Practice Problems and Individual Assignments

Subhadip Chowdhury

- *You must complete and attach an Assignment Cover Sheet with every submission. This sheet can be downloaded from Blackboard.*
- **Recommendation:** Show all work! It's not helpful to you to just write down a final answer to a question. When you're reviewing for exams later, you'll want the *thought* process behind how to approach a problem, not only the answer to the problem. And if you write it out now, it will sink into your brain and you'll remember it better later.
- Homeworks are to be turned in to me at the beginning of class on the specified date.

Practice Problems 1

Do the following before next class on 1/28.

- Read section 1.1 and 1.2 from the book. We haven't covered all of 1.2 yet.
- Watch: Essence of Linear Algebra (ELA) preview
- Watch: ELA Chapter 1: Reminders about vectors
- Work out the problems 1.1.(Practice Problems,7,8,15) and 1.2.(Practice Problems,21,22). You do not have to submit solutions to these problems. But it is a good idea to work them out to supplement the reading.

Assignment 1, due Wednesday Jan 30

Exercise 1

Problems 1.1(24, 30, 31, 32).

Exercise 2

Problems 1.2.(2, 11, 12, 14, 19, 25, 26). You will need the definition of Pivot position on page 14 to work out the last couple of exercises.

Practice Problems 2

Do the following before next class on 1/30.

- Review your multivariable knowledge about vectors. Read section 1.5 from the book. The notations might be a little confusing and that's fine; we will come back to 1.5 again next week.
- Try to work out the problems 1.5.(Example 1, Example 2, Practice Problem 1, Exercise 1). You do not have to submit solutions to these problems. But it is a good idea to work them out to supplement the reading.

Assignment 2, due Friday Feb 1

Exercise 1

Problems 1.2.$(16, 28, 29, 30, 31)$.

Exercise 2

Problems 1.5.$(4, 10, 12, 17)$.

Practice Problems 3

Do the following before next class on 2/1.

- Read section 2.1 from the book. We will finish the 'Properties' section next time.

Assignment 3, due Wednesday Feb 6

Exercise 1

Problems 2.1.$(6, 10, 12, 18, 19)$.

Exercise 2

The following problems are from Handout 1, available online in Blackboard. Problems 1.3 and 2.$(1, 3)$.

Optional Group Project 1, due Feb 8

Available on Blackboard.

Practice Problems 4

Do the following before next class on 2/6.

- Finish reading section 2.1. Finish reading section 1.5.
- Work out the problems 2.1.(Practice Problems, 15, 16) and 1.5.(Practice Problems, 23,24). You do not have to submit solutions to these problems. But it is a good idea to work them out to supplement the reading.

Assignment 4, due Friday Feb 8

Exercise 1

Problems 1.4.$(2, 3, 8, 15)$.

Exercise 2

Problems 1.5.$(27, 28, 35)$.

Practice Problems 5

Do the following before next class on 2/8.

- Read section 1.3. Start reading section 1.7 and 1.4.
- Watch: ELA Chapter 2: Linear combinations, span, basis
- Work out the problems 1.3.(Practice Problems, 23,24). You do not have to submit solutions to these problems. But it is a good idea to work them out to supplement the reading.

Assignment 5, due Wednesday Feb 13

Exercise 1

Problems 2.1.$(23, 24)$.

Exercise 1

Problems 1.3.$(8, 18, 22, 32)$.

Exercise 2

Problems 1.4.$(13, 22, 30)$.

Practice Problems 6

Do the following before next class on 2/11.

- Finish section 1.7 and 1.4.
- Work out the problems 1.7.(Practice Problems, 21,22) and 1.4.(Practice Problems, 23,24). You do not have to submit solutions to these problems. But it is a good idea to work them out to supplement the reading.

Assignment 6, due Wednesday Feb 13

Exercise 1

Problems $1.4.(32 - 34)$.

Exercise 2

Problems $1.7.(14, 16 - 18, 32, 37, 38)$.

Assignment 7, due Friday Feb 15

Exercise 1

Problems from Handout 2. Problems $4, 5(c, d, e, f, g, h)$.

Practice Problems 8

Do the following before next class on 2/15.

