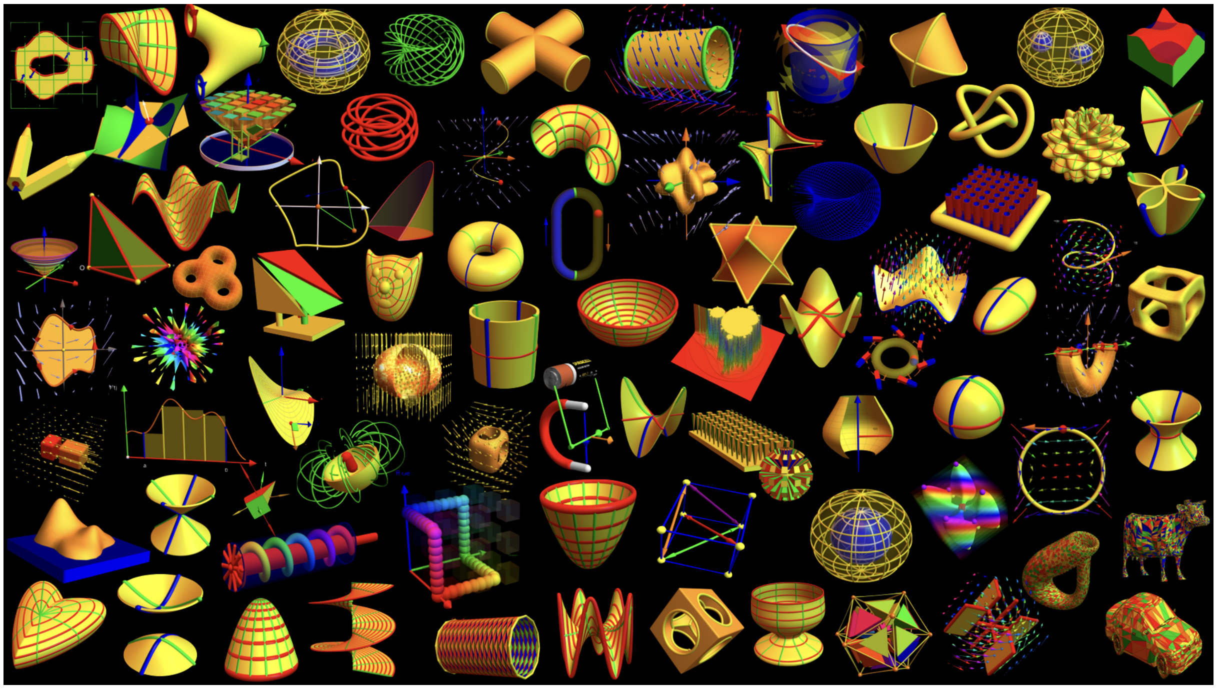

Multivariable Calculus

Math 212 Lecture notes

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THE COLLEGE OF WOOSTER

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Contents

Acknowledgement

Introduction

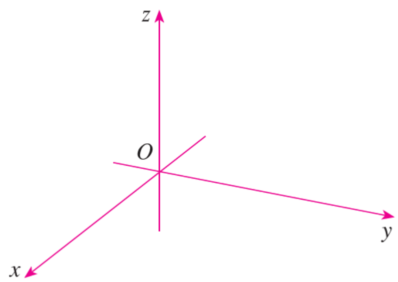
The History of every major Galactic Civilization tends to pass through three distinct and recognizable phases, those of Survival, Inquiry and Sophistication, otherwise known as the How, Why, and Where phases. For instance, the first phase is characterized by the question ‘How can we eat?’ the second by the question ‘Why do we eat?’ and the third by the question ‘Where shall we have lunch?[**adams1995hitchhiker**]

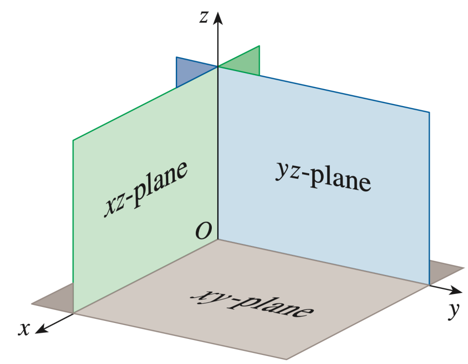
Chapter 1 | Three Dimensional Coordinate Geometry

To locate a point in a plane, we need two numbers. Any point in the plane can be represented as an ordered pair (a, b) of real numbers, where a is the x–coordinate and b is the y–coordinate. For this reason, a plane is called two-dimensional. To locate a point in space, three numbers are required. We represent any point in space by an ordered triple (a, b, c) of real numbers. For this reason, we will refer to the space as the 3–space or $\mathbb{R}^3$.

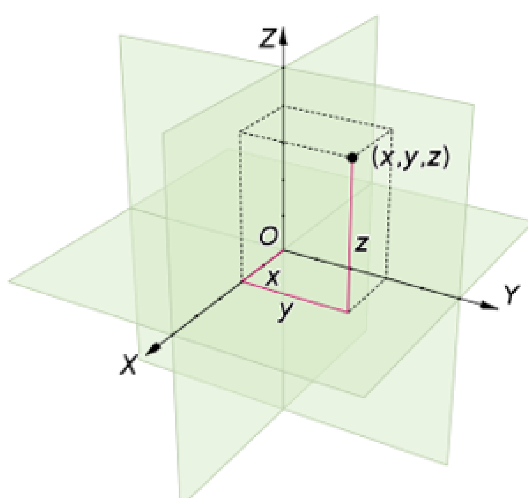
§1.1 Coordinate Axes and Points in 3-space

The three coordinate axes in 3-space are drawn using a right-hand-thumb rule as follows. It is important that you understand how to draw the axes in different orientations.



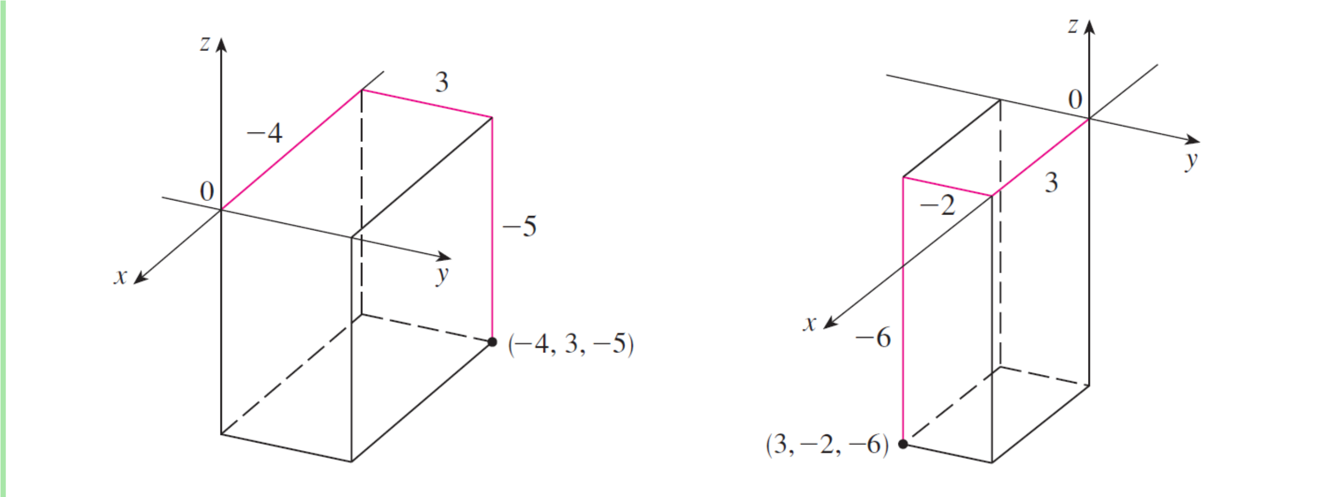


A point in three dimension has three coordinates (x, y, z) that denote respectively how far in/out, left/right, up/down a point is from the origin.



Example 1.1

hoose one of the corners of this classroom as the origin. Where are the points P$(-4, 3, -5)$ and Q$(3, -2, -6)$?



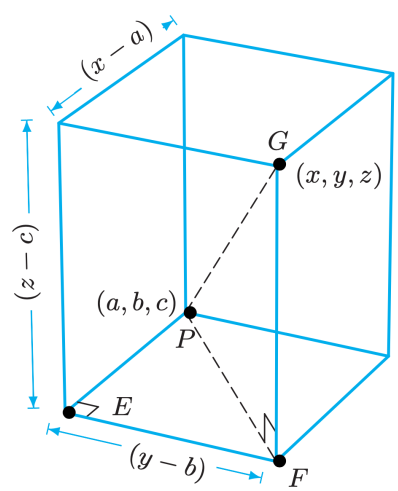
■ **Question 1.** □

You awaken one morning to find that you have been transferred onto a grid which is set up like a standard right-hand coordinate system. You are at the point $(-1,3,-3)$, standing upright, and facing the xz-plane. You walk 2 units forward, turn left, and walk for another 2 units. What is your final position?

§1.2 Distance between two points

The distance between two points (a,b,c) and (x,y,z) in space is given by

$$\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}.$$



■ **Question 2.** □

What is the distance from $(1,2,1)$ to $(-3,1,2)$?

■ **Question 3.** □

Which of the following points lies closest to the xy-plane?

(i) $(3,0,3)$ *(ii)* $(0,4,2)$ *(iii)* $(2,4,1)$ *(iv)* $(2,3,4)$

1.2.1 Equation of a Sphere

We can define a sphere to be collection (locus) of points that are equidistant to a fixed point called the **center**. Suppose the center is at (a,b,c) and the radius is r. Then the distance formula tells us that the required points (x,y,z) satisfy

$$\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}=r \iff (x-a)^2+(y-b)^2+(z-c)^2=r^2$$

§1.3 Sets of points described using coordinates

Question 4. □

Assume the origin is at the bottom, left, front corner of the classroom. Describe the set of points (x,y,x) with

- $x>0, y>0$ *and* $z\leq 0$
- $z=3$
- $x=0, y=0$

Question 5. □

Describe/draw the set of points with $0\leq x\leq 1, 0\leq y\leq 1-x$, *and* $z\leq 0$.

Question 6. □

Find any points where the sphere $(x-1)^2+(y+3)^2+(z-2)^2=4$ *intersects the* y *-axis.*

Question 7. □

Consider the set of points P *given by the equation* $z=5$. *Also, let* Q *be the set of points that satisfy* $x^2+y^2=1$.

(a) *Sketch* P,Q, *and their intersection.*

(b) *Find an equation for the intersection.*

Question 8. □

How would you describe the points making up the ***solid*** *cube with sides of length 2 and centered at the origin?*

Question 9. □

An equilateral triangle is standing vertically in 3-space with a vertex above the xy-plane and its two other vertices at $(7,0,0)$ *and* $(9,0,0)$. *What are the coordinates of the third vertex?*

Chapter 2 | Vectors in 3D

§2.1 Definition

Vectors in two or three dimensions are quantities that have both **magnitude** and **direction**. Quantities that are not vectors are called **scalars**.

Question 10. □

Which of the following are vectors?

(a) *The cost of a movie ticket.*

(b) *The volume of the Bowdoin polar bear.*

(c) *The weight of the Bowdoin polar bear.*

(d) *The number of students in our class.*

(e) *The velocity of a car.*

(f) *The speed of a car.*

§2.2 Graphical representation of vectors

The vector $\vec{v} = \overrightarrow{\mathbf{PQ}}$ can be drawn as an arrow with "tail" at the point P and "tip" at the point Q. It represents a vector whose magnitude is equal to the length of PQ and its direction is from Q towards P. This vector is defined entirely by its direction and length, and can be moved around the space.

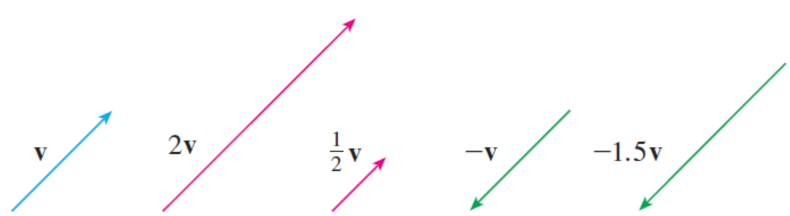
§2.3 Vector Arithmetic

2.3.1 Addition

To add two vectors $\vec{u}$ and $\vec{v}$, draw them one after the other, tip to tail. The vector from the base of the pair to the final tip is the sum of the vectors.

2.3.2 Scalar Multiplication

If λ is a scalar i.e. a real number, then $\lambda\vec{v}$ is a vector whose magnitude is λ times the magnitude of $\vec{v}$. It has the same direction as $\vec{v}$ if $\lambda > 0$ and the opposite direction if $\lambda < 0$.



Warning: If $\lambda = 0$, we get the **Zero Vector**, a vector whose magnitude is zero and is **omnidirectional**!

2.3.3 Subtraction

$\vec{u} - \vec{v}$ is defined as $\vec{u} + (-1) \times \vec{v}$.

■ **Question 11.** □

Suppose the three sides of a triangle $\triangle ABC$ are denoted by vectors as $\vec{c} = \overrightarrow{AB}$, $\vec{a} = \overrightarrow{BC}$, and $\vec{b} = \overrightarrow{CA}$. What is $\vec{a} + \vec{b} + \vec{c}$?

■ **Question 12.** □

Consider $\triangle ABC$ as above. Let D be the mid-point of BC and let $\vec{m} = \overrightarrow{AD}$ be one of the medians of the triangle. Find $\vec{m}$ in terms of $\vec{a}, \vec{b}$ and $\vec{c}$.

