Math 2208: Ordinary Differential Equations

Assignment 4

Fall 2019 | **Subhadip Chowdhury** | **Due: Feb 19**

Reading

Section 1.9 from the textbook.

Exercises

Don't forget to be neat and thorough. No fringe, and please use the cover page.

Question 1.

Book problems **1.9.(4, 8)**. These problems require that you use the integrating factor method.

Additional Problems

Question 2.

Recall that in your second assignment you did a qualitative analysis of the ODE

$$\frac{dy}{dt} = \frac{\sin(2\pi t) - y}{RC}$$

with $R = 0.5$ and $C = 1$. You may recall that *no matter what the initial value*, all the solutions seemed to converge to a single solution. We would like to show that this is indeed the case by explicitly solving the ODE.

Find the general solution of the ODE above by using the integrating factor method. You may evaluate the integrals using software (like Wolfram alpha), but please show the ODE part of your work. Use your analytical formula to describe the long term behavior (what happens when $t \to \infty$?) of the solutions and compare this with your observation from the previous assignment.

Solution. To use the method of integrating factor, first we rewrite the ODE to find $P(t)$ and $Q(t)$ as usual. Thus,

$$P(t) = \frac{1}{RC} = \frac{1}{0.5} = 2$$

and

$$Q(t) = 2\sin(2\pi t)$$

Then the general solution looks like

$$y(t) = \frac{1}{e^{2t}}\left[\int 2\sin(2\pi t)e^{2t}\,dt + c\right]$$

We need the method of integral-by-parts to calculate the right hand side. Note that you were allowed to use Wolfram Alpha for this part. Observe that,

$$\begin{aligned}
I &= \int 2\sin(2\pi t)e^{2t}\,dt \\
&= 2\sin(2\pi t)\int e^{2t}\,dt - 2\int 2\pi\cos(2\pi t)\left(\int e^{2t}\,dt\right)dt \\
&= \sin(2\pi t)e^{2t} - \int 2\pi\cos(2\pi t)e^{2t}\,dt
\end{aligned}$$

And

$$\int 2\pi\cos(2\pi t)e^{2t}\,dt = \pi\left(\cos(2\pi t)e^{2t} + \int 2\pi\sin(2\pi t)e^{2t}\,dt\right)$$
$$= \pi\left(\cos(2\pi t)e^{2t} + \pi I\right)$$

So we have

$$I = \sin(2\pi t)e^{2t} - \pi\left(\cos(2\pi t)e^{2t} + \pi I\right) \implies I = \frac{\sin(2\pi t)e^{2t} - \pi\cos(2\pi t)e^{2t}}{1+\pi^2}$$

Hence

$$y(t) = \frac{1}{e^{2t}}(I + c) = ce^{-2t} + \frac{\sin(2\pi t) - \pi\cos(2\pi t)}{1+\pi^2}$$

Now the value of the free constant c above depends on the initial condition. But as $t \to \infty$, we get $ce^{-2t} \to 0$ regardless of the value of c. So for large values of t, the solutions all converge towards the periodic curve

$$Y(t) = \frac{\sin(2\pi t) - \pi\cos(2\pi t)}{1+\pi^2}$$

of period 1. If you have done up to this point, you get full credit.

Instructor's Note: We could push this a little further and say if $\alpha = \arccos\frac{1}{\sqrt{1+\pi^2}}$, then $\sin\alpha = \frac{\pi}{\sqrt{1+\pi^2}}$. So

$$Y(t) = \frac{\cos\alpha\sin(2\pi t) - \sin\alpha\cos(2\pi t)}{\sqrt{1+\pi^2}} = \frac{\sin(2\pi t - \alpha)}{\sqrt{1+\pi^2}}$$

is a periodic function of amplitude $\frac{1}{\sqrt{1+\pi^2}}$ and period 1.

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■ Question 3.

