Math 2208: Ordinary Differential Equations

Lecture 5 Worksheet

Spring 2020	Subhadip Chowdhury	Feb 5

Title: Phase Line and Equilibria

Summary: We will continue our qualitative analysis of differential equations by learning how to use phase lines and the classification of equilibrium points of autonomous, first-order ODEs.

§A. Motivation

Consider the differential equation

$$\frac{dy}{dt} = -0.5y + 0.1$$

This model comes from a drug dosing problem where your body removes 50% of the drug every hour and an IV drip administers 0.1mg of drug each hour. The value of y is the number of milligrams of drug in the body at time t. If done correctly, the amount of drug in your system will stabilize at a constant level. What is that level? Be sure to fully explain your thinking.

§B. Definitions

Equilibrium point

An **equilibrium point** of an autonomous ODE $y' = f(y)$ is a real number c where the rate of change is 0. If c is an equilibrium point of an autonomous ODE, then the constant curve $y(t) = c$ is a solution of the DE.

Phase Portrait

A one dimensional phase portrait of an autonomous DE $y' = f(y)$ is a diagram which indicates the values of the dependent variable for which y is increasing, decreasing or constant. Sometimes the vertical version of the phase portrait is called a **phase line.**

§C. Algorithm For Drawing A Phase Line

Step 1. Draw a vertical line.

Step 2. Find the equilibrium points (i.e. values such that $\frac{dy}{dt} = 0$) and mark them on the line.

Step 3. Find intervals for which $\frac{dy}{dt} > 0$ and mark them with up arrows $\uparrow$.

Step 4. Find intervals for which $\frac{dy}{dt} < 0$ and mark them with down arrows $\downarrow$.

■ Question 1.

Consider the autonomous differential equation $\frac{dy}{dt} = y(1-3y)(y-2)$.

(a) Find the equilibrium points of the DE.

(b) Determine the values of y for which $y(t)$ is increasing and decreasing.

(c) Draw the phase line for this ODE.

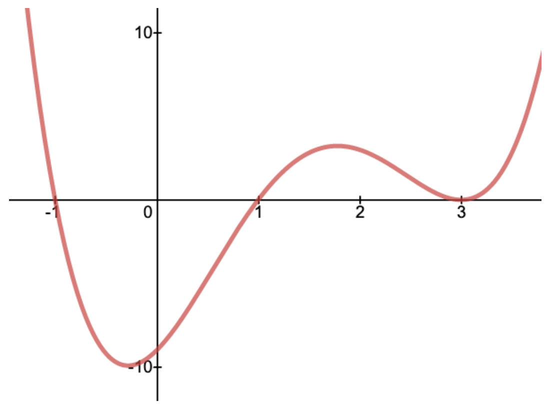
§D. **Obtaining Solution Information from Phase Lines**

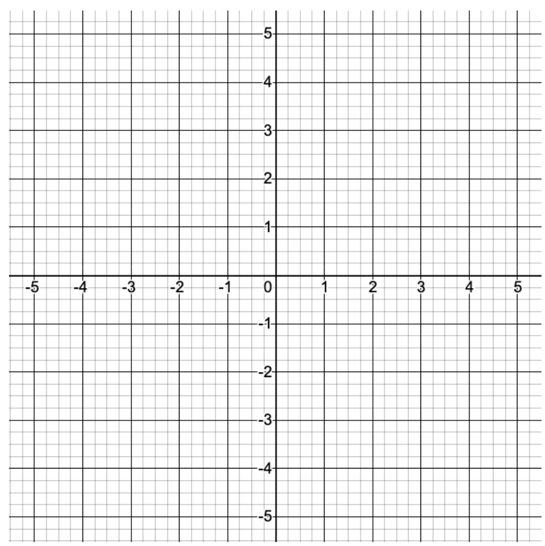
Consider $y' = f(y)$ where $f(y)$ is a **continuously differentiable** function (why do we need this condition?) and $y(t)$ is a solution to an autonomous ordinary differential equation. The following conclusions are consequences of the EUT.

