Assignment 6 Solutions

Chapter 2.1

15. Suppose $y = 1$. If we can find a value of x such that $dy/dt = 0$, then for this x and $y = 1$ the predator population is constant. (This point may not be an equilibrium point because we do not know if $dx/dt = 0$.) The required value of x is $x = 0.05$ in system (i) and $x = 20$ in system (ii). Survival for one unit of predators requires 0.05 units of prey in (i) and 20 units of prey in (ii). Therefore, (i) is a system of inefficient predators and (ii) is a system of efficient predators.

17. **(a)** For the initial condition close to zero, the pest population increases much more rapidly than the predator. After a sufficient increase in the predator population, the pest population starts to decrease while the predator population keeps increasing. After a sufficient decrease in the pest population, the predator population starts to decrease. Then, the population comes back to the initial point.

(b) After applying the pest control, you may see the increase of the pest population due to the absence of the predator. So in the short run, this sort of pesticide can cause an explosion in the pest population.

Chapter 2.2

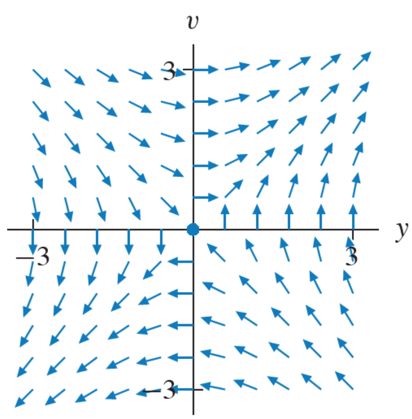
7. **(a)** Let $v = dy/dt$. Then

$$\frac{dv}{dt} = \frac{d^2y}{dt^2} = y.$$

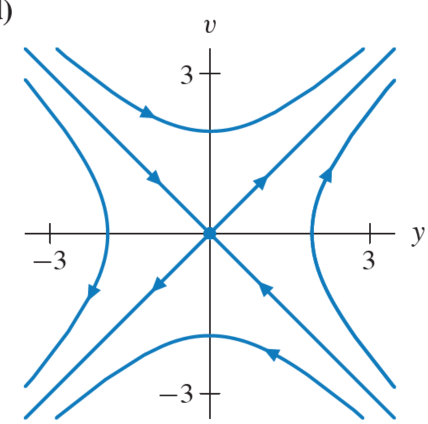
Thus the associated vector field is $\mathbf{V}(y, v) = (v, y)$.

(b) See part (c).

(c)



(d)



Chapter 3.1

9. $$\frac{dx}{dt} = \beta y$$
$$\frac{dy}{dt} = \gamma x - y$$

19. Letting $v = dy/dt$ and $w = d^2y/dt^2$ we can write this equation as the system

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= \frac{d^2y}{dt^2} = w \\ \frac{dw}{dt} &= \frac{d^3y}{dt^3} = -ry - qv - pw. \end{aligned}$$

In matrix notation, this system is

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

where

$$\mathbf{Y} = \begin{pmatrix} y \\ v \\ w \end{pmatrix} \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -r & -q & -p \end{pmatrix}.$$

26. **(a)** Substitute $\mathbf{Y}_1(t)$ into the differential equation and compare the left-hand side to the right-hand side. On the left-hand side, we have

$$\frac{d\mathbf{Y}_1}{dt} = \begin{pmatrix} -3e^{-3t} \\ -3e^{-3t} \end{pmatrix},$$

and on the right-hand side, we have

$$\mathbf{AY}_1(t) = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} e^{-3t} \\ e^{-3t} \end{pmatrix} = \begin{pmatrix} -2e^{-3t} - e^{-3t} \\ 2e^{-3t} - 5e^{-3t} \end{pmatrix} = \begin{pmatrix} -3e^{-3t} \\ -3e^{-3t} \end{pmatrix}.$$

Since the two sides agree, we know that $\mathbf{Y}_1(t)$ is a solution.

For $\mathbf{Y}_2(t)$,

$$\frac{d\mathbf{Y}_2}{dt} = \begin{pmatrix} -4e^{-4t} \\ -8e^{-4t} \end{pmatrix},$$

and

$$\mathbf{A}\mathbf{Y}_2(t) = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} e^{-4t} \\ 2e^{-4t} \end{pmatrix} = \begin{pmatrix} -2e^{-4t} - 2e^{-4t} \\ 2e^{-4t} - 10e^{-4t} \end{pmatrix} = \begin{pmatrix} -4e^{-4t} \\ -8e^{-4t} \end{pmatrix}.$$

Since the two sides agree, the function $\mathbf{Y}_2(t)$ is also a solution.

