Calculus & Analytical Geometry II

Lecture 15-16 Worksheet

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So far in Differential and Integral Calculus, we have seen that a function's derivative and integral relates to information about its change. For example, the derivative tells us the rate at which the function is changing. On the other hand, the integral, as a consequence of the Fundamental Theorem of Calculus, helps us determine the total change of a function over an interval from the function's rate of change. This might lead you to believe that many real-world phenomena that change over time, can be modeled by mathematical equations. In fact, as long as a system changes according to a fixed rule, we should be able to describe the process using functions, and predict the future behavior using calculus!

This is the study of Differential Equations and Mathematical Modeling, which we will cover over the next couple of lectures.

§A. Growth and Decay Models

Let's start with a population model. Suppose we have a Petri dish full of bacteria and we wish to examine the population of these cells as a function of time, say $\mathrm{P}(t)$. If we have an initial population of 100 cells and every 1 hour each cell divides into 3 more cells, then we can determine an expression for $\mathrm{P}(t)$ as follows:

$$\begin{aligned}
&\mathrm{P}(0) = 100\\
&\mathrm{P}(1) = 3 \cdot 100 = 300\\
&\mathrm{P}(2) = 3 \cdot 300 = 3^2 \cdot 100 = 900
\end{aligned}$$

Assuming nothing inhibits the growth of the cells and the pattern continues, we can see that the values of $\mathrm{P}(t)$ can be written as $\mathrm{P}(t) = 100 \cdot 3^t$, an exponential function.

In this section, we will look at several phenomena that can be modeled via exponential functions. Examples include population growth, radioactice decay, compounding interest etc.

Definition A.1

Systems that exhibit exponential growth or decay can be modeled according to the mathematical equation

$$y(t) = y_0 e^{kt}$$

where y_0 represents the initial state of the system (i.e. the value $y(0)$) and k is a constant, called the growth constant or decay constant, depending on whether it is positive or negative.

Usually, the initial value of y at $t = 0$, i.e. $y_0 = y(0)$ is provided to us in the model and we have to determine k experimentally or algebraically. Let's take a look at an example.

Example A.2

Let $y(t)$ represent the population of human beings on Earth (in billions) since 2000 (i.e. so t is in years, with $t = 0$ corresponding to the year 2000). We can model the population growth using an exponential model, the reasons are explained later in this worksheet.

The population was recorded to be about 6.1 billion in 2000. Hence, we could write $y(0) = y_0 = 6.1$

and model the population via the function

$$y(t) = y_0 e^{kt} = 6.1e^{kt}$$

(a) Given that the world population was around 7.4 billion in 2015, determine the growth constant k for our population model. Give an approximate value up to 4 decimal places.

(b) Once you have k, use your completed model to calculate $y(20)$ (i.e., the world population in 2020). How close is this to the recorded population in 2020? Use the internet to check!

(c) Predict the world population for 2050 using the model. How close is this to other predictions? The UN has some world population prediction numbers, so you could try doing an internet search to see how things compare!

Question 1.

Global temperatures have been rising, on average, for more than a century, sparking concern that the polar ice will melt and sea levels will rise. With t in years since 1880, we can give an approximate model for the average global temperature in Celsius as a function of t as follows

$$h(t) = 13.63e^{kt}$$

(a) The average global temperature in 2020 is 13.86° C. Find an approximae value of k up to 5 decimal points.

(b) Florida will face chronic floods when the average global temperature becomes 14° C. Assuming the current trend of global warming continues, estimate the year when this will happen.

§B. Differential Equations

Before moving on with other modeling examples, let's take a look at something called differential equations. Differential equations are equations that involve one or more derivatives of a function. Usually, we study a differential equation by trying to determine what function(s) could satisfy the equation.

Example B.3

For example,

$$y''(x) + y'(x) = 0$$

is an example of a differential equation. A solution would be a function $y(x)$ such that its derivatives y'' and y' add up to zero.

Question 2.

Verify that the function $y(x) = e^{-x} + 1$ is a solution to the differential equation above.

Question 3.

Verify that $y = 7\cos(t) + 5\sin(t)$ is a solution to the differential equation

$$y'' + y = 0.$$

Can you identify another solution?

Question 4.

Check that all functions of the form $y = \frac{k}{x}$ satisfy the differential equation $xy' + y = 0$. Which one of these satisfies $y(1) = 2$?

End of Question 4

Now consider the exponential growth function $y(t)$ given by $y(t) = y_0 e^{kt}$. Let's examine the derivative of y:

$$\begin{aligned} y'(t) &= k \cdot y_0 e^{kt} \\ &= k \cdot y(t) \end{aligned}$$

So a feature of the exponential functions is that the derivative is proportional to the original function. As we will show next week, this is in fact *unique* to exponential functions and serves as a defining characteristic. For now, note that this explains why we usually model population growth using exponential model, as one can argue the rate of change of population over time should be proportional to the current population at the time.

We conclude that, a function modeling exponential growth or decay is a solution to a differential equation of the form:

$$y' = ky, \quad \text{where } k \text{ is a constant.}$$

Since differential equations essentially describe the rate of change of a function, we can model all natural phenomenon in the universe(!) using appropriate sets of differential equations. In fact, there is a whole Math course at Wooster, Math 221, dedicated to Differential Equations!

§C. Doubling Time and Half-life

Suppose we have an exponential process that starts at time $t = 0$ with y_0 and changes over time according to $y(t) = y_0 e^{kt}$. Assuming $k > 0$, i.e. $y(t)$ is increasing, how long will it take for $y(t)$ to be double the initial value?

