MATH 2208: ORDINARY DIFFERENTIAL EQUATIONS

ASSIGNMENT 6

Spring 2020 | **Subhadip Chowdhury** | **Due: Mar 25**

Reading

Section 2.1(The Predator Prey System), 2.2(The Predator Prey Vector Field, Examples of Systems and Vector Fields, Equilibrium Solutions), 3.1 (skip the harmonic oscillator part) from the textbook .

Linear Algebra Practice

The following problems are things you have seen in linear algebra. Perhaps you need to refresh your memories, which is the purpose of these problems. You do NOT need to submit solutions to these problems, but it is highly recommended that you work these out.

Question -1.

1. Let $\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, and $a = .5$ and $b = 2$.
    (a) Compute $\mathbf{A}\vec{v}$.
    (b) Compute $a\vec{v}$.
    (c) Draw the vector $\vec{v}$ in the (x, y) plane with the tail centered at the origin.
    (d) Draw the vector $a\vec{v}$ in the (x, y) plane with the tail centered at the origin. How does this compare to the drawing for $\vec{v}$?
    (e) Draw the vector $b\vec{v}$ in the (x, y) plane with the tail centered at the origin. How does this compare to the drawing for $\vec{v}$?
2. Compute the determinant of $\mathbf{A}$. Show your work.
3. Compute the eigenvalues and eigenvectors of $\mathbf{A}$. Show your work.

Question 0.

Book problems $\mathbf{3.1.(31, 32)}$. You will need to use the conclusions of these two problems to answer problem **3.1.35** below.

Exercises

Don't forget to be neat and thorough. No fringe, and please use the cover page.

Question 1 - 2 points, 4 points.

Book problems $\mathbf{2.1.(15, 17)}$. or problem 17, the 'essay' doesn't need to be long! Just two or three sentences will suffice.

Question 2 - (2+2) points, 0.5*4 points .

Book problems $\mathbf{2.2.(7}\boldsymbol{a}\mathbf{, 7}\boldsymbol{b}\mathbf{, 11)}$.

■ **Question 3 - 1 point, 2 point, 4 points.**

Book problems **3.1.(9, 19, 26)**.

■ **Question 4 - (2+2+2+1) points each.**

Book problems **3.2.(2(a,b,d,e), 6(a,b,d,e), 8(a,b,d,e))**.
Note: For problem 2, **Y** is the same thing as the vector which we have been calling $\vec{r}$. We will learn how to do part d and e of these problems on Monday Mar 23.

Additional Problems

■ **Question 5 - 4 points.**

Consider the first order linear ODE

$$\frac{dy}{dt} + 2y = Q(t)$$

where $Q(t)$ is a function such that $-1 < Q(t) < 2$ for all t. Show that in the long-term (i.e. as $t \to \infty$), all solutions become bounded between $\frac{-1}{2}$ and 1.
Note: This is a problem about Integrating factors, not system of ODEs.

Solution. Suppose $F(t)$ is an antiderivative of $\int Q(t)e^{2t}dt$. Then the general solution to the ODE looks like

$$y(t) = \frac{1}{e^{2t}}[F(t) + C] = [F(t) + C]e^{-2t}$$

for some arbitrary constant C. Note that $y(0) = F(0) + C \implies y(t) = [F(t) + y(0) - F(0)]e^{-2t} \implies$

$$y(t) = y(0)e^{-2t} + [F(t) - F(0)]e^{-2t} \tag{$\star$}$$

Now $-1 < Q(t) < 2$ and $e^{2t} > 0$ for all t; so we get $-e^{2t} < Q(t)e^{2t} < 2e^{2t}$ and consequently,

$$-\int_a^b e^{2t}dt < [F(b) - F(a)] < 2\int_a^b e^{2t}dt$$

for any a and b. In particular for $a = 0$ and $b = t$, we have

$$-\left[\frac{e^{2t}}{2} - \frac{1}{2}\right] < [F(t) - F(0)] < 2\left[\frac{e^{2t}}{2} - \frac{1}{2}\right]$$