In your first homework, you derived the following IVP to describe a population subject to a carrying capacity N,

$$\frac{dy}{dt} = k\left(1 - \frac{y}{N}\right)y, \quad y(0) = y_0$$

where k, N are positive constants and y_0 is some real number. You will now analyze this ODE using analytical, qualitative, and numerical approaches.

(a) **Analytical Approach:** Assuming that $k = 0.5$ and $N = 10$ and $y_0 = 1$, solve the IVP using the separation of variable method. Show your work.

Solution. We have

$$\frac{dy}{dt} = 0.5\left(1 - \frac{y}{10}\right)y$$
$$\implies 20\int \frac{dy}{y(10-y)} = \int dt$$
$$\implies 2\left(\int \frac{dy}{y} + \int \frac{dy}{10-y}\right) = t + c \quad \text{for some constant } c$$
$$\implies \ln|y| - \ln|10-y| = \frac{t+c}{2}$$
$$\implies \ln\frac{|y|}{|10-y|} = \frac{t+c}{2}$$
$$\implies \frac{10-y}{y} = \frac{1}{e^{\frac{t+c}{2}}} = Ae^{-t/2} \quad \text{for some constant } A$$

We know that at $t = 0, y(0) = 1$. So $\frac{10-1}{1} = A \implies A = 9$. Substituting the value of A in above equation,

$$\begin{aligned} &\frac{10-y}{y} = 9e^{-t/2} \\ \implies &\frac{10}{y} - 1 = \frac{1}{e^{\frac{t+c}{2}}} = 9e^{-t/2} \\ \implies &\frac{y}{10} = \frac{1}{1+9e^{-t/2}} \\ \implies &y = \frac{10}{1+9e^{-t/2}} \end{aligned}$$

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(b) **Numerical Approach:** Assuming that $k = 0.5$ and $N = 10$ and $y_0 = 1$, solve the IVP using Octave/Matlab's ODE45 routine. Use the time domain $[0, 30]$ and a time step size of $\Delta t = 0.5$. Make a picture showing your answer from ODE45 (as solid circles) and the exact solution you obtained in the previous problem (shown as a solid line) in the same graph. Include a printout of your graph, which should have appropriate and legible labels.

Solution. See figure below. Ignore the typo in the title (it should be Question 3b from Assignment 4). ■

