MATH 221 - Differential Equations

Lecture 8 Worksheet

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Title: Integrating Factors

Summary: We will learn an analytical technique to solve *any* linear first order ODE that has a closed form solution.

Related Reading: Chapter 2.3

§A. Integrating Factor

Consider a linear ODE of the form $y' = \varphi(t)y + \psi(t)$. To use the technique of **Integrating Factors**, we will first rewrite it into the following form:

$$\frac{dy}{dt} + P(t)y = Q(t) \tag{1}$$

What's the idea?

Think about the product rule for differentiating the function $\mu(t)y(t)$.

$$\frac{d(\mu(t)y(t))}{dt} = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y(t)$$

So if we stare at both of the last two equations hard enough and long enough, we might think about rewriting the ODE from (1) as

$$\mu(t)\frac{dy}{dt} + \mu(t)P(t)y = \mu(t)Q(t) \tag{2}$$

whose left hand side 'sort of' looks like the product rule. So if we could find a function $\mu(t)$ such that

$$\mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y(t) = \mu(t)\frac{dy}{dt} + \mu(t)P(t)y,$$

we would be able to rewrite the ODE from (2) as

$$\frac{d(\mu(t)y(t))}{dt} = \mu(t)Q(t) \tag{3}$$

which can be easily solved as

$$\mu(t)y(t) = \int \mu(t)Q(t)dt \implies y(t) = \frac{1}{\mu(t)}\int \mu(t)Q(t)dt$$

So what's $\mu(t)$?

So our goal is to find a function $\mu(t)$ such that

$$\frac{d\mu}{dt} = \mu(t)P(t)$$

■ Question 1.

Find $\mu(t)$.

Theorem A.1

We call $\mu(t)$ the **integrating factor**. With the formula for $\mu(t)$ you obtained above, the complete formula for $y(t)$ is given by

$$y(t) = \frac{1}{e^{\int P(t)dt}} \int \left(Q(t)e^{\int P(t)dt}\right) dt$$

■ Question 2.

Solve $y' = -2ty + 4e^{-t^2}$.

■ Question 3.

Solve the Initial Value Problem using Integrating Factor method.

$$(t^2+1)\frac{dy}{dt} + 3ty = 6t, \quad y(0) = 3$$

[Hint: You will need to rewrite the ODE in the required form first.]

Look in the textbook for more examples.

§B. **Another Mixing Problem**

■ Question 4.

In this problem, we are going to try to model the quantity of salt in a **1500** gallon water tank. Assume that at time $t = 0$ hr, the tank contains **600** gallons of water with **5** lbs of salt dissolved in it. Water enters the tank at a rate of **9** gal/hr and the water entering the tank has a salt concentration of **0.1** lbs/gal. Water exits the tank at a rate of **6** gal/hr. So the volume of water in the tank changes over time!

The main assumption that we'll be using is that the concentration of the salt in the water is uniform throughout the tank. So we will assume that at any given moment, the water that is leaving the tank has the same salt concentration as in the tank. Suppose $Q(t)$ lb gives the amount of salt dissolved in the water in the tank at any time t hr. Set up an Initial Value Problem that, when solved, will give us an expression for $Q(t)$. Solve it using integrating factors and find the interval of definition. Use calculators as necessary.