- Read 2.2. Then start reading section 2.3. We will cover the part about elementary matrices and linear transformations after the midterm.
- Work out the problems 2.2.(Practice Problems, 9,10). You do not have to submit solutions to these problems. But it is a good idea to work them out to supplement the reading.

Assignment 8, due Wednesday Feb 20

Today in class, we defined a matrix $A_{n\times n}$ to be invertible if there exists a matrix B such that $AB = I_n$. Then we claimed the following statement to be true:

Lemma $(\star)$**.** If A and B are square matrices such that $AB = I$ where I is the identity matrix, then $BA = I$.

Note that with our definition, we need to prove $(\star)$ to show that $(A^{-1})^{-1} = A$.

The book on the other hand defines inverse to be a matrix B such that $AB = BA = I_n$, bypassing the need to prove $(\star)$ above. For now we will take $(\star)$ as granted, since we need to learn a bit more about range of a matrix and basis before we can prove it. We will come back to this after the Midterm. For now, you should use the definition from the book i.e. $AB = BA = I$.

Exercise 0

(You do not have to submit the solution to this problem. This is theorem 6b in the book. Try proving it yourself first.)

If A and B are invertible square $n \times n$ matrices, show that $(AB)^{-1} = B^{-1}A^{-1}$.

Exercise 1

Problems 2.2.$(16, 20, 25, 31, 32, 38)$.

Couple of hints:

- In 16, it is not known whether A is invertible to begin with. So we can't use theorem 6b directly with A and B. You should try to find two matrices to which you can apply theorem 6b.
- In 20a, write B as product of two invertible matrices and use theorem 6b. In 20b, pay attention to the comment in the problem. You will need to use the result of exercise 16.

Optional Group Project 2, due Mar 1

Available on Blackboard. You can still submit group project 1 before Feb 25.

Practice Problems 9

Do the following before next class on 2/18.

- Finish reading section 2.3. We will cover the part about elementary matrices and linear transformations after the midterm.
- Work out the problems 2.3.(Practice Problems, 11,12). You do not have to submit solutions to these problems. But it is a good idea to work them out to supplement the reading.

Assignment 9, due Wednesday Feb 20

Exercise 1

Unless otherwise specified, assume that all matrices in these exercises are $n \times n$. Pay attention to the special instruction in 31.

Problems 2.3.$(6, 13, 15, 17, 19, 22, 31)$.

Practice Problems 10

Do the following before next class on 2/27.

- Read the Elementary Matrices part from 2.2. Then read 3.1 and start reading 3.2. We will use all the theorems in section 3.2 without proof.
- Work out the problems 3.1.(Practice Problems, 1,2,3,39,40). You do not have to submit solutions to these problems. But it is a good idea to work them out to supplement the reading.

Assignment 10, due Friday Mar 1

Exercise 1

Problems 2.2.$(33, 34, 35)$.

Exercise 2

Problems 3.1.$(13, 19-21, 25-32)$.

Practice Problems 11

Do the following before next class on 3/1.

- Read section 3.2. Start reading 4.1. We will finish 4.1 and parts of 4.2 next class.
- Work out the problems 3.2.(Practice Problems, 27,28). You do not have to submit solutions to these problems. But it is a good idea to work them out to supplement the reading.

Assignment 11, due Wednesday Mar 6

Exercise 1

Problems 3.2.$(9, 10, 11, 12, 22, 31, 34, 35)$.

Exercise 2

Problems 4.1.$(16, 18, 22)$.

Practice Problems 12

Do the following before next class on 3/4.

- Read section 4.1 and start reading 4.2.
- Work out the problems 4.1.(Practice Problems, 23,24). You do not have to submit solutions to these problems. But it is a good idea to work them out to supplement the reading.

Assignment 12, due Wednesday Mar 6

Exercise 1

Problems 4.1.$(6, 8, 33)$.

Exercise 2

Problems 4.2.(12).

Exercise 3

Consider an $m \times p$ matrix A and a $p \times n$ matrix B such that $\mathrm{Nul}(A) = \{0\}$ and $\mathrm{Nul}(B) = \{0\}$. Find $\mathrm{Nul}(AB)$.

Exercise 4

Consider an $m \times p$ matrix A and a $p \times n$ matrix B. We are told that the columns of A and the columns of B are linearly independent. Are the columns of the product AB linearly independent as well? [Hint: Use the result of exercise 3.]

