Math 2208: Ordinary Differential Equations

Project 3: Higher-Order Linear Systems

Spring 2020 **Subhadip Chowdhury** **Due: Apr 22**

§A. **Higher-Order Linear Systems**

Consider a $n-$dimensional linear system of ODEs of the form $\vec{\mathbf{R}}'(t) = \mathbf{A}\vec{\mathbf{R}}(t)$ where $\mathbf{A}$ is a $n \times n$ matrix whose (i,j)th element is a_{ij} and $\vec{\mathbf{R}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ is a $n \times 1$ vector.

The strategy for finding solutions to the system $\vec{\mathbf{R}}'(t) = \mathbf{A}\vec{\mathbf{R}}(t)$ is the same as for systems of two equations. In particular, if λ is an eigenvalue of $\mathbf{A}$ with eigenvector $\vec{v}$ and $\vec{r}(t) = e^{\lambda t}\vec{v}$, we can still show that $\vec{r}(t)$ is a solution to our system.

Solution. We observe that

$$\mathbf{A}\vec{r}(t) = e^{\lambda t}\mathbf{A}\vec{v} = e^{\lambda t}\lambda\vec{v} = \frac{de^{\lambda t}}{dt}\vec{v} = \frac{d}{dt}\vec{r}(t)$$

Hence $\vec{r}(t)$ is a solution. ■

It is easy to check that the **Linearity Principle** also holds in higher dimension. So a general solution can be found as a linear combination of linearly independent solutions.

■ Question 1. (3 points)

Find the general solution to the system of ODEs

$$\begin{aligned} x' &= -5x - 8y - 2z \\ y' &= 5x + 12y + 4z \\ z' &= -11x - 19y - 5z \end{aligned}$$

You can use `WolframAlpha` to find the eigenvalues and eigenvectors of matrices. Type

```
Eigenvalues[{-5,-8,-2},{5,12,4},{-11,-19,-5}]
```

into the query field.

Solution.

$$\vec{r}(t) = k_1 e^{2t}\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} + k_2 e^{-t}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + k_3 e^{t}\begin{bmatrix} 5 \\ -7 \\ 13 \end{bmatrix}$$

■

§B. The Geometry of Solutions

Last week, we classified all of the geometry of the solutions for planar systems using the trace-determinant plane. Another way to classify equilibrium points is by their stability.

Definition 2.1

If every solution that starts close to an equilibrium stays close to that equilibrium for all time, then the equilibrium solution is called **stable**. Otherwise, it is called **unstable**.

This definition is a bit vague but we can formalize this if needed. Here 'close' means within some neighborhood of bounded radius.

■ Question 2. (2+2+0 points)

(a) Identify the types of equilibrium solutions in $\mathbb{R}^2$ that are stable according to the above definition from the list below:

Nodal source, Spiral source, Saddle, Nodal Sink, Spiral Sink, Center

Solution. nodal sink, spiral sink, center ■

(b) Fill in the blanks with either the word "**stable**" or "**unstable**":

If an equilibrium solution has eigenvalues with real parts that are non-positive, then the equilibrium solution is ________________.

Solution. stable ■

If an equilibrium solution has at least one eigenvalue with a positive real part, then the equilibrium solution is ________________.

Solution. unstable ■

(c) It turns out, the conclusion you made in part (b) holds even for systems with more than two dependent variables. For example, if we have a $n \times n$ matrix $\mathbf{A}$ and a $n-$dimensional system $\vec{\mathbf{R}}' = \mathbf{A}\vec{\mathbf{R}}$, we can compute the eigenvalues of $\mathbf{A}$. By inspecting the real parts of the eigenvalues only, we can determine the stability. Does this make sense, yes or no? ________________.

If yes, please continue. If not, call me (post in your channel).

Although the stability of equilibrium solutions for linear systems in more than two variables is easy to determine, the geometry is a bit more complicated. For a system in three dimension, the solution curves live in $\mathbb{R}^3$, and there is simply a lot more room to move around in three dimensions than in two dimensions! Note that the origin is still the only equilibrium solution for a non-degenerate system of linear differential equations in three variables. In the rest of this section, we look at possible geometry of equilibrium solutions in three dimensions.

CASE 1: THREE REAL DISTINCT EIGENVALUES.

Suppose the eigenvalues are $\lambda_1, \lambda_2, \lambda_3$. All $\lambda_i \in \mathbb{R}$.

■ Question 3. (1+1+1 points)

What conditions on λ_i would make the origin a nodal **sink**? What about a nodal **source** or a **saddle**?

Solution. nodal sink: $\lambda_i < 0$ for all i

nodal source: $\lambda_i > 0$ for all i

saddle: Either two negative, one positive; or one negative, two positive ■

In the case of a three dimensional saddle, we could have, for example, a **stable straight line** of solution (parallel to corresponding eigenvector) and an **unstable plane** of solutions (or vice versa). In this case, all solutions of the system with initial point on the stable line would get closer to the origin as $t \to \infty$, but all solutions with initial point on the unstable plane would move away from the origin.

