MATH 221 - Differential Equations

Lecture 26 Worksheet

Fall 2020 **Subhadip Chowdhury** **Oct 16**

Title: Trace-Determinant Plane

Summary: We'll summarize all the possible qualitative behavior one can get with a 2×2 linear system of ODEs into one big picture!

Related Reading: From The ODE Project - Section 3.7.

As you read through this worksheet, you should treat it as a study guide only. It is not a replacement for your textbook. Please make sure to follow along in the textbook (or other online resources from the related reading section above) side-by-side as you read through the topics. You are not expected to be able to answer the questions in here immediately after reading the synopsis from the worksheet. They are designed as more of an exploration directives; as you try to find the answers yourself, you will also learn the topic. You should use whatever resources are available to you, including the internet, to accomplish that task.

§A. Summarizing the possibilities

Recall that a system of linear ODEs with associated matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has the **characteristic polynomial**

$$p_{\mathbf{A}}(\lambda) = \mathbf{det}(\mathbf{A} - \lambda\mathbf{I}) = (a-\lambda)(d-\lambda) - bc = \lambda^2 - \lambda(a+d) + (ad-bc) = \lambda^2 - \mathbf{tr}(\mathbf{A})\lambda + \mathbf{det}(\mathbf{A})$$

with roots

$$\lambda = \frac{\mathbf{tr}(\mathbf{A}) \pm \sqrt{\mathbf{tr}(\mathbf{A})^2 - 4\,\mathbf{det}(\mathbf{A})}}{2}$$

We will use the notation $\mathbf{T} = \mathbf{tr}(\mathbf{A})$ and $\mathbf{D} = \mathbf{det}(\mathbf{A})$ for convenience of writing.

Theorem A.1

If a 2×2 matrix $\mathbf{A}$ has eigenvalues λ_1 and λ_2, then $\mathbf{tr}(\mathbf{A}) = \lambda_1 + \lambda_2$ and $\mathbf{det}(\mathbf{A}) = \lambda_1\lambda_2$.

■ Question 1. □

Fill out the following table. For the last three columns find whether the given quantity is positive or negative or zero.

Eigenvalues	Type of Equilibrium	$\mathbf{T}$	$\mathbf{D}$	$\mathbf{T}^2 - 4\mathbf{D}$
$\lambda_i \in \mathbb{R}, \lambda_1 < \lambda_2 < 0$				
$\lambda_i \in \mathbb{R}, \lambda_1 > 0 > \lambda_2$				
$\lambda_i \in \mathbb{R}, \lambda_1 > \lambda_2 > 0$				
$\lambda_i \in \mathbb{C}, \mathfrak{Re}(\lambda_i) > 0$				
$\lambda_i \in \mathbb{C}, \mathfrak{Re}(\lambda_i) = 0$				
$\lambda_i \in \mathbb{C}, \mathfrak{Re}(\lambda_i) < 0$				

§B. The Trace-Determinant Plane

Above table shows that the condition on what kind of eigenvalues we will have depends on the sign of $\mathbf{T}$, $\mathbf{D}$ and the discriminant $\mathbf{T}^2 - 4\mathbf{D}$. Consider the following picture where we have drawn the $\mathbf{T}$-axis horizontally and the $\mathbf{D}$-axis vertically. We have also drawn the curve $\mathbf{T}^2 - 4\mathbf{D} = 0$, a parabola.

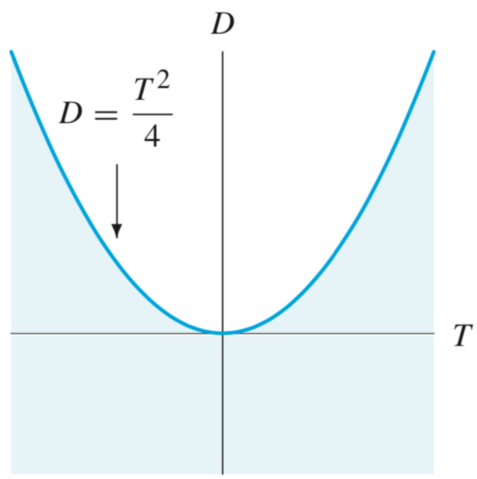

Figure 1: Shaded region corresponds to $\mathbf{T}^2 - 4\mathbf{D} > 0$

This is known as the **Trace-Determinant Plane**. As the matrix $\mathbf{A}$ changes, it has different values of $\mathbf{T}$ and $\mathbf{D}$ and the linear system $\frac{d\vec{\mathbf{R}}}{dt} = \mathbf{A}\vec{\mathbf{R}}$ corresponding to that matrix will be located at a different location in $(\mathbf{T}, \mathbf{D})$-plane.

■ **Question 2.** □

Find the regions in the picture of $(\mathbf{T},\mathbf{D})$-plane that correspond to each of the six cases above.

■ **Question 3.** □

What kind of phase portraits will exist in $(\mathbf{T},\mathbf{D})$-plane along the $\mathbf{D}$ axis?

§C. The degenerate and the defective cases

We are missing a couple of more cases in our summary above: for example, what happens along the $\mathbf{T}$-axis and what happens on the curve $\mathbf{T}^2 = 4\mathbf{D}$. Open the applet on the following webpage:

http://mathlets.org/mathlets/linear-phase-portraits-matrix-entry/

Turn on the `Eigenvalue` option. Move your cursor around on the $(\mathbf{T},\mathbf{D})$-plane and answer the questions below.

■ **Question 4.** **Zero Eigenvalue**

(a) Fix the $\mathbf{D}$ value to $\mathbf{0}$ and move the $\mathbf{T}$ slider. Check that matrices corresponding to points on the $\mathbf{T}$-axis have at least one zero eigenvalue. Can you prove this mathematically?

(b) When $\mathbf{T} > \mathbf{0}, \mathbf{D} = \mathbf{0}$, we call the equilibrium a **degenerate source**. Draw a representative phase portrait in your notebook. How is it different from a nodal source.

(c) The other possibility is $\mathbf{T} < \mathbf{0}, \mathbf{D} = \mathbf{0}$, called a **degenerate sink**. Can you see the difference between the two phase portraits? Draw them in your notebook to make sure you memorize them. Note that every solution curve is a straight line solution in this case.

■ **Question 5.** **Repeated Eigenvalue**

(a) *In terms of eigenvalues, $\lambda_1 = \lambda_2$ is the border line case between real distinct eigenvalues and complex conjugate eigenvalues.* Justify this last statement by inspecting the eigenvalues corresponding to points on the curve $\mathbf{T}^2 = 4\mathbf{D}$.

(b) What can you say about the discriminant when $\lambda_1 = \lambda_2$? Check that in this case, $\lambda_i = \frac{\mathbf{T}}{2} \in \mathbb{R}$.

(c) Again there are two cases: $\lambda_1 = \lambda_2 > \mathbf{0}$ and $\lambda_1 = \lambda_2 < \mathbf{0}$. The first case is called a **defective source** and the second one a **defective sink**. Which part of the parabola does each case correspond to? Make sure to draw them in your notebook to understand the difference.

(d) How many straight line solutions does the system have?

Digression

The words 'degenerate' and 'defective' are not interchangeable. They have specific meaning that has to do with Matrix algebra. A matrix is 'Defective' if it does not have n linearly independent eigenvectors. Whereas, a matrix is called 'Degenerate' if it is not invertible.