Thus from equation ($\star$) above, we get

$$y(0)e^{-2t} - \left[\frac{e^{2t}}{2} - \frac{1}{2}\right]e^{-2t} < y(t) < y(0)e^{-2t} + 2\left[\frac{e^{2t}}{2} - \frac{1}{2}\right]e^{-2t}$$

$$\implies [y(0) + 1/2]e^{-2t} - \frac{1}{2} < y(t) < [y(0) - 1]e^{-2t} + 1$$

So in particular, as $t \to \infty$, we get

$$\frac{-1}{2} \le y(t) \le 1$$

Thus graphs of all solutions must approach the strip $-1/2 \le y(t) \le 1$ in the ty-plane as t increases. More precise information about the long-term behavior is difficult to obtain without specific knowledge of $Q(t)$.

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■ **Question 6 - 4 points.**

Find the equation of the nullclines of the following system, draw them in **XY**-plane, and find the equilibrium points.

$$\frac{dx}{dt} = x(2-x) - xy$$
$$\frac{dy}{dt} = xy - y$$

Draw a rough diagram of the direction field on the phase plane $0 \le x, y \le 2$, using the table from section D. Here are some instructions: find an easily identifiable point in the plane which is not on one of the nullclines. Then evaluate (f, g) at this point and use this to draw a direction at that specific point. Repeat the process.

Can you draw some sample solution curves?

Solution. **Instructor's Note**: I have given two plots separately so that you can identify all the features, but you may have drawn them on the same graph. Since you are supposed to plot the nullclines and direction field by hand, as long as the first picture is correct and the direction of the arrows in the regions are roughly in correct angle (N,S,E,W,NE,NW,SE, or SW), you get full credit.

