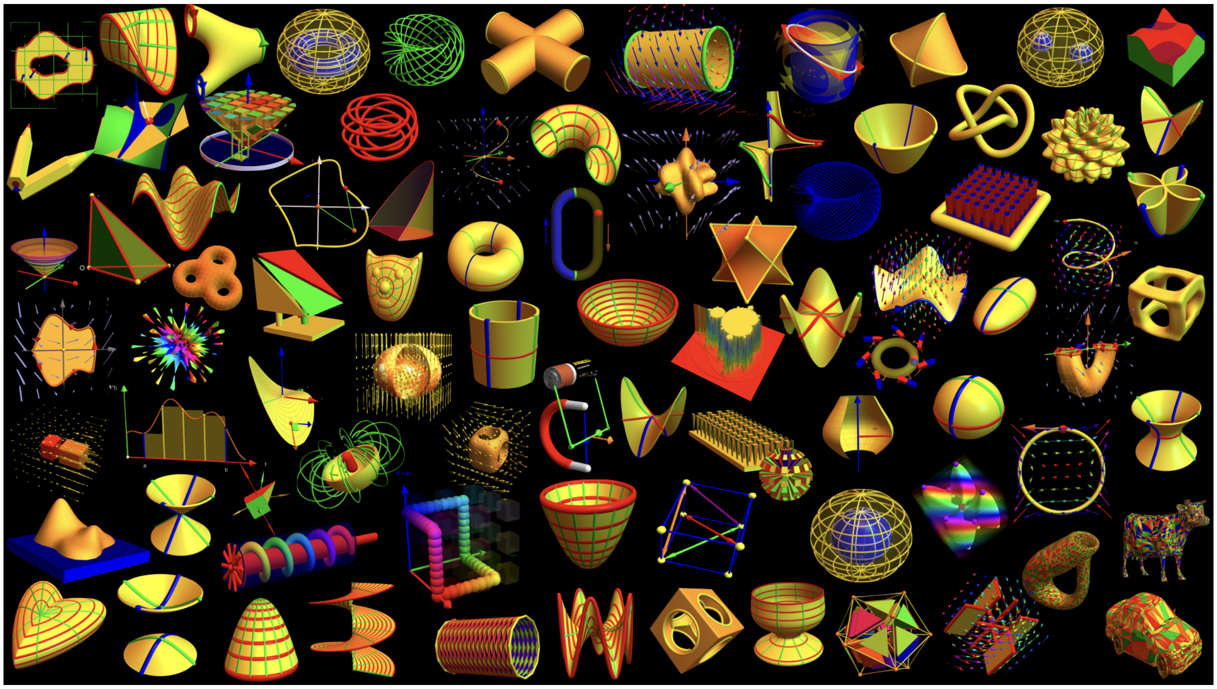

Multivariable Calculus

Math 212 Lecture notes

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THE COLLEGE OF WOOSTER

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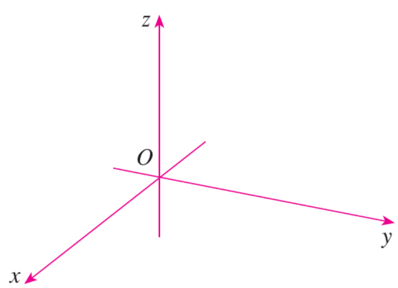
Contents

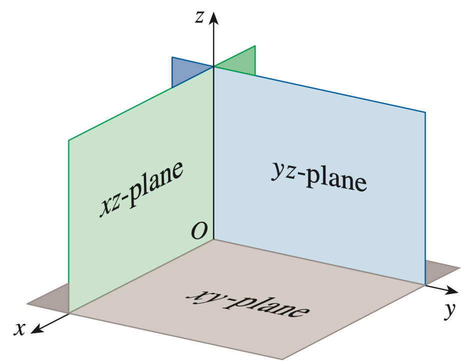
Chapter 1 | Three Dimensional Coordinate Geometry

To locate a point in a plane, we need two numbers. Any point in the plane can be represented as an ordered pair (a,b) of real numbers, where a is the x–coordinate and b is the y–coordinate. For this reason, a plane is called two-dimensional. To locate a point in space, three numbers are required. We represent any point in space by an ordered triple (a,b,c) of real numbers. For this reason, we will refer to the space as the 3–space or $\mathbb{R}^3$.

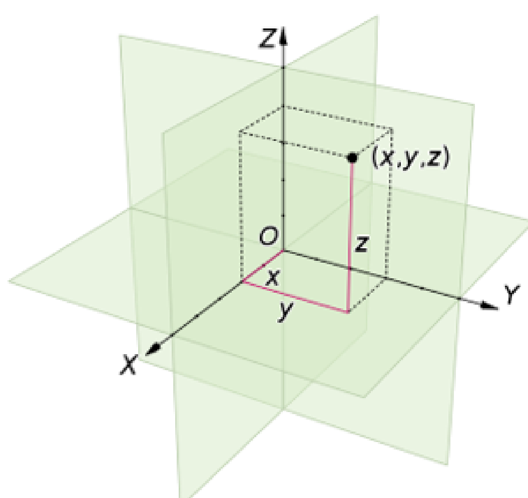
§1.1 Coordinate Axes and Points in 3-space

The three coordinate axes in 3-space are drawn using a right-hand-thumb rule as follows. It is important that you understand how to draw the axes in different orientations.



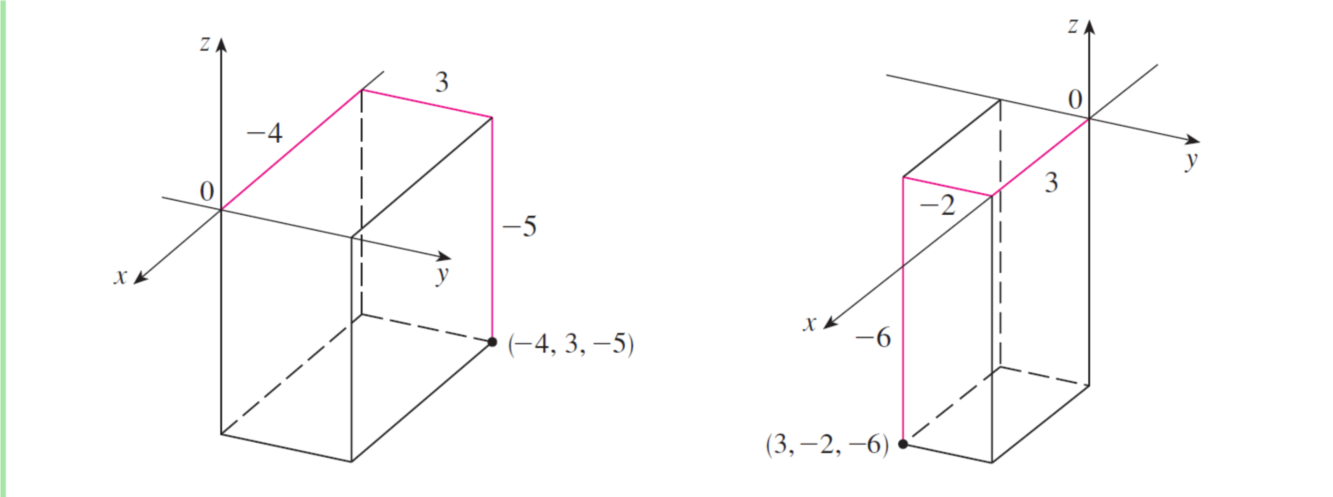


A point in three dimension has three coordinates (x,y,z) that denote respectively how far in/out, left/right, up/down a point is from the origin.



