Math 2208: Ordinary Differential Equations

Lecture 2 Worksheet

Spring 2020 | **Subhadip Chowdhury** | **Jan 27**

Title: Separation of Variables and Slope Fields

Summary: First we will learn an analytical technique for solving separable ODEs. Then we will analyze how much information about an unknown solution to an ODE one can obtain without being able to obtain an explicit formula for the solution itself, by using a technique called slope fields.

§A. Separable Differential Equations

Definition 1.1

A first-order differential equation of the form $\dfrac{dy}{dt} = g(t)h(y)$ is called a separable DE.

The technique for solution is to separate the variables in the equation by placing everything with an independent variable on one side, and everything with a dependent variable on the other. Then integrate both sides with respect to corresponding variables.

Question 1.

Consider the Malthusian model of population $\frac{d\mathrm{P}}{dt} = k\mathrm{P}, \mathrm{P}(0) = \mathrm{P}_0$.

(a) Identify the dependent and independent variables and the parameter.

(b) Obtain the solution by separation of variables.

Question 2.

Which one(s) of the following differential equations is/are

a) autonomous? b) linear? c) separable?

(I) $y' = 1/t$ (II) $y' = 1/y$ (III) $y' = e^{-t^2}$ (IV) $y' = y^2 - 1$ (V) $y' = \frac{t+y}{t-y}$ (VI) $y' = \sin(t)\sin(y)$

§B. Slope Fields

Definition 2.1

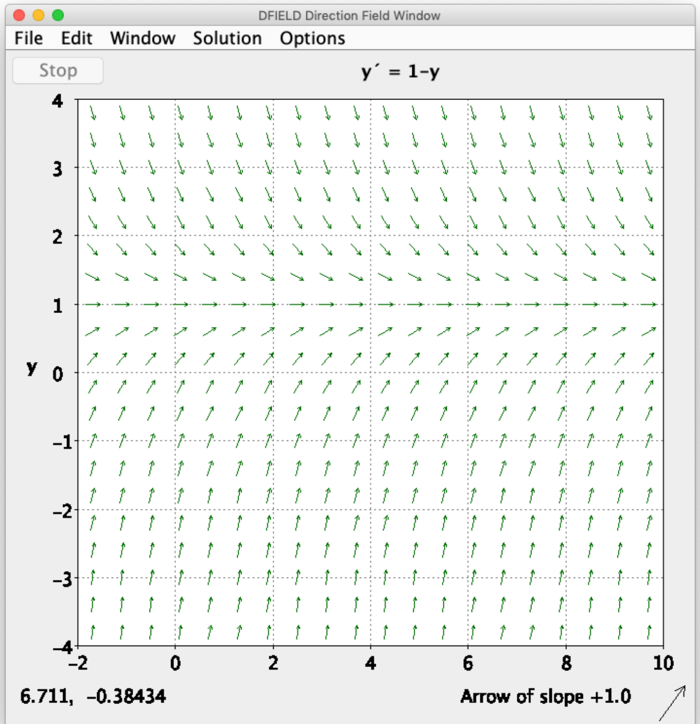
A collection of short, oriented line segments (arrows) placed at each point (t, y) over a rectangular grid, whose slopes at each point is equal to $f(t, y)$ is called the *direction field* or *slope field* of the first order ODE $\frac{dy}{dt} = f(t, y)$.

Question 3. Why do we need slope fields?.

Consider the separable ODE $y' = \sin(y^2)$. Can you solve it using separation of variables? How about $y' = \frac{\sin y}{y}$ or $y' = e^{y^2}$?

Question 4.

Consider the direction field of $y' = 1 - y$. We will use the software DFIELD to plot the direction field. There are several interesting things we can notice about the slope field.



(a) All the arrows are angled the same way as we move from left to right on a horizontal path. Explain this analytically.

(b) When $y = 1$ all the arrows are horizontal. Explain this analytically. Can you guess a solution to the initial value problem $y' = 1 - y$, $y(0) = 1$?

(c) When $y > 1$ all solution curves appear to be decreasing and concave up. When $y < 1$ all solution curves appear to be increasing and concave down. Explain these analytically.

