Midterm 2 part 2

DUE DATE: APRIL 17, 2019
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Instructions:
This is a take-home exam. As such, your written arguments will be held to a higher standard than on a sit-down in-class exam. Please submit clear and carefully composed solutions, and explain the concepts you are using and the connections among them. As always, points may be deducted for any unjustified steps, and generous partial credit will be given if you explain your thought process to me.
You may consult and use our course materials while taking this exam, including the textbook, class notes, your problem sets, and any of the handouts or Mathematica code on Blackboard. If you use Mathematica, please print and attach your commands and output to your exam. If you are not sure if some resource is allowed, please ask! You may NOT consult the internet or discuss problem specifics with other people. You may email me to ask questions.
When submitting your exam, staple this packet on top, and sign the ''Honor Signature'' to indicate that you followed Bowdoin's Honor Code with respect to this exam.

Full Name: ______________________________

Honor Signature: ______________________________

Total Points Available: 90

§1. Null Space and Column Space

Consider the matrix $A = \begin{bmatrix} 1 & 2 & a \\ 0 & 1 & b \\ 1 & 1 & c \end{bmatrix}$.

Question 1. *[4+3+3 points]*

(a) *Find some condition on* (a, b, c) *that will tell you whether or not* $\mathbf{Col}(A)$ *is all of* $\mathbb{R}^3$.

(b) *Suppose* $\mathbf{Col}(A)$ *is not all of* $\mathbb{R}^3$. *Find a vector* $\vec{\mathbf{u}} \in \mathbb{R}^3$ *with nonzero entries that is linearly independent from any vector in* $\mathbf{Col}(A)$.

(c) *Find a triple* (a, b, c) *such that* $\vec{\mathbf{v}} = \begin{bmatrix} 4 \\ -6 \\ 2 \end{bmatrix}$ *is in* $\mathbf{Nul}(A)$.

§2. Rank

Question 2. *[6 points] Show that any $m \times n$ matrix of rank 1 can be written in the form $\vec{v}(\vec{w}^T)$ for some choice of column vectors $\vec{v} \in \mathbb{R}^m$ and $\vec{w} \in \mathbb{R}^n$.*

Hint: What can you say about the columns?

§3. Determinant

Question 3. *[6 points]*

Compute the determinant of the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 & 6 \\ 0 & 1 & 13 & 0 \\ 0 & 0 & -2 & 5 \\ 3 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 7 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Please do not multiply the matrices to find A explicitly. Use properties of determinant instead.

§4. Geometry of Linear Transformations

Question 4. *[6 points] Find the matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that represents reflection about the straight line $y = 3x$.*

Question 5. *[6 points] Let $A = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$. Compute A^{2019}.*

Question 6. *[7 points] Consider a nonzero vector $\vec{v}$ in $\mathbb{R}^3$. Arguing geometrically, describe the image and the kernel of the linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by*

$$T\left(\vec{x}\right) = \vec{v} \times \vec{x}.$$

Here $\times$ denotes cross product.

§5. Kernel and Image of a Linear Transformation.

Let $\mathbb{P}_2$ be the space of all polynomials in t that have degree ≤ 2.

$$\mathbb{P}_2 = \left\{a + bt + ct^2 \,\middle|\, a, b, c \in \mathbb{R}\right\}$$

Define a linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^2$ as

$$T(f(t)) = \begin{bmatrix} f(5) \\ f(7) \end{bmatrix}$$

Recall that the kernel of T is defined as

$$\ker(T) = \left\{f(t) \in \mathbb{P}_2 \,\middle|\, T(f(t)) = \vec{0} \in \mathbb{R}^2\right\}$$

Question 7. *[6+2+1 points]*

(a) *Describe the kernel of the transformation T.*

(b) *Find a basis of $\mathbf{ker}(T)$.*

(c) *What is the dimension of $\mathbf{ker}(T)$?*

Recall that the image of T is defined as

$$\mathbf{im}(T) = \left\{\vec{v} \in \mathbb{R}^2 \,\middle|\, \exists f(t) \in \mathbb{P}_2 \text{ such that } T(f(t)) = \vec{v}\right\}$$

Question 8. *[6+2+1 points]*

(a) *Explain why given any two real numbers α and β, we can find a polynomial $f(t) = a + bt + ct^2$ such that $f(5) = \alpha$ and $f(7) = \beta$.*

Hint: You might want to think about consistent system of linear equations.

(b) *What is the image of T?*

(c) *Is T an isomorphism?*

§6. An Abstract Vector Space

Let $\mathbb{R}^+$ denote the set of positive real numbers. On $\mathbb{R}^+$, we define "exotic addition" of x and y as

$$x \oplus y = xy (\text{usual multiplication})$$

and "exotic scalar multiplication" by a real number k as

$$k \otimes x = x^k$$

Question 9. *[4+4+3 points]*

(a) *Show that $\mathbb{R}^+$ endowed with above definition of "addition" and "scalar multiplication" is a vector space.*

Hint: Check that it satisfies the ten defining properties of a vector space.

