MATH 221 - Differential Equations

Lecture 8 Worksheet

Fall 2020 **Subhadip Chowdhury** **Sep 4**

Title: Euler's Method

Summary: Then we will learn about a simple numerical technique called Euler's Method[1] which approximates solutions to ODEs quantitatively. During class, we will learn to use Python and implement Euler's method using Python.

Related Reading: Chapter 2.6

§A. Euler's Method

Suppose our goal is to solve the IVP

$$y' = f(t,y), \qquad y(t_0) = y_0 \tag{1}$$

and to determine $y(\mathrm{T})$ for some value of T. We will describe a method to find a numerical approximation of the value by discretizing the interval $[t_0, \mathrm{T}]$ into a partition of N subintervals

$$t_0 \le t_1 \le t_2 \le t_3 \le \ldots \le t_\mathrm{N} = \mathrm{T}$$

and using linear approximation to recursively calculate $y(t_{i+1})$ from $y(t_i)$ using tangent lines.

Assume $t_{i+1} - t_i = \Delta t = \frac{\mathrm{T}-t_0}{\mathrm{N}}$. Recall from your Calculus class that the equation of the tangent line to the curve $y(t)$ at $t = t_0$ is

$$y = y_0 + y'(t_0)(t - t_0) \overset{\text{By eqn. 1}}{=\!=\!=} y_0 + f(t_0, y_0)(t - t_0)$$

If Δt is small enough, then $y_1 = y_0 + f(t_0, y_0)\Delta t$ is a good approximation for $y(t_1)$. Refer to figure 1 in the next page. We can then continue the process with

$$y_{i+1} = y_i + f(t_i, y_i)\Delta t = y_i + \Delta y_i$$

for $i = 1, 2, \ldots, \mathrm{N} - 1$.

[1] https://www.youtube.com/watch?v=v-pbGAts_Fg

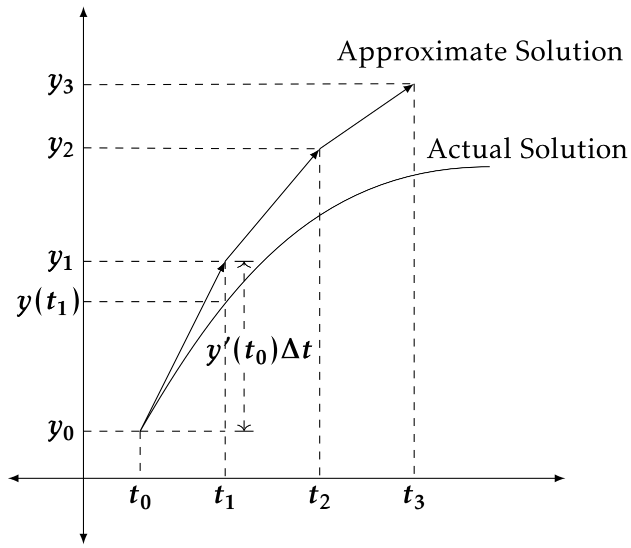

Figure 1: Euler's Method

■ Question 1.

Consider the differential equation $y' = -t/y$ with initial condition $y(0) = 1$. Recall that we found earlier in the Ant Path problem that the exact solution[2] is $y(t) = \sqrt{1-t^2}$.

(a) Use DFIELD to draw the slope field and the solution curve that starts at $(0, 1)$.

(b) Visually estimate the value of $y(1/2)$ from the output picture.

(c) Compare your estimate with the exact value of $y(1/2)$ using the formula $y(t) = \sqrt{1-t^2}$.

(d) Next use Euler's Method with $\Delta t = .25$ to estimate $y(1/2)$. You can use a table as below to keep track of the values as you calculate by hand.

(e) Is your Euler's Method estimate an over-estimate or an under-estimate? Explain why.

i	t_i	y_i	$f(t_i, y_i)$	Δy_i
0				
1				
2				

■ Question 2.

(a) Consider the IVP

$$\frac{dy}{dx} = (3-y)(y+1), \qquad y(0) = 4$$

Use Euler's method with the step size $\Delta t = 1$ to fill out a table of the approximate values as above over the time interval $0 \leq t \leq 2$. Sketch a picture of the approximate solution curve. Use a calculator as necessary.

[2] Why are we are considering the positive square root only? Think about interval of definition.

(b) Now suppose we change the initial condition and have the new IVP

$$\frac{dy}{dx} = (3-y)(y+1), \qquad y(0) = 0$$

Do the same with $\Delta t = 0.5$ over the interval $0 \le t \le 2$. Your table will have at 4 rows.

(c) Do a qualitative analysis of the solution of the ODE using a Phase Line and compare your conclusions with your results in the last two parts. What's wrong with the approximate solutions given by Euler's method?

(d) What can you do to have a better approximation for the solution curves?