Math 2208: Ordinary Differential Equations

Assignment 7

Spring 2020 | **Subhadip Chowdhury** | **Due: Apr 1**

Reading

Section 3.3 and 3.4 from the textbook .

Homework Problems

Question 1 (3 × 3 points).

Draw the phase portrait for 3.2.(2,6,8) by hand (you found the eigenvalues and eigenvectors for these problems in assignment 6).

Question 2 (3 × 3 points).

For each of three systems in question 1, find the solution that satisfies the initial value $\vec{r}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Solution.

1. **Problem 3.2.2.** We are solving

$$k_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The answer is $k_1 = -1$, $k_2 = 1$.

2. **Problem 3.2.6.** We are solving

$$k_1 \begin{bmatrix} 4 \\ -9 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The answer is $k_1 = -\frac{1}{13}$, $k_2 = \frac{17}{13}$.

3. **Problem 3.2.8.** We are solving

$$k_1 \begin{bmatrix} 2 \\ 1-\sqrt{5} \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ 1+\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The answer is $k_1 = \frac{5-3\sqrt{5}}{20}$, $k_2 = \frac{5+3\sqrt{5}}{20}$.

■

Question 3 (2+4+4 points).

Book problems 3.4.(2, 4, 6). For part (e) of problem 4 and 6, use `pplane`.

Assignment 7 Solutions

Chapter 3.2

2. **(a)** The characteristic polynomial is

$$(-4-\lambda)(-3-\lambda)-2=\lambda^2+7\lambda+10=0,$$

and therefore the eigenvalues are $\lambda_1=-2$ and $\lambda_2=-5$.

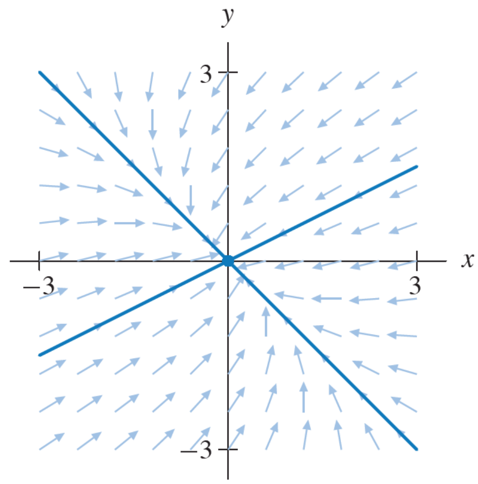
(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1=-2$, we solve the system of equations

$$\begin{cases} -4x_1-2y_1=-2x_1 \\ -x_1-3y_1=-2y_1 \end{cases}$$

and obtain $y_1=-x_1$.

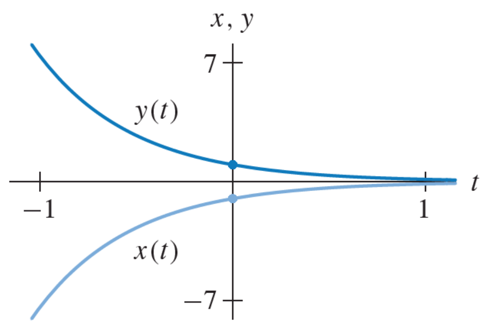
Using the same procedure, we obtain the eigenvectors (x_2, y_2) where $x_2=2y_2$ for $\lambda_2=-5$.

(c)

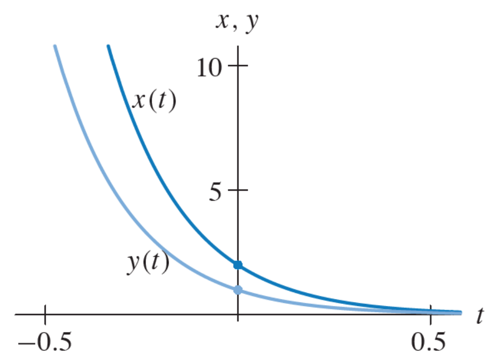


(d) One eigenvector $\mathbf{V}_1$ for λ_1 is $\mathbf{V}_1 = (1, -1)$, and one eigenvector $\mathbf{V}_2$ for λ_2 is $\mathbf{V}_2 = (2, 1)$. Given the eigenvalues and these eigenvectors, we have two linearly independent solutions

$$\mathbf{Y}_1(t) = e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{-5t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$.