Example C.4

Let T be the amount of time it takes for the quantity $y(t)$ to double. That is,

$$y(\mathrm{T}) = 2y(0)$$

Using the equation for $y(t)$, solve for T.

Check that with this value of T, we have $y(t + \mathrm{T}) = 2y(t)$ for any t. That means the quantity y doubles after every T time interval.

This value you found above is defined as the **doubling time** of the quantity y(t). Note that the doubling time is a function of k. So given the doubling time for a specific quantity, you can actually determine the growth constant k. Try to do that in the problem below.

Question 5.

Suppose a colony of cells doubles in size in 3 hours. How long will it take the colony to reach 10 times its initial population? (Note: just use y_0 for the initial population, as you don't need to know what it is exactly to solve this problem).

End of Question 5

The same expression you found for doubling time is known as the **half-life** for a quantity if it is modeled by exponential *decay*. In other words, if a quantity decays exponentially, the half-life, denoted $t_{1/2}$ is the amount of time it takes the quantity to be reduced by half. Write down the formula below before doing the next problem.

$t_{1/2}$ = ______________________________

Question 6.

The half-life of caffeine is, on average, 5.5 hours. We can model the amount of caffeine* in an average human body (in milligrams, mg) via the function $C(t)$, where t is in hours. Over time, the amount $C(t)$ decreases as the body processes the caffeine. This process is actually one of exponential decay and can be applied to many other drugs in the human bloodstream.

Thus, $C(t)$ should satisfy the differential equation:

$$\frac{dC}{dt} = kC$$

where $k < 0$. Thus we can write $C(t) = C_0 e^{kt}$, where C_0 is the initial amount of caffeine.

(a) Use the half-life of caffeine to determine the decay constant k.

(b) Suppose I started the class after having X mg of caffeine. How long will it take before 90% of the caffeine is gone from my body?

§D. Another Mathematical Model - Newton's Law of Cooling

Newton's law of cooling is a differential equation that predicts the cooling of a warm object when placed in a cold environment. We all are aware that hot things will cool down to room temperature, and if you have hot soup in a ceramic bowl versus a glass bowl, your soup might cool down faster in one over the other.

Theorem D.5: Newton's Law of Cooling

The rate at which an object cools is proportional to the temperature difference between the object and its surroundings.

Let's see if we can translate the verbal description to a differential equation. Consider, for example, you take a cake out of an oven and place it in a room where the temperature is 20° C. Let $T(t)$ represent the temperature of the object, in ° C, after t minutes.

Then

$$\begin{aligned} \frac{dT}{dt} &= \text{ rate of change of temperature (° C per minute)} \\ 20 - T &= \text{ temperature difference between room and cooling object} \end{aligned}$$

and the modeling Differential Equation is

$$\frac{dT}{dt} = k(20 - T)$$

where $k > 0$. Note that usually $20 - T$ would be negative, which agrees with the fact that T decreases over time.

We will show next week how to solve this differential equation. For now, you can check that all functions of the form

$$T(t) = 20 + (T_0 - 20)e^{-kt}$$

satisfy the differential equation.

*An 8 oz. cup of coffee contains approximately 95 mg of caffeine.

Question 7.

Our room temperature is about 20° C. Experiments have shown that a ceramic coffee mug has a cooling constant of $k = 0.12\frac{1}{\text{min}}$.

(a) If my coffee begins class the class at about 80° C, then how cold is it after 50 minutes?

(b) Suppose coffee is only enjoyable for temperatures greater than 38° C. How long do we have to drink this mug of coffee before it is no longer enjoyable?

End of Question 7

When applying differential equations, we are usually not as interested in finding a family of solutions (the general solution) as we are in finding a solution that satisfies some additional requirement. In many physical problems we need to find the particular solution that satisfies a condition of the form $y(0) = y_0$. This is called an **initial condition**, and the problem of finding a solution of the differential equation that satisfies the initial condition is called an **initial-value problem**.

Question 8.

An IVP example

(a) Show that every member of the family of functions

$$y(t) = \frac{1+ce^t}{1-ce^t}$$

is a solution of the differential equation $y' = \frac{1}{2}\left(y^2 - 1\right)$.

(b) Find a solution of the initial value problem

$$y' = \frac{1}{2}\left(y^2 - 1\right), \qquad y(0) = 2.$$

§E. A Mixing Model

Consider the system of Great Lakes: Lake Huron feeding into Lake Erie, which then feeds into Lake Ontario. We are interested in determining how the pollution concentrations change in the lakes over time. For simplicity, we will model pollution concentration in Lake Erie alone, by treating it like a well-mixed container that receives inflow and outflow.

Assume the volume of Lake Erie remains constant at 480 km^3, and that the rate of water inflow (from Lake Huron) and outflow (into Lake Ontario) is a constant 350 km^3 per year. Suppose the concentration of pollutant in the inflow from Lake Huron is a constant 20 kg/km^3.

Suppose $Q(t)$ (in kg) is the total quantity of pollutant in Lake Erie at time t (in years). Then we can write,

- Rate of change of $Q(t)$ = Rate at which $Q(t)$ enters the lake − Rate at which $Q(t)$ exits the vat
- Rate at which $Q(t)$ enters/exits the lake = (flow rate of water entering/exiting) × (concentration of pollutant in water entering/exiting)
- Concentration of pollutant in the lake at time $t = \dfrac{\text{Amount of pollutant in the lake at time } t}{\text{Volume of water in the lake at time } t}$

Question 9.

Assume the initial pollution concentration is 3 kg/km^3. Use above steps to translate the word problem to a initial value problem that models the pollution concentration in Lake Erie. Keep track of the units to check your work.