Assignment 5 Solutions

1.7.5

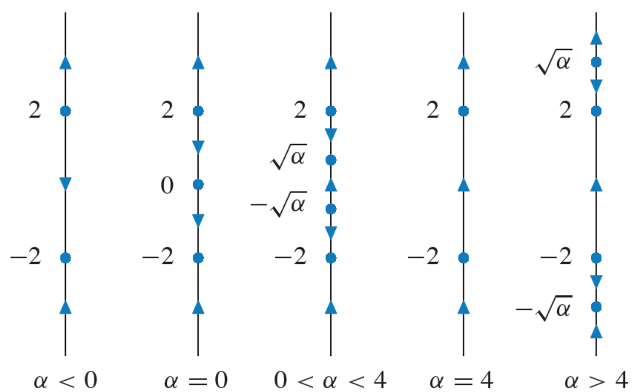
5. To find the equilibria we solve

$$(y^2 - \alpha)(y^2 - 4) = 0,$$

obtaining $y = \pm 2$ and $y = \pm\sqrt{\alpha}$ if $\alpha \geq 0$. Hence, there are two bifurcation values of α, $\alpha = 0$ and $\alpha = 4$.

For $\alpha < 0$, there are only two equilibria. The point $y = -2$ is a sink and $y = 2$ is a source. At $\alpha = 0$, there are three equilibria. There is a sink at $y = -2$, a source at $y = 2$, and a node at $y = 0$. For $0 < \alpha < 4$, there are four equilibria. The point $y = -2$ is still a sink, $y = -\sqrt{\alpha}$ is a source, $y = \sqrt{\alpha}$ is a sink, and $y = 2$ is still a source.

For $\alpha = 4$, there are only two equilibria, $y = \pm 2$. Both are nodes. For $\alpha > 4$, there are four equilibria again. The point $y = -\sqrt{\alpha}$ is a sink, $y = -2$ is now a source, $y = 2$ is now a sink, and $y = \sqrt{\alpha}$ is a source.



1.7.13

13. **(a)** Each phase line has an equilibrium point at $y = 0$. This corresponds to equations (i), (iii), and (vi). Since $y = 0$ is the only equilibrium point for $A < 0$, this only corresponds to equation (iii).

(b) The phase line corresponding to $A = 0$ is the only phase line with $y = 0$ as an equilibrium point, which corresponds to equations (ii), (iv), and (v). For the phase lines corresponding to

$A < 0$, there are no equilibrium points. Only equations (iv) and (v) satisfy this property. For the phase lines corresponding to $A > 0$, note that $dy/dt < 0$ for $-\sqrt{A} < y < \sqrt{A}$. Consequently, the bifurcation diagram corresponds to equation (v).

(c) The phase line corresponding to $A = 0$ is the only phase line with $y = 0$ as an equilibrium point, which corresponds to equations (ii), (iv), and (v). For the phase lines corresponding to $A < 0$, there are no equilibrium points. Only equations (iv) and (v) satisfy this property. For the phase lines corresponding to $A > 0$, note that $dy/dt > 0$ for $-\sqrt{A} < y < \sqrt{A}$. Consequently, the bifurcation diagram corresponds to equation (iv).

(d) Each phase line has an equilibrium point at $y = 0$. This corresponds to equations (i), (iii), and (vi). The phase lines corresponding to $A > 0$ only have two nonnegative equilibrium points. Consequently, the bifurcation diagram corresponds to equation (i).

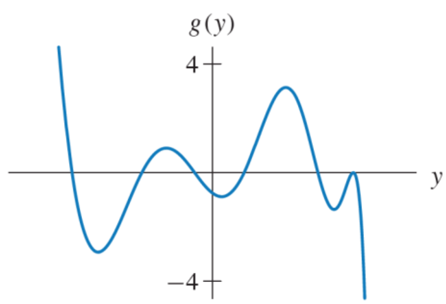
1.7.16

16. The graph of g can only intersect horizontal lines above 4 once, and it must go from above to below as y increases. Then there is exactly one sink for $\alpha \leq -4$.

Similarly, the graph of g can only intersect horizontal lines below -4 once, and it must go from above to below as y increases. Then there is exactly one sink for $\alpha \geq 4$.

Finally, the graph of g must touch the y-axis at exactly six points so that there are exactly six equilibria for $\alpha = 0$.

The following graph is the graph of one such function.



1.7.21

21. If $C < kN/4$, the differential equation has two equilibria

$$P_1 = \frac{N}{2} - \sqrt{\frac{N^2}{4} - \frac{CN}{k}} \quad \text{and} \quad P_2 = \frac{N}{2} + \sqrt{\frac{N^2}{4} - \frac{CN}{k}}.$$

The smaller one, P_1, is a source, and the larger one, P_2, is a sink. Note that they are equidistant from $N/2$. Also, note that any population below P_1 tends to extinction.

If C is near $kN/4$, then P_1 and P_2 are near $N/2$. Consequently, if the population is near zero, it will tend to extinction. As C is decreased, P_1 and P_2 move apart until they reach $P_1 = 0$ and $P_2 = N$ for $C = 0$.

Once P is near zero, the parameter C must be reset essentially to zero so that P will be greater than P_1. Simply reducing C slightly below $kN/4$ leaves P in the range where $dP/dt < 0$ and the population will still die out.