Math 2000-B Handout 2

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■ Exercise 1.

Let A be an $(n \times n)$ matrix. Prove that A is nonsingular iff for any $\vec{\mathbf{b}} \in \mathbb{R}^n$, the system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a *unique* solution.

Sketch of the proof. We will show each implication direction separately. Backwards implication is immediate (why?). For the forward implication, first prove that the system is consistent by showing that the columns of the augmented matrix $[A|\vec{\mathbf{b}}]$ must be linearly dependent. Prove uniqueness by subtracting two solutions.

■ Exercise 2.

Suppose that a 5×3 matrix A has linearly independent columns. What is the REF of A?

■ Exercise 3.

Show that if the columns of B are linearly dependent, then so are the columns of AB.

■ Exercise 4.

Suppose A is an $m \times n$ matrix and there exist $n \times m$ matrices C and D such that $CA = I_n$ and $AD = I_m$. Prove that $m = n$ and $C = D$.

Solution. Consider the matrix CAD. Observe that $CAD = (CA)D = I_nD = D$ and $CAD = C(AD) = CI_m = C$. Hence $C = D$.

The part about proving $m = n$ is a bit harder. We need to use the result of exercise 2.1.23 and 2.1.24 from assignment 5. Exercise 23 claimed that if $C_{n\times m}A_{m\times n} = I_n$, then $m \geq n$. Exercise 24 claimed that if $A_{m\times n}D_{n\times m} = I_m$, then $m \leq n$. Combining the two inequalities, we get that $m = n$.

■ Exercise 5.

For each of the following statements, find out whether it is '**always true**', '**sometimes true**', or '**always false**'. Give a brief explanation for your answer.

(a) If A and B are $n \times n$ matrices then $(A + B)(A - B) = A^2 - B^2$.

(b) If A is an $m \times n$ matrix, if the equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has at least two different solutions, and if the equation $A\vec{\mathbf{x}} = \vec{\mathbf{c}}$ is consistent, then the equation $A\vec{\mathbf{x}} = \vec{\mathbf{c}}$ has infinitely many solutions.

(c) For three nonzero vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$, and $\vec{\mathbf{v}}_3$ if $\vec{\mathbf{v}}_1 \in \mathbf{Span}\{\vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ then $\vec{\mathbf{v}}_2 \in \mathbf{Span}\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_3\}$.

(d) If $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2\}$ is linearly independent then so is $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2\}$.

(e) If $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_4 \in \mathbb{R}^4$ and $\vec{\mathbf{v}}_3 = \mathbf{0}$, then $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4\}$ is linearly dependent.

(f) If $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_4 \in \mathbb{R}^4$ and $\vec{\mathbf{v}}_4$ is not a linear combination of the other three vectors, then $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ is linearly independent.

(g) A plane in three dimension can be thought of as $\mathbf{Span}\{\vec{\mathbf{u}}, \vec{\mathbf{v}}\}$ for some vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in \mathbb{R}^2$.

(h) Suppose that $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$, and $\vec{\mathbf{v}}_3$ are in $\mathbb{R}^5$, $\vec{\mathbf{v}}_2$ is not a multiple of $\vec{\mathbf{v}}_1$, and $\vec{\mathbf{v}}_3$ is not a linear combination of $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$. Then $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ is linearly independent.

Solution.

(a) **Sometimes True**. Since matrix multiplication is not commutative in general, AB is not necessarily equal to BA.

(b) **Always True**. If $\vec{\mathbf{p}}_1$ and $\vec{\mathbf{p}}_2$ are solutions to $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ and $\vec{\mathbf{q}}$ is a solution to $A\vec{\mathbf{x}} = \vec{\mathbf{c}}$, then $\vec{\mathbf{q}} + \lambda(\vec{\mathbf{p}}_1 - \vec{\mathbf{p}}_2)$ is also a solution to $A\vec{\mathbf{x}} = \vec{\mathbf{c}}$ for any real number λ.

(c) **Sometimes True**. The claim is false if $\vec{\mathbf{v}}_1 = \vec{\mathbf{v}}_3$, but $\vec{\mathbf{v}}_2$ is linearly independent from $\vec{\mathbf{v}}_1$.

(d) **Always True**.

$$c_1\vec{\mathbf{v}}_1 + c_2(\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2) = 0 \implies (c_1 + c_2)\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 = 0 \implies c_1 + c_2 = c_2 = 0$$

since $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ are linearly independent. But then $c_1 = c_2 = 0$ and hence $\vec{\mathbf{v}}_1$ and $(\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2)$ are linearly independent.

(e) **Always True**. If a set of vectors includes the zero vector, then the set is linearly dependent, regardless of the dimension or the number of vectors.

(f) **Sometimes True**. There are no restrictions on $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$. They can be whatever they want to be!

(g) **Sometimes True** When we think of the vectors in $\mathbb{R}^2$ as column vectors, we are considering them as position vectors that start at the origin. As such two such vectors can only span planes passing through the origin. Another way to see this is that the zero vector must always be in the span of two such vectors. So in particular a plane that doesn't contain the origin cannot be written in the form $\mathbf{Span}\{\vec{\mathbf{u}}, \vec{\mathbf{v}}\}$ for some vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in \mathbb{R}^2$.

(h) **Sometimes True**. The only case when this is not true is if $\vec{\mathbf{v}}_1 = \vec{\mathbf{0}}$. If $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ are all nonzero, then this statement is true.