Math 2208: Ordinary Differential Equations

Lecture 4 Worksheet

Spring 2020 **Subhadip Chowdhury** **Feb 3**

Title: Existence and Uniqueness of Solutions

Summary: We will investigate the conditions which guarantee existence and/or uniqueness of solutions to the initial value problem $y' = f(t,y), y(t_0) = y_0$.

§A. Do Problems Always Have Solutions?

Think about the equation $2x^5 - 10x + 5 = 0$. Does it have a solution? How do we know? Discuss!

§B. Existence and Uniqueness Of Particular Solutions

The main questions we would like to be able to answer when analyzing IVPs are:

1. **Existence:** Does the differential equation possess solutions which pass through the given initial condition? and

2. **Uniqueness:** If such a solution does exist, can we be certain that it is the only one?

Luckily, there's a theorem that answers these questions for us.

Theorem B.1: Existence of a unique solution

Let $\mathcal{R}$ be a rectangular region in the ty-plane defined by

$$\mathcal{R} = \{(t,y) \mid a \le t \le b, c \le y \le d\}$$

that contains the point (t_0, y_0) in its interior. IF $f(t,y)$ and $\partial f/\partial y$ are continuous on $\mathcal{R}$, THEN **there exists** some interval $\mathbf{I_0}$ defined as $(t_0 - \epsilon, t_0 + \epsilon)$, for some $\epsilon > 0$, contained in (a,b) and a **unique** function $y(t)$ defined on $\mathbf{I_0}$ that is a solution of the initial value problem $y' = f(t,y), \quad y(t_0) = y_0$.

§C. Theorems Have Hypotheses and Conclusions

The existence and uniqueness theorem is actually two different theorems with different hypotheses and conclusions.

1. **Existence:** IF $f(t,y)$ is continuous on a square containing (t_0, y_0), THEN there exists a solution on an interval $(t_0 - \epsilon, t_0 + \epsilon)$ for some $\epsilon > 0$.

2. **Uniqueness:** IF $f(t,y)$ and $\frac{\partial f}{\partial y}$ are **both** continuous on a square containing (t_0, y_0), THEN there exists a unique solution on an interval $(t_0 - \epsilon, t_0 + \epsilon)$ for some $\epsilon > 0$.

■ Question 1.

(a) Show that the initial value problem

$$\frac{dy}{dt} = t\sqrt{y}, \quad y(0) = 0$$

has at least two solutions since the equilibrium solution $y(t) = 0$ and the solution $y(t) = \frac{1}{16}t^4$ both satisfy the IVP.

(b) Using the Existence and Uniqueness Theorem, we look at the functions $f(t,y) = t\sqrt{y}$ and $\frac{\partial f}{\partial y} = \frac{t}{2\sqrt{y}}$. At the origin $(0,0)$ what can we say about $f(t,y)$ and $f_y(t,y)$?

(c) What can we say about $f(t,y)$ and $f_y(t,y)$ at $(2,4)$? What does this imply about existence and uniqueness of the corresponding IVP

$$y' = t\sqrt{y}, \quad y(2) = 4?$$

■ Question 2.

Does the initial value problem

$$y' = \frac{t}{t^2 + y^2}, \quad y(-1) = 3$$

have a unique solution in a neighborhood around $t = -1$?

§D. Implications of the Existence & Uniqueness Theorem

Extendability

■ Question 3.

Consider the IVP $\frac{dy}{dt} = 1 + y^2, \quad y(0) = 0$. What is $y(2)$? Discuss!

Role of equilibrium solutions

Lemma D.1: When solution curves do not intersect

IF $y' = f(t,y)$ is a first-order differential equation with f and $\partial f/\partial y$ both continuous for all values of t and y in some region $\mathbf{S}$ in the ty-plane, THEN inside the region $\mathbf{S}$, the solution curves of the differential equation will form a non-intersecting space-filling family of curves.

■ Question 4.

(a) Show that $y_1(t) = t^2$ and $y_2(t) = t^2 + 1$ are both solutions to

$$\frac{dy}{dt} = -y^2 + y + 2yt^2 + 2t - t^2 - t^4$$

(b) Explain why if $y(t)$ is a solution to the differential equation in part (a) and if $0 < y(0) < 1$, then $t^2 < y(t) < t^2 + 1$ for all t.

Uniqueness and Numerical Approximation

■ Question 5.

(a) Use the Euler's method in `dfield` to plot the solution to the IVP

$$\frac{dy}{dt} = e^t \sin y, \quad y(0) = 5$$

(b) Check that the constant function $y(t) = n\pi$ is a solution to the ODE $y' = e^t \sin y$ for any integer n.

(c) Explain why we should not believe the numerical result from part (a).