MATH 221 - Differential Equations

Lecture 2 Worksheet

Fall 2020 **Subhadip Chowdhury** **Aug 21**

Title: Why Differential Equations?

Summary: We will continue our discussion of differential equations as mathematical modeling tools.

§A. The Exponential Population Model

In last class we performed an analysis of the population model $\frac{dP}{dt} = kP$. We guessed that a solution is $P(t) = ae^{kt}$. However, if we couldn't make the guess, how would we have analyzed our equation? Can you answer any of the following questions?

- When is $P(t)$ increasing/decreasing? [Hint: Look at sign of $\frac{dP}{dt}$]
- When is $P(t)$ concave up/down? [Hint: Look at sign of $\frac{d^2P}{dt^2}$]

Only using these qualitative information, can you draw the graph of $P(t)$?

§B. The Logistic Population Model

Another modeling approach consists of modifying existing models, which we will do here. We want to modify the population model to account for an environment with limited (finite) amount of resources (in some simple way).

1. Using data from the US population in the 1800s, we estimated $k \approx 0.03$. Suppose for England at that same time, we had data that suggested that $k = 0.005$. Which population is growing faster?

   Solution. Since k denotes the growth rate of population, higher k implies faster growth. Hence US population is growing faster. ■

2. As the population increases exponentially in an environment with limited amount of resource, there will be a point when P becomes too large for the resources to sustain the population. In that scenario (when limited resources is an issue), should k be smaller or larger?

   Solution. When P becomes unsustainable, the growth rate decreases. Hence k should be smaller when P is large. ■

3. The previous two parts suggest that the growth rate constant k is not actually a constant, and instead should actually depend on the population P. So let's say that the growth rate is a function of P, called $K(P)$. Thus by replacing k with the (yet to be determined) function $K(P)$, our model becomes:

$$\frac{dP}{dt} = K(P) \times P \tag{1}$$

   Keeping in mind the model building guidelines (or suggestions) from our first lecture, we should first try a linear formula for $K(P)$. So let's assume $K(P) = mP + c$ for some unknown c and m. Let's find c and m.

(a) We can assume that if $\mathbf{P}$ is very small (i.e. close to $\mathbf{0}$), limited resources is not an issue (makes sense, right?), and thus the population should be described roughly by $\frac{d\mathbf{P}}{dt} = k\mathbf{P}$ as before. Assume we have found this k by some numerical means as in class.

Under these assumptions, what can you say about the value of c? [Remember that $c = \mathbf{K}(\mathbf{0})$.]

Solution. When $\mathbf{P}$ is small, $\mathbf{K}(\mathbf{0}) = k$ because growth rate doesn't depend on $\mathbf{P}$. ■

(b) In a system with limited resources, we will assume that there is some maximum population the environment can support. We call this number the 'carrying capacity', $\mathbf{N}$. What can you say about $\mathbf{K}(\mathbf{N})$?

Find m in terms of k and $\mathbf{N}$.

Solution. As $\mathbf{P} \to \mathbf{N}$, the growth decreases and theoretically when $\mathbf{P}$ reaches $\mathbf{N}$, the growth should stop. Hence $\mathbf{K}(\mathbf{N}) = \mathbf{0}$. ■

4. Combining (a) and (b), write down your function $\mathbf{K}(\mathbf{P})$, and then your ODE model. Keep in mind that $\mathbf{K}(\mathbf{P})$ should be a decreasing function of $\mathbf{P}$. Your model will depend on two parameters: k and $\mathbf{N}$. Do NOT try to guess a formula for $\mathbf{P}(t)$.

Solution. From part (a), we get $c = k$. From part (b), we get $m\mathrm{N} + c = \mathbf{0} \implies m\mathrm{N} + k = \mathbf{0} \implies m = -k/\mathrm{N}$. Hence the ODE model is

$$\frac{d\mathrm{P}}{dt} = \left(k - \frac{k\mathrm{P}}{\mathrm{N}}\right)\mathrm{P}$$

■

5. Now we are going to conduct a qualitative analysis of our model. The goal is for you to discover whatever qualitative information you can find from above DE, without solving it analytically. Try to answer the following questions:

- When is $\mathbf{P}(t)$ increasing/decreasing? [Hint: Look at sign of $\frac{d\mathbf{P}}{dt}$]
- When is $\mathbf{P}(t)$ concave up/down? [Hint: Look at sign of $\frac{d^2\mathbf{P}}{dt^2}$]
- Does the graph of $\mathbf{P}(t)$ have any horizontal or vertical asymptote? [Hint: What happens to $\frac{d\mathbf{P}}{dt}$, the slope, as the graph becomes almost horizontal?]