Example 1.1

Choose one of the corners of this classroom as the origin. Where are the points $P(-4,3,-5)$ and $Q(3,-2,-6)$?



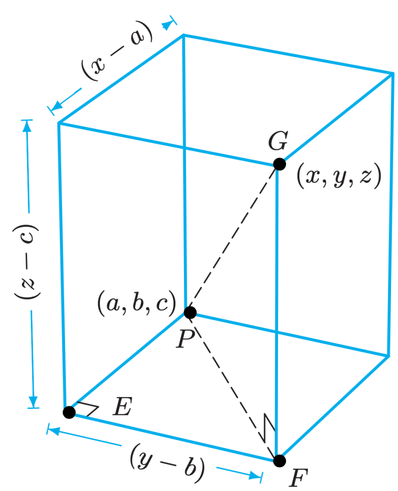
Question 1. □

You awaken one morning to find that you have been transferred onto a grid which is set up like a standard right-hand coordinate system. You are at the point $(-1,3,-3)$*, standing upright, and facing the xz-plane. You walk 2 units forward, turn left, and walk for another 2 units. What is your final position?*

§1.2 Distance between two points

The distance between two points (a,b,c) and (x,y,z) in space is given by

$$\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}.$$



Question 2. □

What is the distance from $(1,2,1)$ *to* $(-3,1,2)$*?*

Question 3. □

Which of the following points lies closest to the xy-plane?

(i) $(3,0,3)$ *(ii)* $(0,4,2)$ *(iii)* $(2,4,1)$ *(iv)* $(2,3,4)$

1.2.1 Equation of a Sphere

We can define a sphere to be collection (locus) of points that are equidistant to a fixed point called the **center**. Suppose the center is at (a,b,c) and the radius is r. Then the distance formula tells us that the required points (x,y,z) satisfy

$$\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}=r \iff (x-a)^2+(y-b)^2+(z-c)^2=r^2$$

§1.3 Sets of points described using coordinates

Question 4. □

Assume the origin is at the bottom, left, front corner of the classroom. Describe the set of points (x,y,x) with

- $x>0, y>0$ *and* $z\leq 0$
- $z=3$
- $x=0, y=0$

Question 5. □

Describe/draw the set of points with $0\leq x\leq 1, 0\leq y\leq 1-x$, and $z\leq 0$.

Question 6. □

Find any points where the sphere $(x-1)^2+(y+3)^2+(z-2)^2=4$ intersects the y -axis.

Question 7. □

Consider the set of points P given by the equation $z=5$. Also, let Q be the set of points that satisfy $x^2+y^2=1$.

(a) Sketch P,Q, and their intersection.

(b) Find an equation for the intersection.

Question 8. □

*How would you describe the points making up the **solid** cube with sides of length 2 and centered at the origin?*

Question 9. □

An equilateral triangle is standing vertically in 3-space with a vertex above the xy-plane and its two other vertices at $(7,0,0)$ and $(9,0,0)$. What are the coordinates of the third vertex?

Chapter 2 | Vectors in 3D

§2.1 Definition

Vectors in two or three dimensions are quantities that have both **magnitude** and **direction**. Quantities that are not vectors are called **scalars**.

Question 10. □

Which of the following are vectors?

(a) *The cost of a ticket to the zoo.*

(b) *The number of crocodiles in a pond.*

(c) *The volume of an elephant.*

(d) *The weight of a giraffe.*

(e) *The velocity of a cheetah.*

(f) *The speed of a gazelle.*

§2.2 Graphical representation of vectors

The vector $\vec{v} = \overrightarrow{\mathbf{PQ}}$ can be drawn as an arrow with "tail" at the point P and "tip" at the point Q. It represents a vector whose magnitude is equal to the length of PQ and its direction is from Q towards P. This vector is defined entirely by its direction and length, and can be moved around the space.

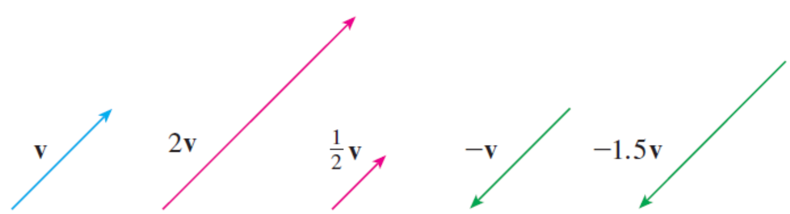
§2.3 Vector Arithmetic

2.3.1 Addition

To add two vectors $\vec{u}$ and $\vec{v}$, draw them one after the other, tip to tail. The vector from the base of the pair to the final tip is the sum of the vectors.

2.3.2 Scalar Multiplication

If λ is a scalar i.e. a real number, then $\lambda\vec{v}$ is a vector whose magnitude is λ times the magnitude of $\vec{v}$. It has the same direction as $\vec{v}$ if $\lambda > 0$ and the opposite direction if $\lambda < 0$.



Warning: If $\lambda = 0$, we get the **Zero Vector**, a vector whose magnitude is zero and is **omnidirectional**!

Definition 3.2

Two vectors are called *parallel* if they point in the same direction. That means $\vec{u}$ and $\vec{v}$ are parallel if there exists a *positive* constant λ such that $\vec{u} = \lambda\vec{v}$.

The vectors are called *antiparallel* if λ is negative.

2.3.3 Subtraction

$\vec{u} - \vec{v}$ is defined as $\vec{u} + (-1) \times \vec{v}$.

Question 11.

Suppose the three sides of a triangle $\triangle$ABC are denoted by vectors as $\vec{c} = \overrightarrow{\mathbf{AB}}$, $\vec{a} = \overrightarrow{\mathbf{BC}}$, and $\vec{b} = \overrightarrow{\mathbf{CA}}$. What is $\vec{a} + \vec{b} + \vec{c}$?

Question 12.

Consider $\triangle$ABC as above. Let D be the mid-point of BC and let $\vec{m} = \overrightarrow{\mathbf{AD}}$ be one of the medians of the triangle. Find $\vec{m}$ in terms of $\vec{a}, \vec{b}$ and $\vec{c}$.

Digression

It is in fact also possible to find the angle bisector vector in terms of the sides. However that requires the law of sines in a triangle. In case you are interested, here is the precise problem. Suppose D is a point on BC such that $\angle$BAD = $\angle$CAD. Find $\overrightarrow{\mathbf{AD}}$ in terms of $\vec{a}, \vec{b}$ and $\vec{c}$.

§2.4 Symbolic representation of vectors

A vector of magnitude 1 is called a **unit vector**. The unit vectors along X−, Y− and Z−axes are called $\hat{i}, \hat{j}$ and $\hat{k}$ respectively.

Consider the vector $\vec{v} = \overrightarrow{\mathbf{PQ}}$ where P = (x_1, y_1, z_1) and Q = (x_2, y_2, z_2). Then it can be also written as

$$\vec{v} = \underbrace{(x_2 - x_1)}_{\Delta x}\hat{i} + \underbrace{(y_2 - y_1)}_{\Delta y}\hat{j} + \underbrace{(z_2 - z_1)}_{\Delta z}\hat{k} = \langle \Delta x, \Delta y, \Delta z \rangle$$

The projection of a vector on to the axes are called its **components**. Thus the X−component of the vector $\vec{v}$ above is $(\Delta x)\hat{i}$ etc. Clearly a vector is the sum of its components.

Addition or scalar multiplication of a vector can be done component-wise.