Digression

It is in fact also possible to find the angle bisector vector in terms of the sides. However that requires the law of sines in a triangle. In case you are interested, here is the precise problem. Suppose D is a point on BC such that $\angle BAD = \angle CAD$. Find $\overrightarrow{AD}$ in terms of $\vec{a}, \vec{b}$ and $\vec{c}$.

§2.4 Symbolic representation of vectors

A vector of magnitude 1 is called a **unit vector**. The unit vectors along X–, Y– and Z–axes are called $\hat{i}, \hat{j}$ and $\hat{k}$ respectively.

Consider the vector $\vec{v} = \overrightarrow{PQ}$ where $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$. Then it can be also written as

$$\vec{v} = \underbrace{(x_2 - x_1)}_{\Delta x}\hat{i} + \underbrace{(y_2 - y_1)}_{\Delta y}\hat{j} + \underbrace{(z_2 - z_1)}_{\Delta z}\hat{k} = \langle \Delta x, \Delta y, \Delta z \rangle$$

The projection of a vector on to the axes are called its **components**. Thus the X–component of the vector $\vec{v}$ above is $(\Delta x)\hat{i}$ etc. Clearly a vector is the sum of its components.

Addition or scalar multiplication of a vector can be done component-wise.

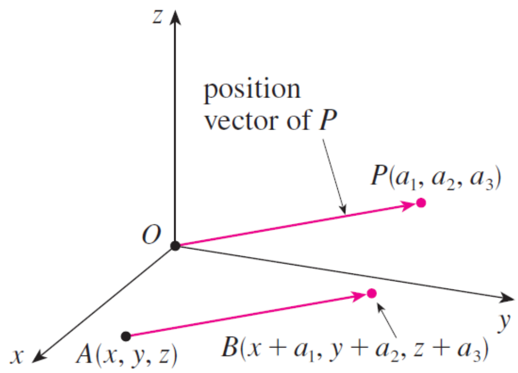
$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle \qquad c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

2.4.1 Position vector - components vs coordinates

If we write $\vec{v} = \langle a_1, a_2, a_3 \rangle$ or equivalently $\vec{v} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, we mean a vector with tail at $(0,0,0)$ and tip at $P(a_1, a_2, a_3)$. In this way every point (a_1, a_2, a_3) corresponds to the unique vector $\langle a_1, a_2, a_3 \rangle$ called its **position vector**. The length of the components of the position vector are equal to the coordinates of the point.

§2.5 Magnitude

The magnitude of a vector $\vec{v} = \langle a, b, c \rangle$ is denoted by $\|\vec{v}\|$, pronounced "norm" or $\vec{v}$, and is equal to $\sqrt{a^2 + b^2 + c^2}$. This is an easy consequence of Pythagoras theorem!



Question 13.

How can you use $\|\vec{v}\|$ to create a unit vector in the same direction as $\vec{v}$?

§2.6 Calculating components in 2D

Suppose a vector $\vec{v}$ makes an angle θ with the positive X–axis. Then we can the magnitude $\|\vec{v}\|$ to express a vector $\vec{v}$ in terms of trigonometric functions as follows:

$$\vec{v} = (\Delta x)\hat{i} + (\Delta y)\hat{j} = (\|\vec{v}\|\cos\theta)\hat{i} + (\|\vec{v}\|\sin\theta)\hat{j} = \|\vec{v}\|\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right)$$

Question 14.

Suppose a three dimensional vector $\vec{v}$ makes angle α, β and γ with positive X–, Y– and Z–axis respectively. Then show that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

§2.7 Word problems

Question 15.

Bert and Ernie are trying to drag a large box on the ground. Bert pulls the box toward the north with a force of 30 N, while Ernie pulls the box toward the east with a force of 40 N. What is the resultant force on the box?

Question 16.

An airplane at altitude is flying NE with airspeed 700 km/hr, with wind from the West at 60 km/hr. Use vectors to determine the resulting direction and ground speed of the plane.

Chapter 3 | Dot Product of Vectors

§3.1 Definition

3.1.1 Algebraic

The **dot product** or scalar product of two vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is defined to be the sum of the product of the components.

$$\vec{u} \cdot \vec{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle := u_1 v_1 + u_2 v_2 + u_3 v_3$$

Question 17. □

What is $\langle 1, 2, 3 \rangle \cdot \langle 4, -5, 6 \rangle$? How about $\langle 1, 2 \rangle \cdot \langle 3, 4, 5 \rangle$?

3.1.2 Geometric

If the angle between $\vec{u}$ and $\vec{v}$ is θ then

$$\vec{u} \cdot \vec{v} := \|\vec{u}\| \|\vec{v}\| \cos\theta$$

Question 18. □

Does it matter whether the angle θ is calculated from $\vec{u}$ to $\vec{v}$ or in the other order?

Question 19. □

What is $\vec{u} \cdot \vec{v}$ if

(a) $\vec{u} \perp \vec{v}$?

(b) $\vec{u} = \vec{v}$?

(c) $\vec{u} \parallel \vec{v}$?

Theorem 1.2

Two vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\vec{u} \cdot \vec{v} = 0$.

§3.2 Basic properties

From the definitions the following basic properties of the dot product are easy to prove. If $\vec{u}, \vec{v}$ and $\vec{w}$ are vectors of the same dimension and c is a scalar, then

(a) $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

(b) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

(c) $(c\vec{v}) \cdot \vec{w} = c(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (c\vec{w})$

§3.3 Angle between two vectors

We can use both definitions of dot product together to calculate angle between two given vectors.

Question 20.

Find the angle between $\langle 1,2,1\rangle$ and $\langle 1,-1,1\rangle$.

Question 21.

In a molecule of Methane (CH_4) you have four hydrogen atoms bonded to a carbon atom. The four hydrogen atoms form the corner of a regular tetrahedron and the carbon atom lies in the center. What is the angle between any two of the C – H bonds?

Hint: *Think of the four hydrogen atoms lying on four corners of a cube.*

§3.4 Vector projections

Take a vector $\vec{u}$ and resolve it into two components, one along another given vector $\vec{v}$ and another perpendicular to $\vec{v}$. We call these two components $\vec{u}_{\|}$ and $\vec{u}_{\perp}$ respectively. Thus

$$\vec{u} = \vec{u}_{\|} + \vec{u}_{\perp}$$

The component $\vec{u}_{\|}$ is called the **projection** of $\vec{u}$ on to $\vec{v}$, and is denoted $\text{Proj}_{\vec{v}}\,\vec{u}$.

Geometrically speaking, if we take a screen along $\vec{v}$ and shine a light perpendicular to it from above, the shadow cast by $\vec{u}$ would be $\text{Proj}_{\vec{v}}\,\vec{u}$.

Theorem 4.3: Projection Formula

$$\text{Proj}_{\vec{v}}\,\vec{u} = \vec{u}_{\|} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v}$$

Question 22.

Prove above theorem.

HINT: Use the dot product formula for $\cos\theta$.

Question 23. 3

The vertices of a triangle $\triangle ABC$ *are* $A=(4,3,2), B=(1,3,1)$, *and* $C=(-5,5,-2)$. *Let* D *be the foot of the perpendicular from* A *to the side* $\overline{BC}$. *Find the vector* $\overrightarrow{\mathbf{AD}}$.

HINT: Find $\overrightarrow{\mathbf{BD}}$ *first.*

Chapter 4 | Cross Product of Vectors

§4.1 Definition

4.1.1 Geometric

The cross product of two vectors $\vec{u}$ and $\vec{v}$ is the **vector** $\vec{u} \times \vec{v}$ whose

- magnitude is equal to $\|\vec{u}\|\|\vec{v}\|\sin\theta$, where θ is the angle from $\vec{u}$ to $\vec{v}$ and
- direction is perpendicular to the both $\vec{u}$ and $\vec{v}$, determined by the right-hand-thumb rule.

Question 24. □

Does it matter whether the angle θ is calculated from $\vec{u}$ to $\vec{v}$ or in the other order?

Question 25. □

What is $\hat{i} \times \hat{j}$? What is $\hat{i} \times \hat{i}$? What is $\hat{j} \times \hat{i}$?

4.1.2 Algebraic

If $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$, then we define $\vec{u} \times \vec{v}$ to be

$$\langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle.$$

There is a handy way of remembering this definition: the cross product $\vec{u} \times \vec{v}$ is equal to the determinant

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{k}$$

Warning: The cross product is only defined for three-dimensional vectors.

§4.2 Basic Properties

From the definitions the following basic properties of the cross product are easy to prove. If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors of the same dimension and c is a scalar, then

(a) $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$

(b) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

(c) $(c\vec{v}) \times \vec{w} = c(\vec{v} \times \vec{w}) = \vec{v} \times (c\vec{w})$

Theorem 2.4

Two vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u} \times \vec{v} = 0$.

§4.3 Cross product as area

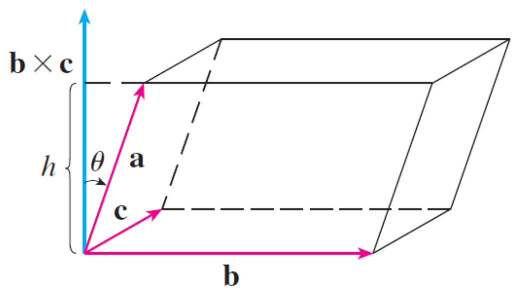
From the geometric definition, it is easy to see that $\|\vec{u} \times \vec{v}\|$ is equal to the area of the parallelogram whose two adjacent sides are given by $\vec{u}$ and $\vec{v}$.

Question 26. □

Show this. Use the fact that the area of the parallelogram is twice the area of the triangle whose two sides are given by $\vec{u}$ and $\vec{v}$. Then use the area formula for a triangle.

4.3.1 Volume of a parallelepiped - Coplanarity

Extending this geometric idea, we can show that the volume of a parallelepiped whose three adjacent sides are given by $\vec{a}, \vec{b}$, and $\vec{c}$ is equal to $\left|\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)\right|$.



Consequently, we have the following theorem

Theorem 3.5

Three vectors $\vec{a}, \vec{b}$, and $\vec{c}$ are coplanar iff $\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = 0$.

Digression

$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)$ is called the **scalar triple product** or **box product** of the three vectors, as also denoted by $[\vec{a}, \vec{b}, \vec{c}]$. An interesting observation here is that since the box product relates to the volume of the parallelepiped, we get

$$[\vec{a}, \vec{b}, \vec{c}] = [\vec{b}, \vec{c}, \vec{a}] = [\vec{c}, \vec{a}, \vec{b}]$$

So if our goal is to only calculate the volume, it doesn't matter which vectors we choose as $\vec{a}, \vec{b}$ and $\vec{c}$ as long they are the three adjacent sides.

Question 27. $\boxed{3}$

Suppose λ and μ are real numbers such that

- *the three vectors*
$$\vec{u} = 2\hat{i} + 3\hat{j} + \hat{k}, \qquad \vec{v} = \hat{i} + \lambda\hat{j} + \mu\hat{k}, \qquad \vec{w} = 7\hat{i} + 3\hat{j} + 2\hat{k}$$
are coplanar, and
- *The vector $\vec{v}$ has magnitude $\sqrt{2}$.*

Find all possible values of λ and μ.

Chapter 5 | Lines and Planes

§5.1 Lines in space

The equation of a line through a point (x_0, y_0, z_0) and parallel to the vector $\vec{u} = \langle a, b, c\rangle$ can be expressed in many ways:

- as parametric scalar equations:

$$x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc;$$

- as a parametric vector equation:

$$\vec{r}(t) = \vec{r}_0 + t\vec{u}, \quad \text{where } \vec{r} = \langle x, y, z\rangle \text{ and } \vec{r}_0 = \langle x_0, y_0, z_0\rangle$$

- or by symmetric equations:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Warning: Observe that the symmetric equation form doesn't make sense in case one of a, b or c is zero.

Question 28. □

Let $\mathscr{L}$ be the line which passes through the points $(1, -2, 3)$ and $(4, -4, 6)$. Find its equation in all three forms.

§5.2 Planes in space

The equation of a plane through the point (x_0, y_0, z_0) and perpendicular (or normal or orthogonal) to the vector $\vec{n} = \langle a, b, c\rangle$ also has many (equivalent) equations:

- as a vector equation:

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \quad \text{or} \quad \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

  where again $\mathbf{r} = \langle x, y, z\rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$; or equivalently

- as a scalar equation:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \text{or} \quad ax + by + cz = d$$

  where $d = ax_0 + by_0 + cz_0$ is a constant.

Question 29. □

Find an equation describing the plane which goes through the point $(1, 3, 5)$ and is perpendicular to the vector $\langle 2, 1, -3\rangle$.

Question 30. □

Find an equation describing the plane which passes through the points $\mathrm{P}(2, 2, 1), \mathrm{Q}(3, 1, 0)$, and $\mathrm{R}(0, -2, 1)$.

§5.3 Types of problems

One of the fundamental topic in Multivariable Calculus is to learn how to find equations of straight lines and planes in three dimensions using ideas from vectors (dot and cross product).The following lists an incomplete but fairly diverse types of problems that you should be able to solve using these ideas.

5.3.1 Finding a Plane

You should know how to find equation of a plane from the following data.

(a) Plane through a given point and perpendicular to a given vector.

(b) Plane through a given point and parallel to a given plane.

(c) Plane containing a given line and parallel to a given plane.

(d) Plane passing through three given points.

(e) Plane with specified $x-$, $y-$ and $z-$intercepts.