■ Question 4. (4 points)

(a) Go back to the example of question (1). It has a three-dimensional saddle equilibrium. Identify the equation of the stable line and the unstable plane.

(b) Describe qualitatively what you think would be the long-term behavior of a solution curve that starts outside the stable line or the unstable plane.

Solution. The line is the one corresponding to the negative eigenvalue. Its parametric equation is $\vec{r}(t) = t\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. You do not have to find the exact equation, as long as you correctly identify it.

The plane is the span of the other two eigenvectors. An exact equation (not necessary) can be found by calculating the normal vector using cross product. The normal is $\begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}$. The plane is $-4x - y + z = 0$.

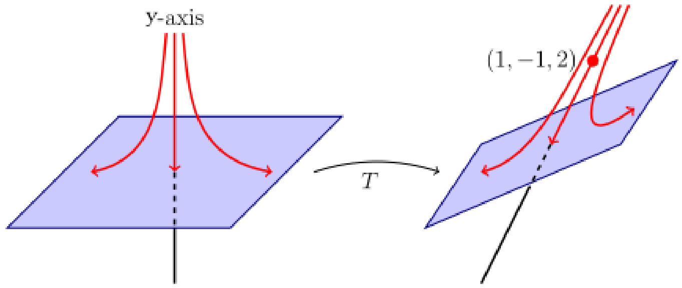
Solution curves that start outside the line or the plane, approaches the plane asymptotically in the long term. At the same time then move away from the line, towards infinity. ■

■ Question 5. (2+2 points)

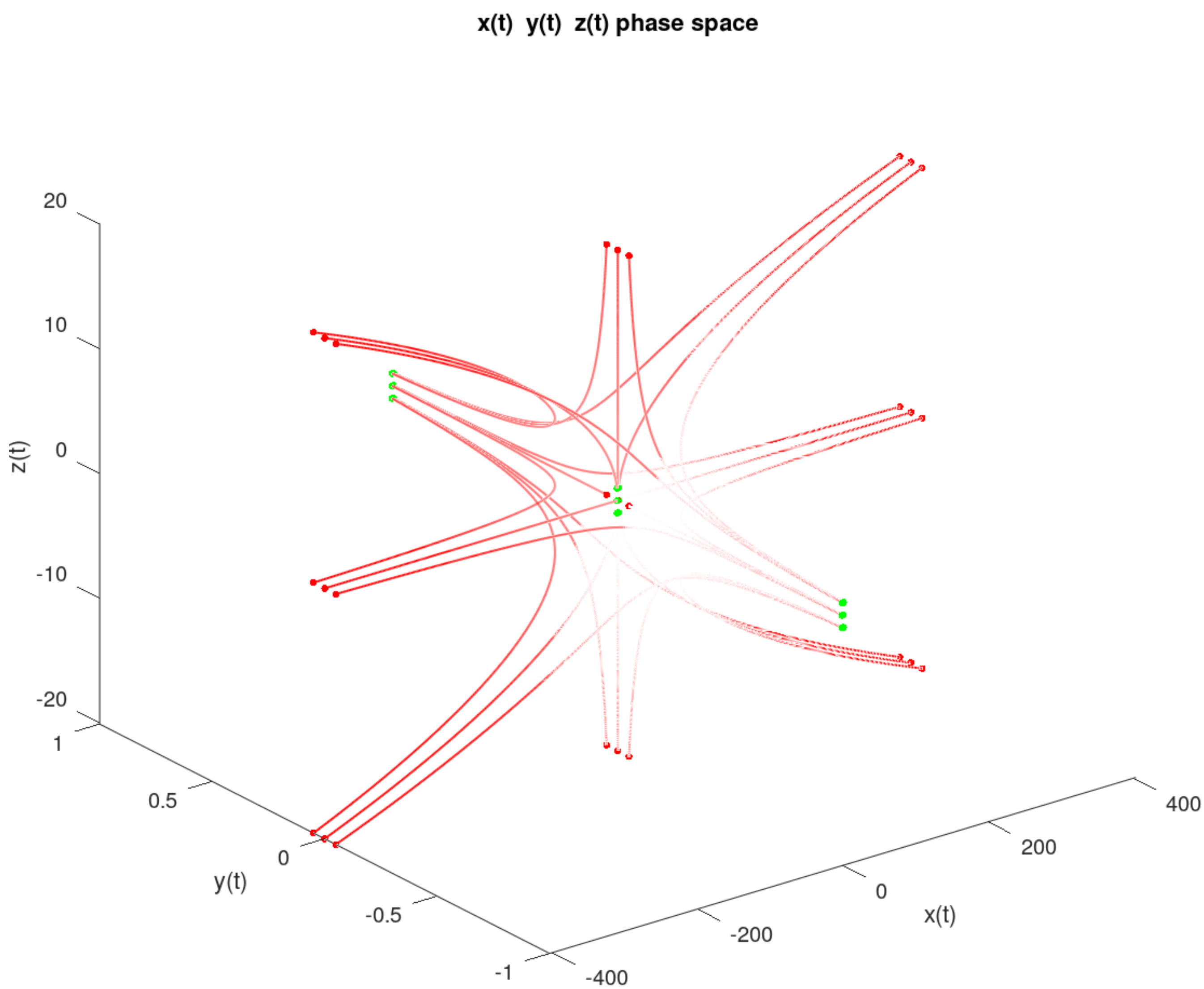
(a) Look up your Linear Algebra notes. Recall that a matrix $\mathbf{A}$ with (real) eigenvalues $\lambda_1, \lambda_2, \lambda_3$ is similar to the diagonal matrix $\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$. Explain why this means the phase portrait of the ODE $\vec{\mathbf{R}}'(t) = \mathbf{A}\vec{\mathbf{R}}(t)$ is qualitatively the same as the phase portrait of the ODE $\vec{\mathbf{R}}'(t) = \mathbf{D}\vec{\mathbf{R}}(t)$, up to a change of basis.

