Math 2208: Ordinary Differential Equations

Assignment 1

Spring 2020 | **Subhadip Chowdhury** | **Due: Jan 29**

Reading

Section 1.1 from the textbook.

Exercises

Question 0.

Your first "instruction" is to not wait until the night before the homework is due to start working on it. Please sign indicating that you acknowledge my recommendation.

Question 1.

Book problems 1.1.(4,5).

Note: Some homework problems (like the ones below) might be more challenging than the standard text book problems involving computations. For some, it is because I do not outline explicitly what to do. The open ended nature of some of the problems might confuse you initially, but the problems are designed to make you think. If you are unsure what a problem is asking, don't hesitate to ask me!

Additional Problems

Question 2.

In class we performed a comprehensive analysis using the population model $\frac{dP}{dt} = kP$. Another modeling approach consists of modifying existing models, which we will do here. We want to modify this model to account for an environment with limited (finite) amount of resources (in some simple way).

(a) Using data from the US population in the 1800s, we estimated $k \approx 0.03$. Suppose for England at that same time, we had data that suggested that $k = 0.005$. Which population is growing faster?

Solution. Since k denotes the growth rate of population, higher k implies faster growth. Hence US population is growing faster. ■

(b) As the population increases exponentially in an environment with limited amount of resource, there will be a point when P becomes too large for the resources to sustain the population. In that scenario (when limited resources is an issue), should k be smaller or larger?

Solution. When P becomes unsustainable, the growth rate decreases. Hence k should be smaller when P is large. ■

(c) The previous two parts suggest that the growth rate constant k should actually depend on the population P. Let's call the growth rate "function" $K(P)$. Thus, by replacing k with the (yet to be determined) function $K(P)$, our model becomes:

$$\frac{dP}{dt} = K(P)P \tag{1}$$

Keeping in mind the model building guidelines (or suggestions) from our first lecture, we should first try a linear formula for $K(P)$. So let's say $K(P) = mP + c$.

(i) Let us assume that if P is very small (i.e. close to 0), limited resources is not an issue (makes sense, right?), and thus the population should be described roughly by $\frac{dP}{dt} = kP$. Using this information, what is $K(0)$? [Evaluate your function $K(P)$ at $P = 0$.]

Solution. When P is small, $K(0) = k$ because growth rate doesn't depend on P. ■

(ii) In a system with limited resources, we will assume there is some maximum population the environment can support. Let's call this number the 'carrying capacity', N. What is $K(N)$?

Solution. As $P \to N$, the growth decreases and theoretically when P reaches N, the growth should stop. Hence $K(N) = 0$. ■

(iii) Combining (i) and (ii), write down your function $K(P)$, and then your ODE model.

Solution. From part (i), we get $c = k$. From part (ii), we get $mN + c = 0 \implies mN + k = 0 \implies m = -k/N$. Hence the ODE model is

$$\frac{dP}{dt} = \left(k - \frac{kP}{N}\right)P$$

■

(iv) Conduct a qualitative analysis of your model.

This question is open ended and vague on purpose. The goal is for you to discover whatever qualitative information you can find from above DE, without solving it analytically. This means describing the solution curve $P(t)$ graphically (increasing/decreasing, concave up/down, asymptotes etc.) without finding for an explicit formula for $P(t)$.

Solution. Here are some information we can find by analyzing above DE.

- There are two equilibrium solutions where dP/dt becomes zero. These are $P = 0$ and $P = N$.
- When $P > N$, the right hand side is negative. Hence dP/dt is negative or in other words, P is decreasing. This means if the population starts at a level higher than the carrying capacity, the population gradually decreases. Since two solution curves can never cross each other, the line $P = N$ acts as a horizontal asymptote to these solution curves.
- If $0 < P < N$, then $dP/dt > 0$. This means if the initial population is positive (it cannot be negative anyway) and less than the maximum capacity, population will gradually increase over time. the line $P = N$ again acts as a horizontal asymptote to these solution curves.
- Note that

$$\begin{aligned}\frac{d^2P}{dt^2} &= k\frac{dP}{dt} - \frac{k}{N}2P\frac{dP}{dt}\\ &= k\frac{dP}{dt}\left(1 - \frac{2P}{N}\right)\end{aligned}$$

