Math 2208: Ordinary Differential Equations

Lecture 1 Worksheet

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Title: What is an Ordinary Differential Equation?

Summary: We will go through various basic definitions and terminology associated with the study of differential equations in order to introduce you to the language we will be using throughout the course this semester. We'll also discuss ODEs as mathematical models.

§A. Definitions and Terminology

Definition 1.1: Dependent and Independent Variables

In a function of the form $y = f(t)$, the variable t is called *independent* and y is called *dependent*.

Definition 1.2: Differential Equation

An equation containing the derivative of one or more dependent variables, with respect to one or more independent variables is called a *differential equation*.

Differential Equations are classified by type, order and linearity.

Definition 1.3: ODE vs PDE

An *ordinary differential equation* (ODE) contains derivatives with respect to only one independent variable (though there may be multiple dependent variables). A *partial differential equation* contains partial derivatives with respect to multiple independent variables.

Question 1.

Consider the following differential equations: Classify them as either ODEs or PDEs.

$$(A)\frac{d^2u}{dt^2} + \lambda e^u = 0 \qquad (B)\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (C)\frac{du}{dt} + \frac{dv}{dt} + \frac{dw}{dt} = 0$$

Definition 1.4: Order

The *order* of a differential equation is the order of the highest derivative found in the DE.

Question 2.

What is the order of each of the following differential equations?

(a) $\frac{d^2u}{dt^2} + \left(\frac{du}{dt}\right)^3 + 4u(t)\sin(t) = 0$

(b) $\left(\frac{du}{dt}\right)^2 - \frac{u^2}{t^2+1} + \ln(t) = 0$

(c) $y'y''' - 2e^t y'' + 5\cos(t)y' = 20$

The most general form of an $n-$th order ordinary differential equation is

$$\mathbf{F}\left(t, y, y', y'', y''', \dots, y^{(n)}\right) = 0$$

where $\mathbf{F}$ is a real-valued function of $n+2$ variables $t, y(t), y'(t), ..., y^{(n)}(t)$.

Definition 1.5: Normal Form

The normal form of an n-th order differential equation involves solving for the highest derivative and placing all the other terms on the other side of the equation, i.e.

$$\frac{d^n y}{dt^n} = f\left(t, y, y', y'', \dots, y^{(n-1)}\right)$$

For example, the normal form of first order ordinary differential equations looks like:

$$y' = f(t, y)$$

Definition 1.6: Linearity

An $n-$th order differential equation is said to be linear if the function f is linear in the variables $y, y', y'', \dots, y^{(n-1)}$.

To be specific, an $n-$th order linear ODE looks like:

$$a_n(t)\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \dots + a_2(t)\frac{d^2 y}{dt^2} + a_1(t)\frac{dy}{dt} + a_0(t)y = g(t)$$

A nonlinear ODE is one that is not linear in the *dependent variable (or its derivatives)*.

Definition 1.7: Autonomy

An ODE is called *autonomous* if the function f is an explicit function of the dependent variable only, i.e. $y' = f(y)$. Otherwise it's called *nonautonomous*.

Question 3.

Write down an example of one linear non-autonomous ODE and one nonlinear autonomous ODE. Compare your answers with at least one of your neighbors.

§B. Solutions of ODEs

Definition 2.1: Solution

A solution φ of an n-th order ODE is a function $y = \varphi(t)$ which is at least n times differentiable on an interval $\mathbf{I}$ and satisfies

$$\mathbf{F}\left(t, \varphi(t), \varphi'(t), \varphi''(t), \dots, \varphi^{(n)}(t)\right) = 0$$

for all $t \in \mathbf{I}$. The interval $\mathbf{I}$ is called the *interval of definition* for solution to the ODE. A graph of the function φ in y vs. t plane is called the *Solution Curve*.

Definition 2.2: Initial Value Problem

An *initial value problem* or IVP is a problem which consists of an n-th order ordinary differential equation combined with n initial conditions defined at a point t_0 found in the interval of definition.

For example, a first-order IVP looks like

$$y' = f(t, y), \quad y(t_0) = y_0$$

and a second-order IVP looks like

$$y'' = f(t, y, y'), \quad y(t_0) = y_0, \quad y'(t_0) = y_1$$

Note that the 'interval of definition' of a solution to an ODE is not necessarily the same as the 'domain of definition' of the function φ.

Question 4.

Consider the IVP

$$ty' + y = 0, \quad y(1) = 1$$

Confirm that $y = 1/t$ is the solution of this IVP. What is the domain of definition of the function $y = 1/t$? What is the interval of definition of the solution of the ODE? Are these two sets identical?

§C. ODEs as Mathematical Models

Question 5 (Group Work).

Consider the following US population data from $1790 - 1850$. Assume that the year 1790 corresponds to $t = 0$.

Year	t	Population (in millions)
1790	0	3.9
1800	10	5.3
1810	20	7.2
1820	30	9.6
1830	40	12.8
1840	50	17
1850	60	23

Can you estimate the population in 1870 based on this data?

Definition 3.1: Mathematical Model

A mathematical model is a mathematical description of a system or phenomenon. Many physical systems often involve time (the variable t) so that the mathematical description of the model involves the rate of change of a variable with respect to time which can be mathematically represented using differential equations. The solution of the model produces a state of the system at certain points in time: the past, present or future.

Differential equations are used to map all sorts of physical phenomena, from chemical reactions, disease progression, motions of objects, electronic circuits, etc. Most mathematical models of real-world situations do not have analytical solutions.