MATH 111 - Calculus and Analytic Geometry I

Lecture 23 Worksheet

Fall 2020 **Subhadip Chowdhury** **Oct 9**

Title: Shape of a Graph I - Maxima, Minima, and FDT

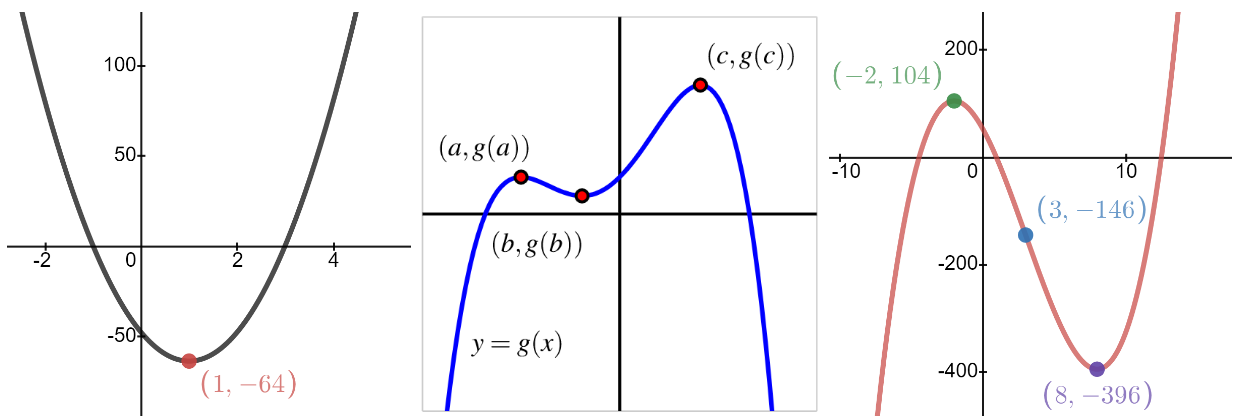
Summary: We will learn about how the first derivative of a function reveals important information about the behavior of the function, including the function's extreme values.

Related Reading: Chapters 4.3 and 4.5 from the textbook.

§A. Motivating Examples and Definitions

In many different settings, we are interested in knowing where a function achieves its least and greatest values. These can be important in applications — say to identify a point at which maximum profit or minimum cost occurs — or in theory to characterize the behavior of a function or a family of related functions.

Consider, for example, the three following graphs of functions.



■ **Question 1.** □

For each of the functions, answer the following.

(a) What is the maximum value of the function? What is the minimum value of the function?

(b) For what values of x, is the function increasing and decreasing? What is the sign of the derivative on the corresponding intervals?

Definition A.1

Given a function f, we say that $f(c)$ is a **absolute** or **global maximum** of f on an interval $\mathbf{I}$ if $f(c) \geq f(x)$ for all x in $\mathbf{I}$.

Similarly we call $f(c)$ a **absolute** or **global minimum** of f on an interval $\mathbf{I}$ whenever $f(c) \leq f(x)$ for all x in $\mathbf{I}$.

If $\mathbf{I}$ is not specified, we take $\mathbf{I}$ to be $(-\infty, \infty)$, the set of all real numbers.

Question 2.

Which of the above three functions has a global maximum or minimum? What are the values?

Definition A.2

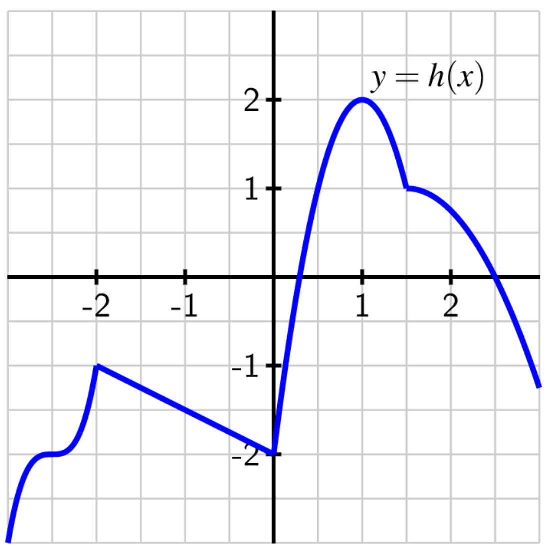
We say that $f(c)$ is a **local maximum** of f provided that $f(c) \geq f(x)$ for all x *near* c. Similarly, $f(c)$ is called a **local minimum** of f whenever $f(c) \leq f(x)$ for all x near c.

