Final Exam Part 1

Linear Algebra -- Math 2000

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Due: May 17, 2019, 8:30 AM

Instructions:

- This is a take-home exam. As such, your written arguments will be held to a higher standard than on a sit-down in-class exam. Please submit clear and carefully composed solutions, and explain the concepts you are using and the connections among them. As always, points may be deducted for any unjustified steps, and generous partial credit will be given if you explain your thought process to me.
- You may consult and use our course materials while taking this exam, including the textbook, class notes, your problem sets, and any of the handouts or Mathematica code on Blackboard. If you use Mathematica, please print and attach your commands and output to your exam. If you are not sure if some resource is allowed, please ask! You may NOT consult the internet or discuss problem specifics with other people. You may email me to ask questions.
- when submitting your exam, staple this packet on top, and sign the ''Honor Signature'' to indicate that you followed Bowdoin's Honor Code with respect to this exam.

Full Name: ______________________________

Honor Signature: ______________________________

Total Points Available: 125

Section Number	1	2	3	4	5	6	7	Total
Available Points	**18**	**20**	**12**	**20**	**20**	**8**	**27**	**125**
Your Score								

§1. The Pell Sequence

The Fibonacci sequence is arguably the most famous integer sequence in Mathematics, making surprise appearances in everything from seashell patterns to the Parthenon. It is defined recursively as follows: each term of the sequence is equal to the sum of the previous two. A generalization of Fibonacci Sequence called $c-$Fibonacci sequence is defined as follows:

$$\begin{aligned} p_0 &= 1 \\ p_1 &= 1 \\ p_n &= cp_{n-1} + p_{n-2},\ n \geq 2 \end{aligned}$$

where c is any natural number. The $2-$Fibonacci sequence is also known as the *Pell Sequence*. In this section, we will try to figure out the long term behaviour of the Pell sequence by calculating an explicit formula for p_n. In the following questions, *we will assume* $c = 2$.

Question 1 (3 points). *For $n \geq 1$, define the vector $\vec{\mathbf{u}}_n$ as*

$$\vec{\mathbf{u}}_n = \begin{bmatrix} p_{n-1} \\ p_n \end{bmatrix}$$

Find a 2×2 matrix A such that

$$\vec{\mathbf{u}}_{n+1} = A\vec{\mathbf{u}}_n \text{ for all } n \geq 1.$$

We can conclude that

$$\begin{bmatrix} p_{n-1} \\ p_n \end{bmatrix} = \vec{\mathbf{u}}_n = A^{n-1}\vec{\mathbf{u}}_1 = A^{n-1}\begin{bmatrix} p_0 \\ p_1 \end{bmatrix} = A^{n-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So if we can calculate A^{n-1}, we can easily give a formula for p_n.

Question 2 ((3+4+3+3+2) points). *(a) Find the eigenvalue(s) of A.*

(b) Find an eigenbasis.

(c) Write down the diagonalization $A = SDS^{-1}$ i.e. find $S, D,$ and S^{-1}.

(d) Find A^{n-1}.

(e) Find $\vec{\mathbf{u}}_n$ and a formula for p_n.

Question 3 (Bonus 2 points). *Show that*

$$\lim_{n\to\infty} \frac{p_{n+1}}{p_n} = 1 + \sqrt{5}$$

The number $1 + \sqrt{5}$ is called the *Silver Ratio*.

§2. Nul space, Col space and Rank

Question 4 ((3+3+4+3) points). *For some real matrix A, the following vectors form a* basis *for its column space and null space:*

$$\mathbf{Col}(A) = \mathbf{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\}$$

$$\mathbf{Nul}(A) = \mathbf{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

(a) *What is the size $m \times n$ of A? What is* $\mathbf{rank}(A)$*?*

(b) *What are the dimensions of* $\mathbf{Col}(A^T)$, $\mathbf{Nul}(A^T)$, *and* $\mathbf{Row}(A)$*?*

(c) *Give a matrix A with this* $\mathbf{Col}(A)$ *and* $\mathbf{Nul}(A)$*. There are many possible answers.*

(d) *Give a vector $\vec{\mathbf{b}}$ for which $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has at least one solution, and give all the solutions $\vec{\mathbf{x}}$ for your A from the previous part.*

Hint: you should not have to do Row-reduction.

Question 5 ((4+3) points). *Let A be a $m \times n$ matrix and let B be a $n \times p$ matrix.*

(a) *Explain why* $\mathbf{rank}(AB) \leq \mathbf{rank}(A)$.

Hint: What is the relation between the column space of AB and that of A?

(b) *Explain why* $\mathbf{rank}(AB) \leq \mathbf{rank}(B)$.

