MATH 221 - Differential Equations

Lecture 14 Worksheet

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Title: System of First Order ODEs

Summary: We will learn how about models that involve a system of first order differential equations. The most famous of these is the Predator-Prey model of Lotka and Volterra.

Related Reading: Chapter 3.3

As you read through this worksheet, you should treat it as a study guide only. It is not a replacement for your textbook. Please make sure to follow along in the textbook side-by-side as you read through the topics. You are not expected to be able to answer the questions in here immediately after reading the synopsis from the worksheet. They are designed as more of an exploration directives; as you try to find the answers yourself, you will also learn the topic. You should use whatever resources are available to you, including the internet, to accomplish that task.

After completing our discussion of First order ODEs with a single dependent variable, we turn towards the task of analyzing more complicated ODEs. The complication can arise in one of two ways - either the number of dependent variables increases (where we have to keep track of two or more things that change over time and interact with each other) or the degree of the ODE increases (whereby we have more complicated equations involving a single dependent variable). We will start with the first case and then show that the second case can be analyzed easily by changing it into a problem from the first case.

§A. General Form

The number of dependent variables for a *system* of differential equation is called its **dimension**[1]. The general form for a two-dimensional system of first order ODEs is

$$\begin{aligned}\frac{dx}{dt} &= f(x,y,t),\\ \frac{dy}{dt} &= g(x,y,t).\end{aligned}$$

When f and g are linear in the variables x and y− that is, f and g have the forms

$$f(x,y,t) = c_1x + c_2y + \varphi(t) \quad \text{and} \quad g(x,y,t) = c_3x + c_4y + \psi(t)$$

where the coefficients c_i could depend on t - then the system is called a **linear** system. A system of differential equations that is not linear is said to be nonlinear.

(One dimensional) Second order ODE ⟶ Two dimensional system of First order ODEs

A second-order ODE can be converted to a two-dimensional system of first-order ODEs via some change of variables. For example, consider the second order ODE

$$y'' = f(t,y,y')$$

[1] The nomenclature will become clearer once we study a little bit of Linear Algebra later in this course.

We can make the substitution $u_2(t) = y'(t)$ and $u_1(t) = y(t)$ to convert the ODE into a system of 2 first-order ODEs

$$\frac{du_1}{dt} = u_2$$
$$\frac{du_2}{dt} = f(t, u_1, u_2)$$

Example A.1

The second order ODE $y'' + 5y' + 6y = t$ can be converted into a two-dimensional system of first order ODE by defining a new variable $v = y'$. The system looks like:

$$y' = v$$
$$v' = t - 5v - 6y$$

Question 1.

Show that the third-order linear ODE

$$y''' + 3y'' + 2y' - 5y = \sin(t)$$

can be written as a system of three first-order linear ODEs.

§B. Lotka-Volterra Model

Probably the most famous system of ordinary differential equations of all time is the Lotka-Volterra predator-prey model. We will study a very special case of interaction between exactly two species, one of which -- the predators -- eats the other -- the prey. Such pairs exist throughout nature: lions and gazelles, birds and insects, pandas and eucalyptus trees, foxes and rabbits. To keep our model simple, we will make some assumptions:

- the predator species is totally dependent on a single prey species as its only food supply,
- the prey species has an unlimited food supply, and
- there is no threat to the prey other than the specific predator.

Let $x(t)$ denote the population of predators (foxes) and let $y(t)$ denote the population of their prey (rabbits). Let a, b, c and d be **nonnegative** parameters. One system of differential equations that might govern the changes in the population of these two species is

$$\frac{dx}{dt} = -ax + bxy$$
$$\frac{dy}{dt} = -cxy + dy$$

Question 2.

1. What do the constants a, b, c and d represent physically?
2. What is the significance of the xy terms in the model? Is the significance same for rabbits and foxes?

