Transition to Advanced Mathematics

Expository Paper

Fall 2021 **Subhadip Chowdhury** **Math 215**

Writing an expository paper is the biggest writing assignment of Math 215 course, and I want you to choose something that interests you. You will be presenting a self contained discussion of a mathematical topic, with your peers as the intended audience, including at least one proof involving appropriate sophistication, complete with an annotated bibliography.

You will be given the opportunity to conduct a thorough drafting process for this paper by participating in peer reviews. That is, each of you will help your classmates by reviewing your final drafts leading up to final paper submission. I will also help you throughout the process by having regular check-ins.

I have included some project ideas at the end of this document, the topics were chosen to include areas that we do not normally get to discuss in traditional mathematics courses. This is by no means an exhaustive list. If you have an idea for a topic that is not on the list, please discuss it with me. Note that you may not choose a topic on which you have previously written a paper for another course or experience.

§A. Learning Objectives

- Find online or offline resources to research on a particular topic.
- Distinguish academic, peer-reviewed, and authentic references from incomplete or misleading ones.
- Read a proof beyond the level of introductory Mathematics and understand the notations and the logic behind it.
- Show that you understand the proof by explaining it in your own words, on paper and verbally through class presentations.
- Explain how your chosen topic relates to other ideas in 'advanced' mathematics and put it in context.
- Give constructive feedback to your peers' presentations and incorporate others' suggestion into your own work.

§B. The Timeline

Step 1. Topic Choice

You have an option of either **working on one topic by yourself** or **working together in a group of two on two separate topics**. In either case, no topic may be repeated, and as such, topics will be assigned on a first come, first served basis.

On **Monday, Oct 18** I will post a sign-up link in Moodle. Please indicate your choice by choosing the topic you wish to write about. If you wish to add/change a topic, discuss it with me beforehand, and then choose 'other' from the list. *If you do not choose a topic by the end of that week, you will fail the course.*

Step 2. Summary and Discussion

You will need to have a short meeting (5-8 minutes) with me between **Monday, Oct 18** and **Friday, Oct 29** to discuss your topic. *You should come prepared to this meeting with at least one source, and a few ideas* about what you would like to include in your paper. Ideally, You should prepare about half a page summary of what your plans are before the meeting, but you do not need to submit anything in Moodle.

> **Note:** Wikipedia is *not* a valid source, but there is often a list of valid sources at the bottom of an article. You should find a book or peer-reviewed journal as a reference before the meeting.

Step 3. Outline and Annotated Bibliography

The next step is to submit an LaTeX outline of your paper along with an annotated bibliography. **In your outline, you need to highlight the main theorem you wish to prove in your paper.** You can change your choice later, but talk to me first if you plan to do so. The deadline for this step is **Friday, Nov 5**.

The entries in an annotated bibliography give all of the bibliographic information about the book, article, or webpage, as well as a brief description of the source.

- Your bibliography must include at least two sources, *at least one of which should be an academic journal or (text)book.*
- In order to write your annotated bibliography, you will have to do more than just find your sources. You will also have to describe how and why you used your source.

Below is an example of an entry in an annotated bibliography:

> Laura Taalman. *Taking Sudoku Seriously.* Math Horizons, pages 5--7, September 2007.
>
> This is an introductory article that gives an overview of the Sudoku puzzle, complete with concise terminology and the rules of the puzzle. It discusses the number of possible Sudoku boards and puzzles, as well as remaining open questions and generalizations of Sudoku. This article provides most of the introductory material needed for the paper.

Step 4. Full First Draft and Presentation

A full first draft (doesn't need to be final version, can have grammatical mistakes etc.) is due by the end of the Thanksgiving week, **Friday, Nov 26**. In the week following that, you will be required to give a short (5 min) presentation of your topic in front of your peers.

What will be the format of the presentation? You should create a slideshow or screen recording (e.g. on a tablet) of **at most 5 min length** that will be played during class. You must upload this video recording (or a link to the video) to the "EP Presentations" channel in our class Team on MS Teams. If you record on Teams, your video is saved on MS Stream, get that link.

