MATH 221 - Differential Equations

Lecture 28 Worksheet

Fall 2020 **Subhadip Chowdhury** **Oct 21**

TITLE: Harmonic Oscillators

SUMMARY: We will examine the standard second order constant-coefficient ODE $y'' + py' + qy = 0$ more closely now that we have completed the analysis of the first order system of 2 linear ODEs.

Related Reading: Section 5.1 from the textbook. We will spend multiple lectures on it.

From The ODE Project - Section 4.1

As you read through this worksheet, you should treat it as a study guide only. It is not a replacement for your textbook. Please make sure to follow along in the textbook (or other online resources from the related reading section above) side-by-side as you read through the topics. You are not expected to be able to answer the questions in here immediately after reading the synopsis from the worksheet. They are designed as more of an exploration directives; as you try to find the answers yourself, you will also learn the topic. You should use whatever resources are available to you, including the internet, to accomplish that task.

§A. Second Order Linear ODE

Consider the second-order linear ODE

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$$

where p and q are positive real numbers. We saw last week how to rewrite this as a system of first order ODEs. We also found out that the characteristic polynomial for the associated matrix is $p_A(\lambda) = \lambda^2 + p\lambda + q$.

■ **Question 1.** □

(a) Recall that the eigenvalues are roots of the characteristic polynomial i.e. when $\lambda^2 + p\lambda + q = 0$. Compare this to the ODE above. Do you see any similarities?

(b) Suppose λ is an eigenvalue of the matrix corresponding to above linear system. Show that the corresponding eigenvector is $\begin{bmatrix} 1 \\ \lambda \end{bmatrix}$.

(c) If λ_1 and λ_2 are the two eigenvalues, write the general solution to the system of linear ODEs and get a formula for $\vec{R}(t) = \begin{bmatrix} y(t) \\ v(t) \end{bmatrix}$.

(d) Isolate $y(t)$ to get the general (real-valued) solution to the second order linear ODE.

§B. Harmonic Motion

The equations for the harmonic oscillator come from Newton's second law

$$\text{force} = \text{mass} \times \text{acceleration}.$$

Suppose we apply it to the motion of a mass attached to a spring, sliding on a table. We let $y(t)$ denote the position (displacement) of the mass m at time t, with $y = 0$ the rest position. The forces on the mass are the spring force, $-ky$, (this is called Hooke's law) and the damping force, $-b\dfrac{dy}{dt}$ (e.g. friction). The negative signs represent that the forces are in the opposite direction to that of $y(t)$. Substituting into Newton's law gives

$$-ky - b\frac{dy}{dt} = m\frac{d^2y}{dt^2}$$

which is often written as

$$my'' + by' + ky = 0 \iff y'' + \frac{b}{m}y' + \frac{k}{m}y = 0$$

The parameters are $m > 0, k > 0,$ and $b \geq 0$. This type of motion is known as a harmonic motion.

■ Question 2. □

First consider the case $b = 0$. This is called a **Simple Harmonic Motion**, also known as the **Undamped** Harmonic Motion.

(a) Check that the eigenvalues of associated linear system are complex numbers whose real parts are equal to **0**.

(b) Write down the general formula for $y(t)$.

(c) Check that it is a periodic function of t. What is the period?

■ Question 3. 　　　　　　　　　　 Making a clock using Mass-Spring System

Suppose we wish to make a clock using a mass and a spring sliding on a table. We arrange for the clock to "tick" whenever the mass crosses $y = 0$. We use a spring with spring constant $k = 2$. If we assume there is no friction or damping ($b = 0$), then what mass m must be attached to the spring so that its natural period is one time unit?

■ Question 4. □

If $b \neq 0$, we get a **Damped** Harmonic Motion. Then depending on different values of $b, k,$ and m we will have different behavior for the solution curves as the discriminant of the characteristic polynomial changes.

(a) Find the discriminant of the characteristic polynomial of the associated system of linear ODEs in terms of $m, b,$ and k.

(b) Fill out the following table. We are considering three cases. Open the applet at

https://mathlets.org/mathlets/damped-vibrations/

Discriminant	negative	zero	positive
Conditions on m, b, k (Note that $m > 0$)			
Eigenvalues are Real/Complex?			
Number of Eigenvalues			
Does the solution curve y vs. t oscillate? If yes, what's the period?			
Kind of Damping	**Underdamped**	**Critically damped**	**Overdamped**
Equilibrium Type of Phase Portrait in (y, y')-phase plane			