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle \qquad c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

2.4.1 Position vector - components vs coordinates

If we write $\vec{v} = \langle a_1, a_2, a_3 \rangle$ or equivalently $\vec{v} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, we mean a vector with tail at $(0,0,0)$ and tip at $P(a_1, a_2, a_3)$. In this way every point (a_1, a_2, a_3) corresponds to the unique vector $\langle a_1, a_2, a_3 \rangle$ called its **position vector** (see fig. 2.1). The length of the components of the position vector are equal to the coordinates of the point.

§2.5 Magnitude

The magnitude of a vector $\vec{v} = \langle a, b, c \rangle$ is denoted by $\|\vec{v}\|$, pronounced "norm" or $\vec{v}$, and is equal to $\sqrt{a^2 + b^2 + c^2}$. This is an easy consequence of Pythagoras theorem!

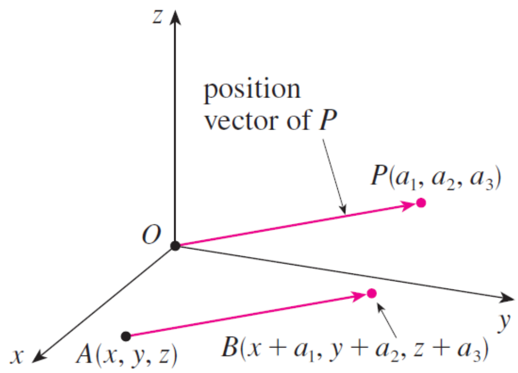


Figure 2.1

Question 13.

How can you use $\|\vec{v}\|$ to create a unit vector parallel to $\vec{v}$?

☐

§2.6 Calculating components in 2D

Suppose a vector $\vec{v}$ makes an angle θ with the positive X–axis. Then we can the magnitude $\|\vec{v}\|$ to express a vector $\vec{v}$ in terms of trigonometric functions as follows:

$$\vec{v} = (\Delta x)\hat{i} + (\Delta y)\hat{j} = (\|\vec{v}\|\cos\theta)\hat{i} + (\|\vec{v}\|\sin\theta)\hat{j} = \|\vec{v}\|\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right)$$

Question 14.

Suppose a three dimensional vector $\vec{v}$ makes angle α, β and γ with positive X–, Y– and Z–axis respectively. Then show that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

☐

§2.7 Word problems

Question 15.

Bert and Ernie are trying to drag a large box on the ground. Bert pulls the box toward the north with a force of 30 N, while Ernie pulls the box toward the east with a force of 40 N. What is the resultant force on the box?

☐

Question 16.

An airplane at altitude is flying NE with airspeed 700 km/hr, with wind from the West at 60 km/hr. Use vectors to determine the resulting direction and ground speed of the plane.

☐

Chapter 3 | Dot Product of Vectors

§3.1 Definition

3.1.1 Algebraic

The **dot product** or scalar product of two vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is defined to be the sum of the product of the components.

$$\vec{u} \cdot \vec{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle := u_1 v_1 + u_2 v_2 + u_3 v_3$$

Question 17. □

What is $\langle 1,2,3 \rangle \cdot \langle 4,-5,6 \rangle$? How about $\langle 1,2 \rangle \cdot \langle 3,4,5 \rangle$?

3.1.2 Geometric

If the angle between $\vec{u}$ and $\vec{v}$ is θ then

$$\vec{u} \cdot \vec{v} := \|\vec{u}\| \|\vec{v}\| \cos\theta$$

Question 18. □

Does it matter whether the angle θ is calculated from $\vec{u}$ to $\vec{v}$ or in the other order?

Question 19. □

What is $\vec{u} \cdot \vec{v}$ if

(a) $\vec{u} \perp \vec{v}$?

(b) $\vec{u} = \vec{v}$?

(c) $\vec{u} \parallel \vec{v}$?

Theorem 1.3

Two vectors $\vec{u}$ and $\vec{v}$ are orthogonal (i.e. perpendicular) if and only if $\vec{u} \cdot \vec{v} = 0$.

§3.2 Basic properties

From the definitions the following basic properties of the dot product are easy to prove. If $\vec{u}, \vec{v}$ and $\vec{w}$ are vectors of the same dimension and c is a scalar, then

(a) $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

(b) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

(c) $(c\vec{v}) \cdot \vec{w} = c(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (c\vec{w})$

§3.3 Angle between two vectors

We can use both definitions of dot product together to calculate angle between two given vectors.

■ **Question 20.** □

Find the angle between $\langle 1,2,1\rangle$ *and* $\langle 1,-1,1\rangle$.

■ **Question 21.** □

In a molecule of Methane (CH_4) *you have four hydrogen atoms bonded to a carbon atom. The four hydrogen atoms form the corner of a regular tetrahedron and the carbon atom lies in the center. What is the angle between any two of the* $\mathrm{C}-\mathrm{H}$ *bonds?*

Hint: *Think of the four hydrogen atoms lying on four corners of a cube.*

§3.4 Vector projections

Take a vector $\vec{u}$ and resolve it into two components, one along another given vector $\vec{v}$ and another perpendicular to $\vec{v}$. We call these two components $\vec{u}_{\parallel}$ and $\vec{u}_{\perp}$ respectively. Thus

$$\vec{u} = \vec{u}_{\parallel} + \vec{u}_{\perp}$$

The component $\vec{u}_{\parallel}$ is called the **projection** of $\vec{u}$ on to $\vec{v}$, and is denoted $\mathrm{Proj}_{\vec{v}}\,\vec{u}$.

Geometrically speaking, if we take a screen along $\vec{v}$ and shine a light perpendicular to it from above, the shadow cast by $\vec{u}$ would be $\mathrm{Proj}_{\vec{v}}\,\vec{u}$.

Theorem 4.4: Projection Formula

$$\mathrm{Proj}_{\vec{v}}\,\vec{u} = \vec{u}_{\parallel} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v}$$

■ **Question 22.** □

Prove above theorem.

HINT: Use the dot product formula for $\cos\theta$.

■ **Question 23.** □

The vertices of a triangle $\triangle ABC$ *are* $A=(4,3,2), B=(1,3,1)$, *and* $C=(-5,5,-2)$. *Let* D *be the foot of the perpendicular from* A *to the side* $\overline{BC}$. *Find the vector* $\overrightarrow{\mathbf{AD}}$.

HINT: Find $\overrightarrow{\mathbf{BD}}$ *first.*

Chapter 4 | Cross Product of Vectors

§4.1 Definition

4.1.1 Geometric

The cross product of two vectors $\vec{u}$ and $\vec{v}$ is the **vector** $\vec{u} \times \vec{v}$ whose

- magnitude is equal to $\|\vec{u}\|\|\vec{v}\|\sin\theta$, where θ is the angle from $\vec{u}$ to $\vec{v}$ and
- direction is perpendicular to the both $\vec{u}$ and $\vec{v}$, determined by the right-hand-thumb rule.

Question 24. □

Does it matter whether the angle θ is calculated from $\vec{u}$ to $\vec{v}$ or in the other order?

Question 25. □

What is $\hat{i} \times \hat{j}$? What is $\hat{i} \times \hat{i}$? What is $\hat{j} \times \hat{i}$?

4.1.2 Algebraic

If $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$, then we define $\vec{u} \times \vec{v}$ to be

$$\langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1\rangle.$$

There is a handy way of remembering this definition: the cross product $\vec{u} \times \vec{v}$ is equal to the determinant

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{k}$$

Warning: The cross product is only defined for three-dimensional vectors.

