppose the figure below shows a graph of data traffic (in Mbps) against hour on a link, and we wish to know, or at least approximate, the total amount of transmitted data in one day.



In both cases we wish to find the best possible approximation, up to some error tolerance, of the definite integrals. You may have already seen one such method using rectangles. We will start by formally describing the process once more and look at other variations.

2.1.1 The Area Problem

Consider the following problem. We wish to find the are under the function $y = f(x)$ from $x = a$ to $x = b$ over the X-axis. This is the region S is fig. 2.1.

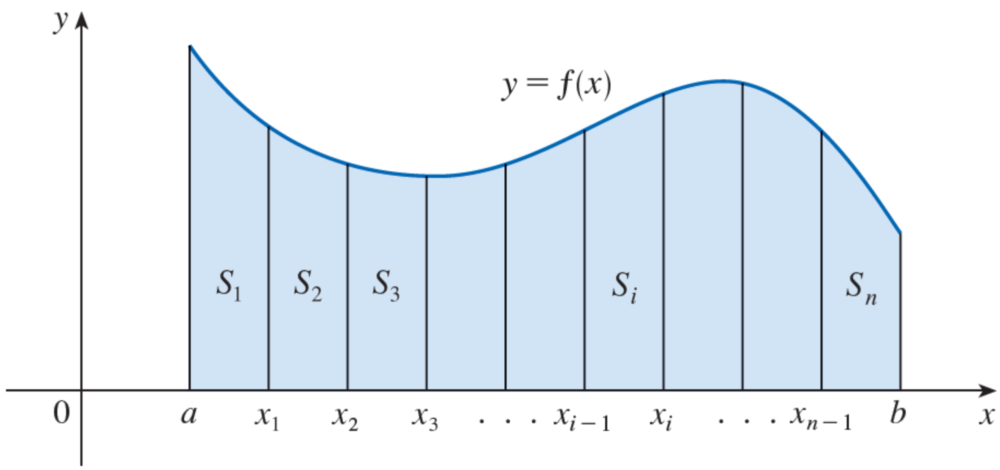


Figure 2.1

Partition and Sample Points

We start by subdividing the region S into n strips $S_1, S_2, S_3, \ldots, S_n$ of equal width as in fig. 2.1. Since the total width of the interval is $b - a$, we find that the width of each of the n strips is $\Delta x =$ ______________ .

These strips divide the interval $[a, b]$ into n subintervals $[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]$ where $x_0 = a$ and $x_n = b$. Check that the strip S_i corresponds to the interval $[x_{i-1}, x_i]$, for $i = 1, 2, 3, \ldots, n$.

Now we are going to try to approximate the are of each strip S_i by a rectangle of width Δx. Intuitively, we should choose the height of the rectangle to be the "average" value of $f(x)$ over the interval $[x_{i-1}, x_i]$.

But recall that we might not know the exact formula of the function f. So, the best we can hope for, is an arbitrary choice of x_i^* in the interval $[x_{i-1}, x_i]$ and take the value of f at that point to be the height of the rectangle (see fig. 2.2).

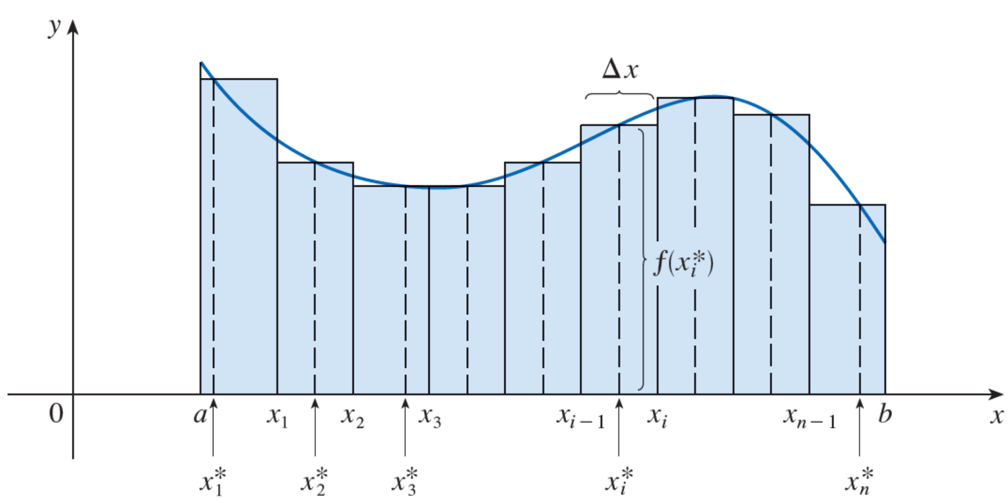


Figure 2.2

Limit of Total Area

What we think of intuitively as the "area" of the region S is thus approximated by the sum of the areas of the rectangles in fig. 2.2, which is

$$A_n^* = f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + \ldots + f(x_n^*)\Delta x$$

Definition 2.1.7

The sum that appears in the above formula, also abbreviated as $\displaystyle\sum_{i=1}^{n} f(x_i^*)\Delta x$, is called a **Riemann sum** after the German mathematician Bernhard Riemann.

Digression

If you are not familiar with the sigma notation for summation, see this wiki link.

Now, as n increases, the subintervals get thinner, and the value of $f(x_i^*)$ becomes closer to the actual average of f over the interval $[x_{i-1}, x_i]$. So it stands to reason that the limiting value of A_n as $n \to \infty$ gives the area A of the region S. In other words,

Definition 2.1.8

If $f(x) \geq 0$ on $[a,b]$, we *define* the **area under the graph of** f **over** $[a,b]$ to be

$$A = \lim_{n\to\infty} A_n^*$$

Note: The choice of x_i^* here is arbitrary, as long as it is in the interval $[x_{i-1}, x_i]$. Here are some choices that are used in practice:

- If $x_i^* = x_{i-1}$, the left end-point, we denote the Riemann sum as L_n.
- If $x_i^* = x_i$, the right end-point, we denote the Riemann sum as R_n.
- If $x_i^* = \dfrac{x_{i-1} + x_i}{2}$, the mid-point, we denote the Riemann sum as M_n.
- If x_i^* is the point where $f(x)$ achieves its maximum in the interval $[x_{i-1}, x_i]$, we call that Riemann sum the "upper sum", denoted $\mathcal{U}_n$.
- If x_i^* is the point where $f(x)$ achieves its minimum in the interval $[x_{i-1}, x_i]$, we call that Riemann sum the "lower sum", denoted $\mathcal{L}_n$.

Look at this applet for an interactive visualization of different Riemann sums.

https://webspace.ship.edu/msrenault/GeoGebraCalculus/integration_riemann_sum.html

Digression

Another approximation, called the Trapezoidal Rule, results from averaging the left-endpoint and the right-endpoint Riemann sums:

$$\begin{aligned} A_n^* &= \frac{1}{2}\left[\sum_{i=1}^{n} f(x_{i-1}^*)\Delta x + \sum_{i=1}^{n} f(x_i^*)\Delta x\right] \\ &= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right] \end{aligned}$$

This essentially replaces the rectangles with trapezoids in each strip to evaluate the area.

Sum to Integral

Now that we have formally defined what "area" means, we are in a position to finally talk about how integration comes into picture.

Definition 2.1.9: Definition of a Definite Integral

If f is a function defined for $a \le x \le b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0(= a), x_1, x_2, \ldots, x_n(= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in these subintervals, so x_i^* lies in the i th subinterval $[x_{i-1}, x_i]$.

Then the **definite integral** of f from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is integrable on $[a, b]$.

Notice the caveat about "all possible choice of sample points". That means when f is integrable,

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n = \lim_{n\to\infty} M_n = \lim_{n\to\infty} \mathcal{U}_n = \lim_{n\to\infty} \mathcal{L}_n$$

Each of the different sampling choices gives a different approximation for the actual integral, depending on how large n is. But as n increases, all choices go towards the same limit.

So the integral is actually a limit! And from differential calculus, you know that limits sometimes don't exist. Fortunately, we have the following theorem:

Theorem 2.1.10

If f is continuous on $[a, b]$ or has only a finite number of jump discontinuities, then f is integrable (i.e. the limit exists) on $[a, b]$.

Digression

On the other hand, it is possible to define functions $f(x)$ such that the limit doesn't exist. For example, the Dirichlet function. This is different from saying that there is no antiderivative formula. Rather we are saying that it is impossible to define $\int f(x)\,\mathrm{d}x$ in any meaningful way and even a computer cannot approximate a value for it. We will see other examples in section 2.2.

Question 12.

(a) *Express each limit as a definite integral:*

(i) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}(4x_i^* - (x_i^*)^2)\,\Delta x$ *over* $[4,7]$

(ii) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}(x_i^*)^2\sin(x_i^*)\,\Delta x$ *over* $[0,\pi]$

(iii) $\displaystyle\lim_{k\to\infty}\sum_{i=1}^{k}\left(3\tan(x_i^*) - \frac{1}{x_i^*}\right)\Delta x$ *over* $\left[\frac{\pi}{6},\frac{\pi}{3}\right]$

(b) *Given* L_n *or* R_n *as indicated, express the limit as* $n\to\infty$ *as a definite integral, identifying the correct interval of integration.*

(i) $\displaystyle\mathrm{R}_n = \frac{1}{n}\sum_{i=1}^{n}\frac{i}{n}$

(ii) $\displaystyle\mathrm{L}_n = \frac{2\pi}{n}\sum_{i=1}^{n}2\pi\frac{i-1}{n}\sin\left(\frac{i-1}{n}\right)$

(iii) $\displaystyle\mathrm{L}_n = \frac{3}{n}\sum_{i=1}^{n}4\left(5+\frac{3(i-1)}{n}\right)\ln\left(5+\frac{3(i-1)}{n}\right)$

§2.2 Improper Integrals

In defining a definite integral $\int_a^b f(x)\,\mathrm{d}x$, we dealt with a function f defined on a **finite** interval $[a,b]$ and we assumed that f does not have an **infinite discontinuity**. In this section we extend the concept of a definite integral to the case where the interval is infinite and also to the case where f has an infinite discontinuity in $[a,b]$. In either case the integral is called an **improper integral**.

2.2.1 Introduction to Infinity

What is Infinity? Wikipedia defines infinity as *"something that is boundless or endless, or else something that is larger than any real or natural number"*. In particular, note that infinity is not a number, it is merely the **opposite of finite.**

So in Mathematical terms, when we write $x \to \infty$, what we means is that x grows larger and larger without bound, but never really approaches anything. On the other hand, writing something like

$$\lim_{x\to\infty} f(x) = \mathrm{L}$$

makes perfect sense, because $f(x)$ can *'approach'* a finite value L even when x gets larger and larger. In other words, this is the same thing as saying the line $y = \mathrm{L}$ is a horizontal asymptote for the graph of $f(x)$.

For the rest of this semester, we will keep dealing with infinity a lot. Let's see an example on how they show up in applications of integrals, specifically as probability distributions.

Example 2.2.11

A company with a large customer base has a call center that receives thousands of calls a day. After studying the data that represents how long callers wait for assistance, they find that the function $p(t) = 0.25e^{-0.25t}$ models the time customers wait in the following way: the **fraction** of customers who wait between $t = a$ and $t = b$ minutes is given by

$$\int_a^b p(t)\,\mathrm{d}t$$

Use this information to answer the following questions.

(a) Determine the fraction of callers who wait between 5 and 10 minutes.

....

(b) Determine the fraction of callers who wait between 10 and 20 minutes.

....

(c) Next, let's study the fraction who wait up to a certain number of minutes:

(i) What is the fraction of callers who wait between 0 and 5 minutes?

....

(ii) What is the fraction of callers who wait between 0 and 10 minutes?

....

(d) Let $F(b)$ represent the fraction of callers who wait between 0 and b minutes. Find a formula for $F(b)$ that involves a definite integral and evaluate it.

(e) What is the value of the limit $\lim_{b\to\infty} F(b)$? What is its meaning in the context of the problem?

Question 13.

Consider the function $g(x) = \dfrac{1}{x^2}$.

(a) Let $b > 1$. *What is the area under* $g(x)$ *on the interval* $[1, b]$? *Write down the corresponding definite integral and draw a diagram.*

(b) Determine the area under $g(x)$ *for* $b = 10, 100$, *and* 1000.

(c) As $b \to \infty$, *what is happening to the values* $\int_1^b g(x)\,dx$? *To answer this, evaluate the exact value of the limit*

$$\lim_{b\to\infty} \int_1^b g(x)\,dx$$

2.2.2 Improper Integrals of Type I

Suppose we have a function $f(x)$ that is continuous on the interval $[a, \infty)$. Then we can denote the area under the graph of $f(x)$ from a to infinity as $\int_a^\infty f(x)\,dx$ and define it as

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

Warning: Suppose the antiderivative of $f(x)$ is $F(x)$. Then it is tempting to write something like

$$\int_a^\infty f(x)\,dx = F(\infty) - F(a)$$

But remember that ∞ is not a number, it is a different kind of object. So $F(\infty)$ doesn't make mathematical sense. We should instead interpret it as $\lim_{t\to\infty} F(t)$.

Depending on the direction of the limit, we can similarly define the following improper integrals.

Definition 2.2.12

(a) Let $f(x)$ be continuous over an interval of the form $[a, +\infty)$. Then

$$\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty} \int_a^t f(x)dx$$

provided this limit exists.

(b) Let $f(x)$ be continuous over an interval of the form $(-\infty, b]$. Then

$$\int_{-\infty}^{b} f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$$

provided this limit exists.

In each case, if the limit exists, then the improper integral is said to **converge**. If the limit does not exist, then the improper integral is said to **diverge**.

(c) Let $f(x)$ be continuous over $(-\infty, +\infty)$. Then

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^{+\infty} f(x)dx$$

provided that $\int_{-\infty}^{a} f(x)dx$ and $\int_a^{+\infty} f(x)dx$ both converge. If either one or both of these two integrals diverge, then $\int_{-\infty}^{+\infty} f(x)dx$ diverges. (It can be shown that the choice of a is arbitrary).

■ Question 14.

This exercise will be of vital importance later on. So make sure to complete it carefully.

Suppose $p > 0$ is some real number. Consider three cases: $p < 1$, $p = 1$, and $p > 1$. In each case,

(a) *Evaluate the integral* $\int_1^{\infty} \frac{1}{x^p}\,dx$.

(b) *Does your integral converge or diverge?*

■ Question 15.

Evaluate the following improper integrals:

(a) $\int_{-\infty}^{0} xe^x\,dx$.

(b) $\int_1^{\infty} \frac{1}{x[1+(\ln x)^2]}\,dx$.

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{e^x}{1+e^x}\,\mathrm{d}x.$ *(d)* $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x.$

■ Question 16. □

To see why improper integrals over $(-\infty,\infty)$ are defined the way they are, evaluate the following

$$\lim_{t\to\infty}\int_{-t}^{t} \frac{1}{1+x^2}\,\mathrm{d}x.$$

How does your answer compare to that in the previous question?

■ Question 17. □

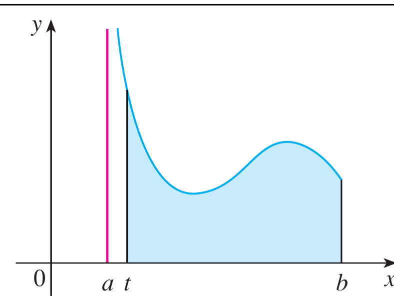
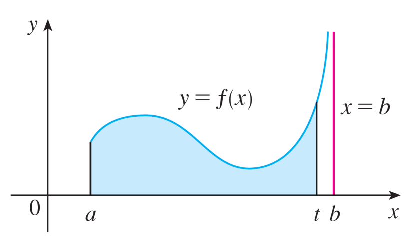
Here is another example to see why improper integrals over $(-\infty,\infty)$ are defined as they are.

