MATH 221 - Differential Equations

Lecture 6 Worksheet

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Title: Direction Field

Summary: we will analyze how much information about an unknown solution to an ODE one can obtain without being able to obtain an explicit formula for the solution itself, by using a technique called direction fields.

Related Reading: Chapter 2.1.1

Motivation

Not all differential equations can be solved analytically. Consider the separable ODE $y' = \sin(x^2)$. Can you actually solve it using separation of variables? How about $y' = \frac{\sin x}{x}$ or $y' = e^{-x^2}$? Some functions don't have closed form integral formula! (Look at the end of section 2.2 in the textbook for an interesting example.) So barring any analytical approach, we need to come up with tools to analyze these ODEs qualitatively. Below we discuss the first of such techniques.

The following technique only works for first order ODEs.

§A. Direction Fields

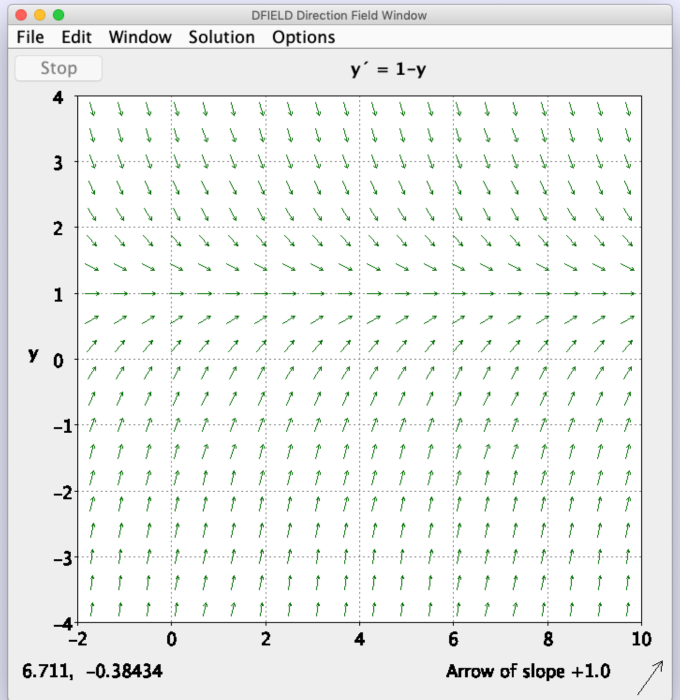
Recall that dy/dx, the derivative is equal to slope of the tangent. using this idea, if we are given an ODE $\frac{dy}{dx} = f(x,y)$, we can tell what is the slope of the tangent to the solution curve, without even knowing what the equation of the curve is. So one way to draw/find a solution curve is to collect the data about all these slopes, and try to fit a curve whose tangents have the correct slopes.

Definition A.1

Given a first order ODE $\frac{dy}{dx} = f(x,y)$, if we evaluate $f(x,y)$ at each point (x,y) over a rectangular grid, and draw short, oriented line segments (arrows) placed at (x,y) whose slope is equal to $f(x,y)$, then the collection of line elements is called the **direction field** or **slope field** of the ODE.

Example A.1

For example, below is the direction field of the ODE $y' = 1 - y$.



We have used a software called DFIELD to plot it. It is very intuitive to use. You can find it in Moodle. Let me know if are having trouble using it.

Question 1.

There are several interesting things we can notice about the slope field.

(a) All the arrows are angled the same way as we move from left to right on a horizontal path. Explain this analytically.

(b) When $y = 1$ all the arrows are horizontal. Explain this analytically. Can you guess a solution to the initial value problem $y' = 1 - y$, $y(0) = 1$?

(c) When $y > 1$ all solution curves appear to be decreasing and concave up. When $y < 1$ all solution curves appear to be increasing and concave down. Explain these analytically.

Question 2.

Consider the ODE $y' = (t-1)^2$. Draw the direction field by hand. Do you notice anything interesting about the arrows along a vertical line?

Find the analytical solution by integration. Is it consistent with the direction field?

Question 3.

Use DFIELD to draw the direction field of a random first order ODE of your choice. Click on the screen to draw some solution curves. Each time you click, you are specifying an initial condition. All the curves that can be drawn are general solutions to your ODE. Each of the curve separately is a particular solution to the IVP which is specified by your click.

■ Question 4.

Use DFIELD to investigate the ODEs we have encountered so far in class:

- $y' = ky$, exponential population model.
- $y' = \frac{k}{N}y(N-y)$, logistic population model.
- $y' = -x/y$, the ant path
- $y' = 1 + y^2$, the tan curve
- $y' = \sqrt{1-y^2}$, the sine curve. Note that this is a different ODE than $y^2 + y'^2 = 1$, which we encountered in last worksheet. The second one includes both the positive and the negative square root.

In each case, draw several particular solution curves, and try to identify the interval of definition of each of them.