MATH 221 - Differential Equations

Lecture 11 Worksheet

Fall 2020 **Subhadip Chowdhury** **Sep 11**

Title: Bifurcation Theory

Summary: We will learn about a modern analytical technique which allows one to characterize how solutions of differential equations which contain parameters changes when the parameter values vary.

Most of this worksheet is copied from http://faculty.sfasu.edu/judsontw/ode/.

§A. Parameter Sensitivity

In quiz 1, we observed how the equilibrium of a population model is affected by a harvesting parameter. In this worksheet we will generalize the idea and explore how changing a paramter changes the qualitative behavior of solution curves.

Definition A.1: Bifurcation

When a small change in the value of a parameter leads to a drastic change in the qualitative nature of the phase line or long-term behavior of the solution of a differential equation, the phenomenon is called a **bifurcation** of the ODE. The value of the parameter at which such changes occur is known as a bifurcation value of the ODE.

Example A.1

Recall the constant harvesting model with harvesting parameter $\mathbf{H}$ from your quiz.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - H$$

Such a family of differential equations indexed by a parameter is called a **one-parameter family** of differential equations. For each value of $\mathbf{H}$, We obtain an autonomous differential equation, and for each value of $\mathbf{H}$, we have a different phase line to examine.

To get started with our model, we will first find the equilibrium solutions. If we will let

$$f_H(P) = kP\left(1 - \frac{P}{N}\right) - H$$

each equilibrium solution must satisfy $f_H(P) = 0$ or

$$-kP^2 + kNP - HN = 0$$

Therefore, our equilibrium solutions are given by

$$P = \frac{-kN \pm \sqrt{k^2N^2 - 4kHN}}{-2k} = \frac{N}{2} \pm \sqrt{\frac{N^2}{4} - \frac{HN}{k}}$$

The explanation of how our model behaves lies in the discriminant,

$$\frac{\mathbf{N}^2}{4} - \frac{\mathbf{HN}}{k}$$

If

$$\frac{\mathbf{N}^2}{4} - \frac{\mathbf{HN}}{k} < 0$$

or, equivalently if $\mathbf{H} > k\mathbf{N}/4$, there are no equilibrium solutions and

$$\frac{d\mathbf{P}}{dt} = f_{\mathbf{H}}(\mathbf{P}) < 0$$

for all values of $\mathbf{P}$. In particular, all solutions of $d\mathbf{P}/dt = f_{\mathbf{H}}(\mathbf{P})$ tend towards negative infinity as $t \to \infty$. In this case, the population is doomed to extinction no matter how large the initial population is. Since negative populations do not make sense, we say that the population is extinct when $\mathbf{P} = 0$. On the other hand, if $\mathbf{H} < k\mathbf{N}/4$, we have equilibrium solutions at

$$\mathbf{P}_1 = \frac{\mathbf{N}}{2} + \sqrt{\frac{\mathbf{N}^2}{4} - \frac{\mathbf{HN}}{k}}$$

and

$$\mathbf{P}_2 = \frac{\mathbf{N}}{2} - \sqrt{\frac{\mathbf{N}^2}{4} - \frac{\mathbf{HN}}{k}}$$

The first equilibrium solution, $\mathbf{P}_1$ is a sink, while the second, $\mathbf{P}_2$ is a source. Finally, if $\mathbf{H} = k\mathbf{N}/4$, then we will have exactly one equilibrium solution at $\mathbf{P} = \mathbf{N}/2$. Although $d\mathbf{P}/dt < 0$ for all $\mathbf{P} \neq \mathbf{N}/2$, we see that $\mathbf{P} \to \mathbf{N}/2$ as $t \to \infty$ for all initial values of $\mathbf{P}$ greater than $\mathbf{N}/2$. For initial values of $\mathbf{P}$ less than $\mathbf{N}/2$ solutions tend towards $-\infty$ as $t \to \infty$. Thus, the initial population of fish must be at least $k\mathbf{N}/4$; otherwise, the fish will go extinct.

Thus we observe that a small change in $\mathbf{H}$ can have a dramatic effect on how the solutions of the differential equation behave. Changing the value of $\mathbf{H}$ from $\frac{k\mathbf{N}}{4}$ to $\frac{k\mathbf{N}}{4} + 0.001$ will doom the population of fish to extinction no matter what the initial population is. As we increase the value of $\mathbf{H}$ from 0, the number of equilibrium solutions changes from two to one and then to none. This change occurs exactly at $\mathbf{H} = \frac{k\mathbf{N}}{4}$. We say that a bifurcation occurs at the bifurcation value $\mathbf{H}_0 = \frac{k\mathbf{N}}{4}$ or at the bifurcation point $(\mathbf{H}_0, \mathbf{P}_0) = \left(\frac{k\mathbf{N}}{4}, \frac{\mathbf{N}}{2}\right)$.

