MATH 221 - Differential Equations

Lecture 32 Worksheet

Fall 2020 | **Subhadip Chowdhury** | **Oct 30**

Title: Equilibria in Nonlinear Systems

Summary: We will analyze equlibrium points of non-linear systems using a technique called linearization which transforms the behavior of nonlinear systems of ODEs back into our familiar linear systems of ODEs.

From The ODE Project - Section 5.1.

As you read through this worksheet, you should treat it as a study guide only. It is not a replacement for your textbook. Please make sure to follow along in the textbook (or other online resources from the related reading section above) side-by-side as you read through the topics. You are not expected to be able to answer the questions in here immediately after reading the synopsis from the worksheet. They are designed as more of an exploration directives; as you try to find the answers yourself, you will also learn the topic. You should use whatever resources are available to you, including the internet, to accomplish that task.

§A. Motivation

It is often impossible to write down explicit solutions of nonlinear (systems of) differential equations. The one exception to this occurs when we are trying to find equilibrium solutions. Provided we can solve the algebraic equations of the nullclines, we can write down the equilibria explicitly. Often, these are the most important solutions of a particular nonlinear system. More importantly, given our extended work on linear systems, we can usually use the technique of **linearization** to determine the behavior of solutions near equilibrium points. We describe this process in detail over the next few lectures.

§B. Some Illustrative Examples

In the following examples, we consider several planar nonlinear systems of differential equations. Our goal is to see that the solutions of the nonlinear system near the equilibrium point resemble those from the linear systems cases.

Example B.1

As a first example, consider the system

$$\frac{dx}{dt} = x - y$$
$$\frac{dy}{dt} = 1 - x^2$$

The x-nullcline is given by $x = y$ and the y–nullcline is $x^2 = 1$. So the equilibrium points are $(-1, -1)$ and $(1, 1)$.

(a) What happens to solution curves that start near the point $(1,1)$? Take a close-up look of the phase portrait near the point $(1,1)$ by changing the plot range. What kind of equilibrium does it resemble most closely?

(b) Do the same for the point $(-1,-1)$.

Example B.2

An important nonlinear second order ODEs which occurs in Physics is the **Van der Pol Oscillator Equation**

$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = 0$$

which can be written as a non-linear system of first order ODEs as

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x + \mu(1-x^2)y$$

For this lecture, we will assume $\mu = 1$.

Draw the phase portrait for the Van der Pol system using PPLANE.

(a) Is $(0,0)$ an equilibrium point?

(b) What happens to solution curves that start near the origin at $(0,0)$?

(c) What about solution curves that start (relatively) far away at $(3,3)$?

(d) Take a close-up look of the phase portrait near the point $(0,0)$ by changing the plot range. What kind of equilibrium does it resemble most closely?

§C. Linear Approximations

Let's use the idea of linear approximations to explain the behavior near the origin of the Van der Pol system. Suppose x and y are very small, for example say less that 0.01, then the nonlinear term x^2y will be less than 10^{-6} in magnitude, much less than either x or y. We can therefore write an **approximate** linearized version of the Van der Pol system near $(0,0)$ as follows:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x + y$$

The associated matrix is $\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ which has eigenvalues

$$\lambda_i = \frac{\mathrm{T} \pm \sqrt{\mathrm{T}^2 - 4\mathrm{D}}}{2} = \frac{1 \pm \sqrt{-3}}{2}$$

Consequently, the linearized system has spiral source at $(0,0)$.

Let's take a look at another example. Consider the system

$$\frac{dx}{dt} = x + y^2$$
$$\frac{dy}{dt} = -y$$

■ **Question 1.** □

There is a single equilibrium point at the origin. Draw the phase portrait using PPLANE. What kind of equilibrium does it resemble most closely? Could we have predicted this by looking at the approximate linearized system?

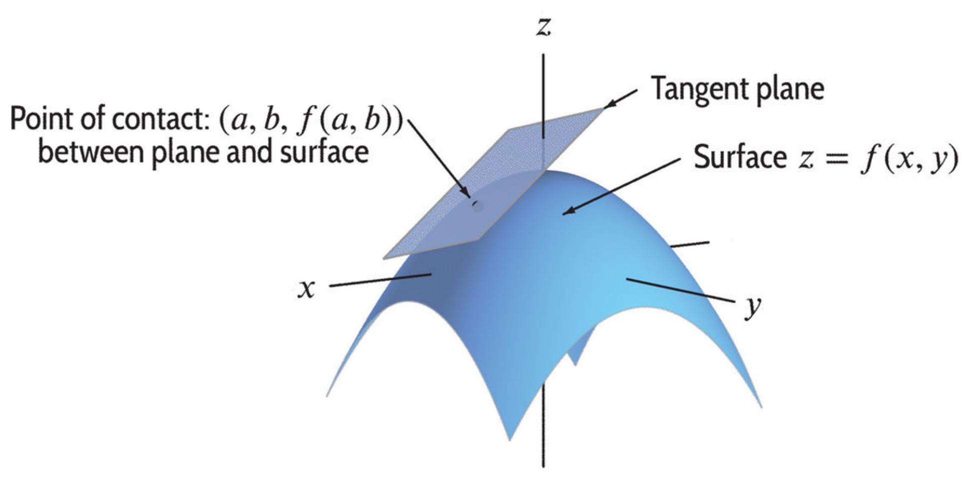
End of Question 1

Can we always just ''drop'' the nonlinear terms in a system? The answer is, as we shall see below, it depends! In particular, it is unclear what to drop when the equilibrium point is not at the origin, because we can't argue that the nonlinear terms are small. Similarly, what should we drop if the functions are not polynomial in nature? To explore further, we need to understand the notion of local linearization for a nonlinear function.