Solution. First observe that the eigenvectors of $\mathbf{D}$ form the standard basis. In that basis, the effect of the linear transformation corresponding to multiplication by $\mathbf{D}$ can be described as dilation by a factor of λ_i along corresponding axes. Consider a change of basis where the new basis is given by the eigenvectors of $\mathbf{A}$ and corresponding change of coordinate systems. Recall that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, where the columns of $\mathbf{P}$ are eigenvectors of $\mathbf{A}$. That means, in this new coordinate system, the effect of the linear transformation corresponding to multiplication by $\mathbf{A}$ can be described as dilation by a factor of λ_i along corresponding eigenvectors, which are the new axes. So we can conclude that the phase portrait of the ODE (which is the flow lines of the vector field defined using the linear transformation) for the two coordinate systems qualitatively look the same, only multiplied by a factor of $\mathbf{P}$.

■



(b) Draw a rough sketch of the three dimensional phase portrait of $\vec{\mathbf{R}}'(t) = \mathbf{D}\vec{\mathbf{R}}(t)$, where λ_i are as in question (4). Don't spend too much time on this, the picture doesn't have to be up to scale. We are only trying to highlight the typical features of a saddle equilibrium in a generic case. Draw a couple of curves that show different long-term behaviors.



Solution. ■

Case 2: Two complex and one real eigenvalue.

■ Question 6. (Spiral Center)

Consider the system

$$\vec{\mathrm{R}}'(t) = \begin{bmatrix} 0 & 4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \vec{\mathrm{R}}(t)$$

(a) **(3 points) Analytical Approach:** Find the general solution to the system.

You can use `WolframAlpha` to find the eigenvalues and eigenvectors of matrices.

Solution.

$$\vec{r}(t) = k_1 \begin{bmatrix} \sin(4t) \\ \cos(4t) \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -\cos(4t) \\ \sin(4t) \\ 0 \end{bmatrix} + k_3 e^{-t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

■

(b) **(1+1+1+1+1 points) Qualitative Approach:** Use the eigenvectors and eigenvalues to answer the following questions:

(i) What happens to solution curves that start in the (x, y)-plane?

Solution. They go in a circle with center at the origin. ■

(ii) What about solution curves starting on the z-axis?

Solution. They flow inwards towards the origin in a straight line. ■

(iii) Describe using words what other solution curves look like?

Solution. Other solution curves look like spirals of constant radius that flow towards **XY**-plane with progressively smaller distance between consecutive loops. ■

(iv) Consider the solution curve that starts at $(1, 1, 1)$. Find equation of a surface of revolution that contains the curve.

Solution. For the solution curve that starts at $(1, 1, 1)$, we have $k_1 = 1, k_2 = -1, k_3 = 1$. So $x(t) = \sin(4t) + \cos(4t)$ and $y(t) = \cos(4t) - \sin(4t)$. The surface of revolution is $x^2 + y^2 = 1$. ■

Draw a rough sketch of the phase portrait in the (x, y, z) plane by hand using your observations.

(c) **(1+2 points) Numerical Approach:** Download the file `threedim.m` from Blackboard. This file draws a total of 27 solution curves from $0 \leq t \leq 2$ where the initial conditions are $x, y, z = -1, 0,$ or 1.

Execute the file. A plot of the solutions in the three dimensional phase space (x, y, z) will appear. The green dots are the initial values, and the red dots are the solution at $t = 2$.

