Final Exam Part 2

MAY 17, 2019
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INSTRUCTIONS:

- Please show ALL your work! Answers without supporting justification will not be given credit.
- Answer the questions in the blue books provided.
- Write legibly and clearly mark the answer.
- Please note that use of any books or notes is not allowed. You are allowed to use the two pages of handwritten letter-sized note that you brought. Use of calculators are not allowed.
- If you write down the correct formula/procedure to find an answer, you will get some partial credit regardless of whether you evaluated the exact values or not.
- Unless otherwise specified, you may use any valid method to solve a problem.

Full Name: ______________________________

Question	Points	Score
1	12	
2	12	
3	16	
4	4	
5	4	
6	4	
Total:	52	

This exam has 6 questions, for a total of 52 points.
The maximum possible point for each problem is given on the right side of the problem.

1. For each of the following statements, find out whether it is '**always true**', '**sometimes true**', or '**always false**'. Give a brief explanation for your answer. If you think an answer is 'sometimes true', give an example or criterion when it's false.

   (a) If we have **2020** vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_{2020} \in \mathbb{R}^{2019}$, then each $\vec{\mathbf{v}}_i$ can be written as a linear combination of the other **2019** vectors. $\boxed{3}$

   (b) If V_k denotes the set consisting of all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ such that $xyz = k$ for some real constant k, then V_k is a subspace of $\mathbb{R}^3$. $\boxed{3}$

   (c) If all the diagonal entries of an $n \times n$ matrix A are odd integers and all other entries are even integers, then A is an invertible matrix. $\boxed{3}$

   (d) If A and B are invertible $n \times n$ matrices then AB and BA are similar matrices. $\boxed{3}$

2. Is it possible or impossible to find examples of each of the following? If possible, please provide an example. If impossible, please explain why.

   (a) A real number t such that $A = \begin{pmatrix} 2 & t & 0 \\ t & 2 & t \\ 0 & t & 2 \end{pmatrix}$ has eigenvalue 0. $\boxed{3}$

   (b) A square matrix whose nullspace is $\mathbf{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\}$. $\boxed{3}$

   (c) Two real invertible 3×3 matrices A and S such that $S^T A S = -A$. $\boxed{3}$

   (d) Two 3×3 nonzero diagonalizable matrices that have same eigenvalues but are not similar. $\boxed{3}$

3. Choose the correct answer(s) from the given choices. No explanation is necessary.

   There may be more than one correct choice. You will get partial credit for circling correct but incomplete set of choices, full credit for correct and complete set of choices, and zero credit if you circle any incorrect choice.

   (a) Consider the following operations on $\mathbb{P}_2$. Which of them is/are linear transformation(s)? $\boxed{4}$

      α. $T(p(x)) = p(x) - p(1)$

      β. $T(p(x)) = p(x) - 1$

      γ. $T(p(x)) = x - p(1)$

      δ. $T(p(x)) = x^2\, p\left(\frac{1}{x}\right)$

(b) Which of the following statements is/are true for all $n \times n$ matrices A and B? $\boxed{4}$

α. $\mathbf{Col}(A) = \mathbf{Col}(AB)$

β. $\mathbf{Col}(A) = \mathbf{Col}(BA)$

γ. $\mathbf{Nul}(A) = \mathbf{Nul}(AB)$

δ. $\mathbf{Nul}(A) = \mathbf{Nul}(BA)$

(c) Suppose A and B are two 4×4 matrices and let $\mathbf{rank}(A) = 2$, $\mathbf{rank}(B) = 3$. Then which of the following are possible values of $\mathbf{rank}(A + B)$? $\boxed{4}$

α. 0

β. 2

γ. 3

δ. 5

(d) Consider the system of linear equations $\boxed{4}$

$$\begin{cases} x + y + z = 1 \\ 2x \quad\;\; + z = 2 \\ -x + y + az = b \end{cases}$$

Then,

α. If $a \neq 0$ then the system has exactly one solution regardless of the value of b.

β. If $a = 0$ then the system has no solution regardless of the value of b.

γ. If $a = 0$ then the system has no solution for exactly one value of b.

δ. If $a = 0$ then the system has infinitely many solutions for exactly one value of b.

4. If $A = Q^T DQ$ for a diagonal matrix D and an orthogonal matrix Q, then explain why the eigenvectors of A are the rows of Q. $\boxed{4}$

5. Suppose A is a 3×3 matrix with eigenvalues $\lambda = 0, 1, -1$ and corresponding eigenvectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$. Find a basis of $\mathbf{Col}(A)$ in terms of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$, and $\vec{\mathbf{v}}_3$. $\boxed{4}$

6. The matrix $\boxed{4}$

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

is diagonalizable and $H^2 = 4I$. Find the eigenvalues of H and their algebraic multiplicities.

Hint: You should not need to calculate any determinant.