(f) Plane through a given point and containing a given straight line.

(g) Plane through a given point and containing the line of intersection of two other given planes.

(h) Plane through a given point and perpendicular to two other given planes.

(i) Plane passing through two points and perpendicular to a given plane.

(j) Plane containing the line of intersection of two other given planes and perpendicular to a given plane.

5.3.2 Finding a Line

You should know how to find equation of a straight line from the following data.

(a) Line through two given points.

(b) Line through one given point and in the direction of a given vector.

(c) Line through one given point and parallel to a given straight line.

(d) Line of intersection of two given planes.

(e) Line through one given point and perpendicular to a given plane.

(f) Line through a given point, that is perpendicular to a given straight line and intersects this second line.

(g) Line through a given point, that is parallel to (i.e. lies in) a given plane and perpendicular to a given straight line.

§5.4 Practice problems

■ **Question 31.** □

Below is a list of vectors and a list of properties. Match the two sets in such a way that each entry in left column matches a different entry in right column.

A. $\langle 3,-2,8\rangle$	*I. is parallel to the straight line* $\frac{x-1}{2}=y-3=z$
B. $\langle 4,2,2\rangle$	*II. is perpendicular to the plane* $z-2y-x=3$
C. $\langle 3,1,-1\rangle$	*III. is perpendicular to both* $\langle 2,3,0\rangle$ *and* $\langle -2,5,2\rangle$
D. $\langle 1,2,-1\rangle$	*IV. is parallel to the plane* $x-y+2z=3$

■ Question 32. □

(a) *Find parametric equations for the line through the points* $(6,1,1)$ *and* $(9,1,4)$*. Call this line* L_1.

(b) *Find parametric equations for the line through the points* $(-4,4,0)$ *and* $(-6,5,1)$*. Call this line* L_2.

(c) *Find parametric equations for the line through the points* $(6,-1,-5)$ *and* $(2,1,-3)$*. Call this line* L_3.

(d) *Verify that* L_2 *and* L_3 *are parallel. (Their direction vectors should be parallel.) Are they the same line? How could you tell?*

(e) *Do lines* L_1 *and* L_2 *intersect? If so, where?*

(f) *Find the intersection of* L_1 *with the plane given by the equation* $2x+y+3z=7$.

(g) *(3 points) Find the point on the plane* $2x+y+3z=7$ *which is closest to the origin.*

(h) *(3 points) Find the point on* L_2 *closest to the origin.*

■ Question 33. □

Find a vector parallel to the intersection of the two planes $2x-3y+5z=2$ *and* $4x+y-3z=7$.

Find the equation of the line of intersection.

■ Question 34. □

Find the distance of the point $P=(1,0,1)$ *from the plane* $x+y-z=1$.

■ Question 35. □

Let L_1 *be the line with parametric vector equation* $\vec{r}_1(t)=\langle 7,1,3\rangle+t\langle 1,0,-1\rangle$ *and* L_2 *be the line described parametrically by* $x=5, y=1+3t, z=t$*. How many planes are there that contain* L_2 *and are parallel to* L_1*? Find an equation describing one such plane.*

■ Question 36. □

Find an equation for the plane that contains the line in the XY-pane where $y=1$*, and the line in the XZ-pane where* $z=2$.

Chapter 6 | Functions of Two Variables

§6.1 Examples of functions of two variables

- Household lobster consumption is a function of income and the price of lobster.
- The density of cars along a highway is a function of position and time.
- The daily temperature across the United States is a function of latitude and longitude.
- Volume of a cylinder is a function of its radius and height.

§6.2 Representations of two-variable functions

6.2.1 Numerical

The body mass index (BMI) is a value that attempts to quantify a person's body fat based on their height h and weight w.

Height h (inches) \ Weight w (lbs)	120	140	160	180	200
60	23.4	27.3	31.2	35.2	39.1
63	21.3	24.8	28.3	31.9	35.4
66	19.4	22.6	25.8	29.0	32.3
69	17.7	20.7	23.6	26.6	29.5
72	16.3	19.0	21.7	24.4	27.1
75	15.0	17.5	20.0	22.5	25.0

■ **Question 37.** □

What is the BMI of a person who is 72 inches tall and weighs 160 lb?

6.2.2 Algebraic

A solid cylinder with closed ends has radius r and height h. Its volume V is given by

$$V = f(r,h)$$

■ **Question 38.** □

What does $V(10,h)$ *mean? What does* $V(r,10)$ *mean?*

■ **Question 39.** □

Can you give a formula for the function $f(r,h)$ *? What about the surface area* $A = g(r,h)$*?*

6.2.3 Visual - Using Graphs

The **graph** of a function of two variables, f, is the set of all points (x,y,z) such that $z = f(x,y)$. In general, the graph of a function of two variables is a surface in 3−space.

Question 40.

Sketch the graph of the function $f(x,y) = 6 - 3x - 2y$.

Question 41.

Sketch the graph of $g(x,y) = \sqrt{9 - x^2 - y^2}$.

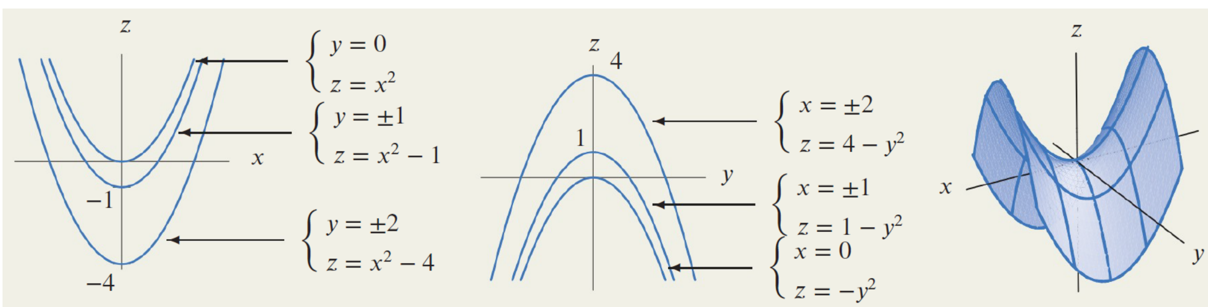
§6.3 Analyzing Graphs using Cross-sections

One way to visualize surfaces that are three-dimensional graphs is to view pieces of them as two-dimensional graphs. If we intersect the graph of $z = f(x,y)$ with a plane (such as $x = k$ or $y = k$), we get a graph in a two-dimensional plane (the kind we're used to). This is called a **cross-section** (or a trace).

Question 42.

Describe the cross-sections of the function $g(x,y) = x^2 - y^2$ with y fixed and then with x fixed. Use these cross-sections to describe the shape of the graph of g.

Solution.



§6.4 Group Problems

Question 43.

Here are six functions whose graphs we ought to be able to visualize in space:

(a) $z = f(x,y) = 6 - 3x - 2y$

(b) $z = f(x,y) = x^2 + y^2$

(c) $z = f(x,y) = x^2 - y^2$

(d) $z = f(x,y) = x^2 + y + 1$

(e) $z = f(x,y) = (x - y)^2$

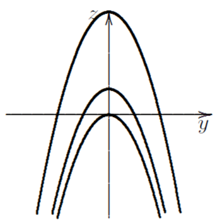
(f) $z = f(x,y) = \frac{1}{1+x^2+y^2}$

Here are cross-sections for two of the surfaces above. Your job is to:

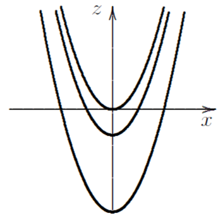
(a) *identify which graph these cross-sections belong with,*

(b) *graph cross-sections of the remaining graphs of surfaces, and*

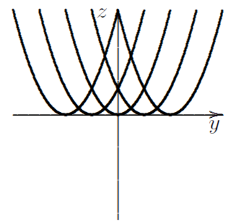
(c) *try to visualize the original surface.*



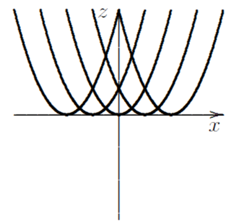
$x = k$



$y = k$



$x = k$

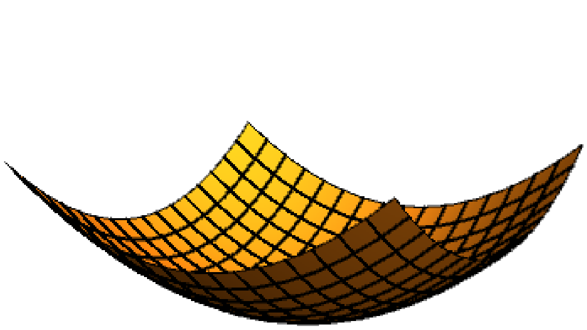


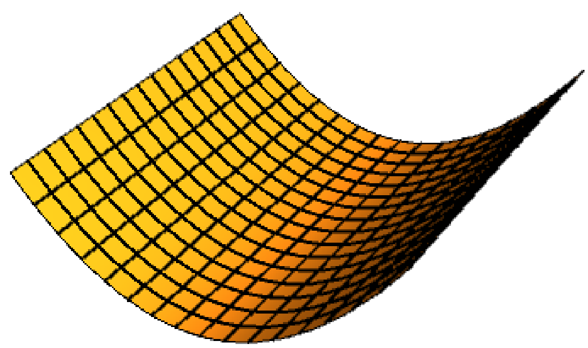
$y = k$

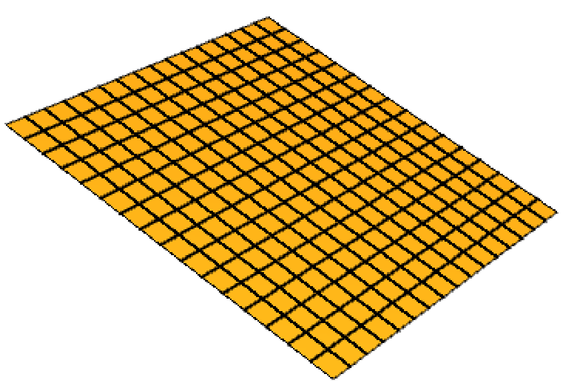
■ Question 44.

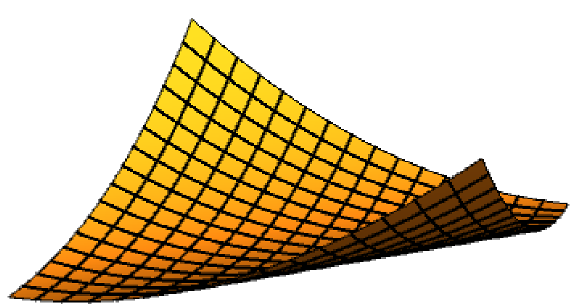
□

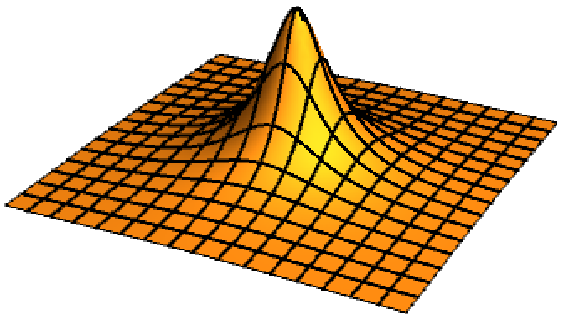
Now that you have a good idea of what each of these graphs look like, you should have no problem identifying which of the following (axes-less) graphs go with each equation for the previous page. Your reasoning should involve the cross-sections you drew as well.

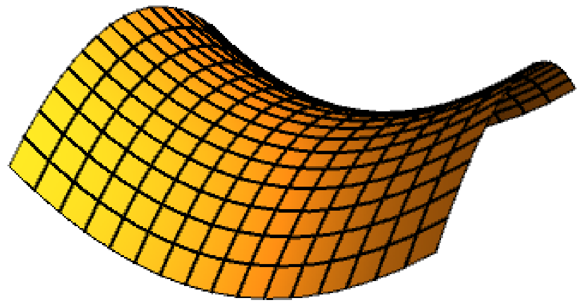












Question 45.

