MATH 221 - Differential Equations

Lecture 7 Worksheet

Fall 2020 | **Subhadip Chowdhury** | **Sep 2**

Title: Phase Line and Equilibria

Summary: We will continue our qualitative analysis of differential equations by learning how to use phase lines and the classification of equilibrium points of autonomous, first-order ODEs.

Related Reading: Chapter 2.1.2

§A. Phase Line and Equilibria

The fact that for an autonomous ODE $y' = f(y)$, all arrows in a horizontal line have same slope in the direction field, tells us that we don't need to draw the full plane to get the qualitative description of an autonomous ODE. Instead we can summarize the information in one vertical straight line as follows.

Definition A.1: Phase Portrait

A one dimensional phase portrait of an autonomous DE $y' = f(y)$ is a diagram which indicates the values of the dependent variable for which y is increasing, decreasing or constant. Sometimes the vertical version of the phase portrait is called a **phase line.**

Definition A.2: Equilibrium point

An **equilibrium point** of an autonomous ODE $y' = f(y)$ is a real number c where the rate of change is 0 i.e. $f(c) = 0$. If c is an equilibrium point of an autonomous ODE, then the constant curve $y(t) = c$ is a solution of the DE.

§B. Algorithm For Drawing A Phase Line

This only applies to autonomous ODEs.

Step 1. Draw a vertical line.

Step 2. Find the equilibrium points (i.e. values such that $\frac{dy}{dx} = 0$) and mark them on the line with a dot.

Step 3. Find intervals for which $\frac{dy}{dx} > 0$ and mark them with upward arrows $\uparrow$.

Step 4. Find intervals for which $\frac{dy}{dx} < 0$ and mark them with downward arrows $\downarrow$.

Question 1.

Consider the autonomous differential equation $\frac{dy}{dx} = y(1-3y)(y-2)$.

(a) Find the equilibrium points of the DE.

(b) Determine the values of y for which $y(x)$ is increasing and decreasing.

(c) Draw the phase line for this ODE.

See the textbook for more examples of phase lines.

§C. **Obtaining Solution Information from Phase Lines**

Consider the autonomous ordinary differential equation $y' = f(y)$ where $f(y)$ is a **continuously differentiable** function (i.e. $f'(y)$ is continuous [1]) and let $y(t)$ be a solution to the ODE. The following conclusions are consequences of the **EUT**.

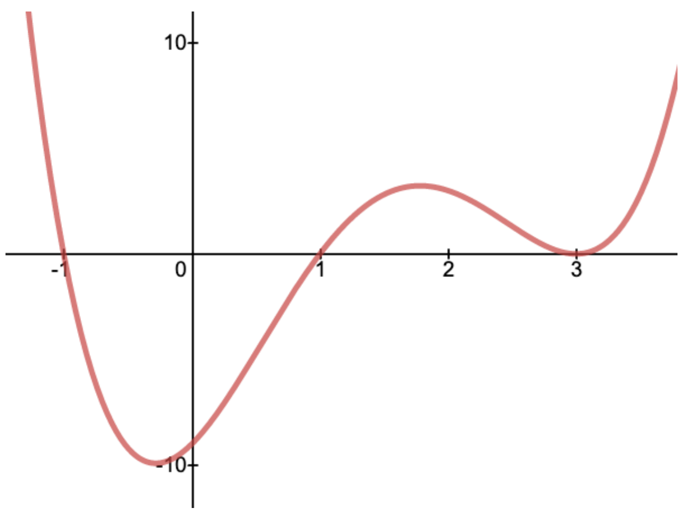
- If $f(y(0)) = 0$ then $y(x) = y(0)$ for all x and $y(0)$ is an equilibrium point.
- If $f(y(0)) > 0$ then $y(x)$ is increasing for all x and either $y(x) \to \infty$ as x increases or $y(x)$ tends to the first equilibrium point larger than $y(0)$.
- If $f(y(0)) < 0$ then $y(x)$ is decreasing for all x and either $y(x) \to -\infty$ as x increases or $y(x)$ tends to the first equilibrium point smaller than $y(0)$.

■ Question 2. (Drawing Phase Lines from Qualitative Information Alone)

Draw the phase line for the ODE $y' = f(y)$ where the graph of $f(y)$ vs. y looks like the graph below.

Then in an xy−plane, draw some particular solution curves starting at

- $y(0) = -2$,
- $y(0) = 0$,
- $y(0) = 1$,
- $y(0) = 2$ and
- $y(0) = 4$.



■ Question 3.

Draw the phase line for the ODE $y' = \frac{1}{1-y}$. Skecth a couple of solution curves.

[1] why do we need this condition?

§D. **Classifying Equilibrium Points: Sink, Source or Node**

An equilibrium point c splits the phase line into two different regions. So there are four possible scenarios for the behavior of y' near c: $(+,0,+),(+,0,-),(-,0,+)$ and $(-,0,-)$ in the direction of increasing y.

When both arrowheads on either side of the dot labeled c point toward c, i.e in the scenario $(+,0,-)$, all solutions $y(x)$ of that start from an initial point in a neighborhood of c have the asymptotic property $\lim_{x\to\infty} y(x) = c$. For this reason the critical point c is said to be **asymptotically stable** or an **attractor** or a **sink**.

Similarly, when both arrowheads on either side of the dot labeled c point away from c, i.e in the scenario $(-,0,+)$, all solutions $y(x)$ of that start from an initial point in a neighborhood of c have the asymptotic property $\lim_{x\to-\infty} y(x) = c$. For this reason the critical point c is said to be **asymptotically unstable** or a **repeller** or a **source**.

Finally in the case of $(+,0,+)$ or $(-,0,-)$, the critical point c is said to be **semi-stable** or a **node**.

■ Question 4.

Draw a sketch of the graph of the function $f(y)$ near equilibrium points classified as a sink, source or node. How does the sign of $f'(y)$ correspond to the classification?