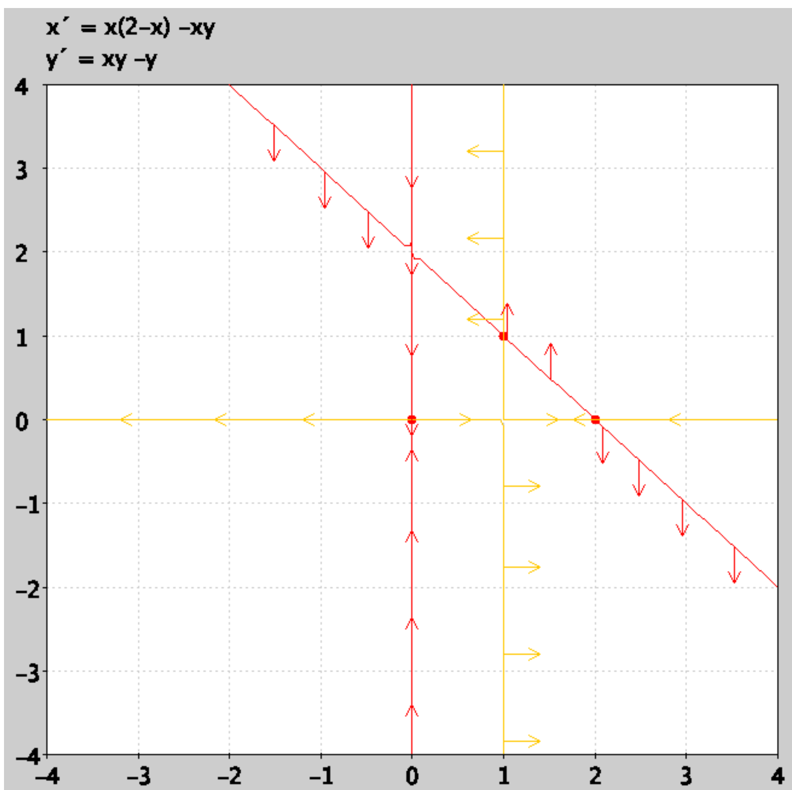

Figure 1: Nullclines and Equilibrium Points

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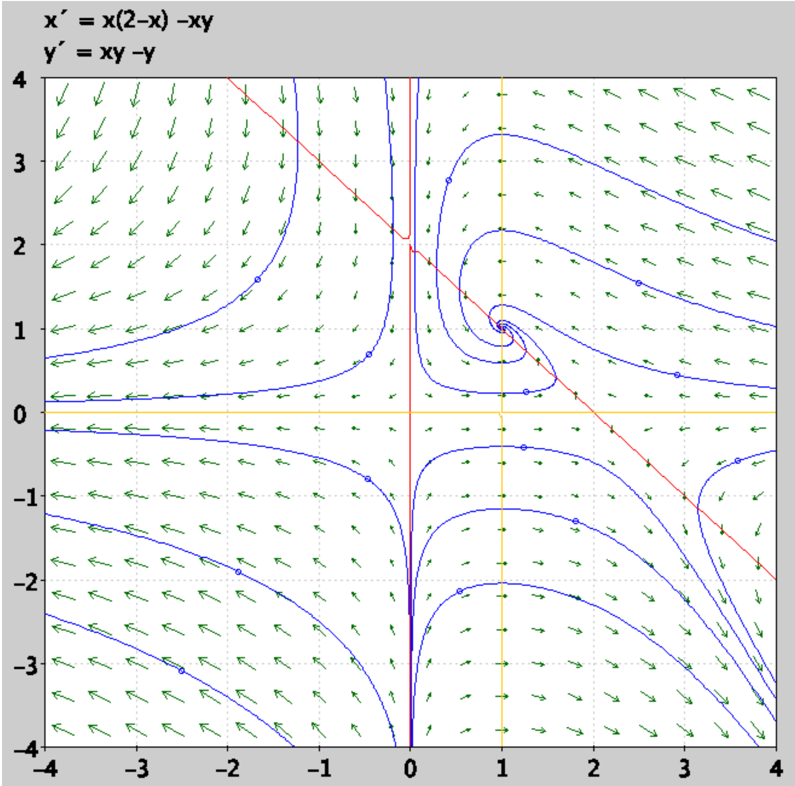


Figure 2: Sample Orbits

Question 7 (Book Problem 3.1.35) - (1+1+1+2) points.

Before you work on the problem 3.1.35 from the book, read the following. It gives a context of what we are trying to achieve.

In class we showed the following result:

Theorem 1. *Suppose $\vec{\mathbf{R}}_1(t)$ and $\vec{\mathbf{R}}_2(t)$ are two solutions to the ODE*

$$\frac{d\vec{\mathbf{R}}}{dt} = \mathbf{A}\vec{\mathbf{R}}.$$

If $\vec{\mathbf{R}}_1(0)$ and $\vec{\mathbf{R}}_2(0)$ are linearly independent, then given any initial condition $\vec{\mathbf{R}}(0)$, we can find constant k_1 and k_2 such that

$$\vec{\mathbf{R}}(0) = k_1\vec{\mathbf{R}}_1(0) + k_2\vec{\mathbf{R}}_2(0)$$

By the Existence and Uniqueness Theorem for systems, we know that each initial-value problem for a linear system has exactly one solution.

By the Linearity principle, $k_1\vec{\mathbf{R}}_1(t) + k_2\vec{\mathbf{R}}_2(t)$ is a solution to the ODE for all constants k_1 and k_2.

Hence by combining both ideas, given any two solutions $\vec{\mathbf{R}}_1(t)$ and $\vec{\mathbf{R}}_2(t)$ of a linear system with linearly independent initial conditions $\vec{\mathbf{R}}_1(0)$ and $\vec{\mathbf{R}}_2(0)$, every general solution of the system belongs to the two-parameter family $k_1\vec{\mathbf{R}}_1(t) + k_2\vec{\mathbf{R}}_2(t)$.