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$.

(e) The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{-5t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

6. **(a)** The characteristic polynomial is

$$(5-\lambda)(-\lambda)-36=0,$$

and therefore the eigenvalues are $\lambda_1=-4$ and $\lambda_2=9$.

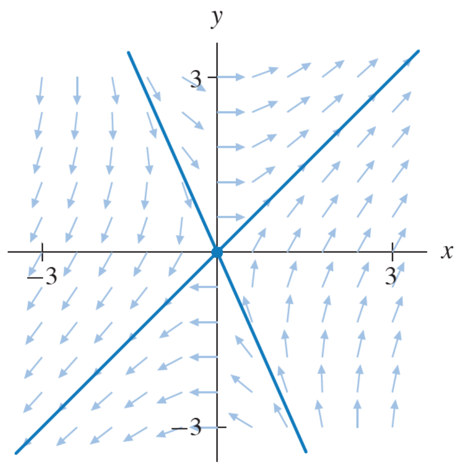
(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1=-4$, we solve the system of equations

$$\begin{cases} 5x_1+4y_1=-4x_1 \\ 9x_1=-4y_1 \end{cases}$$

and obtain $9x_1=-4y_1$.

Using the same procedure, we see that the eigenvectors (x_2, y_2) for $\lambda_2=9$ must satisfy the equation $y_2=x_2$.

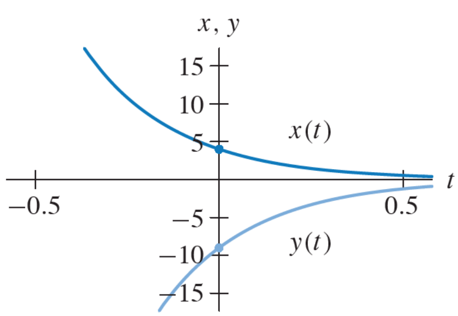
(c)



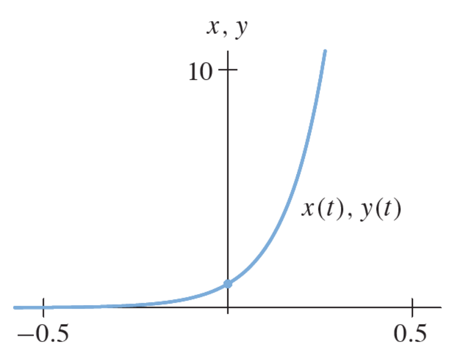
(d) One eigenvector $\mathbf{V}_1$ for λ_1 is $\mathbf{V}_1=(4,-9)$, and one eigenvector $\mathbf{V}_2$ for λ_2 is $\mathbf{V}_2=(1,1)$.

Given the eigenvalues and these eigenvectors, we have the two linearly independent solutions

$$\mathbf{Y}_1(t)=e^{-4t}\begin{pmatrix}4\\-9\end{pmatrix} \quad\text{and}\quad \mathbf{Y}_2(t)=e^{9t}\begin{pmatrix}1\\1\end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$.



The (identical) $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$.

