Math 2208: Ordinary Differential Equations

Lecture 14 Worksheet

Spring 2020 | **Subhadip Chowdhury** | **Apr 1**

Title: Linear Systems with Complex Eigenvalues

Summary: We'll continue to explore the various scenarios that occur with linear systems of ODEs; this time dealing with those that possess complex eigenvalues. Corresponding Book Chapter - 3.4.

§A. Two Complex Eigenvalues

Recall that the general solution to the ODE

$$\frac{d\vec{r}}{dt} = \mathbf{A}\vec{r}$$

can be written as

$$\vec{r}(t) = k_1 e^{\lambda_1 t}\vec{v}_1 + k_2 e^{\lambda_2 t}\vec{v}_2$$

where λ_is are the eigenvalues of $\mathbf{A}$ and $\vec{v}_i$ are the eigenvectors corresponding to λ_i. What happens if λ_1 and λ_2 are $\mathbb{C}$omplex numbers?

Question 1.

Consider the ODE $\frac{d\vec{r}}{dt} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\vec{r} = \mathbf{A}\vec{r}$.

(a) Find the eigenvalues and corresponding eigenvectors of $\mathbf{A}$. By an exercise from the last worksheet, we know they will be of the form $\alpha \pm i\beta$ for some real numbers α and β.

Solution. Eigenvalues are $1 \pm 2i$. Eigenvectors are $\begin{bmatrix} \pm i \\ 1 \end{bmatrix}$. ∎

(b) Let $\lambda = \alpha + i\beta$ and name the corresponding eigenvector $\vec{v}$. Then $\vec{r}_0(t) = e^{(\alpha+i\beta)t}\vec{v}$ is a (complex-valued) solution to our ODE. Use Euler's formula to rewrite your solution in the form

$$\vec{r}_0(t) = \vec{r}_{\mathfrak{Re}}(t) + i\vec{r}_{\mathfrak{Im}}(t)$$

where $\vec{r}_{\mathfrak{Re}}(t)$ and $\vec{r}_{\mathfrak{Im}}(t)$ are real-valued functions of t.

Solution.

$$\begin{aligned} e^{(1+2i)t}\begin{bmatrix} i \\ 1 \end{bmatrix} &= e^t\left(\cos(2t) + i\sin(2t)\right)\begin{bmatrix} i \\ 1 \end{bmatrix} \\ &= e^t\begin{bmatrix} -\sin(2t) + i\cos(2t) \\ \cos(2t) + i\sin(2t) \end{bmatrix} \\ &= e^t\begin{bmatrix} -\sin(2t) \\ \cos(2t) \end{bmatrix} + ie^t\begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix} \end{aligned}$$

∎

(c) Check that $\vec{r}_{\mathfrak{Re}}(t)$ and $\vec{r}_{\mathfrak{Im}}(t)$ are also (real-valued) solutions to the ODE.

Our goal is to express the general solution to the ODE in terms of real-valued functions. Recall from assignment 6 that the solution space is two-dimensional. Hence the expression

$$\vec{r}(t) = k_1\vec{r}_{\mathfrak{Re}}(t) + k_2\vec{r}_{\mathfrak{Im}}(t)$$

will represent a general real-valued solution as long as $\vec{r}_{\mathfrak{Re}}(t)$ and $\vec{r}_{\mathfrak{Im}}(t)$ are linearly independent for all t (in which case, they form a basis). So let's prove the linear independence of $\vec{r}_{\mathfrak{Re}}(t)$ and $\vec{r}_{\mathfrak{Im}}(t)$.

Answer the following two questions for a general matrix **A**.

Question 2.

Suppose a matrix **A** with real entries has the complex eigenvalue $\lambda = \alpha + i\beta, \beta \neq 0$. Let $\vec{v}$ be an eigenvector for λ and write $\vec{v} = \vec{v}_1 + i\vec{v}_2$, where $\vec{v}_1 = \langle x_1, y_1\rangle$ and $\vec{v}_2 = \langle x_2, y_2\rangle$ have real entries. Show that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.
[Hint: Suppose they are not linearly independent. Then $\langle x_2, y_2\rangle = k\langle x_1, y_1\rangle$ for some constant k. Then $\vec{v} = (1+ik)\vec{v}_1$. Then use the fact that $\vec{v}$ is an eigenvector of **A** and that $\mathbf{A}\vec{v}_1$ contains no imaginary part.]

Solution.

$$\mathbf{A}\vec{v} = \lambda\vec{v} \implies (1+ik)\mathbf{A}\vec{v}_1 = \lambda(1+ik)\vec{v}_1 \implies \mathbf{A}\vec{v}_1 = \lambda\vec{v}_1$$

This is a contradiction since the LHS $\in \mathbb{R}^2$ but the RHS $\notin \mathbb{R}^2$. ■

Question 3.

