MATH 111 - Calculus and Analytic Geometry I

Lecture 40-42 Worksheet

Fall 2020 | **Subhadip Chowdhury** | **Nov 18,20,23**

Title: Integration By Substitution

Summary: We will learn a method to find antiderivatives when the integrand is the result of a chain-rule derivative.

Related Reading: Chapter 5.(5,6) from the textbook.

§A. Recap

We learned in the last lectures that for a function $f(x)$ which is continuous on $[a,b]$,

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

where $F(x)$ is an antiderivative of $f(x)$. We call this a definite integral. Definite integrals help us determine area under a curve.

If we do not have limits of integration, we say we have an indefinite integral and the answer is a family of functions:

$$\int f(x)dx = F(x) + C, \text{ where } C \text{ is an arbitrary constant.}$$

Example A.1

For example,

$$\int (x^3 + \sin x)dx = \frac{x^4}{4} - \cos x + C,$$

where C is an arbitrary constant.

The Fundamental Theorem of Calculus gives us a method to evaluate integrals without using Riemann sums. The drawback of this method, though, is that we must be able to find an antiderivative, and this is not always easy. In this section we examine a technique, called integration by substitution, to help us find antiderivatives.

§B. Reversing the Chain Rule

Suppose we want to find $\int e^{5x}\,dx$. What is the antiderivative $F(x)$ of the function $f(x) = e^{5x}$? We did the following **Guess and Check** method before:

Guess and Check

$$\begin{aligned} \text{Guess:} \quad & F(x) = e^{5x} + C \\ \text{Check:} \quad & F'(x) = 5e^{5x} \\ \text{How do we fix this?} \quad & \frac{1}{5}F'(x) = e^{5x} \\ \text{We divide by } 5. \quad & F(x) = \frac{1}{5}e^{5x} + C \\ \int e^{5x}\,dx = \quad & \frac{1}{5}e^{5x} + C \end{aligned}$$

Now let's do it in another way!

Substitution

We start by creating a new variable for our "inner function". Let's call it $\heartsuit$.

$$\begin{gathered} \text{Let } \heartsuit = 5x \\ \text{Then } d\heartsuit = 5dx \\ \Longrightarrow \frac{1}{5}d\heartsuit = dx \end{gathered}$$

Now we use our new code to convert our integral to hearts.

$$\begin{aligned} \text{Substitute:} \quad & \int e^{5x}\,dx = \int e^{\heartsuit} \cdot \frac{1}{5}d\heartsuit \\ \text{Pull the constant to the outside:} \quad & \frac{1}{5}\int e^{\heartsuit} d\heartsuit \\ \text{Integrate:} \quad & \frac{1}{5}e^{\heartsuit} + C \\ \text{Plug in the original variable:} \quad & \frac{1}{5}e^{5x} + C \end{aligned}$$

Example B.1

Whenever f is a familiar function whose antiderivative is known and $u(x)$ is a linear function, it is straightforward to integrate a function of the form $h(x) = f(u(x))$.

Suppose we want to find $\int (2x-3)^6\,dx$. First, let's assume $\diamondsuit = 2x-3$. Then $d\diamondsuit = 2dx$.

So we can write

$$\int (2x-3)^6\,dx = \int \diamondsuit^6 \cdot \frac{1}{2}d\diamondsuit = \frac{1}{2}\int \diamondsuit^6\,d\diamondsuit = \frac{1}{2}\frac{\diamondsuit^7}{7} + C = \frac{(2x-3)^7}{14} + C$$

■ Question 1. □

Find $\int \sin(4x-3)\,dx$.

What do we do if the inner function is not linear? For example, what is $\int xe^{x^2}\, dx$? Let's take a look at a bit more complicated examples next. Recall that the Chain Rule states:

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

So equivalently, restating this relationship in terms of an indefinite integral,

$$\int f'(g(x)) \cdot g'(x)\, dx = f(g(x)) + C$$

How do we use this in practice?

Theorem B.1

Let $u = g(x)$, where $g'(x)$ is continuous over an interval I, and let $f(x)$ be continuous over $g(I)$. Let $F(x)$ be an antiderivative of $f(x)$. Then $du = g'(x)dx$, and we can write

$$\begin{aligned} \int f(g(x)) \cdot g'(x)\, dx &= \int f(u)\, du \\ &= F(u) + C \\ &= F(g(x)) + C \end{aligned}$$

Example B.2

Let's find $\int xe^{x^2}\, dx$.

The inside function is x^2, with derivative $2x$. The integrand has a factor of x, and since the only thing missing is a constant factor, we try $u = x^2$ to get

$$du = u'(x)dx = 2xdx \implies xdx = \frac{1}{2}du$$

Thus,

$$\int xe^{x^2}\, dx = \int e^u \cdot \frac{1}{2}\, du = \frac{1}{2}\int e^u\, du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C$$

Example B.3

Let's find $\int x^3\sqrt{x^4+5}\,dx$.

The inside function is x^4+5, with derivative $4x^3$. The integrand has a factor of x^3, and since the only thing missing is a constant factor, we try $u = x^4+5$ to get

$$du = u'(x)dx = 4x^3dx \implies x^3dx = \frac{1}{4}du$$

Thus,

$$\int x^3\sqrt{x^4+5}\,dx = \int \sqrt{u}\cdot\frac{1}{4}\,du = \frac{1}{4}\int u^{1/2}\,du = \frac{1}{4}\frac{u^{3/2}}{3/2} + C = \frac{1}{6}(x^4+5)^{3/2} + C$$

We saw in the preceding example that we can apply the substitution method when a constant factor is missing from the derivative of the inside function. However, we may not be able to use substitution if anything other than a constant factor is missing. For example, setting $u = x^4+5$ to find

$$\int x^2\sqrt{x^4+5}\,dx$$

does us no good because x^2dx is not a constant multiple of $du = 4x^3dx$. Substitution works if the integrand contains the derivative of the inside function, to within a constant factor.

