Math 2208: Ordinary Differential Equations

Assignment 8

Spring 2020 **Subhadip Chowdhury** **Due: Apr 8**

Reading

Section 3.3, 3.4, 3.7 from the textbook.

Homework

The following problems appear in the DE book by Steven Strogatz. For the purpose of this whole assignment, you can ignore the degenerate and defective equilibrium cases. Feel free to use PPLANE *whenever you want. You might want to give worksheet 15 a quick look before going through the problems, although it is not necessary.*

To arouse your interest in the classification of linear systems, we now discuss a simple model for the dynamics of love affairs (Strogatz 1988). The following story illustrates the idea.

Romeo is in love with Juliet, but in our version of this story, Juliet is a fickle lover. The more Romeo loves her, the more Juliet wants to run away and hide. But when Romeo gets discouraged and backs off, Juliet begins to find him strangely attractive. Romeo, on the other hand, tends to echo her: he warms up when she loves him, and grows cold when she hates him.

Let

$$\mathbf{R}(t) = \text{Romeo's love / hate for Juliet at time } t$$
$$\mathbf{J}(t) = \text{Juliet's love / hate for Romeo at time } t$$

Positive values of $\mathbf{R}, \mathbf{J}$ signify love, negative values signify hate. Then a model for their star-crossed romance is

$$\frac{d\mathbf{R}}{dt} = a\mathbf{J}$$
$$\frac{d\mathbf{J}}{dt} = -b\mathbf{R}$$

where the parameters a and b are positive, to be consistent with the story.

Question 1 (2 points).

What kind of equilibrium do you observe here? The sad outcome of this affair is, of course, a neverending cycle of love and hate. At least they manage to achieve simultaneous love one-quarter of the time.

Solution. The corresponding matrix has $\mathbf{T} = 0, \mathbf{D} = ab > 0$. So it is a center equilibrium. ■

Now consider the forecast for lovers governed by the general linear system

$$\frac{d\mathbf{R}}{dt} = a\mathbf{R} + b\mathbf{J}$$
$$\frac{d\mathbf{J}}{dt} = c\mathbf{R} + d\mathbf{J}$$

where the parameters a, b, c, d may have either sign. A choice of signs specifies the romantic styles.

For example, As named by one of Strogatz' students, the choice $a > 0, b > 0$ means that Romeo is an ''eager beaver'' - he gets excited by Juliet's love for him, and is further spurred on by his own affectionate feelings for her. Similarly we will say $a < 0, b > 0$ means Romeo is a ''Cautious Lover'', he tries to avoid throwing himself at Juliet, but gets excited by the Juliet's advances.

It's entertaining to name the other two romantic styles, and to predict the outcomes for the various pairings.

■ Question 2 (4 points).

What happens when two identically cautious lovers get together? The system is $\frac{dR}{dt} = aR + bJ, \frac{dJ}{dt} = bR + aJ$ with $a < 0, b > 0$.

1. Show that if $a^2 > b^2$, the relationship always fizzles out to mutual indifference. The lesson seems to be that excessive caution can lead to apathy.
2. Show that if $a^2 < b^2$, the lovers are more daring, or perhaps more sensitive to each other. Now the relationship is explosive. Depending on their feelings initially, their relationship either becomes a love fest or a war. In either case, all trajectories approach the line $R = J$, so their feelings are eventually mutual.

Solution. The corresponding matrix has $T = 2a < 0, D = a^2 - b^2$. We also check that $T^2 - 4D = 4a^2 - a^2 + b^2 > 0$. So the equilibrium is either a nodal sink, or a saddle depending on D.

1. If $a^2 > b^2$, we get $D > 0$ i.e. a nodal sink. So over a long time, both R and J go to zero, i.e. the relationship fizzles out!
2. If $a^2 < b^2$, we get a saddle point. The eigenvalues of the matrix $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ are given by $a - b$ and $a + b$. If $b > 0 > a$, the larger eigenvalue is $a + b$, and the corresponding eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So over time, all solution curves will become asymptotic towards the line $R = J$ containing the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Depending on initial condition, they either both go to $+\infty$ or they both go to $-\infty$.

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■ Question 3 (Name Calling, 0 points).

Suggest names for the other two romantic styles, determined by the signs of a and b in $\frac{dR}{dt} = aR + bJ$.

Solution. **Instruction to Grader:** Pick your most favorite one! ■

■ Question 4 (1+(2+1)+(1+1) points).

Consider the affair described by $a = 0, b = 1, c = -1, d = 1$.

(a) Characterize the romantic styles of Romeo and Juliet.

(b) Classify the fixed point at the origin. What does this imply for the affair?

