MATH 221 - Differential Equations

Lecture 30 Worksheet

Fall 2020 | **Subhadip Chowdhury** | **Oct 26**

Title: Forcing and Resonance

Summary: We will examine the phenomenon of Resonance in the case of a forced undamped harmonic motion.

Related Reading: Section 5.1 from the textbook. We will spend multiple lectures on it.

From The ODE Project - Section 4.4.

As you read through this worksheet, you should treat it as a study guide only. It is not a replacement for your textbook. Please make sure to follow along in the textbook (or other online resources from the related reading section above) side-by-side as you read through the topics. You are not expected to be able to answer the questions in here immediately after reading the synopsis from the worksheet. They are designed as more of an exploration directives; as you try to find the answers yourself, you will also learn the topic. You should use whatever resources are available to you, including the internet, to accomplish that task.

§A. Undamped Forcing

Certain systems have one or more natural frequencies at which they oscillate. If you force such a system by an input that oscillates at close to a natural frequency, the response may be very large in amplitude. In this section we explore this phenomenon.

Consider a mass-spring apparatus lying on a table and moving back and forth periodically (with frequency ν). Now we force the table to tilt periodically as well with some constant frequency (μ). We get the following equation of motion (up to some scaling)

$$y'' + \nu^2 y = \cos(\omega t)$$

Definition A.1

The constant ν is called the **resonant frequency** and ω is called the **forcing frequency**.

Question 1.

Write down the general solution to the **associated homogeneous equation**.

Question 2.

Find a particular solution to the nonhomogeneous system using the method of undetermined coefficients and conclude that the general solution looks like

$$y(t) = \frac{\cos(\omega t)}{\nu^2 - \omega^2} + k_1 \sin(\nu t) + k_2 \cos(\nu t)$$

Question 3.

Qualitative Analysis.

We will analyze how the solution curves change as we fix ν and vary ω as a parameter. Open the following link

https://mathlets.org/mathlets/forced-damped-vibration/

and set $m = 1, k = 4, b = 0, A = 1$. This sets $\nu = 2$ (why?). Check the `Show Trajectory` and `Relate Diagram` options. Use the top left slider to zoom out/in as necessary.

Question 4.

Suppose we have the initial condition $y(0) = y'(0) = 0$. Solve for k_1 and k_2 and find the solution to this IVP.

Question 5.

Set $\omega = 0.5$. What does the solution curve look like? You may have to increase the time range to see the full picture.

Can you explain the behavior physically? Is this a periodic solution curve?

■ Question 6.

Are solution curves periodic for any value of ω? What happens when $\omega = \sqrt{2}$?

■ Question 7.

Now set $\omega = 1.9$, a value very close to ν. You may have to zoom out to see the full picture. What do you observe?

■ Question 8.

As $\omega \to \nu$, how does the amplitude of $y(t)$ change? Does the solution curve have a more 'regular' pattern?

The phenomenon you observe here is called **Beating** which occurs when the resonant frequency and forcing response have approximately the same value. You can hear the phenomenon of beating when listening to a piano or a guitar that is slightly out of tune.

§B. Resonance

What happens when $\omega = \nu$? Above algebraic calculations don't make sense in that case since the denominator becomes zero.

■ Question 9.

Find the limit

$$\lim_{\omega \to \nu} \frac{\cos(\omega t) - \cos(\nu t)}{\nu^2 - \omega^2}$$

Check that your answer is **the** solution to the IVP

$$y'' + \omega^2 y = \cos(\omega t), \qquad y(0) = y'(0) = 0$$

Question 10. □

What does the solution curve look like? Draw the curve using Desmos or some other graphing tool. Compare it to the solution curve when $\omega = 1.9$.

Question 11. □

Watch this video: https://www.youtube.com/watch?v=2cuXbpXRvJ0. What's going on here? How would extra damping help?

Question 12. □

Another video **(Loud Noise Warning)**: https://www.youtube.com/watch?v=17tqXgvCN0E. Why does the glass break? Explain it using Physics (Physical Chemistry?).

