Math 2208: Ordinary Differential Equations

Lecture 23 Worksheet

Spring 2020 | **Subhadip Chowdhury** | **May 4**

Title: Miscellaneous Topics

Summary: We are going to work on some practice problems to review the semester.

§A. Two Dimensional Linear system of First Order ODEs

Question 1. (Straight Line Solutions)

Consider the system of ODEs

$$\frac{d\vec{r}}{dt} = \mathbf{A}\vec{r}, \qquad \vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

where $\mathbf{A}$ is a 2x2 real matrix with eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 2$ which have associated eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(a) Suppose that $\vec{r}_1(t) = \begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix}$ is a solution (but not an equilibrium solution) such that $\vec{r}_1(0)$ and $\vec{v}_2$ are linearly **independent**. Explain why

$$\lim_{t\to\infty} \frac{dy_1(t)}{dx_1(t)} = -2$$

(b) Suppose that $\vec{r}_2(t) = \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix}$ is a solution (but not an equilibrium solution) such that $\vec{r}_2(0)$ and $\vec{v}_2$ are linearly **dependent**. Explain why

$$\lim_{t\to\infty} \frac{dy_2(t)}{dx_2(t)} = 1$$

Note that you should answer this question without finding explicit formula for $\vec{r}_1(t)$ and $\vec{r}_2(t)$.

Question 2. (EUT)

Suppose that a first-order, two-variable ODE system has unique solutions for any initial value and one of its solution curve has parametric equation $x(t) = \sin(t)$, $y(t) = \sin(2t)$.

(a) Sketch this orbit in the xy phase plane. You can use Desmos or other graphing tools.

(b) Explain why the ODE system cannot be autonomous.

Question 3. (Bonus)

Solve the initial-value problem

$$\frac{d\vec{\mathrm{R}}}{dt} = \begin{bmatrix} \pi^2 & 37.4 \\ \sqrt{555} & 8.01234 \end{bmatrix}\vec{\mathrm{R}}, \quad \vec{\mathrm{R}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

§B. Second Order Autonomous Linear ODE

Question 4. (Boundary Value Problems)

We know that the solution of a second-order linear differential equation is uniquely determined by two initial conditions. In particular, the solution of the initial value problem

$$y'' + py' + qy = 0, \qquad y(a) = 0, y'(a) = 0 \tag{1}$$

has a unique solution $y(t) \equiv 0$. The situation is quite radically different for a problem such as

$$y'' + py' + qy = 0, \qquad y(a) = 0, y(b) = 0 \tag{2}$$

The difference between the problems in equations (1) and (2) is that in (2) the two conditions are imposed at two different points a and b with (say) $a < b$. In (2) we are to find a solution of the differential equation on the interval (a, b) that satisfies the conditions $y(a) = 0$ and $y(b) = 0$ at the endpoints of the interval. Such a problem is called an endpoint or **boundary value problem**.

Consider the Boundary Value Problem (BVP)

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y(\pi) = 0$$

(a) Let $\lambda = 3$. Show that the only solution to above BVP in this case is the trivial solution $y(t) = 0$ for all t.

(b) Show that if $\lambda = 0$, then the only solution to the BVP is the trivial solution.

(c) Show that if $\lambda < 0$, then the only solution to the BVP is the trivial solution.

(d) Show that if $\lambda > 0$, then the BVP has a non-trivial solution if and only if λ is of the form

$$\lambda = n^2, \qquad n = 1, 2, 3, \ldots$$

§C. Second Order Non-autonomous Linear ODE

Question 5. (Damped Harmonic Oscillator with Sinusoidal Forcing)

In most physical oscillator systems, we have nonzero damping. We wish to analyze the analogue of the resonance phenomenon for such a damped harmonic oscillator when subjected to sinusoidal forcing. The relevant equation is given by

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + y = \cos(\omega t) \tag{$\star$}$$

where

- $y(t)$ denotes the displacement at time t,
- $p > 0$ is the damping constant, and
- $\omega > 0$ is the forcing frequency.

(a) A bit of background on $\mathbb{C}$omplex numbers first.

(i) Suppose the polar form of the complex number $z = a + ib$ is given by $re^{i\theta}$. What's the polar form of $a - ib$ in terms of r and θ?

(ii) Use Euler's formula and calculate the real part of $(a - ib)e^{i\omega t}$ in two different ways to show that

$$a\cos(\omega t) + b\sin(\omega t) = r\cos(\omega t - \theta)$$

where (r, θ) is the polar coordinate of (a, b).

(b) Find a particular solution $y_0(t)$ to above differential equation (★) using the **method of undetermined coefficient**. Your solution will involve parameters ω and p. Once you find the particular solution, Change it into the polar form using the formula from part (a.ii).

(c) Explain why all general solutions of the differential equation (★), regardless of initial conditions, converge to $y_0(t)$ for large values of t.

[HINT: What can you say about the long term behavior of solutions to the **associated homogeneous equation**?]

(d) We conclude that if damping is present, in the long-term, every solution of (★) oscillates with same frequency and amplitude. Write the amplitude as a function of the parameters ω and p.