Practice Problems 13

Do the following before next class on 3/6.

- Read section 4.2.
- Work out the problems 4.2.(Practice Problems, 25,26). You do not have to submit solutions to these problems. But it is a good idea to work them out to supplement the reading.

Assignment 13, due Friday Mar 8

Exercise 1

Problems 4.2.$(10, 19, 24, 28)$.

Exercise 2

Consider a square matrix A.

1. What is the relationship between $\mathrm{Nul}(A)$ and $\mathrm{Nul}(A^2)$? Are they necessarily equal? Is one of them necessarily contained in the other? More generally, what can you say about $\mathrm{Nul}(A), \mathrm{Nul}(A^2), \mathrm{Nul}(A^3), \mathrm{Nul}(A^4), \ldots$?
2. What can you say about $\mathrm{Col}(A), \mathrm{Col}(A^2), \mathrm{Col}(A^3), \ldots$?

Solution:

(a) If a vector $\vec{\mathbf{x}}$ is in $\text{Nul}(A^k)$, that is if $A^k\vec{\mathbf{x}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{x}}$ is also in $\text{Nul}(A^{k+1})$, since

$$A^{k+1}\vec{\mathbf{x}} = A(A^k\vec{\mathbf{x}}) = A\vec{\mathbf{0}} = \vec{\mathbf{0}}$$

Therefore $\text{Nul}(A) \subseteq \text{Nul}(A^2) \subseteq \text{Nul}(A^3) \subseteq \text{Nul}(A^4) \subseteq \dots$.

They are not equal in general. Here is an example. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Then

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and hence

$$\text{Nul}(A) = \text{span}\left(\vec{\mathbf{e}}_1\right), \text{Nul}\left(A^2\right) = \text{span}\left(\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2\right), \text{Nul}\left(A^3\right) = \mathbb{R}^3.$$

(b) If a vector $\vec{\mathbf{y}}$ is in $\text{Col}(A^{k+1})$, that is if $\vec{\mathbf{y}} = A^{k+1}\vec{\mathbf{x}}$ for some $\vec{\mathbf{x}}$, then $\vec{\mathbf{y}}$ is also in $\text{Col}(A^k)$, since we can write

$$\vec{\mathbf{y}} = A^k(A\vec{\mathbf{x}})$$

Therefore, $\text{Col}(A) \supseteq \text{Col}(A^2) \supseteq \text{Col}(A^3) \supseteq \text{Col}(A^4) \supseteq \dots$.

Again they are not equal in general. With the same example as above, note that

$$\text{Col}(A) = \text{span}\left(\vec{e}_1, \vec{e}_2\right), \text{Col}\left(A^2\right) = \text{span}\left(\vec{e}_1\right), \text{Col}\left(A^3\right) = \{\vec{0}\}$$

Exercise 3 (Extra Credit)

Consider a square-matrix A with $\text{Nul}(A^2) = \text{Nul}(A^3)$. Is $\text{Nul}(A^3) = \text{Nul}(A^4)$? Justify your answer.

Solution: From the last exercise, we know that $\text{Nul}(A^3) \subseteq \text{Nul}(A^4)$. We claim that the converse is true as well i.e. $\text{Nul}(A^4) \subseteq \text{Nul}(A^3)$.

To see this, observe that if $\vec{\mathbf{x}}$ is in $\text{Nul}(A^4)$, then

$$\vec{\mathbf{0}} = A^4\vec{\mathbf{x}} = A^3(A\vec{\mathbf{x}})$$

so that $A\vec{\mathbf{x}}$ is in $\text{Nul}(A^3)$. So $A\vec{\mathbf{x}} \in \text{Nul}(A^2)$, which implies that

$$A^2(A\vec{\mathbf{x}}) = \vec{\mathbf{0}} \implies A^3\vec{\mathbf{x}} = \vec{\mathbf{0}}$$

i.e. $\vec{\mathbf{x}}$ is in $\text{Nul}(A^3)$. So everything in $\text{Nul}(A^4)$ is also in $\text{Nul}(A^3)$ i.e. $\text{Nul}(A^4) \subseteq \text{Nul}(A^3)$. Hence we have shown that $\text{Nul}(A^3) = \text{Nul}(A^4)$.