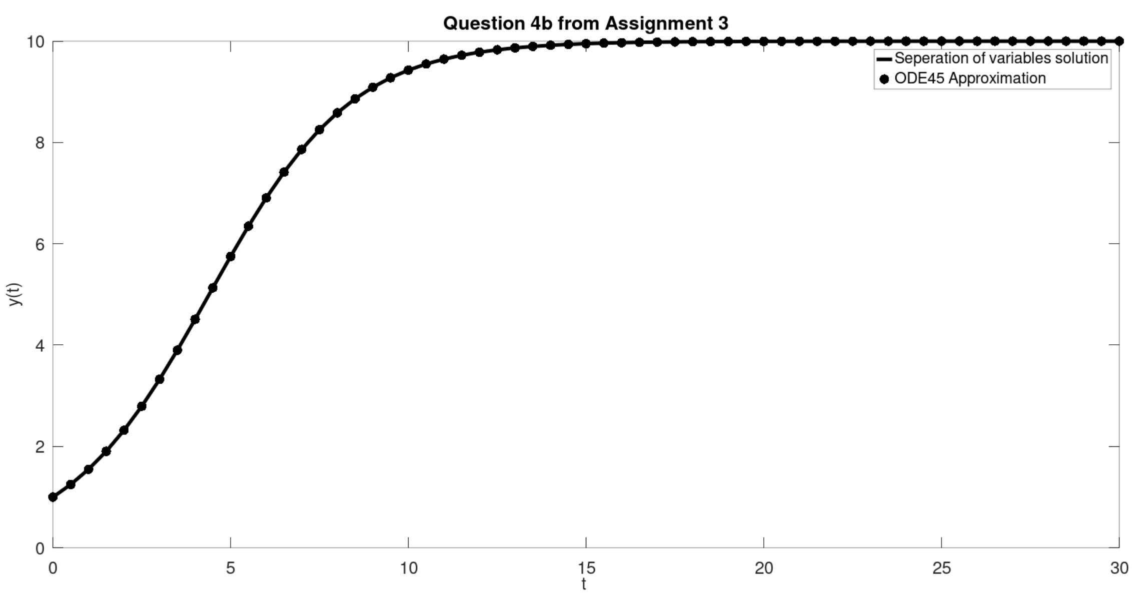


Figure 1

(c) **Qualitative Approach:** Assuming that $k = 0.5$ and $N = 10$, draw the phase line. Use your phase line to describe the long term behavior for the following situations:

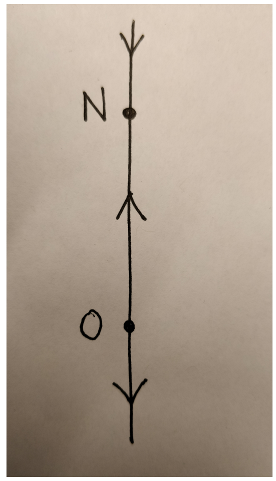
(i) $y_0 < 0$

(ii) $y_0 = 0$

(iii) $0 < y_0 < N$

(iv) $y_0 = N$

(v) $y_0 > N$



Solution. (i) tends to $-\infty$

(ii) stays at a constant value of $\mathbf{0}$

(iii) tends towards $\mathbf{N = 10}$

(iv) stays at a constant value of $\mathbf{N = 10}$

(v) tends towards $\mathbf{N = 10}$

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(d) Briefly describe the pros and cons of each of the above approaches in (a)-(c).

Solution. (a) **Analytic Approach:**

Pro: We get an exact solution. So we can tell the exact value of the population at a specified time t (which may or may not agree with observed data).

Con: The integral is tricky and time consuming to calculate by hand.

(b) **Numerical Approach:**

Pro: The numerical method is faster and provides excellent approximation in this case. It also helps us visualize the data points.

Con: The numerical method is often not so precise.

(c) **Qualitative Approach:**

Pro: The solution was extremely quick and provides valuable information about long term behaviors for different initial values.

Con: It doesn't say anything specific about the solution curve. In particular, we can't use this to make any numerical prediction other than what happens as $t \to \infty$.

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We have learned a number of tools to study first order ODEs. One challenge is deciding what tool is the best one for a given problem. In problems **4, 5, 6, 7** below, you should address the following:

(a) Decide which tools we could apply to the problem.

(b) Classify each tool as analytical, numerical, or qualitative.

(c) Solve the problem with the tool of your choice (some may have multiple tools that work). You may use any computer software of your choice if necessary. If using qualitative method, use the new techniques e.g. phase lines when appropriate.

Question 4.

What is $y(5)$ if y satisfies

$$\frac{dy}{dt} = -2ty + 4t, \quad y(0) = 1$$

Solution. (a) Since we can easily apply the analytical method of *separation of variable* here, and the problem asks for a specific value of y, that would be the best approach.

We could also use a numerical method like ODE45/Euler's method to get an answer as well, but they would only provide an approximation, not the exact answer.

A qualitative approach does not give enough information to answer this question.

(b) Analytical

(c)

$$\begin{aligned} \frac{dy}{dt} &= -2ty + 4t \\ \implies \int \frac{dy}{-y+2} &= \int 2t\,dt \\ \implies -\ln(2-y) &= t^2 + c \\ \implies 2 - y &= de^{-t^2} \\ \implies y &= 2 - de^{-t^2} \end{aligned}$$

Since $y(0) = 1$, we get $d = 1$. So $y(t) = 2 - e^{-t^2}$ and $y(5) = 2 - e^{-25} \approx 2$.