(c) Sketch $R(t)$ and $J(t)$ as functions of t, assuming $R(0) = 1, J(0) = 0$.

Solution. (a) Romeos affection grows or decays depending on Juliets state and size of affection. The more Juliet loves him, the faster his love for her grows and vice versa. In contrary, the more Romeo loves Juliet, the more Juliet's love is decaying. Additionally, Juliet's growth of affection is depending on her actual state of love. Thus, they pull and push each other in infinitely growing (change of) love and hate.

(b) $T = 1, D = 1 \implies T^2 - 4D < 0$. So the origin is a spiral source.
Their feeling towards each other grows stronger over time (since $\alpha = \frac{1}{2} > 0$); but oscillates periodically between love and hate (period of $\frac{2\pi}{\beta} = \frac{4\pi}{\sqrt{3}}$).

(c)

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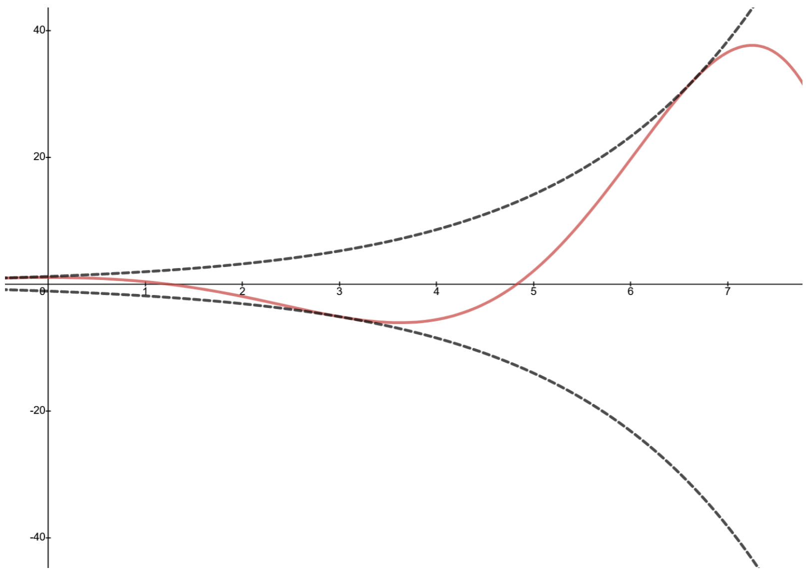

Figure 1: $\mathbf{R}(t)$ vs t

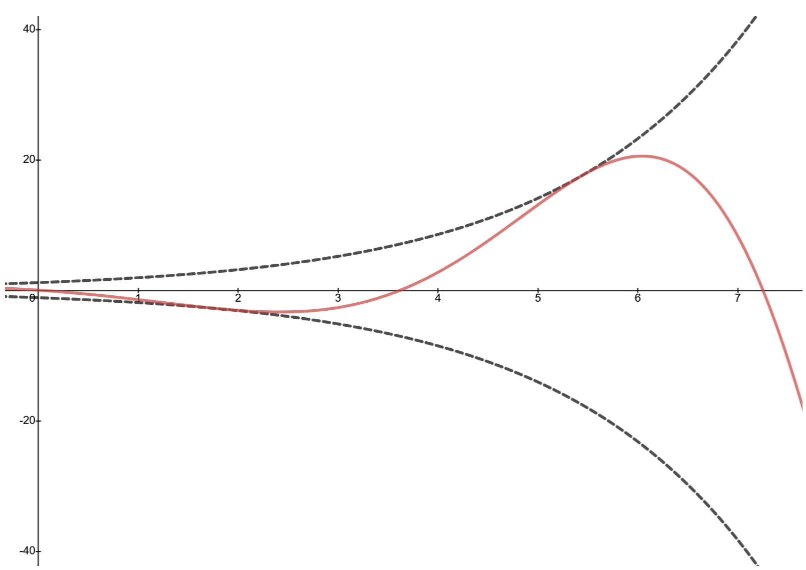

Figure 2: $\mathbf{J}(t)$ vs t

In each of the following problems, predict the course of the love affair, depending on the signs and relative sizes of a and b.

Question 5 (Out of touch with their own feelings).

Suppose Romeo and Juliet react to each other, but not to themselves: $\frac{d\mathrm{R}}{dt} = a\mathrm{J}, \frac{d\mathrm{J}}{dt} = b\mathrm{R}$. What happens?

Solution. $\mathrm{T} = 0, \mathrm{D} = -ab$. So the equilibrium point is either a saddle or center, depending on whether $ab > 0$ or $ab < 0$, respectively.

