MATH 221 - Differential Equations

Project 3: Relationship Dynamics!

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To arouse your interest in the classification of linear systems, we will discuss different possible dynamics of a two-person romantic relationship -- as a linear two dimensional system of equations[1]. The overall aim of this project is threefold:

- to learn how the differential equation of a modeling scenario can be modified to obtain the differential equation of a related scenario;
- to illustrate how to use the phase portraits of a system of differential equations to analyze short term and long term behaviors of the dynamics of a real-world scenario;
- to assess the pros and cons of using analytical and numerical techniques to visualize and explain different behavior.

We start with a synopsis of the *Romeo and Juliet* story as stated in [1]. Romeo is in love with Juliet, but in our version of this story, Juliet is a fickle lover. The more Romeo expresses his love for Juliet, the more Juliet wants to run away and hide. But when Romeo gets discouraged and backs off, Juliet begins to find him strangely attractive. Romeo, in comparison, desires Juliet when she shows interest in him and becomes disinterested when Juliet no longer shows him attention.

Let

$$\mathbf{R}(t) = \text{Romeo's love/hate for Juliet at time } t$$

$$\mathbf{J}(t) = \text{Juliet's love/hate for Romeo at time } t$$

Note that positive values for $\mathbf{R}$ and $\mathbf{J}$ signify love while negative values signify hate. Then a model for their star-crossed romance can be given as the following system of differential equations:

$$\frac{d\mathbf{R}}{dt} = a\mathbf{J}, \qquad \frac{d\mathbf{J}}{dt} = -b\mathbf{R} \tag{1}$$

where the parameters a and b are positive real numbers, to be consistent with the story. For the purpose of this project, you may use whatever tools you have at your disposal (excluding the internet), including our textbooks, Jupyter notebooks posted on Moodle, online apps linked from your Worksheets, and PPLANE, that best serves your purpose.

§A. Exploration of The Model

Question 1. (1+(2+1)+(1+2)+0) points

(a) Why do the right hand sides of the equations have different signs?

(b) With $a = 1$, $b = 1$, and initial conditions $\mathbf{R}(0) = 2$ and $\mathbf{J}(0) = 0$, plot the component graphs of $\mathbf{R}(t)$ and $\mathbf{J}(t)$ against t on the same set of axes. Explain why your graphs match the behavior of Romeo and Juliet's relationship as described in the statement of the problem.

(c) With $a = 1$ and $b = 1$, sketch the phase portrait of system. Can you use your sketch to conclude that the behavior of Romeo and Juliet's relationship does not depend on how much love or hate Romeo initially has for Juliet?

(d) What name would you give to Romeo and Juliet's relationship?

■ Question 2. 4 points

With different positive values for a and b, repeat questions $1(b)$ and $1(c)$. How do your results differ from the results you obtained from question 2 and 3 above, if at all? What kind of equilibrium do you observe here?

End of Question 2

The sad outcome of this affair is, of course, a never-ending cycle of love and hate. At least they manage to achieve simultaneous love one-quarter of the time...

§B. Modifying the Model

Assume that the right hand sides are now both positive i.e

$$\frac{dR}{dt} = aJ, \qquad \frac{dJ}{dt} = bR \tag{2}$$

where a and b are both positive real numbers.

■ Question 3. (2+2+4+2+0) points

(a) Describe, in your own words, the behavior of Romeo and Juliet's relationship captured by this system of equations.

(b) With $a = 1$ and $b = 1$, sketch the phase portrait of R and J and explain how your sketch explains the behavior of Romeo and Juliet's relationship you described in part (a).

(c) Use your phase portrait to come up with **three** scenarios with different initial feelings that result in **distinct** outcomes for the long-term relationship of Romeo and Juliet.

(d) What name would you give to this type of Romeo and Juliet's relationship?

§C. Extending the Model

Suppose Romeo's love for Juliet depends **entirely** on Juliet's feelings in the following manner: the more Juliet loves Romeo, the more Romeo loves Juliet; and the more Juliet hates Romeo, the more Romeo hates Juliet. On the other hand, suppose Juliet's love for Romeo depends on **both** his feelings towards her and her own feelings; the more Romeo loves Juliet, the less interested Juliet is.

■ Question 4. (2+4+2) points

(a) Write a system of equation that captures this scenario. Specifically, mention which coefficients are zero, positive, negative, or we don't have enough information about.