§D. Multivariable Calculus Basics II - Tangent Plane and Local Linearization

Tangent Plane

The graph of a function $z = f(x,y)$ can be viewed as a surface in 3D. If we zoom in on the graph, at most points it seems to flatten out and become planar. Geometrically, the plane we see is the **tangent plane** at that point, which very closely approximates the function $f(x,y)$ at the point of tangency.

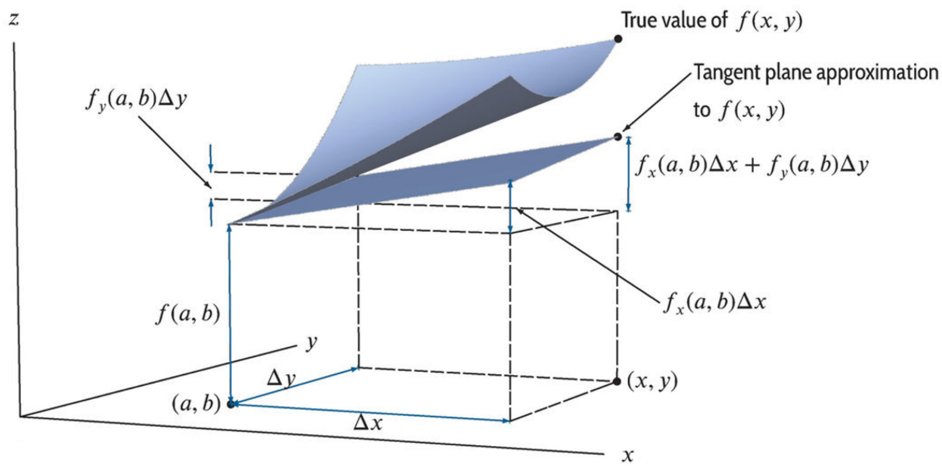


Theorem D.1

The equation of the tangent plane of a differentiable function $f(x,y)$ at a point (a,b) is given by

$$z = \mathrm{L}_f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Here is picture that explains above theorem geometrically.



■ **Question 2.** □

Find the equation of the tangent plane to the elliptic paraboloid $z = x^2 + 2y^2$ at the point $(x, y, z) = (1, 1, 3)$.

Local Linearization

Since the tangent plane function matches f (and its partial derivatives) at (a, b) and is a linear function, it is called a **local linearization** of f near (a, b). We can use the local linearization to approximate a differentiable function at a point. It is essentially the first order **Taylor Series approximation** of $f(x, y)$ at (a, b).

§E. Local Linearization at an isolated Equilibrium Point

Let's assume (x_e, y_e) is an equilibrium of the ODE system

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

and f and g are continuously differentiable in a neighborhood of (x_e, y_e).

■ **Question 3.** □

If $(x, y) \to (x_e, y_e)$, the function $f(x, y)$ can be well approximated by the local linearization $\mathbf{L}_f(x, y)$. Similarly, we can approximate $g(x, y)$ by $\mathbf{L}_g(x, y)$ near the the point (x_e, y_e). Replace $f(x, y)$ and $g(x, y)$ by $\mathbf{L}_f(x, y)$ and $\mathbf{L}_g(x, y)$ respectively, and rewrite the system.

Question 4.

Consider the change of variable $u = x - x_e$ and $v = y - y_e$. Check that $u' = x'$ and $v' = y'$.

Rewrite the linearized system in terms of the variables u and v.

Question 5.

Show that the linearization of the nonlinear system at (x_e, y_e) is a linear system of the form

$$\vec{\mathbf{U}}'(t) = \mathbf{J}\vec{\mathbf{U}}(t)$$

where $\vec{\mathbf{U}}(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ and the coefficient matrix $\mathbf{J}$ is

$$\mathbf{J}(x_e, y_e) = \begin{bmatrix} f_x(x_e, y_e) & f_y(x_e, y_e) \\ g_x(x_e, y_e) & g_y(x_e, y_e) \end{bmatrix} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}\Bigg|_{(x_e, y_e)}$$

This is called the **Jacobian** matrix of f and g, evaluated at (x_e, y_e).

Question 6.

Go back to the non-linear system

$$\frac{dx}{dt} = x - y$$
$$\frac{dy}{dt} = 1 - x^2$$

(a) Find the Jacobian at each equilibrium point.

(b) Calculate the eigenvalue of the Jacobian matrices to find out the type of equilibrium at each point.

(c) Compare your answer to the observation you made using PPLANE in example B.1.

§F. Suggested Homework Problem

Question 7.

□

Consider the non-linear system

$$\frac{dx}{dt} = (x-1)(y+1)$$
$$\frac{dy}{dt} = (y-2)(x+2)$$

(a) What are the equilibrium points?

(b) Find the Jacobian at each equilibrium point.

(c) Calculate the eigenvalue of the Jacobian matrices to find out the type of equilibrium at each point.

(d) Draw the phase portrait using PPLANE to confirm your observation.

Question 8.

□

Find the Jacobian at the origin for the three nonlinear ODEs from project 4. Find the eigenvalues to classify the type of equilibrium for the linearizations.

Question 9.

□

Recall the SIR model. The point $(1,0)$ in the (s,i)-phase plane is called the Disease Free Equilibrium. Linearize the system at the DFE and explain why $i(t)$ looks almost exponential for small values of t.