MATH 221 - Differential Equations

Lecture 4 Worksheet

Fall 2020 **Subhadip Chowdhury** **Aug 26**

Title: Separation of Variables and EUT

Summary: First we will learn an analytical technique for solving separable ODEs. Then we will investigate the conditions which guarantee existence and/or uniqueness of solutions to the initial value problem $y' = f(t,y), y(t_0) = y_0$.

Related Reading: Chapter 2.2, 1.2

---

§A. Motivating Example

Suppose an ant is moving along a curve in the plane such that for every point (x,y) on the curve, the tangent line to the curve is perpendicular to the line joining (x,y) to the origin $(0,0)$. What does the path of the ant look like? Construct a ODE that gives the equation of the path. We will see below how to analytically solve it.

[Hint: Check question 5 from last worksheet.]

§B. Separable Differential Equations

Definition B.1

A differential equation is called separable if it can be written in the form

$$\frac{dy}{dx} = f(x)g(y).$$

Note that either of the function can be equal to $\mathbf{1}$.

§C. Algorithm for solving a Separable DE

Step 1. Write the ODE in normal form and factor the right hand side of the ODE as a product $f(x)g(y)$.

Step 2. The values of y for which $g(y) = 0$ are called equilibrium solutions. Make a list of these.

Step 3. Separate the 'x's and the 'y's to two different sides and rewrite the ODE as an equality of differentials. Then integrate both sides. Your answer may be an implicit curve in x and y.

Step 4. If an initial condition is given, use it to find the constant of integration.

Example C.1

Here is an worked out example:

Consider the ODE

$$x^2y' = e^y$$

We can rewrite this in the form $\frac{dy}{dx} = \frac{1}{x^2} \times e^y$. So this is separable. Separating x and y to two different sides, we get the equality of differentials:

$$e^{-y} dy = \frac{dx}{x^2}$$

Integrating both sides, we get

$$-e^{-y} = -\frac{1}{x} + c$$

We can rearrange this to get $e^{-y} = \frac{1}{x} - c$ or equivalently, $y = -\ln\left(\frac{1}{x} - c\right)$ as an explicit solution, where c can be any arbitrary constant. Note that there is no equilibrium solution since e^y is never zero.

Question 1.

Solve the other separable DEs from the first question in the quiz. Don't forget the '+c' term when integrating. Make sure to review integration formula and techniques if you are having any difficulty.

Question 2.

Solve the ODE

$$y^2 + \left(\frac{dy}{dx}\right)^2 = 1$$

Question 3.

Consider the logistic population model $\frac{dP}{dt} = \frac{k}{N} P(N - P)$, $P(0) = P_0$.

(a) Identify the dependent and independent variables and the parameters.

(b) What are the equilibrium solutions? Why do we call them 'equilibrium'?

(c) Obtain the solution by separation of variables. You will need to use the partial fraction integration technique.

§D. Do Problems Always Have Solutions?

Think about the equation $2x^5 - 10x + 5 = 0$. Does it have a solution? How do we know? Discuss!

§E. Existence and Uniqueness Of Particular Solutions

The main questions we would like to be able to answer when analyzing ODEs are:

(i) **Existence:** Does the differential equation $\frac{dy}{dx} = f(x, y)$ possess solutions? Do any of the solution curves pass through A given point (x_0, y_0)? and

(ii) **Uniqueness:** If such a solution does exist, can we be certain that it is the only one?

Question 4.

Solve the initial value problem

$$\frac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0$$

Show that there is at least one solution curve passing through $(0,0)$ that is nontrivial, i.e. other than the equilibrium solution $y(x) = 0$.

Luckily, there's a theorem that answers the above two questions for us.

Theorem E.1: Existence of a unique solution

Consider the IVP

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

Let $\mathcal{R}$ be a rectangular region in the xy-plane defined by

$$\mathcal{R} = \{(x,y) \mid a \le x \le b, c \le y \le d\}$$

that contains the point (x_0, y_0) in its interior. IF $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous on $\mathcal{R}$, THEN **there exists** some interval $\mathbf{I}_0$ defined as $(x_0 - \epsilon, x_0 + \epsilon)$, for some $\epsilon > 0$, contained in (a,b) and a **unique** function $y(x)$ defined on $\mathbf{I}_0$ that is a solution of the initial value problem $y' = f(x,y), \quad y(x_0) = y_0$.

The statement of this theorem might be a bit confusing for you, but you don't need to memorize it. For most purposes it says that, as long as $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous at a given point, everything "nice" happens i.e. we get a unique solution curve passing through the point. We will focus on ODEs that are "nice" in this sense, and try to understand what the consequences of the theorem are.

Digression

What is $\frac{\partial f}{\partial y}$? It's called the **partial derivative** of f with respect to y. To calculate it, simply take the derivative of $f(x,y)$ with respect to y but treat x as a constant.

Question 5.

Does the initial value problem

$$y' = \frac{x}{x^2 + y^2}, \quad y(-1) = 3$$

have a unique solution in a neighborhood around $x = -1$?

[Hint: What can you say about the continuity of $f(x,y)$ and $\partial f / \partial y$ near $(-1,3)$?]

§F. Implications of the Existence & Uniqueness Theorem

Question 6.

Suppose $y = \varphi(x)$ is a solution to the IVP $\frac{dy}{dx} = 1 + y^2, \quad y(0) = 0$. What is $\varphi(2)$? Discuss!

Lemma F.1: When solution curves do not intersect

IF $y' = f(x,y)$ is a first-order differential equation with f and $\partial f / \partial y$ both continuous for all values of x and y in some region $\mathbf{S}$ in the xy-plane, THEN inside the region $\mathbf{S}$,

- No two solution curves of the differential equation intersect each other.

- the family of solution curves together fill out all of **S**.

Question 7.

(a) Show that $y_1(x) = x^2$ and $y_2(x) = x^2 + 1$ are both solutions to

$$\frac{dy}{dx} = -y^2 + y + 2yx^2 + 2x - x^2 - x^4$$

(b) Suppose $y = \varphi(x)$ is a solution to the differential equation in part (a) such that $0 < \varphi(0) < 1$. Then explain why we must have $x^2 < \varphi(x) < x^2 + 1$ for all x.