MATH 111 - Calculus and Analytic Geometry I

Lecture 31 Worksheet

Fall 2020 | **Subhadip Chowdhury** | **Oct 21-23**

Title: L'Hôpital's Rule - Growth and Dominance

Summary: We will learn how to use derivatives to help us evaluate indeterminate limits.

Related Reading: Chapter 4.8 from the textbook.

§A. Motivation

Differential calculus is based on derivative, and the definition of the derivative involves a limit, so one can say that all of calculus rests on limits. Recall that the definition of a derivative is given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

An interesting thing to note here is that the limit on the right hand side has a $\frac{0}{0}$ form, not only does $h \to 0$, but also $f(x+h) - f(x) \to 0$ since f is continuous. Remember, saying that a limit has an indeterminate form only means that we don't yet know its value and have more work to do: indeed, limits of the form $\frac{0}{0}$ can take on any value.

Example A.1

Consider a very simple example where $f(x) = 2x^2$ and $g(x) = x^2$. Note that as $x \to 0$, both f and g approach 0. However, clearly $f(x)$ is always twice the amount of $g(x)$**.** So when $x \to 0$, the quantity $\frac{f(x)}{g(x)} \to 2$.

We learned different ways of calculating limits for functions in indeterminate form by factorizing, multiplying by conjugate etc. Now let's take a look at an example where those strategies may not work.

Example A.2

Consider the following limit which has a $\frac{0}{0}$ form

$$\lim_{x\to 0} \frac{\sin(2x)}{e^x - 1}.$$

There is no further simplification to be done here, so how do we calculate the limit?

Let's try to look at the graphs of the two functions $f(x)$ and $g(x)$ near the point $x = 0$. Open the `DESMOS` link at

https://www.desmos.com/calculator/upwozqfmno

The idea here is that that we can evaluate an indeterminate limit of the form $\frac{0}{0}$ by replacing each of the numerator and denominator with their 'tangents' at $x = 0$.

That clearly shows that the limit of the quotient is equal to 2 as $x \to 0$.

§B. L'Hôpital's Rule

In general, we can use the tangent line formula at $x = a$ to replace both $f(x)$ and $g(x)$ as follows. Suppose we need to calculate $\lim_{x \to a} \frac{f(x)}{g(x)}$ where $f(a) = g(a) = 0$. Recall that the equation of the tangent line to a function $y = f(x)$ at $x = a$ is given by $y = f(a) + f'(a)(x - a)$. Then,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(a) + f'(a)(x-a)}{g(a) + g'(a)(x-a)} = \lim_{x \to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} = \lim_{x \to 0} \frac{f'(a)}{g'(a)} = \frac{f'(a)}{g'(a)}$$

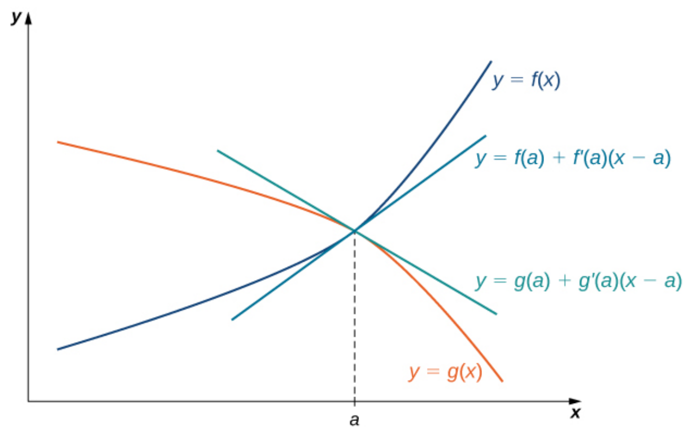

Figure 1: If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, then the ratio $f(x)/g(x)$ is approximately equal to the ratio of their linear approximations near a.

This result holds as long as $g'(a)$ is not equal to zero. The formal name of the result is L'Hôpital's Rule.

Theorem B.1

Let f and g be differentiable at $x = a$, and suppose that $f(a) = g(a) = 0$ and that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

■ Question 1.

□

Find the following limits:

(a) $\lim_{x \to 0} \frac{\sin x}{x}$

(b) $\lim_{x \to 1} \frac{x^5 + x - 2}{x^2 - 1}$

(c) $\lim_{x \to 1} \frac{2\ln(x)}{1 - e^{x-1}}$

(d) $\lim_{x \to 1} \frac{\sin(\pi x)}{\ln(x)}$

What if both $f'(a)$ and $g'(a)$ are also 0? Then we continue the process. A more general form of L'Hôpital's rule says,

Theorem B.2

Let f and g be differentiable at $x = a$, and suppose that $f(a) = g(a) = 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided that the limit on the right-hand side exists.

■ Question 2. □

Find the following limits:

(a) $\lim_{x \to 0} \frac{\sin(x) - x}{\cos(2x) - 1}$

(b) $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$

§C. L'Hôpital's Rule - Other Indeterminate Forms

The expressions $\frac{\infty}{\infty}$, $\infty - \infty$, $0 \times \infty$, 1^∞, ∞^0, and 0^0 are all considered indeterminate forms. These expressions are not real numbers. Rather, they represent forms that arise when trying to evaluate certain limits.

L'Hôpital's rule can be also applied to these indeterminate forms.

■ Question 3. □

Find the following limits:

(a) $\displaystyle\lim_{x \to \frac{\pi}{2}^-} \frac{\tan(x)}{x - \frac{\pi}{2}}$

(b) $\displaystyle\lim_{x \to 0^+} x \ln(x)$

(c) $\displaystyle\lim_{x \to 0} x \cot(x)$

(d) $\displaystyle\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$

(e) $\displaystyle\lim_{x \to \infty} \frac{e^x + x}{2e^x + x^2}$

(f) $\displaystyle\lim_{x \to \infty} \frac{(\ln x)^3}{x^2}$

■ Question 4.

Indeterminate form 0^0

Consider the function $g(x) = x^{2x}$, which is defined for all $x > 0$. Observe that $\lim_{x\to 0^+} g(x)$ is indeterminate due to its form of 0^0. (Think about how we know that $0^k = 0$ for all $k > 0$, while $b^0 = 1$ for all $b \neq 0$, but that neither rule can apply to 0^0.)

(a) Let $h(x) = \ln(g(x))$. Explain why $h(x) = 2x\ln(x)$.

(b) Use L'Hôpital's Rule to compute $\lim_{x\to 0^+} h(x)$.

(c) Based on the value of $\lim_{x\to 0^+} h(x)$, determine $\lim_{x\to 0^+} g(x)$.

■ Question 5.

Indeterminate form ∞^0

Find $\lim_{x\to\infty} x^{1/x}$.

§D. **Growth and Dominance**

Suppose the functions f and g both approach infinity as $x \to \infty$. Although the values of both functions become arbitrarily large as the values of x become sufficiently large, sometimes one function is growing more quickly than the other.

Definition D.1

We say that a function g **dominates** a function f provided that $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} g(x) = \infty$, and

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

Question 6.

(a) Which function dominates the other: $\ln(x)$ or $\sqrt{x}$?

(b) Which function dominates the other: $\ln(x)$ or $\sqrt[n]{x}$? (n can be any positive integer)

(c) Explain why e^x will dominate any polynomial function.

(d) Compare the growth rates of x^{100} and 2^x.

(e) Explain why x^n will dominate $\ln(x)$ for any positive integer n.

(f) Give any example of two nonlinear functions such that neither dominates the other.