MATH 221 - Differential Equations

Lecture 35 Worksheet

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Title: Almost Linear Systems

Summary: We are going to show that if an almost linear system is used to model a physical situation, then -- apart from two sensitive cases -- the qualitative behavior of the system near an equilibrium point can be determined by examining its linearization.

As you read through this worksheet, you should treat it as a study guide only. It is not a replacement for your textbook. Please make sure to follow along in the textbook (or other online resources from the related reading section above) side-by-side as you read through the topics. You are not expected to be able to answer the questions in here immediately after reading the synopsis from the worksheet. They are designed as more of an exploration directives; as you try to find the answers yourself, you will also learn the topic. You should use whatever resources are available to you, including the internet, to accomplish that task.

§A. Almost Linear Systems

Definition A.1

An equilibrium point (x_e, y_e) is called **isolated** if we can find a neighborhood of the point that contains no other equilibrium point. In other words, if we zoom in far enough it is the only equilibrium point we can see.

Example A.1

The two equilibrium points for the system

$$\frac{dx}{dt} = x - y$$
$$\frac{dy}{dt} = 1 - x^2$$

are isolated.

If on the other hand there would be a whole curve of equilibrium points, then it would not be isolated. For example, the equilibrium point $(1, 0)$ for the system

$$\frac{ds}{dt} = -\beta si$$
$$\frac{di}{dt} = \beta si - \gamma i$$

is not isolated because all points of the form $(s, 0)$ is an equilibrium point for the system.

■ Question 1. □

Show that a 2D linear system has an isolated equilibrium point at the origin if and only if both of its eigenvalues are non-zero.

Definition A.2

A nonlinear system is called **almost linear** at an equilibrium point $\mathbf{P} = (x_e, y_e)$ if the equilibrium point $\mathbf{P}$ is isolated **and** its linearization at $\mathbf{P}$ also has an isolated equilibrium point at the origin $(0, 0)$.

By the last question, the second requirement is equivalent to saying that the determinant of the Jacobian is nonzero.

■ Question 2. □

Check that the system

$$\frac{dx}{dt} = x\left(x^3 - 2y^3\right)$$
$$\frac{dy}{dt} = y\left(2x^3 - y^3\right)$$

has an isolated equilibrium at the origin, but is not an almost linear system.

Fortunately, most often equilibrium points are isolated, and the system is almost linear at the equilibrium points. So if we learn what happens here, we have figured out the majority of situations that arise in applications.

§B. Classification of Isolated Equilibrium Points

Theorem B.1

Consider a **almost linear system** whose linearization at a point $\mathbf{P} = (x_e, y_e)$ is a linear system with associated matrix $\mathbf{J}$. Let λ_1 and λ_2 be the eigenvalues of $\mathbf{J}$ (we know that $\det(\mathbf{J}) = \lambda_1\lambda_2 \neq 0$). Then the equilibrium point $\mathbf{P}$ is identical in type to that of $(0,0)$ in its linearization, except in two cases:

1. If $\lambda_1 = \lambda_2 \in \mathbb{R}$, then the linearization has a defective equilibrium at the origin, but the original almost linear system has either a node or a spiral point at $\mathbf{P}$.
2. If λ_1 and λ_2 are purely imaginary (i.e. $\mathfrak{Re}(\lambda_i) = 0$ and $\lambda_i \in \mathbb{C}$), then the linearization has a center at the origin, but the original almost linear system has either a center or a spiral.

Question 3. The Trouble With Centers

Recall the following system from your Active Shock Absorber project:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -y^5 - x$$

(a) Check that the system has an isolated equilibrium point at $(0,0)$.

(b) Check that the corresponding linearization has a center at the origin.

On the other hand, we are going to show that the nonlinear system actually has a spiral sink at the origin.

(c) Consider the quantity $r = x^2 + y^2$. It measures the square of the distance between the point (x,y) and the origin. Find $\dfrac{dr}{dt}$ and show that it's negative.

Thus we have shown that all solution curves spiral slowly toward the origin! Consequently, the actual equilibrium type for the nonlinear system is a spiral sink.

