Math 2208: Ordinary Differential Equations

Assignment 9

Spring 2020 **Subhadip Chowdhury** **Due: Apr 15**

Reading

Section 3.7 and 3.6 from the textbook.

Homework

Bifurcation in Trace-Determinant Plane

Consider the one-parameter family of linear system with real number a as the parameter:

$$\frac{d\vec{r}}{dt} = \begin{bmatrix} a & \sqrt{5-a^2} \\ -1 & 0 \end{bmatrix} \vec{r}$$

Question 1 (2 points).

Sketch the **D** vs. **T** curve corresponding to the family in the trace-determinant plane.

Solution. We have $\mathbf{T} = a, \mathbf{D} = \sqrt{5-a^2}$. So $\mathbf{T}^2 + \mathbf{D}^2 = 5$. The **D** vs. **T** curve Looks like a semi-circle of radius $\sqrt{5}$ in the upper half-plane $\mathbf{D} > 0$. ■

Question 2 (3 points).

Write down the values of a where the qualitative behavior of the system changes. These are the bifurcation values of a.

Solution. The behavior changes when $\mathbf{D} = 0$ or $\mathrm{T}^2 = 4\mathrm{D}$ or $\mathbf{T} = 0, \mathbf{D} > 0$. The corresponding values of a are $\pm\sqrt{5}, \pm 2,$ and 0. ■

Question 3 (8 points).

In a couple of sentences, discuss different types of behaviors exhibited by the system as a increases from $-\sqrt{5}$ to $\sqrt{5}$. Include the boundary cases as well. If your solution curve spirals, find out whether it's clockwise or counterclockwise. Include pictures of sample phase portraits in each case.

You are being asked to identify the types of equilibria only, no analytical calculation is needed. You should not use a computer or any graphing tools other than pen and paper.

Solution.

(i) $a = -\sqrt{5}$ - Degenerate sink.

(ii) $-\sqrt{5} < a < -2$ - Nodal sink.

(iii) $a = -2$ - Defective sink.

(iv) $-2 < a < 0$ - Spiral sink.

(v) $a = 0$ - Center

(vi) $0 < a < 2$ - Spiral source

(vii) $a = 2$ - Defective Source.

(viii) $2 < a < \sqrt{5}$ - Nodal Source.

(ix) $a = \sqrt{5}$ - Degenerate source.

when $\vec{r} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we get $\frac{d\vec{r}}{dt} = \begin{bmatrix} a \\ -1 \end{bmatrix}$. So the spirals are always clockwise.

■

■ **Question 4 (4 points).**

Book problem 3.7.11. Be sure to read the instructions for this problem. This is a problem about bifurcation in (T,D)-plane.

Active Shock Absorbers

■ **Question 5.**

Book problem 3.6.36. Note that this problem does not require any analytical calculation. It asks you to make a qualitative decision based on observations you made about shape of the solution curves in different cases of damped harmonic motion.

Inspired by problem 3.6.36 above, we are going to investigate a modified harmonic oscillator equation with the damping constant b replaced by a function of the velocity $b(v)$. The intent is to model active shock absorbers used in the suspension system of trucks or school bus seats.

Schematically, we can think of a truck seat as being attached to the rest of the truck by a spring and a dashpot (see Figure 1). For the perfect ride, we would want the spring to have spring constant $k = 0$ and the dashpot to have damping coefficient $b = 0$. In this case, the seat would float above the truck. For obvious reasons, the seat does have to be connected to the truck, so at least one of the two constants must be nonzero. The springs are chosen so that k is large enough to hold the seat firmly to the truck, and the damping coefficient b is chosen with the comfort of the driver in mind.

