Math 2000-B Handout 4: The Invertible Matrix Theorem

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The Invertible Matrix Theorem (found in chapter 2.3 of the book) combines several ideas we have covered so far in the form of a list of equivalent statement. The statement is as follows.

Theorem 1 (Invertible Matrix Theorem)**.** *Let A be an $n \times n$ matrix. Then the following are equivalent:*

(a) *A is an invertible matrix.*

(b) *A is row equivalent to the identity matrix I_n.*

(c) *A has n pivot positions.*

(d) *The homogeneous equation $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has only the trivial solution i.e. A is nonsingular.*

(e) *The n columns of A form a linearly independent set of $n \times 1$ vectors.*

(f) *The equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is consistent for all $\vec{\mathbf{b}} \in \mathbb{R}^n$.*

(g) *The equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a unique solution for all $\vec{\mathbf{b}} \in \mathbb{R}^n$.*

(h) *The columns of A span $\mathbb{R}^n$ i.e. any vector $\vec{\mathbf{b}} \in \mathbb{R}^n$ can be written as a linear combination of the columns of A.*

(i) *There is an $n \times n$ matrix C such that $CA = I_n$.*

(j) *There is an $n \times n$ matrix D such that $AD = I_n$.*

(k) *The transpose A^T is an invertible matrix.*

Many of these equivalences are perhaps unexpected, initially, such as $(e) \iff (h)$. This happens only because A is a *square matrix*.We can explain this by breaking the theorem into two sets of equivalences for matrices that are not necessarily square. Pay special attention to the use of m and n in the following theorems.

Theorem 2. *Let A be an $m \times n$ matrix. Then the following statements are equivalent:*

(a) *A has a pivot in every* column.

(b) *The homogeneous equation $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has only the trivial solution.*

(c) *The n columns of A form a linearly independent set of $m \times 1$ vectors.*

(d) *There exists an $n \times m$ matrix C such that $CA = I_n$.*

When a matrix A satisfies any of the above conditions, we must have $n \leq m$.

Proof. We have already proved $(a) \iff (b) \iff (c)$ before. Exercise 2.1.23 from assignment 5 says that $(d) \implies n \leq m$ and also $(d) \implies (b)$. So we only need to show that one of $(a), (b)$ or (c) implies (d). The proof of that requires the idea of Elementary matrices which we haven't covered yet. So we will finish that part after the midterm. However, you are free to use the equivalences above in the midterm without proof. □

Theorem 3. *Let A be an $m \times n$ matrix. Then the following statements are equivalent:*

(a) A has a pivot in every row.

(b) The equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is consistent for all $\vec{\mathbf{b}} \in \mathbb{R}^m$.

(c) The columns of A span $\mathbb{R}^m$.

(d) There exists an $n \times m$ matrix D such that $AD = I_m$.

When a matrix A satisfies any of the above conditions, we must have $m \leq n$.

Proof. Again, statements $(a), (b)$ and (c) are equivalent according to what we have learned already. The fact that $(d) \implies (b)$ and that $m \leq n$ both follow from Exercise 2.1.24. To complete the equivalence we show that $(b) \implies (d)$.

Assume (b) holds. For each of the columns $\vec{\mathbf{e}}_1, \dots, \vec{\mathbf{e}}_m$ of the identity matrix I_m, statement (b) implies there exists a solution to $A\vec{\mathbf{x}} = \vec{\mathbf{e}}_i$. Call this solution $\vec{\mathbf{x}}_i \in \mathbb{R}^n$, so that $A\vec{\mathbf{x}}_i = \vec{\mathbf{e}}_i$ for $i = 1, 2, \dots, m$. Set $D = \left[\vec{\mathbf{x}}_1\, \vec{\mathbf{x}}_2\, \dots, \vec{\mathbf{x}}_m\right]$ which is an $n \times m$ matrix. Then

$$\begin{aligned} AD &= A \begin{bmatrix} \vec{\mathbf{x}}_1 & \vec{\mathbf{x}}_2 & \cdots & \vec{\mathbf{x}}_m \end{bmatrix} \\ &= \begin{bmatrix} A\vec{\mathbf{x}}_1 & A\vec{\mathbf{x}}_2 & \cdots & A\vec{\mathbf{x}}_m \end{bmatrix} \\ &= \begin{bmatrix} \vec{\mathbf{e}}_1 & \vec{\mathbf{e}}_2 & \cdots & \vec{\mathbf{e}}_m \end{bmatrix} \\ &= I_m \end{aligned}$$

So we have constructed our matrix D as desired. □

Remark 1. Observe that in the case $n = m$, theorem **2.b** and **3.c** together imply **1.g**. Similarly **2.a** and **3.a** together imply **1.b** and **1.c**. The other statements of theorem **1** are just the statement of **2** and **3** together in one list.

Remark 2. The statement of theorem **2.b** does not say that A is nonsingular. To be nonsingular, A must be a square matrix by definition. So only in the case $m = n$, statement **2.b** is equivalent to nosingularity.

Remark 3. In the case $m = n = 2$, we can add another equivalent statement to theorem **1**:

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then A is invertible iff $\det A = ad - bc \neq 0$.