Math 2208: Ordinary Differential Equations

[Optional] Project 2: Recipe for an All-nighter

Spring 2020 **Subhadip Chowdhury** **Due: Apr 8**

Caffeine is the most widely used drug among most college students across America to help them stay awake at late night working on ODE assignments before the day they are due. About half of American adults take about 300 mg of caffeine per day, in form of coffee or soft drinks. Although it is well known for improving cognitive performance in the short run, it is not a substitute for sleep. In fact, irritability, dehydration, and anxiety sets in if an excess amount of caffeine is sent to the brain. Mood swings may also occur if the level of caffeine fluctuates too much.

Experiments and studies show that the optimal level of caffeine in blood is somewhere between $200-300$ mg. One math students decides to pull an all-nighter to complete her assignment by staying awake continuously overnight with the help of Starbucks coffee and Caffeine Pills from the vending machine. She checks beforehand that

- One large-mug of drip-brewed coffee (a venti cappuchino in Strabucks) contains about 300 mg caffeine. The student needs about 30 minutes to finish a cup of coffee.
- One generic store brand caffeine pill has about 100 mg caffeine. Each pill digests almost instantaneously.

She also has access to a caffeine intravenous drip that can deliver caffeine into her body at a specified rate. Before choosing the optimal strategy, she wishes to model the amount of caffeine in her blood over time with an ODE so that she can figure out how to maintain an optimal level of caffeine in blood over a long period.

She makes a reasonable assumption that when one is not drinking more coffee, the caffeine level $c(t)$ in the blood is governed by the equation

$$c' = -\lambda c,$$

where λ is a positive rate constant. This means that $c(t)$ decays exponentially. The decay rate is often expressed in terms of the half-life $t_{1/2}$, the time it takes to reduce the initial amount of caffeine by half. Experiments show that $t_{1/2}$ is approximately 7 hours in adults (although the rate can vary a great deal from person to person).

Question 1.

Find the value of λ when $t_{1/2} = 7$. You can assume $\ln(2) \approx 0.7$.

We are going to look at several different intake methods next and try to decide which would work best for a longer-lasting effect.

§A. Constant drip

Suppose the student decides to hook up to an intravenous drip supplying caffeine at a constant rate of 50 mg/hr. We will call the intake rate function $I_1(t)$. Thus $I_1(t) = 50$ for all t, where t is measured in hours and I_1 is measured in mg/hr.

■ **Question 1.**

Set up an IVP that models the caffeine level in her body. Can you give a general formula for caffeine level in blood over time?

■ **Question 2.**

Use `dfield` to plot the slope field. What is the long term caffeine level in the body if the process continues for several days?

■ **Question 3.**

Around what time does the student start to feel the effect of the caffeine (i.e. 200 mg of caffeine in blood)?

§B. **Periodic Drip**

After the calculation of the last problem, the student thinks it might not be a good idea to have a constant intake of caffeine into her body. So she decides to regulate the intake to be a periodic function of time. The intake function is now given by

$$I_2(t) = 50(1 + \sin(2\alpha\pi t))$$

where α determines the period of the drip ($\alpha = 1$ corresponds to a period of 1 hr.).

■ **Question 4.**

Set up an IVP that models the caffeine level in her body.

■ **Question 5.**

Assume $\alpha = 1$. At what value of t is the intake rate maximum and minimum?

■ **Question 6.**

Use `dfield` to plot the caffeine level in blood over time with α as a parameter. Around what time does the student start to feel the effect of the caffeine (i.e. 200 mg of caffeine in blood)? Is this a significant change than last method?

■ **Question 7.**

Try different values of α between 0 and 1 and explain why it might actually be a better idea to in fact use a higher frequency (i.e. lower α) or constant drip ($\alpha = 0$).

■ **Question 8.**

What can you say about the long term caffeine level in the body if the process continues for several days?

§C. **Discontinuous Intake Function**

Most sane students will either decide against a caffeine IV drip or don't have access to such medical equipment. There is also an issue with the earlier model that the optimal level of caffeine isn't reached quickly enough. The student doesn't want to increase the drip amount either in fear of caffeine poisoning. So she wants to look into the possibility of having a couple of small intakes (in form of a cup of coffee or a caffeine pill) in regular time-intervals to keep going at a steady level for longer period of time.

The challenge we face this time when modeling the caffeine level using an IVP is that the intake rate is a discontinuous step function. For example, if the student drinks a large mug of coffee, the intake rate is 600 mg/hr for $1/2$ hr and 0 otherwise. So we will need to use numerical methods to get an idea of how the graph of $c(t)$ looks like.

Question 10.

The first option is to just have a coffee at the start and nothing else. So we can define the intake rate $I_3(t)$ as

$$I_3(t) = \begin{cases} 600, & 0 < t < 0.5 \\ 0, & t > 0.5 \end{cases}$$

The easiest way to write discontinuous intake function in `dfield` is as follows:

```
600*(t>0)*(t<0.5)
```

Computer Programs treat the construct '$(t > T)$' as a binary check. The computer reads it as 0 if the statement is False (i.e. if $t \leq T$) and 1 if the statement is True (i.e. if t is in fact bigger than T). So above code output 600 if both $t > 0$ and $t < 0.5$, and outputs 0 otherwise.

Use `dfield` to plot $c(t)$ vs. t and explain why this method is not sufficient to keep energy up overnight.

Question 11.

Since one cup of coffee isn't enough, the student decides to drink one cup of coffee at $T = 0$ and another cup at $T = 5$. This time

$$I_3(t) = 600*(t>0)*(t<0.5)+600*(t>5)*(t<5.5)$$

Use `dfield` to find $c(10)$ (Assuming over-night means about 10 hours).