In other words, this shows that the solution space is $\mathbf{span}\left\{\vec{\mathbf{R}}_1(t), \vec{\mathbf{R}}_2(t)\right\}$. Now, in problem 3.1.35, we are going to prove the following theorem:

Theorem 2. *Suppose that $\vec{\mathbf{R}}_1(t)$ and $\vec{\mathbf{R}}_2(t)$ are solutions to the system*

$$\frac{d\vec{\mathbf{R}}}{dt} = \mathbf{A}\vec{\mathbf{R}}.$$

If $\vec{\mathbf{R}}_1(0)$ and $\vec{\mathbf{R}}_2(0)$ are linearly independent, then $\vec{\mathbf{R}}_1(t)$ and $\vec{\mathbf{R}}_2(t)$ are also linearly independent for every t.

Then theorem 2, along with the fact that the solution space is $\mathbf{span}\left\{\vec{\mathbf{R}}_1(t), \vec{\mathbf{R}}_2(t)\right\}$ will prove that the solution space is *two dimensional* with $\left\{\vec{\mathbf{R}}_1(t), \vec{\mathbf{R}}_2(t)\right\}$ as a *basis*.

Now work out problem 3.1.35. You will need to use the results from problem 31 and 32.

Assignment 6 Solutions

Chapter 2.1

15. Suppose $y = 1$. If we can find a value of x such that $dy/dt = 0$, then for this x and $y = 1$ the predator population is constant. (This point may not be an equilibrium point because we do not know if $dx/dt = 0$.) The required value of x is $x = 0.05$ in system (i) and $x = 20$ in system (ii). Survival for one unit of predators requires 0.05 units of prey in (i) and 20 units of prey in (ii). Therefore, (i) is a system of inefficient predators and (ii) is a system of efficient predators.

17. **(a)** For the initial condition close to zero, the pest population increases much more rapidly than the predator. After a sufficient increase in the predator population, the pest population starts to decrease while the predator population keeps increasing. After a sufficient decrease in the pest population, the predator population starts to decrease. Then, the population comes back to the initial point.

(b) After applying the pest control, you may see the increase of the pest population due to the absence of the predator. So in the short run, this sort of pesticide can cause an explosion in the pest population.

Chapter 2.2

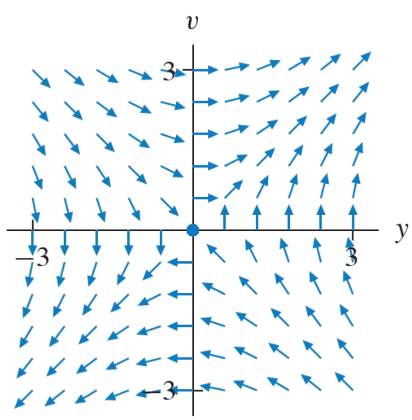
7. **(a)** Let $v = dy/dt$. Then

$$\frac{dv}{dt} = \frac{d^2y}{dt^2} = y.$$

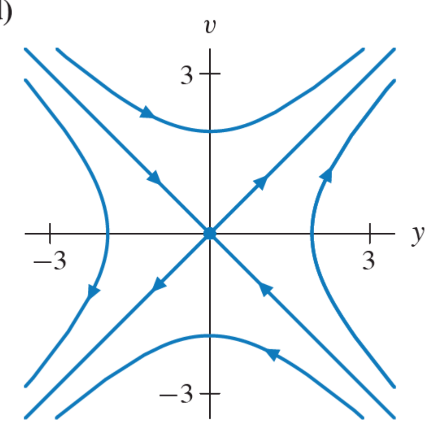
Thus the associated vector field is $\mathbf{V}(y, v) = (v, y)$.

(b) See part (c).

