Math 2208: Ordinary Differential Equations

Lecture 15 Worksheet

Spring 2020 | **Subhadip Chowdhury** | **Apr 6**

Title: The Trace-Determinant Plane

Summary: We'll summarize all the possible qualitative behavior one can get with a 2×2 linear system of ODEs into one big picture! Corresponding Book Chapter - 3.7.

§A. Summarizing the possibilities

Recall that a system of linear ODEs with associated matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has the **characteristic polynomial**

$$p_{\mathbf{A}}(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}) = (a-\lambda)(d-\lambda) - bc = \lambda^2 - \lambda(a+d) + (ad-bc) = \lambda^2 - \mathrm{tr}(\mathbf{A})\lambda + \det(\mathbf{A})$$

with roots

$$\lambda = \frac{\mathrm{tr}(\mathbf{A}) \pm \sqrt{\mathrm{tr}(\mathbf{A})^2 - 4\det(\mathbf{A})}}{2}$$

We will use the notation $\mathbf{T} = \mathrm{tr}(\mathbf{A})$ and $\mathbf{D} = \det(\mathbf{A})$ for convenience of writing.

Question 1. 1

Fill out the following table. For the last three columns find whether the given quantity is positive or negative or zero.

Eigenvalues	Type of Equilibrium	$\mathbf{T}$	$\mathbf{D}$	$\mathbf{T}^2 - 4\mathbf{D}$
$\lambda_i \in \mathbb{R}, \lambda_1 < \lambda_2 < 0$				
$\lambda_i \in \mathbb{R}, \lambda_1 > 0 > \lambda_2$				
$\lambda_i \in \mathbb{R}, \lambda_1 > \lambda_2 > 0$				
$\lambda_i \in \mathbb{C}, \mathfrak{Re}(\lambda_i) > 0$				
$\lambda_i \in \mathbb{C}, \mathfrak{Re}(\lambda_i) = 0$				
$\lambda_i \in \mathbb{C}, \mathfrak{Re}(\lambda_i) < 0$				

§B. The Trace-Determinant Plane

Above table shows that the condition on what kind of eigenvalues we will have depends on the sign of $\mathbf{T}$, $\mathbf{D}$ and the discriminant $\mathbf{T}^2 - 4\mathbf{D}$. Consider the following picture where we have drawn the $\mathbf{T}$-axis horizontally and the $\mathbf{D}$-axis vertically. We have also drawn the curve $\mathbf{T}^2 - 4\mathbf{D} = 0$, a parabola.

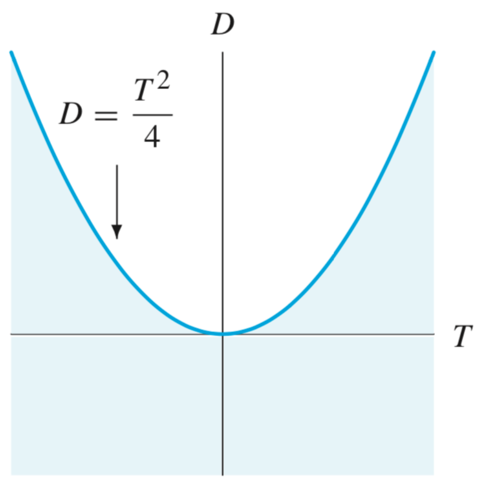

Figure 1: Shaded region corresponds to $\mathbf{T^2 - 4D > 0}$

This is known as the **Trace-Determinant Plane**. As the matrix **A** changes, it has different values of **T** and **D** and the linear system $\frac{d\vec{\mathbf{R}}}{dt} = \mathbf{A}\vec{\mathbf{R}}$ corresponding to that matrix will be located at a different location in $(\mathbf{T}, \mathbf{D})$-plane.

■ Question 2.

Find the regions in the picture of $(\mathbf{T}, \mathbf{D})$-plane that correspond to each of the six cases above.

■ Question 3.

What kind of phase portraits will exist in $(\mathbf{T}, \mathbf{D})$-plane along the **D** axis?

§C. The degenerate and the defective cases

We are missing a couple of more cases in our summary above: for example, what happens along the **T**-axis and what happens on the curve $\mathbf{T^2 = 4D}$. Open the applet on the following webpage:

`http://mathlets.org/mathlets/linear-phase-portraits-matrix-entry/`

Turn on the `Eigenvalue` option. Move your cursor around on the $(\mathbf{T}, \mathbf{D})$-plane and answer the questions below.

■ Question 4. (Zero Eigenvalue)

(a) Fix the **D** value to **0** and move the **T** slider. Check that matrices corresponding to points on the **T**-axis have at least one zero eigenvalue. Can you prove this mathematically?

(b) When $\mathbf{T > 0, D = 0}$, we call the equilibrium a **degenerate source**. Draw a representative phase portrait in your notebook. How is it different from a nodal source.

(c) The other possibility is $\mathbf{T < 0, D = 0}$, called a **degenerate sink**. Can you see the difference between the two phase portraits? Draw them in your notebook to make sure you memorize them. Note that every solution curve is a straight line solution in this case.

■ **Question 5. (Repeated Eigenvalue)**

(a) In terms of eigenvalues, $\lambda_1 = \lambda_2$ is the border line case between real distinct eigenvalues and complex conjugate eigenvalues. Justify this last statement by inspecting the eigenvalues corresponding to points on the curve $\mathbf{T}^2 = 4\mathbf{D}$.

(b) Check that if $\lambda_1 = \lambda_2$, then $\lambda_i = \frac{\mathbf{T}}{2} \in \mathbb{R}$.

(c) Again there are two cases: $\lambda_1 = \lambda_2 > \mathbf{0}$ and $\lambda_1 = \lambda_2 < \mathbf{0}$. The first case is called a **defective source** and the second one a **defective sink**. Which part of the parabola does each case correspond to? Make sure to draw them in your notebook to understand the difference.

(d) How many straight line solutions does the system have?

Note: The words 'degenerate' and 'defective' are not interchangeable. They have specific meaning that has to do with Matrix algebra. A matrix is 'Defective' if it does not have $\boldsymbol{n}$ linearly independent eigenvectors. Similarly a matrix is called 'Degenerate' if it is not invertible.

§D. Bifurcation in a family of system

■ **Question 6.**

(a) Suppose we have a family of system of ODEs where we keep $\mathbf{tr}(\mathbf{A}) = \mathbf{T}$ fixed at $\mathbf{T} = 2$ and gradually change $\mathbf{det}(\mathbf{A}) = \mathbf{D}$ from -2 to 2. This corresponds to moving along the straight line $\mathbf{T} = 2$ in the $(\mathbf{T}, \mathbf{D})$-plane. Use the applet to describe the changes in qualitative behavior along the path.

(b) Identify the points where the qualitative behavior of the system changes. These are the bifurcation points.

■ **Question 7.**

Consider the one-parameter family of linear system $\dfrac{d\vec{\mathbf{R}}}{dt} = \begin{bmatrix} a & -1 \\ 2 & 0 \end{bmatrix} \vec{\mathbf{R}}$ where the parameter $\boldsymbol{a}$ is a real number.

(a) Sketch the corresponding curve in the $(\mathbf{T}, \mathbf{D})$-plane.

(b) In a couple of sentences, discuss different types of behaviors exhibited by the system as $\boldsymbol{a}$ increases from -4 to 4.

(c) Identify the bifurcation values of $\boldsymbol{a}$.