§4.2 Basic Properties

From the definitions the following basic properties of the cross product are easy to prove. If $\vec{u}, \vec{v}$ and $\vec{w}$ are vectors of the same dimension and c is a scalar, then

(a) $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$

(b) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

(c) $(c\vec{v}) \times \vec{w} = c(\vec{v} \times \vec{w}) = \vec{v} \times (c\vec{w})$

Theorem 2.5

Two vectors $\vec{u}$ and $\vec{v}$ are parallel or antiparallel if and only if $\vec{u} \times \vec{v} = 0$.

§4.3 Cross product as area

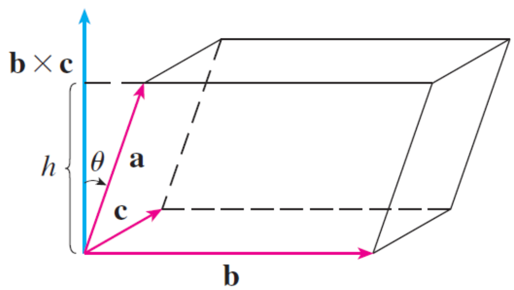
From the geometric definition, it is easy to see that $\|\vec{u} \times \vec{v}\|$ is equal to the area of the parallelogram whose two adjacent sides are given by $\vec{u}$ and $\vec{v}$.

Question 26. □

Show this. Use the fact that the area of the parallelogram is twice the area of the triangle whose two sides are given by $\vec{u}$ and $\vec{v}$. Then use the area formula for a triangle.

4.3.1 Volume of a parallelepiped - Coplanarity

Extending this geometric idea, we can show that the volume of a parallelepiped whose three adjacent sides are given by $\vec{a}, \vec{b}$, and $\vec{c}$ is equal to $\left|\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)\right|$.



Consequently, we have the following theorem

Theorem 3.6

Three vectors $\vec{a}, \vec{b}$, and $\vec{c}$ are coplanar iff $\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = 0$.

Digression

$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)$ is called the **scalar triple product** or **box product** of the three vectors, as also denoted by $[\vec{a}, \vec{b}, \vec{c}]$. An interesting observation here is that since the box product relates to the volume of the parallelepiped, we get

$$[\vec{a}, \vec{b}, \vec{c}] = [\vec{b}, \vec{c}, \vec{a}] = [\vec{c}, \vec{a}, \vec{b}]$$

So if our goal is to only calculate the volume, it doesn't matter which vectors we choose as $\vec{a}, \vec{b}$ and $\vec{c}$ as long they are the three adjacent sides.

Question 27. $\boxed{3}$

Suppose λ and μ are real numbers such that

- *the three vectors*

$$\vec{u} = 2\hat{i} + 3\hat{j} + \hat{k}, \qquad \vec{v} = \hat{i} + \lambda\hat{j} + \mu\hat{k}, \qquad \vec{w} = 7\hat{i} + 3\hat{j} + 2\hat{k}$$

  are coplanar, and

- *The vector $\vec{v}$ has magnitude $\sqrt{2}$.*

Find all possible values of λ and μ.

Chapter 5 | Lines and Planes

§5.1 Lines in space

The equation of a line through a point (x_0, y_0, z_0) and parallel to the vector $\vec{u} = \langle a, b, c\rangle$ can be expressed in many ways:

- as parametric scalar equations:

$$x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc;$$

- as a parametric vector equation:

$$\vec{r}(t) = \vec{r}_0 + t\vec{u}, \quad \text{where } \vec{r} = \langle x, y, z\rangle \text{ and } \vec{r}_0 = \langle x_0, y_0, z_0\rangle$$

- or by symmetric equations:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Warning: Observe that the symmetric equation form doesn't make sense in case one of a, b or c is zero.

Question 28. □

Let $\mathscr{L}$ be the line which passes through the points $(1, -2, 3)$ and $(4, -4, 6)$. Find its equation in all three forms.

§5.2 Planes in space

The equation of a plane through the point (x_0, y_0, z_0) and perpendicular (or normal or orthogonal) to the vector $\vec{n} = \langle a, b, c\rangle$ also has many (equivalent) equations:

- as a vector equation:

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \quad \text{or} \quad \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

  where again $\mathbf{r} = \langle x, y, z\rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$; or equivalently

- as a scalar equation:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \text{or} \quad ax + by + cz = d$$

  where $d = ax_0 + by_0 + cz_0$ is a constant.

Note: Two planes $\mathrm{A}x + \mathrm{B}y + \mathrm{C}z = \mathrm{D}$ and $ax + by + cz = d$ are parallel iff there exists a constant λ such that $\langle \mathrm{A}, \mathrm{B}, \mathrm{C}\rangle = \lambda\langle a, b, c\rangle$. The planes are identical if they are parallel and $\mathrm{D} = \lambda d$.

Question 29. □

Find an equation describing the plane which goes through the point $(1, 3, 5)$ and is perpendicular to the vector $\langle 2, 1, -3\rangle$.

■ **Question 30.** □

Find an equation describing the plane which passes through the points $P(2,2,1), Q(3,1,0)$, *and* $R(0,-2,1)$.

§5.3 Types of problems

One of the fundamental topic in Multivariable Calculus is to learn how to find equations of straight lines and planes in three dimensions using ideas from vectors (dot and cross product).The following lists an incomplete but fairly diverse types of problems that you should be able to solve using these ideas.

5.3.1 Finding a Plane

You should know how to find equation of a plane from the following data.

(a) Plane through a given point and perpendicular to a given vector.

(b) Plane through a given point and parallel to a given plane.

(c) Plane containing a given line and parallel to a given plane.

(d) Plane passing through three given points.

(e) Plane with specified $x-$, $y-$ and $z-$intercepts.

(f) Plane through a given point and containing a given straight line.

(g) Plane through a given point and containing the line of intersection of two other given planes.

(h) Plane through a given point and perpendicular to two other given planes.

(i) Plane passing through two points and perpendicular to a given plane.

(j) Plane containing the line of intersection of two other given planes and perpendicular to a given plane.

5.3.2 Finding a Line

You should know how to find equation of a straight line from the following data.

(a) Line through two given points.

(b) Line through one given point and in the direction of a given vector.

(c) Line through one given point and parallel to a given straight line.

(d) Line of intersection of two given planes.

(e) Line through one given point and perpendicular to a given plane.

(f) Line through a given point, that is perpendicular to a given straight line and intersects this second line.

(g) Line through a given point, that is parallel to (i.e. lies in) a given plane and perpendicular to a given straight line.

§5.4 Practice problems

■ **Question 31.** □

Below is a list of vectors and a list of properties. Match the two sets in such a way that each entry in left column matches a different entry in right column.

A. $\langle 3,-2,8\rangle$	*I. is parallel to the straight line* $\frac{x-1}{2}=y-3=z$
B. $\langle 4,2,2\rangle$	*II. is perpendicular to the plane* $z-2y-x=3$
C. $\langle 3,1,-1\rangle$	*III. is perpendicular to both* $\langle 2,3,0\rangle$ *and* $\langle -2,5,2\rangle$
D. $\langle 1,2,-1\rangle$	*IV. is parallel to the plane* $x-y+2z=3$

■ Question 32. □

(a) *Find parametric equations for the line through the points* $(6,1,1)$ *and* $(9,1,4)$*. Call this line* L_1.

(b) *Find parametric equations for the line through the points* $(-4,4,0)$ *and* $(-6,5,1)$*. Call this line* L_2.

(c) *Find parametric equations for the line through the points* $(6,-1,-5)$ *and* $(2,1,-3)$*. Call this line* L_3.

