MATH 221 - Differential Equations

Lecture 39 Worksheet

Fall 2020 **Subhadip Chowdhury** **Nov 16**

Title: Initial and Boundary Value Problems of Second Order Differential Equations

Summary: Often the mathematical description of a physical system demands that we solve a linear differential equation subject to boundary conditions—that is, conditions specified on the unknown function, or on one of its derivatives, or even on a linear combination of the unknown function and one of its derivatives at two (or more) different points. In this section, we will look at some examples of this type of problems.

Relevant Book Chapter: 5.2

As you read through this worksheet, you should treat it as a study guide only. It is not a replacement for your textbook. Please make sure to follow along in the textbook (or other online resources from the related reading section above) side-by-side as you read through the topics. You are not expected to be able to answer the questions in here immediately after reading the synopsis from the worksheet. They are designed as more of an exploration directives; as you try to find the answers yourself, you will also learn the topic. You should use whatever resources are available to you, including the internet, to accomplish that task.

§A. IVP vs. BVP

Consider a second-order linear ODE with constant coefficients:

$$y'' + py' + qy = 0$$

We have learned how to find the general solution to such DEs. In an **initial value problem** for such a differential equation, we are provided the value of the solution and value of the first derivatives at the same point (collectively called **initial conditions**). The **Existence and Uniqueness Theorem** tells us that a particular solution to above ODE can be uniquely determined by two initial conditions.

■ Question 1.

□

Find the solution to the IVP

$$y'' - 3y' + 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0$$

■ Question 2. □

Find the solution of the initial value problem

$$y'' + py' + qy = 0, \qquad y(a) = 0, y'(a) = 0 \tag{1}$$

The situation is quite radically different for a problem such as

$$y'' + py' + qy = 0, \qquad y(a) = 0, y(b) = 0 \tag{2}$$

The difference between the problems in equations (1) and (2) is that in (2) the two conditions are imposed at two different points a and b with (say) $a < b$. In (2), we have to find a solution of the differential equation on the interval (a, b) that satisfies given conditions at the endpoints of the interval. Such a problem is called an endpoint or **boundary value problem**.

In a boundary value problem for a second order differential equation, we will specify the values of the function and/or derivatives at different points, which we'll call **boundary conditions**. Any of the following pairs can be used:

$$\begin{aligned}
&\text{First Kind:} && y(a) = \gamma, \quad y(b) = \delta && (3)\\
&\text{Second Kind:} && y'(a) = \gamma, \quad y'(b) = \delta && (4)\\
&\text{Third or Mixed Kind:} && \alpha_1 y(a) + \alpha_2 y'(a) = \gamma, \quad \beta_1 y(b) + \beta_2 y'(b) = \delta && (5)\\
&\text{Periodic:} && y(a) = y(b), \quad y'(a) = y'(b) && (6)
\end{aligned}$$

Boundary Value Problems do not behave as nicely as Initial value problems. There are BVPs for which solutions do not exist; and even if a solution exists there might be many more. Thus existence and uniqueness generally fail for BVPs. The following example illustrate all the three possibilities.

■ Question 3. □

Consider the DE

$$y'' + y = 0 \tag{7}$$

(a) Find the general solution to the DE.

(b) Show that the BVP for equation 7 with the boundary conditions $\boldsymbol{y(0) = 1, y\left(\frac{\pi}{2}\right) = 1}$ has a uniques solution.

(c) Show that the ODE with the boundary conditions $\boldsymbol{y(0) = 1, y(\pi) = 1}$ has a no solution.

(d) Show that the ODE with the boundary conditions $\boldsymbol{y(0) = 1, y(2\pi) = 1}$ has an infinite number of solutions.

§B. **Eigenfunctions**

■ Question 4. ☐

Consider the Boundary Value Problem (BVP)

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y(\pi) = 0$$

(a) Show that if $\lambda \leq 0$, then the only solution to the BVP is the trivial solution.

(b) Show that if $\lambda > 0$, then the BVP has a non-trivial solution if and only if λ is of the form

$$\lambda = n^2, \qquad n = 1, 2, 3, \ldots$$

Definition B.1

Conssider a BVP with underlying differential equation of the form

$$y'' + \lambda y = 0$$

The values of λ that give nontrivial solutions are called **eigenvalues** for the BVP and the nontrivial solutions are called **eigenfunctions** for the BVP corresponding to the given eigenvalue.

■ Question 5. □

Find all the eigenvalues and eigenfunctions for the following BVPs.

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y(2\pi) = 0$$