Practice Problems 14

Do the following before next class on 3/8.

- Start reading section 4.3.

Practice Problems 15

Do the following over Spring break.

- Finish reading section 4.3.
- Work out the problems 4.3.(Practice Problems, 21,22). You do not have to submit solutions to these problems. But it is a good idea to work them out to supplement the reading.

Assignment 14, not due

Work through handout 5 at home that summarizes the results and ideas we have learned so far. Look it over during spring break so that you don't forget what we learned before when you come back. You do not have to submit the solutions.

Assignment 15, due Friday Mar 29

Exercise 1

Problems 4.3.$(4, 6, 14, 15, 20, 34)$.

Exercise 2

Consider an $m \times p$ matrix A and a $p \times n$ matrix B.

(a) What is the relationship between $\mathrm{Nul}(AB)$ and $\mathrm{Nul}(B)$? Are they always equal? Is one of them always contained in the other?

(b) What is the relationship between $\mathrm{Col}(A)$ and $\mathrm{Col}(AB)$?

Practice Problems 16

Do the following before next class on 4/1.

- Read your class notes from last three classes. We covered parts of section 4.4,4.5 and 4.6.
- Work out the problems 4.5.(Practice Problems,19,20), 4.6.(Practice Problems, 17,18). You do not have to submit solutions to these problems. But it is a good idea to work them out to supplement the reading.

Assignment 16, due Wednesday Apr 3

Exercise 1

Problems 4.4.$(7, 13, 19)$.

Exercise 2

Problems $4.5(12, 24, 25, 26)$

Exercise 3

Problems $1.8.(15, 18, 25, 26)$.

Practice Problems 17

Do the following before next class on 4/3.

- Read section 1.8 and 1.9.
- Work out the problems 1.8.(Practice Problems,21,22), 1.9.(Practice Problems,23,24). You do not have to submit solutions to these problems. But it is a good idea to work them out to supplement the reading.

Assignment 17, due Friday Apr 5

Exercise 1

Problems $4.6.(20, 23, 25, 29)$.

Exercise 2

Problems $1.8.(31, 34)$.

Assignment 18, due Wednesday Apr 10

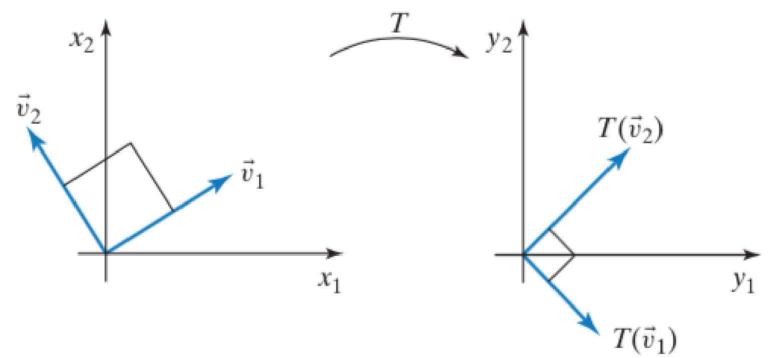
Exercise 1

Problems $1.9(2, 4, 8, 12, 13, 22, 29, 30, 31, 32, 35)$. We did 31,32 and 35 in handout 6. So you can just write the answer for these without explanation.

Exercise 2

In this exercise, we are going to prove the following remarkable result:

If $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ is any linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, then there exist two perpendicular unit vectors $\vec{\mathbf{u}}_1$ and $\vec{\mathbf{u}}_2$ in $\mathbb{R}^2$ such that the vectors $T(\vec{\mathbf{u}}_1)$ and $T(\vec{\mathbf{u}}_2)$ are perpendicular as well, in the sense that the dot product $T(\vec{\mathbf{u}}_1) \cdot T(\vec{\mathbf{u}}_2) = 0$.



First thing to note is that this isn't intuitively obvious: Think about the case of a shear, for example.