(d) *Verify that* L_2 *and* L_3 *are parallel. (Their direction vectors should be parallel.) Are they the same line? How could you tell?*

(e) *Do lines* L_1 *and* L_2 *intersect? If so, where?*

(f) *Find the intersection of* L_1 *with the plane given by the equation* $2x+y+3z=7$.

(g) *(3 points) Find the point on the plane* $2x+y+3z=7$ *which is closest to the origin.*

(h) *(3 points) Find the point on* L_2 *closest to the origin.*

■ Question 33. □

Find a vector parallel to the intersection of the two planes $2x-3y+5z=2$ *and* $4x+y-3z=7$.

Find the equation of the line of intersection.

■ Question 34. □

Find the distance of the point $P=(1,0,1)$ *from the plane* $x+y-z=1$.

■ Question 35. □

Let L_1 *be the line with parametric vector equation* $\vec{r}_1(t)=\langle 7,1,3\rangle+t\langle 1,0,-1\rangle$ *and* L_2 *be the line described parametrically by* $x=5, y=1+3t, z=t$*. How many planes are there that contain* L_2 *and are parallel to* L_1*? Find an equation describing one such plane.*

■ Question 36. □

Find an equation for the plane that contains the line in the XY-pane where $y=1$*, and the line in the XZ-pane where* $z=2$.

Chapter 6 | Functions of Two Variables

§6.1 Examples of functions of two variables

- Household lobster consumption is a function of income and the price of lobster.
- The density of cars along a highway is a function of position and time.
- The daily temperature across the United States is a function of latitude and longitude.
- Volume of a cylinder is a function of its radius and height.

§6.2 Representations of two-variable functions

6.2.1 Numerical

The body mass index (BMI) is a value that attempts to quantify a person's body fat based on their height h and weight w.

Height h (inches)	Weight w (lbs) 120	140	160	180	200
60	23.4	27.3	31.2	35.2	39.1
63	21.3	24.8	28.3	31.9	35.4
66	19.4	22.6	25.8	29.0	32.3
69	17.7	20.7	23.6	26.6	29.5
72	16.3	19.0	21.7	24.4	27.1
75	15.0	17.5	20.0	22.5	25.0

■ **Question 37.** □

What is the BMI of a person who is 72 inches tall and weighs 160 lb?

6.2.2 Algebraic

A solid cylinder with closed ends has radius r and height h. Its volume V is given by

$$V = f(r, h)$$

■ **Question 38.** □

Write a formula for f. What does $V(10, h)$ mean? What does $V(r, 10)$ mean?

■ **Question 39.** □

Can you give a formula for the function $f(r,h)$? What about the surface area $A = g(r,h)$?

6.2.3 Visual - Using Graphs

Before we give the visual description, let's answer the following question first:

■ Question 40.

☐

What is the domain of the function $f(x,y) = \dfrac{\sqrt{y^2 - x}}{\ln(x-2)}$?

The **graph** of a function of two variables, f, is the set of all points (x,y,z) such that $z = f(x,y)$ for all (x,y) in the domain of f. In general, the graph of a function of two variables is a surface in 3–space.

■ Question 41.

☐

Sketch the graph of the function $f(x,y) = 6 - 3x - 2y$.

■ Question 42.

☐

Sketch the graph of $g(x,y) = \sqrt{9 - x^2 - y^2}$.

§6.3 Analyzing Graphs using Cross-sections

One way to visualize surfaces that are three-dimensional graphs is to view pieces of them as two-dimensional graphs. If we intersect the graph of $z = f(x,y)$ with a plane (such as $x = k$ or $y = k$), we get a graph in a two-dimensional plane (the kind we're used to). This is called a **cross-section** (or a trace).

■ Question 43.

☐

Describe the cross-sections of the function $g(x,y) = x^2 - y^2$ *with* y *fixed and then with* x *fixed. Use these cross-sections to describe the shape of the graph of* g.

§6.4 Group Problems

■ Question 44.

☐

Here are six functions whose graphs we ought to be able to visualize in space:

(a) $z = f(x,y) = 6 - 3x - 2y$

(b) $z = f(x,y) = x^2 + y^2$

(c) $z = f(x,y) = x^2 - y^2$

(d) $z = f(x,y) = x^2 + y + 1$

(e) $z = f(x,y) = (x-y)^2$

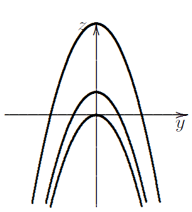
(f) $z = f(x,y) = \frac{1}{1+x^2+y^2}$

Here are cross-sections for two of the surfaces above. Your job is to:

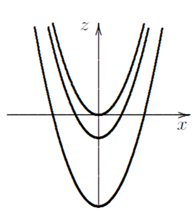
(a) *identify which graph these cross-sections belong with,*

(b) *graph cross-sections of the remaining graphs of surfaces, and*

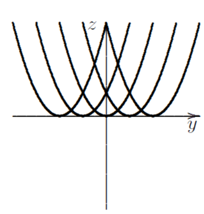
(c) *try to visualize the original surface.*



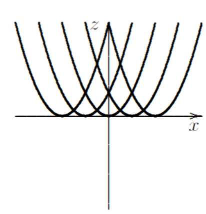
$x = k$



$y = k$



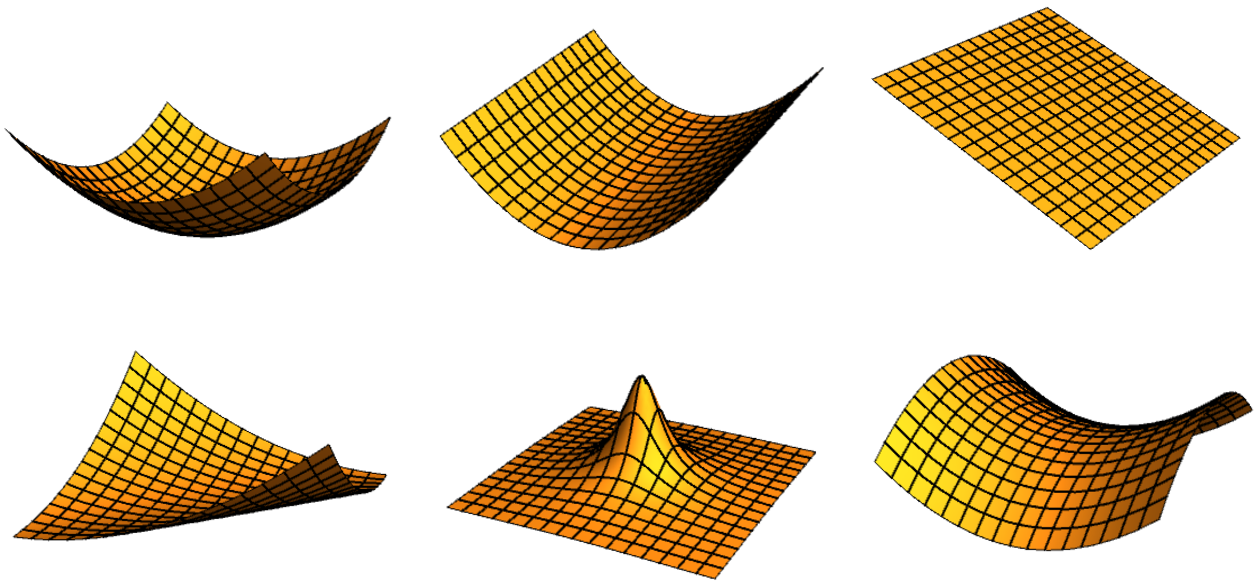
$x = k$



$y = k$

Question 45.

