MATH 221 - Differential Equations

Lecture 29 Worksheet

Fall 2020 | **Subhadip Chowdhury** | **Oct 23**

Title: Forced Harmonic Oscillators

Summary: We will examine the second order constant-coefficient linear, non-homogeneous ODE $y'' + py' + qy = f(t)$.

Related Reading: Section 5.1 from the textbook. We will spend multiple lectures on it.

From The ODE Project - Section 4.2

As you read through this worksheet, you should treat it as a study guide only. It is not a replacement for your textbook. Please make sure to follow along in the textbook (or other online resources from the related reading section above) side-by-side as you read through the topics. You are not expected to be able to answer the questions in here immediately after reading the synopsis from the worksheet. They are designed as more of an exploration directives; as you try to find the answers yourself, you will also learn the topic. You should use whatever resources are available to you, including the internet, to accomplish that task.

§A. Forced Harmonic Oscillators - Nonhomogeneous Equations

If we apply an external force to the harmonic oscillator system, the differential equation governing the motion becomes

$$y'' + py' + qy = f(t)$$

where $p = \dfrac{b}{m} \geq 0$ and $q = \dfrac{k}{m} > 0$ and $f(t)$ measures the external force. As a system, the forced harmonic oscillator equation becomes

$$\frac{d\vec{\mathbf{R}}(t)}{dt} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \vec{\mathbf{R}}(t) + \begin{bmatrix} 0 \\ f(t) \end{bmatrix} \tag{1}$$

where $\vec{\mathbf{R}}(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$.

Question 1.

If $\vec{\mathbf{R}}_1(t)$ and $\vec{\mathbf{R}}_2(t)$ are any two solutions to system (1), show that $\vec{\mathbf{R}}_1(t) - \vec{\mathbf{R}}_2(t)$ is a solution to the system

$$\frac{d\vec{\mathbf{R}}(t)}{dt} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \vec{\mathbf{R}}(t) \tag{2}$$

This is called the **associated homogeneous system**.

■ Question 2. □

Suppose $\vec{\mathbf{R}}_h(t)$ is the general solution to the associated homogeneous system (2) obtained using eigenvalues and eigenvectors of the matrix. On the other hand, suppose $\vec{\mathbf{R}}_p(t)$ is one particular solution to the system (1) obtained by some other method. Then explain why all solutions to the system (1) must be of the form $\vec{\mathbf{R}}_p(t) + \vec{\mathbf{R}}_h(t)$.

■ Question 3. □

Suppose

$$y'' + by' + ky = f(t),$$

where $b > 0$ and $k > 0$ has general solutions of the form $y(t) = y_p(t) + y_h(t)$. Then show that

$$y(t) \to y_p(t) \quad \text{as } t \to \infty.$$

In other words, all solutions of a damped harmonic oscillator with nonzero damping are essentially the same for large values of t.

§B. Method of Undetermined Coefficient

Since we have already learned how to find $\vec{\mathbf{R}}_h(t)$, and consequently $y_h(t)$, all that remains is to find **just one particular solution** to the non-homogeneous equation. Often we can get such a solution by simply making an educated guess! The guessing method is usually called the **method of undetermined coefficients**.

We are going to look at some cases when $f(t)$ is sufficiently simple and try to make some intelligent guess for the particular solution $y_p(t)$ for the system

$$y'' + by' + ky = f(t) \tag{3}$$

Example B.1

Suppose $f(t)$ is an exponential function of the form

$$f(t) = me^{\omega t}$$

Note that any derivative of $e^{\omega t}$ is a constant multiple of $e^{\omega t}$. So we make a reasonable guess that the particular solution $y_p(t)$ is of the form

$$y_p(t) = \mathbf{M}e^{\omega t}$$

which is of the same form but with as yet **undetermined coefficients** $\mathbf{M}$.

To find $\mathbf{M}$, we substitute this y_p in Eq. (3), and then -- by equating coefficients of $e^{\omega t}$ on both sides of the resulting equation -- attempt to determine the coefficient $\mathbf{M}$ so that y_p will, indeed, be a particular solution.

Question 4. □

Find a particular solution of $y'' + 3y' - 4y = 2e^{3t}$.

Question 5. □

Find the general solution to the ODE $y'' + 3y' - 4y = 0$. Use that to write down the general solution to the ODE $y'' + 3y' - 4y = 2e^{3t}$.

Example B.2

If $f(t)$ is sinusoidal function

$$f(t) = m\sin(\omega t) + n\cos(\omega t)$$

then it is reasonable to expect a particular solution

$$y_p(t) = \mathbf{M}\sin(\omega t) + \mathbf{N}\cos(\omega t)$$

which is of the same form but with as yet **undetermined coefficients** **M** and **N**. The reason is that any derivative of such a linear combination of $\cos(\omega t)$ and $\sin(\omega t)$ has the same form.

To find **M** and **N**, we substitute this form of y_p in Eq. (3), and then -- by equating coefficients of $\cos(\omega t)$ and $\sin(\omega t)$ on both sides of the resulting equation -- attempt to determine the coefficients **M** and **N** so that y_p will, indeed, be a particular solution.

Question 6.

Find a particular solution of $3y'' + y' - 2y = 2\cos(t)$.

Question 7.

Find a particular solution of $y'' + \nu^2 y = \cos(\omega t)$.