MATH 221 - Differential Equations

Lecture 15 Worksheet

Fall 2020 **Subhadip Chowdhury** **Sep 21**

Title: Nullclines and Phase Plane

Summary: We will learn about some tools - direction fields and phase portraits - used to perform qualitative and quantitative analysis of system of ODEs. We will use them to analyze the Lotka-Volterra model further.

Related Reading: Chapter 3.3 from the textbook.

Additionally, as I mentioned during our Bifurcation discussion, we will be using some other free online resources apart from our textbook to supplement our reading from time-to-time. Below is one such link.

From The ODE Project - Section 2.1.1, Section 2.2.1. We will cover the rest of the ideas in those two sections on Wednesday.

As you read through this worksheet, you should treat it as a study guide only. It is not a replacement for your textbook. Please make sure to follow along in the textbook (or other online resources from the related reading section above) side-by-side as you read through the topics. You are not expected to be able to answer the questions in here immediately after reading the synopsis from the worksheet. They are designed as more of an exploration directives; as you try to find the answers yourself, you will also learn the topic. You should use whatever resources are available to you, including the internet, to accomplish that task.

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§A. Phase Portrait

Recall from last class that one of the ways to graph the solution of the system is to form the pair $(x(t), y(t))$ and think of it as a point in the xy-plane. In other words, the coordinates of the point are the values of the two populations at time t. As t varies, the pair $(x(t), y(t))$ sweeps out a parametric curve in the xy-plane. This curve is called the **solution curve**.

The xy-plane is called the **phase plane**, and it is analogous to the phase line for an autonomous first-order differential equation.

Observe that the solution 'curves' that correspond to equilibrium solutions of a system are really just *points* in the phase plane. So we refer to them as **equilibrium points**.

Definition A.1: Nullclines

For a system

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

the $x-$**nullcline** is the set of points

$$\left\{(x,y)\middle|\frac{dx}{dt}=0\right\}\equiv\{(x,y)|f(x,y)=0\}$$

i.e. the curve in the phase plane where $f(x,y)=0$. Similarly the $y-$nullcline is the curve(s) $g(x,y)=0$.

At an equilibrium point, both $f(x,y)$ and $g(x,y)$ must be zero, hence

Theorem A.1

The intersection of the nullclines are the equilibrium points.

A **Phase Portrait** of a system consists of the following information on the phase plane.[1]

- the nullclines,
- the equilibrium points
- several solution curves corresponding to different initial conditions.

Question 1.

Consider the following Lotka-Volterra model:

$$\frac{dx}{dt}=2x-1.2xy$$
$$\frac{dy}{dt}=0.9xy-y$$

Draw the $x-$ and $y-$nullclines of above system and check that the equilibrium point(s) are the intersection points of an $x-$nullcline with an $y-$nullcline. We will see below how to draw solution curves.

§B. Multivariable Calculus Basics II - Vectors and Fields

Recall from last time that with the new notation of vectors, we can use $\vec{\mathbf{R}}(t)$ to denote the vector

$$\vec{\mathbf{R}}(t)=\begin{bmatrix}x(t)\\y(t)\end{bmatrix}$$

Then the vector-valued function $t\mapsto\vec{\mathbf{R}}(t)$ (i.e. the input is time, and the output is the vector $\vec{\mathbf{R}}(t)$) corresponds to the solution curve $(x(t),y(t))$ in the xy-plane.

To compute the derivative of the vector-valued function $\vec{\mathbf{R}}(t)$, we compute the derivatives of each component. That is,

$$\frac{d\vec{\mathbf{R}}}{dt}=\begin{bmatrix}x'(t)\\y'(t)\end{bmatrix}$$

Using this notation, we can rewrite the predator-prey system as the single vector equation

$$\frac{d\vec{\mathbf{R}}}{dt}=\begin{bmatrix}-ax+bxy\\-cxy+dy\end{bmatrix}$$

[1] Compare this to the case of phase line in one-dimension.

So far we have only introduced more notation. We have converted our first-order system consisting of two equations into a single vector equation involving vectors with two components.

The advantages of the vector notation start to become evident once we consider the right-hand side of this system as a 'function' of $\vec{\mathbf{R}}(t)$. Consider a 'function' $\vec{\mathbf{V}}$ defined as

$$\vec{\mathbf{V}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -ax + bxy \\ -cxy + dy \end{bmatrix}$$

It takes a vector as an input and gives a vector as an output. Another way to think of it is that it takes a point in the xy-plane as an input and gives a vector as an output. For example, at the point $(x, y) = (2, 1)$,

$$\vec{\mathbf{V}}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2a + 2b \\ -2c + d \end{bmatrix}$$

Similarly, at the point $(3, 4)$,

$$\vec{\mathbf{V}}\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -3a + 12b \\ -12c + 4d \end{bmatrix}$$

Clearly, these types of 'function's are different than the usual kind, and deserve a special name - they are called **Vector Fields**.