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Question 5.

Consider the population model

$$\frac{dP}{dt} = -(\sin(P) + 2)\left(P^2 - P\right)$$

What is the population at time $t = 2$ if $P(0) = 0.5$.

Solution. (a) Even though the ODE is separable, the integration is hard to compute and may not have a closed form solution.

A qualitative approach does not give enough information to answer this question.

So our best approach is to use a numerical method like *ODE45*/Euler's method to get an approximate answer.

(b) Numerical

(c) Using ODE45, $P(2)$ is approximately $\mathbf{0.99599}$.

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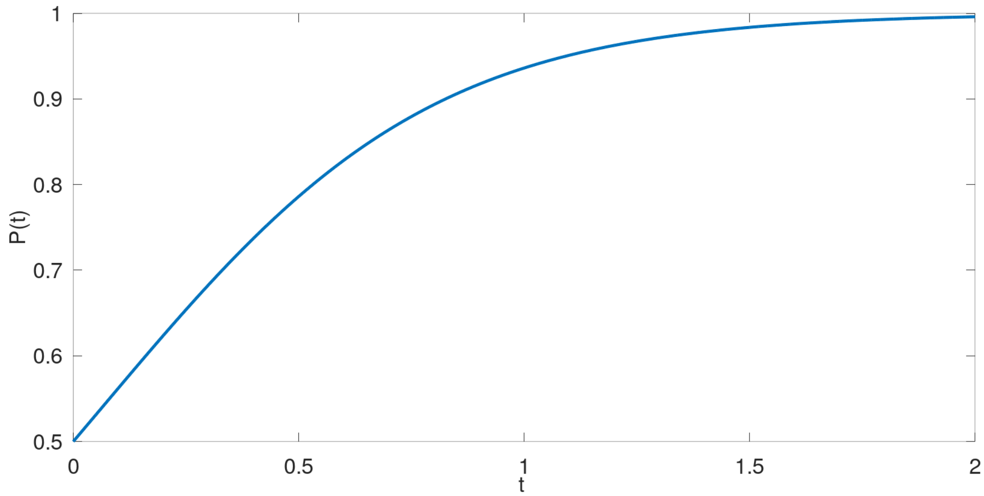


Question 6.

Consider the ODE

$$\frac{dP}{dt} = (P + 1 + \cos(t))(P - 2)$$

Describe, for large values of t, all solutions that have an initial value satisfying $-3 > P(0) > -5$.