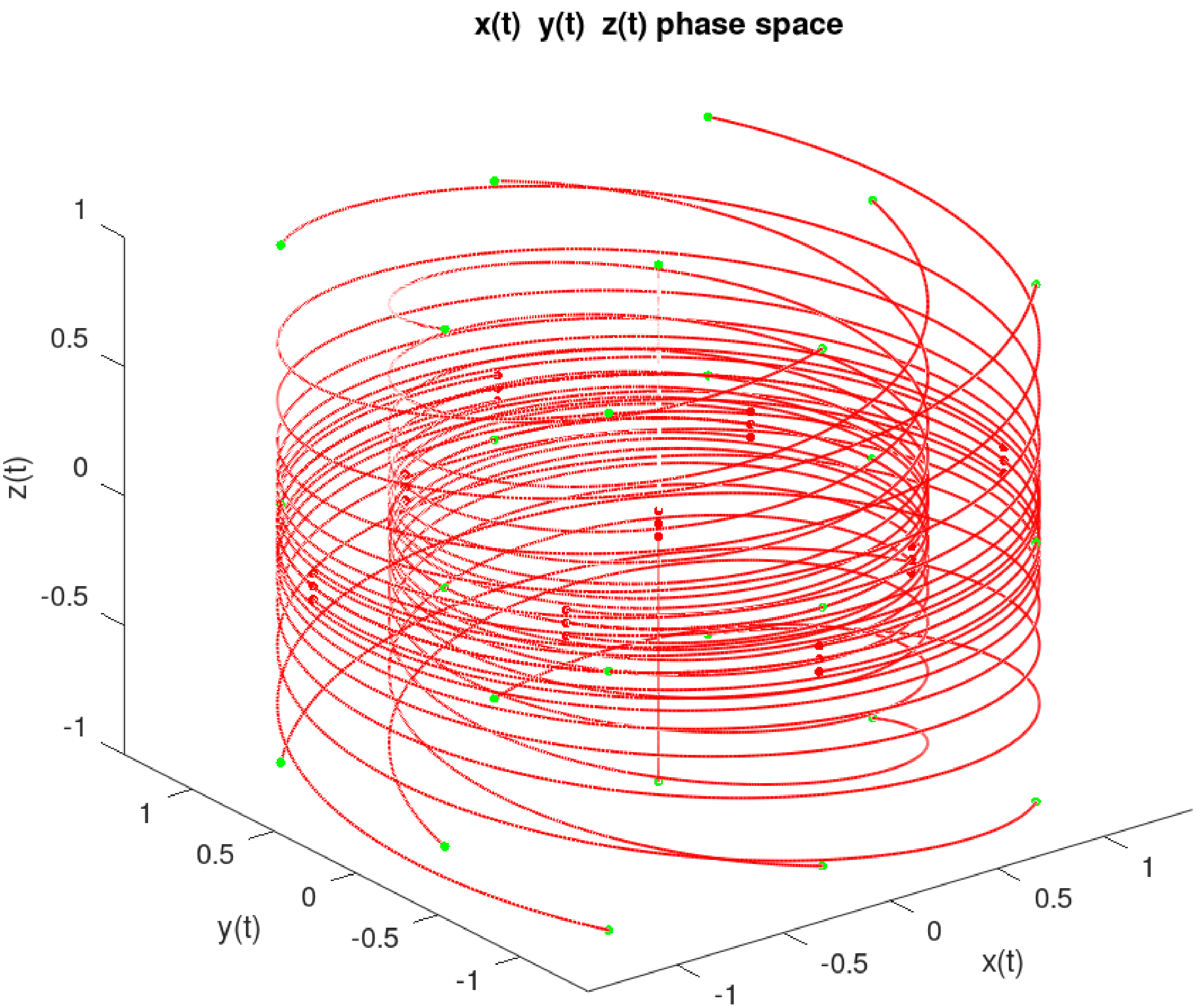
Click the left-most icon on the toolbar labelled "Rotate". Now you can change the angle how you view the three dimensional plot. Note, the values of x,y, and z is printed on the command line at the final time.

(i) Is the result consistent with your analysis in part (b)?

Solution. Yes! ■

(ii) Include a picture of the Octave output.

Solution. ■



(d) **(1 points)** Would you classify the origin as a stable or unstable equilibrium?

Solution. Stable. ■

■ Question 7. (Spiral Saddle - 12 points)

Repeat all parts of question (6) for the system below:

$$\vec{\mathbf{R}}'(t) = \begin{bmatrix} -1 & 4 & 0 \\ -4 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{\mathbf{R}}(t)$$

(a) **(3 points) Analytical Approach:** Find the general solution to the system.

You can use `WolframAlpha` to find the eigenvalues and eigenvectors of matrices.

