MATH 221 - Differential Equations

Lecture 20 Worksheet

Fall 2020 | **Subhadip Chowdhury** | **Oct 2**

Title: Eigenvalues, Eigenvectors and Straight Line Solutions

Summary: We will look at straight line solutions to systems of the form $\frac{d\vec{\mathbf{R}}}{dt} = \mathbf{A}\vec{\mathbf{R}}$.

Related Reading: Section 8.1 from the textbook, but we only care about 2×2 matrices.

From The ODE Project - Section 3.1,3.2.

As you read through this worksheet, you should treat it as a study guide only. It is not a replacement for your textbook. Please make sure to follow along in the textbook (or other online resources from the related reading section above) side-by-side as you read through the topics. You are not expected to be able to answer the questions in here immediately after reading the synopsis from the worksheet. They are designed as more of an exploration directives; as you try to find the answers yourself, you will also learn the topic. You should use whatever resources are available to you, including the internet, to accomplish that task.

§A. Motivation

Consider the system $\frac{d\vec{\mathbf{R}}}{dt} = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix}\vec{\mathbf{R}}$ from last lecture. We found that $\vec{\mathbf{R}}_1(t) = \begin{bmatrix} 2e^{-3t} \\ e^{-3t} \end{bmatrix}$ and $\vec{\mathbf{R}}_2(t) = \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix}$ are solutions to this system. Observe that

$$\vec{\mathbf{R}}_1(t) = e^{-3t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } \vec{\mathbf{R}}_2(t) = e^{2t}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Question 1.

Let's see if we can find anything special about the column vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ in relationship to the matrix $\mathbf{A} = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix}$. How does each vector change if you multiply it by $\mathbf{A}$? Can you see a pattern between your answer and how the solution curves were defined above?

End of Question 1

Suppose, for a general system $\frac{d\vec{\mathbf{R}}}{dt} = \mathbf{A}\vec{\mathbf{R}}$, we are lucky enough to have the following situation: There is a non-zero vector $\vec{v}$ and a real number λ such that $\mathbf{A}\vec{v} = \lambda\vec{v}$. Then imitating the example above, we might guess that the system has **a** solution curve of the form

$$\vec{\mathbf{R}}(t) = e^{\lambda t}\vec{v}$$

Digression

Compare this with the fact that a one-dimensional linear autonomous homogeneous equation has an exponential solution.

$$\frac{dy}{dt} = ay \implies y(t) = Ce^{at}$$

Let's check if it works. The derivative of $e^{\lambda t}$ is $\lambda e^{\lambda t}$ and $\vec{v}$ is a constant. Thus if $\vec{R}(t) = e^{\lambda t}\vec{v}$, then we get

$$\frac{d\vec{R}}{dt} = \lambda e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda\vec{v}\right) = e^{\lambda t}A\vec{v} = A\left(e^{\lambda t}\vec{v}\right) = A\vec{R}(t)$$

It works! So now we have a simple thing to try.

§B. Linear Algebra Basics III - Eigenvalues and Eigenvectors

Definition B.1

Given a $n \times n$ matrix A, a $n \times 1$ nonzero vector $\vec{v}$ is called an **eigenvector** of A if there exists a $\lambda \in \mathbb{R}$ such that

$$A\vec{v} = \lambda\vec{v}.$$

If such a λ exists, we say that λ is the **eigenvalue** associated to $\vec{v}$.

With this definition, the vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ from above example are eigenvectors of the matrix $\begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix}$ with eigenvalues -3 and 2 respectively. Since eigenvectors and eigenvalues give us a way to find solutions to homogeneous linear systems as above, let's look into how we can find them from the matrix.

How to find Eigenvalues?

First observe that finding $\vec{v}$ is equivalent to solving a linear system.

$$A\vec{v} = \lambda\vec{v} \iff (A - \lambda I)\vec{v} = \vec{0},$$

where I is the identity matrix and $\vec{0}$ is the origin. We learned last lecture that the linear system has a non-zero solution iff $\det(A - \lambda I) = 0$. This gives us a way to solve for λ.

Definition B.2

The **characteristic polynomial** of a $n \times n$ matrix A is defined as

$$p_A(\lambda) = \det(A - \lambda I).$$

We will refer to it sometimes as $\mathbf{charpoly}_A(\lambda)$.

If the matrix A is 2×2, the determinant is a quadratic polynomial in λ. So we can use the quadratic formula to solve it and find its roots, which will be the eigenvalues of our matrix!

Example B.1

Suppose $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. Then

$$\mathbf{A} - \lambda \mathbf{I} = \mathbf{A} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}$$

and consequently,

$$p_{\mathbf{A}}(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}) = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1)$$

So the eigenvalues are $\lambda = 5$ and $\lambda = -1$.

■ **Question 2.** **Two Distinct Real Roots**

Consider the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. What conditions on $a, b, c,$ and d would ensure that the characteristic polynomial has two real distinct roots?

How to find Eigenvectors?

Once we have found an eigenvalue, we need to solve the actual equation $\mathbf{A}\vec{v} = \lambda\vec{v}$ to find the eigenvector. Fortunately, as we will observe, the task is simpler than solving a system of two linear equations. Let's take look at an example.

Example B.2

We will work with the matrix from example B.1. We are trying to solve for $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ when $\lambda = 5$. We have

$$\mathbf{A}\vec{v} = \lambda\vec{v} \implies \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 5 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \implies \begin{bmatrix} v_1 + 2v_2 \\ 4v_1 + 3v_2 \end{bmatrix} = \begin{bmatrix} 5v_1 \\ 5v_2 \end{bmatrix}$$

which simplifies to

$$\begin{cases} v_1 + 2v_2 & = 5v_1 \\ 4v_1 + 3v_2 & = 5v_2 \end{cases}$$

$$\implies \begin{cases} 2v_2 = 4v_1 \\ 4v_1 = 2v_2 \end{cases}$$

So we make the interesting observation that both equations are infact the same! So any vector of the form $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, i.e. any scalar multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector with eigenvalue 5.