Solution. Here are some information we can find by analyzing above DE.

- There are two equilibrium solutions where $d\mathbf{P}/dt$ becomes zero. These are $\mathbf{P} = \mathbf{0}$ and $\mathbf{P} = \mathbf{N}$.
- When $\mathbf{P} > \mathbf{N}$, the right hand side is negative. Hence $d\mathbf{P}/dt$ is negative or in other words, $\mathbf{P}$ is decreasing. This means if the population starts at a level higher than the carrying capacity, the population gradually decreases. Since two solution curves can never cross each other, the line $\mathbf{P} = \mathbf{N}$ acts as a horizontal asymptote to these solution curves.
- If $\mathbf{0} < \mathbf{P} < \mathbf{N}$, then $d\mathbf{P}/dt > \mathbf{0}$. This means if the initial population is positive (it cannot be negative anyway) and less than the maximum capacity, population will gradually increase over time. the line $\mathbf{P} = \mathbf{N}$ again acts as a horizontal asymptote to these solution curves.
- Note that

$$\begin{aligned}\frac{d^2\mathrm{P}}{dt^2} &= k\frac{d\mathrm{P}}{dt} - \frac{k}{\mathrm{N}}2\mathrm{P}\frac{d\mathrm{P}}{dt} \\ &= k\frac{d\mathrm{P}}{dt}\left(1 - \frac{2\mathrm{P}}{\mathrm{N}}\right)\end{aligned}$$

So when $\mathbf{0} \le \mathbf{P} \le \mathbf{N}$, since $\frac{d\mathbf{P}}{dt} \neq \mathbf{0}$, we get that $\frac{d^2\mathbf{P}}{dt^2} = \mathbf{0} \implies \mathbf{P} = \mathbf{N}/2$. This means the solution curve has an inflection point when $\mathbf{P} = \mathbf{N}/2$. In other words, the population growth curve is concave up upto $\mathbf{N}/2$, and then concave downwards.

■

6. Based on your qualitative analysis, try to sketch the graph of $\mathbf{P}$ vs. t. How does the curve depend on $\mathbf{P}(0)$?

   Solution. Based on above observations, a picture of $\mathbf{P}(t)$ vs t for different initial values of $\mathbf{P}(0)$ look like figure 1.

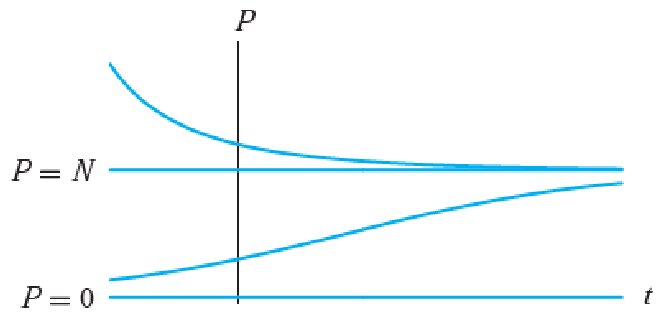

Figure 1: From our textbook page 11

■

§C. Another Population Modeling Example

The rhinoceros is now extremely rare. Suppose enough game preserve land is set aside so that there is sufficient room for many more rhinoceros territories than there are rhinoceroses. Consequently, there will be no danger of overcrowding or limitation of resource. However, if the population $\mathbf{R}$ is too small, fertile adults have difficulty finding each other when it is time to mate. So we assume that there is a minimum threshold that the population must exceed if it is to survive.

1. With these assumptions in mind, what can you about the growth rate $\dfrac{d\mathbf{R}}{dt}$ when the population $\mathbf{R}$ is close to $\mathbf{0}$?
2. Write a differential equation that models the rhinoceros population based on these assumptions. Note that there is more than one reasonable model that fits above assumptions. Consequently, there are many possible 'correct' answers.
3. Conduct a qualitative analysis of your model.