(c)



(d)



Chapter 3.1

9. $$\frac{dx}{dt} = \beta y$$
$$\frac{dy}{dt} = \gamma x - y$$

19. Letting $v = dy/dt$ and $w = d^2y/dt^2$ we can write this equation as the system

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= \frac{d^2y}{dt^2} = w \\ \frac{dw}{dt} &= \frac{d^3y}{dt^3} = -ry - qv - pw. \end{aligned}$$

In matrix notation, this system is

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

where

$$\mathbf{Y} = \begin{pmatrix} y \\ v \\ w \end{pmatrix} \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -r & -q & -p \end{pmatrix}.$$

26. **(a)** Substitute $\mathbf{Y}_1(t)$ into the differential equation and compare the left-hand side to the right-hand side. On the left-hand side, we have

$$\frac{d\mathbf{Y}_1}{dt} = \begin{pmatrix} -3e^{-3t} \\ -3e^{-3t} \end{pmatrix},$$

and on the right-hand side, we have

$$\mathbf{AY}_1(t) = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} e^{-3t} \\ e^{-3t} \end{pmatrix} = \begin{pmatrix} -2e^{-3t} - e^{-3t} \\ 2e^{-3t} - 5e^{-3t} \end{pmatrix} = \begin{pmatrix} -3e^{-3t} \\ -3e^{-3t} \end{pmatrix}.$$

Since the two sides agree, we know that $\mathbf{Y}_1(t)$ is a solution.

For $\mathbf{Y}_2(t)$,

$$\frac{d\mathbf{Y}_2}{dt} = \begin{pmatrix} -4e^{-4t} \\ -8e^{-4t} \end{pmatrix},$$

and

$$\mathbf{A}\mathbf{Y}_2(t) = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} e^{-4t} \\ 2e^{-4t} \end{pmatrix} = \begin{pmatrix} -2e^{-4t} - 2e^{-4t} \\ 2e^{-4t} - 10e^{-4t} \end{pmatrix} = \begin{pmatrix} -4e^{-4t} \\ -8e^{-4t} \end{pmatrix}.$$

Since the two sides agree, the function $\mathbf{Y}_2(t)$ is also a solution.

Both $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are solutions, and we proceed to the next part of the exercise.

(b) Note that $\mathbf{Y}_1(0) = (1, 1)$ and $\mathbf{Y}_2(0) = (1, 2)$. These vectors are not on the same line through the origin, so the initial conditions are linearly independent. If the initial conditions are linearly independent, then the solutions must also be linearly independent. Since the two solutions are linearly independent, we proceed to part (c) of the exercise.

(c) We must find constants k_1 and k_2 such that

$$k_1\mathbf{Y}_1(0) + k_2\mathbf{Y}_2(0) = k_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

In other words, the constants k_1 and k_2 must satisfy the simultaneous system of linear equations

$$\begin{cases} k_1 + k_2 = 2 \\ k_1 + 2k_2 = 3. \end{cases}$$

It follows that $k_1 = 1$ and $k_2 = 1$. Hence, the required solution is

$$\mathbf{Y}_1(t) + \mathbf{Y}_2(t) = \begin{pmatrix} e^{-3t} + e^{-4t} \\ e^{-3t} + 2e^{-4t} \end{pmatrix}.$$

31. **(a)** If $(x_1, y_1) = (0, 0)$, then (x_1, y_1) and (x_2, y_2) are on the same line through the origin because (x_1, y_1) is the origin. So (x_1, y_1) and (x_2, y_2) are linearly dependent.