Make sure that your video is playable after you upload it. If you believe your video needs subtitles, you should add them. If you record on Teams, subtitles are automatically generated.

What to Include in your presentation? You may include in your presentation any compelling/relevant material. You will not have time to give all of the details of a proof. So only provide an outline or important key ideas.

The presentations should be understandable to the class who have never seen the material before, so majority of your time should spent on describing the terminology. This may require pictures or tables. See the rubric to check what to focus on.

If creating a slideshow, keep the number of slides low, and make sure your slides are not too dense with information. Your audience should learn about your topic mostly by listening, not by reading the slides. However, there should be still enough keywords in the slides to follow the train of thought. As a general rule of thumb, avoid writing full English sentences in your slides and only write phrases.

One of your primary goals is to engage your audience! It may be the case that you do not get to talk about the proof of the main question at all. Instead, you should give enough context to the problem so that the audience finds it something to care or be excited about.

Step 5. Peer Review

During the presentation week, you will be required to review the draft and presentation of two of your peers. Obviously, for you to give and receive meaningful feedback, you will need to have a full first draft of your own paper submitted before the start of the presentation week.

Since your partners will rely on your feedback in order to improve their papers, you will need to complete your review of their papers in a timely manner. The deadline for this step is **Wednesday, Dec 8, 5PM EST**.

You will be provided an assessment form for each of your partners' papers you are reviewing. You do not need to assign a numerical score. Note that you are not grading or being graded by your classmates. Instead, I will be grading how meaningful your feedbacks to your partners are. You will receive full credit for this phase of the expository paper if you provide a meaningful review of your peers' paper during the workshop.

Step 6. Response to Peer Review and Final Submission

The last phase of the EP writing process is to submit the final version of your EP after you have made any necessary adjustments according to any feedback you received.

Your submission packet will consist of the following files:

- Your responses to the peer reviews you received.
- The final version of your EP.

The deadline for this step is **Wednesday, December 15, 5PM EST**.

§C. Grading

The grading for the final presentation and the expository paper will be according to the rubric on the following page.

MATH 215: Presentation Grading Rubric

Mathematical Content	**/(2+2)**
• Did the speaker demonstrate adequate understanding of the content? (2=adequate, 1=marginal, 0=unsatisfactory) • Was the amount and sophistication of content presented appropriate for the task? (2=adequate, 1=marginal, 0=unsatisfactory)	
Presentation Style	**/(1+1+1+1+1+1)**
• Was the voice of the speaker of appropriate volume and clear? (1=yes, 0=no) • Was the pace of the presentation appropriate? (1=yes, 0=no) • Did the speaker make good use of the board and/or prudent use of media/slides? (1=yes, 0=no) • Was the time allotted for the presentation used judiciously? (1=yes, 0=no) • Did the speaker demonstrate sufficient preparation and practice for the presentation? (1=yes, 0=no) • Did the speaker engage appropriately with the audience? (1=yes, 0=no)	
Clarity and Organization	**/(1+1+1+1)**
• Was there a clear overall organization to the presentation? (1=yes, 0=no) • Were sufficient and clear examples given when appropriate? (1=yes, 0=no) • Was sufficient motivation for the mathematics given when appropriate? (1=yes, 0=no) • Were the explanations of terminology and the statement of theorems clearly presented as appropriate? (1=yes, 0=no)	
Total	**/14**