Both $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are solutions, and we proceed to the next part of the exercise.

(b) Note that $\mathbf{Y}_1(0) = (1, 1)$ and $\mathbf{Y}_2(0) = (1, 2)$. These vectors are not on the same line through the origin, so the initial conditions are linearly independent. If the initial conditions are linearly independent, then the solutions must also be linearly independent. Since the two solutions are linearly independent, we proceed to part (c) of the exercise.

(c) We must find constants k_1 and k_2 such that

$$k_1\mathbf{Y}_1(0) + k_2\mathbf{Y}_2(0) = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

In other words, the constants k_1 and k_2 must satisfy the simultaneous system of linear equations

$$\begin{cases} k_1 + k_2 = 2 \\ k_1 + 2k_2 = 3. \end{cases}$$

It follows that $k_1 = 1$ and $k_2 = 1$. Hence, the required solution is

$$\mathbf{Y}_1(t) + \mathbf{Y}_2(t) = \begin{pmatrix} e^{-3t} + e^{-4t} \\ e^{-3t} + 2e^{-4t} \end{pmatrix}.$$

31. (a) If $(x_1, y_1) = (0, 0)$, then (x_1, y_1) and (x_2, y_2) are on the same line through the origin because (x_1, y_1) is the origin. So (x_1, y_1) and (x_2, y_2) are linearly dependent.

(b) If $(x_1, y_1) = \lambda(x_2, y_2)$ for some λ, then (x_1, y_1) and (x_2, y_2) are on the same line through the origin. To see why, suppose that $x_2 \neq 0$ and $\lambda \neq 0$. (The $\lambda = 0$ case was handled in part (a) above.) In this case, $x_1 \neq 0$ as well. Then the slope of the line through the origin and (x_1, y_1) is y_1/x_1, and the slope of the line through the origin and (x_2, y_2) is y_2/x_2. However, because $(x_1, y_1) = \lambda(x_2, y_2)$, we have

$$\frac{y_1}{x_1} = \frac{\lambda y_2}{\lambda x_2} = \frac{y_2}{x_2}.$$

Since these two lines have the same slope and both contain the origin, they are the same line. (The special case where $x_2 = 0$ reduces to considering vertical lines through the origin.)

(c) If $x_1y_2 - x_2y_1 = 0$, then $x_1y_2 = x_2y_1$. Once again, this condition implies that (x_1, y_1) and (x_2, y_2) are on the same line through the origin. For example, suppose that $x_1 \neq 0$, then

$$y_2 = \frac{x_2y_1}{x_1} = \frac{x_2}{x_1}y_1.$$

But we already know that

$$x_2 = \frac{x_2}{x_1}x_1,$$

so we have

$$(x_2, y_2) = \frac{x_2}{x_1}(x_1, y_1).$$

By part (b) above (where $\lambda = x_2/x_1$), the two vectors are linearly dependent.

If $x_1 = 0$ but $y_1 \neq 0$, it follows that $x_2y_1 = 0$, and thus $x_2 = 0$. Thus, both (x_1, y_1) and (x_2, y_2) are on the vertical line through the origin.

Finally, if $x_1 = 0$ and $y_1 = 0$, we can use part (a) to show that the two vectors are linearly dependent.

32. If $x_1y_2 - x_2y_1$ is nonzero, then $x_1y_2 \neq x_2y_1$. If both $x_1 \neq 0$ and $x_2 \neq 0$, we can divide both sides by x_1x_2, and we obtain

$$\frac{y_2}{x_2} \neq \frac{y_1}{x_1},$$

and therefore, the slope of the line through the origin and (x_2, y_2) is not the same as the slope of the line through the origin and (x_1, y_1).

If $x_1 = 0$, then $x_2 \neq 0$. In this case, the line through the origin and (x_1, y_1) is vertical, and the line through the origin and (x_2, y_2) is not vertical.