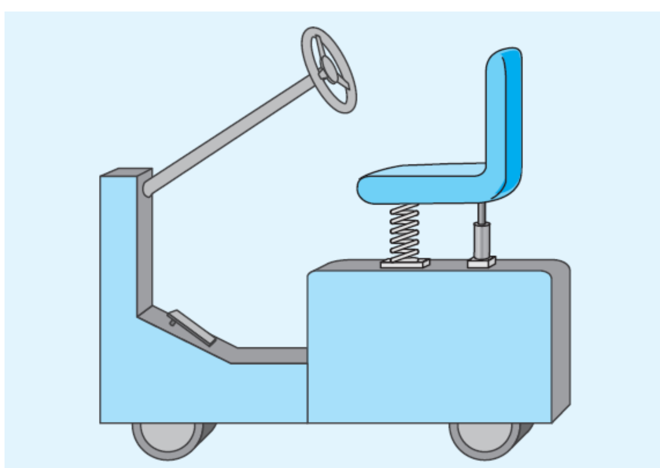


Figure 1

If b is very large, the seat is rigidly attached to the truck, which makes the ride very uncomfortable. On the other hand, if b is too small, the seat may ''bottom out'' when the truck hits a large bump. That is, the spring compresses so much that the seat violently strikes the base. This response is both dangerous and uncomfortable. In practice, designers compromise between having b small (a smooth ride that has danger from large bumps) and b large (protection from large bumps but a rough ride).

Active damping allows adjustment of the damping coefficient according to the state of the system. That is, the damping coefficient b can be replaced by a function of y, the vertical displacement from the rest position and $v = y'$, the vertical velocity of the seat. As a first step in studying the possibilities in such a system, we consider a modification of the harmonic oscillator of the form

$$my'' + b(v)y' + ky = 0$$

where m is the mass of the driver. In this case, the damping coefficient $b(v)$ is assumed to be a function of the velocity v . For this problem, we assume that the units of mass and distance are chosen so that $k = m = 1$, and we study the equation

$$y'' + b(v)y' + y = 0$$

We would like the following requirements to be satisfied by $b(v)$:

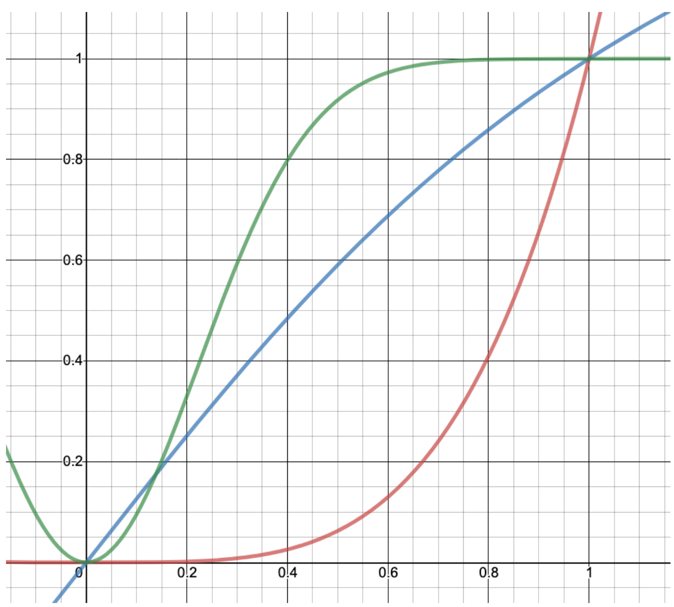
- When the vertical velocity of the seat, i.e. y' is near zero, we want small damping so that small bumps are not transmitted to the seat.
- When the vertical velocity of the seat is large (in our scale, 'large' means close to 1), we want damping to be large to protect from "bottoming out" (and "topping out").

We will consider the behavior of a truck seat for three possible choices of the damping function $b(v)$ as follows:

$$b_1(v) = v^4 \qquad b_2(v) = 1 - e^{-10v^2} \qquad b_3(v) = \frac{4\arctan v}{\pi}$$

Question 6.

What do the three functions have in common and where are they different? You should especially consider whether they satisfy our requirements. Include a picture (hand-drawn is sufficient) of all three functions plotted on the same graph (feel free to use **Desmos** or a graphing calculator to do this).



Solution. All three of the $b(v)$ functions satisfy our requirement that $b(v)$ is small when v is small and $b(v)$ is large when v is large. The only difference they have is how fast or how slow $b(v)$ approaches 1 when v approaches 1. In particular, b_2 (green) is faster than b_3 (blue) and b_3 is faster than b_1 (red). However, just from this information, it is unclear which would be the best choice for the shock absorber. ■

Question 7.

Convert the 2^{nd}-order oscillator equation into a two dimensional system of first order ODEs. Use PPLANE to investigate the solution curves in each case.

- Include a screenshot of the (y, v) phase portrait in the region $-1 \le y \le 1, -1 \le v \le 1$ as well as a component graph of y versus t as $0 \le t \le 100$.
- Your goal is to describe the behavior of the solution $y(t)$ for each $b_i(v)$ and for different initial conditions $y(0)$ and $v(0)$. Be as specific as possible in your answers but keep the end goal in mind. The interpretation of what these pictures reveal is the key.
- Don't overload your answer with large numbers of graphs that all tell the same story.