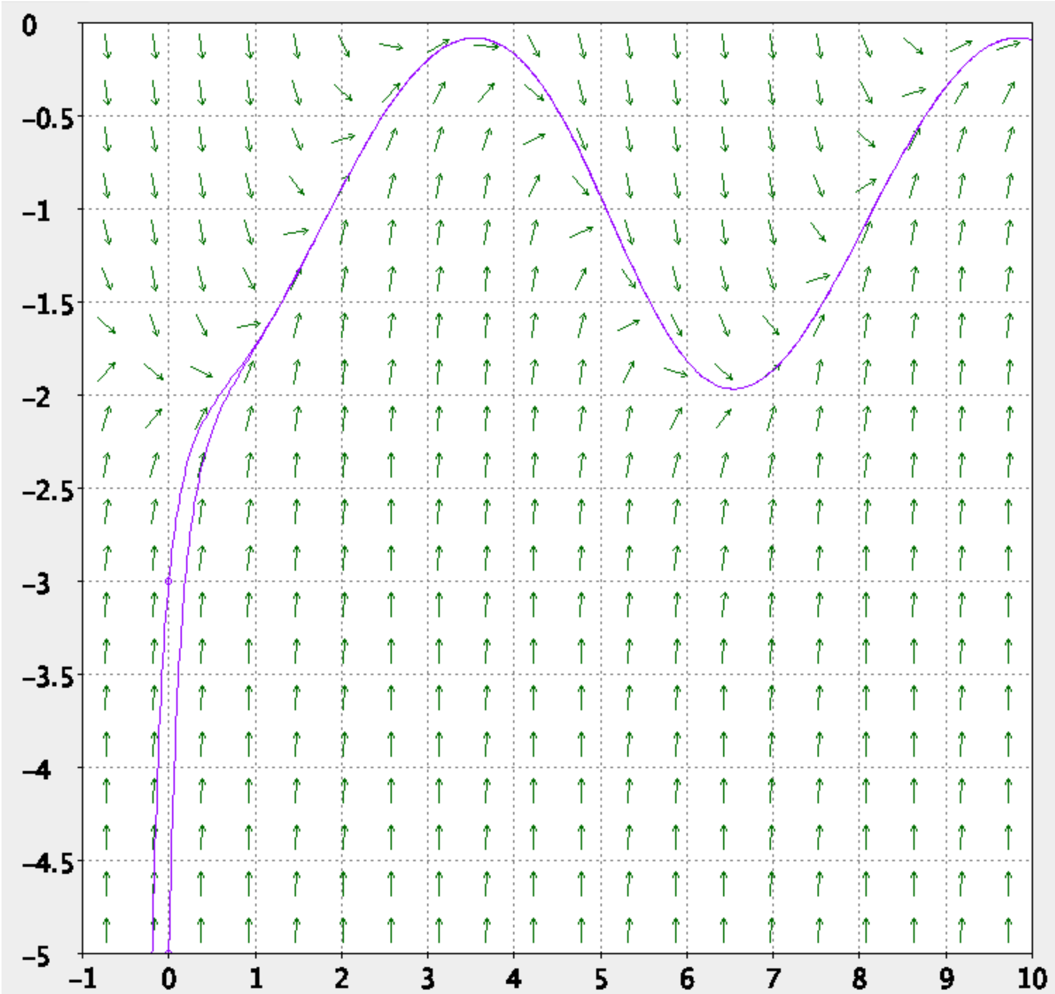
Solution. (a) The ODE is neither separable nor linear, so we don't really have a analytical method of solving this.

Note that the ODE is not autonomous, so we should not be thinking about equilibrium solutions (although it has one at $P = 2$). Instead, as a qualitative approach, we could do the following. Existence and Uniqueness theorem would tell us that The solution curve would be bound in between the solution curve that starts at $(0, -3)$ and $(0, -5)$ in the tP-plane. So our next question becomes what do those two solution curves look like. If we draw a *slope field* by hand, we can see that solution curves seem to go upward and converge toward a sinusoidal oscillating curve around $P = -1$ line. To be sure of this fact, we would need to use a computer. Using `dfield` (so using a combination of qualitative and numerical method), we could confirm our guess.

(b) Qualitative and Numerical together

(c) The solution curve goes upward and converges toward a sinusoidal oscillating curve around $P = -1$ line. See next page for a picture.

■



Question 7.

Consider the population model

$$\frac{dP}{dt} = -(\sin(P) + 2)\left(P^2 - P\right)$$

If $P(0) = 0.5,$ does the population go toward extinction?

Solution. (a) Even though the ODE is separable, the integration is hard to compute and may not have a closed form solution.

Since the ODE is autonomous, we know how the *phase lines* would behave. We don't need the exact value of P at any time, so we don't need to use numerical methods. In fact, since we only need to find what happens as $t \to \infty$, an analysis using Uniqueness theorem and slope field will be enough.

(b) Qualitative

(c) The equilibrium solutions of this ODE are $P = 0, 1$, since $sin(P) \neq 2$ for any P. Also $\frac{dP}{dt} \geq 0$ at $P = 0.5$ So the solution curve goes upward and converges towards the $P = 1$ equilibrium line. In particular, the population does not go towards extinction.

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