Math 2000-B Handout 5: Putting It All Together!

Column Space / Image / Span / Pivots / Rank, and

Nullspace / Kernel / Linear Independence / Free Variables / Nullity

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§1. It's All A Matter of Perspective

The fundamental object of linear algebra is the matrix, an array of numbers. We've been considering either $(m \times n)$ coefficient matrices or augmented matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{or} \quad [\,A \mid \vec{\mathbf{b}}\,] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

As we've seen by now, we can think about these arrays of numbers from (at least) five perspectives:

■ The Row Perspective.

Each row of the augmented matrix can be converted to a linear equation in n variables, e.g.:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \end{array}\right] \rightarrow a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1.$$

i. Thus the augmented matrix corresponds to a set of m linear equations in n variables. We can ask how many solutions this system has: 0, 1, or infinitely many.

ii. If we consider only the coefficient matrix A, we can ask a more general question: which augmented columns $\vec{\mathbf{c}}$ lead to a consistent linear system, and if so, does the system have a unique solution or infinitely many?

iii. For the specific case where the augmented column is $\vec{\mathbf{0}}$, we get the homogeneous system $[\,A \mid \vec{\mathbf{0}}\,]$, and we can ask: are there any homogeneous solutions besides the trivial one (where all the x_i's equal zero)? The set of all homogeneous solutions is called the **nullspace** of A.

■ The Column Perspective.

We can consider each column of A as a vector in its own right:

$$A = [\,\vec{\mathbf{a}}_1\; \vec{\mathbf{a}}_2\; \cdots\; \vec{\mathbf{a}}_n\,]$$

i. For the augmented matrix, we can then ask: is the vector $\vec{\mathbf{b}}$ a linear combination of the vectors $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \ldots, \vec{\mathbf{a}}_n$? Writing this question as trying to find scalars $x_1, x_2, \ldots, x_n$ such that

$$x_1\vec{\mathbf{a}}_1 + x_2\vec{\mathbf{a}}_2 + \cdots + x_n\vec{\mathbf{a}}_n = \vec{\mathbf{b}}$$

shows that this question is equivalent to whether the corresponding system of equations (from the row perspective) has solutions.

ii. If we consider only the coefficient matrix A, we can ask a more general question: which vectors $\vec{\mathbf{c}}$ are a linear combination of $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \ldots, \vec{\mathbf{a}}_n$? In other words, what is the span of the columns of A? This span is called the **column space** of A.

iii. For the specific case where the augmented column is $\vec{\mathbf{0}}$, we get the question: is $\vec{\mathbf{0}}$ a nontrivial linear combination of $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \ldots, \vec{\mathbf{a}}_n$? In other words, are these vectors linearly dependent?!

■ The Matrix Algebra Perspective.

We can define the matrix equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$, where the product of a matrix A times a vector $\vec{\mathbf{x}}$ is given by the left-hand side of the system of equations (row perspective) or the linear combination of the columns of A weighted by the x_i (column perspective).

This equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is often simply a shorthand for writing a system of equations or a linear combination of column vectors. However, in the case when A is square, we can ask if A is invertible. If so,

$$A\vec{\mathbf{x}} = \vec{\mathbf{b}} \qquad \text{has the unique solution} \qquad \vec{\mathbf{x}} = A^{-1}\vec{\mathbf{b}} \text{ (!)}$$

It's also sometimes useful to write (and think of) a sequence of linear transformations (see below) as a product of matrices.

■ The Linear Transformation Perspective:.

The $m \times n$ matrix A defines the linear transformation

$$T : \mathbb{R}^n \to \mathbb{R}^m \qquad \text{given by} \qquad T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}.$$

i. For the augmented matrix $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$, we can then ask: is there an input vector $\vec{\mathbf{x}}$ that leads to the output $\vec{\mathbf{b}}$?

ii. We can ask a more general question: which vectors $\vec{\mathbf{c}}$ are output by the transformation T? The set of all such $\vec{\mathbf{c}}$ is called the **image** of T.

iii. For the specific case where the augmented column is $\vec{\mathbf{0}}$, we get $T(\vec{\mathbf{x}}) = \vec{\mathbf{0}}$. Certainly $T(\vec{\mathbf{0}}) = \vec{\mathbf{0}}$, but we can ask: are there any other inputs $\vec{\mathbf{x}}$ that lead to the output $\vec{\mathbf{0}}$? The set of all $\vec{\mathbf{x}} \in \mathbb{R}^n$ such that $T(\vec{\mathbf{x}}) = \vec{\mathbf{0}}$ is called the **kernel** of T.