Let $\lambda = \alpha + i\beta$ (where $\beta \neq 0$) be an eigenvalue of **A** with associated eigenvector $\vec{v} = \vec{v}_1 + i\vec{v}_2$, where $\vec{v}_1$ and $\vec{v}_2$ are real valued vectors. Let $\vec{r}_{\mathfrak{Re}}(t)$ and $\vec{r}_{\mathfrak{Im}}(t)$ be the real and imaginary parts of the complex valued solution

$$\vec{r}_0(t) = e^{\lambda t}\vec{v}.$$

Prove that $\vec{r}_{\mathfrak{Re}}(t)$ and $\vec{r}_{\mathfrak{Im}}(t)$ are linearly independent for all t.
[Hint: Observe that $\vec{r}_{\mathfrak{Re}}(0) = \vec{v}_1$ and $\vec{r}_{\mathfrak{Im}}(0) = \vec{v}_2$. Then use a theorem from assignment 6.]

Solution. We have

$$\vec{r}_0(t) = e^{\lambda t}\vec{v} = \vec{r}_{\mathfrak{Re}}(t) + i\vec{r}_{\mathfrak{Im}}(t)$$

Recall the following theorem from assignment 6:

Theorem A.1

If $\vec{r}_{\mathfrak{Re}}(t)$ and $\vec{r}_{\mathfrak{Im}}(t)$ are both solutions to the system and if $\vec{r}_{\mathfrak{Re}}(0)$ and $\vec{r}_{\mathfrak{Im}}(0)$ are linearly independent, then $\vec{r}_{\mathfrak{Re}}(t)$ and $\vec{r}_{\mathfrak{Im}}(t)$ are linearly independent.

Observe that

$$\frac{d\vec{r}_0}{dt} = \mathbf{A}\vec{r}_0 \implies \frac{d\vec{r}_{\mathfrak{Re}}}{dt} + i\frac{d\vec{r}_{\mathfrak{Im}}}{dt} = \mathbf{A}\vec{r}_{\mathfrak{Re}} + i\mathbf{A}\vec{r}_{\mathfrak{Im}}$$

Since $\vec{r}_{\mathfrak{Re}}$ and $\vec{r}_{\mathfrak{Im}}$ are both real-valued, comparing the real and imaginary parts of both sides of above equality we get

$$\frac{d\vec{r}_{\mathfrak{Re}}}{dt} = \mathbf{A}\vec{r}_{\mathfrak{Re}}$$

and

$$\frac{d\vec{r}_{\mathfrak{Im}}}{dt} = \mathbf{A}\vec{r}_{\mathfrak{Im}}$$

In other words, $\vec{r}_{\mathfrak{Re}}(t)$ and $\vec{r}_{\mathfrak{Im}}(t)$ are both solutions to the system.

Next, note that $\vec{r}_{\mathfrak{Re}}(0) = \vec{v}_1$ and $\vec{r}_{\mathfrak{Im}}(0) = \vec{v}_2$. Question 2 above concludes that $\vec{v}_1$ and $\vec{v}_2$ are linearly independent for any eigenvalue $\vec{v}_1 + i\vec{v}_2$ of a complex eigenvalue λ. Hence $\vec{r}_{\mathfrak{Re}}(0)$ and $\vec{r}_{\mathfrak{Im}}(0)$ are linearly independent.

Thus by theorem A.1, $\vec{r}_{\mathfrak{Re}}(t)$ and $\vec{r}_{\mathfrak{Im}}(t)$ are linearly independent for all t.

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§B. Classification of Solutions in case of Complex Eigenvalues

Now write

$$e^{\lambda t}\vec{v} = e^{\alpha + i\beta}(\vec{v}_1 + i\vec{v}_2)$$

and simplify using the Euler's formula.

Solution. $e^{\alpha+i\beta}(\vec{v}_1 + i\vec{v}_2) = e^{\alpha}(\cos\beta + i\sin\beta)(\vec{v}_1 + i\vec{v}_2) = e^{\alpha}\left[(\cos\beta\vec{v}_1 - \sin\beta\vec{v}_2) + i(\sin\beta\vec{v}_1 + \cos\beta\vec{v}_2)\right]$ ■

■ Question 4.

Can you justify the following statement:

> The effect of the exponential term on solutions depends on the sign of α whereas β determines the periodic nature of the solutions.

We are going to discuss the nature of the solution curves in the following three cases:

a) Case 1: $\alpha < 0$ **(Spiral Sink)**. b) Case 2: $\alpha > 0$ **(Spiral Source)**. c) Case 3: $\alpha = 0$ **(Center)**.

In each case,

(i) Come up with your own examples of 2×2 matrices $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ that have complex eigenvalues and the corresponding ODE for each of the three cases named above.

Hint: It may be helpful to recall that the eigenvalues are given by the formula

$$\lambda = \alpha \pm i\beta = \frac{\mathrm{tr}(\mathbf{A}) \pm \sqrt{\mathrm{tr}(\mathbf{A})^2 - 4\det(\mathbf{A})}}{2}$$

(ii) Find the period and frequency of the solution curves using the analytical formula.

Hint: What is period of $\cos(\beta t)$?

(iii) Determine the direction of the oscillations in the phase plane (do the solutions go clockwise or counterclockwise around the origin?)

(iv) Are there any straight line solutions?

(v) Use PPLANE to sketch the phase portrait for each case. Also use PPLANE to draw the $x(t)$ vs t and $y(t)$ vs. t graphs for some initial condition and check that they are consistent with your answers above. Do you see a justification for the names of the equilibria.