3 Match each function with its graph.

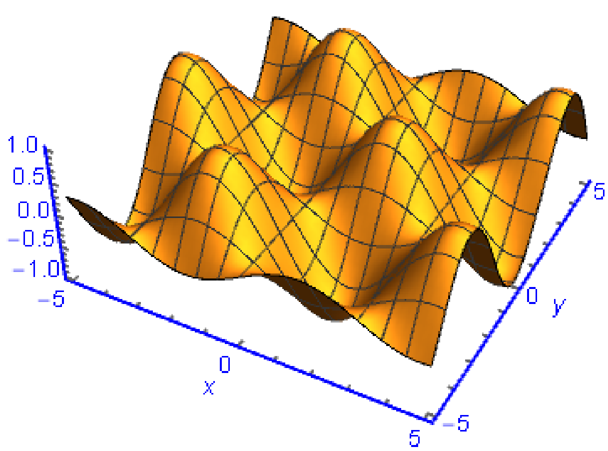
(a) $f(x,y) = \frac{x^2}{x^2+y^2}$

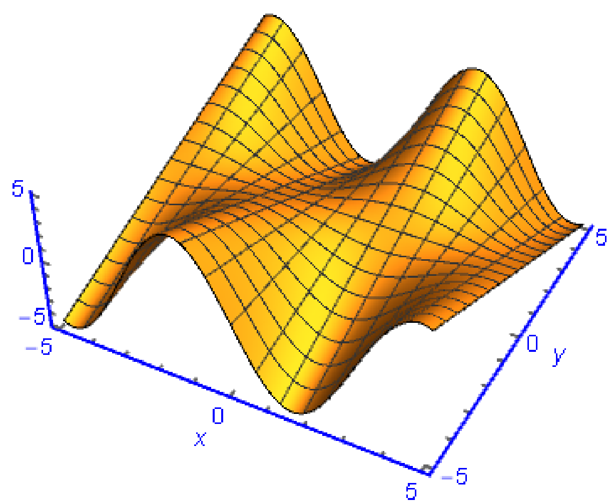
(b) $f(x,y) = y\sin x$

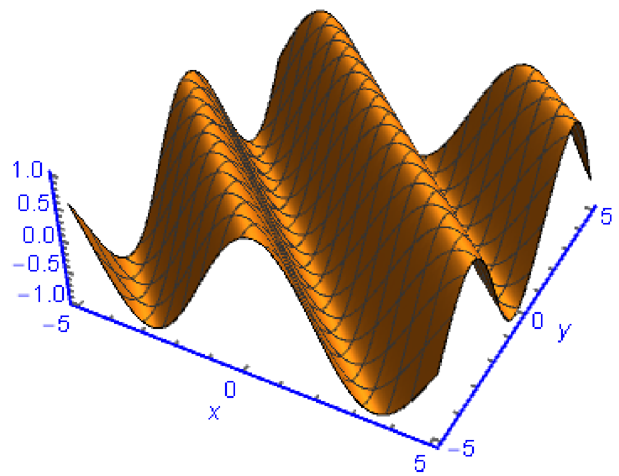
(c) $f(x,y) = \sin(x+y)$

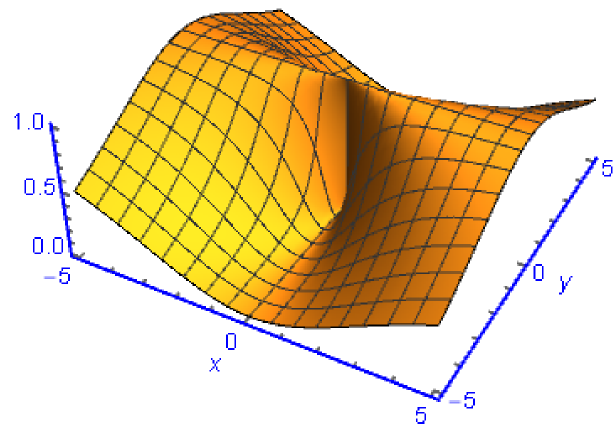
(d) $f(x,y) = \sin x \cos y$

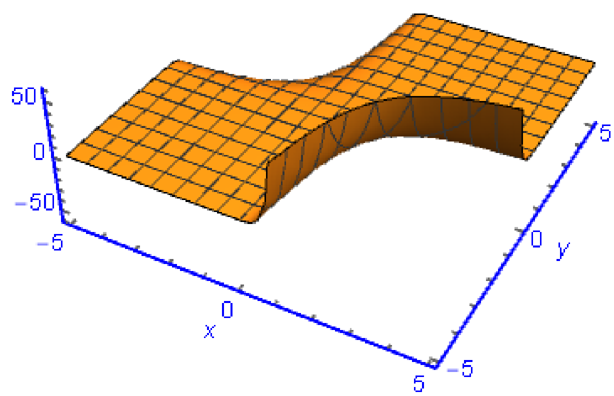
(e) $f(x,y) = xe^{-xy}$






1.0
0.5
0.0
−0.5
−1.0
−5
0
x
5
−5
0
y
5


1.0
0.5
0.0
−5
0
x
5
−5
0
y
5


50
0
−50
−5
0
x
5
−5
0
y
5

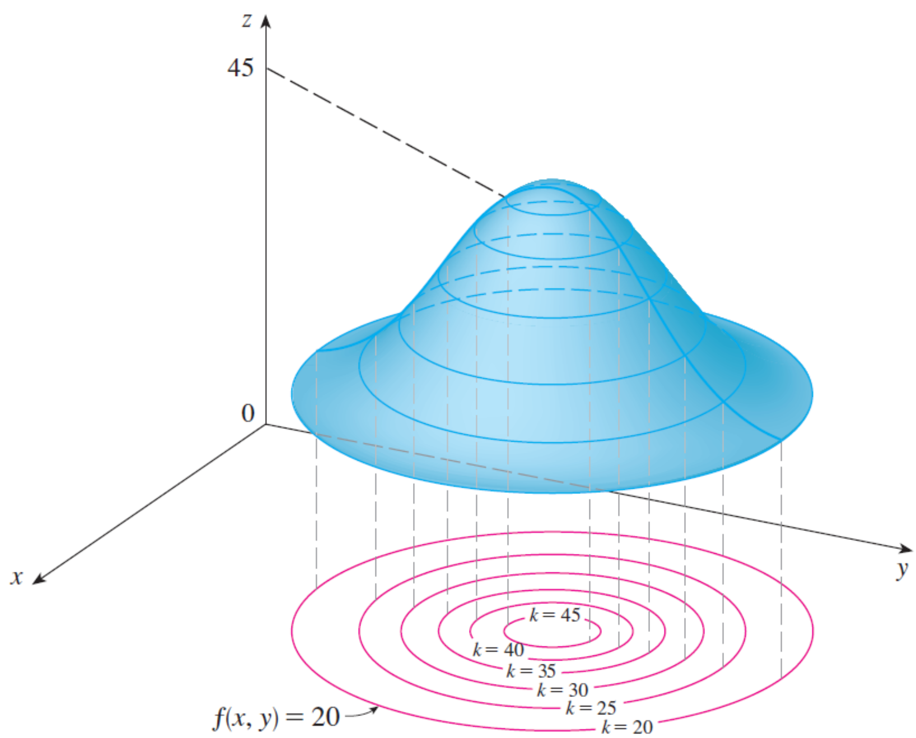
Chapter 7 | Contour Plots - Level Curves and Level Surfaces

§7.1 Level Curves

7.1.1 Definition

The **level curves** of a function f of two variables are the curves with equations $f(x,y) = c$, where c is a constant (in the range of f).

These are essentially the z–cross-sections of the graph of $f(x,y)$. The collection of all the level curves is called a **contour plot**.

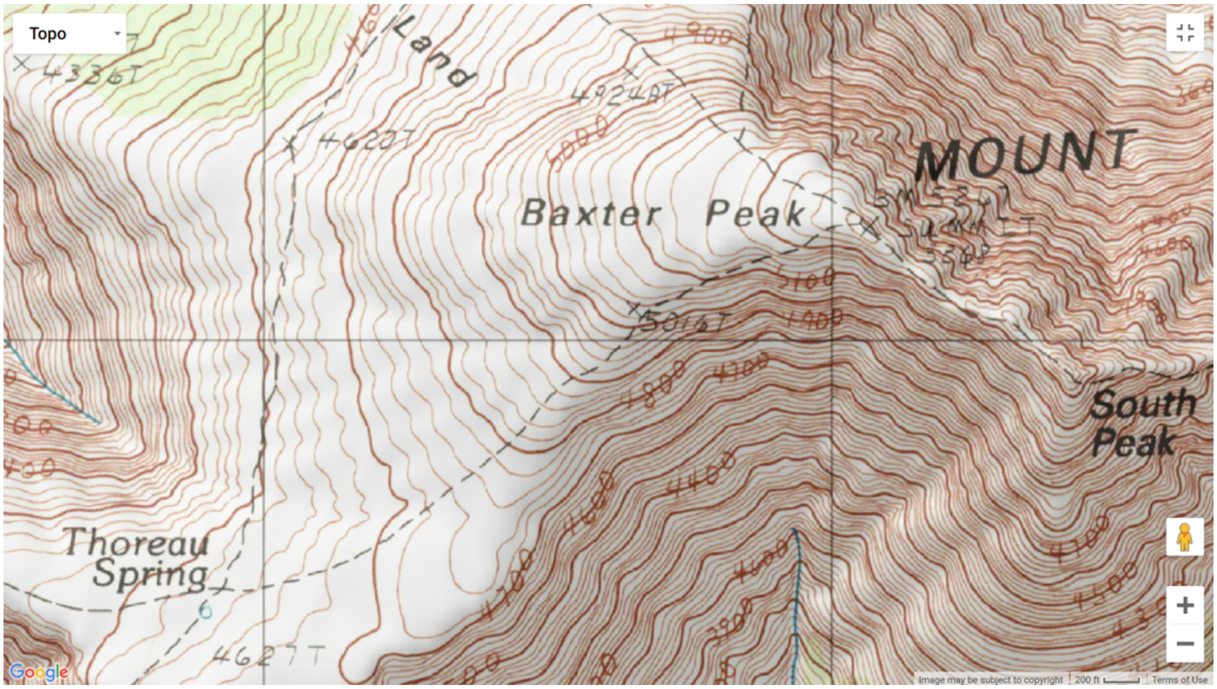


7.1.2 Basic Facts

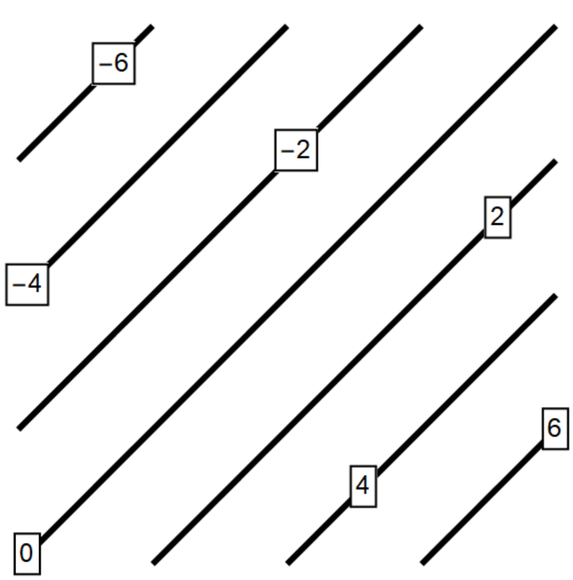
- The level curve for fixed output c is called the c -level of $f(x,y)$.
- unless otherwise indicated, contours are drawn at regular z–increments.
- We can build contour plots by hand following the same procedure we used for x and y cross-sections, but now fixing the z-values and plotting in the xy-plane.
- Two different level curves cannot cross.

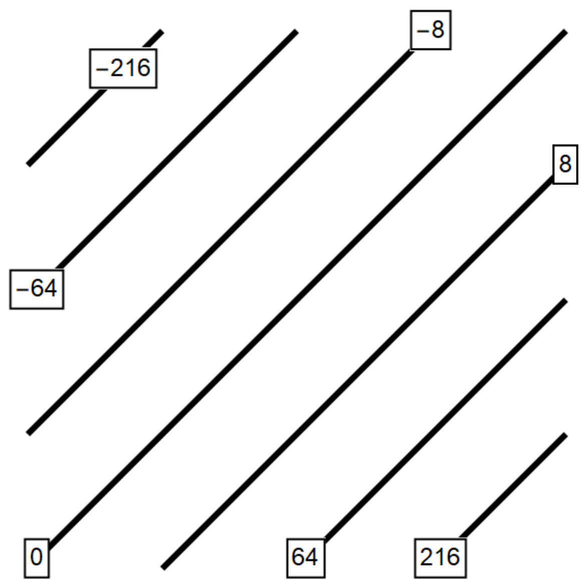
§7.2 Practice Problems

Consider two functions $f(x,y) = x - y$ and $g(x,y) = (x-y)^3$. Observe the difference in the contour plots for each of the graphs below.



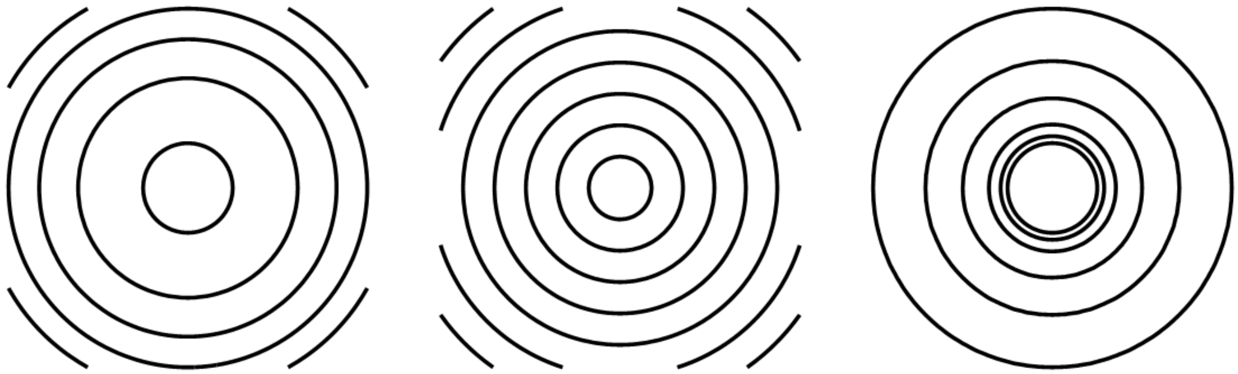
A height contour plot of Mount Katahdin





■ **Question 46.** □

Assuming regular z–increments, describe what the corresponding graphs might look like.

