MATH 221 - Differential Equations

Lecture 21 Worksheet

Fall 2020 | **Subhadip Chowdhury** | **Oct 5**

Title: Phase Portraits of Linear Systems

Summary: We'll explore the various scenarios that occur with linear systems of ODEs that possess two distinct real eigenvalues.

Related Reading: Section 8.2.1 from the textbook, but we only care about 2×2 matrices.

From The ODE Project - Section 3.2, 3.3.

As you read through this worksheet, you should treat it as a study guide only. It is not a replacement for your textbook. Please make sure to follow along in the textbook (or other online resources from the related reading section above) side-by-side as you read through the topics. You are not expected to be able to answer the questions in here immediately after reading the synopsis from the worksheet. They are designed as more of an exploration directives; as you try to find the answers yourself, you will also learn the topic. You should use whatever resources are available to you, including the internet, to accomplish that task.

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§A. General Solution of a Linear System with Two Distinct Real Eigenvalues

Recall from last time, we proved that if a matrix $\mathbf{A}$ has real eigenvalues λ_1 and λ_2 with eigenvectors $\vec{v}_1$ and $\vec{v}_2$ respectively, then the curves

$$\vec{\mathbf{R}}_1(t) = e^{\lambda_1 t}\vec{v}_1 \quad \text{and} \quad \vec{\mathbf{R}}_2(t) = e^{\lambda_2 t}\vec{v}_2$$

are solutions to the system of ODEs $\dfrac{d\vec{\mathbf{R}}}{dt} = \mathbf{A}\vec{\mathbf{R}}$. Additionally, by the superposition principle, any curve of the form $c_1\vec{\mathbf{R}}_1(t) + c_2\vec{\mathbf{R}}_2(t)$ is also a solution to the system. This time, we wish to prove that all solutions are of this form.

Theorem A.1

Suppose that $\mathbf{A}$ has a pair of **distinct real** eigenvalues, λ_1 and λ_2, with associated eigenvectors $\vec{v}_1$ and $\vec{v}_2$. Then the general solution of the linear system $\vec{\mathbf{R}}' = \mathbf{A}\vec{\mathbf{R}}$ is given by

$$\vec{\mathbf{R}}(t) = c_1\vec{\mathbf{R}}_1(t) + c_2\vec{\mathbf{R}}_2(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

where c_1 and c_2 are arbitrary constants. In other words, all solutions of the system are linear combinations of the two straight line solutions.

Proof. The strategy is as follows. Note that $\vec{\mathbf{R}}_i(0) = \vec{v}_i$. We will first show that given **any** initial condition of the form $\vec{\mathbf{R}}(0) = \begin{bmatrix}\alpha \\ \beta\end{bmatrix}$, we can **always** find constants c_1 and c_2 such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{\mathbf{R}}(0)$$

Then by the Existence and Uniqueness Theorem, $\vec{\mathbf{R}}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$ will be the only solution to that IVP. So that would imply all solution curves are of the same form for different c_1 and c_2. However, to show the first claim, we will need some Linear Algebra basics again.

Linear Algebra Basics IV

Consider two vectors in the plane $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ that are **not multiples of each other**[*] i.e. $\frac{u_1}{u_2} \neq \frac{v_1}{v_2}$.
Consider the system of equation

$$c_1 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + c_2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

where we wish to solve for c_1 and c_2.

From the first equation we get, $c_1 = \frac{\alpha - c_2 v_1}{u_1}$. Substituting this in the second equation, we find,

$$\frac{\alpha - c_2 v_1}{u_1} u_2 + c_2 v_2 = \beta \implies c_2 = \frac{\beta u_1 - \alpha u_2}{v_2 u_1 - v_1 u_2}$$

The fact that $\vec{u}$ and $\vec{v}$ are not multiples of each other tells us that the denominator is nonzero; and in fact, as long as the denominator is nonzero, we can solve for c_2 and hence c_1.

Different eigenvectors are not multiples of each other

From above discussion, it becomes immediate (how?) that if we can show $\vec{v}_1$ and $\vec{v}_2$ are not multiples of each other, then we have our proof. So let's show that $\vec{v}_1$ and $\vec{v}_2$ cannot be multiples of each other..