Solution.

$$\vec{r}(t) = k_1 e^{-t} \begin{bmatrix} \sin(4t) \\ \cos(4t) \\ 0 \end{bmatrix} + k_2 e^{-t} \begin{bmatrix} -\cos(4t) \\ \sin(4t) \\ 0 \end{bmatrix} + k_3 e^{t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(b) **(1+1+1+1+1 points) Qualitative Approach:** Use the eigenvectors and eigenvalues to answer the following questions:

(i) What happens to solution curves that start in the (x, y)-plane?

Solution. They go in a spiral inwards towards the origin. ∎

(ii) What about solution curves starting on the z-axis?

Solution. They flow out from the origin in a straight line.. ∎

(iii) Describe using words what other solution curves look like?

Solution. Other solution curves look like spirals of increasing radius that flow away from **XY**-plane with progressively larger distance between consecutive loops. ∎

(iv) Consider the solution curve that starts at $(1,1,1)$. Find equation of a surface of revolution that contains the curve.

Solution. For the solution curve that starts at $(1,1,1)$, we have $k_1 = 1, k_2 = -1, k_3 = 1$. So $x(t) = e^{-t}(\sin(4t) + \cos(4t)), y(t) = e^{-t}(\cos(4t) - \sin(4t)),$ and $z(t) = e^t$. The surface of revolution is

$$x^2 + y^2 = \frac{2}{z^2}$$

∎

Draw a rough sketch of the phase portrait in the (x, y, z) plane by hand using your observations.

(c) **(1+2 points) Numerical Approach:** Download the file `threedim.m` from Blackboard. This file draws a total of 27 solution curves from $0 \le t \le 2$ where the initial conditions are $x, y, z = -1, 0,$ or 1.

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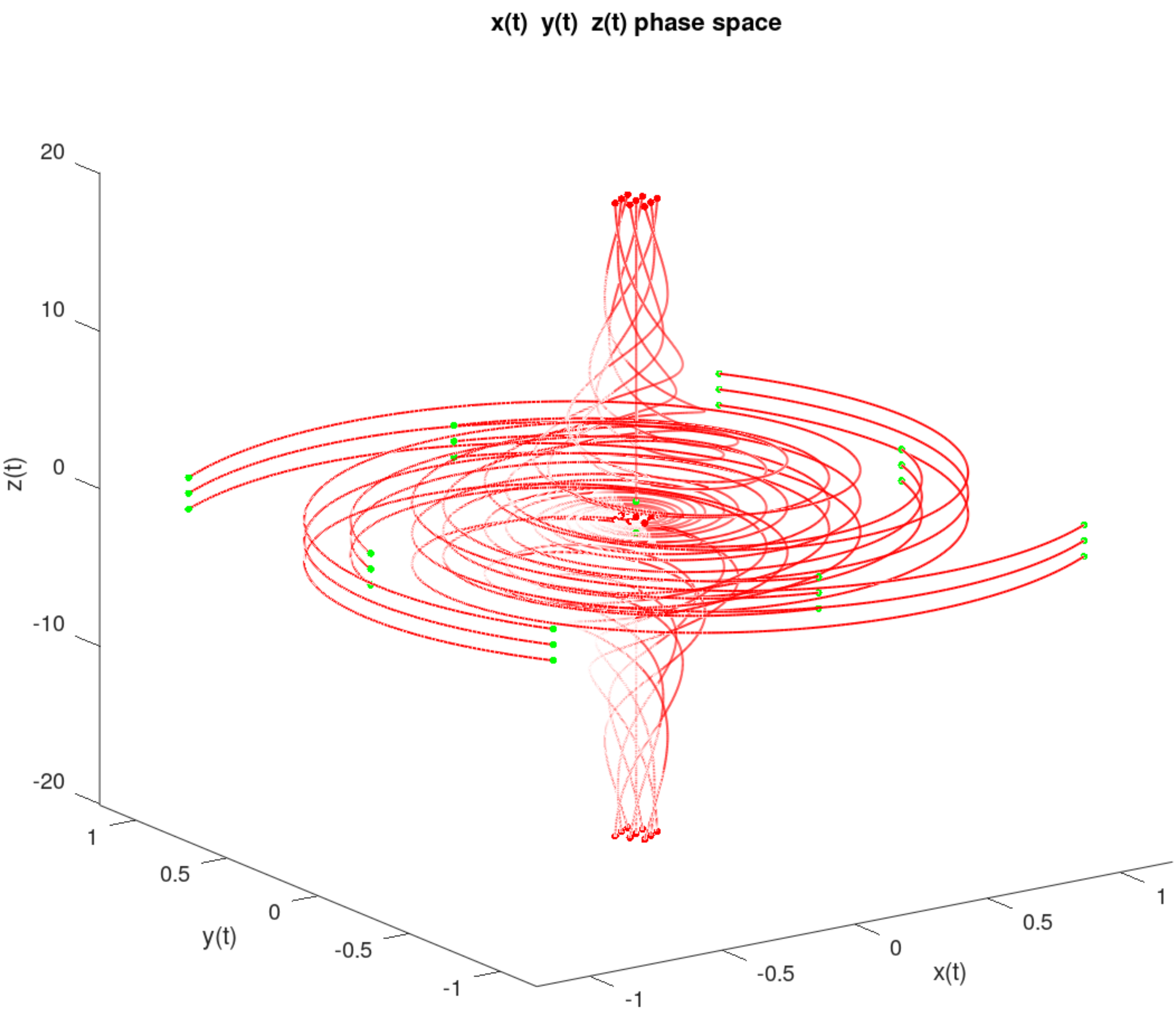
(i) Is the result consistent with your analysis in part (b)?

