MATH 221 - Differential Equations

Lecture 2 Activities

Fall 2020 **Subhadip Chowdhury** **Aug 21**

§A. Air Resistance and Terminal Velocity

A paratrooper jumps out of an airplane at a sufficiently high altitude, and opens her parachute. Assume her mass is m and let $v(t)$ denote her velocity at time t.

Her vertical motion is subject to two forces:

- a downward gravitational force $\mathbf{F_G} = mg$, where g is the acceleration due to gravity, and
- a force $\mathbf{F_R}$ of air resistance that is proportional to momentum (so that $\mathbf{F_R} = \rho m v$) and of course directed opposite to the direction of motion of the body (i.e. upward).

Newton's second law of motion says that the net force acting on the paratrooper is equal to her mass times her acceleration. With the parachute open, ρ is approximately $\mathbf{1.5}$.

Question 1.

Find a differential equation that models the velocity of the paratrooper over time.

Question 2.

Conduct a qualitative analysis of your model and sketch an approximate graph of v vs. t. Demonstrate the following statement using your observations.

> As $t \to \infty$, the paratrooper's velocity does not increase indefinitely. Instead, it approaches a finite limiting velocity, called the **terminal velocity**.

§B. Ant Tunnel

[1] How long does it take an ant to build a tunnel? That seems like a reasonable question. If you ever had an ant colony purchased by a well-meaning aunt for you in grade school you may have watched ants building industriously and you just might have an idea on this. To answer the question we might need some narrowing of scope, some simplification, and certainly some identification of terms and variables before we can get a nice answer. Let us identify some variables and then together make some assumptions which will lead to a mathematical model.

Let x be the length of the tunnel in meters that an ant builds. Let $\mathbf{T}(x)$ be the time in hours it takes the ant to build the tunnel of length x. We can get some idea of our situation by making a sketch in Figure 1.

[1] Source: Brian Winkel, SIMIODE

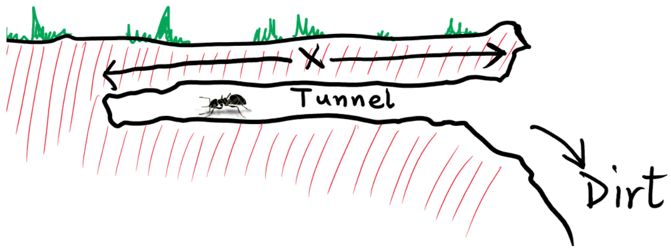

Figure 1: Crude drawing for ant tunnel building model.

■ Question 3.

Try to guess some candidate functions for $\mathbf{T}(x)$ and give one or two statements in each one's defense and one or two statements against each.

We see that attempting to jump right on top of $\mathbf{T}(x)$ can be hard. So, instead of going after $\mathbf{T}(x)$ directly let us examine Figure 2. Consider some assumptions which

(i) reflect the reality of such a situation and

(ii) might make the model simple in a first attempt.

Write these assumptions down in words (no mathematical abstraction needed yet) and put the most useful ones as a prologue to your responses to the next parts of the problem.

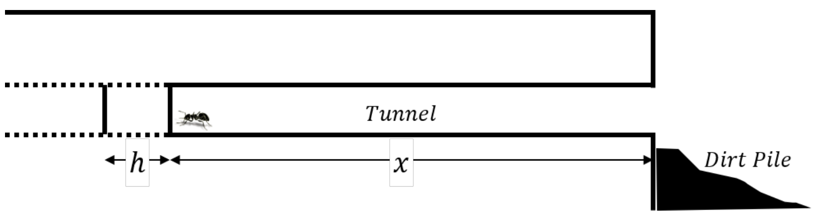

Figure 2: Useful diagram for discovering the time it takes to build a small section of the ant tunnel from distance x to $x+h$.

Now let's see if we can build a model using these assumptions to help us tell how long it might take an ant to extend a tunnel from distance x to distance $x+h$. Thus we seek an expression for

$$\mathbf{T}(x+h) - \mathbf{T}(x) = \underline{\hspace{2em}} \tag{1}$$

Notice that $\mathbf{T}(x+h) - \mathbf{T}(x) \neq \mathbf{T}(h)$, because $\mathbf{T}(h)$ represents the time it takes to dig a small tunnel of length h from the mouth of the tunnel, while $\mathbf{T}(x+h) - \mathbf{T}(x)$ includes the time it takes to extend the tunnel from length x to $x+h$. This latter time must account for the time for the ant to bring the material all they way out along a path of length x from the region from x to $x+h$ which is more than just the time it takes, $\mathbf{T}(h)$, to bring the material a distance of only h to the mouth of the tunnel.

■ Question 4.

List the variables (present or to be introduced) on which the expression (1) might depend.

■ Question 5.

Below are several possible mathematical models for the expression (1). Defend or reject each and offer your reasons. Perhaps modify one or two and make it better. When trying to reject a model, consider some trivial cases and see if it makes sense, e.g., $h = 0$ or $x = 0$ or either h or x very large.

(a) $\mathbf{T}(x+h) - \mathbf{T}(x) = x + h$

(b) $\mathbf{T}(x+h) - \mathbf{T}(x) = x - h$

(c) $\mathbf{T}(x+h) - \mathbf{T}(x) = x^h$

(d) $\mathbf{T}(x+h) - \mathbf{T}(x) = xh$

(e) $\mathbf{T}(x+h) - \mathbf{T}(x) = h^x$

(f) $\mathbf{T}(x+h) - \mathbf{T}(x) = c$, a constant

Question 6.

Convert your expression in (1) to a differential equation using the limit definition of derivative.

Question 7.

Solve the differential equation you create in the last part for $\mathbf{T}(x)$. Don't forget to include '$+c$'. How would you find c?

Question 8.

Use your solution from to determine how much longer it takes to build a tunnel which is twice as long as an original tunnel of length $\mathbf{L}$. What would some of your original function models you set forth in question (3) have told you here?

Question 9.

Suppose we had two ants digging from either side of our sand hill along the same straight line.How would this alter the total time for digging the tunnel? Of course, we can apply these same principles of our model to real tunnel building for engineers.

Question 10.

If we were considering question (9) as related to engineering construction of a long tunnel of length $\mathbf{L}$, outline some of the issues we should be aware of when having two crews (one from each end of the tunnel) working on the tunnel.