The phenomenon that we have just described is known as (pure) **resonance**. Resonance is not always an unwanted phenomenon. Being able to selectively amplify a particular forcing frequency can be extremely useful. Amplifying a particular frequency of a radio signal is perhaps the most common example of the usefulness of resonance. Since we cannot completely eliminate damping from a natural process, we instead use what we call a **Practical Resonance**.

§C. Suggested Homework Problems

Question 13. □

Suppose the suspension system of the average car was modeled by an undamped harmonic oscillator with a natural period of 2 seconds. How far apart should speed bumps be placed so that a car traveling at 10 miles per hour over several bumps will bounce more and more violently with each bump?

Question 14. □

Compute the solution of the given initial-value problem.

(a) $\frac{d^2y}{dt^2} + 4y = 3\cos 2t,\ y(0) = y'(0) = 0$

(b) $\frac{d^2y}{dt^2} + 9y = \sin 3t,\ y(0) = 1, y'(0) = -1$

Question 15. □

In order to use the Method of Undetermined Coefficients to find a particular solution, we must be able to make a reasonable guess (up to multiplicative constants - the undetermined coefficients) of a particular solution. In last class we discussed exponential and sinusoidal forcing. If the forcing is some other type of function, then we must adjust our guess accordingly. For example, for an equation of the form

$$\frac{d^2y}{dt^2} + 4y = -3t^2 + 2t + 3$$

the forcing function is the quadratic polynomial $g(t) = -3t^2 + 2t + 3$. It is reasonable to guess that a particular solution in this case is also a quadratic polynomial. Hence we guess $y_p(t) = at^2 + bt + c$. The constants $a, b,$ and c are determined by substituting $y_p(t)$ into the equation.

(a) Find the general solution of the equation specified in this exercise.

(b) Find the particular solution with the initial condition $y(0) = 2, y'(0) = 0$

■ Question 16.

□

Find one particular solution of the equation

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = c$$

where p, q, c are constants.

The next section is completely optional for the purpose of this course. I will not go over it in class. It is only included in case you are interested in understanding what practical resonance is.

§D. **Practical Resonance**

Damped Harmonic Oscillator with Sinusoidal Forcing

In most physical oscillator systems, we have nonzero damping. We wish to analyze the analogue of the resonance phenomenon for such a damped harmonic oscillator when subjected to sinusoidal forcing. The relevant equation is given by

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + y = \cos(\omega t) \qquad (\star)$$

where

- $y(t)$ denotes the displacement at time t,
- $p > 0$ is the damping constant, and
- $\omega > 0$ is the forcing frequency.

■ Question 17. □

A bit of background on $\mathbb{C}$omplex numbers first.

(i) Suppose the polar form of the complex number $z = a + ib$ is given by $re^{i\theta}$. What's the polar form of $a - ib$ in terms of r and θ?

(ii) Use Euler's formula and calculate the real part of $(a - ib)e^{i\omega t}$ in two different ways to show that

$$a\cos(\omega t) + b\sin(\omega t) = r\cos(\omega t - \theta)$$

where (r, θ) is the polar coordinate of (a, b).

■ Question 18. □

Find a particular solution $y_p(t)$ to above differential equation ($\star$) using the **method of undetermined coefficient**. Your solution will involve parameters ω and p. Once you find the particular solution, Change it into the polar form using the formula from part(ii) of last question.

Recall that all general solutions of the differential equation ($\star$), regardless of initial conditions, converge to the steady-state solution $y_p(t)$ for large values of t. So we can conclude that if damping is present, in the long-term, every solution of ($\star$) oscillates with same frequency and amplitude.

■ Question 19. □

Write the amplitude of the steady state solution as a function of the parameters ω and p.

■ Question 20. □

We are going to fix p and let ω vary as a parameter. Let $r(\omega)$ denote the amplitude of the particular solution (from last part) when the system is forced at frequency ω. We say that **Practical resonance** occurs when $r(\omega)$ achieves its maximum as a function of ω.

(i) Show that if $p < \sqrt{2}$, then practical resonance occurs at $\omega = \sqrt{1 - \frac{p^2}{2}}$.

(ii) Show that if $p > \sqrt{2}$, then no practical resonance occurs.