(b) How similar is your system to equation (1)? Note that we didn't specify whether one of the coefficient is positive or negative. Use this coefficient as a **parameter** in your system to regulate how similar (or

different) the phase portraits of your system are to those in (1). Demonstrate this with two phase portraits that show different behaviors for different values of this parameter.

(c) Assuming $\mathbf{R}(0) = 1, \mathbf{J}(0) = 0$, sketch $\mathbf{R}(t)$ and $\mathbf{J}(t)$ as functions of t in each case.

§D. Generalizing the Model

Now consider the forecast for lovers governed by the general linear system

$$\frac{d\mathbf{R}}{dt} = a\mathbf{R} + b\mathbf{J}, \qquad \frac{d\mathbf{J}}{dt} = c\mathbf{R} + d\mathbf{J} \tag{3}$$

where the parameters a, b, c, d may have either sign. A choice of signs specifies the romantic styles.

For example, As named by one of Strogatz' students, the choice $a > 0, b > 0$ means that Romeo is an ''eager beaver'' - he gets excited by Juliet's love for him, and is further spurred on by his own affectionate feelings for her. Similarly we will say $a < 0, b > 0$ means Romeo is a ''Cautious Lover'', he tries to avoid throwing himself at Juliet, but gets excited by the Juliet's advances.

It's entertaining to name the other two romantic styles, and to predict the outcomes for the various pairings.

■ Question 5. — 6 points

What happens when two identically cautious lovers get together? The system is

$$\frac{d\mathbf{R}}{dt} = a\mathbf{R} + b\mathbf{J}, \frac{d\mathbf{J}}{dt} = b\mathbf{R} + a\mathbf{J}$$

with $a < 0, b > 0$.

(a) Show that if $a^2 > b^2$, the relationship always fizzles out to mutual indifference. The lesson seems to be that excessive caution can lead to apathy.

(b) Show that if $a^2 < b^2$, the relationship is explosive. This time, the lovers are more daring, or perhaps more sensitive to each other. Depending on their feelings initially, their relationship either becomes a love fest or a war. In either case, show that all trajectories approach the line $\mathbf{R} = \mathbf{J}$, so that their feelings are eventually mutual.

■ Question 6. — 4+1 points

What happens when an eager beaver gets together with a cautious lover? Consider the system

$$\frac{d\mathbf{R}}{dt} = a\mathbf{R} + b\mathbf{J}, \frac{d\mathbf{J}}{dt} = -b\mathbf{R} + a\mathbf{J}$$

with $a > 0, b > 0$.

(a) Show that their mutual feelings grow stronger over time but oscillates periodically between love and hate. What's the natural period?

(b) Does the type of equilibrium depend on the relative magnitudes of a and b?

■ Question 7. — Name Calling, 0 points

Suggest names for the other two romantic styles, determined by the signs of a and b in $\frac{d\mathbf{R}}{dt} = a\mathbf{R} + b\mathbf{J}$.

In each of the following problems, predict the course of the love affair, depending on the signs and relative sizes of a and b.

■ Question 8. Fire and water, 4 points

Do opposites attract? Analyze $\frac{d\mathbf{R}}{dt} = a\mathbf{R} + b\mathbf{J}, \frac{d\mathbf{J}}{dt} = -b\mathbf{R} - a\mathbf{J}$. Find the conditions relating a and b that gives distinct possibilities for the type of the equilibrium point. You do not need to analyze the long-term behavior.

■ Question 9. Peas in a pod, 8 points

If Romeo and Juliet are romantic clones: $\frac{d\mathbf{R}}{dt} = a\mathbf{R} + b\mathbf{J}, \frac{d\mathbf{J}}{dt} = b\mathbf{R} + a\mathbf{J},$ should they expect boredom or bliss? We have already looked at the case when $a < 0, b > 0$. This time, analyze all other possible behaviors the phase portrait might demonstrate. Specify the conditions on a and b that distinguishes each case.

In each case, try to answer whether their long-term feelings are mutual or opposite, and/or if the relationship fizzles out or explodes.

■ Question 10. Romeo the robot, 6 points

Nothing could ever change the way Romeo feels about Juliet: $\frac{d\mathbf{R}}{dt} = 0, \frac{d\mathbf{J}}{dt} = a\mathbf{R} + b\mathbf{J}.$ Does Juliet end up loving him or hating him? How does it depend on their initial dispositions towards each other? Check all cases depending on different signs for a and b.

§D. References

[1] Strogatz, Steven H. 1994. *Non-Linear Dynamics and Chaos*. New York: Perseus Books Group. 1994.