Math 2000-B Handout 7: Geometry of Linear Transformations

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As shown in Theorem 10 in Section 1.9, when a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ given, it can be identified with a matrix, and there is an easy way to get a formula for the function as follows. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ a linear transformation and let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ denote the columns of the $n \times n$ identity matrix.

Figure out what each $T(\mathbf{e}_i)$ should be and write each $T(\mathbf{e}_i)$ as a column vector. If you then define the matrix $A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \ldots \quad T(\mathbf{e}_n)]$, then it will be true that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$, and $A\mathbf{x}$ gives a formula for the function. In other words, given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ if you know its values at just the n independent vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ then its value at every point $\mathbf{x}$ is determined!

Example 1. The 2×2 linear transformation that maps $\vec{\mathbf{e}}_1$ to $\vec{\mathbf{e}}_1 + \vec{\mathbf{e}}_2$ and $\vec{\mathbf{e}}_2$ to $\vec{\mathbf{e}}_1 - \vec{\mathbf{e}}_2$ is $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Example 2. The 3×3 matrix transformation that maps $\vec{\mathbf{e}}_1$ to $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$, $\vec{\mathbf{e}}_2$ to $\begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix}$, and $\vec{\mathbf{e}}_3$ to $\begin{bmatrix} 5 \\ 4 \\ -1 \end{bmatrix}$ is $\begin{bmatrix} 3 & 6 & 5 \\ -2 & 0 & 4 \\ 1 & 7 & -1 \end{bmatrix}$.

Exercise 1. *(a) The function that reflects $\mathbb{R}^2$ across the line $y = -x$ is a linear transformation. What are the images of $\vec{\mathbf{e}}_1$ and $\vec{\mathbf{e}}_2$ under this map? What is the matrix of this linear transformation?*

(b) Write a 2×2 matrix that reflects $\mathbb{R}^2$ across the line $y = x$.

A matrix transformation always maps a line onto a line or a point, and maps parallel lines onto parallel lines or onto points. (See exercises 25-28 in Section 1.8.) In the following exercises, you will verify these things for a particular matrix.

Exercise 2. *Let* $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

(a) Explain why the function $T(\mathbf{x}) = M\mathbf{x}$ maps the x-axis onto the line $y = x$, and why it maps the line $y = 2$ onto the line $y = x + 2$.

[Hint: Use the parametric equation form of a straight line.]

(b) Sketch and label the lines and their images on a graph.

Because a matrix transformation maps parallel lines to parallel lines, it will map any parallelogram to another parallelogram. (The parallelogram could be degenerate-one line segment or a single point.) When a linear transformation and parallelogram are given, the easy way to draw the image of the parallelogram is to plot the images of its four vertices and connect those points to make a parallelogram.

Define the *standard unit square* to be the square in $\mathbb{R}^2$ whose vertices are $(0,0), (1,0), (1,1)$, and $(0,1)$. When you want to visualize what a 2×2 matrix transformation does geometrically, it is particularly useful to sketch the image of this standard square. Seeing how this square gets moved or distorted shows what the transformation does to the x-axis and y-axis and thus gives a good idea what the transformation does geometrically to the whole plane. Recall that any linear transformation maps the origin to itself (why?), so you only need to figure out where the transformation maps the other three vertices.

Example 3. The image of the standard unit square under the matrix from exercise 2 looks like figure 1.

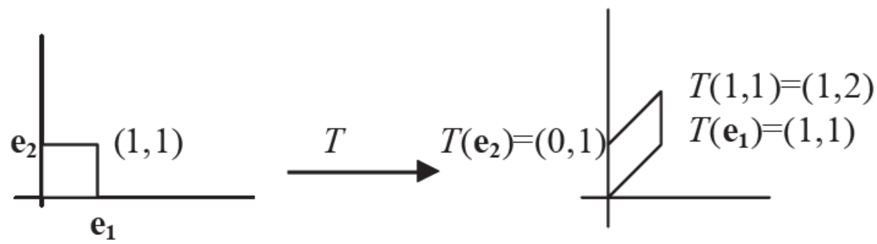


Figure 1

So you can see that the y-axis stays fixed and the x-axis is mapped onto the line $y = x$, causing a vertical *shear* of the plane. See more examples like this in Table 3 in Section 1.9.

