Math 2208: Ordinary Differential Equations

[Optional] Project 3 Extra Reading: Modelling a Double Mass-Spring System

Spring 2020 **Subhadip Chowdhury** **Due: Apr 26**

§A. The Double Mass-Spring Model

Suppose that we have two masses on a table, m_1 and m_2, connected by three springs, with the outside springs connected to two walls (see figure 1), and the masses are free to move horizontally. We will assume that the springs are uniform and all have the same spring constant k. The horizontal displacements of the springs are denoted by x_1 and x_2 for the masses m_1 and m_2, respectively. Assuming that there is no damping, the only forces acting on mass m_1 at time t are those of left and middle springs. The force from the left spring will be $-kx_1$ while the force from middle spring will be $k(x_2 - x_1)$. By Newton's Second Law of motion, we have

$$m_1 x_1'' = -kx_1 + k(x_2 - x_1).$$

Similarly, the only forces acting on the second mass, m_2, will come from middle and right springs. Again using Newton's Second Law,

$$m_2 x_2'' = -k(x_2 - x_1) - kx_2.$$

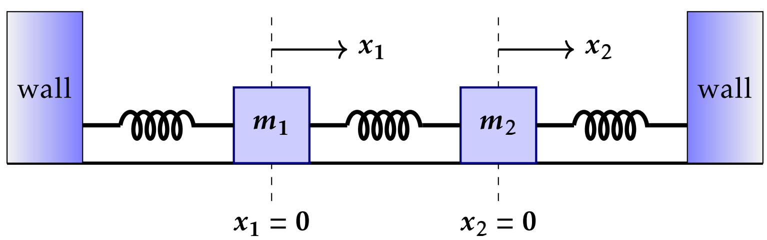

Figure 1: A double mass-spring system

■ Question 1.

First of all, make sure you understand how we got the two second-order differential equations above. If you have any doubt, stop, and contact me.

■ Question 2. (2 points)

Write the two second-order ODEs as a system of four first order linear ODEs. Rewrite the system using matrix notation. You will be using a 4×4 matrix.

For the sake of calculation, we will assume $k = m_1 = m_2 = 1$ for the rest of this project.

■ Question 3. (1 points)

Find the eigenvalues and eigenvectors of the 4×4 matrix by hand or using `WolframAlpha`.

Analytical Approach.

■ **Question 4. (4 points)**

Write down the formula for the general real-valued solution.

■ **Question 5. (3 points)**

Suppose we have the initial conditions

$$x_1(0) = 2p, \quad x_2(0) = 2q, \quad x_1'(0) = 0, \quad x_2'(0) = 0.$$

Then find the formula for $x_1(t)$ and $x_2(t)$. Your answer will involve the constants p and q.

Qualitative Approach.

Generally speaking, the motions of the two weights will not be periodic, since the solutions for x_1 and x_2 are linear combinations of various **sin** and **cos** terms with irrational frequencies.

■ **Question 6. (2+2+2+1+(2+1) points)**

(a) Show that the only cases where the motion of both weights are periodic is when either $p = q$ or $p = -q$.

(b) In each case, determine the period.

(c) Suppose $p = q = 2$. Draw the graph of the functions $x_1(t)$ vs t and $x_2(t)$ vs t in the same picture. Use whatever software you want.

(d) Note that the original system had two dependent variable. So we can try to analyze it using a $(x1, x2)$-phase plane. Draw the solution curve in (x_1, x_2) phase plane that corresponds to $p = q = 2$.

(e) Repeat above two parts for the case $p = 2, q = -2$.

■ **Question 7. (Optional, 2 points)**

Suppose $p = 2$ and $q = 0$. Use `WolframAlpha` or `Mathematica` or `Desmos` to draw the solution curve in (x_1, x_2)-phase plane from $t = 0$ to $t = 10$. Is the system periodic for these values of p and q? Include the picture.

When the system is not periodic, we observe that the system exhibits a **beating phenomenon**, in which the energy is distributed cyclically from one mass to another. This is a typical behavior when the two periods are nearly equal. We will learn more about this in future lectures.

Mass-spring systems are the physical basis for modeling and solving many engineering problems. Such models are used in the design of building structures, or, for example, in the development of sportswear. Of course, the system of equations in real situations can be much more complex. We will look at an example of higher order nonlinear system called **Lorenz Equations** towards the end of this course.