(b) If $(x_1, y_1) = \lambda(x_2, y_2)$ for some λ, then (x_1, y_1) and (x_2, y_2) are on the same line through the origin. To see why, suppose that $x_2 \neq 0$ and $\lambda \neq 0$. (The $\lambda = 0$ case was handled in part (a) above.) In this case, $x_1 \neq 0$ as well. Then the slope of the line through the origin and (x_1, y_1) is y_1/x_1, and the slope of the line through the origin and (x_2, y_2) is y_2/x_2. However, because $(x_1, y_1) = \lambda(x_2, y_2)$, we have

$$\frac{y_1}{x_1} = \frac{\lambda y_2}{\lambda x_2} = \frac{y_2}{x_2}.$$

Since these two lines have the same slope and both contain the origin, they are the same line. (The special case where $x_2 = 0$ reduces to considering vertical lines through the origin.)

(c) If $x_1y_2 - x_2y_1 = 0$, then $x_1y_2 = x_2y_1$. Once again, this condition implies that (x_1, y_1) and (x_2, y_2) are on the same line through the origin. For example, suppose that $x_1 \neq 0$, then

$$y_2 = \frac{x_2y_1}{x_1} = \frac{x_2}{x_1}y_1.$$

But we already know that

$$x_2 = \frac{x_2}{x_1}x_1,$$

so we have

$$(x_2, y_2) = \frac{x_2}{x_1}(x_1, y_1).$$

By part (b) above (where $\lambda = x_2/x_1$), the two vectors are linearly dependent.

If $x_1 = 0$ but $y_1 \neq 0$, it follows that $x_2y_1 = 0$, and thus $x_2 = 0$. Thus, both (x_1, y_1) and (x_2, y_2) are on the vertical line through the origin.

Finally, if $x_1 = 0$ and $y_1 = 0$, we can use part (a) to show that the two vectors are linearly dependent.

32. If $x_1y_2 - x_2y_1$ is nonzero, then $x_1y_2 \neq x_2y_1$. If both $x_1 \neq 0$ and $x_2 \neq 0$, we can divide both sides by x_1x_2, and we obtain

$$\frac{y_2}{x_2} \neq \frac{y_1}{x_1},$$

and therefore, the slope of the line through the origin and (x_2, y_2) is not the same as the slope of the line through the origin and (x_1, y_1).

If $x_1 = 0$, then $x_2 \neq 0$. In this case, the line through the origin and (x_1, y_1) is vertical, and the line through the origin and (x_2, y_2) is not vertical.

35. **(a)** Using the Product Rule we compute

$$\frac{dW}{dt} = \frac{dx_1}{dt}y_2 + x_1\frac{dy_2}{dt} - \frac{dx_2}{dt}y_1 - x_2\frac{dy_1}{dt}.$$

(b) Since $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are solutions, we know that

$$\begin{aligned}\frac{dx_1}{dt} &= ax_1 + by_1\\ \frac{dy_1}{dt} &= cx_1 + dy_1\end{aligned}$$

and that

$$\begin{aligned}\frac{dx_2}{dt} &= ax_2 + by_2\\ \frac{dy_2}{dt} &= cx_2 + dy_2.\end{aligned}$$

Substituting these equations into the expression for dW/dt, we obtain

$$\frac{dW}{dt} = (ax_1 + by_1)y_2 + x_1(cx_2 + dy_2) - (ax_2 + by_2)y_1 - x_2(cx_1 + dy_1).$$

After we collect terms, we have

$$\frac{dW}{dt} = (a + d)W.$$

(c) This equation is a homogeneous, linear, first-order equation (as such it is also separable—see Sections 1.1, 1.2, and 1.8). Therefore, we know that the general solution is

$$W(t) = Ce^{(a+d)t}$$

where C is any constant (but note that $C = W(0)$).

(d) From Exercises 31 and 32, we know that $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are linearly independent if and only if $W(t) \neq 0$. But, $W(t) = Ce^{(a+d)t}$, so $W(t) = 0$ if and only if $C = W(0) = 0$. Hence, $W(t) = 0$ is zero for some t if and only if $C = W(0) = 0$.

Chapter 3.2

2. **(a)** The characteristic polynomial is

$$(-4-\lambda)(-3-\lambda)-2=\lambda^2+7\lambda+10=0,$$

and therefore the eigenvalues are $\lambda_1=-2$ and $\lambda_2=-5$.