MATH 215: Final Paper Grading Rubric

Introduction	**/(2+2)**
• Is the topic introduced in a clear and compelling manner? (2=adequate, 1=marginal, 0=unsatisfactory) • Does the introduction provide a logical framework for the paper? (2=adequate, 1=marginal, 0=unsatisfactory)	
Background	**/(3+3)**
• Is an appropriate amount of background content (definitions, terminology, lemmas etc.) included for the reader to understand the paper? (3=strong, 2=adequate, 1=marginal, 0=unsatisfactory) • Are all the mathematical notation and terminology defined correctly and explained clearly? (3=strong, 2=adequate, 1=marginal, 0=unsatisfactory)	
Proof Structure	**/(1+2+1+3)**
• Does the paper contain at least one proof of a result pertinent to the topic? (1=yes, 0=no) • Is the result being proven clearly stated? (2=adequate, 1=marginal, 0=unsatisfactory) • Is the proof prefaced with a brief description of the proof strategy? (1=yes, 0=no) • Is the proof easy-to-read and written using the correct LaTeX environment? (3=strong, 2=adequate, 1=marginal, 0=unsatisfactory)	
Tables/Figures/Diagrams	**/(2+1+1)**
• Does the paper include sufficient figures, tables, diagrams, equations etc. to make the content clear? (2=adequate, 1=marginal, 0=unsatisfactory) • Are all the above properly labeled and captioned? (1=yes, 0=no) • Are all the above properly cited (if not the author's own)? (1=yes, 0=no)	
Examples	**/(1+2+1)**
• Does the paper include at least one example different from those in the author's sources? (1=yes, 0=no) • Is the choice of examples simple and illuminating enough to make the content clearer? (2=adequate, 1=marginal, 0=unsatisfactory) • Is the example written using the proper LaTeX environment? (1=yes, 0=no)	
Mathematical Content	**/(4+4+4)**
• Is the mathematical content correct? (4=E, 3=M, 2=P, 1=X, 0=N) • Does the author demonstrate a clear understanding of the mathematics? (4=exceptional, 3=strong, 2=adequate, 1=marginal, 0=unsatisfactory) • Is the sophistication of the mathematics discussed appropriately challenging for the course level? (4=exceptional, 3=strong, 2=adequate, 1=marginal, 0=unsatisfactory)	
Annotated Bibliography	**/(2+2)**

• Does the paper acknowledge external sources with appropriate citations? (2=adequate, 1=marginal, 0=unsatisfactory) • Does the bibliography have sufficient and appropriate annotations? (2=adequate, 1=marginal, 0=unsatisfactory)	
Writing and organization	**/(3+3+2+2+2+2)**
• Is the purpose of the paper and the topic clear and consistent throughout? (3=strong, 2=adequate, 1=marginal, 0=unsatisfactory) • Is the paper organized in a logical manner that makes it easy to read? (3=strong, 2=adequate, 1=marginal, 0=unsatisfactory) • Are there smooth transitions between paragraphs and sections? (2=adequate, 1=marginal, 0=unsatisfactory) • Does the paper provide some conclusion or applications of the main result? (2=adequate, 1=marginal, 0=unsatisfactory) • Does the paper use correct grammar, punctuation, and spelling? (2=adequate, 1=marginal, 0=unsatisfactory) • Is the paper of the appropriate length (5-7 pages in the provided template, without including tables, figures, bibliography etc.)? (2=adequate, 1=marginal, 0=unsatisfactory)	
Total	**/56**

§D. General Requirements

(a) The expository paper is to be roughly 5-7 pages in length (not including the annotated bibliography) and written for your fellow students in this class. If your paper contains many diagrams, adjust the length accordingly (i.e. the length increases).

(b) The paper has to be typeset in LaTeX. Use 12 pt. font, double-spacing, with margins of 1'' on the top, bottom, right, and left. (The spacing and margins are already set up for you in the LaTeX template that has been posted on Moodle.)

(c) Your paper should be expository, combining lots of mathematical content (not necessarily new mathematics) with some interesting (but minimal) historical notes.

(d) **Follow the guidelines in "Mathematical Writing Practices"** (Appendix C in the lecture notes). For example, any time you define a concept, introduce a variable, state a result, or present a proof, it should be done clearly and typeset so it is easy to find. Be sure to proof-read your paper many times before turning it in!

(e) The paper must contain the following components:

- **A title and an introduction, in which you explain what the paper is about and why the reader should continue reading.**

  For example, this can include some background information on the problem you are studying. Who has studied the problem? Which mathematicians have contributed to the solution? Why is this problem interesting/important? Are there any connections to history, politics, culture? How are the mathematical conventions of other cultures/times relevant? **Be sure to focus on the Math, not the Mathematician.**

  The prompts vary in the type of background information that is most appropriate. I don't expect you to give an exhaustive account of all mathematical connections. I want you to tell me a compelling story. Give me a reason to care about the problem, or demonstrate that some prominent mathematical figures were interested in the math.

