MATH 221 - Differential Equations

Lecture 23 Worksheet

Fall 2020 | **Subhadip Chowdhury** | **Oct 9**

Title: Phase Portraits of Linear Systems

Summary: We'll explore the various scenarios that occur with linear systems of ODEs that possess two complex eigenvalues.

Related Reading: Section 8.2.3 from the textbook, but we only care about 2×2 matrices.

From The ODE Project - Section 3.3, 3.4.

Solution to this worksheet will be posted on Oct 9.

As you read through this worksheet, you should treat it as a study guide only. It is not a replacement for your textbook. Please make sure to follow along in the textbook (or other online resources from the related reading section above) side-by-side as you read through the topics. You are not expected to be able to answer the questions in here immediately after reading the synopsis from the worksheet. They are designed as more of an exploration directives; as you try to find the answers yourself, you will also learn the topic. You should use whatever resources are available to you, including the internet, to accomplish that task.

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§A. Two Complex Eigenvalues

Recall that the general solution to the ODE

$$\frac{d\vec{\mathbf{R}}}{dt} = \mathbf{A}\vec{\mathbf{R}}$$

can be written as

$$\vec{\mathbf{R}}(t) = k_1 e^{\lambda_1 t}\vec{v}_1 + k_2 e^{\lambda_2 t}\vec{v}_2$$

where λ_is are the eigenvalues of $\mathbf{A}$ and $\vec{v}_i$ are the eigenvectors corresponding to λ_i. What happens if λ_1 and λ_2 are $\mathbb{C}$omplex numbers?

Question 1.

Consider the ODE $\dfrac{d\vec{\mathbf{R}}}{dt} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\vec{\mathbf{R}} = \mathbf{A}\vec{\mathbf{R}}$.

(a) Find the eigenvalues and corresponding eigenvectors of $\mathbf{A}$. You will find that they will be of the form $\alpha \pm i\beta$ for some real numbers α and β.

(b) Let $\lambda = \alpha + i\beta$ and name the corresponding eigenvector $\vec{v}$. Suppose $\vec{v} = \vec{v}_1 + i\vec{v}_2$, where $\vec{v}_i$ have real entries. Then

$$\vec{\mathbf{R}}_0(t) = e^{\lambda t}\vec{v} = e^{(\alpha+i\beta)t}\left(\vec{v}_1 + i\vec{v}_2\right)$$

is a complex-valued solution to our ODE. Unfortunately, we cannot draw a complex-valued solution![*]

[*] That will require 4 dimensions to draw. Why?

Use Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, to simplify, and rewrite your solution in the form

$$\vec{\mathbf{R}}_0(t) = \vec{\mathbf{R}}_{\mathfrak{Re}}(t) + i\vec{\mathbf{R}}_{\mathfrak{Im}}(t)$$

where $\vec{\mathbf{R}}_{\mathfrak{Re}}(t)$ and $\vec{\mathbf{R}}_{\mathfrak{Im}}(t)$ are real-valued functions of t.

(c) Check that $\vec{\mathbf{R}}_{\mathfrak{Re}}(t)$ and $\vec{\mathbf{R}}_{\mathfrak{Im}}(t)$ are real-valued solutions to the ODE.

(d) Is the last claim true in general? Can you show that $\vec{\mathbf{R}}_{\mathfrak{Re}}(t)$ and $\vec{\mathbf{R}}_{\mathfrak{Im}}(t)$ are real-valued solutions to the system $\dfrac{d\vec{\mathbf{R}}}{dt} = \mathbf{A}\vec{\mathbf{R}}$ for any matrix $\mathbf{A}$ with complex conjugate eigenvalues?

End of Question 1

Our goal is to express the general solution to the ODE in terms of **real-valued solutions**. Recall from last lecture that the expression

$$\vec{\mathbf{R}}(t) = k_1\vec{\mathbf{R}}_{\mathfrak{Re}}(t) + k_2\vec{\mathbf{R}}_{\mathfrak{Im}}(t)$$

will represent a general real-valued solution as long as the two vectors $\vec{\mathbf{R}}_{\mathfrak{Re}}(0)$ and $\vec{\mathbf{R}}_{\mathfrak{Im}}(0)$ are linearly independent i.e. not multiples of each other. So let's prove the linear independence of $\vec{\mathbf{R}}_{\mathfrak{Re}}(0)$ and $\vec{\mathbf{R}}_{\mathfrak{Im}}(0)$.

Answer the following two questions for a general matrix $\mathbf{A}$.

■ Question 2. □

Suppose a matrix $\mathbf{A}$ with real entries has the complex eigenvalue $\lambda = \alpha + i\beta, \beta \neq 0$. Let $\vec{v}$ be an eigenvector for λ and write $\vec{v} = \vec{v}_1 + i\vec{v}_2$, where $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ have real entries. Show that $\vec{v}_1$ and $\vec{v}_2$ can not be scalar multiples of each other.

[Hint: If $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = k\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ for some real number k, then $\vec{v} = (1+ik)\vec{v}_1$. Then use the fact that $\vec{v}$ is an eigenvector of $\mathbf{A}$ and that $\mathbf{A}\vec{v}_1$ contains no imaginary part. That should give you a contradiction.]

End of Question 2

■ Question 3. □

Write down the general form of the solution in terms of $t, \alpha, \beta, \vec{v}_1$ and $\vec{v}_2$.

§B. Classification of Solutions in case of Complex Eigenvalues

We are going to discuss the nature of the solution curves in the following three cases:

a) Case 1: $\alpha < 0$ **(Spiral Sink)**. b) Case 2: $\alpha > 0$ **(Spiral Source)**. c) Case 3: $\alpha = 0$ **(Center)**.

■ Question 4. □

In each case,

(i) Recall that the eigenvalues are given by the formula

$$\lambda = \alpha \pm i\beta = \frac{\mathrm{tr}(\mathrm{A}) \pm \sqrt{\mathrm{tr}(\mathrm{A})^2 - 4\det(\mathrm{A})}}{2}$$

Come up with your own examples of 2×2 matrices $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ that have complex eigenvalues (so we need the discriminant to be negative) for each of the three cases named above. Write down the corresponding system of ODEs.

If you are stuck, take a look at examples 3.4.(2,4,6) in the ODE Projects book. If you come up with your own example, use this link to confirm: `https://matrixcalc.org/en/vectors.html`.

(ii) Find the general solution for $\vec{\mathbf{R}}(t)$. Also write down the formula for $x(t)$ and $y(t)$ separately.

(iii) Use PPLANE to sketch the phase portrait for each case. Also use PPLANE to draw the $x(t)$ vs t and $y(t)$ vs. t graphs for some initial condition and check that they are consistent with your answers above. Do you see a justification for the names of the equilibria?

(iv) The solution curve is not periodic in general if $\alpha \neq 0$. However, they do go around the origin in regular time interval. The amount of time taken to cycle once around the origin is called the **natural period** of the system.

The **natural frequency** is the number of cycles that solutions make in one unit of time. Since each cycle takes one period to complete, the product of the natural frequency and the natural period of the system is $\mathbf{1}$, and consequently, frequency is the reciprocal of period.

Find the natural period and natural frequency of your system.

(v) Determine the direction of the spiral in the phase plane (do the solutions go clockwise or counter-clockwise around the origin?). How would you find this without drawing a phase portrait?

(vi) Are there any straight line solutions?

■ Question 5. ☐

Can you justify the following statement:

For the solution curves, the change in magnitude over time depends on the sign of α, whereas β determines the periodic nature of the solutions.