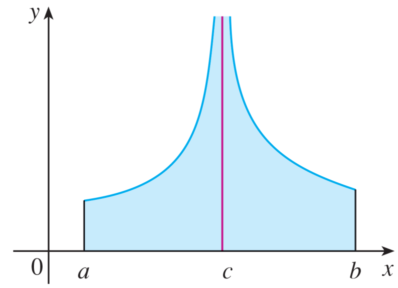
(a) *Evaluate* $\displaystyle\lim_{t\to\infty}\int_{-t}^{t} x\,\mathrm{d}x.$

(b) *Compare the previous answer with the definition for* $\displaystyle\int_{-\infty}^{\infty} x\,\mathrm{d}x.$

2.2.3 Improper Integrals of Type II

Suppose that $f(x)$ is a positive function that is continuous on a closed interval $[a,b]$ except at one point where it has an infinite discontinuity. There are three situations we can examine where a continuous function might be integrated over an interval containing such a discontinuity: discontinuous at a, discontinuous at b, or discontinuous at some point c between a and b.

<table>
<tr><th>Improper Integral</th><th>Definition</th><th>Picture</th></tr>
<tr><td>$\displaystyle\int_a^b f(x)\,\mathrm{d}x$</td><td>$\displaystyle\lim_{t\to a^+}\int_t^b f(x)\,\mathrm{d}x$</td><td>
</td></tr>
<tr><td>$\displaystyle\int_a^b f(x)\,\mathrm{d}x$</td><td>$\displaystyle\lim_{t\to b^-}\int_a^t f(x)\,\mathrm{d}x$</td><td>
</td></tr>
</table>

Improper Integral	Definition	Picture
$\displaystyle\int_a^b f(x)\,dx$	$\displaystyle\lim_{t\to c^-}\int_a^t f(x)\,dx + \lim_{t\to c^+}\int_t^b f(x)\,dx$	

Question 18.

For each of the following, explain why the integral is improper. Then determine whether it converges or diverges. If it does converge, find its value.

(a) $\displaystyle\int_0^1 x\ln x\,dx$

(b) $\displaystyle\int_0^1 \ln x\,dx$

(c) $\displaystyle\int_{-1}^1 \frac{1}{\sqrt[3]{x^4}}\,dx$

(d) $\displaystyle\int_0^{\pi/6} \frac{\cos x}{\sqrt{1-2\sin x}}\,dx$

(e) $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$

(f) $\displaystyle\int_0^3 \frac{1}{x-1}\,dx$ *[WARNING: The answer is not* $\ln(2)$.*]*

L'Hôpital's Rule. You may need to use L'Hôpital's Rule to calculate some of the limits above. Here is a quick recap in case you forgot about it.

Suppose you wish to calculate $\lim_{x\to c}\frac{f(x)}{g(x)}$, where the limit has an indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then we can compute the limit using a ratio of derivatives:

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}.$$

§2.3 Comparison of Improper Integrals

As we saw in earlier sections, sometimes it is difficult to find the exact value of an improper integral by anti-differentiation; and yet it might be important to know whether an integral is convergent or divergent. One of the ways to do so is by **comparing** the given integral to one whose behavior we already know. Before writing the theorem, let's take a look at two examples. Use DESMOS (or your preferred graphing/calculating device) as required and draw your own conclusions.

Example 2.3.13

Consider the improper integral

$$\int_3^\infty \frac{1}{x^2 \ln x}\,dx$$

(a) First use DESMOS[a] to find approximate numerical values of

(i) $\displaystyle\int_3^{100} \frac{1}{x^2 \ln x}\,dx$ (ii) $\displaystyle\int_3^{10000} \frac{1}{x^2 \ln x}\,dx$ (iii) $\displaystyle\int_3^{1000000} \frac{1}{x^2 \ln x}\,dx$

(b) Based on the numerical values, Which of the following statements is closest to what you can conclude regarding the convergence or divergence of the improper integral?

(i) Based on numerical evidence, the improper integral converges.

(ii) Based on numerical evidence, the improper integral diverges.

(iii) Based on numerical evidence, it appears the improper integral converges.

(iv) Based on numerical evidence, it appears the improper integral diverges.

(c) Now draw the graph of $f(x) = \dfrac{1}{x^2 \ln x}$ and DESMOS $g(x) = \dfrac{1}{x^2}$ together and compare the two functions.

(i) Do they intersect? Where and how many times?

(ii) Which function has bigger value?

(d) Using the two graphs you just made,

(i) Do you think $\displaystyle\int_3^\infty \frac{1}{x^2 \ln x}\,dx \leq \int_3^\infty \frac{1}{x^2}\,dx$? Why or why not?

(ii) Do you know whether $\displaystyle\int_3^\infty \frac{1}{x^2}\,dx$ converges or diverges?

(iii) What does this suggest about the convergence/divergence of the two integrals? Explain.

[a] Go here https://www.desmos.com/calculator/2efrlvbdwv if you are feeling lazy!

■ Question 19. □

Would your conclusion change if we consider the improper integral $\displaystyle\int_2^\infty \frac{1}{x^2 \ln x}\,dx$ *instead?*

Question 20.

Repeat the process of above example with the improper integral

$$\int_3^\infty \frac{\ln x}{\sqrt{x}}\,\mathrm{d}x$$

This time, compare it with $\displaystyle\int_3^\infty \frac{1}{\sqrt{x}}\,\mathrm{d}x$ *and explain what you can conclude about its convergence.*

We summarize the conclusion of the above exercises as follows:

Theorem 2.3.14: Comparison Theorem

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x$ is convergent, then $\displaystyle\int_a^\infty g(x)\,\mathrm{d}x$ is convergent.

In fact, if $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x = \mathrm{L}$ and $\displaystyle\int_a^\infty g(x)\,\mathrm{d}x = \mathrm{M}$, then $\mathrm{M} \leq \mathrm{L}$.

(b) If $\displaystyle\int_a^\infty g(x)\,\mathrm{d}x$ is divergent, then $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x$ is divergent.

In other words, if $\displaystyle\int_a^\infty g(x)\,\mathrm{d}x = +\infty$ then $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x = +\infty$.

Note: Finding whether an improper integral is convergent or not using the comparison test involves two stages:

- Guess, by looking at the behavior of the integrand for large x, whether the integral converges or not. (This is to decide whether to test for (a) or for (b)).
- Confirm the guess by comparing with a second **positive** function.

Warning: Note that the reverse is not necessarily true. If $\displaystyle\int_a^\infty g(x)\,\mathrm{d}x$ is convergent, $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x$ may or may not be convergent, and if $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x$ is divergent, $\displaystyle\int_a^\infty g(x)\,\mathrm{d}x$ may or may not be divergent.

In the above example and exercise, we investigated the convergence of an integral by comparing it with an easier integral. How did we pick the easier integral? This is a matter of trial and error, guided by any information we get by looking at the original integrand as $x \to \infty$. We want the comparison integrand to be easy and, in particular, to have a simple antiderivative.

Here are two useful integrals for comparison.

Theorem 2.3.15

Let a be a positive real number. Then

(a) *The integral* $\displaystyle\int_a^\infty \frac{1}{x^p}\,\mathrm{d}x$ *converges if* $p > 1$ *but diverges if* $p \le 1$.

(b) *The integral* $\displaystyle\int_a^\infty \frac{1}{\mathrm{e}^{mx}}\,\mathrm{d}x$ *converges if* $m > 0$ *but diverges if* $m \le 0$.

Question 21.

For each of the following integrals, draw the integrand function in DESMOS, *find a suitable second easier integrand to compare it to, and use the inequality to find whether the improper integral is convergent or divergent.*

(a) $\displaystyle\int_1^\infty \frac{\cos^2 x}{x^2}\,\mathrm{d}x$

(b) $\displaystyle\int_1^\infty \frac{1+\sin^4(3x)}{\sqrt{x}}\,\mathrm{d}x$

(c) $\displaystyle\int_2^\infty \frac{1}{x+e^x}\,\mathrm{d}x$

(d) $\displaystyle\int_2^\infty \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x$

(e) $\displaystyle\int_2^\infty e^{-x^2}\,\mathrm{d}x$

(f) $\displaystyle\int_2^\infty \frac{3}{\sqrt{x^3+x}}\,\mathrm{d}x$

(g) $\displaystyle\int_2^\infty \frac{5+2\sin x}{x^2+2}\,\mathrm{d}x$

(h) $\displaystyle\int_2^\infty \frac{5+2\sin x}{x-2}\,\mathrm{d}x$

Chapter 3 | Introduction to Sequences and Series

§3.1 What is a Sequence?

A **sequence** can be defined as an infinite set of real numbers written in a definite order. There is a more precise definition using functions, we will come to that later.

We can write denote a sequence of real numbers as follows

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

Here a_1 is the first term in the sequence, a_2 is the second etc. and more generally, a_n, the term with **index** n, is called the nth term. We also use the notation

$$\{a_n\}_{n=1}^{\infty}$$

to denote above sequence.

Sequences can be described in a couple of different ways. We can provide an **explicit** "closed-form" formula for the nth term.

Example 3.1.16

Some examples are

(a) $a_n = \dfrac{1}{n}$ for all n. This is the sequence $1, 1/2, 1/3, 1/4, \ldots$

(b) $a_n = \dfrac{n}{n+1}$ for all n. This is the sequence $1/2, 2/3, 3/4, 4/5, \ldots$

Question 22. □

Write down the first 5 *terms of the sequence defined as*

(a) $\{a_n\} = \left\{\dfrac{(-1)^n}{n}\right\}$.

(b) $\{a_n\} = \left\{\cos\left(\dfrac{n\pi}{6}\right)\right\}$

Alternately, we can define a sequence **recursively**. This means, we define the nth term of the sequence using the term(s) before it.

Example 3.1.17

Some examples are

(a) $a_n = \dfrac{a_{n-1}}{2}$ for all $n > 1$ and $a_1 = 1$. This is the sequence $1, 1/2, 1/4, 1/8, \ldots$

(b) $a_1 = 0$ and $a_n = 3a_{n-1} + 4$ for all $n > 1$. This is the sequence $0, 4, 16, 52, \ldots$

(c) The **Fibonacci Sequence** is one of the famous examples of integer sequences. It is defined as:

$$F_1 = F_2 = 1, \quad F_n = F_{n-1} + F_{n-2} \text{ for all } n \geq 3$$

This is the sequence $1, 1, 2, 3, 5, 8, 13, \ldots$.

Note: Observe that we need two pieces of information to define a sequence this way. We need the defining relation and we need the starting points.

Question 23.

Give the first 6 terms of the following sequences and then guess a closed form formula for the nth term. You don't have to prove the formula.

(a) $a_1 = 1, \quad a_2 = 3, \quad a_{n+1} = 2a_n - a_{n-1} \quad$ *for* $n \geq 2$.

(b) $a_1 = 1, \quad a_2 = 3, \quad a_{n+1} = 3a_n - 2a_{n-1} \quad$ *for* $n \geq 2$.

Finally, there are some sequences which don't have a simple defining equation.

Example 3.1.18

Some examples are,

(a) $a_n =$ the nth prime number.

(b) $a_n =$ the nth digit of π.

3.1.1 Arithmetic and Geometric Progressions

An **Arithmetic Progression** (AP) or arithmetic sequence is a sequence of numbers where the **difference** between the consecutive terms is constant. For example,

$$4, 9, 14, 19, 24, 29, \ldots$$

If we denote this sequence by $\{A_n\}$, we can define it recursively by $A_1 = 4$ and $A_n = A_{n-1} + 5$ for $n \geq 2$.

Question 24.

Find an explicit formula for the sequence.

In general, to define an AP, we need two information: the starting value, usually denoted $A_1 = a$; and we need the common difference, usually denoted d. Then the formula for the nth term is given by

$A_n =$

A sequence where the **ratio** of consecutive terms is constant, is called an **Geometric Progression** (GP) or geometric series. For example,

$$2, 4, 8, 16, 32, \ldots$$

$$\frac{-3}{7}, \frac{3}{4}, \frac{-21}{16}, \frac{147}{64}, \ldots$$

Question 25.

Identify the common ratio in the two examples above.

Similar to an AP, we need two pieces of information to define an GP $\{G_n\}$. We need the starting value $G_1 = a$; and we need the common ratio, call it r. Then we have the following formula:

$G_n =$

3.1.2 Limit of a Sequence

A sequence such as $a_n = \dfrac{n}{n+1}$, can be pictured by plotting its graph, as in fig. 3.1.

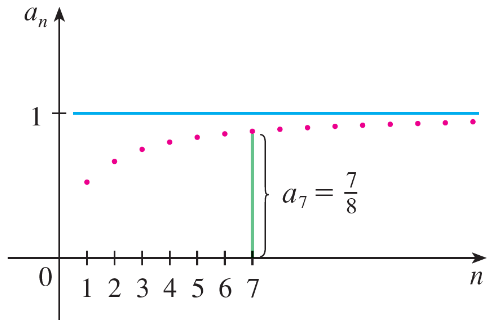


Figure 3.1

Here we are treating a sequence a_n as a function $f(n)$ whose domain is the set of positive integers, range is all real numbers, and its graph consists of isolated points with coordinates

$$(1,a_1) \qquad (2,a_2) \qquad (3,a_3) \qquad \ldots \qquad (n,a_n) \qquad \ldots$$

From fig. 3.1, it appears that the terms of the sequence $a_n = \dfrac{n}{n+1}$ are approaching 1 as n becomes large. In fact, the difference

$$1 - \frac{n}{n+1} = \frac{1}{n+1}$$

can be made as small as we like by taking n sufficiently large. We indicate this by writing

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

Thus the idea of limit of a sequence is the same exact idea as discussed in Calculus I regarding the limit of a function. For a function $y = f(x)$, we said that $\lim\limits_{x\to c} f(x) = L$ if the function values $f(x)$ gets arbitrarily close to L as x approaches c. We can say the same thing for sequences:

Definition 3.1.19

A sequence $\{a_n\}$ has the limit L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim\limits_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is convergent). Otherwise, we say the sequence **diverges** (or is divergent).

Question 26. □

For the four examples given below (available HERE in DESMOS*) make a guess if the sequence converges or diverges using the graphs.*

(i) $a_n = \dfrac{(-1)^n n}{n+1}$

(ii) $b_n = \dfrac{3^n}{3n+1}$

(iii) $c_n = \dfrac{(-1)^n 2\ln(n)}{n}$

(iv) $d_n = \left(\dfrac{2}{3}\right)^n + \left(\dfrac{1}{4}\right)^n$

Of course, the graphs are not enough as proof of convergence. We should be able to find the limits of these sequences algebraically or analytically to confirm our guesses. Fortunately, with the idea that a sequence is just a function defined on the set of Natural numbers, the usual limit theorems all apply.

Example 3.1.20

Suppose $a_n = \dfrac{n^2+3n+1}{3n^2+2}$. Then the sequence is defined using the function $f(x) = \dfrac{x^2+3x+1}{3x^2+2}$, restricted to only natural numbers. We can use L'Hôpital's rule to easily find that $\lim_{x\to\infty} f(x) = \dfrac{1}{3}$. Then we can also conclude that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = \frac{1}{3}.$$

In fact, we have the following theorem;

Theorem 3.1.21

Consider a sequence $\{a_n\}$ such that $a_n = f(n)$ for all $n \geq 1$. If there exists a real number L such that

$$\lim_{x\to\infty} f(x) = \mathrm{L}$$

then $\{a_n\}$ converges and

$$\lim_{n\to\infty} a_n = \mathrm{L}.$$

Question 27.

For each of the following sequences, decide whether they converge or diverge.

(i) $a_n = \dfrac{1-e^{-n}}{1+e^{-n}}$

(ii) $a_n = \dfrac{\ln(n)}{\ln(2n)}$

(iii) $a_n = \sin(n\pi)$

(iv) $a_n = \sin(\pi/n)$

(v) $a_n = n^{1/n}$

(vi) $a_n = \dfrac{5n^2+1}{e^n}$

Example 3.1.22

We can use our knowledge of exponential functions to determine the convergence of some geometric sequences. Let $a_n = r^n$ where r is a real number. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } 0 \leq r < 1 \\ 1 & \text{if } r = 1 \\ \text{undefined} & \text{if } r > 1 \end{cases}$$

Question 28.

Consider the sequence from question 5.(iv) above. Can you use limit laws to determine the limit?

The traditional limit laws all hold for sequences as well:

Theorem 3.1.23

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then:

- $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty}(a_n) \pm \lim_{n\to\infty}(b_n)$
- $\lim_{n\to\infty} c = c \quad , \quad \lim_{n\to\infty} c\cdot a_n = c\cdot \lim_{n\to\infty} a_n$
- $\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty}(a_n)\cdot \lim_{n\to\infty}(b_n)$
- $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty}(a_n)}{\lim_{n\to\infty}(b_n)}$, *provided* $\lim_{n\to\infty} b_n \neq 0$
- $\lim_{n\to\infty}(a_n)^p = \left[\lim_{n\to\infty}(a_n)\right]^p$ *if* $p > 0$ *and* $a_n > 0$.

