MATH 221 - Differential Equations

Lecture 19 Worksheet

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Title: Linear System of ODEs

Summary: We will begin to analyze systems of ODEs which have the form $\frac{d\vec{\mathbf{R}}}{dt} = \mathbf{A}\vec{\mathbf{R}}$.

Related Reading: Section 8.1 from the textbook, but we only care about 2×2 matrices.

From The ODE Project - Section 3.1.

As you read through this worksheet, you should treat it as a study guide only. It is not a replacement for your textbook. Please make sure to follow along in the textbook (or other online resources from the related reading section above) side-by-side as you read through the topics. You are not expected to be able to answer the questions in here immediately after reading the synopsis from the worksheet. They are designed as more of an exploration directives; as you try to find the answers yourself, you will also learn the topic. You should use whatever resources are available to you, including the internet, to accomplish that task.

§A. Linear Algebra Basics II - Matrices and Linear Systems

For this and couple of next lectures, we will be primarily dealing with 2×2 matrices. So I will try to define and explain the theory of Matrix Algebra in that context only. As I mentioned in class, we will try to take the shortest route to the results we need to understand systems of differential equations, without going into too much Linear Algebra.

Matrix Algebra

Addition. Given two matrices **A** and **B** of the same size we can add rather easily. Just add component by component.

Example A.1

For example, if

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ and } \quad \mathbf{B} = \begin{bmatrix} 3 & -1 \\ \pi & 7 \end{bmatrix}$$

then it is easy to compute that

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} 3 & 0 \\ 1+\pi & 7 \end{bmatrix}$$

If your matrices have different sizes, then matrix addition doesn't make any sense. It is 'undefined'.

Vector Addition. Vectors are column matrices. So there is nothing new here.

Scalar Multiplication. When considering vectors and matrices, regular numbers are often called scalars, just to give them a name to distinguish them from the other types of objects running around. Any matrix can be multiplied by any scalar. The idea is to multiply the scalar quantity to all of the entries of the matrix.

Example A.2

If $\mathbf{C} = \begin{bmatrix} 3 & -5 \\ -2 & 1 \end{bmatrix}$, then $7\mathbf{C} = \begin{bmatrix} 21 & -35 \\ -14 & 7 \end{bmatrix}$.

Matrix Subtraction. $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B}$.

Matrix Multiplication. This is the most important operation for us as it is the part most important to our future work with differential equations. We can multiply a pair of matrices only under constraints on their size. Consider an $m \times n$ matrix $\mathbf{A} = [a_{ij}]$ and a $n \times p$ matrix $\mathbf{B} = [b_{jk}]$. We define the matrix product of $\mathbf{A}$ and $\mathbf{B}$ to be the $m \times p$ matrix

$$\mathbf{C} = \mathbf{A} \cdot \mathbf{B} = [c_{ik}]$$

where $c_{ik} = \displaystyle\sum_{j=1}^{n} a_{ij} b_{jk}$. A way to understand this is that the (i,k)-th entry of $\mathbf{C}$ is the 'product' of the i-th row of $\mathbf{A}$ and k-th column of $\mathbf{B}$.

This looks a bit weird the first time, but it isn't so bad. We will demonstrate the process with two 2×2 matrices. I will do more examples in class. Look up more examples online if necessary.

Example A.3

If $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$, then

$$\mathbf{A} \cdot \mathbf{B} = \begin{bmatrix} 1 \times 5 + 2 \times 7 & 1 \times 6 + 2 \times 8 \\ 3 \times 5 + 4 \times 7 & 3 \times 6 + 4 \times 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

The Identity Matrix. The 2×2 identity matrix $\mathbf{I}_2$ is given by $\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$. It's called the identity because for any 2×2 matrix $\mathbf{A}$, we have

$$\mathbf{A} \cdot \mathbf{I} = \mathbf{A} = \mathbf{I} \cdot \mathbf{A}$$

Question 1. **Multiplication Practice**

Start with two random 2×2 matrices. See if you can multiply them. Check your answer here: https://matrixcalc.org/en/. Bookmark that link because we will use it quite often.

Linear System

The wonderful part of what we have set up is that now we have a very compact way of writing systems of linear equations. Any system of two equations in two variables,

$$\begin{aligned} ax + by &= \alpha \\ cx + dy &= \beta \end{aligned}$$

can be written as a matrix equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

Suppose $\alpha = \beta = 0$. Let's try to solve the system

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned}$$

If $a \neq 0$*, we can write $x = \frac{-by}{a}$ from the first equation. Substituting this value of x in the second equation gives

$$c\frac{-by}{a} + dy = 0 \implies -cby + ady = 0 \implies (ad - bc)y = 0$$

So if $ad - bc \neq 0$, we can get $y = 0$, which will imply $x = 0$.