Definition B.1

A **vector field** in 2-dimensions is a function

$$\vec{\mathbf{V}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$$

whose value at a point (x, y) is a 2-dimensional vector.

Then using this notation, we can write the predator-prey system very economically as

$$\frac{d\vec{\mathbf{R}}}{dt} = \vec{\mathbf{V}}(\vec{\mathbf{R}}(t))$$

The vector notation is much more than just a way to save ink. It also gives us a new way to think about and to visualize systems of differential equations. Below we will learn how to utilise this new perspective.

§C. Direction Field - A (normalized) Vector Field

How do we 'draw' a Vector Field? We need a way to depict a two dimensional vector output for every 2-dimensional input! If we treat the input as a point in the **phase plane**, then we only need to figure out how we are going to depict the vector output at every point in the xy-plane.

Without going into too much details about vectors in general, one way to depict a two-dimensional vector[2] $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is to draw an arrow that starts at $(0, 0)$ and ends at (v_1, v_2). (Look at figure 1)

[2]The two entries in a 2D vector are called first (or x) and second (or y) components respectively.

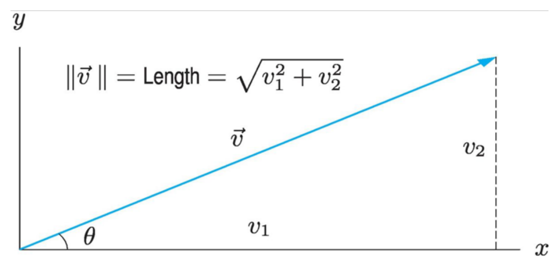


Figure 1: A Two-dimensional Vector

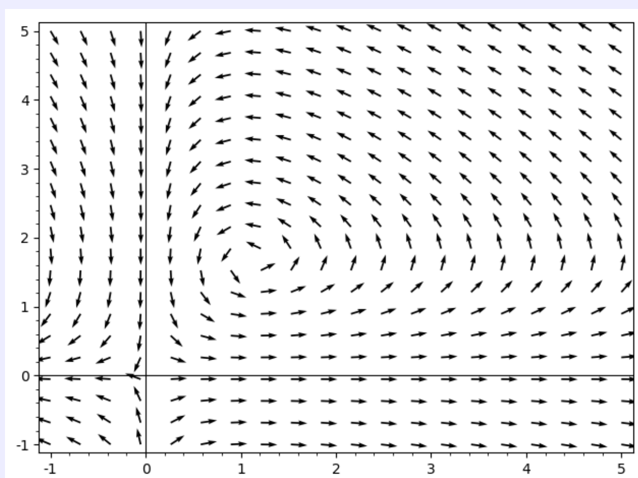
However, for our purposes, we are not going to be too concerned with the length of the vectors and focus mostly on the direction. With that in mind, one way to 'draw' a vector field

$$\vec{\mathbf{V}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$$

would be put small arrows that correspond to the output vector $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$ that start at every point (x,y) in the phase plane.

Example C.1

Below is the direction field corresponding to the Lotka-Volterra model from question 1.



Here an arrow at the point (x,y) denotes the direction of the vector $\begin{bmatrix} 2x - 1.2xy \\ 0.9xy - y \end{bmatrix}$.

For example, if we look at the point $(2,2)$, the arrow at that point corresponds to the $\begin{bmatrix} 2 \times 2 - 1.2 \times 2 \times 2 \\ 0.9 \times 2 \times 2 - 2 \end{bmatrix} = \begin{bmatrix} -0.8 \\ 1.6 \end{bmatrix}$. Check that it is indeed pointing towards the left (negative first component) and up (positive second component).

Since we are forgoing the need for being strict about the magnitude, we also call this picture the **direction field** of the system. Read the linked online book (from related reading above) for more examples of Direction Field.

Solution Curves as Flow Lines

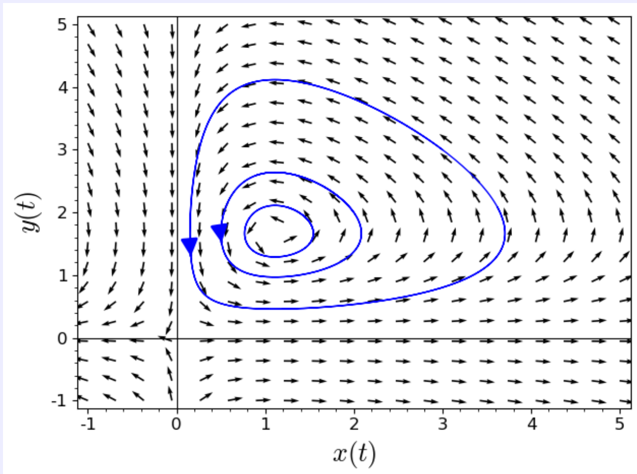
With this interpretation of the vector field, and the observation that we can rewrite our system of ODEs as a vector-valued differential equation, we make the connection that the solution curves on a phase plane are essentially the curves that **flow** along the vector field!

