# MATH 221 - Differential Equations

## Lecture 11 Activities

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## §A. In-class Practice Problems

### Question 1.

Consider the ODE with a single parameter

$$\frac{dy}{dt} = y^2 + \mu y + 1$$

(a) Find the equilibrium point(s) of the ODE in terms of $\mu$.

(b) Using your answer from part $(a)$ or otherwise (e.g. using theorem B.1), find the bifurcation point(s) $(\mu_0, y_0)$.

(c) Sketch the phase lines for values of the parameter $\mu$ slightly smaller, slightly larger than, and at the bifurcation value. Your answer should have at least five phase lines.

(d) Draw the bifurcation diagram.

## §B. Suggested Homework Problems

### Question 2.

Locate the bifurcation values for the one-parameter family and draw the phase lines for values of the parameter slightly smaller than, slightly larger than, and at the bifurcation values.

(a) $\dfrac{dy}{dx} = (y^2 - \alpha)(y^2 - 4)$

(b) $\dfrac{dy}{dx} = \sin y + \alpha$

### Question 3.

Sketch the graph of a function $g(y)$ such that the one-parameter family of differential equations $\dfrac{dy}{dt} = g(y) + \alpha$ satisfies all of the following properties:

- For all $\alpha \leq -4$, the differential equation has one sink and no other equilibria.
- For all $\alpha \geq 4$, the equation has one sink and no other equilibria.
- For $\alpha = 0$, the differential equation has exactly six equilibria.

There are many possible functions $g(y)$ that satisfy these conditions. Sketch just one graph.

## ■ Question 4.

Recall the Ermentrout-Kopell model for the spiking of a neuron introduced in Lecture 7 Activity.

$$\frac{d\theta}{dt} = 1 - \cos\theta + (1 + \cos\theta)I(t)$$

Suppose the input function $I(t)$ is a constant function, that is $I(t) = I$ where $I$ is a constant. Describe the bifurcations that occur as the parameter $I$ varies.

## ■ Question 5.

Using our discussion about the constant harvesting model from the worksheet, we found that even for values of $H < \frac{kN}{4}$, with low enough initial population, fish population goes extinct in the long-term. Obviously we don't want to happen. This suggests that the model needs some serious revision to account for lower initial populations, as it is clear that one cannot keep fishing a fixed number $H$ when the population becomes smaller. To modify the model, we propose a second harvesting scheme.

We will harvest a fixed **proportion** $H$ of the fishes per year. If the population is $P(t)$ at time $t$ in years, we say that the **yield** is $\mathcal{Y} = HP$, and the ODE now becomes

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - HP \tag{1}$$

Let $k = 2$ and $N = 100$ as in the quiz.

(a) Draw phase lines when the value of $H$ equals $0, 1, 2, 3$, and $4$. Identify and classify any and all equilibrium points for each value of $H$.

(b) Draw the bifurcation diagram with $P$ on the vertical axis and $H$ on the horizontal axis.

(c) For what values of $H$, is it possible for the fishes to go extinct with this harvesting scheme?

(d) If you harvest at a per capita rate $H < 2$, what will be the long-term population size $\lim_{t\to\infty} p(t)$? Your answer will depend on $H$.

(e) In above scenario, what will be the long term yield $\lim_{t\to\infty} \mathcal{Y}$?

(f) What value of $H$ maximizes the long term yield?