Warning: It's not always possible to think about a sequence as a function defined on all of $\mathbb{R}$! For example: $a_n = \frac{(-1)^n}{2^n+1}$ or $a_n = \frac{n^5}{n!}$. In such cases, often we might need to use the squeeze theorem.

The Squeeze Theorem for sequences can be formulated as follows.

Theorem 3.1.24: The Squeeze Theorem

If $a_n \le b_n \le c_n$ for $n \geqslant n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Question 29.

Use the squeeze theorem to find the limit of the sequence from question 5.(iii) above. □

Question 30.

Use the squeeze theorem to show that if $\lim_{n\to\infty}|a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$. □

We end this discussion with a couple of definitions.

Definition 3.1.25

A sequence $\{a_n\}_{n=1}^{\infty}$ is called

increasing	if $a_{n+1} \ge a_n$
decreasing	if $a_{n+1} \ge a_n$
monotone	if it is either increasing or decreasing throughout.
bounded above	if there is a number M such that $a_n \le M$ for all $n \ge 1$
bounded below	if there is a number m such that $a_n \ge m$ for all $n \ge 1$
bounded	if it is bounded both above and below.

■ Question 31. □

Prove that if $\lim_{n\to\infty} a_n = 0$ *and* $\{b_n\}$ *is a bounded sequence (not necessarily convergent), then* $\lim_{n\to\infty}(a_n b_n) = 0$.

We will mention a final theorem without proof for the sake of completion. It is pretty straightforward to verify the statement by drawing a graph. Although we will not prove it here, it will be heavily used later to prove convergence of sequences.

Theorem 3.1.26: The Monotone Convergence Theorem

A monotone and bounded sequence is convergent.

So we can see that there are two ways a sequence can diverge:

- Either the sequence is unbounded. For example, $a_n = 2^n$.
- Or the sequence is bounded, but oscillating. For example, $a_n = (-1)^n$.

Here is an optional exercise to demonstrate an application of theorem 26, if you know how to do proof by induction.

■ Question 32. □

A sequence $\{a_n\}$ *is given by* $a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2a_n}$.

(a) *By induction or otherwise, argue that* $\{a_n\}$ *is increasing and bounded above by 2. Apply the monotone convergence Theorem to show that* $\lim_{n\to\infty} a_n$ *exists.*

(b) *Find* $\lim_{n\to\infty} a_n$.

We will see more applications of theorem 26 in the next section.

We will end with some more practice problems that use techniques from this section.

■ Question 33. □

Find the limit of each sequence. If the sequence diverges, say so.

(a) $a_n = \sqrt{n+1} - \sqrt{n}$

(b) $a_n = \dfrac{(\ln n)^2}{n+1}$

(c) $a_n = \dfrac{\sin n}{n^{\frac{1}{3}} + 1}$

(d) $a_n = \dfrac{e^{\frac{1}{n}}}{\cos\left(\frac{2}{n}\right)}$

§3.2 What is a Series?

What do we mean when we express a number as an infinite decimal? For instance, what does it mean to write
$$\pi = 3.14159265358979323846264338327950288\ldots?$$

The convention behind our decimal notation is that any number can be written as an infinite sum. Here it means that
$$\pi = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \frac{6}{10^7} + \frac{5}{10^8} + \cdots$$
where the three dots $(\cdots)$ indicate that the sum continues forever, and the more terms we add, the closer we get to the actual value of π.

In general, if we try to add the terms of an infinite sequence $\{a_i\}_{i=1}^{\infty}$ we get an expression of the form
$$a_1 + a_2 + a_3 + \cdots + a_i + \cdots$$
which is called an infinite series (or just a series) and is denoted, for short, by the symbol
$$\sum_{i=1}^{\infty} a_i \quad \text{or} \quad \sum a_i$$

But does it always make sense to talk about the sum of infinitely many terms? What about
$$1 + 2 + 3 + 4 + 5 + \ldots$$

Clearly that sum is infinite. In other words, the series $\sum_{i=1}^{\infty} i$ diverges. But how about the infinite sum
$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

When we plot it in DESMOS (see here), we can see that the sum converges to 1.

Warning: Be sure to distinguish between the convergence of a sequence vs. the series. The two words do not mean the same thing in Calculus. For example, the sequence $\frac{1}{2^i} \to 0$, whereas the series $\sum \frac{1}{2^i} \to 1$.

Question 34. □

Check in DESMOS *what happens when you change the series to*

(i) $\sum \frac{1}{i}$ (ii) $\sum \frac{(-1)^i}{i}$ (iii) $\sum (-1)^i$

Question 35. □

Let $s_n = \frac{n}{n+1}$. *Investigate the convergence of the sequence* $\{s_i\}_{i=1}^{\infty}$ *vs. the series* $\sum_{i=1}^{\infty} s_i$.

3.2.1 Definitions

First of all, notice that to define a series, we need to start with a sequence $\{a_n\}$. Now, consider a second sequence $\{s_n\}$ constructed from the original sequence as follows:

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ s_4 &= a_1 + a_2 + a_3 + a_4 \end{aligned}$$

and, in general,

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

This is called the sequence of partial sums.

Definition 3.2.27

We define the **infinite series** $\sum_{i=1}^{\infty} a_i$ to be the limit of the sequence $\{s_n\}_{n=1}^{\infty}$ of **partial sums**, where

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

In other words,

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

The limit, if it exists, is called the **sum** of the series and we say that the series **converges**. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

Example 3.2.28

Consider the sequence $a_n = \dfrac{1}{n(n+1)}$ for $n = 1, 2, \ldots$. The first couple of terms in this sequence are

$$\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \cdots$$

We would like to know: does the series $\sum_{i=1}^{\infty} a_i$ converge?

Observe that $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$. So the sum of the first n terms of the series is

$$s_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{i=1}^{n} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$$

Then the sum of the series is the limit of the sequence $\{s_n\}$:

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) = 1$$

Hence the series converges.

Next, we will consider two examples where the series diverges.

Example 3.2.29

Consider the series $\displaystyle\sum_{i=1}^{\infty}\frac{1}{\sqrt{i}}$. We can write

$$\begin{aligned}
s_1 &= \frac{1}{\sqrt{1}}\\
s_2 &= \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}\\
s_3 &= \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}\\
s_n &= \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}\\
&> \underbrace{\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n}}+\ldots+\frac{1}{\sqrt{n}}}_{n \text{ terms in total}}\\
&= \frac{n}{\sqrt{n}} = \sqrt{n}
\end{aligned}$$

So clearly the sequence s_n is increasing and hence, unbounded. Thus $\lim_{n\to\infty} s_n$ does not exist.

Example 3.2.30

Consider the harmonic series $\displaystyle\sum_{i=1}^{\infty}\frac{1}{i}$. We are going to show that it is divergent. Observe that we can write

$$\begin{aligned}
s_2 &= 1+\frac{1}{2}\\
s_4 &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right) > 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right) = 1+\frac{1}{2}+\frac{1}{2} = 1+\frac{2}{2}\\
s_8 &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) > 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right) = 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} = 1+\frac{3}{2}\\
s_{16} &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\ldots+\frac{1}{8}\right)+\left(\frac{1}{9}+\ldots+\frac{1}{16}\right)\\
&> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\cdots+\frac{1}{8}\right)+\left(\frac{1}{16}+\cdots+\frac{1}{16}\right)\\
&= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} = 1+\frac{4}{2}
\end{aligned}$$

So in general, $s_{2^n} > 1 + \frac{n}{2}$. This shows that $s_n \to \infty$, because it's unbounded! Therefore the harmonic series diverges.

3.2.2 Geometric Series

One of the most important examples of an infinite series is the geometric series that arises from a geometric sequence:

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + \cdots = \sum_{i=1}^{\infty} ar^{i-1}, \quad a \neq 0$$

Question 36. ☐

Let $s_n = n^{th}$ be the nth partial sum of above geometric series. Then $s_n =$ ....

$\implies rs_n =$ ....

$\implies rs_n - s_n =$ ....

(a) *Show that if $r \neq 1$, the partial sum sequence $\{s_n\}$ has the formula $s_n = \dfrac{a(1-r^n)}{1-r}$.*

(b) *For what values of r does the limit $\lim_{n\to\infty} s_n$ exist?*

Theorem 3.2.31

The geometric series

$$\sum_{i=1}^{\infty} ar^{i-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \underline{\qquad\qquad} \quad \text{for } |r| < 1$$

If $|r| \geqslant 1$, the geometric series is divergent.

Question 37. ☐

Which of the following series are geometric? For those that are, identify a and r. Which of the geometric series converge, and to what value?

(i) $\displaystyle\sum_{i=1}^{\infty} \frac{1}{2^i}$ (ii) $\displaystyle\sum_{i=1}^{\infty} \frac{2^{2i}}{3^{i-1}}$ (iii) $\displaystyle\sum_{i=1}^{\infty} 2(i-1)^3$ (iv) $\displaystyle\sum_{i=1}^{\infty} 3^{i+1}\frac{1}{4^i}$

Question 38. ☐

Write $0.9 + 0.09 + 0.009 + 0.0009 + \ldots$ as a geometric series and find the sum.

Question 39. ☐

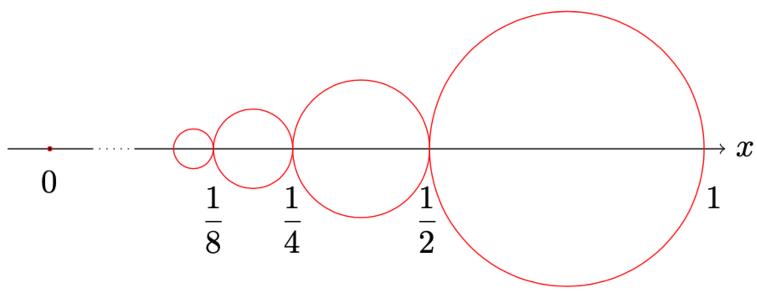
Find a way to write the repeating decimal $0.\overline{123}$ as a geometric series and determine what rational number it is equal to.

3.2.3 Examples of Geometric Series

Next, we will explore several different geometric series by means of geometry, lending some credence to their name!

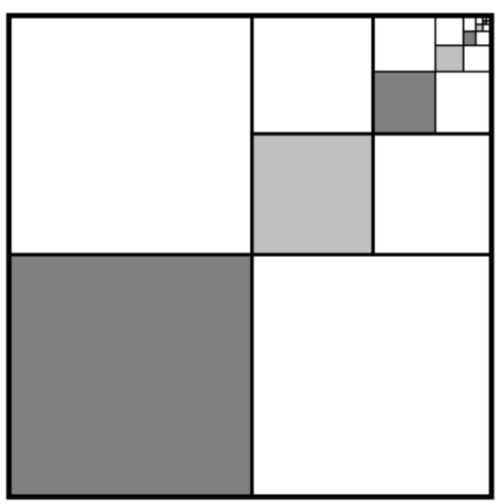
■ Question 40.

Find the total area of the infinitely many circles on the interval $[0, 1]$ *pictured below by determining a geometric series that represents the area.*



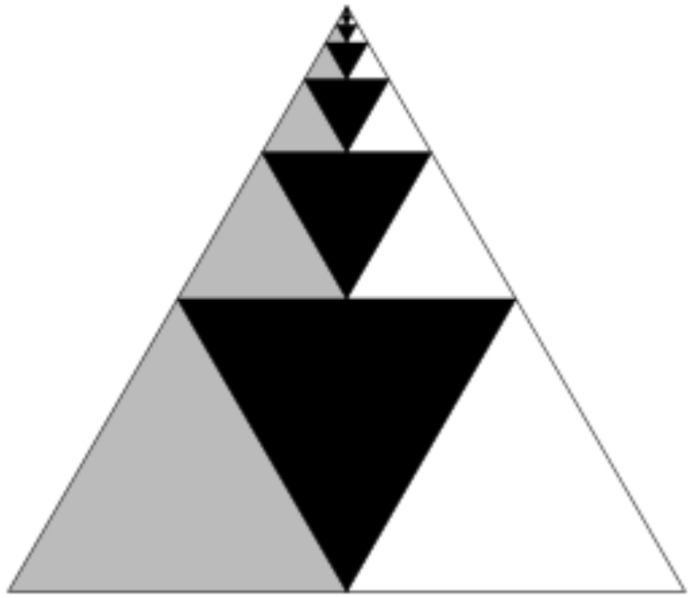
■ Question 41.

Find the total area of the shaded region inside the given unit square (i.e. the sides of the square are length 1*).*



■ Question 42.

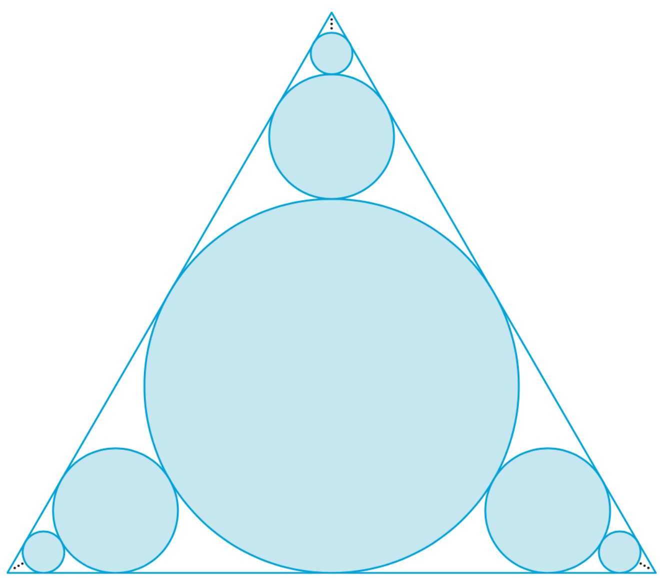
Suppose we take an equilateral triangle and divide it into four smaller triangles of equal area (suppose the area of the whole triangle is 1*). If we continue this process for forever, we get something like what is sketched below. What is the total area outside of the black region?*



■ Question 43.

□

In the figure below there are infinitely many circles approaching the vertices of an equilateral triangle, each circle touching other circles and sides of the triangle. If the triangle has sides of length 1*, find the total area occupied by the circles.*



Chapter 4 | Convergence and Divergence Tests for Series

In the previous chapter, we determined the convergence or divergence of several series by explicitly calculating the limit of the sequence of partial sums s_n. In practice, explicitly calculating this limit can be difficult or impossible. Luckily, several tests exist that allow us to determine convergence or divergence for many types of series.

§4.1 Divergence Test

Let's start with the following claim.

Theorem 4.1.32

If a series $\sum_{i=1}^{\infty} a_i$ *is convergent, then* $\lim_{i\to\infty} a_i = 0$.

Question 44.

Let's prove the theorem. If $\sum_{i=1}^{\infty} a_i$ *is convergent, then* $\lim_{n\to\infty} s_n = L$ *exists. Now write*

$$a_n = s_n - s_{n-1}$$

Does that make sense? Why? Now what can you say about $\lim_{n\to\infty} a_n$*?*

Warning: This theorem does not say that if $\lim_{i\to\infty} a_i = 0$, then $\sum_{i=1}^{\infty} a_i$ is convergent. Harmonic series is a clear counterexample.

For example, the following two statements are not equivalent.

"If I'm not in California, then I'm not in Los Angeles." ⟵ True

"If I'm in California, then I'm in Los Angeles." ⟵ False

So we have the first test for convergence.

Theorem 4.1.33: Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$, *then* $\sum_{i=1}^{\infty} a_i$ *is divergent.*

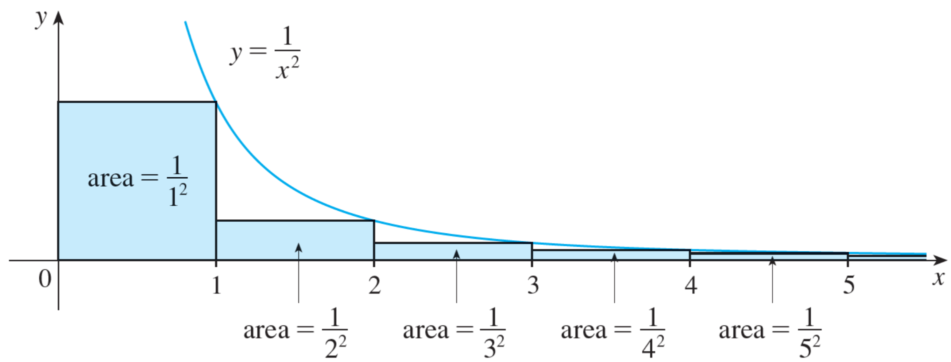
Question 45.