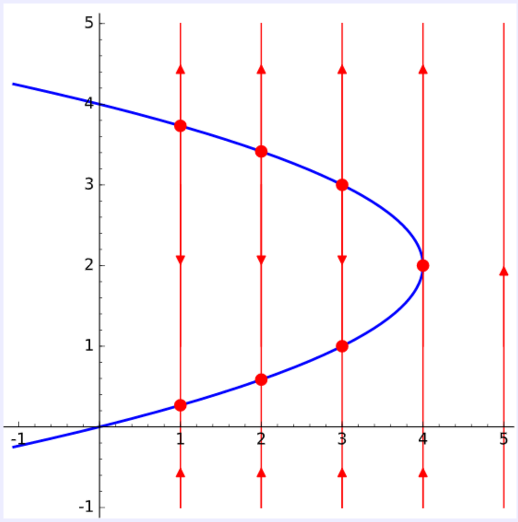
With the above example in mind, another possible way of defining the bifurcation is as follows. We say that the ODE $y' = f_h(y)$ has a bifurcation point (h_0, y_0) if the number of solutions of the equation $f_h(y) = 0$ for y in a neighborhood of (h_0, y_0) is **not** a constant independent of h.

§B. Analysis of Bifurcations

Example B.1

Consider the one parameter family of ODEs $\frac{dy}{dt} = y(y-4) + h$. Check that the number of equilibrium solutions changes at $h = 4$. For $h < 4$, we have two equilibrium solutions $y = 2 \pm \sqrt{4-h}$. For values of $h > 4$, there are no equilibrium solutions. We say that $h = 4$ is a bifurcation value for the differential equation.

We can record all of the information for the various values of h in a graph called the bifurcation diagram. The horizontal axis is h and the vertical axis is y. Over each value of h, we will plot the corresponding phase line. If we do this for all possible value of h, and join all the equilibrium dots together, we get the curve in the graph. The curve represents represents the various equilibrium solutions for the different values of h. It is called the bifurcation diagram for this ODE, and is in fact a parabola in this case.



Definition B.1: Bifurcation Diagram

A **Bifurcation diagram** is a picture of the phase lines near a bifurcation value. It appears as a curve in the plane with the equilibrium points of the autonomous dependent variable on the vertical axis, and the bifurcation parameter on the horizontal axis.

Question 1.

Draw the bifurcation diagram of the problem from the first take-home quiz with p on the vertical axis and h on the horizontal axis. Label which half of the curve corresponds to stable equilibrium (sink) and which half corresponds to unstable equilibrium (source).

To be more precise, we will refer to the curve as the **bifurcation curve**, whereas, the curve with its parts labeled as stable vs unstable will be referred to as the **bifurcation diagram**.

Equation of the Bifurcation Curve

We can make the interesting observation that, since the bifurcation diagram is a picture of equilibrium values vs the parameter value, it is essentially the implicit curve $f_h(y) = 0$ drawn on y vs h axes! Then the bifurcation points are the critical points of the curve where $\frac{d f_h(y)}{dy} = 0$. We can use this to algebraically find the bifurcation point(s).

Example B.2

For example, the bifurcation diagram for $y' = y(y-4)+h$ is the parabola

$$y(y-4)+h = 0 \iff h = y(4-y).$$

The bifurcation point is where

$$\frac{d(y(4-y))}{dy} = 0 \implies 4-2y = 0 \implies y = 2 \implies h = 2(4-2) = 4.$$

Theorem B.1

Given a one-parameter family of differential equations $y' = f_h(y)$, a point (h_0, y_0) is a bifurcation point iff both $f_h(y)\Big|_{(h_0,y_0)} = 0$ and $f_h'(y)\Big|_{(h_0,y_0)} = 0$.

In general, the two requirement together give us a set of two equations in two variables, that can be solved to get h_0 and y_0.

Question 2.

Find the bifurcation values of the one-parameter family

$$\frac{dy}{dt} = y(y-2)^2 + h$$

and draw the bifurcation diagram. Make sure to label the stable and unstable parts clearly.

[Hint: What's the implicit equation of the curve? Can you write it in $h = \varphi(y)$ form? Can you graph $\varphi(y)$ on y vs. h axes? Remember that h has to be horizontal and y has to be vertical.]

