MATH 221 - Differential Equations

Lecture 4 Activities

Fall 2020 | **Subhadip Chowdhury** | **Aug 26**

§A. In-class Practice Problems

Question 1.

Recall the ant path problem. Do you think that $x^2 + y^2 = 1$ is an implicit solution of the initial-value problem $\dfrac{dy}{dx} = \dfrac{-x}{y}, \quad y(1) = 0$?

Question 2.

Use separation of variables to show that $y = \left(\dfrac{x^2}{4} + c\right)^2$ is a solution to the ODE $\dfrac{dy}{dx} = x\sqrt{y}$ for any arbitrary constant c.

Now consider the IVP $\dfrac{dy}{dx} = x\sqrt{y}, \quad y(0) = 1$. Note that $\dfrac{\partial f}{\partial y} = \dfrac{x}{2\sqrt{y}}$ is continuous at $(0, 1)$. So according to the EUT, this IVP should have a unique solution passing through $(0, 1)$, right?

But both $y = \left(\dfrac{x^2}{4} - 1\right)^2$ and $y = \left(\dfrac{x^2}{4} + 1\right)^2$ both satisfy the initial condition. What's going on here?

Question 3.

We have emphasized that the Uniqueness Theorem does not apply to every differential equation. There are hypotheses that must be verified before we can apply the theorem. However, there is a temptation to think that, since models of "real world" problems must obviously have solutions, we don't need to worry about the hypotheses of the Uniqueness Theorem when we are working with differential equations modeling the physical world. The following model illustrates the flaw in this assumption.

Suppose we wish to study the formation of raindrops in the atmosphere. We make the reasonable assumption that raindrops are approximately spherical. We also assume that the rate of growth of the volume of a raindrop is proportional to its surface area.

Let $r(t)$ be the radius of the raindrop at time t, $s(t)$ be its surface area at time t, and $v(t)$ be its volume at time t. From three-dimensional geometry, we know that

$$s = 4\pi r^2 \qquad \text{and} \qquad v = \frac{4}{3}\pi r^3$$

(a) Show that the differential equation that models the volume of the raindrop under these assumptions is

$$\frac{dv}{dt} = kv^{2/3},$$

where k is a proportionality constant.

(b) Why doesn't this equation satisfy the hypotheses of the Uniqueness Theorem?

(c) Give a physical interpretation of the fact that solutions to this equation with the initial condition $v(0) = 0$ are not unique. Does this model say anything about the way raindrops begin to form?

§B. **Suggested Homework Problems**

Question 4.

Determine the region in the xy-plane for which the given differential equation would have a unique solution passing through a given point in the region.

(a) $(1 + y^3)y' = x^2$

(b) $(y - x)y' = y + x$

Question 5.

Let $f(t, y)$ be a function that is continuous for all y and assume that $\dfrac{\partial f}{\partial y}$ is continuous for all y. Suppose that $y_1(t) = 1 + t^2$ is a solution to the ODE

$$\frac{dy}{dt} = f(t, y)$$

Answer the following questions, giving justification for your responses.

(a) Suppose $y_2(t)$ is a solution to the above ODE with initial value $y_2(0) = 0$. What can you say conclusively about $y_2(5)$?

(b) Suppose $y_3(t)$ is a solution to the above ODE with initial value $y_3(0) = 1$. What can you say conclusively about $y_3(5)$?

(c) Suppose $y_4(t)$ is a solution to the above ODE with initial value $y_4(1) = 2$. What can you say conclusively about $y_4(5)$?