35. **(a)** Using the Product Rule we compute

$$\frac{dW}{dt} = \frac{dx_1}{dt} y_2 + x_1 \frac{dy_2}{dt} - \frac{dx_2}{dt} y_1 - x_2 \frac{dy_1}{dt}.$$

(b) Since $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are solutions, we know that

$$\frac{dx_1}{dt} = ax_1 + by_1$$
$$\frac{dy_1}{dt} = cx_1 + dy_1$$

and that

$$\frac{dx_2}{dt} = ax_2 + by_2$$
$$\frac{dy_2}{dt} = cx_2 + dy_2.$$

Substituting these equations into the expression for dW/dt, we obtain

$$\frac{dW}{dt} = (ax_1 + by_1)y_2 + x_1(cx_2 + dy_2) - (ax_2 + by_2)y_1 - x_2(cx_1 + dy_1).$$

After we collect terms, we have

$$\frac{dW}{dt} = (a + d)W.$$

(c) This equation is a homogeneous, linear, first-order equation (as such it is also separable—see Sections 1.1, 1.2, and 1.8). Therefore, we know that the general solution is

$$W(t) = Ce^{(a+d)t}$$

where C is any constant (but note that $C = W(0)$).

(d) From Exercises 31 and 32, we know that $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are linearly independent if and only if $W(t) \neq 0$. But, $W(t) = Ce^{(a+d)t}$, so $W(t) = 0$ if and only if $C = W(0) = 0$. Hence, $W(t) = 0$ is zero for some t if and only if $C = W(0) = 0$.

Chapter 3.2

2. **(a)** The characteristic polynomial is

$$(-4-\lambda)(-3-\lambda)-2=\lambda^2+7\lambda+10=0,$$

and therefore the eigenvalues are $\lambda_1=-2$ and $\lambda_2=-5$.

(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1=-2$, we solve the system of equations

$$\begin{cases} -4x_1-2y_1=-2x_1 \\ -x_1-3y_1=-2y_1 \end{cases}$$

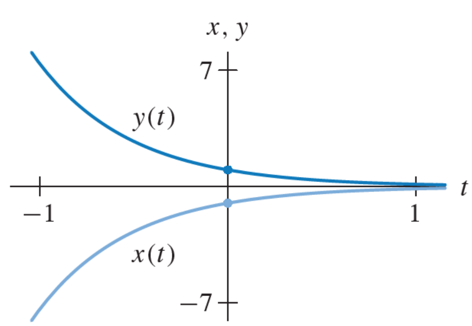
and obtain $y_1=-x_1$.

Using the same procedure, we obtain the eigenvectors (x_2, y_2) where $x_2=2y_2$ for $\lambda_2=-5$.

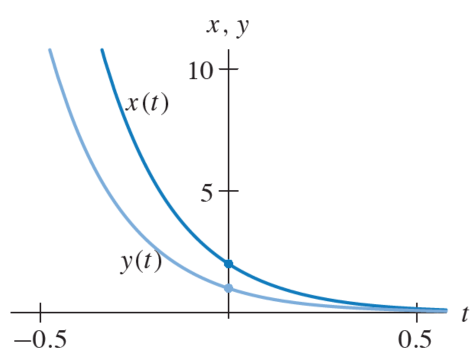
(d) One eigenvector $\mathbf{V}_1$ for λ_1 is $\mathbf{V}_1=(1,-1)$, and one eigenvector $\mathbf{V}_2$ for λ_2 is $\mathbf{V}_2=(2,1)$.

Given the eigenvalues and these eigenvectors, we have two linearly independent solutions

$$\mathbf{Y}_1(t)=e^{-2t}\begin{pmatrix}1\\-1\end{pmatrix} \quad\text{and}\quad \mathbf{Y}_2(t)=e^{-5t}\begin{pmatrix}2\\1\end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$.



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$.

(e) The general solution to this linear system is

$$\mathbf{Y}(t)=k_1e^{-2t}\begin{pmatrix}1\\-1\end{pmatrix}+k_2e^{-5t}\begin{pmatrix}2\\1\end{pmatrix}.$$

6. **(a)** The characteristic polynomial is

$$(5-\lambda)(-\lambda) - 36 = 0,$$

and therefore the eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = 9$.

(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = -4$, we solve the system of equations

$$\begin{cases} 5x_1 + 4y_1 = -4x_1 \\ 9x_1 = -4y_1 \end{cases}$$

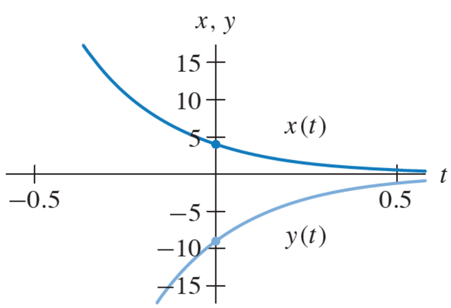
and obtain $9x_1 = -4y_1$.