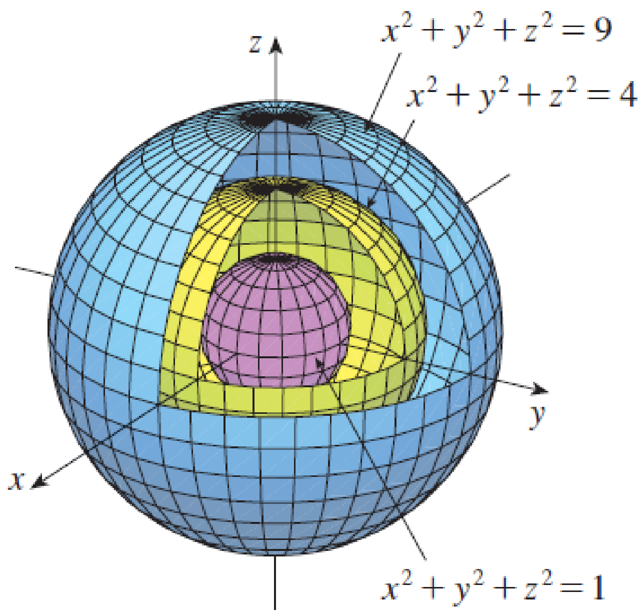


§7.3 Functions of three variables - Level surface

It's very difficult to visualize a function f of three variables by its graph, since that would lie in a four-dimensional space. However, we do gain some insight into f by examining its **level surfaces**, which are the surfaces with equations $f(x,y,z)=c$, where c is a constant. If the point (x,y,z) moves along a level surface, the value of $f(x,y,z)$ remains fixed.

7.3.1 Level surfaces of $f(x,y,z)=x^2+y^2+z^2$

The level surfaces are $x^2+y^2+z^2=k$, where $k\geq 0$. These form a family of concentric spheres with radius $\sqrt{k}$. Thus, as (x,y,z) varies over any sphere with center O the value of $f(x,y,z)$ remains fixed.



■ **Question 47.** $\boxed{3}$

What do the level surfaces of $f(x,y,z)=x^2+y^2-z^2$ look like? Finish project 2 to find out.

Digression

Functions of any number of variables can be considered. A function of n variables is a rule that assigns a number $z = f(x_1, x_2, \ldots, x_n)$ to an n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers. We denote by $\mathbb{R}^n$ the set of all such n-tuples.

Recall that there is a one-to-one correspondence between points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ and their position vectors $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$. So we have three ways of looking at a function f defined on a subset of $\mathbb{R}^n$:

(a) As a function of n real variables $x_1, x_2, \ldots, x_n$

(b) As a function of a single point variable $(x_1, x_2, \ldots, x_n)$

(c) As a function of a single vector variable $\vec{\boldsymbol{x}} = \langle x_1, x_2, \ldots, x_n \rangle$

We will see that all three points of view are useful.

Chapter 8 | Linear Functions

§8.1 Definition

A function of two variables of the form

$$f(x,y) = mx + ny + d$$

where m, n and d are fixed constants is called a linear functions.

Recall that the equation $z = mx + ny + d$ represents a plane in three-dimensions. Thus clearly, the graph of a linear function looks like a plane. The constant m and n respectively represent the slope of the graph in $x-$direction and $y-$direction.

§8.2 Graphical Representation

What does the contour plot of a linear function look like? If we set $f(x,y) = c$, we can rewrite the equation as

$$y = -\frac{m}{n}x + \frac{c-d}{n}$$

which is a line with slope $-\frac{m}{n}$. So no matter what level c we choose, the lines remain parllel. Thus the contour plot of a linear function is a set of evenly-spaced parallel lines (assuming regular $c-$increments).

§8.3 Practice Problems

■ **Question 48.** □

Consider the function $z = f(x,y) = 8x - 4y + 2$.

(a) *Sketch the contour plot for the graph with z-increment value of* 1.

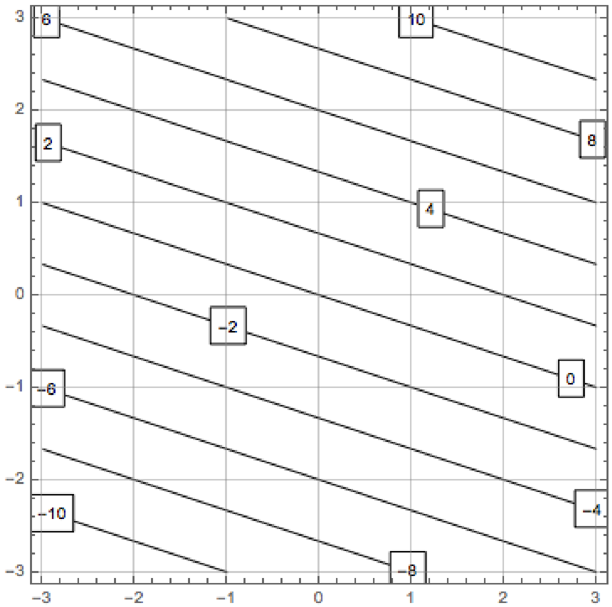
(b) (i) *Starting at any point (x,y), what is the slope of the surface in the $x-$direction?*

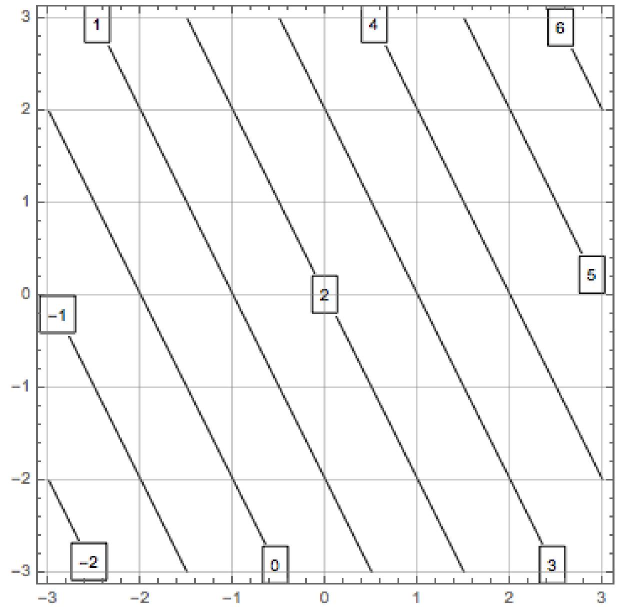
(ii) *What is the slope in $y-$direction?*

(iii) *What is the slope along the line $x = y$?*

■ **Question 49.** □

Find the linear functions whose contour plots are shown below.





■ Question 50.

☐

Fill in the blank with "certainly", "possibly", or "certainly not": *If $f(x,y)$ is a linear function, then the graph of f is* ____ *parallel to the xz-plane.*

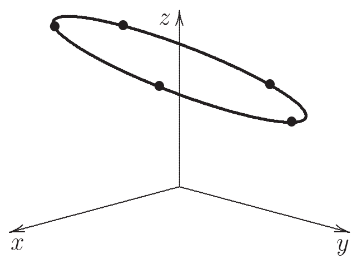
Chapter 9 | Vector Valued Functions and Space Curves

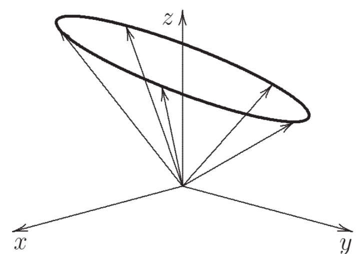
Any curve in 3–space can be described as either a collection of points

$$(x(t), y(t), z(t))$$

for t in some interval (possibly infinite) or as the trace of the heads of the position vectors (which start at the origin)

$$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$





Definition 0.6

$\vec{r}(t)$ is an example of a **vector valued function** of a real variable t. In general a vector valued function $\vec{\mathbf{F}}(t)$ can be written as

$$\vec{\mathbf{F}}(t) = \langle f(t), g(t), h(t)\rangle$$

We will come back to general vector-valued functions in the next chapter.

Example 0.7: Circle

A circle in the xy-plane can be described as the set of pairs $(x(t), y(t))$ generated by the **parameterization**

$$x(t) = \cos(t) \text{ and } y(t) = \sin(t) \quad \text{for } 0 \le t \le 2\pi$$

associated with the position vectors

$$\vec{r}(t) = \cos(t)\hat{i} + \sin(t)\hat{j} \quad \text{for } 0 \le t \le 2\pi$$

where the parameter t is the angle from the x–axis.

When curves are generated in this manner, they are called **parametrized curves**. Notice that **parameterization** carries more information than simply the final curve shape in the xy-plane. In particular, the example above indicates that the curve shape (circle) in the xy-plane is traced exactly once around, and that this curve shape is traced counterclockwise as t increases through its range.

§9.1 Different ways of parametrizing the same curve

We can parametrize the same curve in different ways and interpret each parametrization as the motion of a particle with the parameter t being time.

Question 51.

Explain why all of the parametrized curves below should look like a circle. In each of the parametrization, determine how many times around, and in which direction (clockwise or counterclockwise) the curve is traced.

- $x(t) = \cos(2t)$ *and* $y(t) = \sin(2t)$ *for* $0 \le t \le 2\pi$
- $x(t) = \cos(t^2)$ *and* $y(t) = \sin(t^2)$ *for* $0 \le t \le 2\pi$
- $x(t) = \cos(t)$ *and* $y(t) = -\sin(t)$ *for* $0 \le t \le 2\pi$
- $x(t) = -\cos(t)$ *and* $y(t) = \sin(t)$ *for* $0 \le t \le 2\pi$
- $x(t) = -\cos(t)$ *and* $y(t) = -\sin(t)$ *for* $0 \le t \le 2\pi$

Question 52.

Recall that $\vec{r}(t) = \vec{r}_0 + t\vec{v}$ (for $t \in \mathbb{R}$) gives the parametrization of a straight line passing through $\vec{r}_0$ and parallel to $\vec{v}$. What kind of curve are the following:

- $\vec{r}(t) = \vec{r}_0 + t^2\vec{v}$ *for* $t \in \mathbb{R}$
- $\vec{r}(t) = \vec{r}_0 + t\vec{v}$ *for* $a \le t \le b$

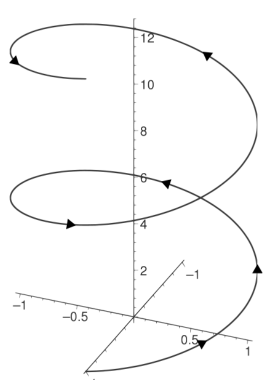
§9.2 3D Curves

9.2.1 Helix

The parametric equation of a helix is given by

$$\vec{r}(t) = \langle \cos t, \sin t, t\rangle$$

When looked on from above (from the positive z direction), this curve is simply a circle in the xy-plane. The $z = t$ component lifts the circle into the helix spinning above the circle in the plane. If we visualize this as a particle at the tip of the position vector $\vec{r}(t)$, then from above it looks like the particle is simply spinning in a circle. But we also know that $z = t$, so the particle is rising at a constant rate. Hence we get the picture below



■ Question 53.

Match each vector-valued function to the curve it parametrizes.

(a) $\vec{r}(t) = \langle t\cos t, t\sin t, t\rangle$

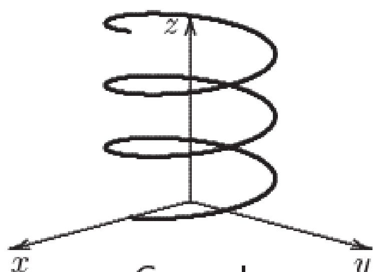
(b) $\vec{r}(t) = \left\langle \cos t, \sin t, t^3 \right\rangle$

(c) $\vec{r}(t) = \langle \cos(t^3), \sin(t^3), t^3\rangle$

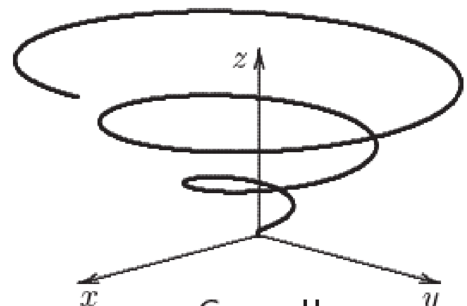
(d) $\vec{r}(u) = \langle \cos u, \sin u, 1 + \sin 4u\rangle$

(e) $\vec{r}(u) = \langle \cos u, \sin u, 1 + 4\sin u\rangle$

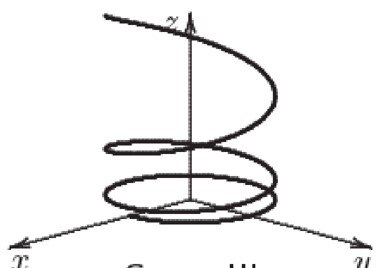
(f) $\vec{r}(t) = \langle 2\cos t, 1 + 4\cos t, 3\cos t\rangle$



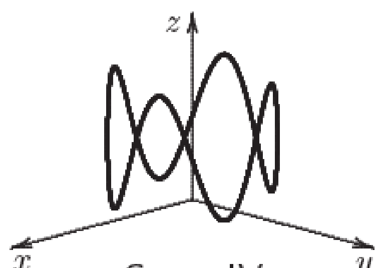
Curve I



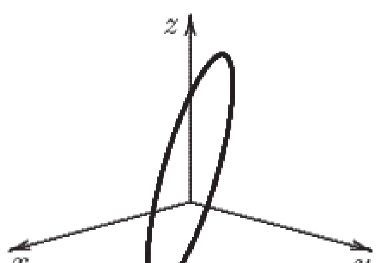
Curve II



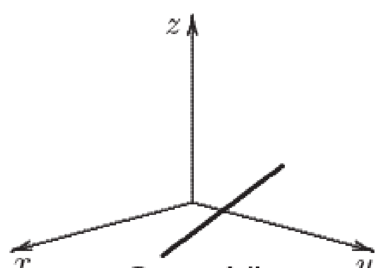
Curve III



Curve IV



Curve V



Curve VI

§9.3 Parametrization from equation of curve

Example 3.8

A parametrization of the parabola $x = 1 - y^2$ in xy- plane can be given by

$$y(t) = t, \quad x(t) = 1 - t^2, \quad t \in \mathbb{R}$$

Question 54.