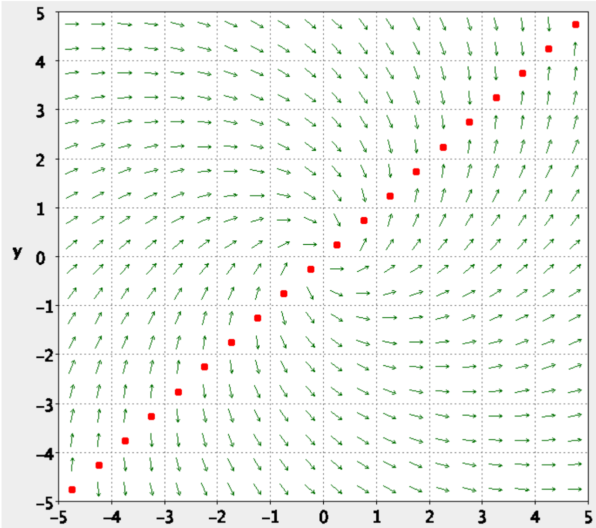
Question 5.

Consider the ODE $y' = (t-1)^2$. What do you think the direction field will look like? Is it consistent with the analytical solution?

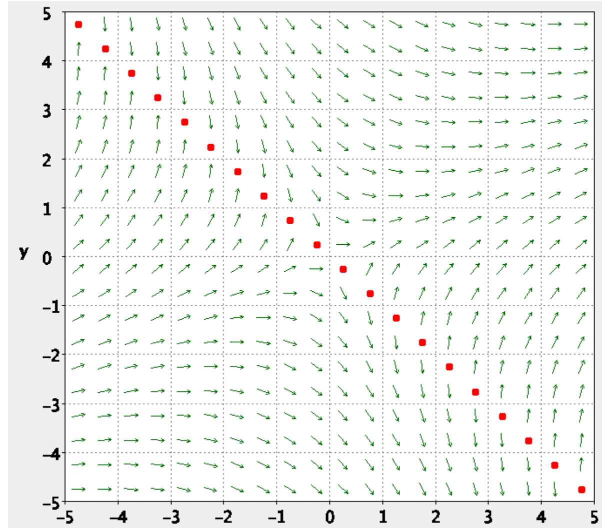
Question 6.

Match the slope fields on the next page to their corresponding ODES.

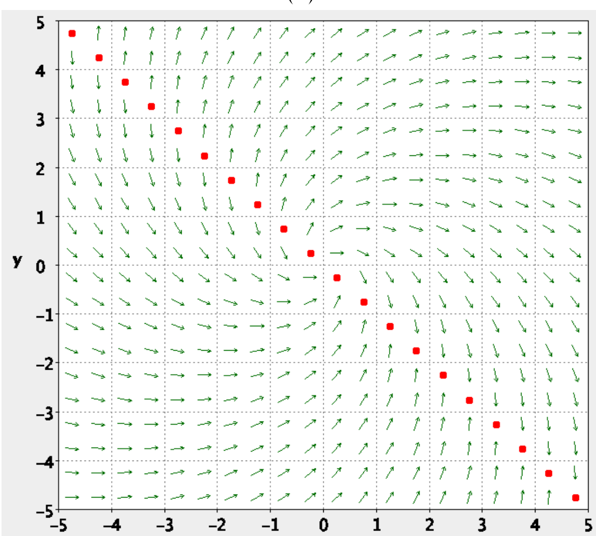
$$\text{(I) } y' = \frac{t-y}{t+y} \qquad \text{(II) } y' = \frac{t+y}{t-y} \qquad \text{(III) } y' = \frac{y+t}{y-t} \qquad \text{(IV) } y' = \frac{y-t}{y+t}$$



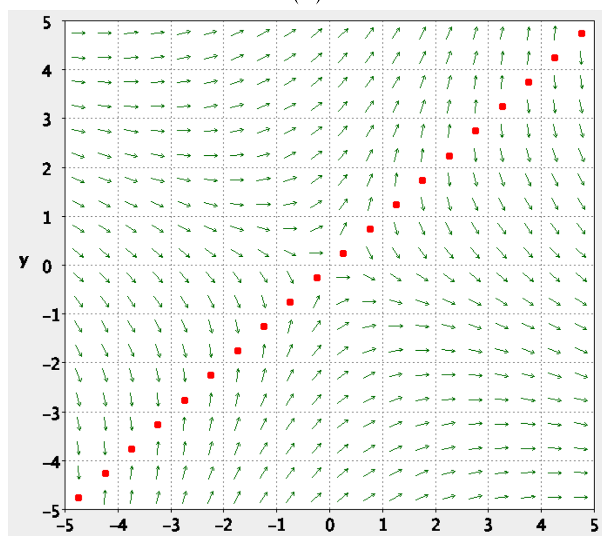
(a)



(b)



(c)



(d)

Question 7.

Try sketching the direction field of the ODE $y' = y(1-y)$. What do you observe?