In other words, at every point on the solution curve, the direction of the tangent (which is the derivative $\frac{d\vec{\mathbf{R}}}{dt}$) is same as the direction of the small arrow at that point (which is $\vec{\mathbf{V}}(x,y) = \vec{\mathbf{V}}(\vec{\mathbf{R}}(t))$) in the vector field picture.

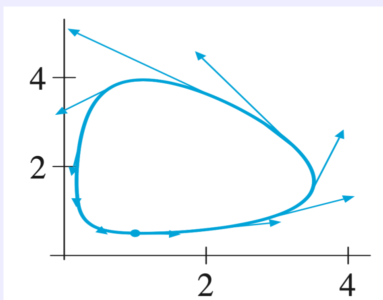
Yet another way to say this is that a solution curve starts at an initial point (initial condition) and then follows the path given by the arrows in the field.

Example C.2

Below are some solution curves for the Lotka-Volterra system from Question 1.



It does seem to be the case that the curves follow along the arrows and an arrow at a point on the curve is tangent to the curve.



Note that the fact that the solution curves are closed loops isn't very evident from the direction field itself. That only becomes evident after using some numerical or analytical tools to plot the curves.

Direction Field along a Nullcline

Along the x-nullcline, the x-component of the vector field is zero, and consequently the arrows in the direction field are vertical. They point either straight up or straight down depending on the sign of $g(x,y)$.

Similarly, on the y-nullcline, the y-component of the vector field is zero, so the vector field is horizontal. Arrows points either left or right depending on the sign of $f(x, y)$.

Direction Field in Regions Separated by Nullclines

Observe that the $x-$nullcline naturally divides the plane into regions where $f(x, y) > 0$ and $f(x, y) < 0$. Since $x' = f(x, y)$, $f(x; y) > 0$ means x is increasing which in turn means the arrows point rightward in the plane. Similarly, the $y-$nullcline shows us where y is increasing or decreasing. The following table will help you with figuring out the direction field once the nullclines are found.

	$f(x,y) < 0$	$f(x,y) = 0$	$f(x,y) > 0$
$g(x,y) < 0$	$\swarrow$	$\downarrow$	$\searrow$
$g(x,y) = 0$	$\leftarrow$	$\cdot$	$\rightarrow$
$g(x,y) > 0$	$\nwarrow$	$\uparrow$	$\nearrow$

Question 2.

(a) Let's go back to the same Lotka-Volterra system again. Check that the arrows are do look horizontal along the $y-$nullclines and vertical along the $x-$nullclines that you found in the last section.

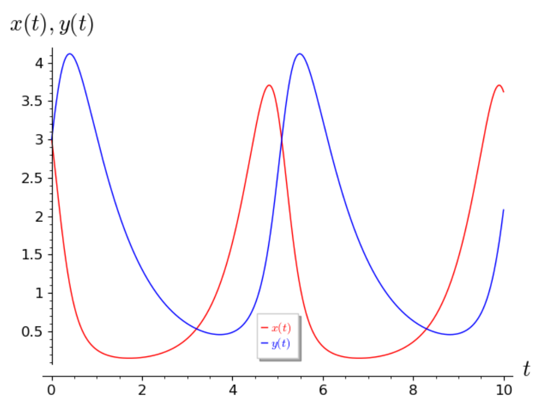
(b) Find the direction of the arrows (up, down, left, right) along the nullclines using algebra, then check your answer using the direction field picture.

(c) What happens to solution curves when the initial condition approaches $(10/9, 5/3)$?

§D. Component Graphs

Question 3.

Consider the solution curve $(x(t), y(t))$ in the direction field above that passes through $(3, 3)$ - it's the outermost one. If we use software to sketch the graphs of $x(t)$ vs. t and $y(t)$ vs. t, they look as follows:



(a) Can you explain the periodic behavior based on the fact that the solution curve in the phase plane is a closed loop?

(b) Can you explain why the two graphs should have the same period?

(c) Why does it makes sense that the peaks in prey graph happen earlier (in time) than the peak in the predator graph? Try to explain it both using the phase plane picture and using biological reasoning!