Math 2208: Ordinary Differential Equations

Lecture 6 Worksheet

Spring 2020 **Subhadip Chowdhury** **Feb 10**

Title: Some analytical techniques for solving first order ODEs
Summary: We will learn about integrating factors, the change of variable method, and exact ODEs.

§A. Integrating Factor

Consider a linear ODE of the form $y' = \varphi(t)y + \psi(t)$. To use the technique of **Integrating Factors**, we will first rewrite it into the following form:

$$\frac{dy}{dt} + P(t)y = Q(t) \tag{1}$$

What's the idea?

Think about the product rule for differentiating the function $\mu(t)y(t)$.

$$\frac{d(\mu(t)y(t))}{dt} = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y(t)$$

So if we stare at both of the last two equations hard enough and long enough, we might think about rewriting the ODE from (1) as

$$\mu(t)\frac{dy}{dt} + \mu(t)P(t)y = \mu(t)Q(t) \tag{2}$$

whose left hand side 'sort of' looks like the product rule. So if we could find a function $\mu(t)$ such that

$$\mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y(t) = \mu(t)\frac{dy}{dt} + \mu(t)P(t)y,$$

we would be able to rewrite the ODE from (2) as

$$\frac{d(\mu(t)y(t))}{dt} = \mu(t)Q(t) \tag{3}$$

which can be easily solved as

$$\mu(t)y(t) = \int \mu(t)Q(t)dt \implies y(t) = \frac{1}{\mu(t)}\int \mu(t)Q(t)dt$$

So what's $\mu(t)$?

So our goal is to find a function $\mu(t)$ such that

$$\frac{d\mu}{dt} = \mu(t)P(t)$$

■ **Question 1.**

Find $\mu(t)$.

Theorem A.1

We call $\mu(t)$ the **integrating factor**. With the formula for $\mu(t)$ you obtained above, the complete formula for $y(t)$ is given by

$$y(t) = \frac{1}{e^{\int P(t)dt}} \int \left(Q(t) e^{\int P(t)dt} \right) dt$$

Question 2.

Solve $y' = -2ty + 4e^{-t^2}$.

Question 3.

Recall the salt-mixing problem from your first assignment. Solve $\frac{dQ}{dt} = 0.9 - 6\frac{Q}{600+3t}, \quad Q(0) = 5.$

Question 4.

For what value(s) of the parameter r is it possible to find explicit formulas (without integrals) for the solution to the ODE $\frac{dy}{dt} = t^r y + 4$.

---

§B. Change of Variable

Often, a first-order ODE that is neither separable nor linear can be simplified to one of these types by making a change of variables. Here are some important examples:

Homogeneous Equation:

If $\frac{dy}{dt} = f(t,y)$ where $f(kt,ky) = f(t,y)$, use the change of variables $z = \frac{y}{t}$ or equivalently, $y = zt$.

Question 5.

Consider the ODE

$$\frac{dy}{dt} = \frac{y-t}{y+t}$$

Change the ODE such that the dependent variable becomes $z = y/t$ instead of y. What do you get? Why is this a better form than what you started with?

Bernoulli Equation:

This is an ODE of the form $\frac{dy}{dt} + P(t)y = Q(t)y^b$, $(b \neq 1)$. This looks almost like a linear ODE but not quite. However, consider the change of variable $z = y^{1-b}$.

Question 6.

Consider the ODE

$$\frac{dy}{dt} + y = e^t y^2$$

Change the ODE such that the dependent variable becomes $z = \frac{1}{y}$ instead of y. What do you get? Why is this a better form than what you started with?