3. What happens to $x(t)$ if $b = c = d = 0$? How does this correspond to our assumptions?
4. What happens to $y(t)$ if $a = b = c = 0$? How does this correspond to our assumptions?
5. What does the model predict will happen if at any time one of the populations of the rabbits or the foxes becomes zero?

Question 3.

Are there any fixed points for the system of equations? In other words, what are the equilibrium solutions for the system? Note that equilibrium here means both x' and y' are equal to $\mathbf{0}$.

We will analyze this model further next week.

§C. Multivariable Calculus Basics I - Parameterized Curves

Before moving forward with our discussion of two-dimensional system of ODEs, we must review some basic concepts from Multivariable Calculus. The first topic of interest is Parameterized Curves.

Think of a curve in the plane as the trajectory of a point that changes position with time. At any moment t, the x and the y-coordinates of the point can be given as functions of time t. So we can describe the curve as a collection of points $(x(t), y(t))$ where t lies in some interval (possibly infinite). This is called a parameterization of the curve. And when curves are generated in this manner, they are called parametrized curves.

Example C.1: Circle

A circle in the xy-plane can be described as the set of pairs $(x(t), y(t))$ generated by the **parameterization**

$$x(t) = \cos(t) \text{ and } y(t) = \sin(t) \qquad \text{for } 0 \leq t \leq 2\pi$$

Here a line joining the origin and the position of the moving point at time t, makes an angle of t radians with the positive x−axis.

The word 'parameter' used here from Multivariable Calculus is different from the 'parameter' we used last week when talking about Bifurcation. It is an unfortunate coincidence. Hopefully the usage will be clear from the context.

Notice that parameterization carries more information than simply the final curve shape in the xy-plane. In particular, in the example above, the parameterization indicates that the curve shape (circle) in the xy-plane is traced exactly once around, and that this curve shape is traced counterclockwise starting from $(1, 0)$ as t increases through its range.

This is particularly important for our discussion of ODEs as we observe that, to find a solution to a system of ODEs is equivalent to finding $x(t)$ and $y(t)$. So instead of plotting x vs. t and y vs. t in separate graphs (which are still important), we can instead plot the parameterized curve $((x(t), y(t))$ in a single xy−plane. We will call this the **solution curve** of the system. A system, along with an initial point, determines a particular solution curve. The parameterization information then tells us how the behavior of the system changes as the (x, y) point moves along the solution curve over time.

We will discuss this in more details next week.

§D. **Linear Algebra Basics I - Matrices and Vectors**

We begin our study of linear algebra with the basics. We shall try to take the shortest route to the results we need to understand systems of differential equations.

Definition D.1

Matrices and vectors are just arrays of numbers. A **$m \times n$ matrix** is an array of numbers which has m rows and n columns. Abstractly, A general $m \times n$ matrix can be written as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & \cdots & a_{mn} \end{bmatrix}$$

The first index on the symbol a_{ij} tells you that the number lives in the i-th row, and the second symbol tells you that it lives in the j-th column. This specifies where in the matrix to put the number.

We will be mostly dealing with matrices where $m, n \le 3$. A **vector** is a matrix which is just a single column with multiple rows. So for example, a 3×1 matrix

$$\vec{v} = \begin{bmatrix} 7 \\ e \\ \pi \end{bmatrix}$$

is called a 3-dimensional vector (or a vector with three components) and we use the arrow on top of v to signify that it is a vector.

Note that every point (x, y) in the plane can be identified by the vector $\vec{r} = \begin{bmatrix} x \\ y \end{bmatrix}$ and vice versa. This gives us a way to use vectors to simplify the notation for parameterized curves, and consequently solutions to differential equations.

With the new notation of vectors in mind, we let $\vec{\mathrm{R}}(t)$ denote the vector

$$\vec{\mathrm{R}}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

Then the vector-valued function $\vec{\mathrm{R}}(t)$ (input is t, output is a vector) corresponds to the parameterized solution curve $(x(t), y(t))$ in the xy-plane. We will discuss how to use this new notation next week.