Question 3.

Which of the above three functions has a local maximum or minimum? What are the values?

Question 4.

Consider the function h given by the graph in the figure below.



(a) Identify all of the values of c in $(-3, 3)$ for which $h(c)$ is a local maximum of h

(b) Identify all of the values of c in $(-3, 3)$ for which $h(c)$ is a local minimum of h.

(c) Does h have a global maximum on the interval $[-3, 3]$? If so, what is the value of this global maximum?

(d) Does h have a global minimum on the interval $[-3, 3]$? If so, what is its value?

(e) Identify all values of c for which $h'(c) = 0$.

(f) Identify all values of c for which $h'(c)$ does not exist.

(g) True or false: every relative maximum and minimum of h occurs at a point where $h'(c)$ is either zero or does not exist.

(h) True or false: at every point where $h'(c)$ is zero or does not exist, h has a relative maximum or minimum.

§B. Critical Points and the First Derivative Test

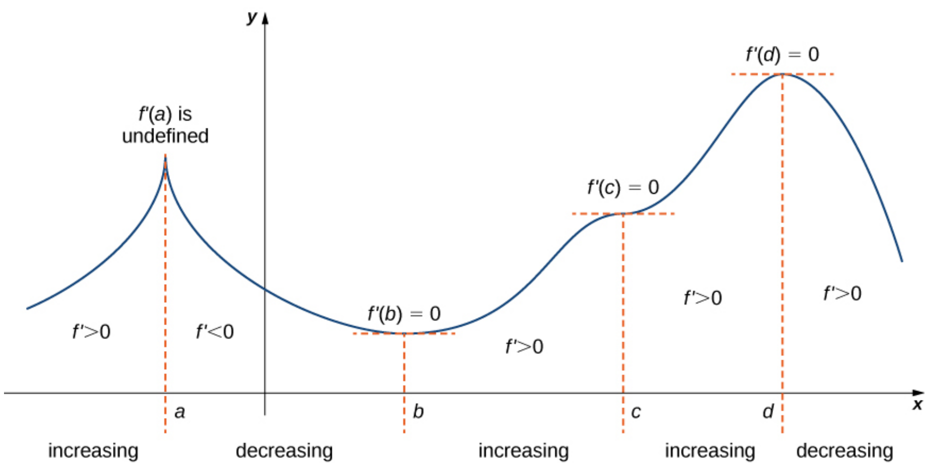
Definition B.1

For any function f, a point c in the *domain* of f where $f'(c)$ is either $\mathbf{0}$ or undefined, is called a **critical point** of the function. In addition, the point $(c, f(c))$ on the graph of f is also called a critical point. A critical value of f is the value, $f(c)$, at a critical point, c.

Theorem B.1: Local Extrema and Critical Points

If f has a local extremum at $x = c$ and f is differentiable at c then $f'(c) = \mathbf{0}$. In other words, every local extremum is a critical point.

The converse is not true. Not every critical point is a local extremum. Consider for example, a function $f(x)$ whose graph is as follows:



The function f has four critical points: $a, b, c,$ and d. The function has local maxima at a and d, and a local minimum at b, The function does not have a local extremum at c.

Perhaps, the most interesting observation to make here is that the sign of f' changes at all local extrema.

Theorem B.2: First Derivative Test

If c is a critical point of a continuous function f that is differentiable near c (except possibly at $x = c$), then f has a relative maximum at c if and only if f' changes sign from positive to negative at $c,$ and f has a relative minimum at c if and only if f' changes sign from negative to positive at c.

Question 5.

Suppose that $g(x)$ is a function continuous for every value of $x \neq 2$ whose first derivative is

$$g'(x) = \frac{(x+4)(x-1)^2}{x-2}.$$

Further, assume that it is known that g has a vertical asymptote at $x = 2$.

(a) Determine all critical points of g.

(b) By developing a carefully labeled first derivative sign chart, decide whether g has as a local maximum, local minimum, or neither at each critical point.

(c) Does g have a global maximum? global minimum? Justify your claims.

(d) What is the value of $\lim_{x \to \infty} g'(x)$? What does the value of this limit tell you about the long-term behavior of g ?

(e) Sketch a possible graph of $y = g(x)$.