Math 2208: Ordinary Differential Equations

Lecture 12 Worksheet

Spring 2020 **Subhadip Chowdhury** **Mar 4**

Title: Autonomous Linear System of ODEs

Summary: We will begin to analyze systems for ODEs which have the form $\frac{d\vec{r}}{dt} = \mathbf{A}\vec{r}$.

§A. Linear Systems and Matrix Notation

The **dimension** of a system of ODEs is equal to the number of dependent variables in the system. A two-dimensional linear system of ODE has the form

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

where $a, b, c,$ and d are some constants. Recall that we can rewrite this system in the form $\frac{d\vec{r}}{dt} = \vec{\mathbf{F}}(\vec{r})$, where $\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ is a column vector and $\vec{\mathbf{F}}$ is the vector field

$$\vec{\mathbf{F}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

If we think of $\vec{\mathbf{F}}$ as a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$, we can then write

$$\frac{d\vec{r}}{dt} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}\vec{r}$$

where $\mathbf{A}$ is the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{A}\vec{r}$ is regular matrix multiplication.

Theorem A.1

If $\mathbf{A}$ is a matrix with nonzero determinant, then the only equilibrium point for the linear system of ODEs $\frac{d\vec{r}}{dt} = \mathbf{A}\vec{r}$ is the origin.

§B. The Linearity Principle

The following two properties are easy to check using the matrix notation.

- Given a solution $\vec{r}(t)$ of the system, $k\vec{r}(t)$ is also a solution for any real number k.
- If $\vec{r}_1(t)$ and $\vec{r}_2(t)$ are solutions of the system, then $\vec{r}_1(t) + \vec{r}_2(t)$ is also a solution.

Question 1.

Consider the linear system $\frac{d\vec{r}}{dt} = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix}\vec{r}$.

(a) Show that it has $\vec{r}_1(t) = \begin{bmatrix} 2e^{-3t} \\ e^{-3t} \end{bmatrix}$ and $\vec{r}_2(t) = \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix}$ as solutions.

(b) Is $\vec{r}_3(t) = 5\vec{r}_1(t) - 3\vec{r}_2(t)$ also a solution?

(c) Is there a solution to the system that solves the following initial value problem?

$$\frac{d\vec{r}}{dt} = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix}\vec{r}, \qquad \vec{r}(0) = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

Theorem B.1: General Solution to 2D Linear System of ODEs

Suppose $\vec{r}_1(t)$ and $\vec{r}_2(t)$ are solutions of the linear equations $\frac{d\vec{r}}{dt} = \mathbf{A}\vec{r}$, where $\mathbf{A}$ is a 2×2 matrix. If $\vec{r}_1(0)$ and $\vec{r}_2(0)$ are linearly independent, then for any initial condition $\vec{r}(0) = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$, we can find a specific solution to the initial value problem

$$\frac{d\vec{r}}{dt} = \mathbf{A}\vec{r}, \qquad \vec{r}(0) = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$$

which looks like $k_1\vec{r}_1(t) + k_2\vec{r}_2(t)$, some linear combination of the two solutions $\vec{r}_1(t)$ and $\vec{r}_2(t)$.

§C. Straight Line Solutions

Consider the system from question 2. Observe that

$$\vec{r}_1(t) = e^{-3t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } \vec{r}_2(t) = e^{2t}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Question 2.

Do you notice anything special about the column vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ in relationship to the matrix $\mathbf{A} = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix}$? What happens if you multiply each vector by $\mathbf{A}$?

Question 3.

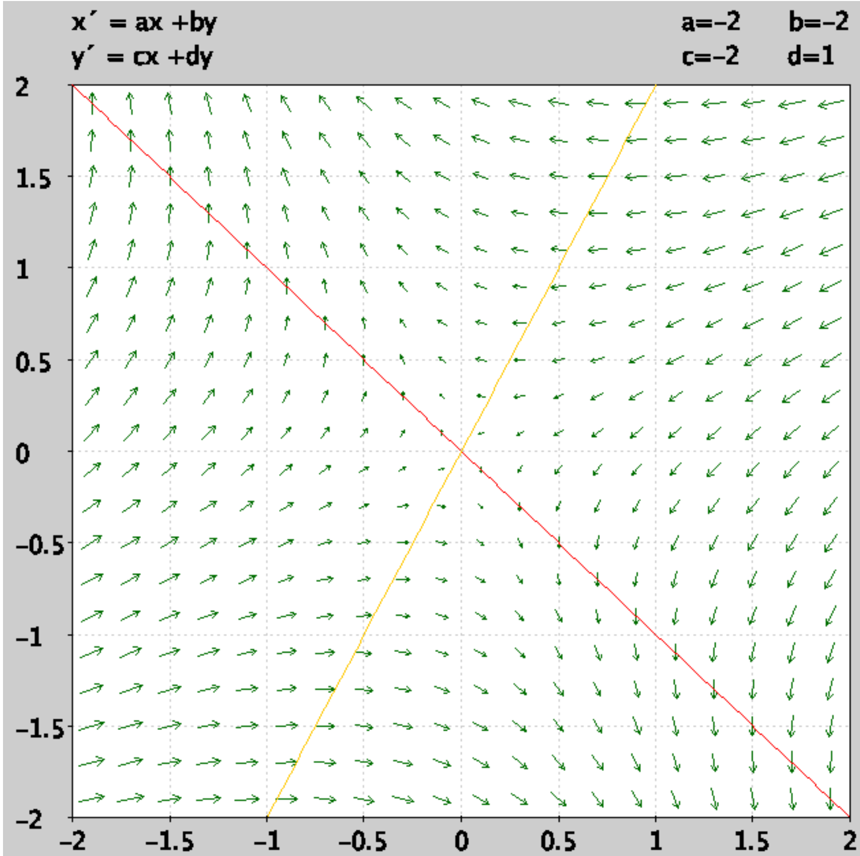
Consider the direction field for the system below.

(a) What do you think the two straight lines represent?

(b) Can you identify the solution curves $\vec{r}_1(t)$ and $\vec{r}_2(t)$ in it?

(c) For those solution curves, does it matter what your initial condition is?

(d) Does one of the solutions seem more "attractive" than the other?



(e) Describe how other solutions are behaving.

(f) Use the general formula of the solution (in terms of some arbitrary constants k_1 and k_2) to find $\frac{dy}{dt}$ and $\frac{dx}{dt}$ and use that to calculate $\frac{dy}{dx}$. Now let $t \to \infty$ and find the limit. Does it depend on whether or not $k_2 = 0$? Is this consistent with your observation about the behavior of the solutions?

(g) What about $t \to -\infty$, (i.e. in the reverse direction of the arrows)? Is that consistent with your observation about the behavior of the solutions?

Question 4.

Consider the system $\frac{d\vec{r}}{dt} = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}\vec{r}$. Find the eigenvalues λ_1 and λ_2 and the corresponding eigenvectors $\vec{v}_1, \vec{v}_2$ of the matrix $\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}$.

Show that the general solution can be written as $\vec{r}(t) = k_1 e^{\lambda_1 t}\vec{v}_1 + k_2 e^{\lambda_2 t}\vec{v}_2$ and confirm that it is actually a solution of $\frac{d\vec{r}}{dt} = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}\vec{r}$.