Solution. The corresponding system of first order ODEs is

$$y' = v \qquad\qquad v' = -b(v)v - y$$

In each case, we will draw the phase portraits using pplane and use the initial conditions $y(0) = 0.1$ and $v(0) = 0.9$ to see what the component graph $y(t)$ vs t looks like.

- In the case of b_1, the phase portrait shows that all solution curves possibly converge toward a stable limit cycle as $t \to \infty$ or towards origin (it might be that the numerical calculations are too computationally difficult after some large amount of time and so the curve doesn't show up).

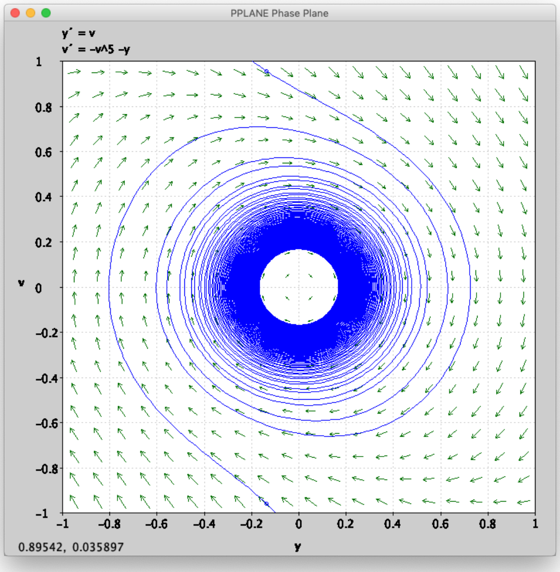
  The $y(t)$ vs t graphs confirm that the the amplitude of oscillation goes down very slowly and it takes a long time to stabilize below a comfortable level.

  The Jacobian at $(0,0)$ *is*

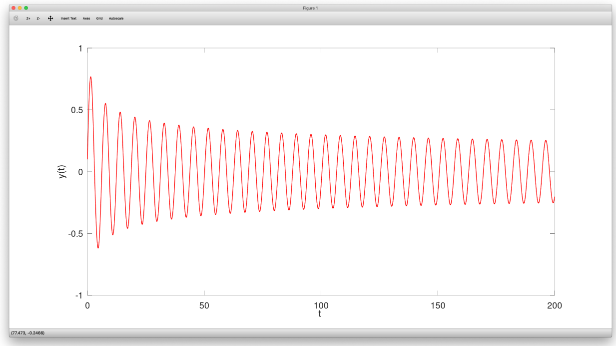
  $$J = \begin{bmatrix} 0 & 1 \\ -1 & -5v^4 \end{bmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

  which has eigenvalue $\pm i$*. So the linearization has a center.*

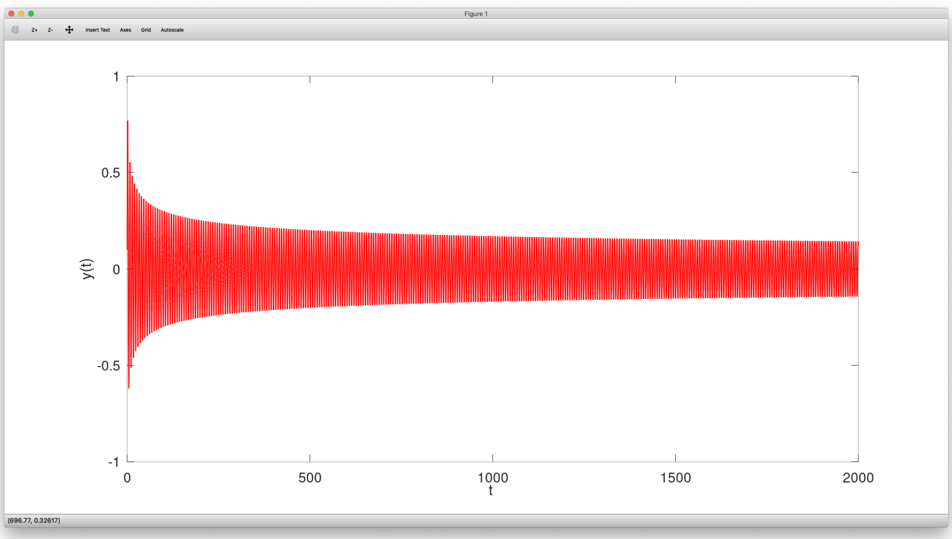
  Realistically, this system experiences minor oscillations from both small and large road bumps.