On the other hand, if $ad - bc = 0$, the value of y can be anything. In that case, $ad = bc \implies \frac{a}{c} = \frac{b}{d}$, so the two equations are actually multiples of each other (so we actually have only one equation) and we say that the system has infinitely many solutions.

The Determinant. The determinant of a 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined as

$$\det \mathbf{A} = ad - bc.$$

§B. Linear Homogeneous Systems of ODEs with Constant Coefficients

Recall that the **dimension** of a system of ODEs is equal to the number of dependent variables in the system. A two-dimensional linear system of ODE has the normal form

$$\begin{aligned} \frac{dx}{dt} &= a(t)x + b(t)y + f_1(t) \\ \frac{dy}{dt} &= c(t)x + d(t)y + f_2(t) \end{aligned}$$

where a, b, c, d, f_1 and f_2 are functions of t. We say that the system is **homogeneous** if $f_1(t) = f_2(t) = 0$. We say it's **autonomous** if a, b, c, d, f_1 and f_2 are constants independent of t. For this section and the following couple of lectures, we will look at homogeneous autonomous linear systems only. So our systems will have $f_1 = f_2 = 0$ and constant coefficients. The normal form will look like

$$\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned}$$

We will denote the constant 2×2 coefficient matrix by $\mathbf{A}$. That is,

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

*At least one of a or c must be nonzero, otherwise we don't have two variables. If $a = 0$ and $c \neq 0$, you can check that you will arrive at similar calculation.

and use $\vec{\mathbf{R}}(t)$ to denote the column vector $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$. Then using matrix notations, we can write

$$\frac{d\vec{\mathbf{R}}}{dt} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}\vec{\mathbf{R}}$$

where $\mathbf{A}\vec{\mathbf{R}}$ is regular matrix multiplication. Recall that at an equilibrium point $\frac{d\vec{\mathbf{R}}}{dt} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. So using our discussion in the last section, we can get the following theorem. Try to prove it yourself.

Theorem B.1

If $\mathbf{A}$ is a matrix with nonzero determinant, then the only equilibrium point for the linear system of ODEs $\frac{d\vec{\mathbf{R}}}{dt} = \mathbf{A}\vec{\mathbf{R}}$ is the origin.

Question 2. — Verification of Solutions

Verify that the two curves

$$\vec{\mathbf{R}}_1(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix} \quad \text{and} \quad \vec{\mathbf{R}}_2(t) = \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{6t} = \begin{bmatrix} 3e^{6t} \\ 5e^{6t} \end{bmatrix}$$

are solutions of

$$\frac{d\vec{\mathbf{R}}}{dt} = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \vec{\mathbf{R}}.$$

This question is worked out as an example in the textbook. So, try it yourself first, but do look it up if you get stuck.

What do you think these two curves look like in xy phase plane?

§C. The Superposition Principle

The following two properties are easy to check using the matrix notation. Check them yourself.

- Given a solution $\vec{\mathbf{R}}(t)$ of the system, $k\vec{\mathbf{R}}(t)$ is also a solution for any real number k.
- If $\vec{\mathbf{R}}_1(t)$ and $\vec{\mathbf{R}}_2(t)$ are solutions of the system, then $\vec{\mathbf{R}}_1(t) + \vec{\mathbf{R}}_2(t)$ is also a solution.

Question 3. — Using the Superposition Principle

Consider the linear system $\frac{d\vec{\mathbf{R}}}{dt} = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} \vec{\mathbf{R}}$.

(a) Show that it has $\vec{\mathbf{R}}_1(t) = \begin{bmatrix} 2e^{-3t} \\ e^{-3t} \end{bmatrix}$ and $\vec{\mathbf{R}}_2(t) = \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix}$ as solutions.

(b) Is $\vec{\mathbf{R}}_3(t) = 5\vec{\mathbf{R}}_1(t) - 3\vec{\mathbf{R}}_2(t)$ also a solution? how about $\vec{\mathbf{R}}_4(t) = 4\vec{\mathbf{R}}_1(t) + 7\vec{\mathbf{R}}_2(t)$? How would you use the Superposition Principle here?

(c) Is there a solution to the system that solves the following initial value problem?

$$\frac{d\vec{\mathbf{R}}}{dt} = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix}\vec{\mathbf{R}}, \qquad \vec{\mathbf{R}}(0) = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

How would you show that the solution you found is the only solution to the IVP? What theorem would you use?

Answer to the last question will lead us to thinking about what the general solution looks like, which we will discuss in next class.