Math 2208: Ordinary Differential Equations

Lecture 16 Worksheet

Spring 2020 **Subhadip Chowdhury** **Apr 8**

Title: Harmonic Oscillators

Summary: We will examine the standard second order constant-coefficient ODE $y'' + py' + qy = 0$ more closely now that we have completed the analysis of the first order system of 2 linear ODEs. Corresponding book chapter is 3.6.

§A. Second Order Linear ODE

Consider the second-order linear ODE

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$$

where p and q are constant real numbers.

Question 1.

(a) Write this equation as a two-dimensional first-order linear system of ODEs.

Hint: Define a new variable $v(t) = y'(t)$.

(b) Suppose λ is an eigenvalue of the matrix corresponding to above linear system. Show that the corresponding eigenvector is $\begin{bmatrix} 1 \\ \lambda \end{bmatrix}$.

(c) If λ_1 and λ_2 are the two eigenvalues, write the general solution to the system of linear ODEs and get a formula for $\vec{r}(t) = \begin{bmatrix} y(t) \\ v(t) \end{bmatrix}$.

Solution. We have

$$\vec{r}(t) = \begin{bmatrix} y(t) \\ v(t) \end{bmatrix} = k_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + k_2 e^{\lambda_2 t} \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$$

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(d) Isolate $y(t)$ to get the general (real-valued) solution to the second order linear ODE.

Solution. If $\lambda_i \in \mathbb{R}$,

$$y(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t}$$

If $\lambda_i = \alpha \pm i\beta \notin \mathbb{R}$,

$$\vec{r}(t) = e^{\alpha t} \begin{bmatrix} \cos(\beta t) + i\sin(\beta t) \\ \ldots \end{bmatrix}$$

So the general real valued solution is

$$y(t) = k_1 e^{\alpha t}\cos(\beta t) + k_2 e^{\alpha t}\sin(\beta t)$$

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■ **Question 2.**

(a) What conditions on p and q guarantee that the eigenvalues are complex numbers?

Solution. The characteristic polynomial is

$$\lambda^2 + p\lambda + q = 0$$

So we need $p^2 < 4q$. ■

(b) What conditions on p and q guarantee that the origin is a spiral sink? What conditions guarantee that the origin is a center? What conditions guarantee that the origin is a spiral source?

§B. Simple Harmonic Motion

The equations for the harmonic oscillator come from Newton's second law

$$\text{force} = \text{mass} \times \text{acceleration}$$

Suppose we apply it to the motion of a mass attached to a spring, sliding on a table. We let $y(t)$ denote the position (displacement) of the mass m at time t, with $y = 0$ the rest position. The forces on the mass are the spring force, $-ky$, (this is called Hooke's law) and the damping force, $-b(dy/dt)$ (e.g. friction). The negative signs represent that the forces are in the opposite direction to that of $y(t)$. Substituting into Newton's law gives

$$-ky - b\frac{dy}{dt} = m\frac{d^2y}{dt^2}$$

which is often written as

$$my'' + by' + ky = 0 \iff y'' + \frac{b}{m}y' + \frac{k}{m}y = 0$$

The parameters are $m > 0, k > 0,$ and $b \geq 0$. This type of motion is known as a harmonic motion. See pages 156-158 in the textbook for a more detailed explanation.

■ **Question 3.**

First consider the case $b = 0$. This is called a **Simple Harmonic Motion**, also known as the **Undamped** Harmonic Motion.

(a) Check that the eigenvalues of associated linear system are complex numbers whose real parts are equal to 0.

(b) Write down the general formula for $y(t)$.

(c) Check that it is a periodic function of t. What is the period?

■ **Question 4. (Making a clock using Mass-Spring System)**

Suppose we wish to make a clock using a mass and a spring sliding on a table. We arrange for the clock to "tick" whenever the mass crosses $y = 0$. We use a spring with spring constant $k = 2$. If we assume there is no friction or damping ($b = 0$), then what mass m must be attached to the spring so that its natural period is one time unit?

■ Question 5.

If $b \neq 0$, we get a **Damped** Harmonic Motion. Then depending on different values of $b, k,$ and m we will have different behavior for the solution curves as the determinant of the characteristic polynomial changes.

(a) Find the discriminant of the characteristic polynomial of the associated system of linear ODEs in terms of $m, b,$ and k.

Solution. The discriminant is

$$p^2 - 4q = \frac{b^2 - 4km}{m^2}$$

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(b) Fill out the following table. We are considering three cases. Open the applet at

https://mathlets.org/mathlets/damped-vibrations/

	D < 0	**D = 0**	**D > 0**
Conditions on m, b, k (Note that $m > 0$)			
Eigenvalues are Real/Complex?			
Number of Eigenvalues			
Does the solution curve y vs. t oscillate? If yes, what's the period?			
Kind of Damping	**Underdamped**	**Critically damped**	**Overdamped**
Equilibrium Type of Phase Portrait in (y, y')-phase plane			