

MATH 2208: ORDINARY DIFFERENTIAL EQUATIONS

ASSIGNMENT 1

Spring 2020

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Due: Jan 29

Reading

Section 1.1 from the textbook.

Exercises

■ Question 0.

Your first “instruction” is to not wait until the night before the homework is due to start working on it. Please sign indicating that you acknowledge my recommendation.

■ Question 1.

Book problems 1.1.(4,5).

Note: Some homework problems (like the ones below) might be more challenging than the standard text book problems involving computations. For some, it is because I do not outline explicitly what to do. The open ended nature of some of the problems might confuse you initially, but the problems are designed to make you think. If you are unsure what a problem is asking, don’t hesitate to ask me!

Additional Problems

■ Question 2.

In class we performed a comprehensive analysis using the population model $\frac{dP}{dt} = kP$. Another modeling approach consists of modifying existing models, which we will do here. We want to modify this model to account for an environment with limited (finite) amount of resources (in some simple way).

- (a) Using data from the US population in the 1800s, we estimated $k \approx 0.03$. Suppose for England at that same time, we had data that suggested that $k = 0.005$. Which population is growing faster?
- (b) As the population increases exponentially in an environment with limited amount of resource, there will be a point when P becomes too large for the resources to sustain the population. In that scenario (when limited resources is an issue), should k be smaller or larger?
- (c) The previous two parts suggest that the growth rate constant k should actually depend on the population P . Let’s call the growth rate “function” $K(P)$. Thus, by replacing k with the (yet to be determined) function $K(P)$, our model becomes:

$$\frac{dP}{dt} = K(P)P \tag{1}$$

Keeping in mind the model building guidelines (or suggestions) from our first lecture, we should first try a linear formula for $K(P)$. So let’s say $K(P) = mP + c$.

- (i) Let us assume that if P is very small (i.e. close to 0), limited resources is not an issue (makes sense, right?), and thus the population should be described roughly by $\frac{dP}{dt} = kP$. Using this information, what is $K(0)$? [Evaluate your function $K(P)$ at $P = 0$.]

- (ii) In a system with limited resources, we will assume there is some maximum population the environment can support. Let's call this number the 'carrying capacity', N . What is $K(N)$?
- (iii) Combining (i) and (ii), write down your function $K(P)$, and then your ODE model.
- (iv) Conduct a qualitative analysis of your model.

This question is open ended and vague on purpose. The goal is for you to discover whatever qualitative information you can find from above DE, without solving it analytically. This means describing the solution curve $P(t)$ graphically (increasing/decreasing, concave up/down, asymptotes etc.) without finding for an explicit formula for $P(t)$.

■ Question 3.

In this problem, we are going to try to model the quantity of salt in a **1500** gallon water tank. Assume that at time $t = 0$ hr, the tank contains **600** gallons of water with **5** lbs of salt dissolved in it. Water enters the tank at a rate of **9** gal/hr and the water entering the tank has a salt concentration of **0.1** lbs/gal. Water exits the tank at a rate of **6** gal/hr.

The main assumption that we'll be using here is that the concentration of the salt in the water is uniform throughout the tank. Clearly this will not be the case in real life, but if we allow the concentration to vary depending on the location in the tank the problem becomes very difficult and will involve partial differential equations, which is not the focus of this course. So we will assume that at any given moment, the water that is leaving the tank has the same salt concentration as in the tank.

Suppose $Q(t)$ lb gives the amount of salt dissolved in the water in the tank at any time t hr. We make the following observations.

- Rate of change of $Q(t) =$ Rate at which $Q(t)$ enters the tank – Rate at which $Q(t)$ exits the tank
- Rate at which $Q(t)$ enters/exits the tank =
(flow rate of water entering/exiting) \times (concentration of salt in water entering/exiting)
- Concentration of salt in the tank at time $t = \frac{\text{Amount of salt in the tank at time } t}{\text{Volume of water in the tank at time } t}$

Set up an Initial Value Problem that, when solved, will give us an expression for $Q(t)$. Do not try to solve it. What is the interval of definition for a solution to your IVP?