This perspective also lets us think of multiplying by a matrix A as geometrically transforming space in some way. We might get some crushing of a larger dimension into a smaller one, or embedding of a smaller dimension into a larger one, or a nice smooth transformation staying in the same dimension, and for any given T or A, we want to figure out how much crushing or expanding there is. This is also connected to whether T is **one-to-one** or **onto**.

■ The Echelon Perspective:.

Last but not least. In fact, we use row-reduction to reduced echelon form to answer pretty much all of the questions raised above in the first four perspectives!

We row-reduce either the coefficient or augmented matrix until we get a matrix in reduced echelon form:

$$A \to E \qquad \text{or} \qquad [\,A \mid \vec{\mathbf{b}}\,] \to [\,E \mid \vec{\mathbf{b}}'\,].$$

We can then ask questions like: what is the number (called r, the **rank**) of pivot columns and the number $(n - r)$ of free variable columns? What is the number (r) of pivot rows and the number $(m - r)$ of zero rows? Is every row a pivot row? Is every column a pivot column? Most importantly, maybe: *which columns are the pivot columns*?

It's still a little fuzzy, but the number of pivot rows vs zero rows or pivot columns vs free variable columns has something to do with how the rows and columns of the original matrix A relate to each other. For example, if we have a matrix with three rows, and the third row is the sum of the first two, then, after row-reducing, the first two rows will be pivot rows and the third will be a zero row. E.g.,

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & -5 \\ 2 & 4 & -3 & 7 & -1 \\ 3 & 6 & 0 & 11 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 4 & -5 \\ 0 & 0 & -9 & -1 & 9 \\ 0 & 0 & -9 & -1 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & * & 0 & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = E.$$

§2. Translating Across Perspectives

We can take a statement in one perspective and almost always reformulate it in terms of another perspective. For example, suppose we have a system of n equations in n variables, corresponding to the augmented matrix $[\,A \mid \vec{\mathbf{b}}\,]$, and this system has a unique solution. This is a statement in the row perspective. The equivalent statements in the other perspectives are:

- **Echelon perspective:** $[\,A \mid \vec{\mathbf{b}}\,]$ reduces to $[\,I_n \mid \vec{\mathbf{x}}_0\,]$, where I_n is the n-by-n identity matrix (in other words, every column is a pivot column), and $\vec{\mathbf{x}}_0$ is the column vector of the solution to the system.
- **Column perspective:** There is one and only one way to write $\vec{\mathbf{b}}$ as a linear combination of the columns of A.
- **Transformation perspective:** There is a unique $\vec{\mathbf{x}} \in \mathbb{R}^n$ such that $T(\vec{\mathbf{x}}) = \vec{\mathbf{b}}$.
- **Matrix algebra perspective:** A is invertible and we can solve for $\vec{\mathbf{x}} = A^{-1}\vec{\mathbf{b}}$.

Exercise 1. *Suppose I tell you the system of equations*

$$\begin{aligned} 2x + 3y &= 1 \\ x - 4y &= 6 \end{aligned}$$

has the unique solution $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right]$. *Rewrite this fact in the column perspective. I.e., you should be able to tell me that a certain vector is a particular linear combination of other vectors. Also, check that the augmented matrix row-reduces how you think it should.*

Exercise 2. *Suppose a given 3-by-5 coefficient matrix A, after row reduction, has a pivot in every row.*

a. *True or False: for any choice of augmented column $\vec{\mathbf{b}}$ (i.e., the choice of the constants after the equal signs), the system associated to $[\,A \mid \vec{\mathbf{b}}\,]$ has solutions. If so, how many?*

b. *Are the columns of A linearly independent or dependent? Hint: it may be helpful to think about the augmented matrix $[\,A \mid \vec{\mathbf{0}}\,]$.*

Exercise 3. *Suppose a given 4-by-4 coefficient matrix A, after row reduction, has a free-variable column.*

a. *True or False: for any choice of augmented column $\vec{\mathbf{b}}$, the system associated to $[\,A \mid \vec{\mathbf{b}}\,]$ has solutions. If so, how many?*

b. *Are the columns of A linearly independent or dependent?*

Exercise 4. *Suppose a system of $\mathbf{3}$ equations in $\mathbf{2}$ variables, with associated augmented matrix $[\,A \mid \vec{\mathbf{b}}\,]$, has a unique solution.*

a. *What is the reduced echelon form of $[\,A \mid \vec{\mathbf{b}}\,]$?*

b. *Is $\vec{\mathbf{b}}$ a linear combination of the columns of A?*

c. *Are the columns of A linearly independent or dependent?*

d. *Does the system $[\,A \mid \vec{\mathbf{c}}\,]$ have solutions for every choice of augmented column $\vec{\mathbf{c}}$?*

e. *Let $\vec{\mathbf{a}}_1$ and $\vec{\mathbf{a}}_2$ be the columns of A. What kind of geometric object is $\mathbf{Span}\{\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \vec{\mathbf{b}}\}$?*