(e) We are going to fix p and let ω vary as a parameter. Let $r(\omega)$ denote the amplitude of the particular solution (from last part) when the system is forced at frequency ω. We say that **Practical resonance** occurs when $r(\omega)$ achieves its maximum as a function of ω.

 (i) Show that if $p < \sqrt{2}$, then practical resonance occurs at $\omega = \sqrt{1 - \frac{p^2}{2}}$.

 (ii) Show that if $p > \sqrt{2}$, then no practical resonance occurs.

§D. **Two dimensional Non-linear system of First Order ODEs**

Question 6. (A Modified Lotka-Volterra Model)

Recall the Lotka-Volterra equations we had used earlier to model predator-prey interaction. Here we are going to look at a modified version of them to understand population dynamics of two competing species, both of which sustain on the same finite resources. We assume that the reproduction rate per individual is adversely affected by high levels of its own species (i.e. a logistic growth) and the other species (i.e. an interaction term) with which it is in competition. Denoting the two populations by x and y, the competing species system can be modeled by the ODE system

$$\frac{dx}{dt} = ax(1-x) - bxy$$
$$\frac{dy}{dt} = cy(1-y) - dxy$$

where $a, b, c,$ and d are positive numbers. For this problem we will assume

$$a = 1, b = 2, c = 1, \text{ and } d = 3$$

(a) Find the equations of the nullclines and the coordinates of the equilibrium point(s).

(b) Use PPLANE to draw the phase plane and find the direction of arrows on the nullclines.

(c) Check that all the arrows in the direction field that start on the straight line $y = 2x$, lie on the straight line. Show this analytically by calculating $\dfrac{dy}{dx}$ when $y = 2x$.

(d) Using above observations, we can make an educated guess that one of the solution curves to this system is a straight line of the form $\vec{r}(t) = f(t)\begin{bmatrix}1\\2\end{bmatrix}$, where $f(t)$ is some function of t. So $x = f(t), y = 2f(t)$ satisfies the Lotka-Volterra system. Use the equations to find $f(t)$ as a function of t.

(e) Use PPLANE to draw the $f(t)$ vs. t graph of the solution curves that start at $(1,2)$ and $(0.1,0.2)$ respectively. What is the long term value? Is this consistent with your observation in the phase plane?

(f) Looking at the phase plane, What can you say about the long term fate of both species when the initial condition satisfies $\frac{y(0)}{x(0)} > 2$? What about $\frac{y(0)}{x(0)} < 2$? Is there anyway the two species can both exist peacefully in the long-term?

The line $y = 2x$ is called a **Separatrix**. It separates the quadrant into two regions each of which corresponds to a different long term behavior of the two populations.

Question 7. (Atlantic Coast Crab population)

The blue crab is native to the US Atlantic coast, but there is concern that the population is on the decline. The European green crab is an invasive species (recently introduced to the US in the ballast waters of ships), that competes with the blue crab. Assume that the interaction between the blue crab, x, and the green crab, y, is modeled by (up to some scaling):

$$\frac{dx}{dt} = x(100 - x) - 2xy$$
$$\frac{dy}{dt} = y(400 - 6y) - xy$$

(a) Use PPLANE to draw the phase portrait. You will need to choose an appropriate range to see all equilibrium points.

(b) Draw the nullclines and find the four equilibrium solutions using PPLANE. Check that three of these have non-negative coordinates, and hence represent biologically relevant values. According to the phase portrait, what is the long-term behavior of the two crab populations?

(c) Suppose an intervention effort is launched to preserve the blue crab by harvesting a proportion h of the green crabs, so the equations modeling the system become (this is up to scaling, so h can be any positive number):

$$\frac{dx}{dt} = x(100 - x) - 2xy$$
$$\frac{dy}{dt} = y(400 - 6y) - xy - hy$$

It would be nice if we could find harvesting values h so that the two species can co-exist. Try different (positive) values of the parameter h and observe that for some values of h, the fourth equilibrium solution can be found at a point in the first quadrant (it will represent co-existence of the species, since both populations will have positive values). We want to find the exact range of these h values.

We can do it in two ways:

- **Analytical Approach:** Solve for the four equilibrium point in terms of h. Find the conditions on h to ensure both coordinates are positive for the fourth equilibrium.
- **Qualitative Approach:** Don't solve for general formula. Argue using the picture of the nullclines. Observe that changing h changes only one of the nullclines. For what h does it intersect at a positive point?

(d) **(Optional)** Suppose the range you found in the previous problem is given as $h_1 < h < h_2$. Show that, if $h_1 < h < h_2$ then the equilibrium solution where both blue and green crab co-exist is the **only** stable equilibrium solution among the four possible options. Thus, no mater what the initial value (as long as it is positive), the green crab and blue crab will co-exist. (yay!)

Try to answer using the $(\mathbf{T}, \mathbf{D})$-plane. Don't calculate the eigenvalues. The algebra will be a little messy in this question. So be patient during your calculation. Use software as necessary.

