Calculus & Analytical Geometry II

Lecture 35-36 Worksheet

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§A. Representations of Functions as Power Series

In this section we learn how to represent certain types of functions as sums of power series by manipulating geometric series or by differentiating or integrating such a series. You might wonder why we would ever want to express a known function as a sum of infinitely many terms. We will see later that this strategy is useful for integrating functions that don't have elementary antiderivatives, for solving differential equations, and for approximating functions by polynomials. Scientists do this to simplify the expressions they deal with; computer scientists do this to represent functions on calculators and computers.

Let's start with the geometric series example we have seen before:

Example A.1

$$F(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 \ldots = \frac{1}{1-x} \text{ for } |x| < 1$$

We can make simple substitutions to this to obtain new power series (we did this last time). Try this simple one: make a substitution in $F(x)$ above and obtain a power series expansion $G(x)$ for the function $\frac{1}{1+x}$ on the interval $(-1, 1)$. Write the series below.

$$\frac{1}{1+x} = \frac{1}{1-(\quad)} = \underline{\hspace{10cm}}$$

Example A.2

Similarly, let's try to find a power series representation for $\frac{1}{x+2}$. The first thing to note is that we can rewrite the function look like $\frac{1}{1-r}$ for some r, and that way we can use the geometric series expansion!

$$\frac{1}{x+2} = \frac{1}{2+x} = \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2\left[1-\left(-\frac{x}{2}\right)\right]} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$$

This series converges when $|-x/2| < 1$, that is, $|x| < 2$. So the interval of convergence is __________.

Question 1.

☐

Starting from the last example, can you find the power series representation for $\frac{x^3}{x+2}$? Make sure to also determine the interval of convergence.

■ Question 2. □

Find the power series representation of $\frac{2}{3-x}$ *and its interval of convergence.*

§B. Differentiation and Integration of Power Series

Let us continue with the power series $G(x)$ obtained above for $\frac{1}{1+x}$. What would happen if we differentiate each term on the right-hand side of $G(x)$ with respect to x? We get

$$\frac{d}{dx}G(x) = \frac{d}{dx}\left(\frac{1}{1+x}\right) = -\frac{1}{(1+x)^2} = \underline{\hspace{10cm}}$$

Rewriting it using sigma notation gives ____.

■ Question 3. □

Use the result above to compute the exact sum of the series $\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{3^n}}$.

We can also try integrating each term of the power series for $\frac{1}{1+x}$. What function is our new power series an expansion of?

$$\int \frac{1}{1+x}\,dx = \qquad\qquad = \underline{\hspace{10cm}}$$

■ Question 4. □

What is the interval of convergence for above series?

Theorem B.3

If the power series $\sum c_n(x-a)^n$ *has radius of convergence* $R > 0$, *then the function* F *defined by*

$$F(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ *and*

(i)

$$F'(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}[c_n(x-a)^n] = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

(ii)

$$\int F(x)\,dx = \int \left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] dx = \sum_{n=0}^{\infty} \int [c_n (x-a)^n]\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$
$$= C + c_0(x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + \cdots$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

The theorem essentially says that we can obtain F′ by differentiating the series for F term-by-term and we can also obtain the antiderivative $\int F(x)\,dx$ by integrating term-by-term.

Note: The radius of convergence R will always be the same for the derivative or integral of a power series - but the interval of convergence may change. We saw this with our second example above! The interval of convergence for $\frac{1}{1+x}$ was $(-1,1)$ but it changed to $(-1,1]$ for $\ln(1+x)$.

Note 2. The idea of differentiating a power series term by term is the basis for a powerful method for solving differential equations. We can also use this idea to anti-differentiate functions we couldn't by usual means before.

Example B.4

We are going to find the power series expansion for $f(x) = \arctan(x)$. Recall that

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$

$(x^2)^3$

Now start with the power series for $\frac{1}{1+x^2}$.

$\frac{1}{1+x^2} =$ $1 - x^2 + x^4 - x^6 + x^8 - x^{10} + x^{12} - \ldots\ldots$.

Then integrate both sides with respect to x.

$\arctan x = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots\ldots$

when $x = 0$, $\arctan(0) = 0 \Rightarrow C = 0$.

$= +x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$

$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} (-1)^n$

even $\rightarrow x^{2n}$

odd $\rightarrow x^{2n+1}$

$n=0 \rightarrow 2n+1=1$

$n=1 \rightarrow 2n+1=3$

$n=2 \rightarrow 2n+1=5$

We can continue to make substitutions to make more power series. We can also use this to approximate values like ln(2) or other inputs to the natural log or inverse tangent functions. Below are two such examples. Also, check out this Desmos page to see how the partial sums of these series approximate the given function (which, although not a proof, gives you strong evidence that these power series expansions are indeed true):

https://www.desmos.com/calculator/cwt347lgkj

Example B.5

We know that the power series of $\ln(1+x)$ converges at $x=1$. Hence

$$\ln(2) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

Thus the alternating harmonic series converges to $\ln 2$.

Example B.6

One great use of power series is to make use of them for integration purposes. Consider the following integral:

$$\int \arctan(x^2)\,\mathrm{d}x$$

We could try doing Integration By Parts like we learned before, the resulting integral would be doable but doesn't look very easy. (Indeed `WolframAlpha` tells me that the integral pretty ugly!) Instead, why don't we come up with a power series expansion for $\arctan(x^2)$ and just integrate that?

Question 5.

Come up with the power series expansion for $\dfrac{1}{1+x^7}$ *and find the integral* $\displaystyle\int \frac{1}{1+x^7}\,\mathrm{d}x$.

Can you approximate the definite integral $\displaystyle\int_0^{0.5} \frac{1}{1+x^7}\,\mathrm{d}x$ *correct to within* 10^{-7}*?*

Question 6.

Consider $\displaystyle\int x\ln(1+x)\,\mathrm{d}x$.

(a) *Use power series to obtain an antiderivative.*

(b) *Use Integration By Parts and substitution to solve the integral and compare the two answers.*

§C. More Practice Problems

Here's a summary of some of the Power Series we derived in this worksheet.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots \qquad \text{for } |x| < 1$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - \dots \qquad \text{for } |x| < 1$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \qquad \text{for } -1 < x \le 1$$

$$\arctan(x) = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \qquad \text{for } -1 \le x \le 1$$

Question 7.

Find a power series representation for the function and determine the radius of convergence.

(a) $f(x) = \ln(5-x)$

(b) $f(x) = x^2 \arctan\left(x^3\right)$

(c) $f(x) = \dfrac{1}{(1+4x)^2}$

(d) $f(x) = \dfrac{x}{(1+4x)^2}$

(e) $f(x) = \dfrac{1}{(1+x)^3}$

(f) $f(x) = \dfrac{1}{(2-x)^3}$

(g) $f(x) = \left(\dfrac{x}{2-x}\right)^3$

(h) $f(x) = \dfrac{1+x}{(1-x)^2}$

(i) $f(x) = \dfrac{x^2+x}{(1-x)^3}$