Show that the series $\sum_{i=1}^{\infty} \frac{i^2}{3i^2 - 6i + 2}$ *is divergent.*

§4.2 Integral Test for Positive Series

Consider the series $\sum_{i=1}^{\infty} \frac{1}{i^2}$. There is no simple formula for s_n, but if you were to use DESMOS to evaluate some of the terms, the sequence s_n would seem to converge towards approximately 1.64... So how do we prove that the series converges?

We will use a test that will involve improper integrals. First note that that there is a clear relationship between this series and the improper integral $\int_1^{\infty} \frac{1}{x^2}\,dx$.



Using the above figure, we conclude that

$$s_n = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} < 1 + \int_1^{n+1} \frac{1}{x^2}\,dx$$

Now we make two crucial obervations:

- We know $\lim_{n\to\infty} \int_1^{n+1} \frac{1}{x^2}\,dx = \int_1^{\infty} \frac{1}{x^2}\,dx = 1$. So the sequence of partial sums for $\sum_{i=1}^{\infty} \frac{1}{i^2}$ is bounded above by $1 + 1 = 2$.
- Additionally, the sequence s_n is clearly increasing since the term $\frac{1}{n^2}$ is always positive.

Hence, using the monotone convergence theorem, we conclude that the series converges.

> **Note:** We could have used the upper Riemann sum to set up an upper bound for the integral as follows:
>
> $$\int_1^{n+1} \frac{1}{x^2}\,dx < 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2}$$
>
> which implies that $\sum_{i=1}^{\infty} \frac{1}{i^2} > 1$ which is not really an useful information. Such a comparison might have been useful if we wanted to test for divergence.

Combining these observations, we get the following test for convergence.

Theorem 4.2.34: Integral Test

Suppose f is a

- *continuous*
- *positive*
- *decreasing*

function on $[1,\infty)$ and let $a_i = f(i)$ for $i = 1, 2, 3, \ldots$.

Then the series $\sum_{i=1}^{\infty} a_i$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,\mathrm{d}x$ is convergent.

Question 46.

Describe the convergence and divergence of the $p-$series for different values of the constant p.

$$\sum_{i=1}^{\infty} \frac{1}{i^p}$$

Question 47.

First make sure that the integral test is applicable for the following series. Then use it to test the convergence of the following series:

(a) $\displaystyle\sum_{i=1}^{\infty} \frac{1}{2i+3}$

(b) $\displaystyle\sum_{i=1}^{\infty} \frac{e^{-i}}{1+e^{-2i}}$

(c) $\displaystyle\sum_{i=1}^{\infty} \frac{\ln(i)}{i^2}$

$\sum_{i=1}^{\infty} \frac{1}{i^p}$

§4.3 Comparison Tests

Thus far, we have seen two "types" of series for which we can easily determine their convergence or divergence:

(a) Geometric Series — conv. if $|r|<1$, div. if $|r|\geq 1$

(b) p-Series — conv. if $p>1$, div. if $p\leq 1$

For Geometric series, we are in fact able to compute the sum of the series. For p-Series, we generally can only determine convergence or divergence.

Additionally, we have the following two tests so far:

(a) Divergence Test ✓

(b) Integral Test ✓

Each test has its limitations. The Divergence Test actually doesn't tell when a given series converges, it's a test of when we can conclude that it diverges. The Integral Test is only applicable to continuous positive decreasing functions. So we can't use it for a series that has terms involving $(-1)^n$ or $\sin(n)$ or $n!$.

We add two more tests to our repertoire in this section: the Direct and Limit Comparison Tests.

4.3.1 Direct Comparison Test

In the comparison tests, the idea is to compare a given series with a series that is known to be convergent or divergent. Let's consider an example.

Example 4.3.35

The series

$$\sum_{i=1}^{\infty} \frac{1}{2^i+1} \tag{4.1}$$

$\left(\frac{1}{2}\right)^i$

looks almost like the series $\sum_{i=1}^{\infty} \frac{1}{2^i}$, which is a geometric series with

$a =$ $\frac{1}{2}$ and $r =$ $\frac{1}{2}$ and is therefore convergent.

However, clearly $\sum_{i=1}^{\infty} \frac{1}{2^i+1} < \sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{a}{1-r}$

Hencve, we can conclude that $s_n = \sum_{i=1}^{n} \frac{1}{2^i+1}$ is bounded above by the sum of the geometric series, which is equal to $\frac{1/2}{1-1/2} = 1$.

Additionally, because each term $\frac{1}{2^i+1}$ $\boxed{\text{is positive}}$ the partial sum sequence $\{s_n\}$ is increasing.

So, by monotone convergence theorem, the series (4.1) converges.

This should remind you of the comparison test for improper integrals. We summarize the comparison test for series as follows. Note that similar to the case of comparison test for improper integrals, *this also has the restriction that all the terms of both the series must be positive.*

Theorem 4.3.36: Direct Comparison Test

Suppose that $\boxed{\sum a_i}$ *and* $\sum b_i$ *are series with positive terms.*

- *If* $\sum b_i$ *is convergent and* $\boxed{a_i \le b_i}$ *for all n, then* $\sum a_i$ *is also convergent.* ✓
- *If* $\sum b_i$ *is divergent and* $a_i \ge b_i$ *for all n, then* $\sum a_i$ *is also divergent.*

Note:

- The purpose of the Comparison Test is to use a "nice" series $\boxed{\sum b_i}$ (whose convergence or divergence behavior we already know) to determine whether or not a second series $\sum a_i$ converges.
- We often choose $\sum b_i$ to be either a p-series or a geometric series which looks most similar to $\sum a_i$.

■ Question 48.

Test each of the following series for convergence.

conv.

✓(a) $\boxed{\displaystyle\sum_{i=1}^{\infty} \frac{5}{3^i+2}} < \displaystyle\sum_{i=1}^{\infty} \frac{5}{3^i}$ → geometric $r = \frac{1}{3}$ ⇝ conv.

div. (b) $\boxed{\displaystyle\sum_{i=1}^{\infty} \frac{4}{\sqrt{3i^2-i-2}}} > \displaystyle\sum \frac{4}{\sqrt{3i^2}} = \sum \frac{4}{\sqrt{3}} \cdot \frac{1}{i} = \frac{4}{\sqrt{3}} \sum \frac{1}{i}$ div.

(c) $\displaystyle\sum_{i=1}^{\infty} \frac{i+\sin^2 i}{i^3+1}$

(d) $\displaystyle\sum_{i=1}^{\infty} \frac{4}{i\sqrt{i}+2^i}$

(e) $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i!}$

(f) $\displaystyle\sum_{i=1}^{\infty} \frac{2^i}{\ln(i)5^{i+1}}$

4.3.2 Limit Comparison Test

Consider the series $\displaystyle\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} \frac{i^2}{i^4-i-1}$. Try a direct comparison. What goes wrong?

In a situation such as this, where the inequality you come up with *goes the wrong direction*, you can try to apply the following test instead.

Theorem 4.3.37: Limit Comparison Test

Let $\sum a_i$ *and* $\sum b_i$ *be series with positive terms. If*

$$\lim_{i\to\infty} \frac{b_i}{a_i} = c$$

for some finite constant c and $c > 0$*, then either both series converge or both diverge.*

Note: The Limit Comparison Test shows that if we have a series $\sum \frac{p(i)}{q(i)}$ of rational functions where $p(i)$ is a polynomial of degree a and $q(i)$ a polynomial of degree b, then the series $\sum \frac{p(i)}{q(i)}$ will behave like the series

$$\sum \frac{i^a}{i^b} = \sum \frac{1}{i^{a-b}}$$

So this test allows us to determine the convergence or divergence of series whose terms are rational functions.

Question 49.

Use the Limit Comparison Test to determine the convergence or divergence of the following series.

(a) $\displaystyle\sum_{i=1}^{\infty} \frac{2i^2+3i}{\sqrt{5+i^5}}$

(b) $\displaystyle\sum_{i=1}^{\infty} \frac{1}{2^i-1}$

(c) $\displaystyle\sum_{i=1}^{\infty} \frac{1}{\sqrt{i}+1}$

(d) $\displaystyle\sum_{i=1}^{\infty} \frac{i^3}{\sqrt{i^8+2i^3+1}}$

(e) $\displaystyle\sum_{i=1}^{\infty} \frac{2^i+1}{5^i+4}$

(f) $\displaystyle\sum_{i=1}^{\infty} \frac{e^i+1}{ie^i+1}$

4.3.3 A Bunch of Practice Problems

Question 50.

For the problems below, first try to find a series to do a direct comparison with to determine convergence or divergence. If your inequality goes the wrong way, try the limit comparison test! If that doesn't seem to work, then try to change the series you are comparing to!

(a) $\displaystyle\sum_{i=1}^{\infty} \frac{\sqrt{i}}{i-3}$

(b) $\displaystyle\sum_{i=1}^{\infty} \frac{\sqrt{i}}{5i^2+2}$

(c) $\displaystyle\sum_{i=1}^{\infty} \frac{\sqrt{i+1}+\sqrt{i}}{i}$

(d) $\displaystyle\sum_{i=1}^{\infty} \frac{1}{\ln(i)}$

(e) $\displaystyle\sum_{i=1}^{\infty} \frac{i!}{(i+2)!}$ *(Simplify first)*

(f) $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^2-i\sin i}$

(g) $\displaystyle\sum_{i=1}^{\infty} \frac{\sqrt[3]{i}}{\sqrt{i^3+4k+3}}$

(h) $\displaystyle\sum_{i=1}^{\infty} \frac{1+\cos i}{e^i}$

(i) $\displaystyle\sum_{i=1}^{\infty} \frac{(2i-1)(i^2-1)}{(i+1)(i^2+4)^2}$

(j) $\displaystyle\sum_{i=1}^{\infty} \frac{i+3^i}{i+2^i}$

4.3.4 Some Useful Inequalities

Plot the following functions in DESMOS to confirm yourself.

- $\sin x \leq x$ for all $x \geq 0$.
- $\ln x \leq x$ for all $x \geq 0$.
- $\ln x \leq x^p$ for sufficiently large x, for any $p > 0$. Try plotting for example, $x^{1/3}$.
- $x^p \leq e^x$ for sufficiently large x, for any $p > 0$. Try plotting for example, x^3.

Question 51. □

(a) *If $\sum a_i$ is a convergent series with positive terms, is it true that $\sum \sin(a_i)$ is also convergent? Try limit comparison test.*

(b) *If $\sum a_i$ and $\sum b_i$ are convergent series with positive terms, is it true that $\sum a_i b_i$ is also convergent? Try direct comparison test.*

§4.4 Alternating Series

The convergence tests that we have looked at so far apply only to series with positive terms. So how do we deal with a series that has both positive and negative terms?

Definition 4.4.38: Alternating Series

An **alternating series** is a series whose terms are alternately positive and negative. Here are two examples:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots=\sum_{i=1}^{\infty}(-1)^{i+1}\frac{1}{i}$$

$$-\frac{1}{2}+\frac{2}{3}-\frac{3}{4}+\frac{4}{5}-\frac{5}{6}+\frac{6}{7}-\cdots=\sum_{i=1}^{\infty}(-1)^{i}\frac{i}{i+1}$$

We see from these examples that the i-th term of an alternating series is of the form

$$a_i=(-1)^{i+1}b_i \quad \text{or} \quad a_i=(-1)^i b_i$$

where b_i is a positive number. (In fact, $b_i=|a_i|$.) The following test says that if the terms of an alternating series decrease toward 0 in absolute value, then the series converges.

Theorem 4.4.39: Alternating Series Test

Let $\{b_n\}$ be a sequence of positive numbers. The alternating series

$$\sum_{i=1}^{\infty}(-1)^{i+1}b_i=b_1-b_2+b_3-b_4+b_5-b_6+\cdots$$

or

$$\sum_{i=1}^{\infty}(-1)^{i}b_i=-b_1+b_2-b_3+b_4-b_5+b_6-\cdots$$

converges if

(a) $b_{n+1}\le b_n$ *for all sufficiently large values of n, i.e. the sequence $\{b_n\}$ is a (eventually) decreasing sequence, and*

(b) $\lim_{n\to\infty} b_n=0$, *i.e. the series passes the divergence test.*

Before we ask why this theorem is true, let's look at some practice problems.

■ Question 52. □

Test whether the following alternating series are convergent.

(a) $\displaystyle\sum_{i=1}^{\infty}\frac{(-1)^i}{i}$

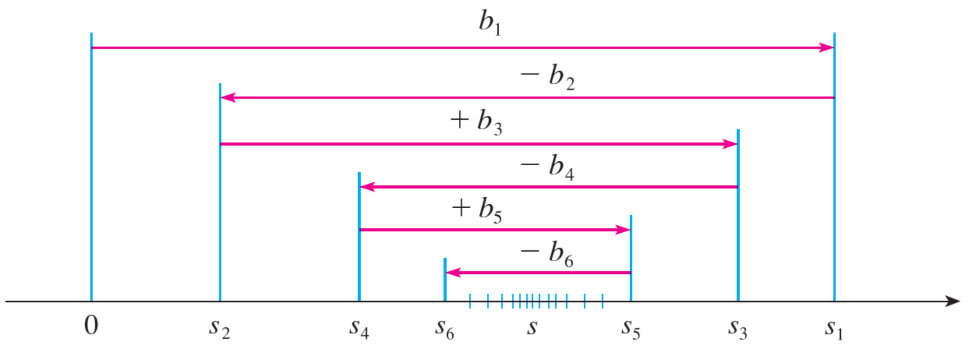
(b) $\displaystyle\sum_{i=1}^{\infty}\frac{(-1)^i 3i}{4i-1}$

(c) $\displaystyle\sum_{i=1}^{\infty}(-1)^i\frac{i^2}{i^3+1}$

(d) $\displaystyle\sum_{i=1}^{\infty}(-1)^i\sin\left(\frac{\pi}{i}\right)$

Digression

The proof of the alternating series test relies on the fact that since $\{b_n\}$ is decreasing, the even partial sums are increasing and bounded, and the odd partial sums are decreasing and bounded. The following picture should give you an idea of what's happening.



Also try the following DESMOS link for an animation that plots the successive partial sums.

https://www.desmos.com/calculator/qxupbepctm

4.4.1 Absolute and Conditional Convergence

Not every alternating series fits the conditions of the alternating series test. In particular, a series could be alternating with $\lim_{n\to\infty} b_n = 0$, but the general term might never be a decreasing sequence. For example,

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{2^3} - \frac{1}{3^3} + \cdots$$

In such cases, we still have one more test we can try.

Theorem 4.4.40

Given any sequence $\{a_n\}$, if $\sum |a_i|$ is convergent, then $\sum a_i$ is convergent.

Digression

The proof of this theorem actually follows from comparison test. We have to use the following inequality:

$$-|a_n| \le a_n \le |a_n| \implies 0 \le a_n + |a_n| \le 2|a_n|$$

Why can't you use the comparison test with the first inequality?

See if you can figure out how to proceed from the second inequality.

Question 53. □

Apply the test to the series above? Is the series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$$

convergent? What does it mean for the original alternating series?

Let's try some more examples.

Question 54.

(a) $\displaystyle\sum_{i=1}^{\infty} \frac{(-1)^i}{i^3}$

(b) $\displaystyle\sum_{i=1}^{\infty} \frac{(-1)^i}{i(\ln i)^2}$

(c) $\displaystyle\sum_{i=1}^{\infty} (-1)^i \frac{\sin\left(\frac{i\pi}{4}\right)}{i^2}$

(d) $\displaystyle\sum_{i=1}^{\infty} \frac{\cos i}{i^2}$

Definition 4.4.41

A series $\sum a_i$ is called **absolutely convergent** if the series of absolute values $\sum |a_i|$ is convergent.

