Math 2208: Ordinary Differential Equations

Lecture 18 Worksheet

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Title: Forced Harmonic Oscillators

Summary: We will examine the second order constant-coefficient linear, nonautonomous ODE $y'' + by' + ky = f(t)$ and explore the idea of Resonance. Corresponding Book chapter 4.1 and 4.3.

§A. Forced Harmonic Oscillation

If we apply an external force to the harmonic oscillator system, the differential equation governing the motion becomes

$$y'' + py' + qy = f(t)$$

where $p = \frac{b}{m} > 0$ and $q = \frac{k}{m} > 0$ and $f(t)$ measures the external force. As a system, the forced harmonic oscillator equation becomes

$$\frac{d\vec{r}(t)}{dt} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \vec{r}(t) + \begin{bmatrix} 0 \\ f(t) \end{bmatrix} \tag{1}$$

where $\vec{r}(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$.

Question 1.

If $\vec{r}_1(t)$ and $\vec{r}_2(t)$ are any two solutions to system (1), show that $\vec{r}_1(t) - \vec{r}_2(t)$ is a solution to the system

$$\frac{d\vec{r}(t)}{dt} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \vec{r}(t) \tag{2}$$

This is called the **associated homogeneous system**.

Question 2.

Suppose $\vec{r}_h(t)$ is the general solution to the associated homogeneous system (2) obtained using eigenvalues and eigenvectors of the matrix. On the other hand, suppose $\vec{r}_p(t)$ is one particular solution to the system (1) obtained by some other method. Then explain why all solutions to the system (1) must be of the form $\vec{r}_p(t) + \vec{r}_h(t)$. You may want to review your notes from Linear Algebra regarding homogeneous and non-homogeneous systems of equations. Look at page 390-391 in your textbook for another perspective.

Since we have already learned how to find $\vec{r}_h(t)$, all that remains is to find **just one particular solution** to the non-homogeneous equation. Often one gets such a solution by simply guessing that solution. The guessing method is usually called the **method of undetermined coefficients**.

§B. Method of Undetermined Coefficient

If $f(t)$ is sufficiently simple, we can make some intelligent guess for the particular solution $y_p(t)$ for the system

$$y'' + by' + ky = f(t) \tag{3}$$

Example B.1

If $f(t)$ is sinusoidal function
$$f(t) = m\sin(\omega t) + n\cos(\omega t)$$
then it is reasonable to expect a particular solution
$$y_p(t) = \mathbf{M}\sin(\omega t) + \mathbf{N}\cos(\omega t)$$
which is of the same form but with as yet **undetermined coefficients** **M** and **N**. The reason is that any derivative of such a linear combination of $\cos(\omega t)$ and $\sin(\omega t)$ has the same form.

To find **M** and **N**, we substitute this form of y_p in Eq. (3), and then -- by equating coefficients of $\cos(\omega t)$ and $\sin(\omega t)$ on both sides of the resulting equation -- attempt to determine the coefficients **M** and **N** so that y_p will, indeed, be a particular solution.

■ **Question 3.**

Find a particular solution of $3y'' + y' - 2y = 2\cos(t)$.

Example B.2

Similarly note that any derivative of $e^{\omega t}$ is a constant multiple of $e^{\omega t}$. So if
$$f(t) = me^{\omega t}$$
we make a reasonable guess that the particular solution $y_p(t)$ is of the form
$$y_p(t) = \mathbf{M}e^{\omega t}$$

■ **Question 4.**

Find a particular solution of $y'' - 4y = 2e^{3t}$.

§C. Undamped Forcing

Certain systems have one or more natural frequencies at which they oscillate. If you force such a system by an input that oscillates at close to a natural frequency, the response may be very large in amplitude. In this section we explore this phenomenon.

Consider a mass-spring apparatus lying on a table and moving back and forth periodically (with frequency ν). Now we force the table to tilt periodically as well with some constant frequency (μ). We get the following equation of motion (up to some scaling)
$$y'' + \nu^2 y = \cos(\omega t)$$

Definition 3.1

The constant ν is called the **resonant frequency** and ω is called the **forcing frequency**.

Question 5.

Write down the general solution to the **associated homogeneous equation**.

Question 6.

Find a particular solution to the nonhomogeneous system using the method of undetermined coefficients and conclude that the general solution looks like

$$y(t) = \frac{\cos(\omega t)}{\nu^2 - \omega^2} + k_1 \sin(\nu t) + k_2 \cos(\nu t)$$

Qualitative Analysis.

We will analyze how the solution curves change as we fix ν and vary ω as a parameter. Open the following link

`https://mathlets.org/mathlets/forced-damped-vibration/`

and set $m = 1, k = 2, b = 0, A = 1$. This sets $\nu = \sqrt{2}$ (why?). Check the `Show Trajectory` and `Relate Diagram` options. Use the top left slider to zoom out/in as necessary.

Suppose we have the initial condition $y(0) = y'(0) = 0$. Solving for k_1 and k_2, we get that the solution to this IVP is

$$y(t) = \frac{\cos(\omega t) - \cos(\sqrt{2}t)}{2 - \omega^2}$$

Question 7.

Set $\omega = 0.5$. What does the solution curve look like? You may have to increase the time range to see the full picture. You can move the slider beside the 'A' slider to change the time range.

Can you explain the behavior physically? Is this a periodic solution curve? Are solution curves periodic for any value of ω?

Question 8.

Now set $\omega = 1.2$, a value very close to ν. You may have to zoom out to see the full picture. What do you observe?

Question 9.

As $\omega \to \nu$, how does the amplitude of $y(t)$ change? Does the solution curve have a more 'regular' pattern? The phenomenon you observe here is called **Beating** which occurs when the resonant frequency and forcing response have approximately the same value. You can hear the phenomenon of beating when listening to a piano or a guitar that is slightly out of tune.

§D. **Resonance**

What happens when $\omega = \nu$? Above algebraic calculations don't make sense in that case since the denominator becomes zero.

Question 10.

Check that

$$y(t) = \frac{t \sin(\sqrt{2}t)}{2\sqrt{2}}$$

is **the** solution to the IVP

$$y'' + 2y = \cos(\sqrt{2}t), \qquad y(0) = y'(0) = 0$$

What does the solution curve look like? Draw the curve using `Desmos` or some other graphing tool. Compare it to the solution curve when $\omega = 1.4$.

Question 11.

Watch this video: https://www.youtube.com/watch?v=2cuXbpXRvJ0. What's going on here?

Question 12.

Another video: https://www.youtube.com/watch?v=17tqXgvCN0E. Why does the glass break?

Resonance is not always an unwanted phenomenon. Being able to selectively amplify a particular forcing frequency can be extremely useful. Amplifying a particular frequency of a radio signal is perhaps the most common example of the usefulness of resonance.