The surfaces $z = \sin(x - y)$ and $y = 2x$ intersect in a curve. Find a parameterization of the curve.

□

Question 55.

The surfaces $x^2 + \frac{y^2}{4} = 1$ and $z = \sin(x - y)$ intersect in a curve. Find a parameterization of the curve.

□

Chapter 10 | Motion, Velocity, Speed and Distance

Think of a parametrized curve as the trajectory of a point that is moving on the curve. At time t, its position vector is given by $\vec{r}(t)$. Then considering the appropriate vector difference quotients, we can build a concept of velocity vector of a parametrized curve.

Question 56. □

*A particle travels along the line $x = 1+t, y = 5+2t, z = -7+t$. **When** and **where** does the particle hit the plane $x+y+z = 1$?*

§10.1 Velocity and Acceleration

The idea is to consider the difference vector $\vec{r}(t+\Delta t) - \vec{r}(t)$ between the position vector $\vec{r}(t+\Delta t)$, a little time Δt beyond t, and the position vector $\vec{r}(t)$ at t. By taking $\Delta t \to 0$, the instantaneous **velocity vector** at time t is given by

$$\begin{aligned}\vec{r}'(t) &= \lim_{\Delta t\to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} \\ &= \lim_{\Delta t\to 0}\left(\frac{x(t+\Delta t)-x(t)}{\Delta t}\right)\hat{i} + \lim_{\Delta t\to 0}\left(\frac{y(t+\Delta t)-y(t)}{\Delta t}\right)\hat{j} + \lim_{\Delta t\to 0}\left(\frac{z(t+\Delta t)-z(t)}{\Delta t}\right)\hat{j} \\ &= x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}\end{aligned}$$

Similarly **acceleration vector** is given by

$$\vec{r}''(t) = x''(t)\hat{i} + y''(t)\hat{j} + z''(t)\hat{k}$$

Question 57. □

A fly is sitting on the wall at the point $(0,1,3)$. At time $t = 0$, he starts flying; his velocity at time t is given by $\vec{v}(t) = \left\langle \cos 2t, e^t, \sin t \right\rangle$. Find the fly's location at time t.

§10.2 Distance - Length of the curve

The magnitude $\|\vec{r}'(t)\|$ of the velocity vector is the **speed**. Hence the value $\|\vec{r}'(t)\|$ approximates the distance traveled from t to $t+\Delta t$ along the curve parameterized by $\vec{r}(t)$. Adding such approximations from $t = a$ to $t = b$ approximates the length of the curve

$$\sum_{t=a}^{t=b} \left\|\vec{r}'(t)\right\| \Delta t \approx \text{Length of curve } \vec{r}(t) \text{ from } t = a \text{ to } t = b$$

Taking $\Delta t \to 0$ gives a perfect approximation in terms of the following integral:

$$s(t) = \int_{t=a}^{t=b} \left\|\vec{r}'(t)\right\| dt = \text{ Length of curve } \vec{r}(t) \text{ from } t = a \text{ to } t = b$$

$s(t)$ is then the arc length of the curve - physically, the distance travelled along the curve.

■ **Question 58.** □

(a) *What is the speed of an object on the circle parameterized by* $\vec{r}(t) = \cos(t)\hat{i} + \sin(t)\hat{j}$*?*

(b) *Compute the length of this* $\vec{r}(t)$ *from* $0 \le t \le 2\pi$.

(c) *Use a dot product to find the orientation of the circle's velocity vectors* $\vec{r}'(t)$ *relative to its position vectors* $\vec{r}(t) = \cos(t)\hat{i} + \sin(t)\hat{j}$.

(d) *Use a dot product to find the orientation of the circle's acceleration vectors* $\vec{r}''(t)$ *relative to its velocity vectors.*

§10.3 Equation of motion

10.3.1 Projectile motion

Suppose x measures horizontal distance in meters, and y measures distance above the ground in meters. At time $t = 0$ in seconds, a projectile starts from a point h meters above the origin with speed v meters/sec at an angle θ to the horizontal. Its path is given by

$$x = (v\cos\theta)t, \quad y = h + (v\sin\theta)t - \frac{1}{2}gt^2$$

■ **Question 59.** □

Suppose a ball thrown off the top of a cliff travels along the path

$$x = 20t, \quad y = 2 + 25t - 4.9t^2$$

(a) *When and where does the ball hit the ground?*

(b) *At what height above the ground does the ball start?*

(c) *What is the value of* g*, the acceleration due to gravity?*

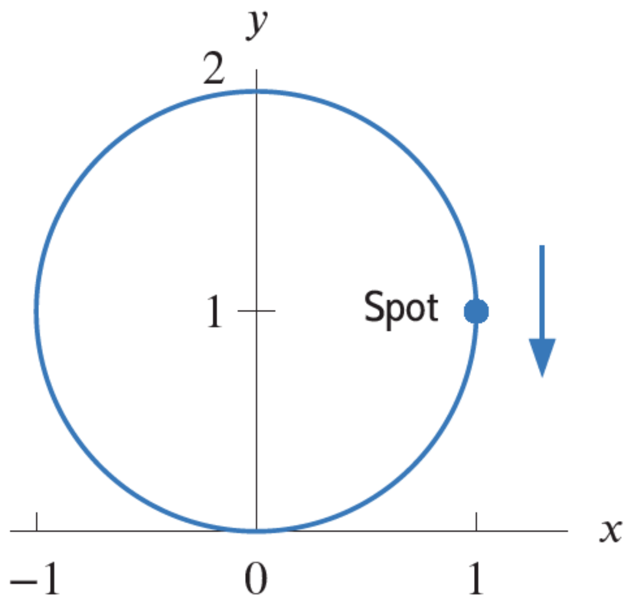
(d) *What are the values of* v *and* θ*?*

10.3.2 Cycloid

A **cycloid** is the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slipping.

■ **Question 60.** □

A wheel of radius 1 *meter rests on the* x*-axis with its center on the* y*-axis. There is a spot on the rim at the point* $(1,1)$*. At time* $t = 0$ *the wheel starts rolling on the* x*-axis in the direction shown at a rate of* 1 *radian per second.*



(a) *Find parametric equations describing the motion of the center of the wheel.*

(b) *Find parametric equations describing the motion of the spot on the rim. Plot its path.*

Chapter 11 | Curvature and Components of Acceleration

§11.1 Calculus of Vector-valued Functions

Consider a vector-valued function $\vec{\mathbf{F}}(t) = \langle x(t), y(t), z(t)\rangle$. In the last chapter, we defined $\vec{\mathbf{F}}'(t)$ and $\vec{\mathbf{F}}''(t)$. What do you think are the formula for the following? Here $p(t)$ is a one-variable dfifferentiable scalar-valued (real) function of t. .

(a) $\displaystyle\int \vec{\mathbf{F}}(t)\,\mathrm{d}t$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}t}[c\vec{\mathbf{F}}(t)]$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}t}[p(t)\vec{\mathbf{F}}(t)]$ (scalar product rule)

(d) $\dfrac{\mathrm{d}}{\mathrm{d}t}[\vec{\mathbf{F}}(t)\cdot\vec{\mathbf{G}}(t)]$ (dot product rule)

(e) $\dfrac{\mathrm{d}}{\mathrm{d}t}[\vec{\mathbf{F}}(t)\times\vec{\mathbf{G}}(t)]$ (cross product rule)

(f) $\dfrac{\mathrm{d}}{\mathrm{d}t}[\vec{\mathbf{F}}(p(t))$ (chain rule)

Question 61. □

If $\|\vec{r}(t)\| = c$, then show that $\vec{r}(t)$ is perpendicular to $\vec{r}'(t)$.

§11.2 Definitions of Curvature

A parametrized curve $\vec{r}(t)$ is called **smooth** if $\vec{r}'(t)$ is continuous and $\vec{r}'(t) \neq \vec{\mathbf{0}}$.

Let C be a smooth curve defined by the vector function $\vec{r}(t)$, the unit tangent vector to the curve $\vec{\mathbf{T}}(t)$ is given by

$$\vec{\mathbf{T}}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}.$$

Recall that te arc length of C is given by

$$s(t) = \int \left\|\vec{r}'(t)\right\| dt$$

Definition 2.9

The curvature of C at a given point κ is a measure of how quickly the curve changes direction at that point. Specifically, we define it to be the magnitude of the rate of change of the unit tangent vector with respect to arc length.

$$\kappa = \left\| \frac{\mathrm{d}\vec{\mathbf{T}}}{\mathrm{d}s} \right\|$$

This definition is not very helpful though. So we are going to do some manipulation using chain rule. Note that if we differentiate both sides of the arc length formula, then by using the Fundamental Theorem of Calculus, we obtain

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \|\vec{r}'(t)\|.$$

Hence, by chain rule,

$$\frac{\mathrm{d}\vec{\mathbf{T}}}{\mathrm{d}s} = \frac{\mathrm{d}\vec{\mathbf{T}}}{\mathrm{d}t}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\frac{\mathrm{d}\vec{\mathbf{T}}}{\mathrm{d}t}}{\frac{\mathrm{d}s}{\mathrm{d}t}} \implies \kappa = \frac{\left\|\vec{\mathbf{T}}'(t)\right\|}{\|\vec{r}'(t)\|}.$$

We can further manipulate the above equation by using the definition of $\vec{\mathbf{T}}$ to get

Theorem 2.10

The curvature of the curve given by the vector function $\vec{r}(t)$ is

$$\kappa = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}.$$

When $\vec{r}(t) = \langle f(t), g(t)\rangle$, we can simplify above formula to

$$\kappa = \frac{\|f'g'' - g'f''\|}{\left\|(f')^2 + (g')^2\right\|^{3/2}}.$$

■ **Question 62.** □

Find the curvature of $\vec{r}(t) = \langle t, ln(cos(t))\rangle$.

§11.3 Components of Acceleration

Projects

§A Distances

In this project, we will use Dot product and Cross product of vectors to derive formula for calculating distances between points, lines and planes. We will use the notation $d(\cdot,\cdot)$ to denote distance.

A.1 Distance between two points

To begin with, the distance between two points P and Q with position vectors $\vec{\mathbf{P}}$ and $\vec{\mathbf{Q}}$ is simply given by

$$d(\mathrm{P},\mathrm{Q}) = \|\vec{\mathbf{Q}} - \vec{\mathbf{P}}\| = \|\overrightarrow{\mathbf{PQ}}\|$$

where $\|\cdot\|$ denotes the magnitude of a vector.

■ **Question 1001.** □

Find the distance between $(-5,2,4)$ *and* $(-2,2,0)$.

A.2 Distance from a point to a plane

The distance of a point P from a plane Σ is defined as the length of the perpendicular from P to Σ. Suppose the plane Σ passes through a point Q and has normal vector $\vec{\boldsymbol{n}}$.

■ **Question 1002.** □

Explain using a picture why the distance from P *to* Σ *is the length of the projection of* $\overrightarrow{\mathbf{PQ}}$ *onto* $\vec{\boldsymbol{n}}$. *Then*

$$d(\mathrm{P},\Sigma) = \frac{|\overrightarrow{\mathbf{PQ}}\cdot\vec{\boldsymbol{n}}|}{\|\vec{\boldsymbol{n}}\|}$$

■ **Question 1003.** □

Find the distance of the point $(7,1,4)$ *from the plane* $2x+4y+5z=9$.

■ **Question 1004.** □

Without the absolute sign in the numerator of the distance formula, your answer in question (2) would have been negative. What does the negative sign signify here?

A.3 Distance from a point to a line

The distance of a point P from a line $\mathscr{L}$ is defined as the length of the perpendicular from P to $\mathscr{L}$. Suppose the line $\mathscr{L}$ passes through a point Q and is parallel to a vector $\vec{\boldsymbol{u}}$ (i.e. its parametric equation looks like $\vec{\boldsymbol{r}}(t) = \vec{\mathbf{Q}} + t\vec{\boldsymbol{u}}$).

■ **Question 1005.** □

Use the definition of cross product to derive the following formula:

$$d(\mathrm{P},\mathscr{L}) = \frac{\|\overrightarrow{\mathbf{PQ}}\times\vec{\boldsymbol{u}}\|}{\|\vec{\boldsymbol{u}}\|}$$

■ **Question 1006.** □

Find the distance of the point $(2,3,1)$ *from the straight line* $\vec{\boldsymbol{r}}(t) = (1,1,2) + t\langle 5,0,1\rangle$.

■ **Question 1007.** □

What is the equation of the plane which contains the point P and the line $\mathscr{L}$?

A.4 Distance between two straight lines

Suppose the two straight lines $\mathscr{L}_1$ and $\mathscr{L}_2$ are given by

$$\vec{r}_1(t) = \vec{\mathbf{P}} + t\vec{u} \quad \text{and} \quad \vec{r}_2(t) = \vec{\mathbf{Q}} + t\vec{v}$$

i.e. the straight lines pass through P (and Q respectively) and is parallel to $\vec{u}$ (and $\vec{v}$ respectively).