Now that you have a good idea of what each of these graphs look like, you should have no problem identifying which of the following (axes-less) graphs go with each equation for the previous page. Your reasoning should involve the cross-sections you drew as well.



Question 46.

3 Match each function with its graph.

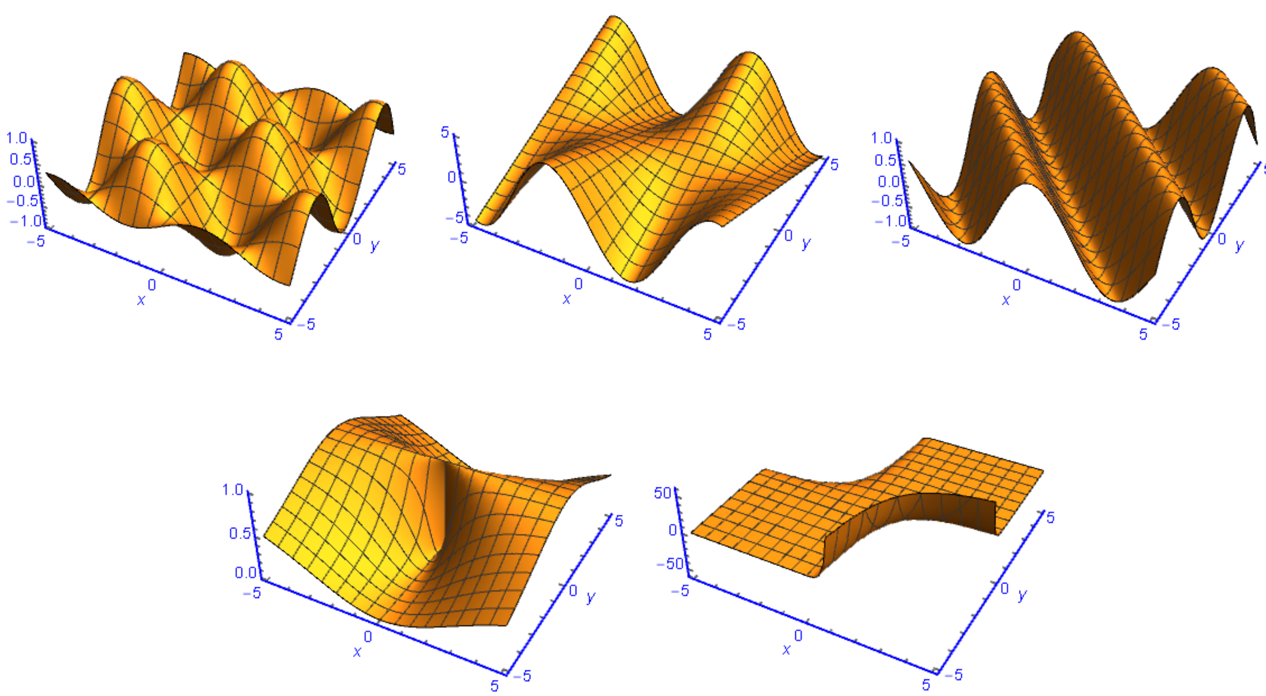
(a) $f(x,y) = \frac{x^2}{x^2+y^2}$

(b) $f(x,y) = y\sin x$

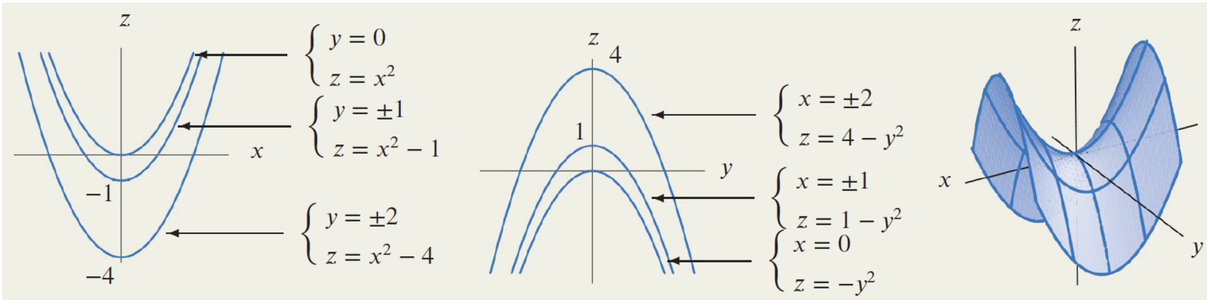
(c) $f(x,y) = \sin(x+y)$

(d) $f(x,y) = \sin x \cos y$

(e) $f(x,y) = xe^{-xy}$



Solution. Solution to question 43.



■

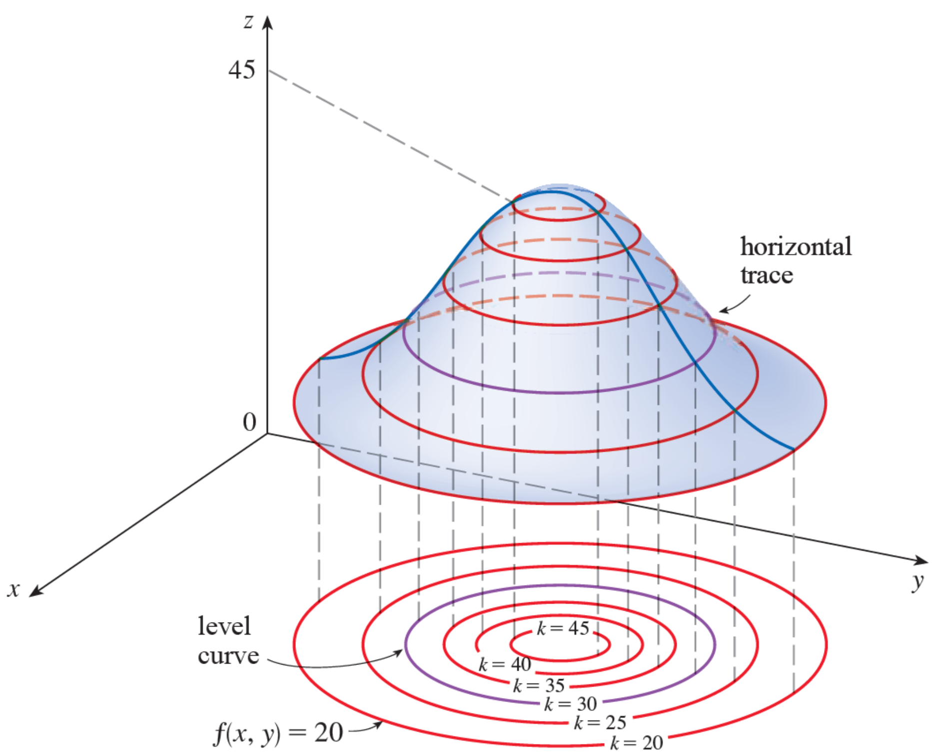
Chapter 7 | Contour Plots - Level Curves and Level Surfaces

