MATH 221 - Differential Equations

Lecture 6 Activities

Fall 2020 | **Subhadip Chowdhury** | **Aug 31**

§A. In-class Practice Problems

Question 1.

Consider the IVP

$$\frac{dy}{dx} = x(y-4)^2 - 2, \quad y(0) = y_0$$

where $y_0 < 4$. Can a solution $y(x) \to \infty$ as $x \to \infty$? Use the direction field to explain your claim.

Question 2.

Suppose that $y(x)$ is a nonconstant solution of the autonomous equation $dy/dx = f(y)$. Can $y(x)$ have a local maximum or minimum?

Question 3. (Mixing in a vat)

Consider a large vat containing sugar water that is to be made into soft drinks (see Figure 1). Suppose:

- The vat contains **100** gallons of liquid. Moreover, the amount flowing in is the same as the amount flowing out, so there are always **100** gallons in the vat.
- The vat is kept well mixed, so the sugar concentration is uniform throughout the vat. Clearly this will not be the case in real life, but if we allow the concentration to vary depending on the location in the vat the problem becomes very difficult and will involve partial differential equations, which is not the focus of this course. As a consequence, we will assume that at any given moment, the water that is leaving the vat has the same sugar concentration as in the vat.
- Sugar water containing **5** tablespoons of sugar per gallon enters the vat through pipe **A** at a rate of **2** gallons per minute. Sugar water containing **10** tablespoons of sugar per gallon enters the vat through pipe **B** at a rate of **1** gallon per minute.
- Sugar water leaves the vat through pipe **C** at a rate of **3** gallons per minute.

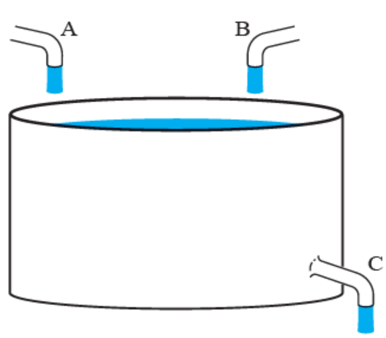

Figure 1: Mixing Vat

Let t be time measured in minutes (the independent variable). Our goal is to model the amount of sugar, $\mathbf{S}(t)$, in the vat at time t measured in tablespoons. We make the following observations.

- Rate of change of $\mathbf{S}(t)$ = Rate at which $\mathbf{S}(t)$ enters the vat – Rate at which $\mathbf{S}(t)$ exits the vat
- Rate at which $\mathbf{S}(t)$ enters/exits the vat = (flow rate of water entering/exiting) × (concentration of sugar in water entering/exiting)
- Concentration of sugar in the vat at time $t = \dfrac{\text{Amount of sugar in the vat at time } t}{\text{Volume of water in the vat at time } t}$

(a) Set up an ODE that, when solved, will give us an expression for $\mathbf{S}(t)$. Don't solve it yet.

(b) Use DFIELD to draw the direction field. Make a conjecture regarding the long term value of $\mathbf{S}(t)$. Does the value depend of the initial amount of sugar in the vat?

(c) Assume the initial amount of sugar in the vat at time $t = 0$ is 500 tablespoons. Solve the IVP using separation of variables and find a formula for $\mathbf{S}(t)$.

§B. Suggested Homework Problems

Question 4.

A function $y(x)$ is called periodic[1] with period T, if for every x in the domain of y the following holds:

$$y(x + \mathrm{T}) = y(x)$$

Let $f(x)$ be a function such that $f(y)$ and $\dfrac{df}{dy}$ are continuous for all y. Show that there is no periodic solution to the ODE $\dfrac{dy}{dx} = f(y)$.

[1] You are familiar with periodic functions. For example $\sin(x)$ or $\cos(3x)$ are periodic functions.