■ **Question 1008.** □

Draw a picture and explain using geometry why the distance between the two straight lines is given by

$$d(\mathscr{L}_1, \mathscr{L}_2) = \frac{|\overrightarrow{\mathbf{PQ}} \cdot (\vec{u} \times \vec{v})|}{\|\vec{u} \times \vec{v}\|}$$

■ **Question 1009.** □

Find the distance between the lines $\vec{r}_1(t) = (2,1,4) + t\langle -1,1,0\rangle$ and $\vec{r}_2(t) = (-1,0,2) + t\langle 5,1,2\rangle$.

A.5 Distance between two planes

Before deriving the formula, observe that the distance between two planes is non-zero iff the two planes are parallel to each other, in which case they have the same normal vector $\vec{n} = \langle a,b,c\rangle$. Suppose the two planes Σ_1 and Σ_2 are given by

$$ax + by + cz = d \quad \text{and} \quad ax + by + cz = e$$

■ **Question 1010.** □

Show that the distance formula is given by

$$d(\Sigma_1, \Sigma_2) = \frac{|d-e|}{\|\vec{n}\|}$$

■ **Question 1011.** □

Find the distance between the planes $5x + 4y + 3z = 8$ and $5x + 4y + 3z = 1$.

■ **Question 1012.** □

Find the distance between the planes $x + 3y - 2z = 2$ and $5x + 15y - 10z = 30$.

§B Conic Sections and Quadric Surfaces

B.1 Conic Sections

A **conic section** (or simply conic) is a curve obtained as the intersection of the surface of a cone with a plane. The three types of conic sections are the **hyperbola**, the **parabola**, and the **ellipse**. The circle is type of ellipse, and is sometimes considered to be a fourth type of conic section.

A cone has two identically shaped parts called **nappes**. One nappe is what most people mean by ‘‘cone", and has the shape of a dunce hat. It can be thought of as the surface of revolution of a straight line around an axis.

- If the intersecting plane is parallel to the axis of revolution of the cone, then the conic section is a hyperbola.
- If the plane is parallel to the generating line, the conic section is a parabola.
- If the plane is perpendicular to the axis of revolution, the conic section is a circle.
- If the plane intersects one nappe at an angle to the axis (other than 90°), then the conic section is an ellipse.

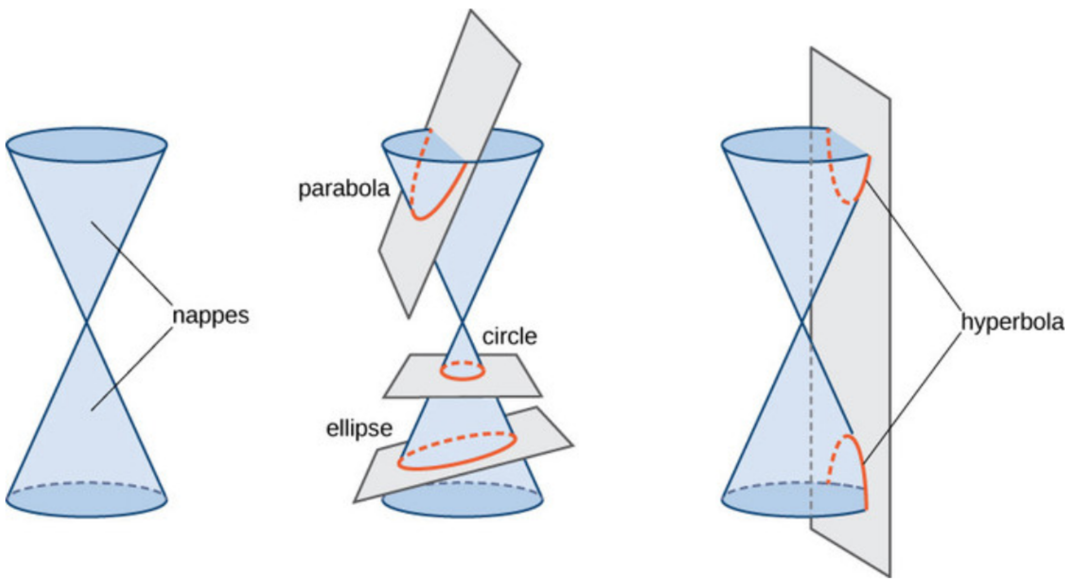


Figure 1: Conic Sections

Observe that when the intersecting plane is parallel to the axis of revolution and passes through the vertex of the cone, the conic section becomes a pair of straight lines (also known as a degenerate hyperbola).

Standard Equations in Cartesian Coordinates

The **Major Axis** is the chord between the two vertices: the longest chord of an ellipse, the shortest chord between the branches of a hyperbola. The **Minor Axis** is the shortest chord of an ellipse.

- In each of the above cases the center of the conic is at the origin. If the curve is translated h units horizontally and k units vertically, its new equation is obtained by replacing x with $(x-h)$ and y with $(y-k)$.

Conic Type	Standard Equation	Major Axis	Minor Axis
Circle	$x^2+y^2=r^2, \quad r\geq 0$	$2r$	$2r$
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad a,b>0$	$2a$	$2b$
Parabola	$y^2=4ax$	N/A	N/A
	$x^2=4ay$	N/A	N/A
Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \quad a,b>0$	$2a$	N/A
	$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1, \quad a,b>0$	$2b$	N/A
Pair of Straight Lines	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0, \quad a,b>0$	N/A	N/A
	$\iff y=\pm\frac{b}{a}x$	N/A	N/A

■ Question 1013.
☐

Use your precalculus memory or your favorite computer graphics software (e.g. DESMOS*) to draw a picture of each of the above conic sections. Clearly denote the center, radius, major axis, minor axis etc. and specify their lengths in terms of a,b etc. as applicable.*

■ Question 1014.
☐

For each of the following curves, find out what kind of conic section it is. See if you can answer without using a computer first.

(a) $(x-3)^2+(y-4)^2=y^2$

(b) $(x-3)^2+(y-4)^2=2y^2$

(c) $(x-3)^2+(y-4)^2=\frac{y^2}{2}$

(d) $(x-3)^2+(y-4)^2=1$

■ Question 1015.
☐

Find the range of value(s) of p for which the curve $\dfrac{x^2}{9-p}+\dfrac{y^2}{p-3}=1$ *looks like a*

(a) *Circle.*

(b) *Ellipse.*

(c) *Hyperbola.*

B.2 Quadric Surface

Equations of surfaces in three dimension are of the form $f(x,y,z)=c$. One special case of interest is when $f(x,y,z)$ is a polynomial which is not linear and quadratic at most, containing terms involving x,y,z,x^2,y^2, and z^2 only. This kind of surface is called a **Quadric Surface**. Quadric surfaces are often used as example surfaces since they are relatively simple. There are nine different basic quadric surfaces listed below. A catalog of the equations and pictures of the quadric surfaces is available on section 2.6 of your openstax textbook.

- **Cylinders:** A cylinder basically has no control over one of the variables. Take some sort of a curve in the plane, and draw a family of parallel lines so that each of the lines intersects the curve in a point.

For example, a cylinder over the line $(0,t,t)$ would be all points of the form (s,t,t) for any values of s and t. This is a plane (a plane is a cylinder over a line!), and has the equation $y = z$.

The most common picture you think of when you hear the term cylinder, is that of a cylinder over a circle. Some basic variations are

- **elliptical cylinder:** A cylinder over an ellipse
- **parabolic cylinder:** A cylinder over a parabola
- **hyperbolic cylinder:** A cylinder over a hyperbola

- **Ellipsoid**, the three-dimensional analogue of the ellipse. A sphere is an uniform ellipsoid.
- **Elliptic paraboloid**, a sort of cup or a bowl
- **Hyperbolic paraboloid**, looks like a horse-saddle or a pringle
- **Cone**, take a straight line intersecting the $z-$axis and consider its surface of revolution around the $z-$axis
- **Hyperboloids:** In three dimensions there are two different analogs of hyperbolas. The word "sheet" is used in an antique, specialized sense with surfaces: it means one connected "piece" of a surface. So a hyperboloid with one sheet is a surface with one (connected) piece, and a hyperboloid with two sheets is a surface with two (connected) pieces.
  - **hyperboloid of one sheet**, obtained by revolving a hyperbola around its minor axis. The surface is connected but there is a hole in it.
  - **hyperboloid of two sheet**, obtained by revolving a hyperbola around its major axis. This surface has two pieces.

Warning: In each case, note the direction of the axes relative to the surfaces and how the corresponding variables show up in the equation. For example, $z = 2y^2 - x^2$ is a hyperbolic paraboloid that goes downward in the $x-$axis direction and upwards in $y-$axis direction. The equation $y = 2x^2 - z^2$ is also a hyperbolic paraboloid that goes downward in the $z-$axis direction and upwards in $x-$axis direction. Similarly, $y = x^2 + z^2$ is an elliptical paraboloid that opens in the $y-$axis direction.

Question 1016.

For each of the following quadric surfaces, use the catalog to pick the term from the list above which seems to most accurately describe the surface.

Then describe what kind of conic sections are obtained by taking its cross-sections parallel to the YZ−plane, XZ−plane and XY−plane. These are called ***traces****.*

(a) $\frac{x^2}{9} - \frac{y^2}{16} = z$.

- *Intersection with $x = k$:*
- *Intersection with $y = k$:*
- *Intersection with $z = k$:*

(b) $\frac{x^2}{4} + \frac{y^2}{25} + \frac{z^2}{9} = 1$.

- *Intersection with $x = k$:*
- *Intersection with $y = k$:*
- *Intersection with $z = k$:*

(c) $\frac{x^2}{4} + \frac{y^2}{9} = \frac{z}{2}$.

- *Intersection with* $x = k$*:*
- *Intersection with* $y = k$*:*
- *Intersection with* $z = k$*:*

(d) $\frac{z^2}{4} - x^2 - \frac{y^2}{4} = 1$.

- *Intersection with* $x = k$*:*
- *Intersection with* $y = k$*:*
- *Intersection with* $z = k$*:*

(e) $x^2 + \frac{y^2}{9} = \frac{z^2}{16}$.

- *Intersection with* $x = k$*:*
- *Intersection with* $y = k$*:*
- *Intersection with* $z = k$*:*

(f) $\frac{x^2}{9} + y^2 - \frac{z^2}{16} = 1$.

- *Intersection with* $x = k$*:*
- *Intersection with* $y = k$*:*
- *Intersection with* $z = k$*:*

Now try identifying the following surfaces without the help of the catalog to see if you remember the names. Try to do this **without** graphing them first.

■ Question 1017. □

Identify the following surfaces.

(a) $9y^2 + 4z^2 = 36$

(b) $y^2 + 2y + z^2 = x^2$

(c) $4x^2 - y^2 + z^2 + 9 = 0$

■ Question 1018. □

(A non-basic Quadric Surface) *Google* `Geogebra 3D calculator`. *Use the website to plot the surface*

$$x^2 - 17z^2 - 2y^2 - 2xz - 12yz - 1 = 0$$

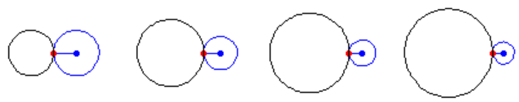
Which of the nine basic quadric surface does this resemble most closely? Can you explain how the equation of the surface might tell us what kind of surface it is, without using a graphing software?

[HINT: Complete the squares.]

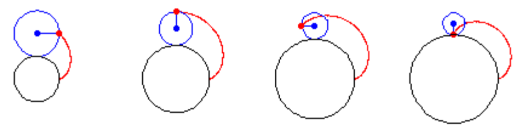
§C Epicycloids and the Rotary Engine

C.1 The Epicycloid

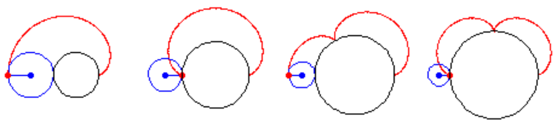
Consider a (black) circle of radius R with its center at the origin O. A (blue) circle of radius r rolls around the **outside** of the circle of radius R. See figure 2 for diagrams of different values of r. A (red) point P is located on the circumference of the rolling circle. The path traced out by P is called an **epicycloid**.



