MATH 221 - Differential Equations

Lecture 3 Worksheet

Fall 2020 | **Subhadip Chowdhury** | **Aug 24**

Title: What is an Ordinary Differential Equation?

Summary: We will go through various basic definitions and terminology associated with the study of differential equations in order to introduce you to the language we will be using throughout the course this semester.

Related Reading: Chapter 1.1

§A. Different Classifications

Definition A.1: Dependent and Independent Variables

In a function of the form $y = f(t)$, the variable t is called **independent** and y is called **dependent**.

Definition A.2: Differential Equation

An equation containing the derivative of one or more dependent variables, with respect to one or more independent variables is called a **differential equation**.

Differential Equations are classified by type, order and linearity.

Definition A.3: ODE vs PDE

An **ordinary differential equation** (ODE) contains derivatives with respect to only one independent variable (though there may be multiple dependent variables). A **partial differential equation** contains partial derivatives with respect to multiple independent variables.

■ **Question 1.**

Do the first problem of the quiz now.

Definition A.4: Order

The **order** of a differential equation is the order of the highest derivative found in the DE.

■ **Question 2.**

Do the second problem of the quiz now.

The most general form of an $n-$th order ordinary differential equation is

$$\mathrm{F}\left(t, y, y', y'', y''', \ldots, y^{(n)}\right) = 0$$

where **F** is a real-valued function of $n+2$ variables $t, y(t), y'(t), ..., y^{(n)}(t)$.

Definition A.5: Normal Form

The normal form of an n-th order differential equation involves solving for the highest derivative and placing all the other terms on the other side of the equation, i.e.

$$\frac{d^n y}{dt^n} = f\left(t, y, y', y'', \ldots, y^{(n-1)}\right)$$

For example, the normal form of first order ordinary differential equations looks like:

$$y' = f(t, y)$$

Definition A.6: Linearity

An $n-$th order differential equation is said to be linear if the function f is linear in the variables $y, y', y'', \ldots, y^{(n-1)}$.

To be specific, an $n-$th order linear ODE looks like:

$$a_n(t)\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_2(t)\frac{d^2 y}{dt^2} + a_1(t)\frac{dy}{dt} + a_0(t)y = g(t)$$

where $a_n(t), a_{n-1}(t), \ldots, a_0(t), g(t)$ are all functions of t. A nonlinear ODE is one that is not linear in the **dependent variable (or its derivatives)**.

Definition A.7: Autonomy

An ODE is called **autonomous** if the function f is an explicit function of the dependent variable only, i.e. $y' = f(y)$. Otherwise it's called **nonautonomous**.

■ Question 3.

Write down an example of one linear non-autonomous ODE and one nonlinear autonomous ODE. Compare your answers with at least one of your group members. We will discuss more examples in class.

§B. Solutions of ODEs

Definition B.1: Solution

A solution φ of an n-th order ODE is a function $y = \varphi(t)$ which is at least n times differentiable on an interval $\mathbf{I}$ and satisfies

$$\mathbf{F}\left(t, \varphi(t), \varphi'(t), \varphi''(t), \ldots, \varphi^{(n)}(t)\right) = 0$$

for all $t \in \mathbf{I}$. The interval $\mathbf{I}$ is called the **interval of definition** for solution to the ODE.

A graph of the function φ in y vs. t plane is called the **Solution Curve**.

Note that the 'interval of definition' of a solution to an ODE is not necessarily the same as the 'domain of definition' of the function φ. Check the textbook for examples. We will do more examples in class.

Question 4.

Consider the ODE

$$ty' + y = 0$$

Check that $y = \dfrac{c}{t}$ is a solution to this ODE for any arbitrary constant c.

When the solution is an explicit function of t, we call them **explicit** solution. We could also have **implicit** solutions for ODEs when the solution curve is not an explicit function.

Question 5.

Use implicit differentiation to check that $x^2 + y^2 = 1$ is an implicit solution curve to $\dfrac{dy}{dx} = -\dfrac{x}{y}$.

A solution containing arbitrary constants is called a **general** solution. Usually, we get a parameterized family of solution to any ODE. Fixing a particular point on the solution curve gives a **particular** solution to the ODE.

Definition B.2: Initial Value Problem

An **initial value problem** or IVP is a problem which consists of an n-th order ordinary differential equation combined with n initial conditions defined at a point t_0 found in the interval of definition.

For example, a first-order IVP looks like

$$y' = f(t, y), \quad y(t_0) = y_0$$

and a second-order IVP looks like

$$y'' = f(t, y, y'), \quad y(t_0) = y_0, \quad y'(t_0) = y_1$$

Question 6.

Consider the IVP

$$ty' + y = 0, \quad y(1) = 1$$

How would you verify that $y = 1/t$ is the **only** solution to this IVP? We will discuss this further in class.