Example 4. Find a matrix M which maps the standard unit square to the parallelogram with vertices $(0,0), (3,1), (2,2), (-1,1)$. To do this, sketch the parallelogram and recognize that $M\vec{e}_1$ and $M\vec{e}_2$ must be $(3,1)$ and $(-1,1)$, or vice versa. (Why?) So either of the matrices $\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ or $\begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$ will work.

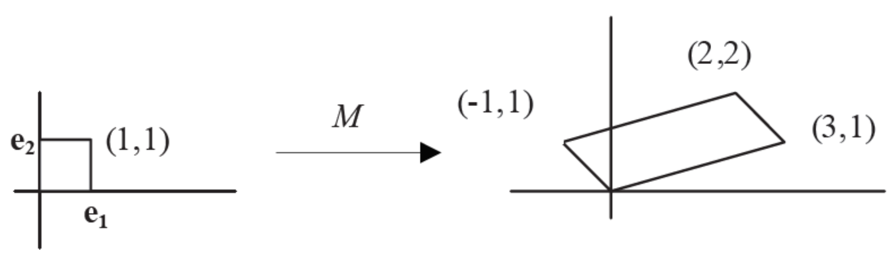


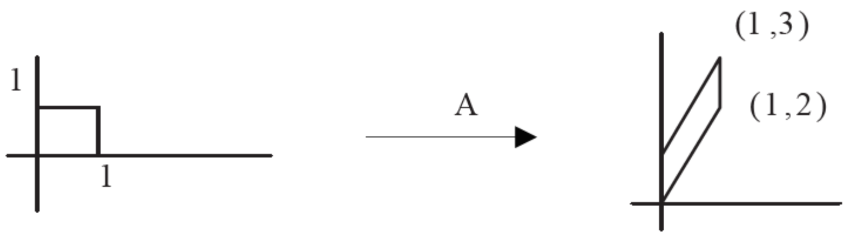
Figure 2

Calculate the image of $(1,1)$ under these matrices and verify that it's $(2,2)$.

Exercise 3. *Each of the following seven matrices is one of the special, simple types described in Sections 1.8 and 1.9. Each determines a linear transformation of* $\mathbb{R}^2$. *For each, sketch the image of the standard unit square, label the vertices of the image, and describe how the matrix is transforming the plane. To get you started, answers are given for the first matrix.*

(A) $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

Description: *A vertical shear. It leaves the y-axis fixed and increases the slope of all other lines through the origin.*



(B) $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

(C) $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(D) $D = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

(E) $E = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(F) $F = \begin{bmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{bmatrix}$

(G) $G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Exercise 4. *(a) Draw the image of the standard unit square under the transformations* EFA *and* FCA. *Here* A, C, E, F *are from the last exercise.*

Exercise 5. *Sketch the parallelogram with vertices* $(0, 0), (4, 2), (0, -4), (4, -2)$ *and write two different* 2×2 *matrices* X *and* Y *which would transform the standard unit square into this parallelogram.*

Exercise 6. *Sketch the parallelogram with vertices* $(1, 1), (1, 2), (3, 1), (3, 2)$. *Explain why no* 2×2 *matrix transformation could map the standard unit square onto this figure.*

Exercise 7. *Let* $T : \mathbb{R}^n \to \mathbb{R}^n$ *a linear transformation. Prove that* T *is completely determined by its values on any* n *linearly independent vectors.*

Hint: *Here is an outline of the proof to get you started. Let* $\vec{v}_1, \vec{v}_2 \ldots, \vec{v}_n$ *be linearly independent vectors in* $\mathbb{R}^n$, *and suppose* $T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_n)$ *are known. Let* $\vec{x}$ *be any element of* $\mathbb{R}^n$. *Explain why* $span\{\vec{v}_1, \vec{v}_2 \ldots, \vec{v}_n\} = \mathbb{R}^n$, *so* $\vec{x}$ *equals some linear combination of* $\vec{v}_1, \vec{v}_2 \ldots, \vec{v}_n$. *Then show* $T(x)$ *can be calculated using* $T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_n)$.