Math 2208: Ordinary Differential Equations

Lecture 19 Worksheet

Spring 2020 **Subhadip Chowdhury** **Apr 20**

Title: Equilibrium Point Analysis

Summary: We will analyze equlibrium points of non-linear systems using a technique called linearization which transforms the behavior of nonlinear systems of ODEs back into our familiar linear systems of ODEs. Relevant Book chapters (parts of) 5.1 and 5.2.

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§A. The Van der Pol Equation

An important nonlinear system of ODEs which occurs in Physics is the Van der Pol Oscillator Equation

$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = 0$$

which can be written as a non-linear system of first order ODEs as

$$\begin{aligned}\frac{dx}{dt} &= y\\ \frac{dy}{dt} &= -x + \mu(1-x^2)y\end{aligned}$$

For this lecture, we will assume $\mu = 1$.

Question 1.

Draw the phase portrait for the Van der Pol system using pplane.

(a) What happens to solutions that start near the origin at $(0,0)$?

(b) What about solutions that start (relatively) far away at $(3,3)$?

(c) Take a close-up look of the phase portrait near the point $(0,0)$ by changing the plot range. Can we call $(0,0)$ an equilibrium point? What kind of equilibrium does it resemble most closely?

Linearization

Let's use the idea of linear approximations to explain the behavior near the origin of the Van der Pol system. Suppose x and y are very small, for example say less that 0.01, then the nonlinear term x^2y will be less than 10^{-6} in magnitude, much less than either x or y. We can therefore write a linearized **approximate** version of the Van der Pol system near $(0,0)$ as follows:

$$\begin{aligned}\frac{dx}{dt} &= y\\ \frac{dy}{dt} &= -x + y\end{aligned}$$

Question 2.

Find the eigenvalues of the associated matrix and use that to classify the type of equilibrium at $(0,0)$.

§B. **Linearization near an Isolated Equilibrium**

Definition 2.1

An equilibrium point (x_e, y_e) is called **isolated** if some neighborhood of it contains no other equilibrium point.

Question 3.

Show that the linearization has an isolated equilibrium point at the origin if and only if both of its eigenvalues are non-zero.

Tangent plane Approximation

Let's assume (x_e, y_e) is an isolated equilibrium of

$$x' = f(x,y) \qquad y' = g(x,y) \tag{1}$$

and f and g are continuously differentiable in a neighborhood of (x_e, y_e).

Question 4.

What is the equation of the tangent plane $\mathbf{L}_f(x,y)$ to the graph of $f(x,y)$ at (x_e, y_e)? You might want to take a look at your Multivariable Calculus notes if you need to refresh your memory.

Question 5.

If $(x,y) \to (x_e, y_e)$, the function $f(x,y)$ can be well approximated by the local linearization $\mathbf{L}_f(x,y)$. Similarly, we can approximate $g(x,y)$ by $\mathbf{L}_g(x,y)$ near the the point (x_e, y_e). Replace $f(x,y)$ and $g(x,y)$ in system (1) by $\mathbf{L}_f(x,y)$ and $\mathbf{L}_g(x,y)$ respectively.

Question 6.

Consider the change of variable $u = x - x_e$ and $v = y - y_e$. Check that $u' = x'$ and $v' = y'$.

Rewrite the system (1) in terms of the variables u and v. This process is called **linearization**.

Question 7.

Show that the linearization of system (1) at (x_e, y_e) is a linear system of the form

$$\vec{\mathbf{U}}'(t) = \mathbf{J}\vec{\mathbf{U}}(t)$$

where $\vec{\mathbf{U}}(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ and the coefficient matrix $\mathbf{J}$ is the **Jacobian**

$$\mathbf{J}(x_e, y_e) = \begin{bmatrix} f_x(x_e, y_e) & f_y(x_e, y_e) \\ g_x(x_e, y_e) & g_y(x_e, y_e) \end{bmatrix} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}\Bigg|_{(x_e, y_e)}$$

■ Question 8.

Consider the non-linear system

$$\frac{dx}{dt} = (x-1)(y+1)$$
$$\frac{dy}{dt} = (y-2)(x+2)$$

(a) What are the equilibrium points?

(b) Find the Jacobian at each equilibrium point.

(c) Calculate the eigenvalue of the Jacobian matrices to find out the type of equilibrium at each point.

(d) Draw the phase portrait using PPLANE to confirm your observation.

§C. Case Study: Nonlinear Pendulum

Consider a pendulum consisting of a mass attached to a rigid rod. When the amplitude of motion of the ball is small enough, we usually make the approximation $\sin(\theta) \approx \theta$. This results in the harmonic motion equation

$$y'' = -\frac{b}{m}y' - \frac{g}{l}y$$

But when the amplitudes get bigger, the physics always becomes nonlinear.

Consider an idealized point mass moving in a circle at the end of a rigid weightless bar. Assume that the corresponding ODE is given by

$$y'' = -0.1y' - \sin(y)$$

where $y(t)$ represents the angle from the vertical in radians at time t. We are going to try to understand the motion using phase portrait in (y, y')-plane. Note that the initial condition $y(0) = 0$ corresponds to the pendulum being vertical at the beginning.

(a) Write down the associated system of two first order ODEs and find the equilibrium points.

(b) Use PPLANE to draw some sample solution curves. What would be the best description of the kind of qualitative behavior you observe locally around the equilibrium points?

(c) Linearize the system at the equilibrium points and classify their types to justify your description.

(d) How does the long term behavior depend on $y'(0)$? Think of a situation where you strike the ball of the pendulum to give it a initial velocity. Then try to think what happens if we strike with larger and larger force. Draw the curves in PPLANE that starts from

- $y(0) = 0, y'(0) = 2$
- $y(0) = 0, y'(0) = 2.5$
- $y(0) = 0, y'(0) = 3$

and give a physical interpretation of the difference between the motions.

(e) Justify the following statement using the phase portrait: "In the absence of damping, a pendulum that swings over once swings over infinitely many times."

(f) Does there exist a initial value of $y'(0)$ (where $y(0) = 0$) such that the pendulum doesn't exhibit a periodic behavior over time? How would you physically interpret this?