Using the same procedure, we see that the eigenvectors (x_2, y_2) for $\lambda_2 = 9$ must satisfy the equation $y_2 = x_2$.

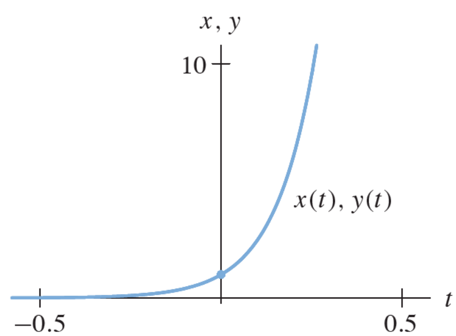
(d) One eigenvector $\mathbf{V}_1$ for λ_1 is $\mathbf{V}_1 = (4, -9)$, and one eigenvector $\mathbf{V}_2$ for λ_2 is $\mathbf{V}_2 = (1, 1)$.

Given the eigenvalues and these eigenvectors, we have the two linearly independent solutions

$$\mathbf{Y}_1(t) = e^{-4t}\begin{pmatrix} 4 \\ -9 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{9t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$.



The (identical) $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$.

(e) The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{-4t}\begin{pmatrix} 4 \\ -9 \end{pmatrix} + k_2 e^{9t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

8. **(a)** The characteristic polynomial is

$$(2-\lambda)(1-\lambda) - 1 = \lambda^2 - 3\lambda + 1 = 0,$$

and therefore the eigenvalues are

$$\lambda_1 = \frac{3+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{3-\sqrt{5}}{2}.$$

(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = (3+\sqrt{5})/2$, we solve the system of equations

$$\begin{cases} 2x_1 - y_1 = \dfrac{3+\sqrt{5}}{2}x_1 \\ -x_1 + y_1 = \dfrac{3+\sqrt{5}}{2}y_1 \end{cases}$$

and obtain

$$y_1 = \frac{1-\sqrt{5}}{2}x_1,$$

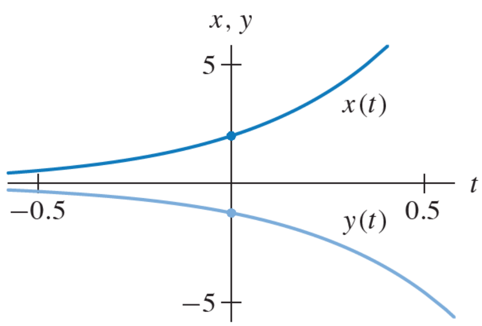
which is equivalent to the equation $2y_1 = (1-\sqrt{5})x_1$.

Using the same procedure, we obtain the eigenvectors (x_2, y_2) where $2y_2 = (1+\sqrt{5})x_2$ for $\lambda_2 = (3-\sqrt{5})/2$.

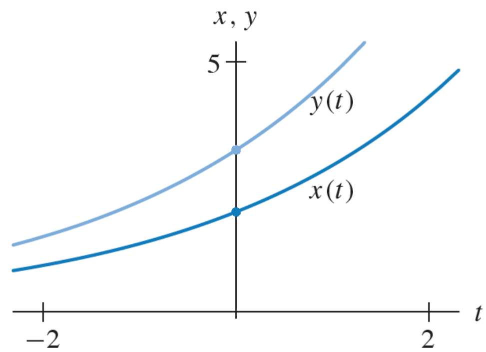
(d) One eigenvector $\mathbf{V}_1$ for the eigenvalue λ_1 is $\mathbf{V}_1 = (2, 1-\sqrt{5})$, and one eigenvector $\mathbf{V}_2$ for the eigenvalue λ_2 is $\mathbf{V}_2 = (2, 1+\sqrt{5})$.

Given the eigenvalues and these eigenvectors, we have two linearly independent solutions

$$\mathbf{Y}_1(t) = e^{(3+\sqrt{5})t/2}\begin{pmatrix} 2 \\ 1-\sqrt{5} \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{(3-\sqrt{5})t/2}\begin{pmatrix} 2 \\ 1+\sqrt{5} \end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$.



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$.

(e) The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{(3+\sqrt{5})t/2} \begin{pmatrix} 2 \\ 1-\sqrt{5} \end{pmatrix} + k_2 e^{(3-\sqrt{5})t/2} \begin{pmatrix} 2 \\ 1+\sqrt{5} \end{pmatrix}.$$