  So when $0 \le P \le N$, since $\frac{dP}{dt} \neq 0$, we get that $\frac{d^2P}{dt^2} = 0 \implies P = N/2$. This means the solution curve has an inflection point when $P = N/2$. In other words, the population growth curve is concave up upto $N/2$, and then concave downwards.
- Based on above observations, a picture of $P(t)$ vs t for different initial values of $P(0)$ look like figure 1.

Instructor's Note: In case you read chapter 1 of the book, the solution to this problem is a summary of page 10-11. ■

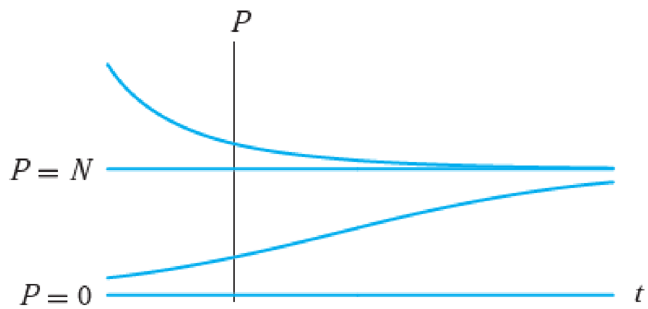

Figure 1: From our textbook page 11

■ Question 3.

In this problem, we are going to try to model the quantity of salt in a $\mathbf{1500}$ gallon water tank. Assume that at time $t = 0$ hr, the tank contains $\mathbf{600}$ gallons of water with $\mathbf{5}$ lbs of salt dissolved in it. Water enters the tank at a rate of $\mathbf{9}$ gal/hr and the water entering the tank has a salt concentration of $\mathbf{0.1}$ lbs/gal. Water exits the tank at a rate of $\mathbf{6}$ gal/hr.

The main assumption that we'll be using here is that the concentration of the salt in the water is uniform throughout the tank. Clearly this will not be the case in real life, but if we allow the concentration to vary depending on the location in the tank the problem becomes very difficult and will involve partial differential equations, which is not the focus of this course. So we will assume that at any given moment, the water that is leaving the tank has the same salt concentration as in the tank.

Suppose $Q(t)$ lb gives the amount of salt dissolved in the water in the tank at any time t hr. We make the following observations.

- Rate of change of $Q(t)$ = Rate at which $Q(t)$ enters the tank – Rate at which $Q(t)$ exits the tank
- Rate at which $Q(t)$ enters/exits the tank = (flow rate of water entering/exiting) × (concentration of salt in water entering/exiting)
- Concentration of salt in the tank at time $t = \dfrac{\text{Amount of salt in the tank at time } t}{\text{Volume of water in the tank at time } t}$

Set up an Initial Value Problem that, when solved, will give us an expression for $Q(t)$. Do not try to solve it. What is the interval of definition for a solution to your IVP?

Solution. According to the setup, the differential equation should look like

$$\frac{dQ}{dt} = 9 \times 0.1 - 6 \times \text{concentration of salt in water exiting the tank}$$

Now according to our assumptions, the concentration of salt in water that is exiting the tank is equal to concentration of salt in the tank at time t. The amount of salt in the tank at time t is $Q(t)$ by assumption. The volume of water in the tank at time t is $600 + 3t$. This is because we start with $\mathbf{600}$ gallons of water and water increases at a net rate of $\mathbf{3}$ gallons per hour. Hence our DE is

$$\frac{dQ}{dt} = 0.9 - 6\frac{Q}{600 + 3t}$$

The IVP is

$$\frac{dQ}{dt} = 0.9 - 6\frac{Q}{600 + 3t}, \qquad Q(0) = 5$$

The IVP model is valid until the tank overflows. That happens when $t = 300$. So the interval of definition is $[0, 300]$. ■