Exercise 5. *Suppose a system of $\mathbf{3}$ equations in $\mathbf{2}$ variables, with associated augmented matrix $[\,A \mid \vec{\mathbf{b}}\,]$, has infinitely many solutions.*

a. *What is the reduced echelon form of $[\,A \mid \vec{\mathbf{b}}\,]$?*

b. *Are the columns of A linearly independent or dependent?*

c. *True or False: $\vec{\mathbf{b}}$ is a multiple of $\vec{\mathbf{a}}_1$, the first column of A.*

Exercise 6. *Suppose we have five vectors $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$, $\vec{\mathbf{a}}_4$, and $\vec{\mathbf{a}}_5$ in $\mathbb{R}^4$, such that $\vec{\mathbf{a}}_2 = -3\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_4 = \vec{\mathbf{a}}_3 + \vec{\mathbf{a}}_5$, and there are no other relationships among these vectors. Let A be the matrix made up of these column vectors.*

a. *Write down the reduced echelon form of A. No asterisks, you can do this exactly! How many pivot columns are there?*

b. *True or False: for any choice of augmented column $\vec{\mathbf{b}}$, the system associated to $[\,A \mid \vec{\mathbf{b}}\,]$ has solutions.*

c. *If, for a given $\vec{\mathbf{b}}$, the system associated to $[\,A \mid \vec{\mathbf{b}}\,]$ does have solutions, how many solutions must it have?*

§3. Column Space, Nullspace, and finding bases for them

The terms "column space" and "nullspace" were mentioned above, but here are more formal definitions. Let A be an $m \times n$ matrix.

Definition 1. The **column space** of A is the span of the columns of A and is often written as $\operatorname{Col} A$. In other words, if $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ are the columns of A, then

$$\begin{aligned}\operatorname{Col} A &= \operatorname{Span}\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\} \\ &= \{\, \vec{c} \in \mathbb{R}^m \mid \vec{c} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n \text{ for some choice of scalars } x_1, x_2, \ldots, x_n \,\}.\end{aligned}$$

Definition 2. The **nullspace** of A is the set of solutions to $A\vec{x} = \vec{0}$, the so-called "homogeneous solutions". It is often written as $\operatorname{Nul} A$. In set language,

$$\operatorname{Nul} A = \{\, \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0} \,\}.$$

Definition 3. A subset V of $\mathbb{R}^m$ is called a **subspace** iff it is closed under the two operations you can do to vectors: addition and scalar multiplication. To be more precise: V is a subspace of $\mathbb{R}^m$ iff for every choice of $\vec{v}, \vec{w} \in V$ and for every $c \in \mathbb{R}$, we have

$$\vec{v} + \vec{w} \in V \qquad \text{and} \qquad c\vec{v} \in V.$$

Exercise 7. *Convince yourself that* $\operatorname{Col} A$ *is a subset of* $\mathbb{R}^m$*, where* m *is the number of rows of* A*. More importantly, show that* $\operatorname{Col} A$ *is a* subspace *of* $\mathbb{R}^m$*.*

Exercise 8. *Convince yourself that* $\operatorname{Nul} A$ *is a subset of* $\mathbb{R}^n$*, where* n *is the number of columns of* A*. More importantly, show that* $\operatorname{Nul} A$ *is a* subspace *of* $\mathbb{R}^n$*.*

Now that we know these are subspaces, we can ask: how do we find a basis (linearly independent spanning set) for each?

Exercise 9. *Show that the pivot columns of* A *-- or, to be more precise, the columns of* A *corresponding to the pivot columns of the (reduced) echelon form of* A *-- form a basis for the column space of* A*. So you need to argue that (a) the pivot columns span* $\operatorname{Col} A$ *and (b) that they are linearly independent.*

Exercise 10. *Illustrate the above result with a* 3×4 *matrix whose 1st and 3rd columns are the pivot columns.*

Since the pivot columns form a basis, the dimension of the column space of A equals the number of pivots, in other words, the rank! I.e. $\dim \operatorname{Col} A = \operatorname{rank} A = r$.

