Math 2000-B Handout 8: Examples of Linear Transformations and Isomorphism

Subhadip Chowdhury

§1. Orthogonal Projections and Reflections

Let $\vec{\mathbf{u}}$ be a unit vector and let L be a line parallel to $\vec{\mathbf{u}}$ passing through the origin. Recall that the formula for projection of a vector $\vec{\mathbf{x}}$ onto L is given by

$$\mathbf{proj}_L\,\vec{\mathbf{x}} = (\vec{\mathbf{x}}\cdot\vec{\mathbf{u}})\vec{\mathbf{u}}$$

Is the transformation $T(\vec{\mathbf{x}}) = \mathbf{proj}_L\,\vec{\mathbf{x}}$ a linear transformation? If so, what is its matrix? Let's first assume both $\vec{\mathbf{x}}$ and $\vec{\mathbf{u}}$ are vectors in $\mathbb{R}^2$. Let

$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad\text{and}\quad \vec{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Exercise 1. *Show that* $\mathbf{proj}_L\,\vec{\mathbf{x}} = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}\vec{\mathbf{x}}$.

Exercise 2. *Find the matrix P of the orthogonal projection onto the line L spanned by* $\vec{\mathbf{w}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Now define $\mathbf{Ref}_L\,\vec{\mathbf{x}}$ to be the reflection of $\vec{\mathbf{x}}$ about L.

Exercise 3. *Show that* $\mathbf{Ref}_L\,\vec{\mathbf{x}} = 2\,\mathbf{proj}_{\vec{\mathbf{u}}}\,\vec{\mathbf{x}} - \vec{\mathbf{x}}$.

Exercise 4. *Show that $\vec{\mathbf{x}} \mapsto \mathbf{Ref}_L\,\vec{\mathbf{x}}$ is a linear transformation and find its matrix.*

§2. Rotations Combined with a Scaling

Recall that the matrix $A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$ represents a scaling by r and a matrix $B = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$ represents a counterclockwise rotation by φ.

Exercise 5. *Let a and b be any two real numbers. Show that*

$$T(\vec{x}) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\vec{x}$$

represents a rotation combined with a scaling.

Hint: *Multiply A and B above and think polar coordinates.*

§3. The Coordinate Mapping

Theorem 6. *Let $\mathscr{B} = \{\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \ldots, \vec{\mathbf{b}}_n\}$ be a basis for a vector space V. Then the map $T : V \to \mathbb{R}^n$ defined as*

$$T(\vec{\mathbf{x}}) = [\vec{\mathbf{x}}]_{\mathscr{B}}$$

is a one-to-one and onto linear transformation.

Sketch of proof. Show that the associated matrix is invertible. □

Definition 1. A *o*ne-to-one linear transformation from a vector space V *o*nto a vector space W is called an isomorphism from V onto W. In that case, V and W are called isomorphic vector spaces.

Exercise 7. *Show that if V and W are isomorphic, then* $\dim V = \dim W$.

Exercise 8. *Let $T : V \to W$ be an isomorphism. Then $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n\} \subset V$ is a linearly independent set of vectors if and only if $\{T(\vec{\mathbf{v}}_1), T(\vec{\mathbf{v}}_2), \ldots, T(\vec{\mathbf{v}}_n)\} \subset W$ is a linearly independent set of vectors. [*Hint*: Show that T is an isomorphism iff T is invertible. Let L be the inverse of T. Then $L(T(\vec{\mathbf{v}}_i)) = \vec{\mathbf{v}}_i$.]*

Example 2. Let $\mathscr{B} = \{1, x, x^2, x^3\}$ be a basis of $\mathbb{P}_3$ = the set of polynomial of degree ≤ 3. Then the coordinate mapping gives an isomorphism from $\mathbb{P}_3$ to $\mathbb{R}^4$.

Exercise 9. *Use coordinate mapping to test the linear independence of the following set of polynomials:*

$$1 - 2x^2 - x^3, \quad x + 2x^3, \quad 1 + x - 2x^2$$

Exercise 10. *Let* $\mathbf{p}_1(t) = 1 + t^2, \mathbf{p}_2(t) = t - 3t^2, \mathbf{p}_3(t) = 1 + t - 3t^2$.

(a) Use coordinate vectors to show that these polynomials form a basis for $\mathbb{P}_2$.

(b) Consider the basis $\mathscr{B} = \left\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\right\}$ for $\mathbb{P}_2$. Find $\mathbf{q}$ in $\mathbb{P}_2$ given that

$$[\mathbf{q}]_{\mathscr{B}} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$