Phase Portrait for b_1



$y(t)$ vs. t graph for b_1



$y(t)$ vs. t graph for b_1 for a longer time

- In the case of b_2, the phase portrait shows that all solution curves converge toward the origin very fast.

  The $y(t)$ vs t graph confirms that the damping is very efficient and the amplitude goes down very fast to a comfortable level.

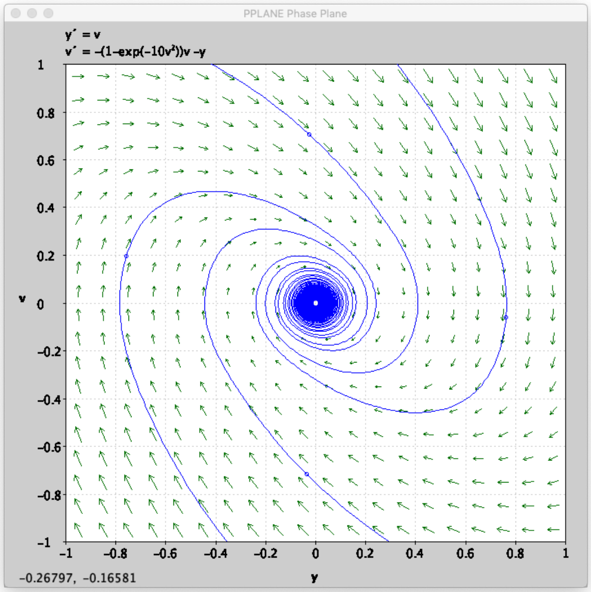
  The following analysis is optional, and will make sense after we learn about Jacobian.

  The Jacobian at $(0,0)$ is

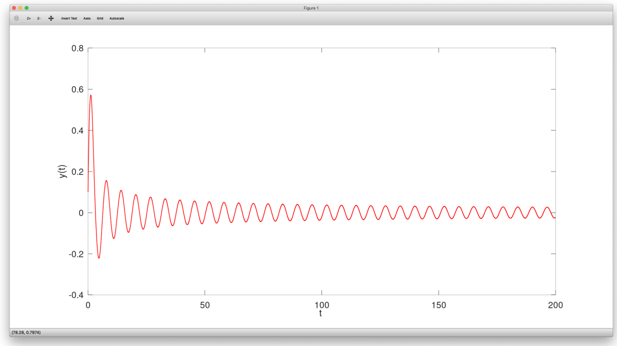
$$J = \left.\begin{bmatrix} 0 & 1 \\ -1 & -e^{-10v^2}\left(20v^2 + e^{10v^2} - 1\right) \end{bmatrix}\right|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

  which has complex eigenvalue with negative real parts. This confirms that the origin is indeed a spiral sink.

  Realistically, for small initial velocities, initial amplitudes are small, so the seat will have its intended effect. However, at large initial velocities and moderate initial displacement, there is