(b) *Show that $T(x) = \ln x$ is a linear transformation from $\mathbb{R}^+$ to $\mathbb{R}$, where $\mathbb{R}$ has ordinary addition and multiplication operations.*

Hint: Check that $T(x \oplus y) = T(x) + T(y)$ and $T(k \otimes x) = kT(x)$.]

(c) *Is T an isomorphism?*

Hint: Check whether T is one-to-one and onto.

§7. Matrix of a Linear Transformation with respect to a Basis

Recall that given a n-dimensional vector space $\mathbb{V}$ and a basis $\mathfrak{B} = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \ldots, \vec{\mathbf{v}}_n\}$, the coordinates of a vector $\vec{\mathbf{v}} \in \mathbb{V}$ with respect to the basis $\mathfrak{B}$, denoted $[\vec{\mathbf{v}}]_{\mathfrak{B}}$, is defined as the unique $n \times 1$ vector $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}$ such that

$$\vec{\mathbf{v}} = c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3 + \ldots + c_n\vec{\mathbf{v}}_n.$$

Suppose $\mathbb{V} = \mathbb{R}^n$ as well. In handout 8, we saw that the coordinate mapping linear transformation $F : \mathbb{R}^n \to \mathbb{R}^n$ defined as

$$F\left(\vec{\mathrm{x}}\right) = \left[\vec{\mathrm{x}}\right]_{\mathfrak{B}}$$

is invertible. Let G be the inverse transformation of F. Thus

$$G\left(\left[\vec{\mathrm{x}}\right]_{\mathfrak{B}}\right) = \vec{\mathrm{x}}$$

Since G is a linear transformation, there is some $n \times n$ matrix C such that

$$\vec{\mathrm{x}} = G\left(\left[\vec{\mathrm{x}}\right]_{\mathfrak{B}}\right) = C\left[\vec{\mathrm{x}}\right]_{\mathfrak{B}}$$

Let's find this matrix C for a particular example.

Question 10. *[4 points] Let* $\vec{\mathbf{v}}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \vec{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, *and* $\vec{\mathbf{v}}_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ *form a basis* $\mathfrak{B}$ *of* $\mathbb{R}^3$*. Find the matrix* C *such that* $\vec{\mathbf{x}} = C\left[\vec{\mathrm{x}}\right]_{\mathfrak{B}}$ *for all* $\vec{\mathbf{x}}$ *in* $\mathbb{R}^n$*.*

Hint: The matrix of a linear transformation consists of column vectors that are images of $\vec{\mathbf{e}}_1$,$\vec{\mathbf{e}}_2$ *and* $\vec{\mathbf{e}}_3$ *under then transformation.*

Now assume T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ and let B be the $n \times n$ matrix that transforms $\left[\vec{\mathrm{x}}\right]_{\mathfrak{B}}$ into $\left[T\left(\vec{\mathrm{x}}\right)\right]_{\mathfrak{B}}$ i.e. $B\left[\vec{\mathrm{x}}\right]_{\mathfrak{B}} = \left[T\left(\vec{\mathrm{x}}\right)\right]_{\mathfrak{B}}$. The matrix B is called the $\mathfrak{B}$-matrix of T.

Question 11. *[8 points] Show that*

$$B = \begin{bmatrix} \left[T\left(\vec{\mathrm{v}}_1\right)\right]_{\mathfrak{B}} & \cdots & \left[T\left(\vec{\mathrm{v}}_n\right)\right]_{\mathfrak{B}} \end{bmatrix}$$

Since we can always find a matrix A such that $T\left(\vec{\mathrm{x}}\right) = A\vec{\mathbf{x}}$, it is natural to ask how A and B are related to each other. We observe that

$$T\left(\vec{\mathrm{x}}\right) = A\vec{\mathrm{x}} = AC\left[\vec{\mathrm{x}}\right]_{\mathfrak{B}}$$

and

$$T\left(\vec{\mathrm{x}}\right) = C\left[T\left(\vec{\mathrm{x}}\right)\right]_{\mathfrak{B}} = CB\left[\vec{\mathrm{x}}\right]_{\mathfrak{B}}$$

Above equations can be summarized in a diagram as follows:

$$\begin{tikzcd} \vec{x} \arrow[r, "A"] & T(\vec{x}) \\ {[\vec{x}]_{\mathfrak{B}}} \arrow[u, "C"] \arrow[r, "B"] & {[T(\vec{x})]_{\mathfrak{B}}} \arrow[u, "C"] \end{tikzcd}$$

Hence $AC = CB$. Since G is an invertible linear transformation, C is an invertible matrix. So we can write

$$\boxed{B = C^{-1}AC}$$

Question 12. *[8 points] Let* $T(\vec{x}) = A\vec{x}$ *be a linear transformation from* $\mathbb{R}^3$ *to* $\mathbb{R}^3$ *where* $A = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$.
Let $\mathfrak{B}$ *and* C *be the same as question 10. Find* B *using the formula from question 11 and check that* $AC = CB$.