(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1=-2$, we solve the system of equations

$$\begin{cases} -4x_1-2y_1=-2x_1 \\ -x_1-3y_1=-2y_1 \end{cases}$$

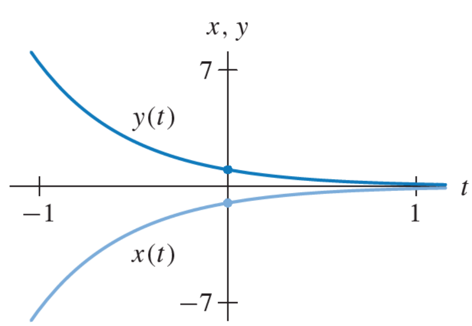
and obtain $y_1=-x_1$.

Using the same procedure, we obtain the eigenvectors (x_2, y_2) where $x_2=2y_2$ for $\lambda_2=-5$.

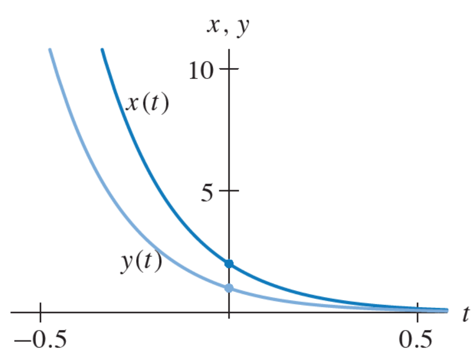
(d) One eigenvector $\mathbf{V}_1$ for λ_1 is $\mathbf{V}_1=(1,-1)$, and one eigenvector $\mathbf{V}_2$ for λ_2 is $\mathbf{V}_2=(2,1)$.

Given the eigenvalues and these eigenvectors, we have two linearly independent solutions

$$\mathbf{Y}_1(t)=e^{-2t}\begin{pmatrix}1\\-1\end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t)=e^{-5t}\begin{pmatrix}2\\1\end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$.



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$.

(e) The general solution to this linear system is

$$\mathbf{Y}(t)=k_1e^{-2t}\begin{pmatrix}1\\-1\end{pmatrix}+k_2e^{-5t}\begin{pmatrix}2\\1\end{pmatrix}.$$

6. **(a)** The characteristic polynomial is

$$(5-\lambda)(-\lambda) - 36 = 0,$$

and therefore the eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = 9$.

(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = -4$, we solve the system of equations

$$\begin{cases} 5x_1 + 4y_1 = -4x_1 \\ 9x_1 = -4y_1 \end{cases}$$

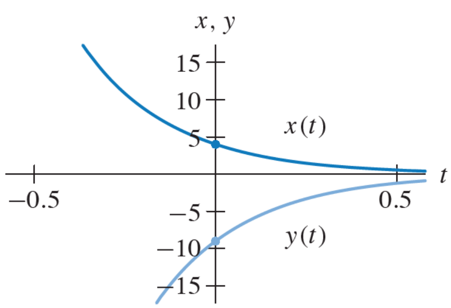
and obtain $9x_1 = -4y_1$.

Using the same procedure, we see that the eigenvectors (x_2, y_2) for $\lambda_2 = 9$ must satisfy the equation $y_2 = x_2$.

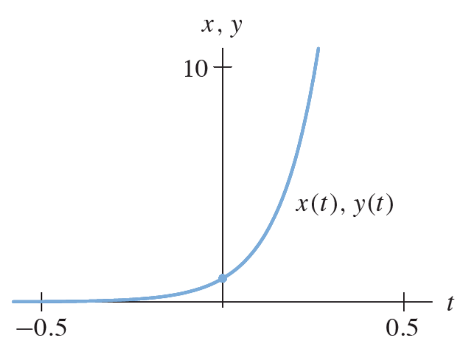
(d) One eigenvector $\mathbf{V}_1$ for λ_1 is $\mathbf{V}_1 = (4, -9)$, and one eigenvector $\mathbf{V}_2$ for λ_2 is $\mathbf{V}_2 = (1, 1)$.