A series $\sum a_i$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

With this definition, the absolute convergence test can be restated as

$$\boxed{\text{Absolute Convergence} \implies \text{Convergence}}$$

Question 55.

For the following series, determine whether they are absolutely convergent, conditionally convergent, or divergent.

(a) $\displaystyle\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i^3+1}$

(b) $\displaystyle\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{\sqrt{i}}$

(c) $\displaystyle\sum_{i=1}^{\infty} \frac{(-1)^i}{5i+1}$

(d) $\displaystyle\sum_{i=1}^{\infty} (-1)^i \ln\left(1+\frac{1}{i}\right)$

Question 56.

Suppose the series $\sum a_i$ is conditionally convergent. Then show that the series $\sum i^2 a_i$ must be divergent.

§4.5 Two ways to test for Absolute Convergence

Last time we learned that *absolute convergence* of a series implies *convergence*. This allows us to test the convergence of a series $\sum a_i$ with positive and negative terms by instead looking at the convergence of a series $\sum |a_i|$ with only positive terms. However, we still need to test the convergence of this second series by other means.

Since the second series has positive terms only, some of the methods would be to apply integral, direct comparison, or limit comparison tests. Consider the following example. What would you apply here?

Example 4.5.42

Consider $\displaystyle\sum_{i=1}^{\infty} \frac{(-1)^i \ln(i)}{i!}$. Does this series converge absolutely? To answer that, we need to test the series $\displaystyle\sum_{i=1}^{\infty} \frac{\ln(i)}{i!}$.

We could try a comparison or limit comparison test for $\displaystyle\sum_{i=1}^{\infty} \frac{\ln(i)}{i!}$. Did you run into any difficulty? Can you apply integral test here?

4.5.1 Ratio Test

The following test is very useful in determining whether a given series is absolutely convergent. It is essentially a way to directly compare a series with a geometric series (where we don't know the common ratio a priori).

Theorem 4.5.43: Ratio Test

Let $\sum a_i$ be a series with nonzero terms and suppose the following limit exists:

$$\mathrm{L} = \lim_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right|$$

(a) If $0 \le \mathrm{L} < 1$, then $\sum a_i$ converges absolutely.

(b) If $\mathrm{L} > 1$ (or if the limit doesn't exist), then $\sum a_i$ diverges.

(c) If $\mathrm{L} = 1$, then the Ratio Test is inconclusive.

Note: The ratio test is ideal for any series involving factorials or exponential expressions.

Warning: Part (c) of the Ratio Test says that if $\lim_{i\to\infty} |a_{i+1}/a_i| = 1$, the test gives no information. For instance, for the convergent series $\sum 1/n^2$ we have

$$\left|\frac{a_{i+1}}{a_i}\right| = \frac{\frac{1}{(i+1)^2}}{\frac{1}{i^2}} = \frac{i^2}{(i+1)^2} = \frac{1}{\left(1+\frac{1}{i}\right)^2} \to 1 \quad \text{as } n \to \infty$$

whereas for the divergent series $\sum 1/n$ we have

$$\left|\frac{a_{i+1}}{a_i}\right| = \frac{\frac{1}{i+1}}{\frac{1}{i}} = \frac{i}{i+1} = \frac{1}{1+\frac{1}{i}} \to 1 \quad \text{as } i \to \infty$$

Therefore, if $\lim_{i\to\infty} |a_{i+1}/a_i| = 1$, the series $\sum a_i$ might converge or it might diverge. In this case the Ratio Test fails and we must use some other test.

Question 57.

Use the Ratio Test to test each of the following series for convergence.

(a) $\sum \frac{(-1)^i \ln(i)}{i!}$.

(b) $\sum \frac{(-1)^i i^2}{i!}$

(c) $\sum \frac{3^i}{i!}$

(d) $\sum \frac{i^i}{i!}$

(e) $\sum \frac{(-1)^i (i!)^2}{(2i)!}$

(f) $\sum \frac{(-1)^i i^3}{3^i}$

4.5.2 Root Test

Another test similar to the ratio test to check absolute convergence is the following. It is almost identical to the ratio test in its conclusion, but you compute a different limit, as per its name.

Theorem 4.5.44: Root Test

Let $\sum a_i$ be a series with nonzero terms and suppose the following limit exists:

$$L = \lim_{i\to\infty} \sqrt[i]{|a_i|} = \lim_{i\to\infty} |a_i|^{1/i}$$

(a) *If $0 \le L < 1$, then $\sum a_i$ converges absolutely.*

(b) *If $L > 1$, then $\sum a_i$ diverges.*

(c) *If $L = 1$, then the Ratio Test is inconclusive.*

You will want to use the root test in only certain circumstances when its application is easier than the ratio test. This will be when a series has general term involving exponentials. In particular, if individual terms a_i look like $(b_i)^i$, an i-th power, then applying the root test allows you to only have to evaluate the (likely much simpler) limit $\lim_{i\to\infty} b_i$.

Note: If $L = 1$ in the Ratio Test, don't try the Root Test because L will again be 1. And if $L = 1$ in the Root Test, don't try the Ratio Test because it will fail too.

Question 58.

Use the Ratio Test to test each of the following series for convergence.

(a) $\sum \left(\frac{i^2+1}{2i^2+3}\right)^i$

(b) $\sum \frac{2^{i^2}}{i!}$ *[Hint:* $2^{i^2} = (2^i)^i$ *and* $i! < i^i$*.]*

(c) $\sum \frac{(-1)^{i-1}}{(\ln i)^i}$

(d) $\sum (\arctan i)^i$

Question 59.

Let $\{b_i\}$ *be a sequence of positive numbers that converges to* $\frac{1}{2}$*. Determine whether the given series is absolutely convergent.*

(a) $\sum_{i=1}^{\infty} \frac{b_i^i \cos i\pi}{i}$

(b) $\sum_{i=1}^{\infty} \frac{(-1)^i i!}{i^i b_1 b_2 b_3 \cdots b_i}$

§4.6 Summary of the chapter

The following is adapted from your Openstax textbook chapter 5.6. The book has a really nice flowchart summarizing strategies for choosing the correct test, as well as a table with summaries of all tests. The book uses a_n instead of a_i. So to be consistent, we will use the notation $\sum a_n$ today.

4.6.1 Problem Solving Strategy

Step 1. First thing to try: the **Divergence Test.** If $\lim_{n\to\infty} a_n \neq 0$, then we know that the series $\sum a_n$ diverges and we are done right away! Otherwise, $\lim_{n\to\infty} a_n = 0$ and we have to try an actual test.

Step 2. Next, try to see if the series looks familiar. Is it of a certain class that we know: a **Geometric Series**, a $p-$**series**, the Harmonic Series, etc. If so, check the ratio r or the power p to determine if the series converges.

Look for variations as well - if the series looks like a sum or difference of two geometric series, or a sum of a $p-$series and a geometric series, etc. Remember, the sum or difference of two converging series converges. (What happens if we add a diverging series with a converging series?)

Step 3. Identify if the series has all positive terms. If it does not, determine if it is an Alternating Series and try the **Alternating Series Test.**

Step 4. If the series has negative terms and is *not alternating*, you can try to determine if the series is absolutely convergent, since absolute convergence implies convergence. Remember, this means determining whether $\sum |a_n|$ converges. You can either use the tests below for positive series, or use the Ratio or Root Test, as these are tests for absolute convergence.

- If the terms in the series contain a factorial or power, try the ratio test first.
- If the terms are powers such that $a_n = (b_n)^n$, try the root test first.

Step 5. Lastly for a non-positive series, make sure you have answered the question! Does it ask for absolute/conditional convergence, or simply asks whether the series converges or diverges? If the former, make sure you fully investigate the series by checking for absolute convergence, especially if it is an alternating series.

Step 6. Now, if your series has only positive terms (or you are examining $\sum |a_n|$), then we can apply the other three tests - the Direct & Limit Comparison Tests and the Integral Test. It's a good idea to try **Direct comparison** *first*. If that fails due to the inequality going the wrong way, then use the **Limit comparison** test.

Step 7. If all of the above has failed you, then we have the **Integral Test** as our backup.

4.6.2 Summary of Series Tests

The following is copied from your textbook.

Series or Test	Conclusions	Comments
Divergence Test For any series $\sum_{n=1}^{\infty} a_n$, evaluate $\lim_{n\to\infty} a_n$.	If $\lim_{n\to\infty} a_n = 0$, the test is inconclusive.	This test cannot prove convergence of a series.
	If $\lim_{n\to\infty} a_n \neq 0$, the series diverges.	
Geometric Series $\sum_{n=1}^{\infty} ar^{n-1}$	If $\lvert r\rvert < 1$, the series converges to $a/(1-r)$.	Any geometric series can be reindexed to be written in the form $a + ar + ar^2 + \cdots$, where a is the initial term and r is the ratio.
	If $\lvert r\rvert \geq 1$, the series diverges.	
***p*-Series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$	If $p > 1$, the series converges.	For $p = 1$, we have the harmonic series $\sum_{n=1}^{\infty} 1/n$.
	If $p \leq 1$, the series diverges.	

Comparison Test For $\sum_{n=1}^{\infty} a_n$ with nonnegative terms, compare with a known series $\sum_{n=1}^{\infty} b_n$.	If $a_n \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.	Typically used for a series similar to a geometric or p-series. It can sometimes be difficult to find an appropriate series.
	If $a_n \geq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	
Limit Comparison Test For $\sum_{n=1}^{\infty} a_n$ with positive terms, compare with a series $\sum_{n=1}^{\infty} b_n$ by evaluating $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.	If L is a real number and $L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.	Typically used for a series similar to a geometric or p-series. Often easier to apply than the comparison test.
	If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.	
	If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	

Integral Test If there exists a positive, continuous, decreasing function f such that $a_n = f(n)$ for all $n \geq N$, evaluate $\int_N^\infty f(x)\,dx$.	$\int_N^\infty f(x)\,dx$ and $\sum_{n=1}^\infty a_n$ both converge or both diverge.	Limited to those series for which the corresponding function f can be easily integrated.
Alternating Series $\sum_{n=1}^\infty (-1)^{n+1} b_n$ or $\sum_{n=1}^\infty (-1)^n b_n$	If $b_{n+1} \leq b_n$ for all $n \geq 1$ and $b_n \to 0$, then the series converges.	Only applies to alternating series.
Ratio Test For any series $\sum_{n=1}^\infty a_n$ with nonzero terms, let $\rho = \lim\limits_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert$.	If $0 \leq \rho < 1$, the series converges absolutely.	Often used for series involving factorials or exponentials.
	If $\rho > 1$ or $\rho = \infty$, the series diverges.	
	If $\rho = 1$, the test is inconclusive.	
Root Test For any series $\sum_{n=1}^\infty a_n$, let $\rho = \lim\limits_{n\to\infty} \sqrt[n]{\lvert a_n \rvert}$.	If $0 \leq \rho < 1$, the series converges absolutely.	Often used for series where $\lvert a_n \rvert = b_n^n$.
	If $\rho > 1$ or $\rho = \infty$, the series diverges.	
	If $\rho = 1$, the test is inconclusive.	

Question 60.

Decide whether each of the following is a p−series, or a geometric series, or neither. If it is one of the two, indicate whether or not the series converges. If it's neither, you do not need to work further.

(a) $\displaystyle\sum_{n=1}^\infty \frac{1}{n\sqrt{n}}$ (b) $\displaystyle\sum_{n=1}^\infty \frac{3}{4^{n-1}}$ (c) $\displaystyle\sum_{n=1}^\infty \frac{n}{(3n)^5}$ (d) $\displaystyle\sum_{n=1}^\infty \frac{2^{2n+1}}{5^{n-1}}$

(e) $\displaystyle\sum_{n=1}^\infty \sqrt{\frac{2}{n}}$ (f) $\displaystyle\sum_{n=1}^\infty \frac{(-1)^n}{n}$ (g) $\displaystyle\sum_{n=1}^\infty \frac{(-2)^n}{e^{2n}}$

Question 61.

For each of the following series, find either a p -series or a geometric series that would be an appropriate candidate for comparison. You need not actually perform the comparison test.

(a) $\displaystyle\sum_{n=1}^\infty \frac{5n^2}{2n^3-1}$ (b) $\displaystyle\sum_{n=1}^\infty \frac{3n}{\sqrt{n^5+n^4+2}}$ (c) $\displaystyle\sum_{n=1}^\infty \frac{3^n+1}{2^n-1}$ (d) $\displaystyle\sum_{n=1}^\infty \frac{4}{n(n+3)}$

(e) $\displaystyle\sum_{n=1}^\infty \frac{3^n}{5^n+n}$ (f) $\displaystyle\sum_{n=1}^\infty \frac{n^2}{n^2\sqrt{6n-1}}$ (g) $\displaystyle\sum_{n=1}^\infty \sqrt{\frac{4^n}{3^{2n}+100}}$ (h) $\displaystyle\sum_{n=1}^\infty \frac{\sqrt{6^n-n}}{4^{2n}+n\sqrt{n}}$

Question 62.

Determine which of the following series converge. Justify your conclusions with the appropriate explanations.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ (b) $\sum_{n=1}^{\infty} 2^{-n}$ (c) $\sum_{n=1}^{\infty} \frac{n+5}{5^n}$ (d) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n(n-1)}}$

(e) $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{n^2}$ (f) $\sum_{n=1}^{\infty} \frac{2n}{8n-5}$ (g) $\sum_{n=1}^{\infty} \frac{3^n}{(2n)!}$ (h) $\sum_{n=1}^{\infty} 2^n$

(i) $\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n^3+1}}$ (j) $\sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^n$ (k) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ (l) $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{\sqrt{n}}$

■ Question 63.

□

Determine if the given series is converging or diverging. If the series has negative terms, determine if the series is absolutely convergent, conditionally convergent, or divergent.

(a) $\sum_{n=1}^{\infty} \frac{n^2+2n}{n^3+3n^2+1}$ (b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(3n+1)}{n!}$ (c) $\sum_{n=1}^{\infty} \frac{e^n}{n^4}$ (d) $\sum_{n=1}^{\infty} \frac{3^n}{(n+1)^n}$

(e) $\sum_{n=1}^{\infty} \frac{2^n-5^n}{7^n}$ (f) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ (g) $\sum_{n=1}^{\infty} \frac{1+(-1)^n}{n}$ (h) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

(i) $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}-(\ln(n))^4}$ (j) $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ (k) $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^{2/3}}$ (l) $\sum_{n=1}^{\infty} \frac{n^5}{5^n}$

(m) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}(\ln(n))^2}$ (n) $\frac{1}{2}-\frac{1}{5}+\frac{1}{4}-\frac{1}{25}+\frac{1}{8}-\frac{1}{125}+\dots$

Chapter 5 | Approximating Functions as Series

In the last section, we saw that

$$\text{Series} \iff \text{Infinite sum of numbers}$$

We now extend this idea by introducing variables, that is, we ask what happens when we add up an infinite number of functions of x? In particular, we are going to add up terms that involve **powers** of x and write it as an infinite sum. This gives us something called a power series.

$$\text{Power Series} \iff \text{Infinite sum of polynomials}$$

As a result, a power series can be used to represent common functions and also to define new functions. We will see that functions like $e^x, \sin(x)$, and $\ln(x+1)$ can all be realized as some power series.

Example 5.0.45

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Note: One might ask what is the benefit of writing e^x as a series? Since e is an irrational number, there is no way to get the exact numerical vale of, say, e^2, by multiplying e twice. Instead to get the value of e^2, your calculator (or a computer) evaluates the sum on the right hand side when $x = 2$, up to finitely many terms, depending on the precision required.

$$e^2 = \sum_{n=0}^{\infty} \frac{2^n}{n!} = 1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \dots$$

This is much easier to do since a finite partial sum on the right hand side involves only evaluating some finite powers of 2.

So our goal in this chapter is twofold:

- Understand when an infinite sum of polynomials makes sense. E.g. given an expression such as

  $$1 + 2x + 3x^2 + 4x^3 + \dots$$

  we seek the values of x for which the expression **converges**.

- Try to find constants c_0, c_1, c_2, etc. so that we can express a function $f(x)$ as a power series

  $$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

  This way we can approximate the value of $f(x)$ numerically at some point $x = a$ up to arbitrary precision.

