Math 2208: Ordinary Differential Equations

[Optional] Project 1 Extra Reading: The Cusp Catastrophe

Spring 2020 **Subhadip Chowdhury** **Due: Mar 2**

§A. Cusp Catastrophe

Recall that the original population model was given by

$$\frac{dp}{dt} = kp\left(1 - \frac{p}{N}\right) - \frac{p^2}{1+p^2}$$

What if instead of fixing the value of N, we allow both k and N to vary, and try to understand how the dependent variable y behaves as it responds to two independent control parameters k and N.

Question 1.

Observe that the ODE is still autonomous. Explain why the equilibrium solutions of this ODE form a surface over the $kN-$plane. What is the equation of the surface?

Question 2.

Try graphing this surface on the computer so that you can rotate it. Set the bounds on k and N to be $0 \leq k \leq 2, 0.01 \leq N \leq 25$ and bounds of p to be $0 \leq p \leq 40$. you can use Mathematica (`ContourPlot3D` command) or Octave (`Plot3` command) or CalcPlot3d (a website) or some other software of your choice. If necessary, you may have to switch from (k, N, p) notation to (x, y, z) notation. The

Question 3.

Draw the plane $N = 10$ and look at the intersection of the plane with the surface. It should look like the bifurcation diagram you had obtained in your project.

Question 4.

Now let's find the bifurcation set of points (k, N) for our two-parameter family of ODE. We can do this analytically, just like we do for a one-parameter family.

(a) What are the analytic conditions for a bifurcation to occur? Note that we can ignore the extinction solution $p = 0$ for the purpose of finding the roots.

(b) We would like to use above analytic conditions to find an equation for the bifurcation set in the (k, N) parameter plane. Check that

$$N = \frac{2p^3}{p^2-1}, \quad k = \frac{2p^3}{(1+p^2)^2}$$

satisfies the analytic conditions. Note that this gives a parametrized curve in $kN-$plane.

(c) Use your favorite software (e.g. Mathematica) to plot the paramterized curve above. This is the bifurcation set! Is it what you expected? Explain how it relates to the full equilibrium surface in **3D**, and what it tells us about the number of equilibrium points of p for different values of k and N.

(d) Sketch the rough shape of the bifurcation diagram in following cases using pen and paper. You don't need to give the exact coordinate. Use the 3D bifurcation diagram for reference.

   (a) $\mathbf{N} = 2$ but $\boldsymbol{k}$ varies

   (b) $\boldsymbol{k} = \mathbf{0.2}$ but $\mathbf{N}$ varies

   (c) $\boldsymbol{k} = \mathbf{0.56}$ but $\mathbf{N}$ varies (recall that this was one the bifurcation points when $\mathbf{N} = \mathbf{10}$ and $\boldsymbol{k}$ varied)

(e) The five dynamical properties that occur together at a **cusp catastrophe** are:

   (i) The behaviour is bimodal for some values of the control factors.

   (ii) Abrupt catastrophic changes are observed between one mode of behaviour and another.

   (iii) There is hysteresis, that is, the abrupt change from one mode of behaviour to another takes place at different values of the control factors depending on the direction of change.

   (iv) There is an inaccessible zone of behaviour for some values of the control factors.

   (v) Similar paths in the control space can lead to divergent behavior.

   Describe how each of these dynamical properties can be understood in terms of equilibria of our two-parameter family of ODEs. Note, what we're asking here is for you to **translate** these sentences about ''behavior'' to sentences about the surface of the equilibria. In other words, to translate from a more ''science'' way of stating and understanding the **5** properties to a more ''math'' way of stating and understanding them. It's an essential skill for applying math!