If $\vec{v}_1 = k\vec{v}_2$, then

$$\mathbf{A}\vec{v}_1 = k\mathbf{A}\vec{v}_2 \implies \lambda_1 \vec{v}_1 = k\left(\lambda_2 \vec{v}_2\right) = \lambda_2\left(k\vec{v}_2\right) = \lambda_2 \vec{v}_1 \implies (\lambda_1 - \lambda_2)\vec{v}_1 = \vec{0}$$

which is not possible since $\lambda_1 \neq \lambda_2$ (this is one of the assumptions in the theorem statement) and $\vec{v}_1$ is nonzero by definition.

So in conclusion, the equation

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{\mathbf{R}}(0)$$

can be always solved for real numbers c_1 and c_2. Since we know that the solution curve $c_1 \vec{\mathbf{R}}_1(t) + c_2 \vec{\mathbf{R}}_2(t)$ also has the same initial value, the linear combination has to be the only solution curve passing through it. Consequently, all general solutions to the ODE system are linear combinations of $\vec{\mathbf{R}}_1(t)$ and $\vec{\mathbf{R}}_2(t)$. ■

§B. Classifying Equilibrium Points

Suppose a linear system has **two real, nonzero, distinct** eigenvalues λ_1 and λ_2. Recall that λ_1 and λ_2 are the solutions to the characteristic polynomial. In what follows, we are going to classify λ_1 and λ_2 into a number of different cases depending on the qualities the eigenvalues possess. In addition, we will also classify the equilibrium at the origin, and a sketch a typical phase portrait for each case.

In the next page, you will find three specific cases, each with an example of an ODE that satisfies the case and a classification of the origin. I suggest coordinating with other students so that you might divide the workload amongst yourselves.

[*] In Linear Algebra, they are said to be **linearly independent**.

■ **Question 1.** □

For each case:

(a) Check that the given matrix will definitely produce real eigenvalues.

(b) Then find the eigenvalues and eigenvectors (by hand) in order to write down the straight line solutions and the general solution of the given system.

(c) Use PPLANE to help you sketch the phase portrait.

(d) Write down a few sentences describing your observations of the phase portrait. What happens to $\vec{\mathbf{R}}(t)$ as $t \to \infty$ or $t \to -\infty$? Note that your answer will depend on the initial condition. Find all possible scenarios. In each case, also find out what happens to the ratio $\frac{y'(t)}{x'(t)}$, i.e. the slope of the solution curve, as $t \to \pm\infty$.

Case 1: $\lambda_1 > \lambda_2 > 0$

Solve $\dfrac{d\vec{r}}{dt} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\vec{r}$.

In this case the origin is called a **nodal source**.

Case 2: $\lambda_1 < \lambda_2 < 0$

Solve $\dfrac{d\vec{r}}{dt} = \begin{bmatrix} -2 & -1 \\ 2 & -5 \end{bmatrix}\vec{r}$.

In this case the origin is called an **nodal sink**.

Case 3: $\lambda_1 > 0 > \lambda_2$

Solve $\dfrac{d\vec{r}}{dt} = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix}\vec{r}$.

In this case the origin is called a **saddle**.

You might be wondering what happens when the roots are not distinct, or one of them is zero. You can try experimenting in PPLANE. We will talk about them in more details next week. The other case remaining is the case when the roots are complex numbers. We will deal with them next class, but here is a refresher/introduction to Complex number before that.

§C. **Introduction to Complex Numbers**

How do we solve the equation $x^2 + 1 = 0$? Clearly, there is no real number that satisfies the equation. However, if we allow ourselves to expand the criteria for being a 'number', we can assume that $\pm\sqrt{-1}$ would be acceptable solutions to this equation. But since $\sqrt{-1}$ does not belong to the set of $\mathbb{R}$eal numbers, we would need pursue a new way of thinking about numbers to understand $\sqrt{-1}$.