Hint: What can you say about $\mathbf{rank}\left((AB)^T\right)$*?*

§3. Characteristic Polynomial

Question 6 (7 points). *Let A be an 5×5 diagonalizable matrix and let*

$$p_A(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + t^5$$

be the characteristic polynomial of A. Show that

$$\mathbf{det}(A) = c_0 \quad \text{and} \quad \mathbf{tr}(A) = -c_4.$$

Question 7 (5 points). *If A is an invertible and diagonalizable $n \times n$ matrix, show that A^{-1} can be written as a linear combination of the matrices $A, A^2, A^3, \ldots, A^{n-1}$.*

Hint: Use the Cayley-Hamilton theorem from assignment 22, chapter 5 supplementary exercise 7.

§4. Matrix of a Linear Transformations with respect to a Basis

Question 8 ((1+5+6) points). *Consider the vector space $\mathbb{V}$ consisting of 3×3 matrices A such that $A^T = -A$. These type of matrices are called* skew-symmetric.

(a) *Show that the diagonal entries of any skew-symmetric matrix are zero.*

(b) *Find the dimension of $\mathbb{V}$ and a basis $\mathfrak{B}$ for $\mathbb{V}$.*

(c) *Let $T : \mathbb{V} \to \mathbb{V}$ be a linear transformation defined by*

$$T(M) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} M - M \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

for all $M \in \mathbb{V}$. Find the $\mathfrak{B}$-matrix of T for your choice of $\mathfrak{B}$ from part (b).

Question 9 ((6+2) points). *Consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined as*

$$T(f(x)) = f(1) + (x-1)f'(1)$$

(a) *Consider the basis $\mathfrak{B}$ of $\mathbb{P}_2$ given by $\mathfrak{B} = \{1, (x-1), (x-1)^2\}$. Find the $\mathfrak{B}$-matrix of T.*

(b) *Is T an isomorphism?*

§5. Orthogonality

Question 10 ((3+2+5+1) points). *Let $\vec{\mathbf{v}}$ be a* unit *vector in $\mathbb{R}^3$ and let $H = I - 2\vec{\mathbf{v}}\vec{\mathbf{v}}^T$.*

(a) *Show that $H^2 = I$.*

(b) *Show that H is an orthogonal matrix.*

(c) *Find the eigenvalues of H and describe the eigenspaces geometrically.*

(d) *Can you give a geometric description of the linear transformation corresponding to H?*

Hint: What is $H\vec{\mathbf{u}}$ for $\vec{\mathbf{u}} \perp \vec{\mathbf{v}}$?

Question 11 ((6+3) points). *Let P be a 3×3 matrix that corresponds to the linear transformation that projects every vector in $\mathbb{R}^3$ orthogonally onto the plane $x + 2y + 2z = 0$.*

(a) *Find an orthogonal basis of $\mathbb{R}^3$ consisting of three eigenvectors of P.*

Hint: Investigate the geometry of the eigenspaces.

(b) *Find the matrix P.*

§6. Vector Space and Dimension

Let $B = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ be a diagonal 3×3 matrix, and let V be the vector space defined as

$$V = \left\{ A \in \mathbb{R}^{3\times 3} \mid AB = BA \right\}$$

i.e. V consists of all 3×3 matrices A that commute with B.

Question 12 (8 points). *Find the possible dimensions of V. In each case, give a basis of V.*

Hint: Consider the following cases separately: (1) when $x = y = z$, (2) any two of x, y, z are equal but the third is not, (3) none of them are equal to each other.

§7. The Least-Square Approximation problem

One of the fundamental problems in Machine Learning is to find the best approximate solution to equations of the form $A\vec{x} = \vec{b}$ where A be a $m \times n$ matrix and $\vec{b}$ is a $m \times 1$ vector and the system is possibly inconsistent. Recall that the space $\mathbf{Col}(A)$ is a subspace of $\mathbb{R}^m$, defined as

$$\mathbf{Col}(A) = \left\{ A\vec{x} \mid \vec{x} \in \mathbb{R}^n \right\}$$

So if $\vec{b} \notin \mathbf{Col}(A)$, then $A\vec{x} = \vec{b}$ is not consistent, but we can still try to find an $\vec{x}$ that will minimize the "error". We define a *least-square solution* $\vec{x}^*$ to be an $\vec{x} \in \mathbb{R}^n$ such that

$$\|A\vec{x}^* - \vec{b}\| \leq \|A\vec{x} - \vec{b}\|$$

for all $\vec{x} \in \mathbb{R}^n$. In other words, $\vec{x}^*$ is such that $\vec{b}^* = A\vec{x}^*$ is the point in $\mathbf{Col}(A)$ that is closest to $\vec{b}$.

Geometrically speaking, $\vec{b}^* = A\vec{x}^*$ is the orthogonal projection of $\vec{b}$ onto $\mathbf{Col}(A)$ i.e

$$\vec{b}^* = \mathbf{proj}_{\mathbf{Col}(A)} \vec{b}.$$

Question 13 ((2+2) points). *(a) Explain why $\vec{b} - \vec{b}^*$ is in $(\mathbf{Col}(A))^\perp$.*

(b) Explain why part (a) is equivalent to the equation $A^T\vec{b} = A^TA\vec{x}^*$.

Hint: What is $(\mathbf{Col}(A))^\perp$ equal to?