- **Appropriate background material, including notation, terminology, and definitions.**

  Remember that this is not talking about historical background. Rather, in this section, you need to introduce and explain the tools you plan to use in your proof.

- **One or more proofs of some results pertinent to the topic, prefaced by a short descriptions of the basic proof strategy.**

  You must give a complete, clear, and understandable proof. **You don't need to come up with the proof on your own;** you should be able to find references that explain the math. But make sure you understand the logic behind the proof and can present it clearly to the audience.

  For some projects it's more obvious what to prove than others. However in some other topics, there are some options as far as which proofs to include. You should look through several references and decide which math is most related to the class/would be best to discuss.

  > **An Important Note:** You will be graded based on whether or not your choice of proof is of an appropriate difficulty level for this class. Talk to me before you begin.

- **Appropriately labeled and captioned mathematical tables/diagrams (at least one), properly cited if not your own.**
- **Appropriate examples (at least one), different from those in your sources.**

  Motivate the math by working out simple and illuminating examples in detail. Try to find examples that demonstrate key aspects of the proof that you will write down abstractly. The abstraction will make more sense if you ground it in concrete numbers.
- **A conclusion.**

  This might include some applications or further reading instructions, generalizations of the results etc.
- **An annotated bibliography**

  Your work should be well cited. See above for details.

(f) Your final submission will include your written response to peer reviews and the final paper. This way it will be clear how you incorporated feedback into the final paper.

§E. EP Topic Ideas

This document is being provided ahead of time so that you can do a little research before deciding on a project! Wikipedia is a good starting place. From there, you should look for any mathematical connection that you could expound upon.

Be careful when choosing a topic, some of these projects are harder than others, and may require additional preliminary ideas from combinatorics, linear algebra, group theory, analysis etc. I will help you with resources and can help you in office hours, but it will be mostly up to you to understand them (think of it as a mini IS). If you are interested, you may choose to work on a variation or only part of the prompt. Please discuss your ideas with me during the 'Summary and Discussion' week if you wish to do so.

Cantor Set and Fractals

Main Objective: How do we construct the Cantor Set? What is a self-similar set? What is the Box-Counting Dimension? How can we use box-counting to define the fractal dimension of the Cantor set? Give some other examples of Fractals and their fractal dimensions.

Further Exploration: What are some interesting properties of the box-counting dimension? What is an Iterated Function System and what does it have to do with fractals?

Lebesgue Measure and Vitali sets

Main Objective: What is Lebesgue measure? What are Vitali sets? How can we use Vitali sets to prove the existence of a nonmeasurable set?

Further Exploration: What is the Banach-Tarski Paradox? Why is it called a paradox? Is it really a paradox? How do we resolve the paradox? Optionally, read this article to get a rough overview of the topic.

Kakeya Needle Problem

Main Objective: What is a Kakeya needle set? Why is a disk of diameter 1 is an example of a Kakeya needle set. Can you find a set that has smaller area? Abram Besicovitch, a Russian mathematician, was able to show that there is no lower bound for the area of such a region. Watch these youtube videos: [1] and [2] to understand the 'sprouting' method construction by Besikovitch and explain the Math behind the proof.

Solving Recurrence Relations

Main Objective: Prove **Binet's formula** for the Fibonacci numbers. What are linear homogeneous recurrence relations? What is a characteristic equation? How do we find the formula for the n^{th} term using the characteristic equation?

There are many other patterns within the Fibonacci sequence. Try to spot some on your own, or read up on them from Wikipedia, and then use induction to prove the patterns continue indefinitely.

Josephus Problem

Main Objective: What is the Josephus number $\mathrm{J}(n)$? Does it satisfy any recurrence relation? Can you give a formula for $\mathrm{J}(n)$ that *does not rely on using a computer or a recurrence relation*? Give a proof using strong induction. What are binary representations? How does the formula for $\mathrm{J}(n)$ relate to binary representation of n? Prove it.

Further Exploration: Write a computer program that find $\mathrm{J}(n,k)$ in the general case. Find the complexity of the algorithm.

