Math 2000 Project 1: The Adjacency Matrix of a Graph*

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- **Purpose:** To learn how powers of a matrix may be used to investigate graphs. Special attention is paid to airline route maps as examples of graphs.
- **Prerequisite:** Matrix multiplication.
- **Resources:** The attached Mathematica Notebook contains the commands needed for some of the exercises. You can also use any other software or online tools.

A *graph* is a finite set of objects called **nodes** (or **vertices**), together with some paths (or **edges**) between some of the nodes, as illustrated below in figure 1. Two vertices connected by an edge are said to be **adjacent**. A *path of length one* is a path that directly connects one node to another. Notice that two vertices may be connected by more than one edge (A and B are connected by **2** distinct edges), that a vertex need not be connected to any other vertex (D), and that a vertex may be connected to itself (F).

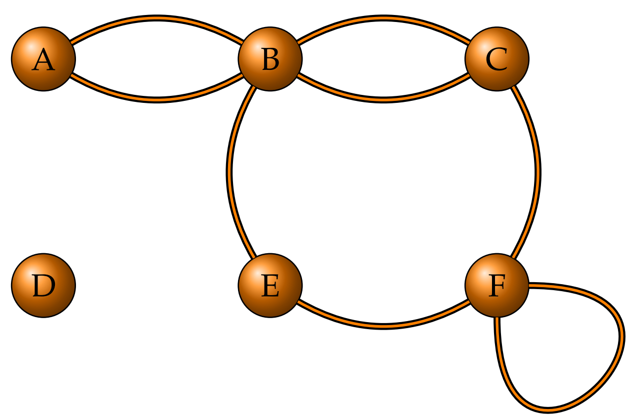


Figure 1

Another example of a graph is the route map that most airlines (or railways) produce. A copy of the northern route map for Cape Air from May 2001 is Figure 2. Here the vertices are the cities to which Cape Air flies, and two vertices are connected if a direct flight flies between them.

*Adapted from a project by Prof. Bill Barker.

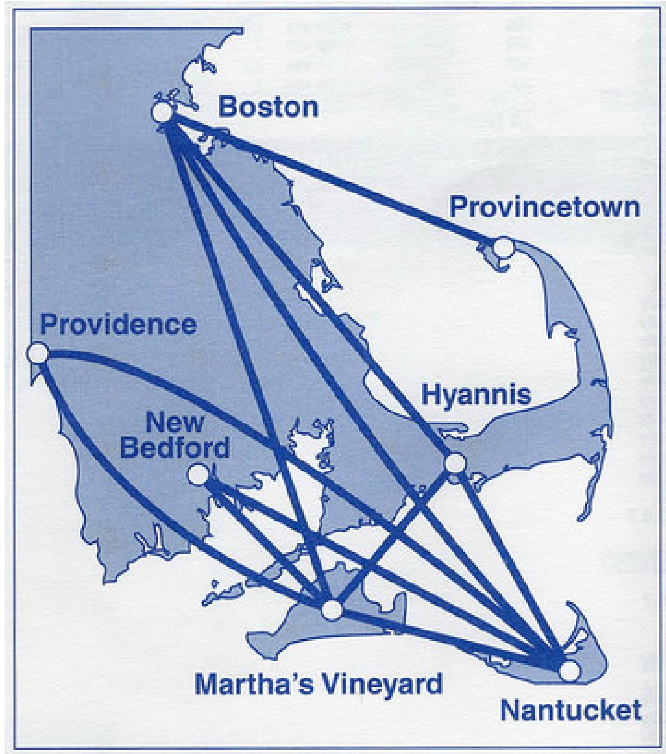


Figure 2: Cape Air Route Map, May 2001

Some natural questions arise about graphs. It might be important to know if two vertices are connected by a sequence of two edges, even if they are not connected by a single edge. In Figure 1, A and C are connected by a two-edge sequence (actually, there are four distinct ways to go from A to C in two steps). In the route map, Provincetown and Hyannis are connected by a two-edge sequence, meaning that a passenger would *have to stop* in Boston while flying between those cities on Cape Air. It might be important to know if it is possible to get from a vertex to another vertex. It is impossible to go from vertex D in Figure 1 to any other vertex, but a passenger on Cape Air can get from any city in their network to any other city given enough flights. But how many flights are enough? This is another issue of interest: what is the minimum number of steps to get from one vertex to another? What is the minimum number of steps to get from any vertex on the graph to any other? While these questions are relatively easy to answer for a small graph, as the number of vertices and edges grows, it becomes harder to keep track of all the different ways the vertices are connected. Matrix notation and computation can help to answer these questions.

When the nodes have been numbered from 1 to n, the *adjacency matrix* A of a graph with n vertices is defined as a $(n \times n)$ matrix whose $(i, j)^{th}$ entry is 1 if there is a path of length one between vertices i and j and 0 otherwise. In other words $A_{ij} = 1$ iff there is an edge of the graph connecting the two vertices i and j.

