

MATH 2208: ORDINARY DIFFERENTIAL EQUATIONS

LECTURE 13 WORKSHEET

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TITLE: Phase Portraits of Linear Systems

SUMMARY: We'll explore the various scenarios that occur with linear systems of ODEs that possess real eigenvalues. Corresponding Book Chapter - 3.3.

§A. Two Real Eigenvalues

Consider a system of linear ODEs with associated matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The **characteristic polynomial** of A is given by

$$p_A(\lambda) = \det(A - \lambda I) = (a - \lambda)(d - \lambda) - bc = \lambda^2 - \lambda(a + d) + (ad - bc) = \lambda^2 - \text{tr}(A)\lambda + \det(A)$$

■ Question 1.

What condition the characteristic polynomial $p_A(\lambda)$ must satisfy in order to produce real eigenvalues?

§B. Classifying Equilibrium Points

Suppose a linear system has **two real, nonzero, distinct** eigenvalues λ_1 and λ_2 . Recall that λ_1 and λ_2 are the solutions to the characteristic polynomial. In what follows, we are going to classify λ_1 and λ_2 into a number of different cases depending on the qualities the eigenvalues possess. In addition, we will also classify the equilibrium at the origin, and sketch a typical phase portrait for each case.

In the next page, you will find three specific cases, each with an example of an ODE that satisfies the case and a classification of the origin. I suggest coordinating with other students so that you might divide the workload amongst yourselves.

For each case:

- Check that the given matrix will definitely produce real eigenvalues.
- Then find the eigenvalues and eigenvectors (by hand) in order to write down the straight line solutions and the general solution of the given system.
- What happens to $\vec{r}(t)$ as $t \rightarrow \infty$ or $t \rightarrow -\infty$? Note that your answer will depend on the initial condition. Find all possible scenarios. In each case, also find out what happens to the ratio $\frac{x(t)}{y(t)}$ as $t \rightarrow \pm\infty$.
- Use **PPLANE** to help you sketch the phase portrait on the given axes. Go to **Solution** menu and sketch the nullclines.
- Write down a few sentences describing your observations of the phase portrait. Are your solution curves consistent with the end behavior you found above?

CASE 1: $\lambda_1 > \lambda_2 > 0$

$$\text{Solve } \frac{d\vec{r}}{dt} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \vec{r}.$$

In this case the origin is called a **nodal source**.

CASE 2: $\lambda_1 < \lambda_2 < 0$

$$\text{Solve } \frac{d\vec{r}}{dt} = \begin{bmatrix} -2 & -1 \\ 2 & -5 \end{bmatrix} \vec{r}.$$

In this case the origin is called an **nodal sink**.

CASE 3: $\lambda_1 > 0 > \lambda_2$

$$\text{Solve } \frac{d\vec{r}}{dt} = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix} \vec{r}.$$

In this case the origin is called a **saddle**.

You might be wondering what happens when the roots are not distinct, or one of them is zero. You can try experimenting in **PPLANE**. We will talk about them in more details next week. The other case remaining is the case when the roots are complex numbers. We will deal with them next class, but here is a refresher/introduction to Complex number before that.

§C. Introduction to Complex Numbers

How do we solve the equation $x^2 + 1 = 0$? Clearly, there is no real number that satisfies the equation. However, if we allow ourselves to expand the criteria for being a ‘number’, we can assume that $\pm\sqrt{-1}$ would be acceptable solutions to this equation. But since $\sqrt{-1}$ does not belong to the set of \mathbb{R} real numbers, we would need pursue a new way of thinking about numbers to understand $\sqrt{-1}$.

Let’s give this new number a name. We will denote $\sqrt{-1}$ by i . Can we use i to solve quadratic equations? Consider the equation $at^2 + bt + c = 0$. Quadratic formula says:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So if $b^2 - 4ac < 0$, let’s say $b^2 - 4ac = -m^2$, we can rewrite above formula as

$$t = \frac{-b \pm \sqrt{-m^2}}{2a} = \frac{-b \pm mi}{2a}$$

Thus if we expand the collection of numbers to include i and in particular numbers of the form $x + iy$ (where $x, y \in \mathbb{R}$), every quadratic equation becomes solvable regardless of the sign of the discriminant. This new collection (set) of numbers is called the set of **Complex Numbers**. If $z = x + iy$ is a complex number, we say that x is the “Real” part of z , and it is denoted as $x = \Re(z)$. Similarly we say y is the “Imaginary” part of z , and we denote it as $y = \Im(z)$ (although there is nothing imaginary about it!).

How do we ‘draw’ complex numbers? Since the \mathbb{R} real number line is evidently not big enough to contain the Complex numbers, we will need to go outside the line to ‘draw’ such a number. We will also need to redefine our rules of arithmetic to allow addition or multiplication of Complex numbers.

We observe that if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

This gives us an idea to represent the complex number $x + iy$ as the point (x, y) (or the vector $\langle x, y \rangle$) in the 2D plane where the X -axis (i.e. the $y = 0$ line) corresponds to the Real number line. We check that the sum of two complex numbers indeed follows the parallelogram law of adding vectors (as we are just adding the two components separately).

What about multiplying two complex numbers? Check that

$$z_1 z_2 = x_1 x_2 + i^2 y_1 y_2 + i x_1 y_2 + i y_1 x_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

This does not look familiar. However, if we change to polar coordinates, this becomes much simpler looking! Recall that the polar coordinate of a point (x, y) is given by (r, θ) where

$$x = r \cos \theta$$

$$y = r \sin \theta$$

Using the new coordinate system, if $z_1 \equiv (r_1, \theta_1)$ and $z_2 \equiv (r_2, \theta_2)$ then the product simplifies due to trigonometric identities and becomes

$$z_1 z_2 \equiv (r_1 r_2, \theta_1 + \theta_2)$$

As a consequence, if $z \equiv (r, \theta)$, then $z^2 \equiv (r^2, 2\theta)$. And similarly, if $z \equiv (r, \theta)$, then $\sqrt{z} \equiv (\sqrt{r}, \frac{\theta}{2})$.

Using this new arithmetic, we observe that since the real number -1 corresponds to the point $(1, \pi)$ in polar coordinates, we must have

$$i = \sqrt{-1} \equiv (1, \pi/2)$$

So we can place the new mystery number i at a distance 1 from the origin in the positive Y -axis direction which is consistent with the idea of associating $z = x + iy$ to the point (x, y) (in Cartesian coordinates) in the plane.

Definition 3.1

The modulus of a complex number $z = x + iy$ is $|z| = \sqrt{x^2 + y^2}$.

The conjugate of a complex number $z = x + iy$ is $\bar{z} = x - iy$.

We will use the following result from Algebra without proof.

Theorem C.1

If the roots of a quadratic polynomial with real coefficients are not real, then they are conjugate complex numbers.

Question 2.

Express the following numbers in the form $a + ib$.

(a) $(1 + 2i)(1 - 2i)$

(b) $\frac{3}{i}$

(c) $(1 + 5i)(i - 2)$

Theorem C.2: Euler's Formula

Euler's formula, named after Leonhard Euler, is a mathematical formula in complex analysis that establishes the fundamental relationship between the trigonometric functions and the complex exponential function. Euler's formula states that for any real number θ :

$$e^{i\theta} = \cos \theta + i \sin \theta$$

When $x = \pi$, Euler's formula evaluates to Euler's identity, the "greatest" Math identity according to some mathematicians:

$$e^{i\pi} + 1 = 0$$

■ Question 3.

Use Euler's formula to show that $(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)$.