For any real number t, the vectors $\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$ and $\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$ will be perpendicular unit vectors. Now we can consider the function

$$\begin{aligned} f(t) &= \left(T\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}\right) \cdot \left(T\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}\right) \\ &= \left(A\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}\right) \cdot \left(A\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}\right) \end{aligned}$$

It is our goal to show that there exists a number c such that

$$f(c) = \left(T\begin{bmatrix} \cos c \\ \sin c \end{bmatrix}\right) \cdot \left(T\begin{bmatrix} -\sin c \\ \cos c \end{bmatrix}\right) = 0.$$

Then the vectors $\vec{\mathbf{u}}_1 = \begin{bmatrix} \cos c \\ \sin c \end{bmatrix}$ and $\vec{\mathbf{u}}_2 = \begin{bmatrix} -\sin c \\ \cos c \end{bmatrix}$ will have the required property that they are perpendicular unit vectors such that $T(\vec{\mathbf{u}}_1) \cdot T(\vec{\mathbf{u}}_2) = 0$.

(a) Show that $f(t)$ is a continuous function of t.

[Hint: Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and simplify $f(t)$.]

(b) Show that $f(\pi/2) = -f(0)$.

(c) Show that there exists a number c with $0 \leq c \leq \pi/2$ such that $f(c) = 0$.

[Hint: Use Intermediate Value Theorem.]

Exercise 3

For the following exercises, a circle is considered an ellipse whose axes are of equal length.

Let T be an invertible linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Show that the image of the unit circle is an ellipse centered at the origin.

Hint: Consider two perpendicular unit vectors $\vec{\mathbf{u}}_1$ and $\vec{\mathbf{u}}_2$ in $\mathbb{R}^2$ such that the vectors $T(\vec{\mathbf{u}}_1)$ and $T(\vec{\mathbf{u}}_2)$ are perpendicular. These exist because of the previous exercise. First explain why we can write the parametric equation of the unit circle as

$$\vec{\mathbf{r}}(t) = \cos(t)\vec{\mathbf{u}}_1 + \sin(t)\vec{\mathbf{u}}_2$$

where t is the parameter. Then show that the parametric curve given by

$$\vec{\mathbf{R}}(t) = T(\vec{\mathbf{r}}(t)) = \cos(t)T(\vec{\mathbf{u}}_1) + \sin(t)T(\vec{\mathbf{u}}_2)$$

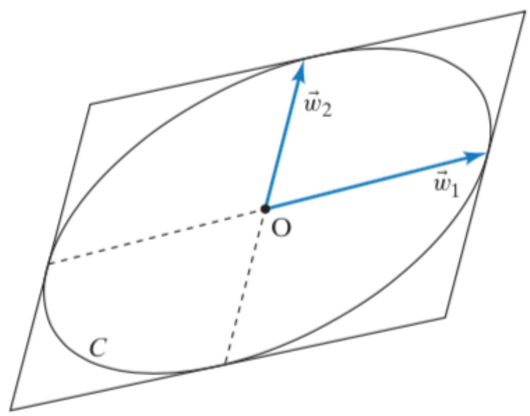
is an ellipse whose axes are along the vectors $T(\vec{\mathbf{u}}_1)$ and $T(\vec{\mathbf{u}}_2)$.

Exercise 4

Let $\vec{\mathbf{w}}_1$ and $\vec{\mathbf{w}}_2$ be two nonparallel (but not necessarily perpendicular) vectors in $\mathbb{R}^2$. Consider the curve C in $\mathbb{R}^2$ whose parametric equation is given by

$$\vec{\mathbf{r}}(t) = \cos(t)\vec{\mathbf{w}}_1 + \sin(t)\vec{\mathbf{w}}_2,$$

where t is the parameter. Show that C is an ellipse.



HINT: You can interpret C as the image of the unit circle under a suitable linear transformation; then use the last exercise.

Exercise 5

Consider an invertible linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$. Let C be an ellipse in $\mathbb{R}^2$. Show that the image of C under T is an ellipse as well.

HINT: Use the last exercise.

Practice Problems 19

Do the following before next class on 4/17.

- Read section 5.1.
- Work out the problems 5.1.(Practice Problems,21,22). You do not have to submit solutions to these problems. But it is a good idea to work them out to supplement the reading.

Assignment 19, due Friday Apr 19

Exercise 1

Problems 5.1.(7, 12, 25, 27, 29, 30).