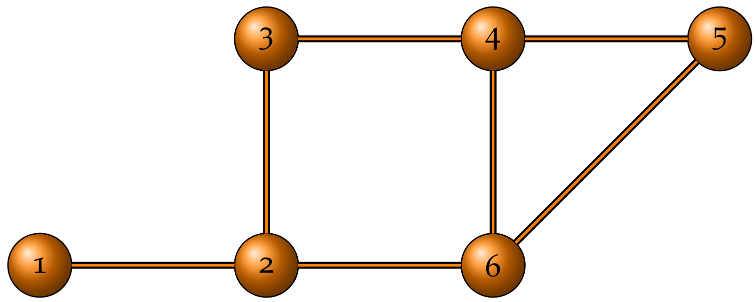


Figure 3

Consider the graph in figure 3. The adjacency matrix of the graph in figure 3 is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Observe that the matrix is symmetric i.e. $A = A^t$. In fact, adjacency matrix of any graph is always a symmetric matrix (Why?).

If the vertices in the Cape Air graph respectively correspond to Boston, Hyannis, Martha's Vineyard, Nantucket, New Bedford, Providence, and Provincetown in numerical order, then the adjacency matrix for Cape Air is

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 1: (5 points)

Find the Adjacency matrix of the graph in figure 1. Note that the entries of the matrix are always 0 or 1.

A *path of length* k is a path made up of k consecutive paths of length one. The same length one path can appear more than once in a longer path; for example, $1 \to 2 \to 1$ is a path of length two from node 1 to itself in figure 3.

Theorem (Interpretation of the powers of an adjacency matrix). *If A is the adjacency matrix of a graph, then the (i, j) entry of A^k is a nonnegative integer which is the number of paths of length k from node i to node j in the graph.*

Exercise 2: (2+2 points)

Consider the matrix A corresponding to figure 3 above. To understand what the theorem says for this example, let us carefully examine the $(6,3)$ entry of A^2. Using the "Row-Column Rule" from Section $\mathbf{2.1}$ in the text, the $(6,3)$ entry of A^2 looks like

$$a_{61}a_{13} + a_{62}a_{23} + a_{63}a_{33} + a_{64}a_{43} + a_{65}a_{53} + a_{66}a_{63}$$

(a) Evaluate each term in the above expression and calculate the sum.

(b) Explain what each term in the sum above tells about paths of length $\mathbf{2}$ from node $\mathbf{6}$ to node $\mathbf{3}$. For example, $a_{62}a_{23} = \mathbf{1} \times \mathbf{1} = \mathbf{1}$. This says that the length one paths $\mathbf{6} \longrightarrow \mathbf{2}$ and $\mathbf{2} \longrightarrow \mathbf{3}$ appear in the graph, and together they give one path from node $\mathbf{6}$ to node $\mathbf{3}$, of length $\mathbf{2}$.

Exercise 3: (4 points)

Use the attached Mathematica Notebook to find A^2 and A^3.

Exercise 4: (2+2 points)

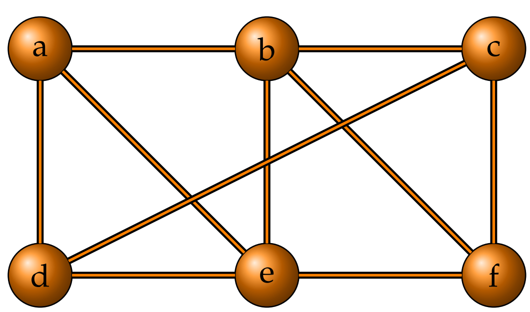
Note that the $(1,2)$ entry of A^2 is zero, so there are no paths of length two from node $\mathbf{1}$ to node $\mathbf{2}$. Verify this by studying the graph. Similarly, notice that the $(6,6)$ entry of A^3 is two, so there are two paths of length three from node $\mathbf{6}$ to itself; study the graph to see that they are $\mathbf{6} \longrightarrow \mathbf{4} \longrightarrow \mathbf{5} \longrightarrow \mathbf{6}$ and $\mathbf{6} \longrightarrow \mathbf{5} \longrightarrow \mathbf{4} \longrightarrow \mathbf{6}$.
In the same way, study the matrices and the graph and answer the following questions.

(a) What are the paths of length two from node $\mathbf{2}$ to itself?

(b) What are the paths of length three from node $\mathbf{3}$ to node $\mathbf{4}$?

Exercise 5: (5 points)

Find the number of paths of length $\mathbf{7}$ between node c and d in the graph below. Use the attached Mathematica notebook for calculation help.