§7.1 Level Curves

7.1.1 Definition

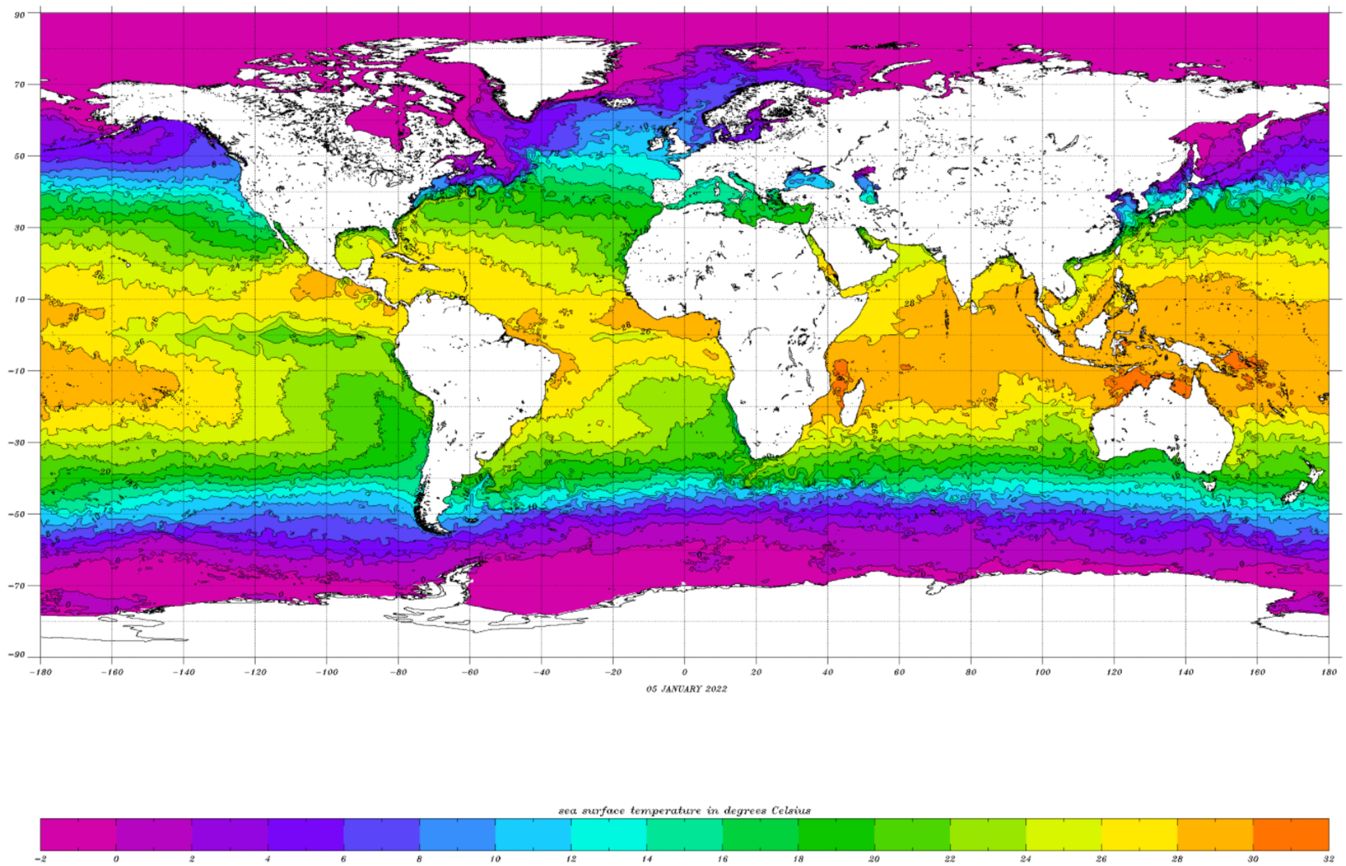
The **level curves** of a function f of two variables are the curves with equations $f(x,y) = c$, where c is a constant (in the range of f).

These are essentially the z–cross-sections of the graph of $f(x,y)$. The collection of all the level curves is called a **contour plot**.



7.1.2 Basic Facts

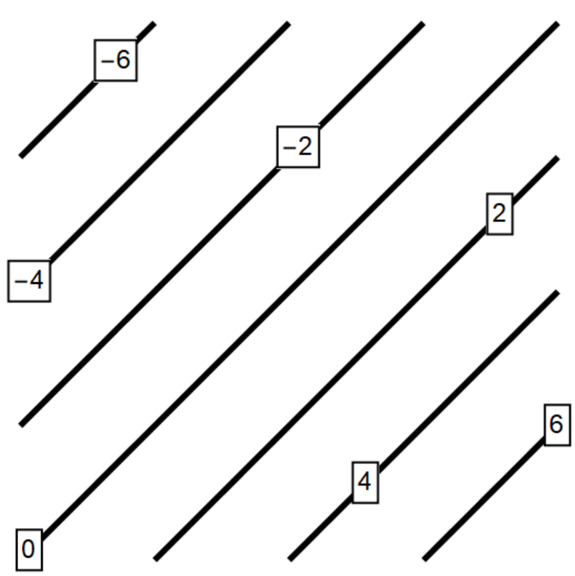
- The level curve for fixed output c is called the c -level of $f(x,y)$.
- unless otherwise indicated, contours are drawn at regular z–increments.
- We can build contour plots by hand following the same procedure we used for x and y cross-sections, but now fixing the z-values and plotting in the xy-plane.
- Two different level curves cannot cross.

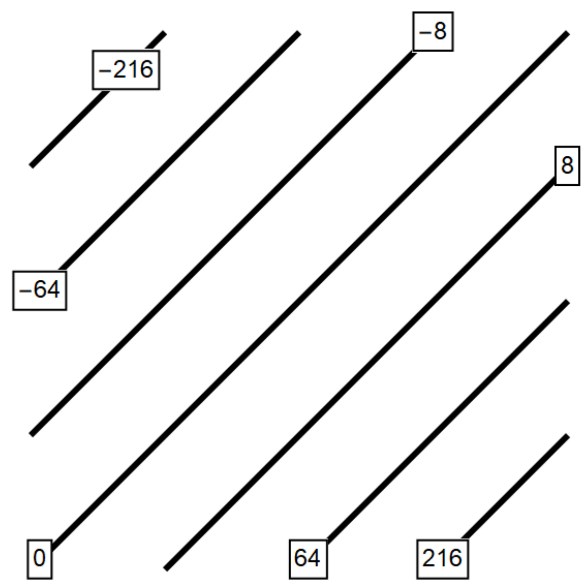


Sea Surface Temperature (SST) Contour Chart from NOAA

§7.2 Practice Problems

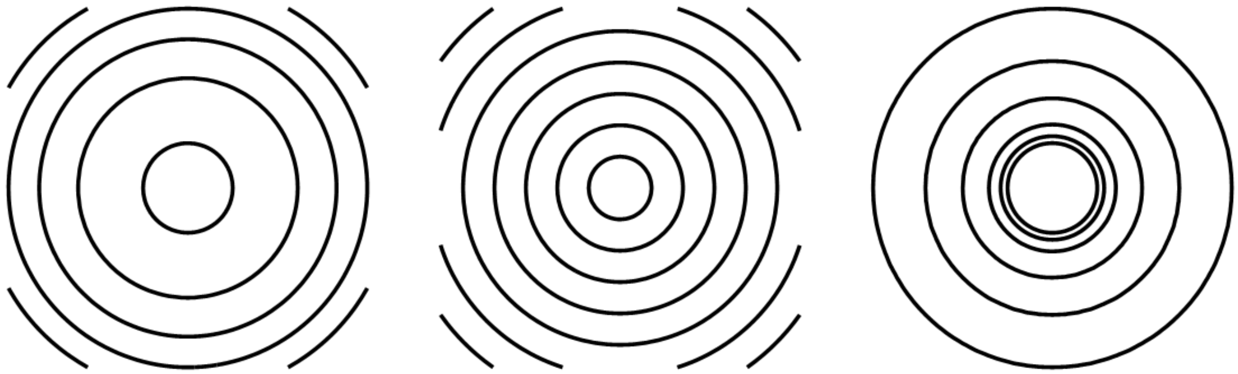
Consider two functions $f(x,y) = x - y$ and $g(x,y) = (x-y)^3$. Observe the difference in the contour plots for each of the graphs below.





