MATH 221 - Differential Equations

Project 5: Motion of a Nonlinear Pendulum

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A pendulum consists of an idealized point mass moving in a circle at the end of a rigid weightless bar. Suppose $y(t)$ represents the angle from the vertical in radians at time t. When the amplitude of motion of the ball is small enough, we usually make the approximation $\sin(y) \approx y$. This results in the harmonic motion equation

$$y'' + \frac{b}{m}y' + \frac{g}{l}y = 0$$

where

- g is the acceleration due to gravity,
- m is the mass,
- l is the length of the rigid rod, and
- b is the damping coefficient.

But when the amplitudes get bigger, the physics always becomes nonlinear.

Assume for a certain pendulum, the corresponding nonlinear equation of motion is given by

$$y'' + 0.1y' + \sin(y) = 0$$

We are going to try to understand the motion using phase portrait in (y, y')-plane. Note that the initial condition $y(0) = 0$ corresponds to the pendulum being vertical at the beginning.

1. Write down the associated system of two first order ODEs and find the equilibrium points.
2. Use PPLANE to draw some sample solution curves. What would be the best description of the kind of qualitative behavior you observe locally around the equilibrium points?
3. Linearize the system at the equilibrium points and classify their types to justify your description.
4. How does the long term behavior of a solution curve depend on the initial condition $y'(0)$? Think of a situation where you strike the ball of the pendulum to give it a initial velocity. Then try to think what happens if we strike with larger and larger force. Draw the curves in PPLANE that starts from
   - $y(0) = 0, y'(0) = 2$
   - $y(0) = 0, y'(0) = 2.5$
   - $y(0) = 0, y'(0) = 3$

   and give a physical interpretation of the difference between the motions.
5. Justify the following statement using PPLANE:

   "In the absence of damping, a pendulum that swings over once swings over infinitely many times."
6. Does there exist a initial value of $y'(0)$ (with $y(0) = 0$) such that the pendulum doesn't exhibit a periodic behavior over time? How would you physically interpret this?