MATH 111 - Calculus and Analytic Geometry I

Lecture 9 Worksheet

Fall 2020 **Subhadip Chowdhury** **Sep 7**

Title: Types of Discontinuity

Summary: We discuss different types of discontinuities of a function. We also talk about how to fix a removable discontinuity by gluing two sides of a piece-wise defined function.

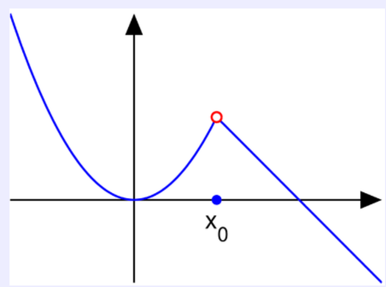
§A. Types of Discontinuity

Recall from our previous discussion about continuity that a function can be discontinuous for several reasons. We can classify the types of discontinuities into three broad categories as follows:

- **Removable Discontinuity** - We say that f has a removable discontinuity at a if $\lim\limits_{x\to a} f(x)$ exists but is not equal to $f(a)$ (which may or may not exist).

Example A.1

Consider the function $f(x) = \begin{cases} x^2 & \text{for } x < 1 \\ 0 & \text{for } x = 1 \\ 2-x & \text{for } x > 1 \end{cases}$

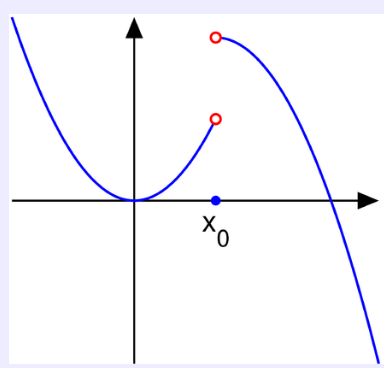


There is a removable discontinuity at $x = 1$. Both the left-hand and right-hand limits are 1.

- **Jump Discontinuity** - We say that f has a jump discontinuity at a if $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist, but $\lim\limits_{x\to a^-} f(x) \neq \lim\limits_{x\to a^+} f(x)$

Example A.2

Consider the function $f(x) = \begin{cases} x^2 & \text{for } x < 1 \\ 0 & \text{for } x = 1 \\ 2-(x-1)^2 & \text{for } x > 1 \end{cases}$

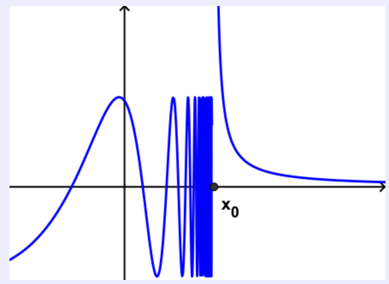


There is a jump discontinuity at $x = 1$. The left-hand limit is 1. The right-hand limit is 2.

- **Essential Discontinuity** - We say that f has an essential discontinuity at a if at least one of the one-sided limits, $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$, does not exist or is infinite.

Example A.3

Consider the function $f(x) = \begin{cases} \sin\dfrac{5}{x-1} & \text{for } x < 1 \\ 0 & \text{for } x = 1 \\ \dfrac{1}{x-1} & \text{for } x > 1 \end{cases}$



There is an essential discontinuity at $x = 1$. The left-hand limit DNE. The right-hand limit is ∞.

§B. Fixing Discontinuities & Gluing

If a function has a removable discontinuity, that implies that we can easily "fix" the discontinuity by filling in the hole. We can do this by using piecewise functions. [1]

Question 1.

The function $f(x) = \dfrac{\sin x}{x}$ has a removable discontinuity at $x = 0$. Determine the value of k necessary to make $\mathbf{F}(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ k & \text{if } x = 0 \end{cases}$ continuous everywhere.

In the next problem, you are trying to determine where exactly you would "glue" the two lines in order to have a continuous function.

Question 2.

Determine what value of k will make the function continuous on the given interval.

$$f(x) = \begin{cases} 2x + 7 & \text{if } 0 \leq x < k \\ 4x - 5 & \text{if } k \leq x \leq 10 \end{cases}.$$

In the next problem, there are removable discontinuities that need to be filled, but we also need to figure out what values of c and r will make sure the functions are "glued" together so as to be made continuous. Start by determining what k must be, then use that information to solve for c and then r.

[1] (Note: these exercises are like 145 - 149 in Section 2.4 of your textbook!)

■ Question 3.

Determine the values of k, c, and r that make the given function continuous everywhere.

$$f(x)=\begin{cases} \dfrac{x^4-16}{x^2-4} & \text{if } x<-2 \\ k & \text{if } x=-2 \\ -x^2+c & \text{if } -2<x<3 \\ \sqrt{rx} & \text{if } x\geq 3 \end{cases}.$$

■ Question 4.

Find the constants a and b, so that the following piece-wise defined function is continuous everywhere.

$$f(x)=\begin{cases} a-bx & \text{if } x\leq 1 \\ x^2 & \text{if } 1<x<2 \\ b+ax & \text{if } x\geq 2 \end{cases}$$