# MATH 111 - Calculus and Analytic Geometry I

## Lecture 25 Worksheet

**Fall 2020** | **Subhadip Chowdhury** | **Oct 14**

**Title:** Global Optimization

**Summary:** We will learn how to find global maximum and minimum of a function.

**Related Reading**: Chapters 4.3 and 4.7 from the textbook.

## §A. Recap of FDT and SDT

### Finding local maximum/minimum of a function $f(x)$

1. Find the critical points of $f(x)$.
2. Find $f''(x)$ and for every critical point $p$ of $f(x)$, check the sign of $f''(p)$.
   (a) If $f''(p) > 0$, then $p$ is a local minimum.
   (b) If $f''(p) < 0$, then $p$ is a local maximum.
   (c) If $f''(p) = 0$ or if $f'(p)$ doesn't exist, then check the sign of $f'(x)$ **near** $p$.
      i. Choose a small number $\epsilon$ e.g. $0.01$ so that $(p-\epsilon, p+\epsilon)$ doesn't contain any other critical point. This is to make sure we are checking sign 'locally' i.e. in a small neighborhood of the critical point $p$.
      ii. If $f'(x)$ changes sign of negative to positive as $x$ passes from left to right of $p$, then $p$ is a local minimum. In other words, if $f'(p-\epsilon) < 0$ and $f'(p+\epsilon) > 0$, then $p$ is a local min.
      iii. If $f'(x)$ changes sign of positive to negative as $x$ passes from left to right of $p$, then $p$ is a local minimum. In other words, if $f'(p-\epsilon) > 0$ and $f'(p+\epsilon) < 0$, then $p$ is a local max.
      iv. If $f'(x)$ does not change sign as $x$ passes from left to right of $p$, then $p$ is neither a local minimum nor a local maximum.

## §B. Motivating Questions for finding Global Extrema

Last time, we learned how to identify local extrema of a function using first and second derivative tests. This time we want to look at global extrema. Some related questions arise.

- How is the process of finding the global maximum or minimum of a function over the function's entire domain different from determining the global maximum or minimum on a restricted domain **I**?
- For a function that has both a global maximum and global minimum on an interval, what are the possible points at which these extreme values occur?

Let's start with an example.

## Question 1.

Let $f(x) = 2 + \dfrac{3}{1+(x+1)^2}$.

(a) Determine all of the critical points of $f$.

(b) Construct a first derivative sign chart for $f$ and thus determine all intervals on which $f$ is increasing or decreasing.

(c) Does $f$ have a global maximum? If so, why, and what is its value and where is the maximum attained? If not, explain why.

(d) Determine $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

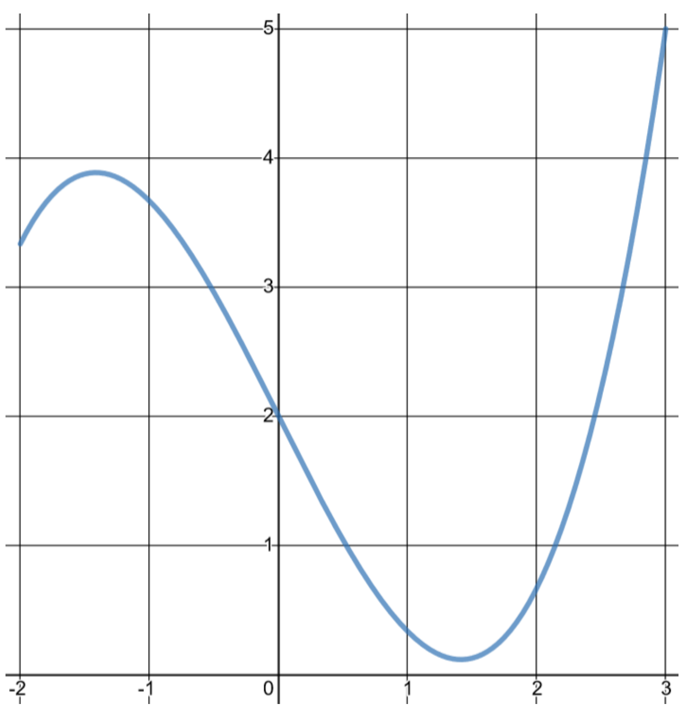
(e) Explain why $f(x) > 2$ for every value of $x$.

(f) Does $f$ have a global minimum? If so, why, and what is its value and where is the minimum attained? If not, explain why.

Next, let's take a look at a problem where we are interested in finding the global minimum and global maximum for $f$ on some restriction of its domain.

## Question 2.

Let $g(x) = \dfrac{x^3}{3} - 2x + 2$.




(a) Find all critical points of $g$ that lie in the interval $-2 \le x \le 3$.

(b) From the graph, determine the $x$-values at which the global minimum and global maximum of $g$ occur on the interval $[-2, 3]$.

(c) How do your answers change if we instead consider the interval $-2 \leq x \leq 2$?

(d) What if we instead consider the interval $-2 \leq x \leq 1$?

We can conclude that the global maximum and global minimum of a function on a closed, bounded interval $[a, b]$ depend not only on the critical numbers of the function, but also on the values of $a$ and $b$. But how do we know that the global maximum and minimum can be always found? That's where our theorem comes in.

## §C. The Extreme Value Theorem

**Theorem C.1**

If $f$ is continuous on the closed interval $a \leq x \leq b$, then $f$ **attains** both a global maximum and a global minimum on that interval.

Note the usage of the word 'attain' here. It specifically says that not only the global max and min exist, but also they are attainable! In other words, we can find some numbers $c$ and $d$ in $[a, b]$ such that $f(c)$ is the global min and $f(d)$ is the global max.

The theorem does not tell us where these extreme values occur, but rather only that they must exist. However, as we saw in the example above, the only possible locations for global extremes are at the endpoints of the interval or at a critical number. So we have the following strategy:

### Finding the global maximum and minimum of a continuous function $f$ on the interval $[a, b]$

Step 1. Find the critical points $f(x)$ that lie inside the interval $(a, b)$. you do not need to check if these are local max/min.

Step 2. Find the value of the function $f(p)$ for every critical point $p$ above.

Step 3. Find the values of $f(x)$ at the endpoints of the interval, i.e. find $f(a)$ and $f(b)$.

Step 4. The largest of the values from Steps 2 and 3 is the global maximum value; the smallest of these values is the global minimum value.

### Question 3.

Find the **exact** global maximum and minimum for each of the functions below on the stated intervals.

(a) $h(x) = xe^{-x}, \quad [0, 3]$

(b) $p(x) = \sin(x) + \cos(x), \quad \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(c) $q(x) = \frac{x^2}{x-2}, \quad [3, 7]$

(d) $r(x) = 2x^3 - 3x^2 - 12x + 1, \quad [-2, 3]$

(e) $s(x) = x + \frac{3}{x}, \quad [1, 4]$

In the next couple of lectures, we are going to use the idea of finding extrema to learn how to **optimize** real life functions. Here's a preview example.

## ■ Question 4.

□

Jared is sick, he is coughing occasionally. When he coughs, his windpipe contracts. The speed, $v$, at which he expels air depends on the radius, $r$, of his windpipe. If $R = 9$ is the normal (rest) radius of his windpipe, then for $0 \le r \le 9$, the speed is given by

$$v = 0.1(9 - r)r^2$$

What value of $r$ maximizes the speed? For what value is the speed minimized?