§5.1 Power Series

Let's start with an example of power series you are already familiar with. Consider a **geometric series** of the form

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + cr^3 + cr^4 + \dots$$

Recall that we know how to get the sum here:

$$\sum_{n=0}^{\infty} cr^n = \begin{cases} \dfrac{c}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$$

So if we consider a function $F(x) = \sum_{n=0}^{\infty} cx^{n-1}$, we can say that the domain of $F(x)$ is ....

Let's assume $c = 1$. Then we can write

$$F(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots = \frac{1}{1-x} \text{ for } \qquad . \tag{$\star$}$$

Practical Interpretation of eq. ($\star$): The above identity is saying that we can *approximate* the numerical value of the function $\dfrac{1}{1-x}$, on the interval $(-1, 1)$, up to arbitrary precision, by evaluating truncated partial sums that are polynomials of larger and larger degree. In the limit, the infinite polynomial is *equal to* the function on this interval. See

https://www.desmos.com/calculator/wkwkrwtdsk

for some graphical evidence.

Once we know that a given function can be written via a power series, we can create many more examples via simple substitutions.

Question 64.

Find the domain of the following power series, and find the function it converges to on that domain.

(i) $\sum_{n=0}^{\infty} x^{2n}$ *(ii)* $\sum_{n=0}^{\infty} 2^n x^n$

You can see graphs of the above examples at the link below. Don't look before you have tried to answer the question yourself.

https://www.desmos.com/calculator/0qyqwojain

Definition 5.1.46

A **power series** is a **function** of the form

$$F(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots + c_n(x-a)^n + \ldots$$

We call a the **center** of the series. A power series centered at $a = 0$ is of the form:

$$F(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots + c_n x^n + \ldots$$

The **domain** of a power series $F(x)$ is all values of x for which the series converges.

Note: In the definition of a power series, x is a variable, a is a constant, and the c_n's are called coefficients.

Example 5.1.47

Let's take a look at a power series that's not a geometric series. Consider

$$F(x) = \sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$$

This power series is centered at $x =$ ________ . We wish to find the domain of $F(x)$, i.e. the values of x for which the series converges.

We will do this in two steps.

Step 1. Use the **Ratio Test**. Let $a_n = \dfrac{(x-3)^n}{n}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \longrightarrow |x-3| \text{ as } n \to \infty$$

By the Ratio Test, the given series is absolutely convergent (and therefore convergent) when $|x-3| < 1$ and divergent when $|x-3| > 1$. Now

$|x-3| < 1 \iff$ ____________________ (write in interval notation).

So the series converges when ________ and diverges when ________ or ________ .

Note that the Ratio Test gives no information when $|x-3| = 1$ so we must consider $x = 2$ and $x = 4$ separately.

Step 2. **Check the endpoints** of the interval for convergence using series tests.

If we put $x = 4$ in the series, it becomes $\sum \frac{1}{n}$, the harmonic series, which is ________ .

If $x = 2$, the series is $\sum \frac{(-1)^n}{n}$, which ____________ by the Alternating Series Test.

Thus the domain of the given power series is ____________________ .

Question 65.

Find the values of x for which the series $\sum_{n=0}^{\infty} n!x^n$ converges.

Question 66.

Define $P(x) = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}$.

(a) *The first couple of partial sums are plotted in* https://www.desmos.com/calculator/0qyqwojain. *Can you guess the function the series appears to be converging towards?*

(b) *Use the Ratio Test to determine the values of x for which $P(x)$ converges.*

5.1.1 Radius and Interval of Convergence

For the power series that we have looked at so far, the set of values of x for which the series is convergent has always turned out to be an interval (a finite interval for the geometric series and the series in example 47), the infinite interval (in question 66), or a collapsed interval $[0,0] = \{0\}$ (in question 65). The following theorem, which we will not prove, says that this is true in general.

Theorem 5.1.48

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are ***only three possibilities****:*

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < \mathrm{R}$ and diverges if $|x-a| > \mathrm{R}$.

The number R in case (iii) is called the **radius of convergence** of the power series. By convention, we will say that the radius of convergence is $\mathrm{R} = 0$ in case (i) and $\mathrm{R} = \infty$ in case (ii).

The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges. In case (i) the interval consists of just a single point $\{a\}$. In case (ii) the interval is $(-\infty, \infty)$.

Note: In case (iii), note that the theorem doesn't say what happens when x is an endpoint of the interval, that is, $x = a \pm \mathrm{R}$,. In fact, depending on the example, anything can happen - the series might converge at one or both endpoints or it might diverge at both endpoints. Thus in case (iii) there are four possibilities for the interval of convergence:

$$(a-\mathrm{R}, a+\mathrm{R}) \qquad (a-\mathrm{R}, a+\mathrm{R}] \qquad [a-\mathrm{R}, a+\mathrm{R}) \qquad [a-\mathrm{R}, a+\mathrm{R}]$$

The case (iii) situation is illustrated in fig. 5.1.

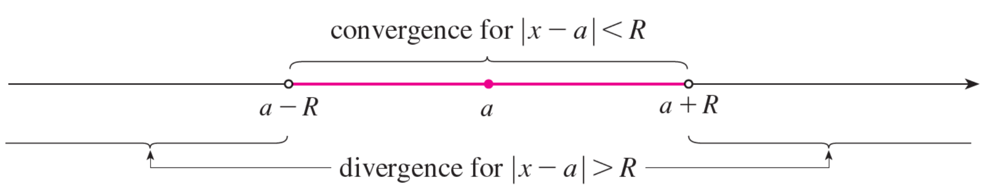


Figure 5.1

Note: In general, the Ratio Test (or sometimes the Root Test) should be used to determine the radius of convergence R. The Ratio and Root Tests always fail when x is an endpoint of the interval of convergence, so the endpoints must be checked with some other test.

Here's a summary of the radius and interval of convergence for examples we have seen so far:

Question 67. □

Find the radius and interval of convergence of the following series.

Series	Radius of Convergence	Interval of Convergence
$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
$\sum_{n=0}^{\infty} (2x)^n$	$R = \frac{1}{2}$	$(-1/2, 1/2)$
$\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
$\sum_{n=0}^{\infty} n!x^n$	$R = 0$	$\{0\}$
$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$R = \infty$	$(-\infty, \infty)$

Table 5.1

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-3)^n}{\sqrt{n}}$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$.

■ Question 68. □

Let p and q be real numbers with $p < q$. Find a power series whose interval of convergence is

(a) (p, q)

(b) $[p, q)$

(c) $(p, q]$

(d) $[p, q]$

■ Question 69. □

Is it possible to find a power series whose interval of convergence is $[0, \infty)$? Explain.

■ Question 70. □

Suppose the series $\sum c_n x^n$ has radius of convergence 2 and the series $\sum d_n x^n$ has radius of convergence 3.

(a) *What is the radius of convergence of the series $\sum (c_n + d_n)x^n$?*

(b) *What is the radius of convergence of the series $\sum c_n x^{2n}$?*

§5.2 Representations of Functions as Power Series

In this section, we will learn how to represent certain types of functions as sums of power series by manipulating geometric series or by differentiating or integrating such a series. You might wonder why we would ever want to express a known function as a sum of infinitely many terms. This strategy is useful

- for integrating functions that don't have elementary antiderivatives,
- for solving differential equations,
- for approximating functions by polynomials, etc.

Scientists do this to simplify the expressions they deal with; computer scientists do this to represent functions on calculators and computers.

5.2.1 Using the Geometric Series

We will obtain power series representations for several functions by manipulating the geometric series.

Example 5.2.49

Let's start with the geometric series example we have seen before:

$$F(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 \ldots = \frac{1}{1-x} \text{ for } |x| < 1$$

We can make simple substitutions to this to obtain new power series (we did this last time). Let's try a new one.

Make a substitution in $F(x)$ above and obtain a power series expansion $G(x)$ for the function $\frac{1}{1+x}$ on the interval $(-1, 1)$. Write the series below.

$\frac{1}{1+x} = \frac{1}{1-(\quad)} =$ ....

What is the interval of convergence? ....

Example 5.2.50

Similarly, let's try to find a power series representation for $\frac{1}{x+2}$. The first thing to note is that we can rewrite the function look like $\frac{1}{1-r}$ for some r, and that way we can use the geometric series expansion!

$\frac{1}{x+2} = \frac{1}{2+x} = \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2\left[1-\left(-\frac{x}{2}\right)\right]} =$ ....

This series converges when .... So the interval of convergence is ....

Question 71.

Starting from the last example, can you find the power series representation for $\dfrac{x^3}{x+2}$? Make sure to also determine the interval of convergence.

Question 72.

Find the power series representation of $\dfrac{2}{3-x}$ and its interval of convergence.

5.2.2 Differentiation and Integration of Power Series

Let us continue with the power series $G(x)$ obtained above for $\dfrac{1}{1+x}$. What would happen if we differentiate both sides of that identity with respect to x? We get

$$\frac{d}{dx}G(x) = \frac{d}{dx}\left(\frac{1}{1+x}\right) = -\frac{1}{(1+x)^2}$$

On the other hand, by differentiating the power series one term at a time, we get

$\dfrac{d}{dx}G(x) =$ ....

Rewriting it using sigma notation gives ....

Question 73.

Use the result above to compute the exact sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{3^n}$.

We can also try integrating each term of the power series for $\dfrac{1}{1+x}$. What function is our new power series an expansion of?

$$\int \frac{1}{1+x}\,dx = \qquad\qquad = \text{..}$$

To find the value of C, we can put $x = 0$ on both sides. What do you conclude?

Question 74.

What is the interval of convergence for the above series?

Theorem 5.2.51

If the power series $\sum c_n(x-a)^n$ *has radius of convergence* $R > 0$, *then the function* F *defined by*

$$F(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ *and*

(i)

$$F'(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}[c_n(x-a)^n] = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

(ii)

$$\int F(x)\,dx = \int\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right]dx = \sum_{n=0}^{\infty}\int [c_n(x-a)^n]\,dx = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$
$$= C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

The theorem essentially says that we can obtain F' by differentiating the series for F term-by-term and we can also obtain the antiderivative $\int F(x)\,dx$ by integrating term-by-term.

Warning: The radius of convergence R will always be the same for the derivative or integral of a power series - but the interval of convergence may change. We saw this with our second example above! The interval of convergence for $\frac{1}{1+x}$ was $(-1,1)$ but it changed to $(-1,1]$ for $\ln(1+x)$.

Note: The idea of differentiating a power series term by term is the basis for a powerful method for solving differential equations. We can also use this idea to anti-differentiate functions we couldn't by usual means before.

Example 5.2.52

We are going to find the power series expansion for $f(x) = \arctan(x)$. Recall that

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

Now start with the power series for $\frac{1}{1+x^2}$.

$\frac{1}{1+x^2} =$ ....

Then integrate both sides with respect to x. What do you get?

arctan(x) =

We can continue to make substitutions to make more power series. We can also use this to approximate values like ln(2) or other inputs to the natural log or inverse tangent functions. Below are two such examples. Also, check out this Desmos page to see how the partial sums of these series approximate the given function (which, although not a proof, gives you strong evidence that these power series expansions are indeed true):

https://www.desmos.com/calculator/xks0pr6hhf

Example 5.2.53

We know that the power series of $\ln(1+x)$ converges at $x = 1$. Hence

$$\ln(2) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$$

Thus the alternating harmonic series converges to $\ln 2$.

One great use of power series is to make use of them for integration purposes. Consider the following integral:

$$\int \arctan(x^2)\,\mathrm{d}x$$

We could try doing Integration By Parts as we learned before, the resulting integral would be doable but doesn't look very easy. (Indeed `WolframAlpha` tells me that the integral is pretty ugly!)

But for all practical purposes, we don't really need the exact formula for the integral, an approximation will suffice as long as we can choose the precision.

So instead, why don't we come up with a power series expansion for $\arctan(x^2)$ and just integrate that?

Question 75.

Find a power series expression for $\int \arctan(x^2)\,\mathrm{d}x$.

Question 76.

Come up with the power series expansion for $\frac{1}{1+x^7}$ *and express the integral* $\int \frac{1}{1+x^7}\,\mathrm{d}x$ *as a power series.*

Can you approximate the definite integral $\int_0^{0.5} \frac{1}{1+x^7}\,\mathrm{d}x$ *correct to within* 10^{-7}?

Question 77.

Consider $\int x\ln(1+x)\,\mathrm{d}x$.

(a) *Use power series to obtain an antiderivative.*

(b) *Use Integration By Parts and substitution to solve the integral and compare the two answers.*

§5.3 More Practice Problems

Here's a summary of some of the Power Series we derived in this section.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots \qquad \text{for } |x| < 1$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - \dots \qquad \text{for } |x| < 1$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \qquad \text{for } -1 < x \le 1$$

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \qquad \text{for } -1 \le x \le 1$$

■ Question 78. □

Find a power series representation for the function and determine the radius of convergence.

(a) $f(x) = \ln(5-x)$

(b) $f(x) = x^2 \arctan\left(x^3\right)$

(c) $f(x) = \dfrac{1}{(1+4x)^2}$

(d) $f(x) = \dfrac{x}{(1+4x)^2}$

(e) $f(x) = \dfrac{1}{(1+x)^3}$

(f) $f(x) = \dfrac{1}{(2-x)^3}$

(g) $f(x) = \left(\dfrac{x}{2-x}\right)^3$

(h) $f(x) = \dfrac{1+x}{(1-x)^2}$

(i) $f(x) = \dfrac{x^2+x}{(1-x)^3}$

■ Question 79. □

A function f is defined by the power series

$$f(x) = 1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + \cdots$$

where the coefficients are alternately 1 and 2.

(a) *Find the interval of convergence of the series.*

(b) *Find a closed form explicit formula for $f(x)$ on the interval of convergence.*

§5.4 Taylor and Maclaurin Series

In the last section, we derived power series representations for a certain restricted class of functions, namely, those that can be obtained from geometric series. In this section, we investigate the more general problems:

- Which functions have power series representations?
- How can we find such representations?

We will answer the second question first.

5.4.1 How to find a Power Series representation?

More precisely, our first main goal is as follows:

Main Goal: Suppose a function $f(x)$ can be represented by a power series

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots \quad \text{on the interval } |x-a| < R$$

Then find the values of c_n for all n in terms of f.

Step 1. Suppose $F(x)$ denotes the right hand side, the power series

$$F(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$$

So we are given that the function $f(x)$ and $F(x)$ agree with each other on the interval $(a-R, a+R)$. In particular, this means $f(a) = F(a)$.

Substitute $x = a$ in the formula for $F(x)$. What does this tell you about c_0?

$c_0 =$ ....

Step 2. Again let's start with the equation $f(x) = F(x)$. Take the derivative of both sides with respect to x.

$f'(x) = F'(x) =$ ....

Now substitute $x = a$ in both sides of above equation. What does this tell you about c_1?

$c_1 =$ ....

Step 3. Take the derivative of both sides with respect to x one more time.

$f''(x) = F''(x) =$ ....

Again substitute $x = a$ in both sides of above equation. What does this tell you about c_2?

$c_2 =$ ....

Step 4. If you repeat the process,

$c_3 =$ ....

Step 5. Can you identify a pattern for the coefficients of the power series $F(x)$? Write down the general formula for c_n.

> **Notation:** The nth derivative of $f(x)$ is denoted as $f^{(n)}(x)$.

$c_n =$ ....

The above formula remains valid even for $n = 0$ if we adopt the conventions that $0! = 1$ and $f^{(0)}(x) = f(x)$. Thus we have proved the following theorem.

Theorem 5.4.54

If $f(x)$ has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad \text{for } |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Definition 5.4.55: Taylor Series of f at a

The **Taylor series** of the function f at a (or about a or centered at a) is given by

$$\begin{aligned} P(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

The Nth partial sum (i.e. sum of the first N terms) of the series, $P_N(x)$, is called the Nth degree Taylor Polynomial of $f(x)$, that is,

$$P_N(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

> **Warning:** There is a reason to give the Taylor series a different name. The Taylor series of a function $f(x)$ is not necessarily equal to $f(x)$ for all x!
>
> Theorem 54 tells us that **if $f(x)$ has a power series expansion**, then the expansion $P(x)$ is the Taylor series of $f(x)$. However, there do exist functions for which the corresponding Taylor series $P(x)$ is **not equal to** $f(x)$. See this page for an example.
>
> We will see a theorem later that tells us under what conditions the Taylor series of a function actually converges to the original function.