(e) The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{-4t} \begin{pmatrix} 4 \\ -9 \end{pmatrix} + k_2 e^{9t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

8. (a) The characteristic polynomial is

$$(2-\lambda)(1-\lambda) - 1 = \lambda^2 - 3\lambda + 1 = 0,$$

and therefore the eigenvalues are

$$\lambda_1 = \frac{3+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{3-\sqrt{5}}{2}.$$

(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = (3+\sqrt{5})/2$, we solve the system of equations

$$\begin{cases} 2x_1 - y_1 = \dfrac{3+\sqrt{5}}{2} x_1 \\[2ex] -x_1 + y_1 = \dfrac{3+\sqrt{5}}{2} y_1 \end{cases}$$

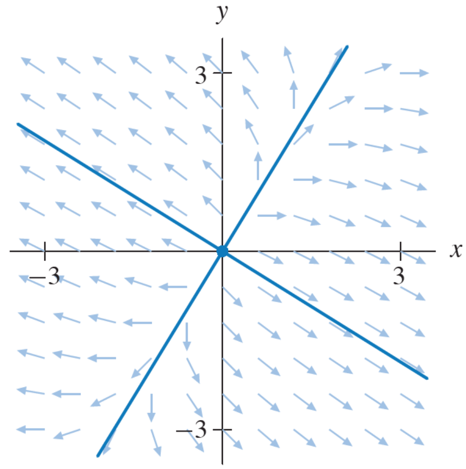
and obtain

$$y_1 = \frac{1-\sqrt{5}}{2} x_1,$$

which is equivalent to the equation $2y_1 = (1-\sqrt{5})x_1$.

Using the same procedure, we obtain the eigenvectors (x_2, y_2) where $2y_2 = (1+\sqrt{5})x_2$ for $\lambda_2 = (3-\sqrt{5})/2$.

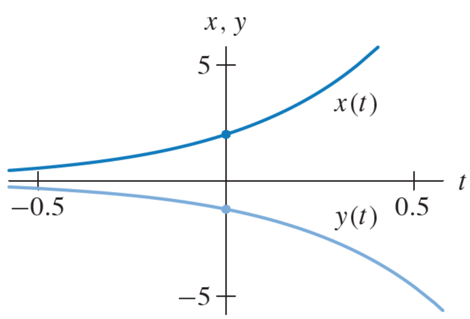
(c)



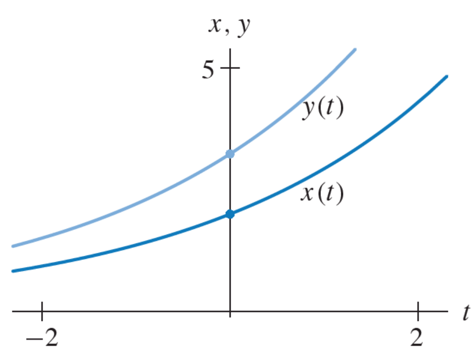
(d) One eigenvector $\mathbf{V}_1$ for the eigenvalue λ_1 is $\mathbf{V}_1 = (2, 1 - \sqrt{5}\,)$, and one eigenvector $\mathbf{V}_2$ for the eigenvalue λ_2 is $\mathbf{V}_2 = (2, 1 + \sqrt{5}\,)$.

Given the eigenvalues and these eigenvectors, we have two linearly independent solutions

$$\mathbf{Y}_1(t) = e^{(3+\sqrt{5}\,)t/2} \begin{pmatrix} 2 \\ 1 - \sqrt{5} \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{(3-\sqrt{5}\,)t/2} \begin{pmatrix} 2 \\ 1 + \sqrt{5} \end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$.



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$.

(e) The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{(3+\sqrt{5}\,)t/2} \begin{pmatrix} 2 \\ 1 - \sqrt{5} \end{pmatrix} + k_2 e^{(3-\sqrt{5}\,)t/2} \begin{pmatrix} 2 \\ 1 + \sqrt{5} \end{pmatrix}.$$