Solution. Yes! ∎

(ii) Include a picture of the Octave output.

Solution. ∎



(d) **(1 points)** Would you classify the origin as a stable or unstable equilibrium?

Solution. Unstable. ■

■ Question 8. (12 points)

Come up with an example of a system that you think corresponds to a **Spiral Sink**. Repeat all parts of question (6) for this system.

Solution. We will consider the system

$$\vec{R}'(t) = \begin{bmatrix} -1 & 4 & 0 \\ -4 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \vec{R}(t)$$

■

(a) **(3 points) Analytical Approach:** Find the general solution to the system.

You can use WolframAlpha to find the eigenvalues and eigenvectors of matrices.

Solution.

$$\vec{r}(t) = k_1 e^{-t} \begin{bmatrix} \sin(4t) \\ \cos(4t) \\ 0 \end{bmatrix} + k_2 e^{-t} \begin{bmatrix} -\cos(4t) \\ \sin(4t) \\ 0 \end{bmatrix} + k_3 e^{t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

■

(b) **(1+1+1+1+1 points) Qualitative Approach:** Use the eigenvectors and eigenvalues to answer the following questions:

(i) What happens to solution curves that start in the (x,y)-plane?

Solution. They go in a spiral inwards towards the origin. ■

(ii) What about solution curves starting on the z-axis?

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Solution. For the solution curve that starts at $(1,1,1)$, we have $k_1 = 1, k_2 = -1, k_3 = 1$. So $x(t) = e^{-t}(\sin(4t) + \cos(4t)), y(t) = e^{-t}(\cos(4t) - \sin(4t)),$ and $z(t) = e^{-t}$. The surface of revolution is

$$x^2 + y^2 = 2z^2$$

■

Draw a rough sketch of the phase portrait in the (x,y,z) plane by hand using your observations.

(c) **(1+2 points) Numerical Approach:** Download the file threedim.m from Blackboard. This file draws a total of 27 solution curves from $0 \leq t \leq 2$ where the initial conditions are $x,y,z = -1, 0,$ or 1.

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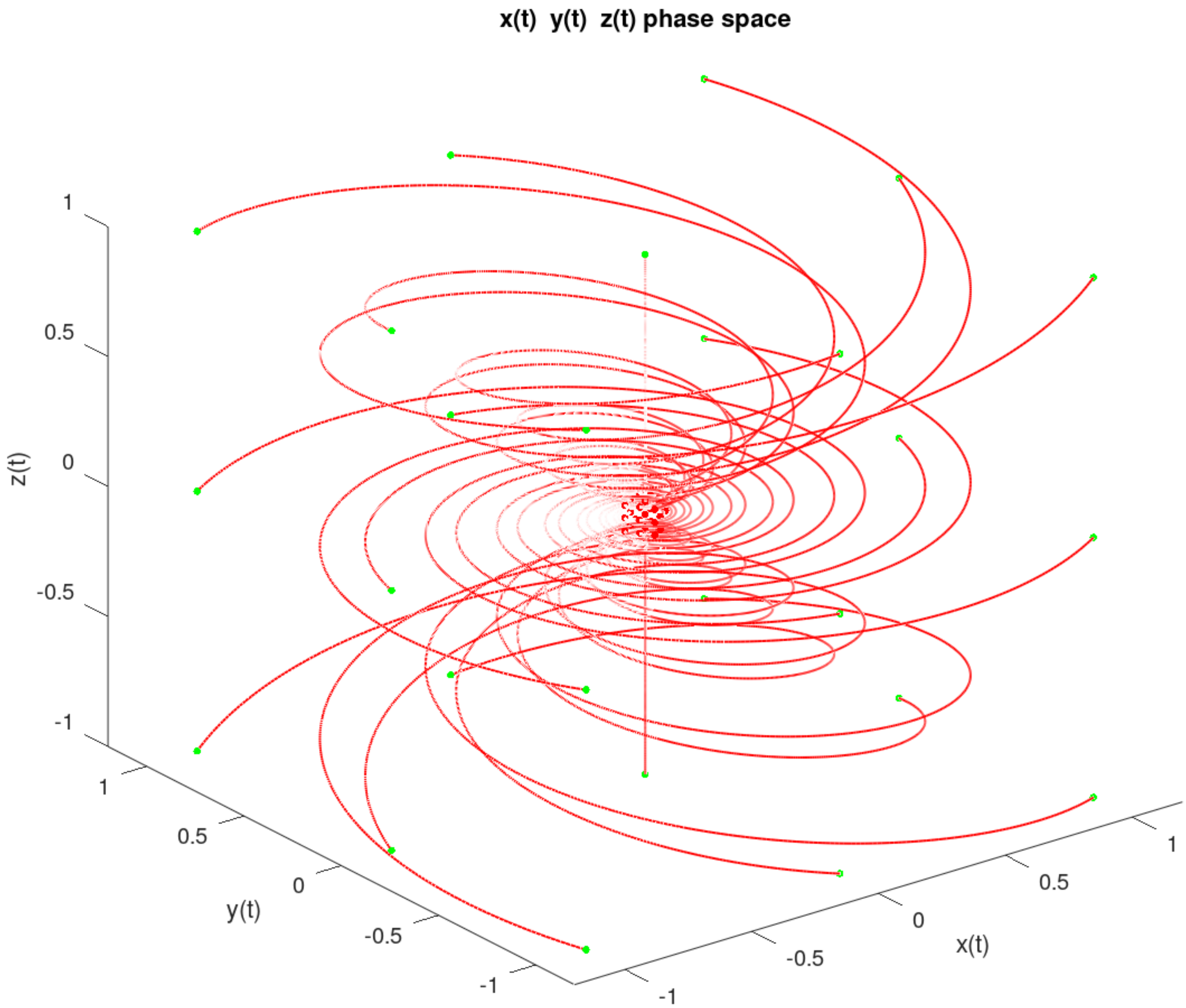
(i) Is the result consistent with your analysis in part (b)?

