MATH 221 - Differential Equations

Lecture 40 Worksheet

Fall 2020 | **Subhadip Chowdhury** | **Nov 18**

Title: Discrete Dynamical Systems

Summary: We will look at some examples of a completely different type of model for processes that evolve in time, namely discrete dynamical systems, or difference equations. Unlike differential equations, these models are well suited to situations in which changes occur at specific times, rather than continuously.

As you read through this worksheet, you should treat it as a study guide only. It is not a replacement for your textbook. Please make sure to follow along in the textbook (or other online resources from the related reading section above) side-by-side as you read through the topics. You are not expected to be able to answer the questions in here immediately after reading the synopsis from the worksheet. They are designed as more of an exploration directives; as you try to find the answers yourself, you will also learn the topic. You should use whatever resources are available to you, including the internet, to accomplish that task.

§A. Kick-Flow Systems

Consider the following scenario. You have your weekly homeworks due tomorrow and it's already 10PM. you decide to pull an all-nighter to complete your all of your assignments by staying awake continuously overnight with help of Caffeine* pills (or Starbucks coffee if you are feeling extravagant!). Caffeine, like any other drug, can be modeled to have an exponential decay in your bloodstream. So to make sure you have a steady (and safe) level of energy in your body, you decide to pop one pill in regular time interval throughout the night. The question we would like to answer is how frequently and in what dosage should you take the pills?

We are going to model the scenario using something called a Kick-Flow system, which is a mix between a continuous and discrete dynamical process.

■ Question 1. □

Consider a first order autonomous ODE of the form

$$y' = f(y).$$

The flow map $\varphi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is defined as follows. Given a time t and an initial value s, define $\varphi(t,s) = y(t)$ where $y(t)$ is value of the solution curve to the ODE at time t that starts with initial condition $y(0) = s$. In other words, $\varphi(t,s)$ is the time-t value associated with the particular solution curve passing through $y = s$ at time $t = 0$.

*It is well known for improving cognitive performance in the short run, it is not a substitute for sleep. In fact, irritability, dehydration, and anxiety sets in if an excess amount of caffeine is sent to the brain. Mood swings may also occur if the level of caffeine fluctuates too much.

Use the fundamental theorem of Calculus to explain why

$$\varphi(t,s) = s + \int_0^t f(y(t))dt$$

We sometimes denote $\varphi(t,s)$ as $\varphi_t(s)$, which corresponds to the idea that it's the value at time t of the solution curve that starts at s.

Now suppose in addition to the flow according to the differential equation, the system is subjected to regular ''kicks''. A **kick** is an instantaneous positive change in position y without regard to the underlying differential equation. We define kick-flow systems to follow a **kick-then-flow** rule; first a kick occurs, then the systems flows undisturbed, according to the underlying differential equation for a given time interval. This process is repeated to produce a dynamical system from the original continuous ODE subjected to discrete discontinuous kicks.

Assume that at time $t = 0$, the system is immediately kicked by an amount $\mathbf{K}$, thus resulting in an instantaneous change in position which takes place over zero time. The system then flows from this position, following the dynamics of the differential equation for time $\mathbf{T}$ until the next kick is applied, in the form of another instantaneous change in position, and the process repeats.

