Math 2208: Ordinary Differential Equations

Assignment 3

Spring 2020 | **Subhadip Chowdhury** | **Due: Feb 12**

Reading

Section 1.(5,6) from the textbook.

Exercises

Don't forget to be neat and thorough. No fringe, and please use the cover page.

Question 1.

Book problem 1.5.(2,12,14,18).

Question 2.

Book problems 1.6.(4, 16, 6, 18, 22, 23, 24, 28, 32, 36).

Additional Problems

Question 3 (5 points).

A function $y(t)$ is called periodic with period $\mathrm{T} > 0$, if for every t in the domain of y the following holds:

$$y(t+\mathrm{T}) = y(t)$$

You are familiar with periodic functions. For example $\sin(t)$ or $\cos(3t)$ are periodic functions. We will assume constant functions are not periodic. So your question is as follows:

Let $f(y)$ be a function such that $f(y)$ and $\dfrac{df}{dy}$ are continuous for all y. Show that there is no periodic solution to the autonomous ODE $\dfrac{dy}{dt} = f(y)$.

Solution. Note that there are lots of ways of showing above statement. I have written down two solutions I believe to be the easiest.

Solution 1. Continuity of $f(y)$ and $\dfrac{df}{dy}$ implies that EUT holds. So we know that two distinct solution curves to the ODE cannot cross each other.

Assume, for the sake of contradiction, that there exists a periodic function $y(t)$ which is solution to the autonomous ODE $y'(t) = f(y)$. Suppose $t = t_0$ is in the domain of $y(t)$. Then there exists some $\mathrm{T} > 0$ such that $y(t_0) = y(t_0 + \mathrm{T})$.

Using the *Mean Value Theorem*, we can find some t_1 such that $t_0 < t_1 < \mathrm{T} + t_0$ and $y'(t_1) = 0$. Suppose $y(t_1) = c$. Then

$$f(c) = f(y(t_1)) = \left.\frac{dy}{dt}\right|_{t=t_1} \equiv y'(t_1) = 0$$

But that means c is an equilibrium value for the ODE $y' = f(y)$, and in particular the straight line $y = c$ is a solution curve to the ODE. However that means the periodic solution touches the equilibrium solution at (t_1, c) which contradicts the fact that two distinct solution curves to the ODE cannot cross each other. Hence no periodic solution $y(t)$ exists.

Solution 2. Assume, for the sake of contradiction, that there exists a periodic function $y(t)$ which is solution to the autonomous ODE $y'(t) = f(y)$. Suppose $t = t_0$ is in the domain of $y(t)$. Then there exists some $\mathbf{T} > \mathbf{0}$ such that $y(t_0) = y(t_0 + \mathbf{T})$.

Since constant functions are not assumed to be periodic, we can find t_1 such that $t_0 < t_1 < \mathbf{T} + t_0$ and $y(t_1) \neq y(t_0) = y(t_0 + \mathbf{T})$. Without loss of generality, we can assume $y(t_1) > y(t_0)$.

Let $\mathbf{K} = \dfrac{y(t_1) + y(t_0)}{2}$. Then by *Intermediate Value Theorem*, we can find t_2 and t_3 such that

$$t_0 < t_2 < t_1 \text{ and } y(t_2) = \mathbf{K} \text{ and } y'(t_2) > \mathbf{0}$$

$$t_1 < t_3 < t_0 + \mathbf{T} \text{ and } y(t_3) = \mathbf{K} \text{ and } y'(t_3) < \mathbf{0}$$

But for an autonomous ODE, the slope field does not depend on t. So $y'(t_2)$ and $y'(t_3)$ should be equal since the y-value in both cases is $\mathbf{K}$. In other words,

$$y'(t_2) = f(y(t_2)) = \mathbf{K} = f(y(t_3)) = y'(t_3)$$

This contradicts our observation above. Hence no periodic solution $y(t)$ exists.

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