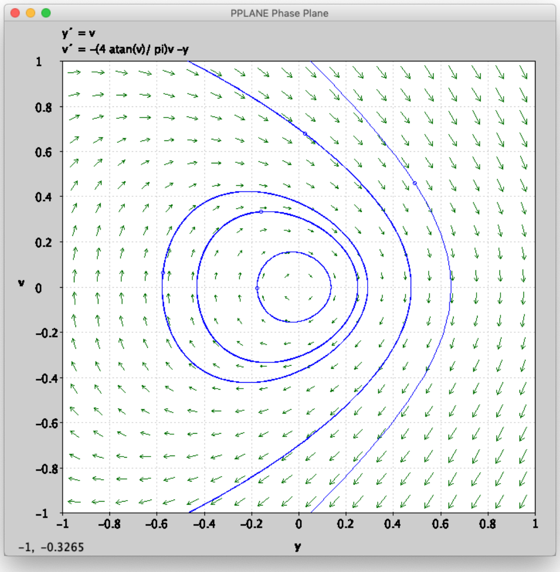
Phase Portrait for b_2



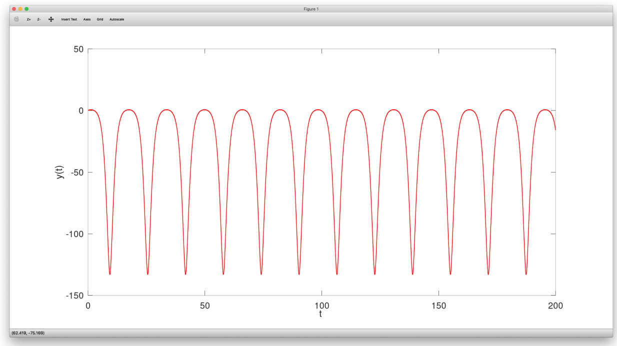
$y(t)$ vs. t graph for b_2

a sharp jump in the y-position, which means the seat has a chance to ''bottom out'', which is a potential safety hazard.

- Finally, for b_3, the solution curves are closed orbits. The $y(t)$ vs t graph confirms that the system follows a periodic cycle and the amplitude of $y(t)$ remains huge over time. The seat never returns



Phase Portrait for b_3



$y(t)$ vs. t graph for b_3

to equilibrium and continuously goes up and down violently every time we hit a pothole! This system should NEVER be used to model an Active Shock Absorber.

■

■ Question 8.

Suppose you are choosing from among the three possible functions $b_i(v)$ above for a truck that drives on relatively smooth roads with an occasional large pothole. In this case, y and v are usually small, but occasionally v suddenly becomes large (i.e. close to 1) when the truck hits a pothole. Which of the functions $b(v)$ above would you choose to control the damping coefficient? Justify your answer in a paragraph.

Solution. Based on our observations above, my personal preference is b_2. Even when $v(0)$ is large, if $y(0)$ is small enough, the sharp jump in $y(t)$ before the seat stabilizes is not too big, so the seat shouldn't bottom out. Obviously damping-wise, b_2 is our best choice. So overall, I would choose b_2 as my option. ■

Instructor's Note: Watch this video to see active shock absorbers in action.

https://www.youtube.com/watch?v=3KPYIaks1UY

Assignment 9 Solutions

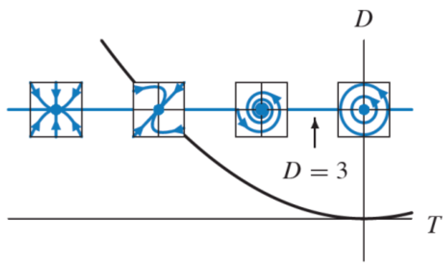
Chapter 3.7.11

11. **(a)** This second-order equation is equivalent to the system

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -3y - bv.$$

Therefore, $T = -b$ and $D = 3$. So the corresponding curve in the trace-determinant plane is $D = 3$.

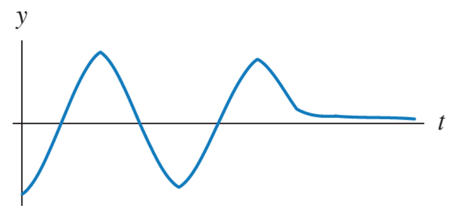
(b)



(c) The line $D = 3$ in the trace-determinant plane crosses the repeated-eigenvalue parabola $D = T^2/4$ if $b^2 = 12$, which implies that $b = 2\sqrt{3}$ since b is a nonnegative parameter. If $b = 0$, we have pure imaginary eigenvalues—the undamped case. If $0 < b < 2\sqrt{3}$, the eigenvalues are complex with a negative real part—the underdamped case. If $b = 2\sqrt{3}$, the eigenvalues are repeated and negative—the critically damped case. Finally, if $b > 2\sqrt{3}$, the eigenvalues are real and negative—the overdamped case.

Chapter 3.6

36. **(a)**



(b) Using the model of a harmonic oscillator for the suspension system, the corresponding system has either real or complex eigenvalues. If it has complex eigenvalues, then solutions spiral in the phase plane and oscillations of $y(t)$ continue for all time. If there are real eigenvalues, then solutions do not spiral, and in fact, they cannot cross the v-axis (where $y = 0$) more than once. Hence, the behavior described is impossible for a harmonic oscillator.

(c) There is room for disagreement in this answer. One reasonable choice is an oscillator with complex eigenvalues and some damping so that the system does oscillate, but the amplitude of the oscillations decays sufficiently rapidly so that only the first two "bounces" are of significant size.