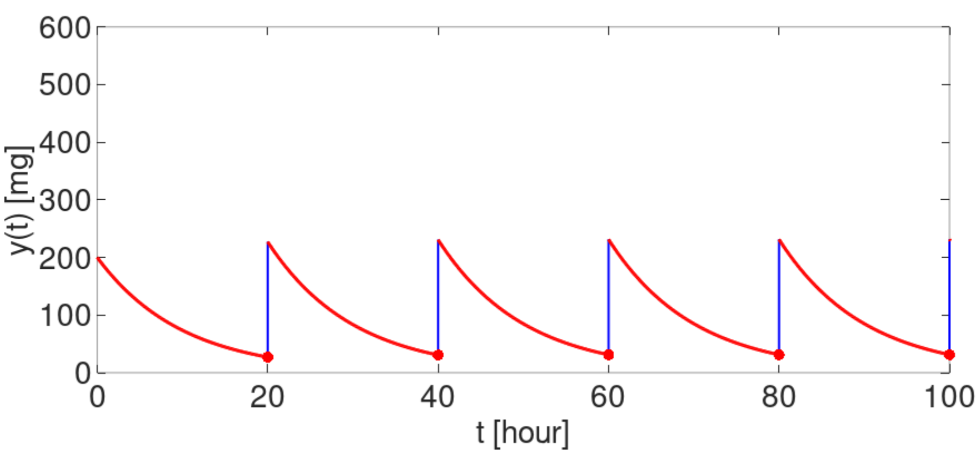

Figure 1: Example of a Kick-Flow System with $\mathbf{K} = 200$ and $\mathbf{T} = 20$

The **Kick-Flow Map** $\psi : \mathbb{R} \to \mathbb{R}$ is defined as

$$\psi(s) = \varphi(\mathbf{T}, s + \mathbf{K})$$

So if we start at y value of s at time $t = 0$, kick it by $\mathbf{K}$, then flow for time $\mathbf{T}$, then $\psi(s)$ is the end value of y. The red dots in figure (1) denote the values $\psi(0), \psi^2(0), \psi^3(0), \dots$ etc.

■ Question 2. □

Use the fact that the ODE is autonomous to show that for any $s \in \mathbb{R}$, we have

$$\psi^2(s) = \varphi(\mathbf{T}, \psi(s) + \mathbf{K})$$

and in general,

$$\psi^n(s) = \varphi\left(\mathbf{T}, \psi^{n-1}(s) + \mathbf{K}\right)$$

Here ψ^n means $\psi \circ \psi \circ \ldots \circ \psi$, a total of n times.

■ Question 3. $\boxed{14}$

We say that a **fixed point** of a function $g(x)$ is a value of x such that $g(x) = x$. Show that $\mathbf{A}$ is a fixed point of ψ if and only if

$$\varphi(\mathbf{T}, \mathbf{A} + \mathbf{K}) = \mathbf{A}.$$

Fixed Point of Caffeine Kick-Flow

In case of Caffeine consumption, the underlying differential equation is $y' = -\lambda y$, where λ is a positive rate constant.

Question 4.

□

Show that the fixed point of Caffeine Kick-Flow system is given by

$$\mathbf{A} = \frac{\mathbf{K}}{e^{\lambda \mathbf{T}} - \mathbf{1}}$$

Question 5.

□

Suppose the optimal caffeine level in blood is $\mathbf{200 - 300}$ mg and the cafeine pills are $\mathbf{100}$ mg each. So you want the fixed point $\mathbf{A}$ to be equal to $\mathbf{200}$. Then use above formula to find out the dosage interval $\mathbf{T}$ that will give the best result in long run.

§B. Discrete Logistic Equation

The population models we studied during the semester so far all have the property that the rate of change of the population is continuous. This is a reasonable assumption for species that reproduce quickly with respect to the time scale we are considering. However, for some species, all births occur in the spring and most deaths occur during the winter; hence the assumption of continuous population change is not valid. Instead, we should think of time as discrete. If we measure the population once a summer, then we will have an accurate estimate of the population for the entire year.

Recall that the continuous logistic population model was given by

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{C}\right)$$

where k is the growth rate and C is the carrying capacity. Now let's think about what the equation would look like if this was a discrete process.

The assumptions we make are:

- The population at the end of the next generation is proportional to the population at the end of the current generation when the population is very small.
- If the population is too large, then all resources will be used and the entire population will die out in the next generation and extinction will result.

If we use P_n to denote the population of the species at the end of the nth time period (e.g. a year), then the assumption says that, for each time n, the population at the end of the $(n+1)$st time step, P_{n+1}, is given by

$$P_{n+1} = kP_n\left(1 - \frac{P_n}{C}\right)$$

Rather than deal with the large numbers that often arise in population models, we will assume that P_n represents the percentage or fraction of this maximum population alive at generation n. That is, we assume that $C = 1$ and that P_n lies between 0 and 1, with $P_n = 0$ (or negative) representing extinction and $P_n = 1$ representing the maximum population level. Thus the model becomes

$$P_{n+1} = kP_n(1 - P_n)$$

which we call the **discrete logistic equation**.

