MATH 221 - Differential Equations

Lecture 41 Worksheet

Fall 2020 | **Subhadip Chowdhury** | **Nov 20**

Title: Lorenz Equations

Summary: We will see an example of **chaos** in three dimensional ODE systems.

§A. Higher-Order Linear Systems

Consider a $n-$dimensional linear system of ODEs of the form $\vec{\mathbf{R}}'(t) = \mathbf{A}\vec{\mathbf{R}}(t)$ where $\mathbf{A}$ is a $n \times n$ matrix whose (i,j)th element is a_{ij} and $\vec{\mathbf{R}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ is a $n \times 1$ vector.

The strategy for finding solutions to the system $\vec{\mathbf{R}}'(t) = \mathbf{A}\vec{\mathbf{R}}(t)$ is the same as for systems of two equations. In particular, if λ is an eigenvalue of $\mathbf{A}$ with eigenvector $\vec{v}$ and $\vec{r}(t) = e^{\lambda t}\vec{v}$, we can still show that $\vec{r}(t)$ is a solution to our system.

It is easy to check that the **Linearity Principle** also holds in higher dimension. So a general solution can be found as a linear combination of linearly independent solutions.

Question 1.

□

Find the general solution to the system of ODEs

$$\begin{aligned} x' &= -5x - 8y - 2z \\ y' &= 5x + 12y + 4z \\ z' &= -11x - 19y - 5z \end{aligned}$$

You can use `WolframAlpha` to find the eigenvalues and eigenvectors of matrices. Type

```
Eigenvalues[{-5,-8,-2},{5,12,4},{-11,-19,-5}]
```

into the query field.

§B. **The Geometry of Solutions**

Over the semester, we classified all possible geometry of an equilibrium point for planar systems using the trace-determinant plane. Another way to classify equilibrium points is by their stability.

Definition B.1

If every solution that starts close to an equilibrium stays close to that equilibrium for all time, then the equilibrium solution is called **stable**. Otherwise, it is called **unstable**.

This definition is a bit vague but we can formalize this if needed. Here 'close' means within some neighborhood of bounded radius.

Question 2.

□

(a) Identify the types of equilibrium solutions in $\mathbb{R}^2$ that are stable according to the above definition from the list below:

Nodal source, Spiral source, Saddle, Nodal Sink, Spiral Sink, Center

(b) Fill in the blanks with either the word "**stable**" or "**unstable**":

If an equilibrium solution has eigenvalues with real parts that are non-positive, then the equilibrium solution is ________________.

If an equilibrium solution has at least one eigenvalue with a positive real part, then the equilibrium solution is ________________.

(c) It turns out, the conclusion you made in part (b) holds even for systems with more than two dependent variables. For example, if we have a $n \times n$ matrix $\mathbf{A}$ and a $n-$dimensional system $\vec{\mathbf{R}}' = \mathbf{A}\vec{\mathbf{R}}$, we can compute the eigenvalues of $\mathbf{A}$. By inspecting the real parts of the eigenvalues only, we can determine the stability. Does this make sense, yes or no? ________________.

Although the stability of equilibrium solutions for linear systems in more than two variables is easy to determine, the geometry is a bit more complicated. For a system in three dimension, the solution curves live in $\mathbb{R}^3$, and there is simply a lot more room to move around in three dimensions than in two dimensions! Note that the origin is still the **only** equilibrium solution for a non-degenerate system of linear differential equations in three variables.

§C. Three Dimensional Systems

We want to analyze the so-called Lorenz equations, which is a famous system of differential equations derived by Edward Lorenz when studying convection rolls in the atmosphere,

$$\begin{aligned} \frac{dx}{dt} &= \alpha(y-x) && (1) \\ \frac{dy}{dt} &= x(\rho-z)-y && (2) \\ \frac{dz}{dt} &= -\beta z + xy && (3) \end{aligned}$$

where α, β and ρ are real parameter values. Note that $(x,y,z) = (0,0,0) \equiv \mathbf{O}$ is an equilibrium solution to this ODE for all parameter values.

$$(x,y,z) = \left(\sqrt{\beta(\rho-1)} \equiv \mathbf{P}, \sqrt{\beta(\rho-1)}, \rho-1\right)$$

and

$$(x,y,z) = \left(-\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}, \rho-1\right) \equiv \mathbf{Q}$$

are equilibrium solutions if $\rho > 1$. There are no other equilibria.

Linearization in three dimensions

■ Question 3. □

(a) Write down the Jacobian matrix in terms of x, y, z. You don't have to evaluate the Jacobian at any equilibrium solution.

(b) Let $\alpha = 10$, $\rho = 5$ and $\beta = 8/3$. Download and open the file `solvelorenz.ipynb`. This file computes the eigenvalues for the Jacobian at each of the three equilibrium points $\mathbf{O}, \mathbf{P}$ and $\mathbf{Q}$.

Determine the type and stability of each of the three equilibrium points.

Numerical Analysis

(c) The Jupyter notebook also computes a solution to the Lorenz equations from $t = 0$ to 100. The green dot is the initial value, and the red dot is the solution at $t = 100$.

(i) Consider a solution curve with the initial value $x(0) = 1$, $y(0) = 0, z(0) = 0$. What is (x, y, z) at $t = 100$? Is this consistent with your observation about the stability of the equilibrium points?

(ii) Consider a solution curve with the initial value $x(0) = 1.000001$, $y(0) = 0, z(0) = 0$. What is (x, y, z) at $t = 100$? Was there a significant change?

(d) Repeat all of the above parts with $\alpha = 10$, $\rho = 28$ and $\beta = 8/3$. Be sure to change the relevant values in the Jupyter notebook. Did you get anything surprising?

What you just observed is exactly what Edward Lorenz did in the 60s. It was something never seen, or even thought of before. The equations are completely deterministic (no statistical variation), yet the solutions can change drastically, even if the initial conditions are changed only slightly. All solutions live on that funny looking butterfly surface, which is now called a strange attractor. This is an example of **Chaos**. Weather systems have this same chaotic property (infinite sensitivity to initial conditions). What does this tell you about long term weather forecasts?