Solution. Yes! ■

(ii) Include a picture of the Octave output.

Solution. ■



(d) **(1 points)** Would you classify the origin as a stable or unstable equilibrium?

Solution. Stable. ■

■ **Question 9. (2 points)**

Is it possible to have a linear system in $\mathbb{R}^3$ with one complex and two real eigenvalues?

Solution. No. Complex roots of a polynomial with real coefficients appear in conjugate pairs. So a characteristic polynomial of degree 3 will never have two real and one complex root. ■

§C. The Degenerate Case (Recycled Brine Tank Cascade)

■ **Question 10. (4 + 1 + 2 + 2 points)**

Consider three brine tanks **A, B, C** with volumes **60** gal, **30** gal, **60** gal, respectively, as in Figure 1.

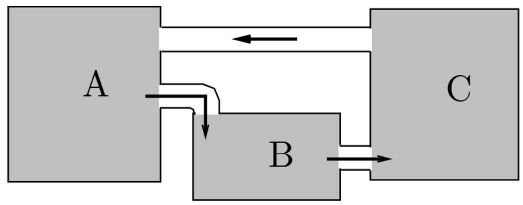


Figure 1: Three brine tanks in cascade with recycling

Saltwater drains from tank A to B at rate 10 gal/hr, drains from tank $\mathbf{B}$ to $\mathbf{C}$ at rate 10 gal/hr, then drains from tank $\mathbf{C}$ to $\mathbf{A}$ at rate 10 gal/hr. The tank volumes remain constant due to constant recycling of fluid. We assume that the concentration of salt is uniform throughout each tank, although it may vary from tank to tank. No salt is lost from the system due to recycling.

(a) Let $x_1(t), x_2(t),$ and $x_3(t)$ denote the amount of salt in tanks A, B, and C, respectively at time t. Write a system of ODEs which we can use to solve for $x_i(t)$ and change it into matrix form.

Solution.

$$x_1' = -\frac{1}{6}x_1 + \frac{1}{6}x_3$$
$$x_2' = \frac{1}{6}x_1 - \frac{1}{3}x_2$$
$$x_3' = \frac{1}{3}x_2 - \frac{1}{6}x_3$$

In matrix form,

$$\vec{\mathbf{X}}'(t) = \begin{bmatrix} -1/6 & 0 & 1/6 \\ 1/6 & -1/3 & 0 \\ 0 & 1/3 & -1/6 \end{bmatrix} \vec{\mathbf{X}}(t)$$

where $\vec{\mathbf{X}} = \langle x_1, x_2, x_3 \rangle$.

■

(b) Use `WolframAlpha` to find the eigenvalues and eigenvectors.

