MATH 221 - Differential Equations

Project 4: Case Study - Active Shock Absorbers

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An automobile's suspension system consists essentially of large springs with damping. When the car hits a bump, the springs are compressed. It is reasonable to use a harmonic oscillator to model the up-and-down motion, where $y(t)$ measures the amount the springs are stretched or compressed and $v(t)$ is the vertical velocity of the bouncing car.

Suppose that you are working for a company that designs suspension systems for cars. One day your boss comes to you with the results of a market research survey indicating that most people want shock absorbers that ''bounce twice'' when compressed, then gradually return to their equilibrium position from above. That is, when the car hits a bump, the springs are compressed. Ideally they should expand, compress, and expand, then settle back to the rest position. After the initial bump, the spring would pass through its rest position three times and approach the rest position from the expanded state.

Question 1. (2+2+2) points

(a) Sketch a graph of the position of the spring after hitting a bump, where $y(t)$ denotes the state of the spring at time t, $y > 0$ corresponds to the spring being stretched, and $y < 0$ corresponds to the spring being compressed.

(b) Explain politely to your boss why the behavior pictured in the figure is impossible with standard suspension systems that are accurately modeled by the harmonic oscillator system.

(c) What is your suggestion for a choice of a harmonic oscillator system that most closely approximates the desired behavior? Justify your answer in a short paragraph.

End of Question 1

Next, we are going to modify the harmonic oscillator equation to more faithfully model **active shock absorbers** used in the suspension system of trucks or school bus seats. In these models, the damping constant b is replaced by a function $b(v)$ that changes depending on the velocity. This is known as **active damping**.

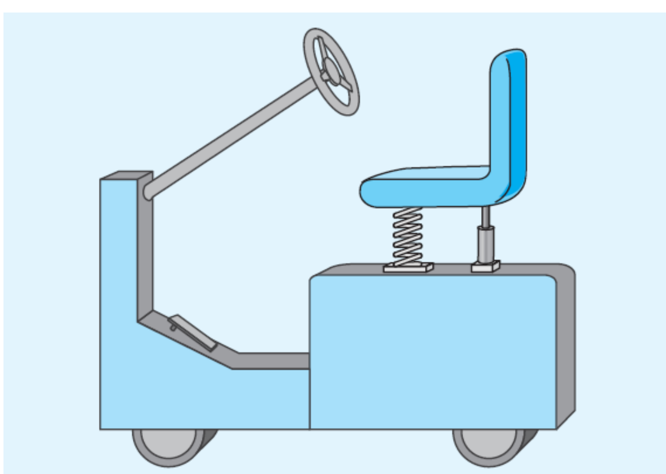

Figure 1

Schematically, we can think of a truck seat as being attached to the rest of the truck by a spring and a dashpot (see Figure 1). For the perfect ride, we would want the spring to have spring constant $k = 0$ and the dashpot to have damping coefficient $b = 0$. But in this case, the seat would float above the truck! For obvious reasons, the seat does have to be connected to the truck, so at least one of the two constants must be nonzero. The springs are chosen so that k is large enough to hold the seat firmly to the truck, and the damping coefficient b is chosen with the comfort of the driver in mind.

If b is very large, the seat is rigidly attached to the truck, which makes the ride very uncomfortable. On the other hand, if b is too small, the seat may "bottom out" when the truck hits a large bump. That is, the spring compresses so much that the seat violently strikes the base. This response is both dangerous and uncomfortable. In practice, designers compromise between having b too small (a smooth ride that has danger from large bumps) or b too large (protection from large bumps but a rough ride).

Active damping allows adjustment of the damping coefficient according to the state of the system. That is, the damping coefficient b can be replaced by a function $v = y'$, the vertical velocity of the seat. As a first step in studying the possibilities in such a system, we consider a modification of the harmonic oscillator equation as follows:

$$my'' + b(v)y' + ky = 0.$$

where m is the mass of the driver. For the rest of this project, we assume that the units of mass and displacement are chosen so that $k = m = 1$, and we study the equation

$$y'' + b(v)y' + y = 0$$

We would like the following requirements to be satisfied by $b(v)$:

- When the vertical velocity of the seat, i.e. v is really small (near zero), we want damping to be small as well, so that small bumps (e.g. potholes) are not transmitted to the seat.
- When the vertical velocity of the seat is large (in our scale, 'large' means close to 1), we want damping to be large as well, so that it protects the seat from "bottoming out" (or "topping out").

We will consider the behavior of a truck seat for three possible choices of the damping function $b(v)$ as follows:

$$b_1(v) = v^4 \qquad b_2(v) = 1 - e^{-10v^2} \qquad b_3(v) = \frac{4 \arctan v}{\pi}$$

Question 2. **3 points**

What do the three functions have in common and where are they different? You should especially consider whether they satisfy our requirements. Include a picture (hand-drawn is sufficient) of all three functions plotted on the same graph (feel free to use **Desmos** or a graphing calculator to do this).

Question 3. **6×3 points**

Convert each 2$^{\text{nd}}$-order oscillator equation into a two dimensional system of first order ODEs. Use PPLANE to investigate the solution curves in each case. Note that the system is not linear.

- Include a screenshot of the (y, v) phase portrait in the region $-1 \le y \le 1, -1 \le v \le 1$ as well as a component graph of y versus t as $0 \le t \le 100$.

- Your goal is to describe the behavior of the solution $y(t)$ for each $b_i(v)$ and for different initial conditions $y(0)$ and $v(0)$. Be as specific as possible in your answers but keep the end goal in mind. The interpretation of what these pictures reveal is the key.
- Don't overload your answer with large numbers of graphs that all tell the same story.

Question 4. **3 points**

Suppose you are choosing from among the three possible functions $b_i(v)$ above for a truck that drives on relatively smooth roads with an occasional large pothole. In this case, y and v are usually small, but occasionally v suddenly becomes large (i.e. close to 1) when the truck hits a pothole. Which of the functions $b(v)$ above would you choose to control the damping coefficient? Justify your answer in a paragraph.

End of Question 4

Watch this video to see active shock absorbers in action.

https://www.youtube.com/watch?v=3KPYIaks1UY