We've seen that we can write the set of solutions to $A\vec{x} = \vec{0}$ in "parametric vector form": there are vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ in $\mathbb{R}^n$ such that

$$A\vec{x} = \vec{0} \implies \vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k \text{ for some choice of scalars } c_1, c_2, \ldots, c_k.$$

Recall that k is the number of free variable columns and the c_i correspond to the free variables.

Exercise 11. *Show that the vectors* $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ *form a basis of* $\operatorname{Nul} A$*. We already know they span* $\operatorname{Nul} A$*, since every homogeneous solution can be written as a linear combination of them. But why are they linearly independent? It's probably helpful to look back at some examples you have done to gain insight.*

We know there are $n-r$ free variables (# of total columns minus # of pivots), thus $\mathbf{dim\,Nul}\,A = n-r$.

Exercise 12. *Suppose A is a square invertible matrix. What does this tell us about the dimensions of the column space and nullspace of A?*

§4. Equivalent Spaces: Col = im, Nul = ker

Given the $m \times n$ matrix of A, as usual we can define the linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ by $T(\vec{x}) = A\vec{x}$.

Exercise 13. *Show the column space of A and the image of T are the same subspace of $\mathbb{R}^m$! So* $\mathbf{Col}\,A = \mathbf{im}\,T$.

Exercise 14. *Show the nullspace of A and the kernel of T are the same subspace of $\mathbb{R}^n$! So* $\mathbf{Nul}\,A = \mathbf{ker}\,T$.

In other words, "column space" and "image" are interchangeable terms, as are "nullspace" and "kernel". On the other hand, it's conceptually clarifying to use the former when talking about matrices and the latter when discussing linear transformations.

Exercise 15. *Show that T is onto if and only if* $\mathbf{im}\,T =$ *all of $\mathbb{R}^m$. (This is pretty much the definition of "onto".)*

Exercise 16. *Show that T is one-to-one if and only if* $\mathbf{ker}\,T = \{\vec{0}\}$. *I.e., the kernel/nullspace only contains the zero vector, which always has to be there.*

§5. The Rank-Nullity Theorem

Here's a statement we are fully confident is true: every column of A is either a pivot column or a free variable column. In other words,

(# of pivot columns) + (# of free variable columns) = # of total columns.

Exercise 17. *Restate this as a theorem relating the dimensions of the column space of A (i.e. the rank of A) and the nullspace of A.*

Exercise 18. *Restate this as a theorem relating the dimensions of the image of T and the kernel of T.*

Exercise 19. *Find the dimensions of the column space and nullspace of each of the matrices in exercises #1-6 and check they satisfy the theorems you just stated. How can you picture the matrix in each problem as a linear transformation?*

§6. Lots and Lots of Equivalences

Exercise 20. *Suppose A is an invertible $n \times n$ matrix. This has many implications in each of the different perspectives we have developed.*

a. *What is the reduced echelon form of A? How many pivots and how many free variables does it have?*

b. *What can you conclude about a system $A\vec{x} = \vec{b}$ of linear equations? What about the homogeneous system $A\vec{x} = \vec{0}$?*

c. *What can you conclude about the columns of A? Are they linearly dependent or independent? What do they span?*

d. *What can you say about the column space and nullspace of A? What about their dimensions?*

e. *What can you conclude about the linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\vec{x}) = A\vec{x}$? What can you say about T's kernel and image? What are their dimensions?*

f. *Confirm the Rank Theorem for this situation. What does it say?*

g. *Now look at the "Invertible Matrix Theorem" in Lay, which is on p. 114 and p. 158 of the textbook and has 18 parts! Compare to your answers to parts (a) through (e) of this problem. You should have pretty much derived all the parts of the Invertible Matrix Theorem! If you missed any of the 18 parts, convince yourself they are true using the logic and equivalences we have developed.*

Pretty satisfying, eh?

Exercise 21. *The previous problem was the case $m = n = r$. What if we had $r = m < n$? What could you say in each of the perspectives now? Use theorem **3** of handout **4** to develop a "Full Row Rank Theorem" analogous to Lay's "Invertible Matrix Theorem".*

Exercise 22. *What if we had $r = n < m$? What could you say in each of the perspectives now? Use theorem **2** of handout **4** to develop a "Full Column Rank Theorem" analogous to Lay's "Invertible Matrix Theorem".*

Exercise 23. *What if we had $r < n = m$? What could you say in each of the perspectives now? Develop a "Not Enough Pivots Theorem" analogous to Lay's "Invertible Matrix Theorem". How many parts does yours have?!*

Okay, enough. Hopefully now you see how everything fits together![1]

[1]Most of this handout is copied from notes written by Prof. King. Thanks!