§C. Four Types of Bifurcations

Up to change of variables (by shifting and scaling) the most important bifurcations in one dimension can be described locally by one of four following ODEs.

$$y' = h - y^2 \text{ (saddle-node bifurcation)}$$

Draw the bifurcation curve. This is probably the most typical kind of bifurcation to arise. In it, a pair of equilibria, one stable and one unstable, coalesce at the bifurcation point, annihilate each other and disappear.

The harvesting examples from before are bifurcations of this kind. There are two saddle-node bifurcations in Question 2.

$$y' = hy - y^2 \text{ (transcritical bifurcation)}$$

Use `DFIELD` to draw some sample direction plane and phase lines for values of h near 0. Then draw the bifurcation curve ($hy - y^2 = 0$) and compare it with the equilibrium points in your phase lines. Can you recognise which is the stable branch and which one is the unstable branch in the curve?

Transcritical bifurcation arises in systems where there is some basic "trivial" solution branch, corresponding here to $y = 0$, that exists for all values of the parameter h. (This differs from the case of a saddle-node bifurcation, where the solution branches exist locally on only one side of the bifurcation point.). There is a second solution branch that crosses the first one at the bifurcation point. When the branches cross one solution goes from stable to unstable while the other goes from stable to unstable. This phenomenon is referred to as an "exchange of stability.".

$$y' = hy - y^3 \text{ (SUPERCRITICAL PITCHFORK)}$$

Look in Lecture 7 activity to find the bifurcation value! Draw the bifurcation curve. Can you identify the stable and the unstable branches?

The reason for the interesting nomenclature should be clear once you draw it. In this, a stable equilibrium solution branch bifurcates into two new stable branches (as well as an unstable branch) as the parameter h is increased.

$$y' = hy + y^3 \text{ (SUBCRITICAL PITCHFORK)}$$

Draw the bifurcation curve. This is called a **subcritical pitchfork**. A supercritical pitchfork bifurcation leads to a "soft" loss of stability, in which the system can go to nearby stable equilibria when the equilibrium $y = 0$ loses stability as h passes through zero. On the other hand, a subcritical pitchfork bifurcation leads to a "hard" loss of stability, in which there are no nearby equilibria and the system goes to some far-off dynamics (or perhaps to infinity) when the equilibrium $y = 0$ loses stability.

Question 3.

Determine the type of bifurcation for each of the following bifurcation diagrams.

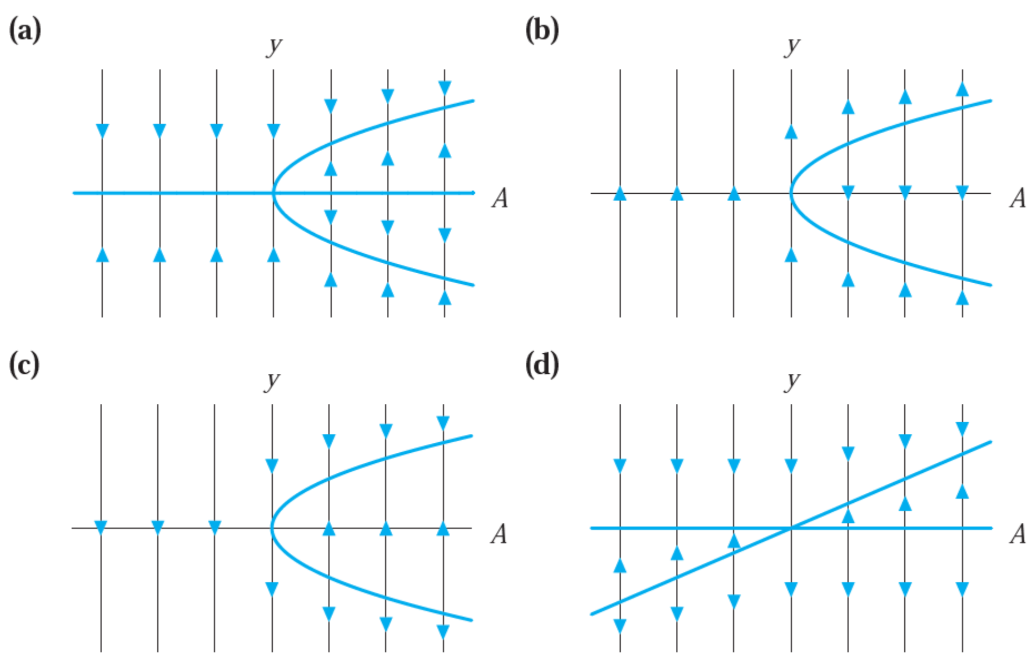


Figure 1