- If $f(y(0)) = 0$ then $y(t) = y(0)$ for all t and $y(0)$ is an equilibrium point.
- If $f(y(0)) > 0$ then $y(t)$ is increasing for all t and either $y(t) \to \infty$ as t increases or $y(t)$ tends to the first equilibrium point larger than $y(0)$.
- If $f(y(0)) < 0$ then $y(t)$ is decreasing for all t and either $y(t) \to -\infty$ as t increases or $y(t)$ tends to the first equilibrium point smaller than $y(0)$.

■ **Question 2 (Drawing Phase Lines from Qualitative Information Alone).**

Draw the phase line in the space below for the ODE $y' = f(y)$ where the graph of $f(y)$ vs. y looks like the graph on the right.
Then draw graphs of various particular solutions (going both forward and backwards in time) starting at $y(0) = -2, y(0) = 0, y(0) = 1, y(0) = 2$ and $y(0) = 4$ in the ty−plane given below.



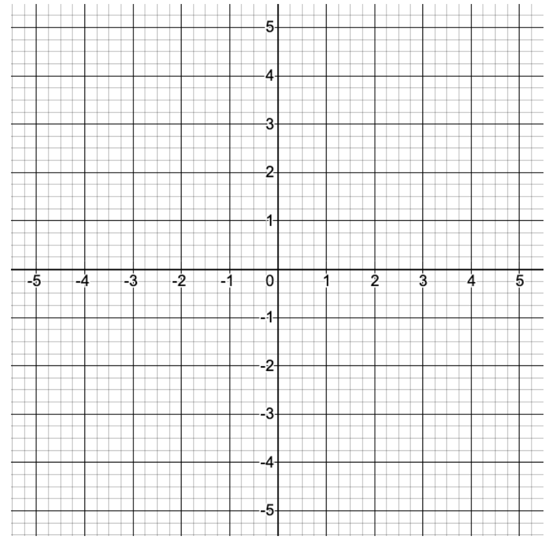


■ **Question 3.**

Draw the phase line for the ODE $y' = \frac{1}{1-y}$. Skecth a couple of solution curves.

§E. **Classifying Equilibrium Points: Sink, Source or Node**

An equilibrium point c splits the phase line into two different regions. So there are four possible scenarios for the behavior of y' near c: $(+, 0, +)$, $(+, 0, -)$, $(-, 0, +)$ and $(-, 0, -)$.



■ Question 4.

In each of the above cases, draw the phase line near the point c and then classify the corresponding critical points as asymptotically stable (aka **attractor** or **sink**), unstable (aka **repellor** or **source**) or neither (aka **node**).

■ Question 5.

Then draw graphs of the function $f(y)$ near equilibrium points classified as a sink, source or node corresponding to the phase line in the example above.

Theorem E.1: Linearization Theorem

Suppose y_0 is an equilibrium point of the differential $y' = f(y)$ where f is a continuously differentiable function. Then,

- if $f'(y_0) < 0$, then y is a sink;
- if $f'(y_0) > 0$, then y is a source; or
- if $f'(y_0) = 0$, then more information is needed to classify the equilibrium point.

■ Question 6.

What can you say about the $y = 0$ equilibrium point of the following ODE?

$$y' = y\left(\cos\left(y^5 + 2y\right) - 27\pi y^4\right)$$

■ Question 7.

Suppose $y' = f(y)$ has an equilibrium point at $y = y_0$ and

(a) $f'(y_0) = 0$ and $f''(y_0) > 0$ **:** Is y_0 a source, a sink, or a node?

(b) $f'(y_0) = 0$ and $f''(y_0) < 0$ **:** Is y_0 a source, a sink, or a node?

(c) $f'(y) = 0, f''(y_0) = 0,$ and $f'''(y) > 0$ **:** Is y_0 a source, a sink, or a node?

(d) $f'(y_0) = 0, f''(y_0) = 0,$ and $f'''(y_0) < 0$ **:** Is y_0 a source, a sink, or a node?