Although this looks like a possible option to keep blood caffeine level higher than 200 mg overnight, the amount of caffeine ingested over time and the fluctuation in blood caffeine amount is huge.

In order to avoid the fluctuation, the student decides that to achieve the optimal caffeine level ($200-300$ mg) in the long run, she is going to have Caffeine pills (100 mg dosage) instead, and wants to figure out what should be the regular dosage interval.

In order to formalize our analysis, we are going to define a dynamical process called the Kick-Flow system.

§D. Kick-Flow System

Given a first order autonomous ODE of the form $y' = f(y)$, the flow map $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined as follows. Given a time t and an initial value s, define $\varphi(t,s) = y(t)$ where $y(t)$ is value of the solution curve to the ODE at time t that starts with initial condition $y(0) = s$. In other words, $\varphi(t,s)$ is the time-t map associated with the particular flow line passing through $y = s$ at time $t = 0$.

Question 12.

Explain why

$$\varphi(t,s) = s + \int_0^t f(y(t))dt$$

In addition to the differential equation, the system will be subjected to regular "kicks". A **kick** is an instantaneous positive change in position y without regard to the underlying differential equation. We define kick-flow systems to follow a **kick-then-flow** rule; first a kick occurs, then the systems flows undisturbed, according to the underlying differential equation for a given time interval. This process is repeated to produce a dynamical system from the original continuous ODE subjected to discrete discontinuous kicks.

Assume that at time $t = 0$, the system is immediately kicked by an amount $\mathbf{K}$, thus resulting in an instantaneous change in position which takes place over zero time. The system then flows from this position, following the dynamics of the differential equation for time $\mathbf{T}$ until the next kick is applied, in the form of another instantaneous change in position, and the process repeats.

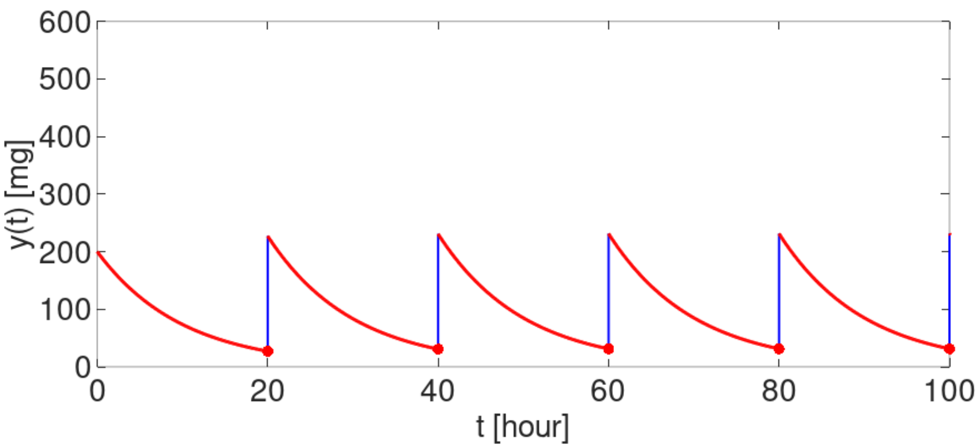

Figure 1: Kick-Flow System with $\mathbf{K} = 200$ and $\mathbf{T} = 20$

The **Kick-Flow Map** $\psi : \mathbb{R} \to \mathbb{R}$ is defined as

$$\psi(s) = \varphi(\mathbf{T}, s + \mathbf{K})$$

So if we start at y value of s at time $t = 0$, kick it by $\mathbf{K}$, then flow for time $\mathbf{T}$, then $\psi(s)$ is the end value of y. The red dots in figure (1) denote the values $\psi(0), \psi^2(0), \psi^3(0), \ldots$ etc.

■ Question 13.

Use the fact that the ODE is autonomous to show that for any $s \in \mathbb{R}$, we have

$$\psi^2(s) = \varphi(\mathbf{T}, \psi(s) + \mathbf{K})$$

and in general,

$$\psi^n(s) = \varphi\left(\mathbf{T}, \psi^{n-1}(s) + \mathbf{K}\right)$$

Here ψ^n means $\psi \circ \psi \circ \ldots \circ \psi$, a total of n times.

■ Question 14.

Recall that a **fixed point** of a function $g(x)$ is a value of x such that $g(x) = x$. Show that $\mathbf{A}$ is a fixed point of ψ if and only if

$$\varphi(\mathbf{T}, \mathbf{A} + \mathbf{K}) = \mathbf{A}.$$

§E. **Fixed Point of Caffeine Kick-Flow**

In case of Caffeine consumption, the underlying differential equation is $\boldsymbol{c' = -\lambda c}$, where $\boldsymbol{\lambda}$ is the value you calculated in question 1.

Question 15.

Show that the fixed point of Caffeine Kick-Flow system is given by

$$\mathbf{A} = \frac{\mathbf{K}}{e^{\lambda t} - \mathbf{1}}$$

Question 16.

Since the optimal caffeine level in blood is $\mathbf{200 - 300}$ mg and the cafeine pills are $\mathbf{100}$ mg each, the student wants the fixed point $\mathbf{A}$ to be equal to $\mathbf{200}$. Then use above formula to find out the dosage interval $\mathbf{T}$ that will give the best result in long run.

Question 17.

Use Octave/Matlab to plot the Kick-Flow system with the optimal $\mathbf{T}$ you found above and $\mathbf{K = 100}$. Attach the plot.