Calculus & Analytical Geometry II

Lecture 20 Worksheet

Spring 2021 **Subhadip Chowdhury** **Math 112**

§A. Comparison of Improper Integrals

Sometimes it is difficult to find the exact value of an improper integral by antidifferentiation and yet it might be important to know whether it is convergent or divergent. One of the ways to do so is by **comparing** the given integral to one whose behavior we already know. Before writing the theorem, let's take a look at two examples. Use DESMOS (or your preferred graphing/calculating device) as required and draw your own conclusions.

Example A.1

Consider the improper integral

$$\int_3^\infty \frac{1}{x^2 \ln x}\,\mathrm{d}x$$

(a) Draw the graph of $f(x) = \dfrac{1}{x^2 \ln x}$ in DESMOS. Then find approximate numerical values of

(i) $\displaystyle\int_3^{100} \frac{1}{x^2 \ln x}\,\mathrm{d}x$ (ii) $\displaystyle\int_3^{10000} \frac{1}{x^2 \ln x}\,\mathrm{d}x$ (iii) $\displaystyle\int_3^{1000000} \frac{1}{x^2 \ln x}\,\mathrm{d}x$

(b) Based on the numerical values, Which of the following statements is closest to what you can conclude regarding the convergence or divergence of the improper integral?

(i) Based on numerical evidence, the above integral converges.

(ii) Based on numerical evidence, the above integral diverges.

(iii) Based on numerical evidence, it appears the above integral converges.

(iv) Based on numerical evidence, it appears the above integral diverges.

(c) Now draw the graph of $g(x) = \dfrac{1}{x^2}$ on the same picture and compare the two functions. Do they intersect? Which function is bigger?

(d) Using the two graphs you just made,

(i) Do you think $\displaystyle\int_3^\infty \frac{1}{x^2 \ln x}\,\mathrm{d}x \le \int_3^\infty \frac{1}{x^2}\,\mathrm{d}x$? Why or why not?

(ii) What does this suggest about the convergence/divergence of the two integrals? Explain.

■ Question 1.

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Would your conclusion change if we consider the imroper integral $\displaystyle\int_1^\infty \frac{1}{x^2 \ln x}\,\mathrm{d}x$ *instead?*

Question 2.

Repeat the process of above example with the improper integral

$$\int_3^\infty \frac{\ln x}{\sqrt{x}}\,\mathrm{d}x$$

This time, compare it with $\int_3^\infty \frac{1}{\sqrt{x}}\,\mathrm{d}x$ *and explain what you can conclude about its convergence.*

Theorem A.2: Comparison Theorem

Suppose that f and g are continuous functions with $f(x) \geqslant g(x) \geqslant 0$ for $x \geqslant a$.

(a) If $\int_a^\infty f(x)\,\mathrm{d}x$ is convergent, then $\int_a^\infty g(x)\,\mathrm{d}x$ is convergent.

(b) If $\int_a^\infty g(x)\,\mathrm{d}x$ is divergent, then $\int_a^\infty f(x)\,\mathrm{d}x$ is divergent.

Warning: Note that the reverse is not necessarily true. If $\int_a^\infty g(x)\,\mathrm{d}x$ is convergent, $\int_a^\infty f(x)\,\mathrm{d}x$ may or may not be convergent, and if $\int_a^\infty f(x)\,\mathrm{d}x$ is divergent, $\int_a^\infty g(x)\,\mathrm{d}x$ may or may not be divergent.

Note: Finding whether an improper integral is convergent or not using the comparison test involves two stages:

- Guess, by looking at the behavior of the integrand $f(x)$ for large x, whether the integral converges or not. (This is the "behaves like" principle.)
- Confirm the guess by comparison with a positive function $g(x)$.

In example 1 and question 2, we investigated the convergence of an integral by comparing it with an easier integral. How did we pick the easier integral? This is a matter of trial and error, guided by any information we get by looking at the original integrand as $x \to \infty$. We want the comparison integrand to be easy and, in particular, to have a simple antiderivative.

Here are two useful integrals for comparison.

Theorem A.3

Let a be a positive real number. Then

(a) The integral $\displaystyle\int_a^\infty \frac{1}{x^p}\,\mathrm{d}x$ *converges if $p > 1$ but diverges if $p \le 1$.*

(b) The integral $\displaystyle\int_a^\infty \frac{1}{\mathrm{e}^{mx}}\,\mathrm{d}x$ *converges if $m > 0$ but diverges if $m \le 0$.*

Question 3.

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For each of the following integrals, draw the integrand function in DESMOS, *find a suitable second easier integrand to compare it to, and use the inequality to find whether the improper integral is convergent or divergent.*

(a) $\displaystyle\int_1^\infty \frac{\cos^2 x}{x^2}\,\mathrm{d}x$

(b) $\displaystyle\int_1^\infty \frac{1+\sin^4(3x)}{\sqrt{x}}\,\mathrm{d}x$

(c) $\displaystyle\int_2^\infty \frac{1}{x+e^x}\,\mathrm{d}x$

(d) $\displaystyle\int_2^\infty \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x$

(e) $\displaystyle\int_2^\infty e^{-x^2}\,\mathrm{d}x$

(f) $\displaystyle\int_2^\infty \frac{3}{\sqrt{x^3+x}}\,\mathrm{d}x$

(g) $\displaystyle\int_2^\infty \frac{5+2\sin x}{x^2+2}\,\mathrm{d}x$

(h) $\displaystyle\int_2^\infty \frac{5+2\sin x}{x-2}\,\mathrm{d}x$