This is expected, because we can easily check that

$$\mathbf{A}\vec{v} = \lambda\vec{v} \implies \mathbf{A}\left(k\vec{v}\right) = \lambda\left(k\vec{v}\right)$$

which means that any scalar multiple of an eigenvector is also an eigenvector with the same eigenvalue.

Question 3. Eigenpractice

Find the other eigenvector of $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ that has eigenvalue -1.

§C. Straight Line Solution

Using the language of Eigenstuff, we can conclude that if the matrix $\mathbf{A}$ has eigenvectors $\vec{v_1}$ and $\vec{v_2}$ with real eigenvalues λ_1 and λ_2 respectively, then the curves

$$\mathbf{R}_1(t) = e^{\lambda_1 t}\vec{v_1} \quad \text{and} \quad \mathbf{R}_1(t) = e^{\lambda_2 t}\vec{v_2}$$

are solutions to the system of ODEs $\dfrac{d\vec{\mathbf{R}}}{dt} = \mathbf{A}\vec{\mathbf{R}}$.

We can also observe that $\vec{\mathbf{R}}_1(t)$and $\vec{\mathbf{R}}_2(t)$ are straight lines, because for any value of t they are scalar multiples of $\vec{v_1}$ and $\vec{v_2}$ respectively.

Additionally, any multiples of $\vec{\mathbf{R}}_1(t)$ and $\vec{\mathbf{R}}_2(t)$ are also solutions to the system because of the Superposition principle. But since $\vec{\mathbf{R}}_1(t)$and $\vec{\mathbf{R}}_2(t)$ are straight lines, their multiples lie on the exact same straight line. These straight lines are thus called the **straight line solutions** of the system.

We haven't yet proved that these are the <u>only</u> straight line solutions for our system. For that, we need to know how other solutions look like, which we will learn next lecture.

Question 4. Practice Problem

Consider the phase plane for the system from question 1. A picture is given below. Use PPLANE on your computer to follow along.

(a) What do you think the red and the yellow straight lines represent?

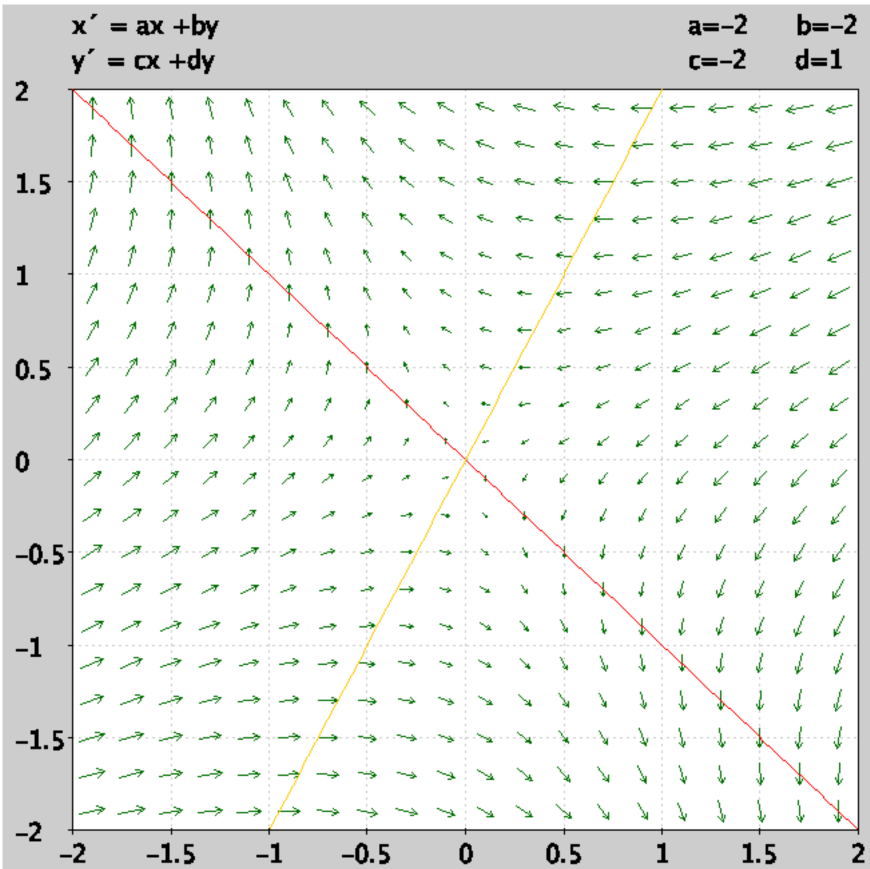
(b) Can you draw the solution curves $\vec{\mathbf{R}}_1(t)$ and $\vec{\mathbf{R}}_2(t)$ in the picture below by hand? They should look like rays that start or end at the origin.

(c) For those solution curves, does it matter what your initial condition is? Use PPLANE to draw the two solutions by providing appropriate initial values.

(d) Does one of the solutions seem more "attractive" than the other?

(e) Describe in words how other solutions are behaving. What can you say about the limit of $\frac{dy}{dx}$ along other solution curves as $t \to \infty$? What about $t \to -\infty$, i.e. in the reverse direction of the arrows?

Guess an answer from the picture and PPLANE, we will prove it next time.



x´ = ax +by
y´ = cx +dy
a=-2
b=-2
c=-2
d=1
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2