Practice Problems 20

Do the following before next class on 4/19.

- Read section 5.2. Note that this section retintroduces determinant.
- Work out the problems 5.2.(Practice Problems,21,22). You do not have to submit solutions to these problems. But it is a good idea to work them out to supplement the reading.

Assignment 20, due Wednesday Apr 24

Exercise 1

Problems 5.1.(16, 26, 32, 35).

Assignment 21, due Wednesday Apr 24

Exercise 1

Problems 5.2.(4, 9, 19, 24).

Optional Group Project 4, due Apr 24

Available on Blackboard.

Assignment 22, due Friday Apr 26

Exercise 1

Problems 5.2.(27). You do not need to read chapter 4.9 to do problem 27.

Exercise 2

Problems from chapter 5 supplementary exercises: 2, 5, 7, 11.

Exercise 2

Problems 4.6.(34). Note that this problem has nothing to do with eigenvalues. This is assigned to make sure you don't forget older stuff.

Exercise 3(Extra Credit)

Problem 4.6.(37, 38). You can use a computer for these.

Exercise 4(Extra Credit)

Problem from chapter 4 supplementary exercises: 16.

Practice Problems 23

Do the following before next class on 4/26.

- Read section 5.3.
- Work out the problems 5.3.(Practice Problems,21,22). You do not have to submit solutions to these problems. But it is a good idea to work them out to supplement the reading.

Assignment 23, due Wednesday May 1

Exercise 1

Problems 5.3.$(4, 6, 10, 13, 14, 20)$.

Exercise 2

Problem 4.5.34 and 4.7.17. You can use a computer to calculate the inverse.

Exercise 3(Extra Credit)

Problems from chapter 5 supplementary exercises: 12.

Exercise 4

Problems from chapter 3 supplementary exercises: 16. Problems from chapter 5 supplementary exercises: $15, 16$.

Practice Problems 24

Do the following before next class on 5/1.

- Read section 5.4.
- Work out the problems 5.4.(Practice Problems). You do not have to submit solutions to these problems. But it is a good idea to work them out to supplement the reading.

Assignment 24, due Friday May 3

Problems 5.4.$(4, 5, 6, 8, 12, 14, 25, 26)$.

Optional Group Project 5, due May 6

Available on Blackboard.

Practice Problems 25

Do the following before next class on 5/6.

- Work out the problems 6.1.(19,20), 6.2.(23,24). You do not have to submit solutions to these problems. But it is a good idea to work them out to supplement the reading.

Assignment 25, due Monday May 6

Exercise 1

Problems 6.1.(24). Write the magnitudes in terms of products of matrices and use properties of matrix multiplication.

Coordinates of a vector with respect to an orthogonal basis: Let $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ be an orthogonal basis of $\mathbb{R}^n$. Then any vector $\vec{\mathbf{v}}$ in $\mathbb{R}^n$ can be written as

$$\vec{\mathbf{v}} = \lambda_1 \vec{\mathbf{b}}_1 + \lambda_2 \vec{\mathbf{b}}_2 + \ldots + \lambda_n \vec{\mathbf{b}}_n$$

for some constants λ_i. Recall that these are called the coordinates of $\vec{\mathbf{v}}$ relative to basis $\mathcal{B}$. In case $\mathcal{B}$ is orthogonal, it is easy to calculate λ_i as follows:

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{b}}_i = \lambda_i \vec{\mathbf{b}}_i \cdot \vec{\mathbf{b}}_i$$

since $\vec{\mathbf{b}}_i \cdot \vec{\mathbf{b}}_j = 0$ for $i \neq j$. So $\lambda_i = \dfrac{\vec{\mathbf{v}} \cdot \vec{\mathbf{b}}_i}{||\vec{\mathbf{b}}_i||^2}$.

Definition: A $n \times n$ matrix U is called an *orthogonal* matrix if its columns form an orthonormal basis.

Using the definition of dot product as a matrix multiplication, we can easily check that U is an orthogonal matrix if and only if $U^T U = I_n$.

Exercise 2

Problems 6.2.(10, 14, 28, 29).

Exercise 2.5 (Optional)

In problem 6.2.10, normalize the vectors to find an orthonormal basis.

Exercise 3

Problems 6.3.(23).

Fin.