Let's give this new number a name. We will denote $\sqrt{-1}$ by i. Can we use i to solve quadratic equations? Consider the equation $at^2 + bt + c = 0$. Quadratic formula says:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So if $b^2 - 4ac < 0$, let's say $b^2 - 4ac = -m^2$, we can rewrite above formula as

$$t = \frac{-b \pm \sqrt{-m^2}}{2a} = \frac{-b \pm mi}{2a}$$

Thus if we expand the collection of numbers to include i and in particular numbers of the form $x + iy$ (where $x, y \in \mathbb{R}$), every quadratic equation becomes solvable regardless of the sign of the discriminant. This new collection (set) of numbers is called the set of **$\mathbb{C}$omplex Numbers**. If $z = x + iy$ is a complex number, we say that x is the ''Real'' part of z, and it is denoted as $x = \mathfrak{Re}(z)$. Similarly we say y is the ''Imaginary'' part of z, and we denote it as $y = \mathfrak{Im}(z)$ (although there is nothing imaginary about it!).

How do we 'draw' complex numbers? Since the $\mathbb{R}$eal number line is evidently not big enough to contain the $\mathbb{C}$omplex numbers, we will need to go outside the line to 'draw' such a number. We will also need to redefine our rules of arithmetic to allow addition or multiplication of Complex numbers.

We observe that if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

This gives us an idea to represent the complex number $x + iy$ as the point (x, y) or the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ in the 2D plane where the $\mathbf{X}-$axis (i.e. the $y = 0$ line) corresponds to the $\mathbb{R}$eal number line. We check that the sum of two complex numbers indeed follows the parallelogram law of adding vectors (as we are just adding the two components separately).

What about multiplying two complex numbers? Check that

$$z_1 z_2 = x_1 x_2 + i^2 y_1 y_2 + i x_1 y_2 + i y_1 x_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

This does not look familiar. However, if we change to polar coordinates, this becomes much simpler looking! Recall that the polar coordinate of a point (x, y) is given by (r, θ) where

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Using the new coordinate system, if $z_1 \equiv (r_1, \theta_1)$ and $z_2 \equiv (r_2, \theta_2)$ then the product simplifies due to trigonometric identities and becomes

$$z_1 z_2 \equiv (r_1 r_2, \theta_1 + \theta_2)$$

As a consequence, if $z \equiv (r, \theta)$, then $z^2 \equiv (r^2, 2\theta)$. And similarly, if $z \equiv (r, \theta)$, then $\sqrt{z} \equiv \left(\sqrt{r}, \frac{\theta}{2}\right)$.

Using this new arithmetic, we observe that since the real number -1 corresponds to the the point $(1, \pi)$ in polar coordinates, we must have

$$i = \sqrt{-1} \equiv (1, \pi/2)$$

So we can place the new mystery number i at a distance 1 from the origin in the positive $\mathbf{Y}-$axis direction which is consistent with the idea of associating $z = x + iy$ to the point (x, y) (in Cartesian coordinates) in the plane.

Definition C.1

The modulus of a complex number $z = x + iy$ is $|z| = \sqrt{x^2 + y^2}$.

The conjugate of a complex number $z = x + iy$ is $\bar{z} = x - iy$.

We will use the following result from Algebra without proof.

Theorem C.1

If the roots of a quadratic polynomial with real coefficients are not real, then they are conjugate complex numbers.

Question 2.

Express the following numbers in the form $a + ib$.

(a) $(1 + 2i)(1 - 2i)$

(b) $\dfrac{3}{i}$

(c) $(1 + 5i)(i - 2)$

Theorem C.2: Euler's Formula

Euler's formula, named after Leonhard Euler, is a mathematical formula in complex analysis that establishes the fundamental relationship between the trigonometric functions and the complex exponential function. Euler's formula states that for any real number θ:

$$e^{i\theta} = \cos\theta + i \sin\theta$$

When $x = \pi$, Euler's formula evaluates to Euler's identity, the "greatest" Math identity according to some mathematicians:

$$e^{i\pi} + 1 = 0$$

Question 3.

Use Euler's formula to show that $(\cos\theta + i \sin\theta)^n = \cos(n\theta) + i \sin(n\theta)$.