Fermat's Little Theorem

Main Objective: What is Pierre de Fermat's little theorem? What are binomial coefficients? What is Euler's totient function? What is Euler's theorem? How is it used in RSA encryption?

Cauchy Functional Equation

Main Objective: Equations for unknown functions are called functional equations. One of the famous example is as follows.

Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R} \tag{$\star$}$$

First suppose we know that the function f satisfies the relation ($\star$) from above and is continuous. What can you say about f? What if we instead assume it's monotonic? Are there functions that satisfy the equation ($\star$) and are not linear? How does this relate to the Axiom of Choice?

Further Exploration: What are the other Cauchy Functional Equations? What are their solutions if we assume differentiability?

Graph Coloring and the Five-color Theorem

Main Objective: What is a planar graph? Explain the connection between coloring countries on a map and Graph Theory. What is the five-color theorem? Give a proof using Kempe chains.

Further Exploration: What is the four-color theorem? What was the controversy regarding the proof?

Knight's Tour

Main Objective: Look up Knight's tour from Wikipedia. What are closed and open Knight's tours? Can you find a closed knight's tour on the 8×8 chessboard? (This is hard, look up the solution online or write a computer program to find one for you!) What are Hamiltonian Paths? Given a $m \times n$ board, when does a closed knight tour exist? Prove Schwenk's Theorem.

Friends and Strangers

Main Objective: Six people meet at a party. Consider any two of them. They might be meeting for the first time-in which case we will call them mutual strangers; or they might have met before-in which case we will call them mutual acquaintances. Show that either at least three of them are (pairwise) mutual strangers or at least three of them are (pairwise) mutual acquaintances. What does this problem have to do with Graph Theory and the Pigeon Hole Principle? What is a complete graph? What is an edge coloring? What are Ramsey numbers? What are some known bounds on Ramsey numbers? Prove at least one other result concerning Ramsey numbers. (Don't prove Ramsey theorem!)

Erdös–Szekeres Theorem

Main Objective: Prove that any sequence of distinct real numbers with length at least $(r-1)(s-1)+1$ contains a monotonically increasing subsequence of length r or a monotonically decreasing subsequence of length s. What is a geometric version of the theorem? How does this relate to Ramsey Theory?

Further Exploration: Discuss how the theorem relates to the following geometric result: for any positive integer n, there exists a sufficiently large number $f(n)$ such that any finite set of at least $f(n)$ points in the plane in general position always has a subset of n points that form the vertices of a convex polygon. What's the related conjecture?

Game Theory

Start by reading about the Prisoner's dilemma and the Nash equilibrium (this is just background material). Game theory is a vast area with many possible topics to discuss, most of which require some preliminary ideas from Multivariable Calculus or Linear Algebra. So be sure to give plenty of background explanations in your EP and include at least one proof of acceptable difficulty level. Here are two possible directions, you will need to learn about partial derivatives for both.

(a) **Main Objective:** What is Auction Theory? What is the least unique integer bid auction? What is the expected payoff and Nash equilibrium of least unique integer bid auctions with 3 players and an upper bound for the bid amount?

Further Exploration: What if there is no upper bound for the bids?

(b) **Main Objective:** What is the Tragedy of the Commons? What is the (pure-strategy) Nash equilibrium in a two-player Tragedy of the Commons? Can we do better? What is a Pareto optimal outcome?

Further Exploration: What is the (symmetric) optimal outcome with n players? And how does society size affect this outcome? Why is there no asymmetric equilibrium in such games?

Pancakes and Ham Sandwiches

Main Objective: The Ham Sandwich theorem states that it is possible, with one straight cut of your knife, to cut a ham sandwich (but not a ham-and-cheese) into two pieces so that each piece contains exactly half the ham and half the bread. Start by proving the two-dimensional version using the Intermediate Value Theorem. Use similar ideas and the statement of the Borsuk-Ulam theorem (without proof) to prove the Ham Sandwich theorem.

Further Exploration: Find more fun applications of Borsuk-Ulam theorem.