Given the eigenvalues and these eigenvectors, we have the two linearly independent solutions

$$\mathbf{Y}_1(t) = e^{-4t}\begin{pmatrix} 4 \\ -9 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{9t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$.



The (identical) $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$.

(e) The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{-4t}\begin{pmatrix} 4 \\ -9 \end{pmatrix} + k_2 e^{9t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

8. **(a)** The characteristic polynomial is

$$(2-\lambda)(1-\lambda)-1=\lambda^2-3\lambda+1=0,$$

and therefore the eigenvalues are

$$\lambda_1=\frac{3+\sqrt{5}}{2}\quad\text{and}\quad\lambda_2=\frac{3-\sqrt{5}}{2}.$$

(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1=(3+\sqrt{5})/2$, we solve the system of equations

$$\begin{cases} 2x_1-y_1=\dfrac{3+\sqrt{5}}{2}x_1 \\[2ex] -x_1+y_1=\dfrac{3+\sqrt{5}}{2}y_1 \end{cases}$$

and obtain

$$y_1=\frac{1-\sqrt{5}}{2}x_1,$$

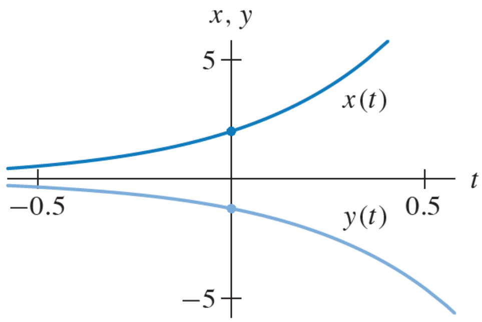
which is equivalent to the equation $2y_1=(1-\sqrt{5})x_1$.

Using the same procedure, we obtain the eigenvectors (x_2, y_2) where $2y_2=(1+\sqrt{5})x_2$ for $\lambda_2=(3-\sqrt{5})/2$.

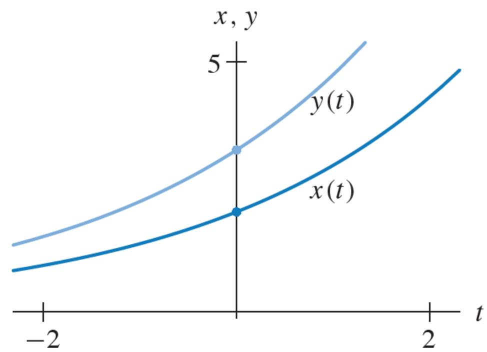
(d) One eigenvector $\mathbf{V}_1$ for the eigenvalue λ_1 is $\mathbf{V}_1=(2, 1-\sqrt{5})$, and one eigenvector $\mathbf{V}_2$ for the eigenvalue λ_2 is $\mathbf{V}_2=(2, 1+\sqrt{5})$.

Given the eigenvalues and these eigenvectors, we have two linearly independent solutions

$$\mathbf{Y}_1(t)=e^{(3+\sqrt{5})t/2}\begin{pmatrix} 2 \\ 1-\sqrt{5} \end{pmatrix}\quad\text{and}\quad\mathbf{Y}_2(t)=e^{(3-\sqrt{5})t/2}\begin{pmatrix} 2 \\ 1+\sqrt{5} \end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$.



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$.

(e) The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{(3+\sqrt{5})t/2} \begin{pmatrix} 2 \\ 1-\sqrt{5} \end{pmatrix} + k_2 e^{(3-\sqrt{5})t/2} \begin{pmatrix} 2 \\ 1+\sqrt{5} \end{pmatrix}.$$