The following theorem follows from the fact that if f has a power series expansion (i.e. a power series that converges to f), then we know the exact values of each coefficient as we found earlier in this section.

Theorem 5.4.56: Uniqueness

If a function f has a power series at a that converges to f on some open interval containing a, then that power series is the Taylor series for f at a.

The case of when the Taylor series of a function is centered at $a = 0$ arises frequently enough that it has its own special name. the Taylor series for $f(x)$ centered at $x = 0$ is called the **Maclaurin Series** for $f(x)$ and has the form

$$\mathrm{P}(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Example 5.4.57

Let's build the Taylor series for $f(x) = e^x$ centered at $x = 0$.

Step 1. $f(x) =$ .... $\implies f(0) =$ ....

Step 2. $f'(x) =$ .... $\implies f'(0) =$ ....

Step 3. $f''(x) =$ .... $\implies f''(0) =$ ....

Step 4. $f^{(n)}(x) =$ .... $\implies f^{(n)}(0) =$ ....

Therefor, the Taylor series of e^x centered at $x = 0$ is given by

$$\mathrm{P}(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots =$$ ....

Example 5.4.58

Let's build the Taylor series for $g(x) = \sin(x)$ centered at $x = 0$.

Step 1. $g(x) =$ .... $\implies g(0) =$ ....

Step 2. $g'(x) =$ .... $\implies g'(0) =$ ....

Step 3. $g''(x) =$ .... $\implies g''(0) =$ ....

Step 4. $g^{(n)}(x) =$ .... $\implies g^{(n)}(0) =$ ....

Therefor, the Taylor series of $g(x) = \sin(x)$ centered at $x = 0$ is given by

$$\mathrm{P}(x) = g(0) + \frac{g'(0)}{1!}x + \frac{g''(0)}{2!}x^2 + \frac{g'''(0)}{3!}x^3 + \cdots =$$ ....

Question 80.

Determine the interval of convergence for the above two power series.

■ Question 81. □

Find the Taylor series of $h(x) = \cos x$, centered at $x = 0$ using theorem 54. Compare this to the derivative of the power series for $\sin(x)$.

Here's a DESMOS page with all three series:

https://www.desmos.com/calculator/oxdwh8kcds

■ Question 82. □

Find the Taylor series for $f(x) = \ln x$ centered at $a = 1$.

■ Question 83. □

Find the Maclaurin series of $f(x) = (1+x)^k$, where k is some real number.

> **Note:** This series is called the **binomial series**. If k is a non-negative integer, then the terms are eventually 0 and so the series becomes finite.

Next, let's answer the question: which functions have a power series expansion?

5.4.2 When is a function equal to its Taylor series?

Let's start with an example. First build the Taylor series for $f(x) = \dfrac{1}{1-x}$ centered at $x = 0$.

Step 1. $f(x) =$ .... $\implies f(0) =$ ....

Step 2. $f'(x) =$ .... $\implies f'(0) =$ ....

Step 3. $f''(x) =$ .... $\implies f''(0) =$ ....

Step 4. $f^{(n)}(x) =$ .... $\implies f^{(n)}(0) =$ ....

We immediately notice that the Taylor series is in fact the geometric series expansion we have seen before. We also know that this series only converges to $\dfrac{1}{1-x}$ when $|x| < 1$.

In general, as with any convergent series, The Taylor series $P(x)$ of a function $f(x)$ will converge to $f(x)$ **if and only if** the sequence of partial sums, i.e. the "Taylor Polynomials" converge to $f(x)$. In other words, if $f(x) = P(x)$, then

$$\lim_{N\to\infty} P_N(x) = f(x)$$

Definition 5.4.59

The Nth **remainder** of the Taylor series $R_N(x)$ is defined as

$$R_N(x) = f(x) - P_N(x)$$

In other words, $R_N(x)$ is the sum of the Taylor series starting at the $(N+1)$th term and it tells us how well the partial sums $P_N(x)$ approximate the function $f(x)$.

Using this new notation,

$$\lim_{N\to\infty} P_N(x) = f(x) \text{ is equivalent to } \lim_{N\to\infty} R_N(x) = 0.$$

So we get the following theorem.

Theorem 5.4.60: Taylor's Theorem

Suppose $f(x)$ has derivatives of all orders on an interval $I = (a-\rho, a+\rho)$, for $\rho > 0$. Then the Taylor series

$$P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

converges to $f(x)$ for all x in I if and only if

$$\lim_{N\to\infty} R_N(x) = 0$$

for all x in I.

Let's use the idea of the remainder to explain how we can find the accuracy of numerical approximation of a function using its Taylor polynomials.

Example 5.4.61

Consider the following problem. We wish to approximate $\sin(1)$ numerically using the Taylor series for $\sin(x)$ centered at $x = 0$. Use the Taylor series from example 58.

Step 1. $P_1(x) =$ .... $\implies P_1(1) =$ ....

Step 2. $P_3(x) =$ .... $\implies P_3(1) =$ ....

Step 3. $P_5(x) =$ .... $\implies P_5(1) =$ ....

There is definitely still some error in this approximation. In fact, the error when you you approximate $\sin(1)$ using $P_N(x)$ is exactly equal to the remainder $R_N(x)$. So if we can somehow give a numerical upper bound for $R_N(x)$, we will have a good idea of how good our approximation is.

Fortunately, we have a theorem that does the job for us. We will not prove it here, but you can check it out in your textbook. It actually follows from the Mean value Theorem.

Theorem 5.4.62: Lagrange's formula

There exists a number z between a and x such that

$$R_N(x) = \frac{f^{(N+1)}(z)}{(N+1)!}(x-a)^{N+1}.$$

Going back to example 61, we can now find the maximum possible error for using P_5 to approximate $\sin(1)$. We look at

$$R_5(1) = \left|\frac{f^{(6)}(z)(1-0)^6}{6!}\right| = \left|\frac{\cos(z)(1)^6}{6!}\right|$$

where z is some number between 0 and 1. But we know that $|\cos z| \leq$ .... Hence we have,

$$R_5(1) = \left|\frac{\cos(z)(1)^6}{6!}\right| \leq \text{......................................}$$

In general, we get the following theorem.

Theorem 5.4.63: Taylor's Inequality

If there exists a real number $M > 0$ such that $\left|f^{(N+1)}(x)\right| \leq M$ for $|x-a| \leq d$, then remainder $R_N(x)$ of the Taylor series satisfies the inequality

$$|R_N(x)| \leq \frac{M}{(N+1)!}|x-c|^{N+1} \quad \textit{for } |x-a| \leq d$$

Question 84.

(a) *Find the fourth order Taylor polynomial for* $\ln(x)$.

(b) *Use it to approximate* $\ln(1.5)$.

Question 85.

(a) *Approximate the function* $f(x) = \sqrt[3]{x}$ *by a Taylor polynomial of degree two at* $a = 8$.

(b) *How accurate is the approximation for* $7 \leq x \leq 9$?

Question 86.

Use the third nonzero Taylor polynomial of e^{-x^2} *to approximate the value of* $\int_0^1 e^{-x^2}\,dx$. *What is the maximum possible error in your approximation?*

Projects

§A Project 1: Fourier Coefficients

A.1 What this project is about

This project introduces Fourier coefficients and Fourier polynomials of simple functions to illustrate an important application of integrals that uses the technique of integration by parts and properties of definite integrals.

A.2 Prerequisites and tech requirements

Before starting this project, you should

- have a solid understanding of Integration by Parts,
- be familiar with the summation (sigma) notation,
- have some knowledge of what the graphs of basic trigonometric functions look like, and
- know how to use `DESMOS` to plot functions.

A.3 Grading criteria

This project will be graded based on the EMPX rubric (see the 'Assessment' document for details). You can check your Moodle gradebook to see your grade and view feedback left by the professor. These appear as text annotations on your PDF submission or as general comments next to the grade. Grades of **E** or **M** may not have much feedback. Grades of **P** or **X** always have feedback, so please look carefully for this.

In order to earn an **E** or **M**, your submission must:

- show all of your work neatly and in a ordered manner.
- back up any claim you make with sufficient proof.
- explain your reasoning in a way that could be understood by a classmate who understands the mathematical concepts but has no familiarity with the particular problem being solved.

In short, readers of your work should not have to fill in any details or guess your thought process.

> **Note:** Completing questions 1001-1004 correctly will earn you an M. You must additionally complete question 1005 correctly to earn an E.

A.4 Project Task

Suppose we have a continuous function $f(x)$ on the interval $[-\pi,\pi]$. Then for each integer $i = 0, 1, 2, \ldots$, we can define the i^{th} **Fourier coefficients** of the function f by the formulae:

$$a_i = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(ix)\,\mathrm{d}x \qquad \text{and} \qquad b_i = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(ix)\,\mathrm{d}x$$

Note that the values a_i and b_i are all constants, not functions. But their values depend on the choice of f.

Definition 0.1.64

We will define the N^{th} **"Fourier polynomial"** of $f(x)$ to be

$$P_N(x) = \frac{a_0}{2} + \sum_{i=1}^{N} [a_i \cos(ix) + b_i \sin(ix)] \qquad \text{for } N = 1, 2, \ldots.$$

Note: The Fourier Polynomial is not actually a polynomial function! It's a misnomer.

Fourier coefficients and polynomials are named for the French mathematician Joseph Fourier. In a speech for the French Academy of Sciences in 1807, **Fourier proposed that these polynomials could be used to approximate arbitrary functions on the interval** $[-\pi, \pi]$. He was led to study these polynomials and make his bold statement by studying heat flow.

Unfortunately, the great French mathematicians of the time did not agree and did not take his ideas seriously. They were wrong. Today, the ideas introduced by Fourier are used to study phenomena that exhibit wavelike (periodic) behavior, such as sound and light. These ideas have been used in physics, engineering, and even economics. This problem is an introduction to some of Fourier's work.

Warning: For the following exercises, any time a problem asks you to plot something in DESMOS, you must include a screenshot of that plot in your final report.

Question 1001.

Let $f(x) = x$. Either using integration by parts, or by using DESMOS *find the first six sets of Fourier coefficients $a_0, b_0, a_1, b_1, \ldots, a_5, b_5$.*

(a) *Do you see any pattern in your answers? Can you guess a general formula for a_i and b_i, in terms of i if necessary.*

(b) *Write down the first three Fourier polynomials P_1, P_2, and P_3, P_4, and P_5 associated to f in expanded form (i.e. not using the sigma notation).*

(c) *Use* DESMOS *to plot the function $f(x)$ and these five Fourier polynomials.*

(d) *What can you conclude from the pictures? Does it look like $P_n(x) \to f(x)$ as n gets larger?*

Question 1002.

Now suppose f is an odd function that is continuous on the interval $[-c, c]$.

(a) *Use the area interpretation of definite integral and the definition of an odd function to explain why*

$$\int_{-c}^{c} f(x)\,dx = 0$$

(b) *If $f(x)$ is an odd function, is $g(x) = f(x)\cos(ix)$ an odd function? Why?*

(c) *If f is any odd function on $[-\pi, \pi]$, what can you say about the Fourier coefficients a_i of f?*

Question 1003.

Next, let's consider the case of even functions that are continuous on an interval $[-c, c]$. After thinking about what happened for the odd functions, what can you conclude about the Fourier coefficients b_i of an even function? Explain.

So far, the domain of our function was chosen to be an interval of the form $[-c, c]$, not the entire real number line. But that's enough if we are working with periodic functions!

Question 1004.

Why did Fourier think that these "polynomials" would be a good technique for approximating ***periodic*** *functions?*

This process of taking a periodic function and approximating it as a sum of sinusoidal functions is called **Fourier Transform**. File formats such as MP3 and JPEG depend on splitting up a function into its component sine waves! Click here to visit an interactive website where you can get a visual idea of what Fourier transform does.

Consider a periodic function $g(x)$ of period 2π, i.e. it repeats after every 2π. Between $[-\pi, \pi)$, the function is defined as

$$g(x) = \begin{cases} -1, & -\pi \le x < 0 \\ 1, & 0 \le x < \pi \end{cases}$$

This is clearly a discontinuous function. The graph of the function for all Real numbers looks like the black curve in fig. 2 and it repeats on both ends (note that the vertical parts are just for drawing purposes, they are not part of the graph, since a graph cannot have a vertical line it). We will call it the **Square Wave Function**.

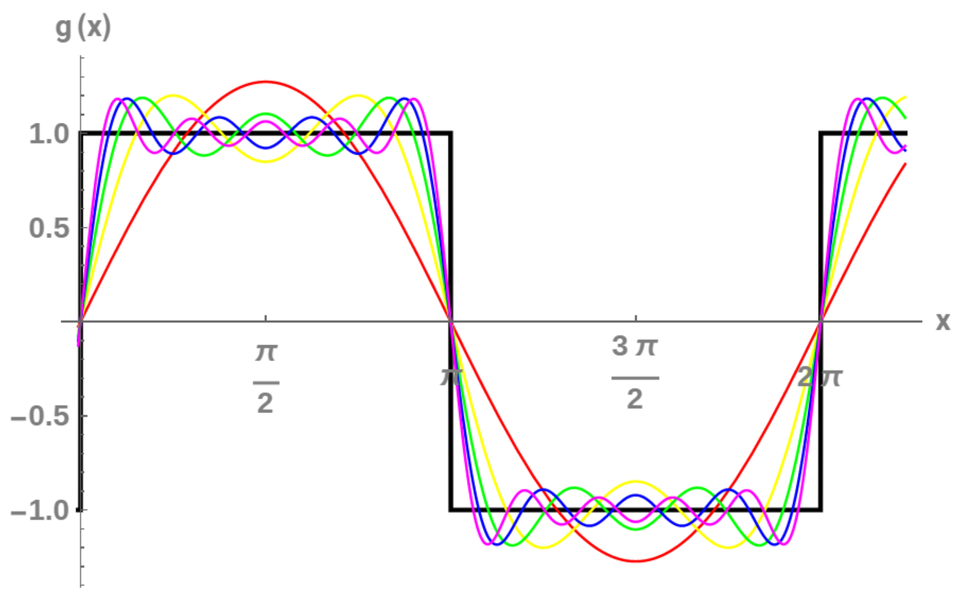


Figure 2: Graph of $g(x)$ and its approximations by Fourier polynomials

If you have checked out the website linked above, you know that this function can be approximated as a sum of sinusoidal functions. The approximations are the Fourier Polynomials! Let's find out mathematically what the coefficients are.

■ Question 1005.

☐

Let $g(x)$ be the square wave function.

(a) *Find a formula for the i^{th} Fourier coefficients a_i and b_i for $g(x)$, in terms of i. Note that $g(x)$ is an odd function, so your work is essentially halved.*

(b) *Plot the fifth Fourier polynomial P_5 associated to g using* `DESMOS`. *It should match one of the colored curves in fig. 2.*

Hint: P_5 for g should be sum of three sine functions.

(c) *What happens when we plot higher order Fourier polynomials? Do you believe they will approximate the Square Wave function better?*

§B Project 2: Laplace Transform

B.1 What this project is about

For this project, we are going to explore an advanced application of integration: integral transforms. Specifically, we describe the **Laplace transform** and some of its properties. The Laplace transform is used in engineering and physics to simplify the computations needed to solve some problems. It takes functions expressed in terms of time and transforms them into functions expressed in terms of frequency. It turns out that, in many cases, the computations needed to solve problems in the frequency domain are much simpler than those required in the time domain.

B.2 Prerequisites and tech requirements

B.3 Grading criteria

This project will be graded based on the EMPX rubric (see the 'Assessment' document for details). You can check your Moodle gradebook to see your grade and view feedback left by the professor. These appear as text annotations on your PDF submission or as general comments next to the grade. Grades of **E** or **M** may not have much feedback. Grades of **P** or **X** always have feedback, so please look carefully for this.

In order to earn an **E** or **M**, your submission must:

- show all of your work neatly and in a ordered manner.
- back up any claim you make with sufficient proof.
- explain your reasoning in a way that could be understood by a classmate who understands the mathematical concepts but has no familiarity with the particular problem being solved.

In short, readers of your work should not have to fill in any details or guess your thought process.