Algorithm for Solving u-substitution Problems

Step 1. Look carefully at the integrand and select an expression $g(x)$ within the integrand to set equal to u. Select $g(x)$ such that $g'(x)$ is also part of the integrand.

Step 2. Substitute $g(x)$ by u and $g'(x)dx$ by du into the integral.

Step 3. We should now be able to evaluate the integral with respect to u. If the integral can't be evaluated we need to go back and select a different expression to use as u.

Step 4. Evaluate the integral in terms of u.

Step 5. Sustitute the expression $g(x)$ back in place of u and write the final answer in terms of x.

■ Question 2.

□

Find the following indefinite integrals and check the results by differentiation.

(a) $\displaystyle\int (\sec^2 x)\sqrt{\tan x}\, dx$

(b) $\displaystyle\int x(x^2+4)^{10}\, dx$

(c) $\displaystyle\int \frac{x^3}{\sqrt{1+x^4}}\, dx$

(d) $\int (e^{5\cos x})(\sin x)\, dx$

(e) $\int \frac{\sin x}{\cos^3 x}\, dx$

(f) $\int 3x^2 \cos(x^3)\, dx$

§C. **Evaluating definite integrals via u-substitution**

Let's solve the following integral using substitution: $\displaystyle\int_0^1 x^3(2x^4+1)^{10}\,dx$

Let $\bigstar = 2x^4+1$, find $d\bigstar =$

Solve for x^3dx:

Now use your "$\bigstar$ code" to translate from x's to $\bigstar$'s:

Now we must find the limits of integration in terms of $\bigstar$'s, instead of x's.

While using x's, the limits of integration were $x=0$ and $x=1$: $\displaystyle\int_{x=0}^{x=1} x^3(2x^4+1)^{10}\,dx$

Find the new lower limit. Let $x=0$ and solve for $\bigstar$:

Find the new upper limit. Let $x=1$ and solve for $\bigstar$:

Plug the limits of integration into our new integral:

Evaluate the integral:

Alternately, we can evaluate the corresponding indefinite integral, as we did on the previous worksheet. Then, once we have x's back in our answer, we can evaluate the integral using the given limits of integration on x. That would proceed as follows:

$$\begin{aligned}\int x^3(2x^4+1)^{10}\,dx &= \frac{1}{8}\int u^{10}\,du\\ &= \frac{1}{8}\frac{u^{11}}{11}+C = \frac{1}{88}(2x^4+1)^{11}+C\end{aligned}$$

And then,

$$\int_0^1 x^3(2x^4+1)^{10}\,dx = \frac{1}{88}(2x^4+1)^{11}\Big|_0^1 = \dots$$

■ Question 3.

Evaluate the following definite integrals using substitution:

(a) $\displaystyle\int_0^1 x\sqrt{1-x^2}\,dx$

(b) $\displaystyle\int_0^1 x^2\cos\left(\frac{\pi}{2}x^3\right)\,dx$

(c) $\displaystyle\int_0^{\sqrt{2}} xe^{-\left(\frac{x^2}{2}\right)}\,dx$

§D. **Antiderivatives with Natural Log**

Recall the antiderivative for $y = \frac{1}{x}$:

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

Why is there an absolute value? Because the domain of $y = \frac{1}{x}$ is $(-\infty, 0) \cup (0, \infty)$, whereas the domain of $y = \ln x$ is only $(0, \infty)$. Although we never explicitly said this, antiderivatives must be defined on the same interval as the function (upto losing endpoints).

■ Question 4. □

The next few problems all involve substitution and this new antiderivative formula. Your goal is to do a substitution and get an integral that looks like $\int \frac{1}{u}\,du$.

(a) $\int \frac{1}{x+5}\,dx$

(b) $\int \frac{x^2}{x^3+2}\,dx$

(c) $\int \frac{1}{x \ln x}\, dx$

(d) $\int \frac{e^x}{5+e^x}\, dx$

(e) $\int \tan x \, dx$

(Hint: write $\tan x = \frac{\sin x}{\cos x}$)

(f) $\int \cot x \, dx$

§E. **Manipulation Substitution**

Sometimes we need to manipulate an integral in ways that are more complicated than just multiplying or dividing by a constant. We need to eliminate all the expressions within the integrand that are in terms of x. When we are done, the new variable u should be the only variable in the integrand. In some cases, this means solving for x in terms of u.

Example E.1

Let's consider the example $\int x\sqrt{3x-4}\,dx$. Let $u = 3x-4$. Then $du = 3\,dx$ and notice that $x = \dfrac{u+4}{3}$.
Now substitute:

$$\int x\sqrt{3x-4}\,dx = \int \left(\frac{u+4}{3}\right)\sqrt{u}\,du = \frac{1}{3}\int \left(u^{3/2} + 4\sqrt{u}\right)\,du$$

Finish the integration.

■ Question 5.

☐

Find the following indefinite integrals and check the results by differentiation. Remember, let your heart be the inside. Also, remember that you can manipulate your substitution equations!

(a) $\displaystyle\int (x+1)\sqrt{2-x}\,dx$

(b) $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx$

(c) $\displaystyle\int \frac{x^2}{\sqrt{x+1}}\,dx$

(d) $\displaystyle\int \cos(\theta)\,(1-\cos(\theta))^{99}\sin(\theta)\,d\theta$

(e) **(Optional)** $\displaystyle\int \sqrt{1+\sqrt{x}}\,dx$