Math 2000-B Handout 1

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§1. Homogeneous Systems and Parametric Vector Form of Solutions

■ Exercise 1.

Suppose we have a general system of m equations in n variables. We can write this as $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$, where A is the m-by-n matrix of coefficients, $\vec{\mathbf{x}} \in \mathbb{R}^n$ is a variable vector, and $\vec{\mathbf{b}} \in \mathbb{R}^m$ is a constant vector.

Here is some standard terminology: the system $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ is called the **associated homogeneous system**, any solution to $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is called a **particular solution**, and any solution to $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ is called a **homogeneous solution**.

(a) Let $\vec{\mathbf{p}}$ be a particular solution (so $A\vec{\mathbf{p}} = \vec{\mathbf{b}}$) and $\vec{\mathbf{h}}$ be a homogeneous solution (so $A\vec{\mathbf{h}} = \vec{\mathbf{0}}$). Show that $\vec{\mathbf{p}} + \vec{\mathbf{h}}$ is another particular solution to the system.

(b) Let $\vec{\mathbf{p}}_1$ and $\vec{\mathbf{p}}_2$ be two particular solutions to the system. Show that $\vec{\mathbf{p}}_1 - \vec{\mathbf{p}}_2$ is a homogeneous solution to the system.

(c) If $\vec{\mathbf{h}}_1$ and $\vec{\mathbf{h}}_2$ are two homogeneous solutions, show that any linear combination $c_1\vec{\mathbf{h}}_1 + c_2\vec{\mathbf{h}}_2$ is also a homogeneous solution. Does the same hold for linear combinations of particular solutions?

(d) Suppose we fix a particular solution $\vec{\mathbf{p}}$ to $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$. Let H be the set of all homogeneous solutions and S be the set of all particular solutions:

$$H = \{\, \vec{\mathbf{h}} \in \mathbb{R}^n \mid A\vec{\mathbf{h}} = \vec{\mathbf{0}} \,\} \qquad \text{and} \qquad S = \{\, \vec{\mathbf{v}} \in \mathbb{R}^n \mid A\vec{\mathbf{v}} = \vec{\mathbf{b}} \,\}.$$

Show that

$$S = \{\, \vec{\mathbf{p}} + \vec{\mathbf{h}}, \text{ where } \vec{\mathbf{h}} \in H \,\}.$$

In other words, we can get *all* solutions to a system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ by "shifting" all the homogeneous solutions by one fixed particular solution. (This follows pretty readily from the previous two questions.)

■ Exercise 2.

Consider the linear system

$$\begin{aligned} 2x_1 - 3x_2 - x_3 + 2x_4 &= -2 \\ x_1 + 3x_3 + x_4 &= 6 \\ 2x_1 - 3x_2 - x_3 + 3x_4 &= -3 \end{aligned}$$

(a) Write it in the form $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

(b) Check that $x_1 = 1, x_2 = 0, x_3 = 2, x_4 = -1$ is a particular solution to this system. The 4-tuple $\langle 1, 0, 2, -1 \rangle$ can be thought of as a point in $\mathbb{R}^4$, or as a **column vector** $\vec{\mathbf{p}} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$.

(c) Find the set of homogeneous solutions of the system.

(d) Using previous exercise, conclude that the solutions to this system can be written in the form

$$\vec{\mathbf{x}} = \vec{\mathbf{p}} + x_3\vec{\mathbf{h}}$$

for a particular solution $\vec{\mathbf{p}}$ and a homogeneous solution $\vec{\mathbf{h}}$.

■ Exercise 3.

(a) Consider $x_1 + 2x_2 - 3x_3 = 6$ as a system of one equation in 3 variables. Show that all the solutions to this system (aka points on the plane) can be written as

$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

What are $\vec{\mathbf{v}} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{\mathbf{w}} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ examples of for this system?

(b) Make up an example of a system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ of 3 equations in 5 variables such that x_2, x_4, and x_5 are free variables. Show that all solutions can be written in the form

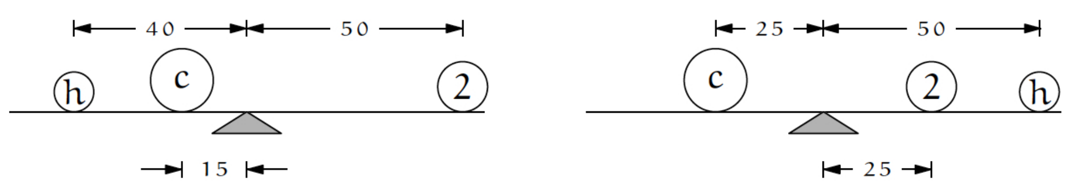
$$\vec{\mathbf{x}} = \vec{\mathbf{p}} + x_2\vec{\mathbf{v}}_1 + x_4\vec{\mathbf{v}}_2 + x_5\vec{\mathbf{v}}_3,$$

for some choice of vectors $\vec{\mathbf{p}}, \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$, and $\vec{\mathbf{v}}_3$.

§2. Practical Applications

■ Exercise 1.

Suppose that we have three objects, we know that one has a mass of 2 kg, and we want to find the two unknown masses. Suppose further that experimentation with a meter stick produces these two balances:



For the masses to balance we must have that the sum of moments on the left equals the sum of moments on the right, where the moment of an object is its mass times its distance from the balance point. Find the two unknown masses.

■ Exercise 2.

We can mix, under controlled conditions, toluene C_7H_8 and nitric acid HNO_3 to produce trinitrotoluene $C_7H_5O_6N_3$ along with the byproduct water (conditions have to be very well controlled -- trinitrotoluene is better known as TNT). In what proportion should we mix them? The number of atoms of each element present before the reaction

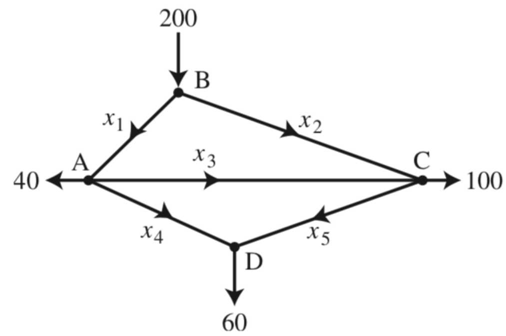
$$x\, C_7H_8 + y\, HNO_3 \rightarrow z\, C_7H_5O_6N_3 + w\, H_2O.$$

must equal the number present afterward. Applying that in turn to the elements C, H, N, and O gives a system of linear equations. Set up and solve the system to find the coefficients in the chemical reaction.

■ Exercise 3.

The basic assumption of a network flow is that the total flow into the network equals the total flow out of the network and that the total flow into a junction equals the total flow out of the junction. The problem of network analysis is to determine the flow in each branch when partial information (such as the flow into and out of the network) is known.

Consider a traffic pattern in the freeway network as shown in the figure below (Flow rates are in cars/minute.)



(a) Find the general flow rates.

(b) Describe the general traffic pattern when the road whose flow is x_4 is closed.

(c) When $x_4 = 0$, what is the minimum value of x_1?