B.4 Project Task

The Laplace transform is an operation that changes a function of time (denoted by the variable t) to a function of frequency (denoted by the variable s). Given an input function $f(t)$, its Laplace transform $\mathcal{L}\{f(t)\}$ is given by $\mathrm{F}(s)$, which is defined in terms of an improper integral as

$$\mathrm{F}(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t.$$

Note that the input is a function of time, $f(t)$, and the output is a function of frequency, $\mathrm{F}(s)$.

Let's start with a simple example. Here we calculate the Laplace transform of $f(t) = t$. We have

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t\,\mathrm{d}t.$$

This is an improper integral, so we express it in terms of a limit, which gives

$$\mathcal{L}\{t\} = \int_0^\infty te^{-st}\,\mathrm{d}t = \lim_{z\to\infty} \int_0^z te^{-st}\,\mathrm{d}t.$$

Now we use integration by parts to evaluate the integral. Note that we are integrating with respect to t, so we treat the variable s as a constant. We have

$$\begin{aligned} u &= t & \mathrm{d}v &= e^{-st}\,\mathrm{d}t \\ \mathrm{d}u &= \mathrm{d}t & v &= -\frac{1}{s}e^{-st}. \end{aligned}$$

Then we obtain

$$\begin{aligned} \lim_{z\to\infty}\int_0^z te^{-st}\,\mathrm{d}t &= \lim_{z\to\infty}\left[\left[-\frac{t}{s}e^{-st}\right]\Big|_0^z + \frac{1}{s}\int_0^z e^{-st}\,\mathrm{d}t\right] \\ &= \lim_{z\to\infty}\left[\left[-\frac{z}{s}e^{-sz} + \frac{0}{s}e^{-0s}\right] + \frac{1}{s}\int_0^z e^{-st}\,\mathrm{d}t\right] \\ &= \lim_{z\to\infty}\left[\left[-\frac{z}{s}e^{-sz} + 0\right] - \frac{1}{s}\left[\frac{e^{-st}}{s}\right]\Big|_0^z\right] \\ &= \lim_{z\to\infty}\left[\left[-\frac{z}{s}e^{-sz}\right] - \frac{1}{s^2}\left[e^{-sz}-1\right]\right] \\ &= \lim_{z\to\infty}\left[-\frac{z}{se^{zz}}\right] - \lim_{z\to\infty}\left[\frac{1}{s^2e^{sz}}\right] + \lim_{z\to\infty}\frac{1}{s^2} \\ &= 0 - 0 + \frac{1}{s^2} \\ &= \frac{1}{s^2} \end{aligned}$$

Thus the Laplace transform of $f(t) = t$ is $\mathrm{F}(s) = \dfrac{1}{s^2}$.

■ Question 2001. □

(a) *Calculate the Laplace transform of $f(t) = 1$.*

(b) *Calculate the Laplace transform of $f(t) = e^{-3t}$.*

(c) *Calculate the Laplace transform of $f(t) = t^2$. (Note, you will have to integrate by parts twice.)*

Laplace transforms are often used to solve differential equations. Differential equations are not covered in this course; but, for now, let's look at the relationship between the Laplace transform of a function and the Laplace transform of its derivative.

Let's start with the definition of the Laplace transform. We have

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,\mathrm{d}t = \lim_{z\to\infty}\int_0^z e^{-st}f(t)\,\mathrm{d}t$$

■ Question 2002. □

Use integration by parts to evaluate $\lim_{z\to\infty}\int_0^z e^{-st}f(t)\,\mathrm{d}t$. *(Let $u = f(t)$ and $\mathrm{d}v = e^{-st}\,\mathrm{d}t$.) After integrating by parts and evaluating the limit, you should see that*

$$\mathcal{L}\{f(t)\} = \frac{f(0)}{s} + \frac{1}{s}\left[\mathcal{L}\{f'(t)\}\right].$$

Then,

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

Thus, differentiation in the time domain simplifies to multiplication by s in the frequency domain.

The final thing we look at in this project is how the Laplace transforms of $f(t)$ and its antiderivative are related. Let $g(t) = \int\limits_0^t f(u)\,\mathrm{d}u$. Then,

$$\mathcal{L}\{g(t)\} = \int\limits_0^\infty e^{-st} g(t)\,\mathrm{d}t = \lim_{z\to\infty} \int\limits_0^z e^{-st} g(t)\,\mathrm{d}t$$

Question 2003.

Use integration by parts to evaluate $\lim\limits_{z\to\infty} \int\limits_0^z e^{-st} g(t)\,\mathrm{d}t$. *(Let* $u = g(t)$ *and* $\mathrm{d}v = e^{-st}\,\mathrm{d}t$*. Note that* $\mathrm{d}u = f(t)\,\mathrm{d}t$*. why?)*

As you might expect, you should see that

$$\mathcal{L}\{g(t)\} = \frac{1}{s} \cdot \mathcal{L}\{f(t)\}$$

Integration in the time domain simplifies to division by s in the frequency domain.

§C Project 3: The Koch Snowflake

C.1 What this project is about

This project gives an example of how geometric series can be used to construct fractal-like objects.

C.2 Prerequisites and tech requirements

Before starting this project,

- you should know how sequences can be defined recursively,
- you should have an understanding of what a geometric sequence and a geometric series is,
- when does a geometric series converge or diverge,
- what is the sum of a geometric series when it converges.

C.3 Grading criteria

To submit this project:

- Create a handwritten or typed document with your solution. Convert the document/picture of the document to PDF format using an app or software. Please do not submit images or MS Word documents.
- Upload the PDF to the appropriate project assignment in the projects area on Moodle.

This project will be graded based on the EMPX rubric. You can check your Moodle gradebook to see your grade and and view feedback left by the professor. These appear as text annotations on your PDF submission or as general comments next to the grade. Grades of **E** or **M** may not have much feedback. Grades of **P** or **X** always have feedback, so please look carefully for this.

In order to earn an **E** or **M**, your submission must:

- show all of your work neatly and in a ordered manner.
- back up any claim you make with sufficient proof.
- explain your reasoning in a way that could be understood by a classmate who understands the mathematical concepts but has no familiarity with the particular problem being solved.

In short, readers of your work should not have to fill in any details or guess your thought process.

C.4 Project Task

Koch Snowflake is an example of an infinitely jagged fractal curve, obtained as a limit of polygonal curves. You start with an equilateral triangle (Stage 0). Then to produce Stage 1, for each side of the triangle, you replace the middle third with two line segments, each a third a length of the side. Then you continue this process indefinitely. See the figures below:

That is, if T_n denotes the polygonal figure at the n-th stage, then the Koch Snowflake is the limit $T = \lim_{n\to\infty} T_n$. This shape is interesting because it has an infinite perimeter, but finite area. Follow the steps below to see why!

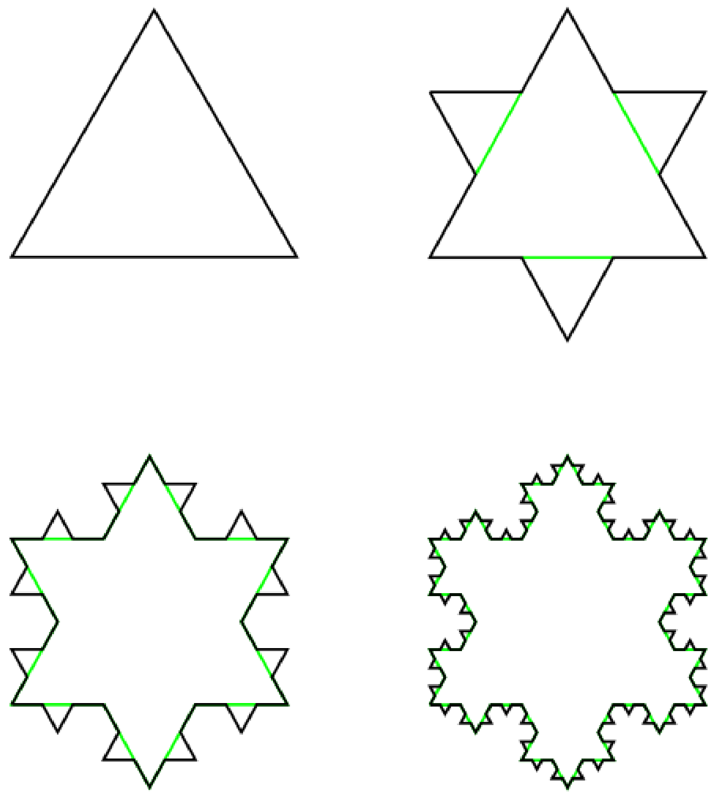


Figure 3: Stages 0, 1, 2, and 3 for the Koch Snowflake construction, from Wikipedia

Perimeter

Question 3001.

First, we examine the perimeter. Let P_n be the perimeter of the polygon at the n-th stage of the construction. Hence, P_0 is the perimeter of the original equilateral triangle.

(a) *Determine a formula for P_1 in terms of P_0.*

(b) *Now, determine a formula for P_2 in terms of P_1. It might help to figure out how many total sides you have in stage 1.*

(c) *Conjecture how you could write P_n recursively in terms of P_{n-1}. Then find a closed-form formula for P_n.*

(d) *The perimeter P of the Koch snowflake is the limit $\lim_{n\to\infty} P_n$. Calculate this limit! Is it finite or infinite?*

Area

Question 3002.

Now we want to compute the area of the Koch snowflake. Let A_n be the area of the polygon at the n-th stage. So A_0 is the area of the original equilateral triangle.

(a) *First we should recall how to determine the area of an equilateral triangle in terms of the length of a side. Suppose an equilateral triangle has side length s. Use your knowledge of geometry or trigonometry to determine the area in terms of s*

[HINT: figure out the height.]

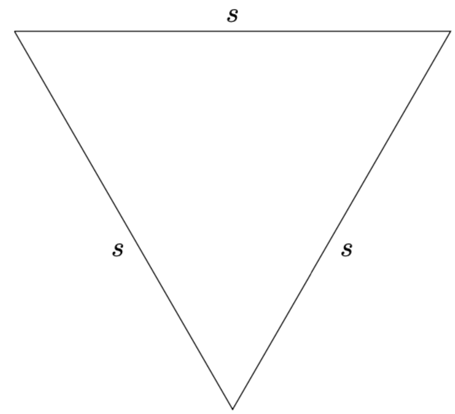


Figure 4: Equilateral triangle of side length s

(b) *Consider one of the new triangles you add in stage* 1*. What is the area of one of these triangles? Determine this in terms of* A_0.

[HINT: a new triangle has side length one-third that of the previous triangle.]

(c) *How many triangles do we add in Stage* 1*? Generalize to determine how many triangles are added in the n-th stage (it should be of the form* $3(\ldots)^{n-1}$*).*

(d) *What is the **total** area of the triangles added in Stage* 1*? Determine this in terms of* A_0.

*Then determine a general formula for the total area of all the **new** triangles formed in Stage n. You should be able to write this in the form* $\frac{3}{4}(\ldots)^n A_0$.

(e) *We can think of the area* A_n *as the previous area* A_{n-1} *plus the area of the newly added triangles in Stage n. Use this idea to write out expressions for* A_1, A_2*, and* A_3 *in terms of* A_0*. Try **not** to simplify too much and collect like terms together or factorize.*

(f) *Use your previous work to write out an explicit formula for* A_n *in terms of* A_0*, using the sigma notation. These should remind you of the partial sums of a series.*

(g) *The area* A *of the Koch snowflake will be the limit* $A = \lim_{n\to\infty} A_n$*. what is the sum of the resulting series? Combine this altogether to show that* $A = \frac{8}{5}A_0$.

§D Project 4: Irrationality of e

D.1 What this project is about

In this project, we will use the Maclaurin series of e^x to prove that Euler's constant, the number e, is irrational.

D.2 Prerequisites and tech requirements

To submit this project:

- Create a handwritten or typed document with your solution. Convert the document/picture of the document to PDF format using an app or software. Please do not submit images or MS Word documents.
- Upload the PDF to the appropriate project assignment in the projects area on Moodle.

D.3 Submission Instructions and Grading criteria

This project will be graded based on the EMPX rubric. You can check your Moodle gradebook to see your grade and and view feedback left by the professor. These appear as text annotations on your PDF submission or as general comments next to the grade. Grades of **E** or **M** may not have much feedback. Grades of **P** or **X** always have feedback, so please look carefully for this.

In order to earn an **E** or **M**, your submission must:

- show all of your work neatly and in a ordered manner.
- back up any claim you make with sufficient proof.
- explain your reasoning in a way that could be understood by a classmate who understands the mathematical concepts but has no familiarity with the particular problem being solved.

In short, readers of your work should not have to fill in any details or guess your thought process.

> **Important Note.** There will be only one chance to submit this project, there will be no chance for revision.

D.4 Project Task

Before we begin, we need to define precisely what it means to be a rational or irrational number.

Definition 0.4.65

A number x is called **rational** if we can find two integers u and v, with $v \neq 0$, such that $x = \dfrac{u}{v}$.

In other words, a number is rational if we can write it as a fraction. For example, $\frac{1}{2}$, 0.3, $\frac{34}{57}$ etc. are rational numbers. A number is called **irrational** if it is **not** a rational number!

Example 0.4.66

Examples of some famous irrational numbers include the ratio π of a circle's circumference to its diameter, Euler's number e, the golden ratio φ, and the square root of two. In fact, all square roots of prime numbers are irrational.

It is however not always easy to **prove** that the examples in the last paragraph are irrational. In fact, the story goes that the first person who actually showed real numbers can be irrational, was thrown overboard from his ship by fellow Mathematicians who thought it was heretical!*

In this project, we are going to give a "proof by contradiction" of the fact that e is irrational. The technique goes as follows:

We begin by asking what it would mean if e **were** in fact a rational number! So in that hypothetical scenario, we are saying that we should be able to find two integers u and v (with $v \neq 0$) such that e is equal to $\frac{u}{v}$. If we can show that this hypothetical scenario leads to obviously erroneous conclusions (e.g. $1 = 2$), then that would imply our assumption must have been wrong!

■ Question 4001. □

Assume e is rational and $e = \frac{u}{v}$, *where u and v are integers and* $v \neq 0$.

(a) *Start with the Maclaurin Series expansion of* e^x. *You do not have to find it yourself, it's in your class notes.*

Write down the Maclaurin polynomials $p_0(x), p_1(x), p_2(x), p_3(x)$, *and* $p_4(x)$ *for* e^x. *Then evaluate the values of* $p_0(1), p_1(1), p_2(1), p_3(1)$, *and* $p_4(1)$. *These should be successive better approximations for* $e^1 = e$.

(b) *Let* $R_n(x)$ *denote the remainder (see* definition 59*) when using* $p_n(x)$ *to estimate* e^x. *Therefore,* $R_n(x) = e^x - p_n(x)$, *and* $R_n(1) = e - p_n(1)$.

Assuming that $e = \frac{u}{v}$ *for integers u and v, write down expressions for* $R_0(1), R_1(1), R_2(1), R_3(1), R_4(1)$.

(c) *We are going to focus on* $R_v(1)$, *where v is the denominator of e as above.*

Write down the formula for the vth Maclaurin polynomial $p_v(x)$ *for* e^x *and the corresponding remainder* $R_v(x)$. *Then substitute* $x = 1$ *and show that the product* $v!R_v(1)$ *is an integer. You must explain why it's a whole number (i.e. the denominator becomes 1 after simplification).*

(d) *Show that* $v!R_v(1)$ *can be simplified to* $\displaystyle\sum_{i=v+1}^{\infty} \frac{v!}{i!}$.

(e) *We will call the above expression* R. *Clearly,* R *is an infinite series. Note that the first term in the sum is* $\frac{v!}{(v+1)!} = \frac{1}{v+1}$. *The next term is* $\frac{v!}{(v+2)!} = \frac{1}{(v+1)(v+2)}$. *It doesn't necessarily look like any series you might be familiar with, but you could compare the series with a geometric series!*

Find a geometric series that you can compare R *to, then use the formula for the sum of the geometric series to show that* $M < \frac{1}{v}$.

> ***Note:*** *This is the main step in the proof. Note that it's not enough to just show that the given series* R *is convergent, we are also finding an upper bound for the actual infinite sum.*

(f) *Recall that we showed in part (c) that* R *is an integer. Can it be less than* $\frac{1}{v}$?

So we find that our initial assumption has led us to an absurd and contradictory conclusion! So we conclude that the initial assumption must have been wrong. Consequently, e must have been an irrational number, to begin with!

*This happened in Ancient Greece.