Question 8. (Phase Portrait of Gradient Systems)

A system of differential equations is called a **gradient system** if there exists a continuously differentiable, scalar-valued function $\mathbf{V}(x, y)$ such that for every (x, y), we have

$$\frac{dx}{dt} = \frac{\partial \mathbf{V}}{\partial x}, \qquad \frac{dy}{dt} = \frac{\partial \mathbf{V}}{\partial y}$$

i.e. $\dfrac{d\vec{\mathbf{R}}}{dt} = \vec{\nabla}\mathbf{V}$, where $\vec{\mathbf{R}} = \begin{bmatrix} x \\ y \end{bmatrix}$. In that case, $\mathbf{V}(x, y)$ is called the **potential function** of the system.

Gradient systems are essentially same as the idea of Gradient Vector fields. You might recall from your multivariable calculus course that Gradient vector fields have some very nice properties, e.g. path-independence, circulation-free, irrotational etc. We can use those properties to describe what the solution curves of the Differential Equations (which are the flow lines of our vector field) would look like!

To begin with, observe that the solution curves (flow lines) of a gradient system are perpendicular to the level curves of $\mathbf{V}(x, y)$ (why?) and the equilibrium points of a gradient system are the stationary critical points of $\mathbf{V}(x, y)$ (why?).

(a) If $(x(t), y(t))$ is a solution curve of the gradient system (and not an equilibrium point), then show that $\dfrac{d\mathbf{V}}{dt} > 0$ along the solution curve. [Hint: Use multivariable calculus chain rule!]

(b) Suppose one of the solution curves in the phase portrait of a gradient system is a closed loop. Denote the closed path by $\mathcal{C}$ and let $\vec{r}(t)$, $0 \le t \le \tau$ be a parametrization of $\mathcal{C}$. We also assume that $\mathcal{C}$ is not just an equilibrium point. Then use part (a) to show that

$$\oint_{\mathcal{C}} \vec{\nabla}\mathbf{V} \cdot d\vec{r} > 0$$

[Hint: Use parameterization of $\mathcal{C}$ to evaluate the line integral.]

This is clearly a contradiction since a gradient vector field is circulation-free! So we can conclude that the phase portrait of a gradient system does not have a closed orbit.

(c) Next calculate the Jacobian matrix and check that it is the same as the Hessian of $\mathbf{V}$ (the matrix that shows up for second derivative test for local max/min/saddle). Show that $\mathbf{T}^2 - 4\mathbf{D} > 0$ for this matrix.

So we can conclude that the phase portrait of a gradient system does not have a spiral source, spiral sink, or center near any equilibrium point.

§E. Three Dimensional Nonlinear System of First Order ODEs

Question 9. (Lorenz Equations and Butterfly Effect)

We want to analyze the so-called Lorenz equations, which is a famous system of differential equations derived by Edward Lorenz when studying convection rolls in the atmosphere,

$$\frac{dx}{dt} = \alpha(y-x) \tag{1}$$

$$\frac{dy}{dt} = x(\rho-z)-y \tag{2}$$

$$\frac{dz}{dt} = -\beta z+xy \tag{3}$$

where α, β and ρ are real parameter values. Note that $(x,y,z)=(0,0,0)$ is an equilibrium solution to this ODE for all parameter values. If $\rho>1$, then

$$(x,y,z)=\left(\sqrt{\beta(\rho-1)},\sqrt{\beta(\rho-1)},\rho-1\right)$$

and

$$(x,y,z)=\left(-\sqrt{\beta(\rho-1)},-\sqrt{\beta(\rho-1)},\rho-1\right)$$

are the only two other equilibrium solutions.

(a) Write down the Jacobian matrix in terms of x,y,z. You don't have to evaluate the Jacobian at any equilibrium solution.

(b) Let $\alpha=10$, $\rho=5$ and $\beta=8/3$. Download the file `EigenvaluesLorenz.m`. This file computes the eigenvalues for the Jacobian at each of the three equilibrium points. Determine the type and stability of each of the three equilibrium points.

(c) Download the file `solve_lorenz.m`. This file computes a solution to the Lorenz equations from $t=0$ to 100. The green dot is the initial value, and the red dot is the solution at $t=100$.

  (i) Consider a solution curve with the initial value $x(0)=1$, $y(0)=0, z(0)=0$. What is (x,y,z) at $t=100$? Is this consistent with your observation about the stability of the equilibrium points?

  (ii) Consider a solution curve with the initial value $x(0)=1.000001$, $y(0)=0, z(0)=0$. What is (x,y,z) at $t=100$? Was there a significant change?

(d) Repeat all of the above parts with $\alpha=10$, $\rho=28$ and $\beta=8/3$. Be sure to change the relevant values in the two `Octave` files. Did you get anything surprising?

What you just observed is exactly what Edward Lorenz did in the 60s. It was something never seen, or even thought of before. The equations are completely deterministic (no statistical variation), yet the solutions can change drastically, even if the initial conditions are changed only slightly. All solutions live on that funny looking butterfly surface, which is now called a strange attractor. This is an example of Chaos, which you can learn more about in **Math 3208: Advanced Dynamics**. Weather systems have this same chaotic property (infinite sensitivity to initial conditions). What does this tell you about long term weather forecasts?