■ Question 4. — A system that's not Almost Linear

In the case of a two-dimensional system that is not almost linear, the trajectories near an equilibrium point can exhibit a considerably more complicated structure. Consider the system from question 2. A loooong time ago (in Lecture 3 Activity), we found the solution curves for this system. The trajectories are called **folia of Descartes**. Use PPLANE to draw some solution curves. Does the equilibrium at $(0,0)$ resemble any of our familiar types?

■ Question 5. — Another Interesting Example

Consider the following non-linear system

$$\frac{dx}{dt} = (x-2y)x$$
$$\frac{dy}{dt} = (x-2)y$$

(a) Find the equilibrium points. Are they isolated?

(b) Is the system almost linear at both equilibrium points?

(c) Use PPLANE to analyze the behavior of solution curves near the equilibrium points.

§C. Possible Trajectories (Consequence of Poincaré-Bendixson Theorem)

In most generic cases* it can be shown that there are exactly four possible trajectories for a nondegenerate solution curve of the autonomous system

$$x' = f(x,y) \qquad y' = g(x,y)$$

The four possibilities are as follows:

(a) $(x(t), y(t))$ approaches an equilibrium point as $t \to \infty$.

(b) $(x(t), y(t))$ is unbounded with increasing t.

(c) $(x(t), y(t))$ is a periodic solution with a closed trajectory.

(d) $(x(t), y(t))$ spirals toward a closed trajectory, called a **limit cycle**, as $t \to \infty$.

*Check https://en.wikipedia.org/wiki/Poincare-Bendixson_theorem for a complete description.

§D. Van der Pol Equation revisited

Recall the Van der Pol equation from last lecture

$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = 0$$

which can be written as a non-linear system of first order ODEs as

$$\begin{aligned}\frac{dx}{dt} &= y\\ \frac{dy}{dt} &= -x + \mu(1-x^2)y\end{aligned}$$

Note that $(0,0)$ is always an equilibrium to above system. Let's draw the phase portrait for a couple of different values of μ and observe if the stability changes at μ. Use `pplane` and set the window to $-6 \le x, y \le 6$ for a comfortable close-up of the origin.

Question 6. ☐

(a) Find the Jacobian and evaluate it at $(0,0)$.

(b) Suppose $\mu = 3$. What kind of stability do you observe at $(0,0)$?

Visually classify the trajectories of other solution curves into the four cases above according to possible initial conditions. Which of the cases can you find examples of?

(c) Next set $\mu = 1$. What kind of stability do you observe at $(0,0)$? Can you describe the trajectories of other solution curves? Which of above four cases can you find examples of?

(d) Repeat above analysis for $\mu = -1$ and $\mu = -3$.

(e) Clearly all four stability types are distinct. So at least three bifurcations happened as μ increased from -3 to 3. Draw the corresponding curve in $(\mathbf{T},\mathbf{D})$-plane and find the bifurcation values.

§E. Suggested Homework Problems

Question 7. ☐

Consider the system

$$\begin{aligned}\frac{dx}{dt} &= 2x\left(1-\frac{x}{2}\right) - xy\\ \frac{dy}{dt} &= y\left(\frac{9}{4}-y^2\right) - x^2y\end{aligned}$$

Find the equilibria and sketch the nullclines. Use the Jacobian matrix to determine the type and stability of each equilibrium point and sketch the phase portrait.

Question 8. ☐

An oscillator with damping is governed by the equation $y'' + 3ay' + bx = 0$, where a and b are positive parameters. Plot the set of points in the (a,b)-plane (called (a,b) parameter space) where the system is critically damped.

■ Question 9. □

Find a particular solution to the equation $u'' + u' + 2u = \sin^2 t$.

Hint: Use a double angle formula to rewrite the right side.

■ Question 10. □

Given a first order autonomous ODE of the form $y' = f(y)$, the flow map $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined as follows. Given a time t and an initial value s, define $\varphi(t,s) = y(t)$ where $y(t)$ is value of the solution curve to the ODE at time t that starts with initial condition $y(0) = s$. In other words, $\varphi(t,s)$ is the time-t value associated with the particular solution curve passing through $y = s$ at time $t = 0$.

Use the fundamental theorem of Calculus to explain why

$$\varphi(t,s) = s + \int_0^t f(y(t))\,dt$$