Chapter 3.4

2. The complex solution is

$$\mathbf{Y}_c(t) = e^{(-2+5i)t} \begin{pmatrix} 1 \\ 4-3i \end{pmatrix},$$

so we can use Euler's formula to write

$$\begin{aligned}
\mathbf{Y}_c(t) &= e^{(-2+5i)t} \begin{pmatrix} 1 \\ 4-3i \end{pmatrix} \\
&= e^{-2t}e^{5it} \begin{pmatrix} 1 \\ 4-3i \end{pmatrix} \\
&= e^{-2t}(\cos 5t + i \sin 5t) \begin{pmatrix} 1 \\ 4-3i \end{pmatrix} \\
&= e^{-2t} \begin{pmatrix} \cos 5t \\ 4\cos 5t + 3\sin 5t \end{pmatrix} + ie^{-2t} \begin{pmatrix} \sin 5t \\ 4\sin 5t - 3\cos 5t \end{pmatrix}.
\end{aligned}$$

Hence, we have

$$\mathbf{Y}_{\text{re}}(t) = e^{-2t} \begin{pmatrix} \cos 5t \\ 4\cos 5t + 3\sin 5t \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_{\text{im}}(t) = e^{-2t} \begin{pmatrix} \sin 5t \\ 4\sin 5t - 3\cos 5t \end{pmatrix}.$$

The general solution is

$$\mathbf{Y}(t) = k_1 e^{-2t} \begin{pmatrix} \cos 5t \\ 4\cos 5t + 3\sin 5t \end{pmatrix} + k_2 e^{-2t} \begin{pmatrix} \sin 5t \\ 4\sin 5t - 3\cos 5t \end{pmatrix}.$$

4. **(a)** The characteristic equation is

$$(2-\lambda)(6-\lambda) + 8 = \lambda^2 - 8\lambda + 20,$$

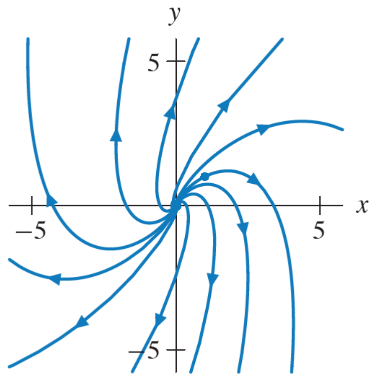
and the eigenvalues are $\lambda = 4 \pm 2i$.

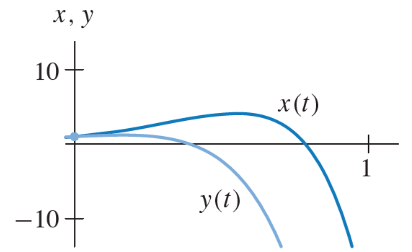
(b) Since the real part of the eigenvalues is positive, the origin is a spiral source.

(c) Since $\lambda = 4 \pm 2i$, the natural period is $2\pi/2 = \pi$, and the natural frequency is $1/\pi$.

(d) At the point $(1, 0)$, the tangent vector is $(2, -4)$. Therefore, the solution curves spiral around the origin in a clockwise fashion.

(e) Since $d\mathbf{Y}/dt = (4, 2)$ at $\mathbf{Y}_0 = (1, 1)$, both $x(t)$ and $y(t)$ increase initially. The distance between successive zeros is π, and the amplitudes of both $x(t)$ and $y(t)$ are increasing.





6. **(a)** The characteristic polynomial is

$$(-\lambda)(-1-\lambda)+4=\lambda^2+\lambda+4,$$

so the eigenvalues are $\lambda = (-1 \pm i\sqrt{15})/2$.

(b) The eigenvalues are complex and the real part is negative, so the origin is a spiral sink.

(c) The natural period is $2\pi/(\sqrt{15}/2) = 4\pi/\sqrt{15}$. The natural frequency is $\sqrt{15}/(4\pi)$.

(d) The vector field at $(1, 0)$ is $(0, -2)$. Hence, solution curves spiral about the origin in a clockwise fashion.

(e) From the phase plane, we see that both $x(t)$ and $y(t)$ are initially increasing. However, $y(t)$ quickly reaches a local maximum. Although both functions oscillate, each successive oscillation has a smaller amplitude.