Math 2208: Ordinary Differential Equations

Lecture 21 Worksheet

Spring 2020 **Subhadip Chowdhury** **Apr 20**

Title: Almost Linear Systems

Summary: We are going to show that if an almost linear system is used to model a physical situation, then -- apart from two sensitive cases -- the qualitative behavior of the system near a critical point can be determined by examining its linearization.

§A. Stability of Almost Linear Systems

Definition 1.1

A nonlinear system is called **almost linear** at an isolated equilibrium point $\mathbf{P} = (x_e, y_e)$ if its linearization at $\mathbf{P}$ has an isolated equilibrium point at the origin $(0, 0)$.

Theorem A.1: Stability of Almost Linear System

Consider a **almost linear system** whose linearization at a point $\mathbf{P} = (x_e, y_e)$ is a linear system with associated matrix $\mathbf{A}$. Let λ_1 and λ_2 be the eigenvalues of $\mathbf{A}$ and assume $\det(\mathbf{A}) \neq 0$ or equivalently, $(0, 0)$ is an isolated equilibrium. Then the equilibrium point $\mathbf{P}$ is identical in type to that of $(0, 0)$ in its linearization, except in two cases:

1. If $\lambda_1 = \lambda_2 \in \mathbb{R}$, then the linearization has a defective node at the origin, but the original almost linear system has either a node or a spiral point at $\mathbf{P}$.
2. If λ_1 and λ_2 are purely imaginary, then the linearization has a center at the origin, but the original almost linear system has either a center or a spiral.

Question 1.

Determine the type of the critical point $(4, 3)$ of the almost linear system

$$\frac{dx}{dt} = 33 - 10x - 3y + x^2 \qquad \frac{dy}{dt} = -18 + 6x + 2y - xy$$

Question 2. (The case of $\mathfrak{Re}(\lambda_i) = 0$)

Next consider the following two systems

$$x' = y - (x^2 + y^2)x, \qquad y' = -x - (x^2 + y^2)y \tag{1}$$

$$\text{and} \qquad x' = y + (x^2 + y^2)x, \qquad y' = -x + (x^2 + y^2)y \tag{2}$$

(a) Check that both of them have isolated equilibrium point at $(0, 0)$ and the same linearization at $(0, 0)$.

(b) Check that the corresponding linearizations in both cases have a center at the origin. On the other hand, we are going to show that system (1) has a spiral sink whereas system (2) has a spiral source at the origin.

(c) Consider the first system. The corresponding vector field can be written as sum of two vector fields as follows:

$$\vec{\mathbf{V}}(x,y) = \langle y-(x^2+y^2)x, \quad -x-(x^2+y^2)y\rangle = \vec{\mathbf{V}}_1(x,y)-(x^2+y^2)\vec{\mathbf{V}}_2(x,y)$$

where $\vec{\mathbf{V}}_1(x,y) = \langle y,-x\rangle$ and $\vec{\mathbf{V}}_2(x,y) = \langle x,y\rangle$.

(d) Check that $\vec{\mathbf{V}}_1(x,y)$ is tangent to the position vector $\langle x,y\rangle$ at any point. Similarly, $\vec{\mathbf{V}}_2(x,y)$ is parallel to the position vector $\langle x,y\rangle$. So the vector field $\vec{\mathbf{V}}(x,y)$ always has a inward pointing radial component.

Consequently, all flow lines spiral slowly toward the origin!

(e) For similar reasons, check that all flow lines of the vector field corresponding to system (2) spiral outward from the origin.

Question 3. (The case of $\lambda_i = 0$)

In the case of a two-dimensional system that is not almost linear, the trajectories near an equilibrium point can exhibit a considerably more complicated structure. Consider the system

$$\frac{dx}{dt} = x\left(x^3-2y^3\right) \qquad \frac{dy}{dt} = y\left(2x^3-y^3\right)$$

that has $(0,0)$ as an equlibrium point.

(a) Is $(0,0)$ an isolated equilibrium for the system? Is $(0,0)$ an isolated equilibrium for the linearization?

(b) We are going to find a formula for the trajectories. Check that the equations imply

$$\frac{dy}{dx} = \frac{y(2x^3-y^3)}{x(x^3-2y^3)}$$

So if we can solve this first order ODE, we would get an implicit formula for the solution curves for this system. Go back to `lec6worksheet` and read the change of variable section. The ODE above is a homogeneous equation. Solve it using the technique described there. Show that the solution curves can be implicitly defined in the form

$$x^3+y^3 = 3cxy$$

where c is an arbitrary constant. These trajectories are called **folium of Descartes**.

(c) Use your favourite software to draw some solution curves. Does the equilibrium at $(0,0)$ resemble any of our familiar types?

§B. Possible Trajectories (Consequence of Poincaré-Bendixson Theorem)

In most generic cases[1] it can be shown that there are four possible trajectories for a nondegenerate solution curve of the autonomous system

$$x' = f(x,y) \qquad y' = g(x,y)$$

The four possibilities are as follows:

(a) $(x(t),y(t))$ approaches an equilibrium point as $t\to\infty$.

(b) $(x(t),y(t))$ is unbounded with increasing t.

(c) $(x(t),y(t))$ is a periodic solution with a closed trajectory.

(d) $(x(t),y(t))$ spirals toward a closed trajectory, called a **limit cycle**, as $t\to\infty$.

[1]Check https://en.wikipedia.org/wiki/Poincare-Bendixson_theorem for a complete description.