Period Three implies Chaos

Main Objective: For a continuous function $f : \mathbb{R} \to \mathbb{R}$, we say that a point x is a periodic point with period n if

$$\underbrace{f \circ f \circ f \circ \ldots \circ f}_{n}(a) = f^n(x) = x$$

The main goal is to prove the following theorem: If f has a periodic point of period 3, then it also has periodic points of all other periods!

What are some consequences of this result in the area of discrete dynamical systems? The term 'chaos' was defined by mathematicians Li and Yorke, who in fact have a paper with the same title as above. How do they define a 'chaotic' function?

Further Exploration: Explain the second result proved by Li and Yorke that shows that period 3 implies chaos. You do not need to prove it.

Non-Euclidean geometry

Main Objective: This EP topic is a bit open-ended. In three dimensions, there are three classes of constant curvature geometries. All are based on the first four of Euclid's postulates, but each uses its own version of the parallel postulate. The "flat" geometry of everyday intuition is called Euclidean geometry (or parabolic geometry), and the non-Euclidean geometries are called hyperbolic geometry (or Lobachevsky-Bolyai-Gauss geometry) and elliptic geometry (or Riemannian geometry).

Choose one of the non-Euclidean geometries and write an expository article that explains the specific axioms of this geometry, explains how this geometry contrasts and compares with Euclidean geometry, and what the practical uses of this geometry are. Start by discussing analogues of lines, angles, triangles etc. and other geometry theorems from Euclidean geometry that you may have learned in high school. In case of hyperbolic geometry, what are some models that depict a hyperbolic space? How is distance defined in this space?

Further Exploration: What is curvature?

Farey sequence and Diophantine Approximations

Main Objective: What are Farey sequences or Farey fractions? How are they constructed? Prove some interesting properties of Farey sequences, such as the mediant property, the neighbors property etc. What is Diophantine approximation? Find a short proof of Dirichlet's theorem on Diophantine approximation that uses Farey fractions.

Further Exploration: Farey sequences show up in lots of other different areas of Mathematics including dynamics, geometry etc. Find one or more interesting applications of the sequence.

Hamming Code

Main Objective: What are error-detecting and error-correcting codes? Watch these two youtube videos: [1] and [2] to get started. Explain the math in the videos. What is Hamming (7,4) code? What does it have to do with how the bar codes on USPS envelopes work?

Further Exploration: What does this have to do with the null space of a matrix over $\mathbb{Z}_2$?

The Fundamental Theorem of Algebra

Main Objective: What is the fundamental theorem of algebra? There are lots of ways to prove this using 'advanced' mathematics. Find one of these proofs that looks the easiest to you and explain the math in it. I suggest trying to understand the proof that uses the Extreme Value theorem for real valued functions defined on closed sets in the plane. You will need basic ideas of what complex numbers are, how we draw them on a plane, and how modulus is defined.

Catalan Numbers

Main Objective: What are Catalan numbers C_n? How are they constructed using recursion? These have lots of application of C_n in various combinatorial problems (e.g. binary trees, paths on a grid, expressions containing matching pairs of parentheses etc.) in Math and CS. For example, the number of full binary trees (A rooted binary tree is full if every vertex has either two children or no children) with $n+1$ leaves is C_n. It's also equal to the number of ways a convex polygon of $n+2$ sides can split into triangles by connecting vertices. Find some more such applications and prove them using induction or otherwise.

Playing Pool With π

Main Objective: Start by watching the following youtube videos: [1] and [2]; and read the paper by Gregory Galperin linked in the description. Explain the math in them (you can skip the Physics explanations such as conservation of momentum or energy etc.).

The Newton-Raphson method and Julia Sets

Main Objective: This project requires some introduction to what complex numbers are and how they are drawn on the Argand plane. What is a holomorphic map? What is a complex dynamical system? What are attracting, indifferent or repelling fixed points? What are Julia sets and Fatou sets? What is the

Newton-Raphson method? What kind of Julia set fractals can you get from the Newton-Raphson iteration for finding roots of a polynomial? See [1] and read [2] to get started.

Knot Theory

Stirling's Approximation

Hall's marriage theorem