Solution. $\lambda_1 = -\frac{1}{3} + \frac{i}{6}$, $v_1 = \left(-\frac{1}{2} - \frac{i}{2}, -\frac{1}{2} + \frac{i}{2}, 1\right)$

$\lambda_2 = -\frac{1}{3} - \frac{i}{6}$, $v_2 = \left(-\frac{1}{2} + \frac{i}{2}, -\frac{1}{2} - \frac{i}{2}, 1\right)$

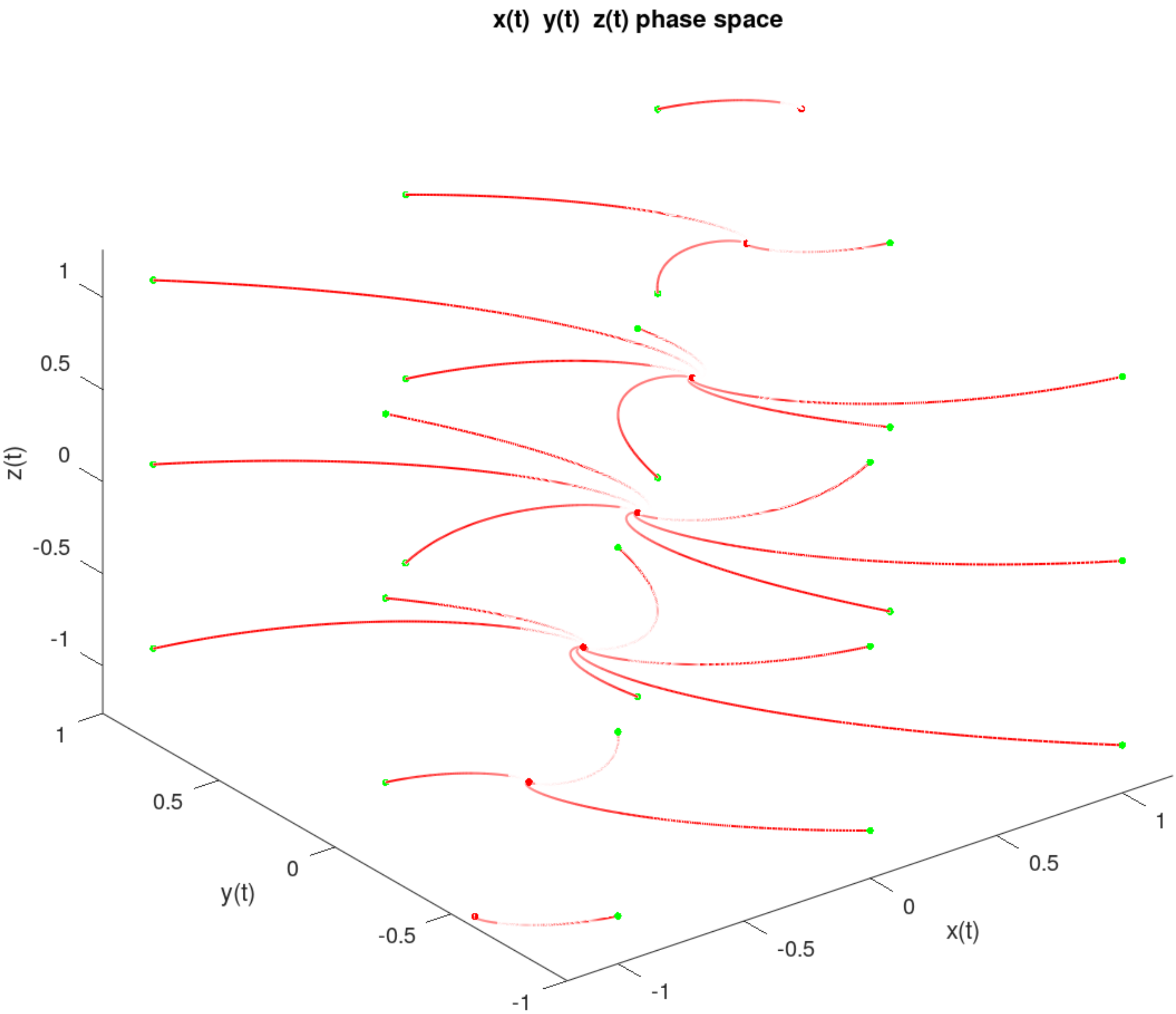
$\lambda_3 = 0$, $v_3 = \left(1, \frac{1}{2}, 1\right)$

■

(c) Use the `threedim.m` file (you will need $0 \le t \le 100$) to draw the phase portrait and qualitatively explain the behavior of solution curves. Include the output picture.

Solution. Since one of the eigenvalues is 0, every point on the line containing the corresponding eigenvector is an equilibrium point. All solution curves that start outside, spiral (because of complex eigenvalues) in the plane perpendicular to $\langle 1, 1/2, 1 \rangle$ and flow inwards to the straight line.

■



(d) Explain why regardless of the initial concentration of salt in each tank, after a long time has passed (i.e. as $t \to \infty$), the total amount of salt is uniformly distributed in the tanks in the ratio $2:1:2$.

Solution. As we observed above, all solution curves eventually converge to a point on the line $t\begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$. So in the long run, $x_1 : x_2 : x_3 = 2:1:2$. ∎