(c) If $A\vec{x} = \vec{b}$, explain why $\vec{b}^* = \vec{b}$.

The equation obtained in part (b) above is called the *Normal Equation* for $A\vec{x} = \vec{b}$. By definition, the Normal Equation is always consistent, however it may have more than one solution i.e. there may be more than one best approximation. One particular case of interest is when $\mathbf{Nul}(A) = \{\mathbf{0}\}$.

Question 14 ((8+3+2) points). *Let A be an $m \times n$ matrix.*

(a) *Show that* $\mathbf{Nul}(A) = \mathbf{Nul}(A^T A)$ *.*

Hint: It is easy to show that $\mathbf{Nul}(A) \subseteq \mathbf{Nul}(A^T A)$*. To show the opposite containment* $\mathbf{Nul}(A^T A) \subseteq \mathbf{Nul}(A)$*, use orthogonal complements.*

(b) *Show that* $\mathbf{rank}(A) = \mathbf{rank}(A^T A)$*.*

(c) *Show that if* $\mathbf{Nul}(A) = \{0\}$*, then* $A^T A$ *is an* $n \times n$ invertible *matrix.*

So if $\mathbf{Nul}(A) = \{0\}$, we get $A^T A$ is invertible and in that case the normal equation has a unique solution

$$\vec{\mathbf{x}}^* = \left(A^T A\right)^{-1} A^T \vec{\mathbf{b}}. \tag{$\star$}$$

Scientists are often interested in fitting a function of a certain type, e.g. a polynomial to experimental data they have gathered. Since physical experiments often have unavoidable errors, there may not be an exact polynomial that satisfies given data-set.

Consider for example an experiment where we know that the quantity V is supposed to be a quadratic polynomial in variable T because of theoretical reasons. Let's write

$$V = a + bT + cT^2 = p(T)$$

During the experiment we note down a list of input versus output (T, V) data-points as follows:

Observation No.	T	V
1	-1	8
2	0	8
3	1	4
4	2	16

Our goal is to find a, b and c that will give the best approximate polynomial with least amount of error. In other words,

Question 15 (10 points). *Use equation (*$\star$*) to find the least-square solution* $\vec{\mathbf{x}}^* = \begin{bmatrix} a^* \\ b^* \\ c^* \end{bmatrix}$ *to the system of equations*

$$V_i = p(T_i) \text{ for } i = 1, 2, 3, 4$$

where (T_i, V_i) *is the data-point from observation* i*.*

A picture of $p^*(T) = a^* + b^*T + c^*T^2$ is given below.

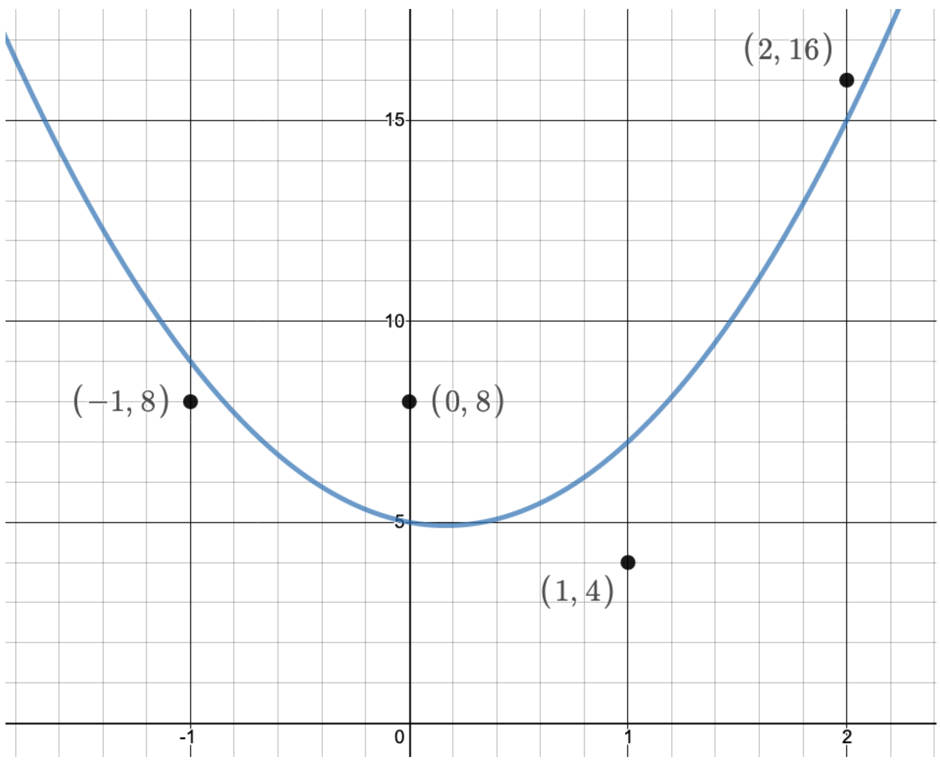

Figure 1