1. If $ab > 0$, we have a saddle equilibrium. The eigenvalues are given by $\pm\sqrt{ab}$.
   (a) If both $a > 0, b > 0$, the eigenvector corresponding to $\sqrt{ab}$ is given by $\begin{bmatrix} \sqrt{a} \\ \sqrt{b} \end{bmatrix}$. So over time, all trajectories approach the line $= \sqrt{\frac{b}{a}}x$. Depending on initial condition, they either both go to $+\infty$ or they both go to $-\infty$. This is similar to question 2.2.
   (b) If both $a < 0, b < 0$, the eigenvector corresponding to $\sqrt{ab}$ is given by $\begin{bmatrix} -\sqrt{a} \\ \sqrt{b} \end{bmatrix}$. So over time, all trajectories approach the line $= -\sqrt{\frac{b}{a}}x$. Depending on initial condition, either $\mathrm{R} \to +\infty, \mathrm{J} \to -\infty$ or $\mathrm{R} \to -\infty, \mathrm{J} \to +\infty$. Thus, one of them grows to love the other more and more, while the other starts to hate the former more and more.
2. If $ab < 0$, we have a center equilibrium. This is the case of question 1.

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Question 6 (Fire and water, 4 points).

Do opposites attract? Analyze $\frac{d\mathrm{R}}{dt} = a\mathrm{R} + b\mathrm{J}, \frac{d\mathrm{J}}{dt} = -b\mathrm{R} - a\mathrm{J}$.

Solution. We have $\mathrm{T} = 0, \mathrm{D} = -a^2 + b^2$.

- If $|b| > |a|$, we have a center. Their mutual feelings oscillate periodically in an endless cycle.
- If $|b| < |a|$, we have a saddle. The larger eigenvalue is $\sqrt{a^2 - b^2}$, with eigenvector $\begin{bmatrix} a - \sqrt{a^2 - b^2} \\ b \end{bmatrix}$. So all cases are possible depending on values of a and b. The values of (R, J) can both go to $+\infty$; or both to $-\infty$; or one of them to $+\infty$ and other to $-\infty$.

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Question 7 (Peas in a pod, 6 points).

If Romeo and Juliet are romantic clones $\frac{d\mathrm{R}}{dt} = a\mathrm{R} + b\mathrm{J}, \frac{d\mathrm{J}}{dt} = b\mathrm{R} + a\mathrm{J}$, should they expect boredom or bliss?

Solution. $\mathrm{T} = 2a, \mathrm{D} = a^2 - b^2, \mathrm{T}^2 - 4\mathrm{D} = 3a^2 + b^2 > 0$. The eigenvalues are $a - b$ and $a + b$, and corresponding eigenvectors are $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, respectively. So there are couple of cases:

1. We are ignoring the case $a^2 = b^2$ as per instructions.
2. If $a^2 < b^2$, we have a saddle. There are two possibilities:

- If $b > 0$, then regardless of whether $a < 0$ or $a > 0$, the larger eigenvalue is $a + b$. we have the case of question 2.2. All trajectories approach the line $\mathbf{R} = \mathbf{J}$, so their feelings are eventually mutual.
- If $b < 0$, then regardless of whether $a < 0$ or $a > 0$, the larger eigenvalue is $a - b$. Their feelings are always opposite of each other. One of them grows to love the other more and more, while the other starts to hate the former more and more.

3. If $a^2 > b^2$, we have two possibilities:

   - If $a > 0$, we have a nodal source, the larger eigenvalue is $a + b$. Over time, all trajectories become parallel to the line $\mathbf{R} = \mathbf{J}$, so their feelings are mutual (but not necessarily equal).
   - If $a < 0$, we have a nodal sink. Over time, the relationship fizzles out.

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■ Question 8 (Romeo the robot, 4 points).

Nothing could ever change the way Romeo feels about Juliet: $\frac{d\mathbf{R}}{dt} = 0, \frac{d\mathbf{J}}{dt} = a\mathbf{R} + b\mathbf{J}.$ Does Juliet end up loving him or hating him?

Solution. $\mathbf{T} = b, \mathbf{D} = 0$. The eigenvalues are 0 and b. So the answer depends on both whether b is positive or negative and on the initial condition.

- If $b > 0$, we have a degenerate source. If the initial condition is $\mathbf{R}(0) + \mathbf{J}(0) > 0$, then Juliet ends up loving Romeo. If $\mathbf{R}(0) + \mathbf{J}(0) < 0$, then Juliet ends up hating Romeo.
- If $b < 0$, we have a degenerate sink. All trajectories end up with $\mathbf{R} = \mathbf{J}$ as $t \to \infty$. Whatever the initial disposition of Romeo towards Juliet is, Juliet ends up having the same disposition towards Romeo in the long run.

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