Math 2208: Ordinary Differential Equations

Lecture 10 Worksheet

Spring 2020 | **Subhadip Chowdhury** | **Feb 24**

Title: Poincare Map and Periodic Solutions
Summary: We will introduce a qualitative tool for analyzing a certain type of non-autonomous ODEs.

§A. Recap on Bifurcation Theory

Question 1.

Recall the Ermentrout-Kopell model for the spiking of a neuron from homeworks:

$$\frac{d\theta}{dt} = 1 - \cos\theta + (1+\cos\theta)\mathbf{I}(t)$$

Let the input function $\mathbf{I}(t)$ be a constant parameter $\mathbf{I}$. Earlier, you had drawn the slope fields for the ODE when $\mathbf{I} = -0.1, \mathbf{I} = 0.0,$ and $\mathbf{I} = 0.1$ and noticed that the number and nature of the equilibrium changed in each case. The observations strongly suggest that some bifurcation happened as $\mathbf{I}$ increased from -0.1 to 0.1. Find the bifurcation value of $\mathbf{I}$.

Question 2.

Consider the family of differential equations $y' = ay + \sin y$ where a is a parameter.

(a) Sketch the phase line when $a = 0$.

(b) Use the graphs of ay and $\sin y$ to determine the qualitative behavior of all of the bifurcations that occur as a increases from -1 to 1.

(c) Sketch the bifurcation diagram for this family of differential equations.

§B. Poincare Map and Periodic Solutions

Consider a first order **non-autonomous** differential equation $y' = f(t,y)$. We are interested in understanding long-term behaviour of solution curves of such an ODE.

Since the ODE is non-autonomous, our usual qualitative tools such as equilibrium solutions, phase lines etc. do not work. However, recall the example of the RC-circuit ODE from homeworks.

$$\frac{dv_c}{dt} = \frac{\sin(2\pi t) - v_c}{\mathrm{RC}}$$

with $\mathrm{R} = 0.5, \mathrm{C} = 1$. We proved by analytical calculations that all solution curves converge to a periodic solution in the long term. So the periodic solution behaves like an 'equilibrium' solution in this case. One might ask, are there any sufficient conditions that will always guarantee the existence of a periodic solution?

Let's consider the following scenario. Suppose the right hand side of the ODE, $f(t,y)$ is a periodic function of t with period 1 i.e.

$$f(t+1,y) = f(t,y) \qquad \text{for all } t \text{ and } y$$

and assume that $f(t,y)$ is continuously differentiable in t and y i.e. the partial derivatives are continuous functions. These conditions somewhat simplify the problem of finding solutions for the ODE.

Question 1.

Suppose that we know the solution of all initial value problems, not for all times, but only for $0 \le t \le 1$. Then explain why we in fact know the solutions for all time.

Consider a solution curve that starts with initial condition $y(0) = y_0$. If we know the value of $y(t)$ at time $t = 1$ for this curve, then we can associate y_0 to $y(1)$. In a similar fashion, to each such initial condition y_0, we can associate the value $y(1)$ of the solution $y(t)$ that satisfies $y(0) = y_0$. This gives us a function φ defined as $\varphi(y_0) = y(1)$.

Definition 2.1

The function φ is called a **Poincare map** for the ODE.

If we compose this function with itself, we derive the value of the solution through y_0 at time 2; that is, $\varphi(\varphi(y_0)) = y(2)$ and in general, $\varphi^n(y_0) = y(n)$. Having such a function allows us to move from the realm of continuous dynamical systems (differential equations) to the often easier-to-understand realm of discrete dynamical systems (iterated functions).

Question 2.

Suppose there exists a constant y_0 such that $\varphi(y_0) = y_0$. Explain why the solution curve with initial condition $y(0) = y_0$ is a periodic solution.

Question 3.

Suppose there are constants p, q with $p < q$ such that

$$f(t,p) > 0, \quad f(t,q) < 0 \qquad \text{for all } t$$

(a) If $y_p(t)$ is a solution curve such that $y_p(0) = p$, then explain why $\varphi(p) > p$.

(b) If $y_q(t)$ is a solution curve such that $y_q(0) = q$, then explain why $\varphi(q) < q$.

Question 4.

Assuming p and q exist that satisfy the conditions of question 2, show that there is a periodic solution $\tilde{y}(t)$ such that $p < \tilde{y}(0) < q$.

Question 5.

Give a qualitative argument for why the RC-circuit ODE must have a periodic solution.

Question 6.

Consider the differential equation

$$y' = y^2 - 1 - \cos(t).$$

What can be said about the existence of periodic solutions for this equation?