Math 2000-B Handout 6

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§1. Extending the Invertible Matrix Theorem

Theorem 1. *Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.*

(l) *The columns of A form a basis of $\mathbb{R}^n$.*

(m) $\mathbf{Col}(A) = \mathbb{R}^n$

(n) $\mathbf{dim\,Col}(A) = \mathbf{rank}(A) = n$

(o) $\mathbf{Nul}(A) = \{0\}$

(p) $\mathbf{dim\,Nul}(A) = 0$

Exercise 2. *Let A be a $m \times n$ matrix. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be one-to-one if each $\vec{\mathbf{b}}$ in $\mathbb{R}^m$ is the image of at most one $\vec{\mathbf{x}}$ in $\mathbb{R}^n$. Explain why $\mathbf{Nul}(A) = \{\vec{0}_{n\times 1}\}$ is equivalent to saying the linear transformation $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ is one-to-one.*

In this case, what is the relationship between m and n?

Exercise 3. *Let A be a $m \times n$ matrix. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto $\mathbb{R}^m$ if each $\vec{\mathbf{b}}$ in $\mathbb{R}^m$ is the image of at least one $\vec{\mathbf{x}}$ in $\mathbb{R}^n$. Explain why $\mathbf{Col}(A) = \mathbb{R}^m$ is equivalent to saying the linear transformation $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ is onto.*

In this case, what is the relationship between m and n?

Definition 1 (Invertible Linear Transformations)**.** A linear transformation $T : V \to W$ is called invertible if there exists a linear transformation $L : W \to V$ such that $T(L(\vec{\mathbf{w}})) = \vec{\mathbf{w}}$ for all $\vec{\mathbf{w}} \in W$ and $L(T(\vec{\mathbf{v}})) = \vec{\mathbf{v}}$ for all $\vec{\mathbf{v}} \in V$. In that case, L is called the inverse of T.

Exercise 4. *Show that if a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is invertible, then $n = m$.*

Exercise 5. *Show that the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined as $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ is invertible iff A is an invertible matrix.*

§2. Rank-Nullity Theorem

Theorem 6. *For a $m \times n$ matrix A,*

$$\mathbf{rank}(A) + \mathbf{nullity}(A) = n$$

Exercise 7. *If A is a 6×4 matrix, what is the smallest possible dimension of $\mathbf{Nul}(A)$?*

Exercise 8. *Suppose the solutions of a homogeneous system of five linear equations in six unknowns are all multiples of one nonzero solution. Will the system necessarily have a solution for every possible choice of constants on the right sides of the equations? Explain.*

Exercise 9. *Suppose a non-homogeneous system of nine linear equations in ten unknowns has a solution for all possible constants on the right sides of the equations. Is it possible to find two nonzero solutions of the associated homogeneous system that are not multiples of each other? Discuss.*

Exercise 10. *Is it possible that all solutions of a homogeneous system of ten linear equations in twelve variables are multiples of one fixed nonzero solution? Discuss.*

Exercise 11. *Show that the equation* $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ *has a solution for all* $\vec{\mathbf{b}} \in \mathbb{R}^m$ *if and only if the equation* $A^T\vec{\mathbf{x}} = \vec{\mathbf{0}}$ *has only the trivial solution.*