In observing the figures, note that some two-step or three step sequences may not be meaningful. On the Cape Air route map, Nantucket is reachable in two steps from Boston (via

Hyannis or Martha's Vineyard), but in reality this does not matter, since there is a direct flight between the two cities. A better question to ask of a graph might be

Question: What is the least number of edges which must be traversed to go from vertex A to vertex B?

To answer this question, consider the matrix sum matrix sum $S_k = M + M^2 + \ldots + M^k$. The (i, j) entry in this matrix tallies the number of ways to get from vertex i to vertex j in k steps or less. If such a trip is impossible, this entry will be zero.

Definition. When we have a graph, we will say that there is a *contact level* k between node i and node j if there is a path of length *less than or equal* to k from node i to node j.

Thus to find the shortest number of steps between the vertices (i.e. to find the smallest contact level between two nodes), continue to compute S_k as k increases; the first k for which the (i, j) entry in S_k is non-zero is the shortest number of steps between i and j. Note that this process is non-constructive; that is, the shortest number of steps may be computed, but the method does not determine what those steps are.

Another question concerned whether it is possible to go from any vertex in a graph to any other. If a graph has the property that each vertex is connected to every other vertex in some number of steps, then the graph is **connected.**

Question: How can you tell whether or not a graph is connected using the Adjacency matrix?

This should be easy to see from a small graph, but is harder to see from the adjacency matrix. However, there is a calculation that can be done. Suppose that the graph contains n vertices. Then the largest number of steps it could take to go from a vertex to any other vertex is n steps. Thus $S_n = M + M^2 + ... + M^n$ can help us. If there are any zeroes in this matrix, it is impossible for some pair of vertices to connect in n steps or less, so this pair will never connect, and the graph is not connected.

Exercise 6: (2 points)

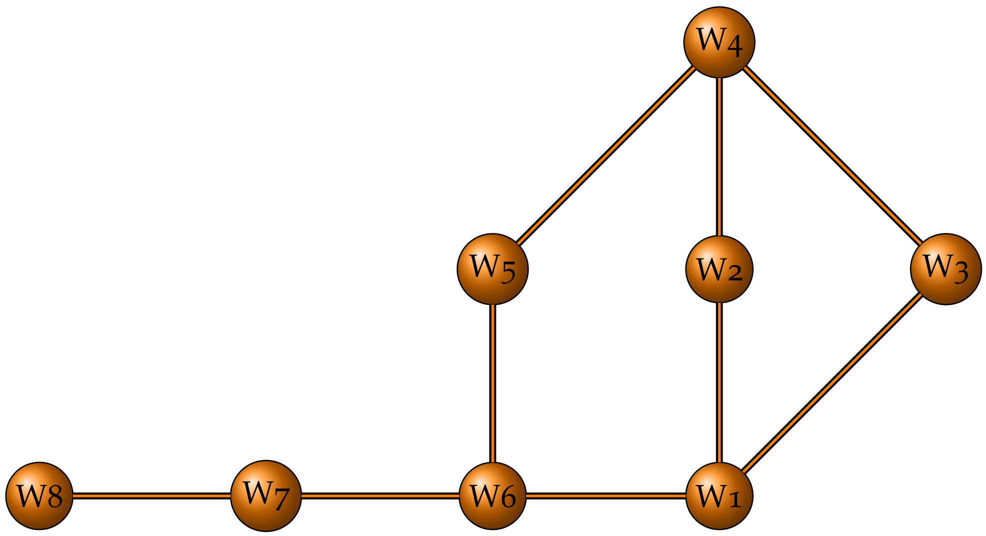
Suppose a graph has the following adjacency matrix.

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Use the attached Mathematica notebook to calculate S_5. Is the graph connected?

Exercise 7: (3+2+2+2+2 points)

Eight workers, denoted W1,..., W8, handle a potentially dangerous substance. Safety precautions are taken but accidents do happen occasionally. It is known that if a worker becomes contaminated, he or she could spread this through contact with another worker. The following graph shows the level one contacts between the workers.



(a) Write the adjacency matrix N for the graph.

(b) Find $S_3 = N + N^2 + N^3$ by hand or using the attached Mathematica notebook.

(c) Which workers have contact level **3** with W3? Which workers have contact level **3** with W7?

(d) Show that **3** is the lowest level of contact between W7 and W4 by observing S_1, S_2 and S_3.

(e) A worker is designated *dangerous* if more that **5** workers are within a contact level **2** (including themselves). Which workers are the most dangerous if contaminated? Which workers are least dangerous?

Exercise 8: (3 + 2 points)

Consider the route map of Spirit Air from 2011 in figure 2.

(a) What is the minimum number of flights necessary to fly between **any** two cities serviced by the airline? [Hint: This is the smallest n such that S_n has no zero entry.]

(b) Give the number of pairs of cities that require this largest minimum number of flights.