MATH 221 - Differential Equations

Lecture 16 Worksheet

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Title: The Competing Species Model

Summary: We will learn about the second Lotka-Volterra System that models the population of two competing species. We will use Phase Plane to get a qualitative interpretation of the species interaction.

Related Reading: Chapter 3.3 from the textbook.

From The ODE Project - Section 2.2.2 and 2.2.3.

As you read through this worksheet, you should treat it as a study guide only. It is not a replacement for your textbook. Please make sure to follow along in the textbook (or other online resources from the related reading section above) side-by-side as you read through the topics. You are not expected to be able to answer the questions in here immediately after reading the synopsis from the worksheet. They are designed as more of an exploration directives; as you try to find the answers yourself, you will also learn the topic. You should use whatever resources are available to you, including the internet, to accomplish that task.

§A. A Modified Lotka-Volterra Model

Recall the Lotka-Volterra equations we had used earlier to model predator-prey interaction. Here we are going to look at a modified version of them to understand population dynamics of two competing species, both of which sustain on the same finite resources. We assume that the reproduction rate per individual is adversely affected by high levels of its own species (i.e. a logistic growth) and the other species (i.e. an interaction term) with which it is in competition. Denoting the two populations by x and y, the competing species system can be modeled by the ODE system

$$\frac{dx}{dt} = ax(1-x) - bxy$$
$$\frac{dy}{dt} = cy(1-y) - dxy$$

where $a, b, c,$ and d are positive numbers. For this problem we will assume

$$a = 1, b = 2, c = 1, \text{ and } d = 3$$

(a) Find the equations of the nullclines and the coordinates of the equilibrium point(s).

(b) Use PPLANE to draw the phase plane and find the direction of arrows on the nullclines.

(c) Check that $y'(t_0) = 2x'(t_0)$ if $y(t_0) = 2x(t_0)$. What does it tell you about the arrows that start on the straight line $y = 2x$? Is it consistent with your observation from PPLANE.

(d) Looking at the phase plane, What can you say about the long term fate of both species when the initial condition satisfies $\dfrac{y(0)}{x(0)} > 2$? What about $\dfrac{y(0)}{x(0)} < 2$? Is there anyway the two species can both exist peacefully in the long-term?

The line $y = 2x$ is called a **Separatrix**. It separates the quadrant into two regions each of which corresponds to a different long term behavior of the two populations.

§B. **Atlantic Coast Crab population**

The blue crab is native to the US Atlantic coast, but there is concern that the population is on the decline. The European green crab is an invasive species (recently introduced to the US in the ballast waters of ships), that competes with the blue crab. Assume that the interaction between the blue crab, x, and the green crab, y, is modeled by (up to some scaling):

$$\frac{dx}{dt} = x(100 - x) - 2xy$$
$$\frac{dy}{dt} = y(400 - 6y) - xy$$

(a) Use PPLANE to draw the phase portrait. You will need to choose an appropriate range to see all equilibrium points.

(b) Draw the nullclines and find the four equilibrium solutions using PPLANE. Check that three of these have non-negative coordinates, and hence represent biologically relevant values. According to the phase portrait, what is the long-term behavior of the two crab populations?

(c) Suppose an intervention effort is launched to preserve the blue crab by harvesting a proportion h of the green crabs, so the equations modeling the system become (this is up to scaling, so h can be any positive number):

$$\frac{dx}{dt} = x(100 - x) - 2xy$$
$$\frac{dy}{dt} = y(400 - 6y) - xy - hy$$

It would be nice if we could find harvesting values h so that the two species can co-exist. Try different (positive) values of the parameter h and observe that for some values of h, the fourth equilibrium solution can be found at a point in the first quadrant (it will represent co-existence of the species, since both populations will have positive values). We want to find the exact range of these h values.

We can do it in two ways:

- **Analytical Approach:** Solve for the four equilibrium point in terms of h. Find the conditions on h to ensure both coordinates are positive for the fourth equilibrium.
- **Qualitative Approach:** Don't solve for general formula. Argue using the picture of the null-clines. Observe that changing h changes only one of the nullclines. For what h does it intersect at a positive point?

When h is within this range, no mater what the initial value (as long as it is positive), the green crab and blue crab will co-exist. (yay!)