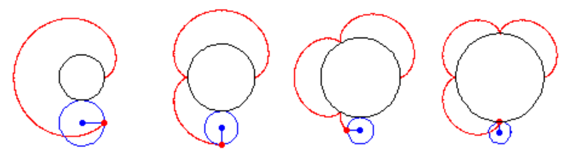
(a) $t = 0$



(b) $t = \pi/2$



(c) $t = \pi$



(d) $t = 3\pi/2$

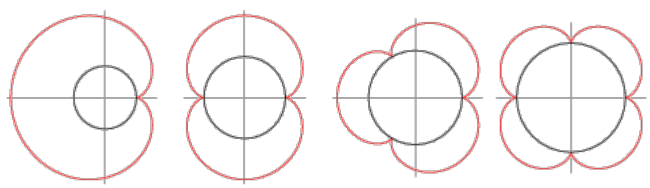


Figure 2

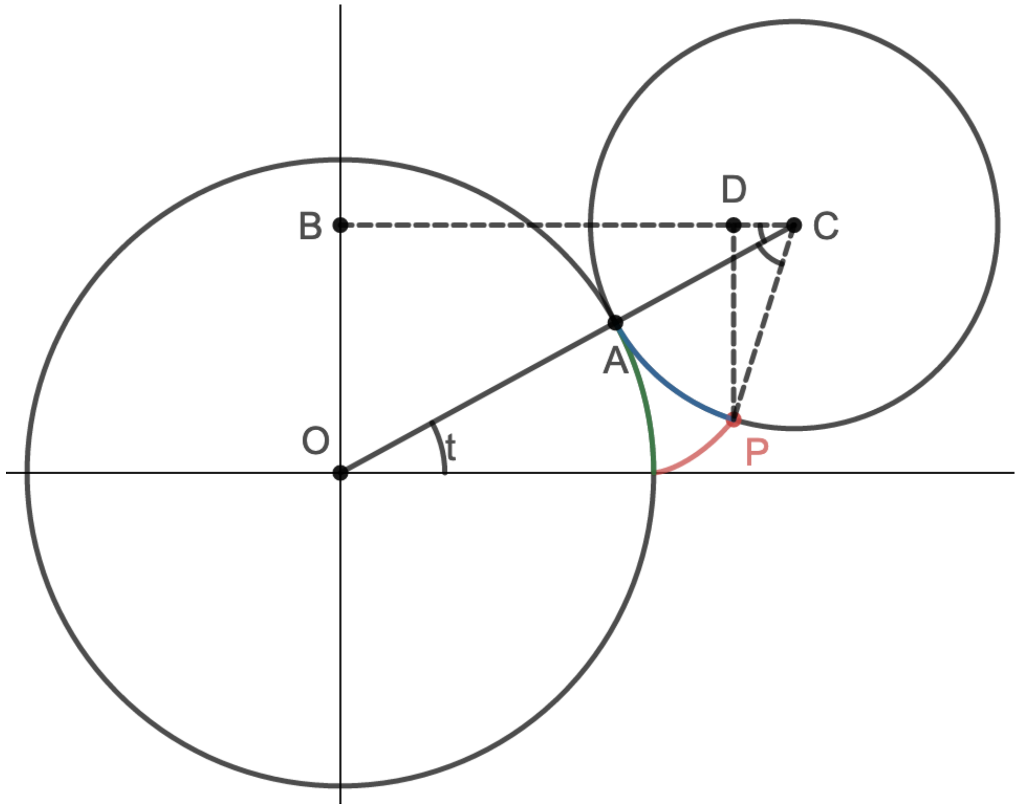
Assume that initially at time $t = 0$, the rolling circle sits to the right of the fixed circle and the point P is located at $(R, 0)$. After the rolling circle has moved a bit, draw a line from the center O of the large circle to the point of contact with the rolling circle and let t be the angle the line makes with the positive X−axis. If the location of P at this moment is given by $(x(t), y(t))$ (see figure 1019), then the parametric equation of the **epicycloid** is given by

$$x(t) = (R + r)\cos t - r\cos\frac{(R + r)t}{r}$$

$$y(t) = (R + r)\sin t - r\sin\frac{(R + r)t}{r}$$

Warning: If the initial location of P and the rolling circle is chosen differently, you will get the same shape with a different orientation and the form of the parametric equations will change slightly. For example, if the sin and cos functions are interchanged ($\sin \leftrightarrow \cos$), we get a vertically oriented epicycloid. If we replace t with $t + \varphi$ we get an epicycloid that has been rotated by angle φ.

Question 1019. □



Use the diagram above to analyze the geometry of the epicycloid. Write the detailed steps for the derivation of the parametric form of the epicycloid.

*[**Hints:** The length of the blue arc and the green arc are equal, why? Use this to derive the angle $\angle$PCA in terms of t. The x–coordinate of P is given by BD which is equal to BC – CD. The y–coordinate is equal to OB – PD.]*

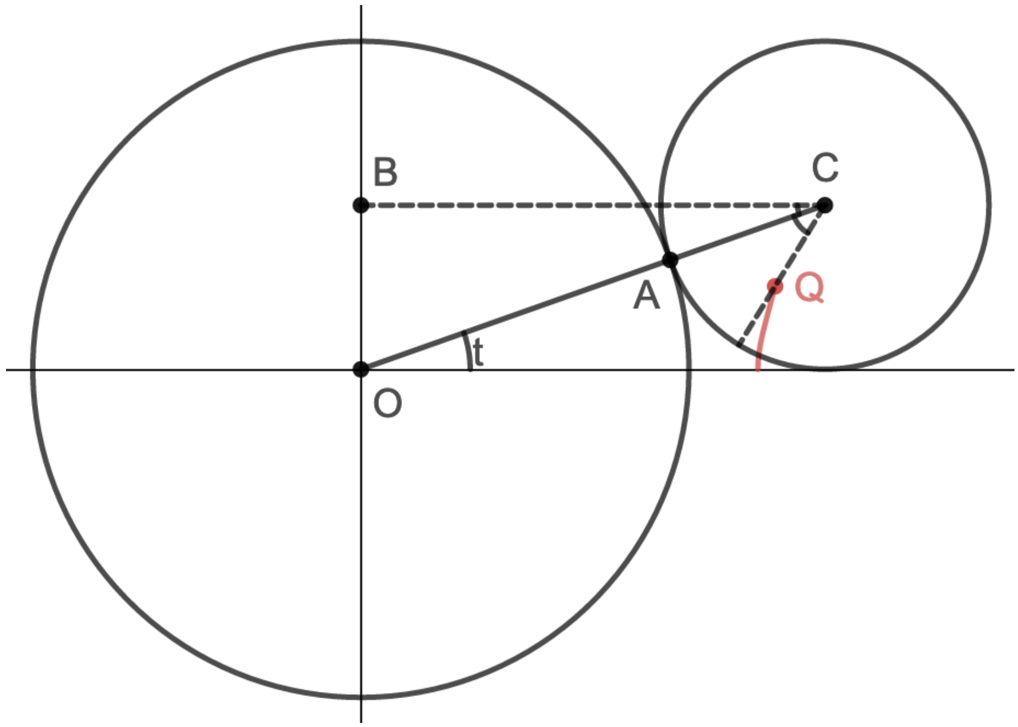
Question 1020.

In figure 2, you have picture of 4 epicycloids. You are told that each of the picture corresponds to a different value of $\frac{R}{r}$. Can you identify the values?

C.2 The Wankel Rotary Engine

We are especially interested in a special case from among the epicycloids as it pertains to the revolutionary rotary engine. First let's slightly modify the starting location of the point we are tracing. Instead of being situated on the circumference of the rolling circle, suppose we trace the point Q located at a distance h from the center of the rolling circle. The curve traced out by Q is called an **epitrochoid**. Epicycloids are epitrochoids with $h = r$.

Question 1021.



Use this figure to show that the parametric equation of the **epitrochoid** *is given by*

$$x(t) = (R + r)\cos t - h\cos\frac{(R + r)t}{r}$$

$$y(t) = (R + r)\sin t - h\sin\frac{(R + r)t}{r}$$

where $CQ = h$.

Now that you have the general form of an epitrochoid, consider the case where the fixed circle is twice the size of the rolling one ($R = 2r$) and $h = 4r/9$. This is the geometry of the **Wankel rotary engine**.

■ Question 1022. □

Give the simplified parametric equations for this special case.

What makes this geometry especially useful is that an equilateral triangle fits perfectly within the epitrochoids no matter how the triangle it is rotated. We will show that an **equilateral triangle** can be inscribed in this epitrochoid independent of the value of t. See figure 3 below for a demonstration.

We will derive this in the following steps. Let

$$Q_0 = (x(t), y(t)), \quad Q_1 = (x(t + 2\pi/3), y(t + 2\pi/3)), \quad Q_2 = (x(t - 2\pi/3), y(t - 2\pi/3))$$

■ Question 1023. □

(a) *Show that the epitrochoid makes one revolution for* $t \in [0, 2\pi]$.

(b) *Since the three points* Q_0, Q_1, Q_2 *are* $2\pi/3$ *apart they are evenly spaced in* $[0, 2\pi]$. *Thus, if their pairwise distances are equal then they must form an equilateral triangle. Show that*

$$d(Q_0, Q_1) = d(Q_1, Q_2) = d(Q_2, Q_0)$$

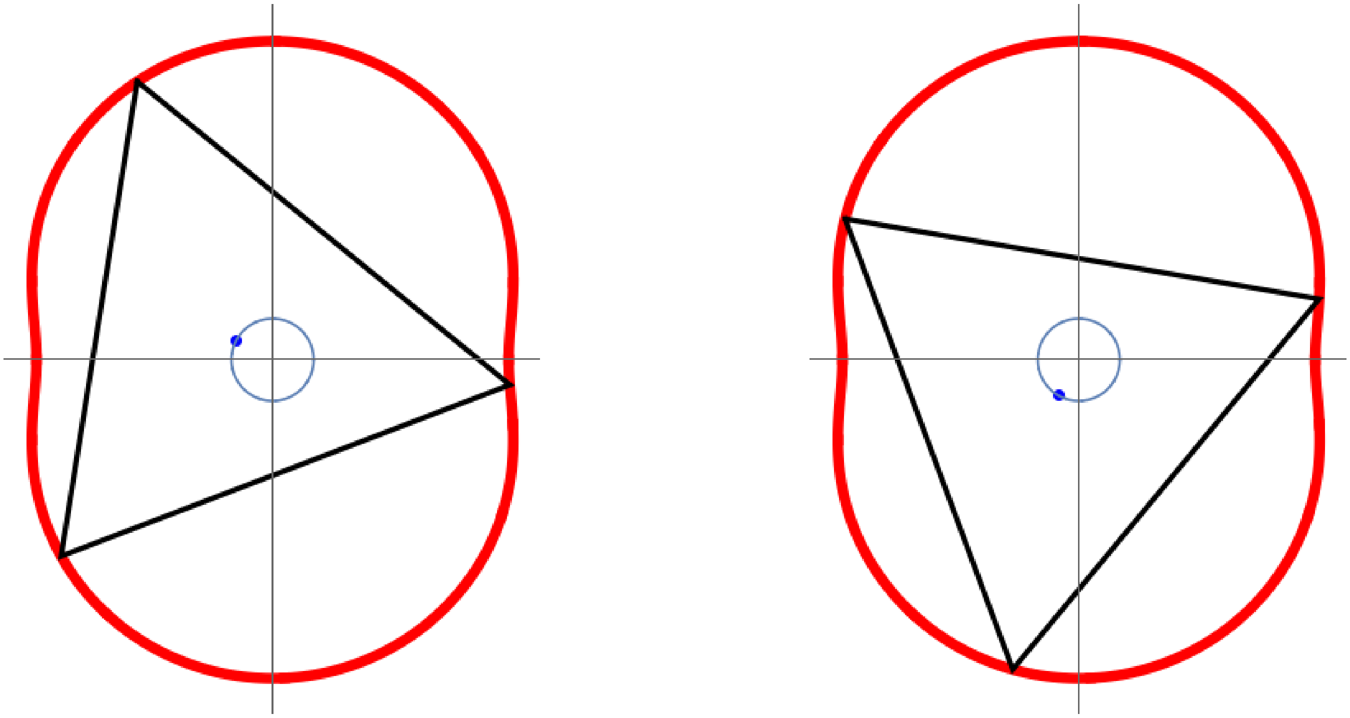


Figure 3

(c) *Furthermore, the centroid of the equilateral triangle always lies on a circle of radius h. Show that the average of Q_0, Q_1 and Q_2 is always at a distance h from the origin.*

This is the principle of the Wankel rotary engine. When the equilateral triangle rotates with its vertices on the epitrochoid, its centroid sweeps out a circle whose center is at the center of the epitrochoid. The space between the triangle and the epitrochoid is the firing chamber. A gif of the engine in action can be found here.

Question 1024. Bonus

How would the shape of the epicycloids (and epitrochoids) change if R/r is a rational number but not an integer? Try plotting the curve with $R = 3, r = 2$. Does it complete a full revolution for $t \in [0, 2\pi]$?

What happens if R/r is an irrational number. Try plotting the curve with $R = e, r = 1$. How long does it take to complete a full revolution?

Exercises

§D Checkpoint Problems

■ Question 2001. □

For the following problems, fill in the blank with either "certainly", "possibly", or "certainly not".

(a) Given three vectors $\vec{u}, \vec{v}$ and $\vec{w}$, if $\vec{u}+\vec{v}=\vec{u}$, then $\vec{w}+\vec{v}$ is ____________________ equal to $\vec{w}$.

(b) $\|\vec{u}-\vec{v}\|$ is ____ less than or equal to $\|\vec{u}+\vec{v}\|$.

■ Question 2002. □

*If $\vec{v}$ and $\vec{w}$ are vectors with the property that $\|\vec{v}+\vec{w}\|^2=\|\vec{v}\|^2+\|\vec{w}\|^2$, then which of the following **must** be true?*

(a) $\vec{v}=\vec{w}$.

(b) $\vec{v}=\vec{0}$.

(c) $\vec{v}$ is orthogonal to $\vec{w}$.

(d) $\vec{v}$ is parallel to $\vec{w}$.

■ Question 2003. □

True or False:

(a) If $\vec{u}\cdot\vec{v}=\vec{w}\cdot\vec{v}$, then $\vec{u}$ is equal to $\vec{w}$.

(b) If $\vec{u}$ and $\vec{v}$ are parallel, then $\|\vec{u}-\vec{v}\|=\|\vec{u}\|-\|\vec{v}\|$.

■ Question 2004. □

True or False:

(a) If $\vec{u}\times\vec{v}=\vec{u}\times\vec{w}$, then $\vec{v}=\vec{w}$.

(b) If $\vec{v}\times\vec{w}=\vec{0}$ and $\vec{v}\cdot\vec{w}=0$, then at least one of $\vec{v}$ and $\vec{w}$ must be $\vec{0}$.

(c) $(\vec{u}\times\vec{v})\times\vec{w}=\vec{u}\times(\vec{v}\times\vec{w})$.