Let's define the function $L_k(x)$ as

$$L_k(x) = kx(1-x)$$

If we know P_0, the initial population, we can determine the population in each succeeding generation by simply computing the expression $L_k(P_n)$ at each stage. There are no integrals to evaluate, no solution curves to approximate. All we have to do is repeatedly calculate the right-hand side of the equation, using the output of the previous calculation as the input for the next. This repetition is what we call **iteration**†, and it gives us a **sequence** $\{P_n\}_{n\in\mathbb{N}}$ defined as

$$P_n = L_k(P_{n-1}) = \underbrace{L_k \circ L_k \circ \ldots \circ L_k}_{n \text{ times}}(P_0) = L_k^n(P_0)$$

The sequence $P_0, P_1, P_2, \ldots, P_n, \ldots$ is called the **orbit** of P_0 under the function L_k.

† We have encountered several types of iterative processes already. For example, Euler's method for solving a differential equation involves iteration.

In discrete dynamics the basic goal is to predict the fate of orbits for a given function. That is, the main question is: What happens to the numbers that constitute the orbit as n tends to infinity? Does it converge? Does become a periodic cycle? Does it grow unbounded to infinity?

Let's take a look at the discrete dynamical system where $\mathbf{F}(x) = x^2 - 1$ and

$$x_n = \mathbf{F}(x_{n-1})$$

What happens if $x_0 = 0$? We get $0, -1, 0, -1, 0, -1, \ldots$, a **cycle of period 2**.

What if we start with $x = \varphi$, the golden ratio? The number φ is a root of the equation $x^2 - x - 1 = 0 \implies \mathbf{F}(\varphi) = \varphi$. So the orbit of φ will be a constant sequence. These are the **fixed points** of the system.

If $x_0 = \sqrt{2}$, the sequence is $\sqrt{2}, 1, 0, -1, 0, -1, \ldots$. We say the orbit is **eventually periodic**.

Finally, if we choose $x_0 = 0.5$, the orbit **tends to the cycle of period 2**!

$$\begin{aligned}
x_0 &= 0.5 \\
x_1 &= (0.5)^2 - 1 = -0.75 \\
x_2 &= -0.4375 \\
x_3 &= -0.8086\ldots \\
&\vdots \\
x_{20} &= 0.00000\ldots \\
x_{21} &= -1.00000\ldots \\
x_{22} &= 0.00000\ldots
\end{aligned}$$

It is important to realize that this orbit never actually reaches the cycle at 0 and -1. Rather, the orbit comes arbitrarily close to these two numbers, alternating between the two values.

The Discrete Logistic Equation

How about fixed points of the discrete logistic equation? These are roots of the equation

$$\mathbf{L}_k(x) = x \implies kx(1-x) = x \implies x = 0 \text{ or } \frac{k-1}{k}$$

We will assume $k > 1$ so that the second fixed point is biologically relevant. How does the orbit of a point look like for various values of k? Can there be a periodic orbit of period n for a certain value of k?

Note that there will be a periodic orbit of period n iff we can find a point x such that

$$\mathbf{L}_k^n(x) = x, \text{ and } \mathbf{L}_k^i(x) \neq x \text{ for } i = 1, 2, \ldots, n-1$$

We can find these points using a Desmos graph as follows:

`https://www.desmos.com/calculator/xda6oqq4sf`

Each fixed point can be attracting or repelling, analogous to the case of sink and source in the continuous case. We can also talk about the change in number and nature of fixed points with respect to k, which gives bifurcations (saddle-node, pitchfork etc.).

The world of discrete dynamical system is vast, mysterious, and **chaotic**; and we really do not have time to even begin to scratch the surface of it. If you are interested in learning more, I will be happy to talk further beyond the scope of this course!