# MATH 111 - Calculus and Analytic Geometry I

## Lecture 14 Worksheet

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**Title:** Product and Quotient Rule

**Summary:** We will talk about two important differentiation rules that might not match your expectations.

## §A. Product rule

Before writing down the formula, let's do a quick exercise.

### Question 1.

Let $f(x) = x^2$ and $g(x) = x^3 + 2x$.

(a) Compute $f'(x)$ and $g'(x)$.

(b) Compute the product $f'(x)g'(x)$.

(c) Form the product $h(x) = f(x)g(x)$ and compute $h'(x)$.

(d) Is it true that $f'(x)g'(x) = h'(x)$?

The takeaway here is that derivative of a product is not equal to the product of the derivatives. Keep this in mind as we write down the actual formula.

**Theorem A.1**

If $f(x)$ and $g(x)$ are two differentiable functions, then

$$(fg)' = f'g + fg'.$$

**In words:** The derivative of a product is the derivative of the first times the second plus the first times the derivative of the second.

### Question 2.

| $x$ | $f(x)$ | $f'(x)$ | $g(x)$ | $g'(x)$ |
|---|---|---|---|---|
| $-2$ | $-6$ | $9$ | $-10$ | $16$ |
| $1$ | $5$ | $-3$ | $3$ | $-2$ |

Table 1: Table for Question 2

Let's practice this formula using a table of values like we did before. Use Table 1 to compute the given derivative value.

(a) Find $h'(1)$ if $h(x) = f(x)g(x)$.

(b) Find $k'(-2)$ if $k(x) = xf(x) - 2g(x)$.

(c) Find $p'(-2)$ if $p(x) = \dfrac{f(x)g(x)}{2} + x^2 f(x)$.

(d) Find $q'(1)$ if $q(x) = f(x)^2$.

Why does the product rule look this way? If we start trying to write out $\dfrac{d}{dx}[f(x)g(x)]$ using the definition, we get the following limit:

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

To help us see how we can go from this limit to the product rule, we will use some geometry.

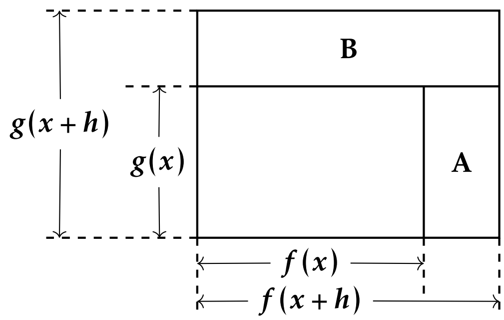


Figure 1: Rectangle Aid for Product Rule Formula

## Question 3.

Use Figure 1 to aid in this problem. Write your answers in terms of $f(x), g(x), f(x+h)$, and $g(x+h)$.

(a) What is the area of the rectangle denoted by **A**?

(b) What is the area of the rectangle denoted by **B**?

(c) What area in the figure does $f(x+h)g(x+h) - f(x)g(x)$ correspond to? Write this area in terms of **A** and **B**.

Using the formulas from the previous problem, we can now see where the product rule comes from. The numerator $f(x+h)g(x+h) - f(x)g(x)$ can be replaced in our expression for $\dfrac{d}{dx}[f(x)g(x)]$ as follows:

$$\begin{aligned}\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \frac{(f(x+h) - f(x))g(x) + (g(x+h) - g(x))f(x+h)}{h}\end{aligned}$$

We can now break this fraction into pieces and use the fact that $f(x)$ and $g(x)$ are both differentiable to get our product formula. But note, we also use the fact that $\lim_{h\to 0} f(x+h) = f(x)$, which is true because $f$ is necessarily continuous since it is differentiable.

$$
\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0}\frac{(f(x+h)-f(x))g(x)+(g(x+h)-g(x))f(x+h)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}g(x)+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}f(x+h)\\
&= f'(x)g(x)+g'(x)f(x)
\end{aligned}
$$

## §B. Quotient Rule

As with the product rule, the derivative formula for the derivative of a quotient of two differentiable functions is anything but what we would expect.

### Question 4.

Come up with an example of two functions $f(x)$ and $g(x)$ such that $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] \neq \frac{f'(x)}{g'(x)}$.

Here is the actual quotient rule formula:

**Theorem B.1**

For $f(x)$ and $g(x)$ differentiable functions,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$$

### Question 5.

Let's practice! Use Table 2 to compute the given derivative value.

(a) Find $h'(1)$ if $h(x) = \dfrac{f(x)}{g(x)}$.

(b) Find $k'(-2)$ if $k(x) = \dfrac{xg(x)}{f(x)}$.

(c) Find $\mathbf{L}'(1)$ if $\mathbf{L}(x) = \dfrac{x^3+4}{f(x)+g(x)}$.

| $x$ | $f(x)$ | $f'(x)$ | $g(x)$ | $g'(x)$ |
|---|---|---|---|---|
| −2 | 4 | 5 | −1 | 2 |
| 1 | 3 | −1 | 2 | −2 |

Table 2: Table for Problem 8

An immediate and useful application of the quotient rule is extending the power rule to include negative exponents.

**Theorem B.2: Power Rule Part II**

For any non-zero integer $n$,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

## Question 6.

Use the quotient rule to prove that the power rule applies for negative integers.

## Question 7.

Calculate $\mathbf{F}'(0)$ for the function $\mathbf{F}(x) = \dfrac{x^9 + x^8 + 4x^5 - 7x}{x^4 - 3x^2 + 2x + 1}$. This problem can be made easier by computing $f(0), g(0), f'(0),$ and $g'(0)$ separately, and then combining the numbers into the quotient rule formula. Once you have $\mathbf{F}'(0)$, write an equation for the tangent line at $x = 0$.

## Question 8.

Find all values of $x = a$ such that the tangent line to $f(x) = \dfrac{x-1}{x+8}$ at $x = a$ passes through the origin.

## Question 9.

Use the quotient rule to find derivatives of the following trigonometric functions.

(a) $\dfrac{d}{dx}[\tan x]$

(b) $\dfrac{d}{dx}[\cot x]$

(c) $\dfrac{d}{dx}[\sec x]$

(d) $\dfrac{d}{dx}[\csc x]$

## Question 10.

See if you can derive the quotient rule formula using the help of Figure 2. Follow the steps below.

(a) Firstly, write out $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right]$ using the definition of the derivative. Then, combine your fractions in the numerator.

(b) Write out the rectangles **C** and **D** in terms of $f(x), g(x), f(x+h)$, and $g(x+h)$.

(c) In part (1), did you get the difference $f(x+h)g(x) - f(x)g(x+h)$? Identify this difference in terms of areas in Figure 2.

(d) Put everything together and you've just proven the quotient rule!!

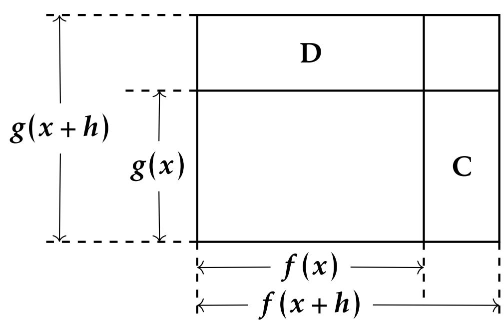


Figure 2: Rectangle Aid for Quotient Rule Formula