■ **Question 47.** □

Assuming regular z–increments, describe what the corresponding graphs might look like.

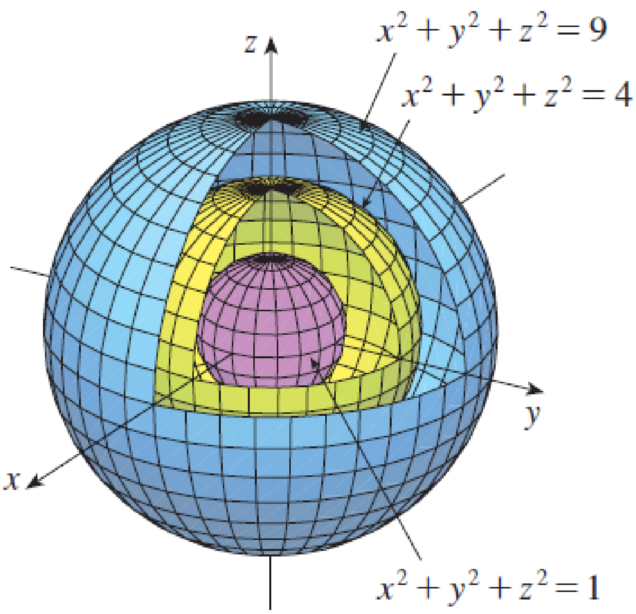


§7.3 Functions of three variables - Level surface

It's very difficult to visualize a function f of three variables by its graph, since that would lie in a four-dimensional space. However, we do gain some insight into f by examining its **level surfaces**, which are the surfaces with equations $f(x,y,z) = c$, where c is a constant. If the point (x,y,z) moves along a level surface, the value of $f(x,y,z)$ remains fixed.

7.3.1 Level surfaces of $f(x,y,z) = x^2 + y^2 + z^2$

The level surfaces are $x^2 + y^2 + z^2 = k$, where $k \geq 0$. These form a family of concentric spheres with radius $\sqrt{k}$. Thus, as (x,y,z) varies over any sphere with center O the value of $f(x,y,z)$ remains fixed.



■ **Question 48.** $\boxed{3}$

What do the level surfaces of $f(x,y,z) = x^2 + y^2 - z^2$ look like? Finish project 2 to find out.

Digression

Functions of any number of variables can be considered. A function of n variables is a rule that assigns a number $z = f(x_1, x_2, \ldots, x_n)$ to an n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers. We denote by $\mathbb{R}^n$ the set of all such n-tuples.

Recall that there is a one-to-one correspondence between points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ and their position vectors $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$. So we have three ways of looking at a function f defined on a subset of $\mathbb{R}^n$:

(a) As a function of n real variables $x_1, x_2, \ldots, x_n$

(b) As a function of a single point variable $(x_1, x_2, \ldots, x_n)$

(c) As a function of a single vector variable $\vec{\boldsymbol{x}} = \langle x_1, x_2, \ldots, x_n \rangle$

We will see that all three points of view are useful.

Chapter 8 | Linear Functions

§8.1 Definition

A function of two variables of the form

$$f(x,y) = mx + ny + d$$

where m, n and d are fixed constants is called a linear functions.

Recall that the equation $z = mx + ny + d$ represents a plane in three-dimensions. Thus clearly, the graph of a linear function looks like a plane. The constant m and n respectively represent the slope of the graph in $x-$direction and $y-$direction.

§8.2 Graphical Representation

What does the contour plot of a linear function look like? If we set $f(x,y) = c$, we can rewrite the equation as

$$y = -\frac{m}{n}x + \frac{c-d}{n}$$

which is a line with slope $-\frac{m}{n}$. So no matter what level c we choose, the lines remain parllel. Thus the contour plot of a linear function is a set of evenly-spaced parallel lines (assuming regular $c-$increments).

§8.3 Practice Problems

■ **Question 49.** □

Consider the function $z = f(x,y) = 8x - 4y + 2$.

(a) *Sketch the contour plot for the graph with z-increment value of* 1.

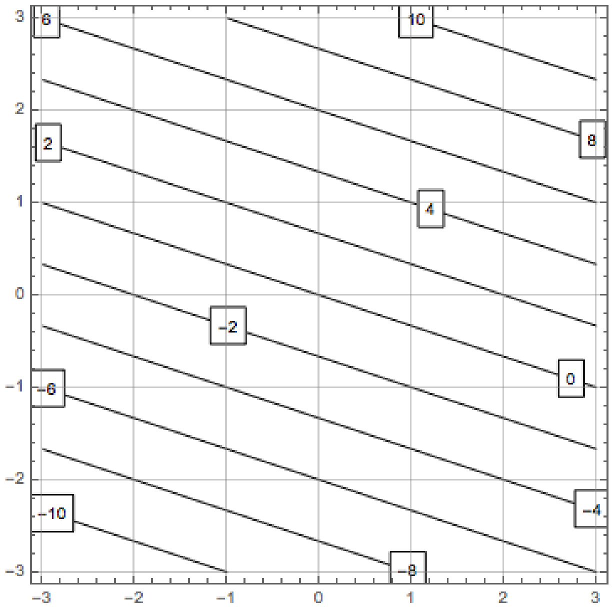
(b) (i) *Starting at any point* (x,y), *what is the slope of the surface in the* $x-$*direction?*

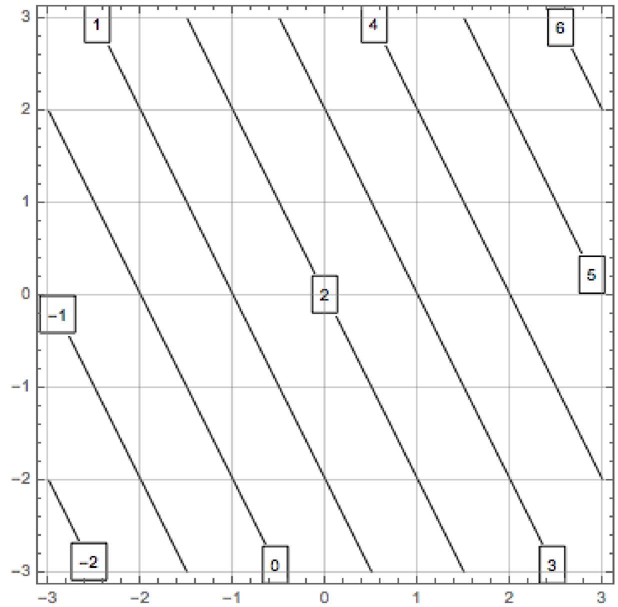
(ii) *What is the slope in* $y-$*direction?*

(iii) *What is the slope along the line* $x = y$?

■ **Question 50.** □

Find the linear functions whose contour plots are shown below.


6
10
2
8
4
-2
0
-6
-10
-4
-8


1
4
6
5
2
-1
-2
0
3

Chapter 9 | Partial Derivatives

§9.1 Motivation

Consider the contour plot for the function $f(x,y) = x^2 + y$.

(a) Sketch the cross-section of the graph with the plane $x = 4$.

(b) Compute the rate of change of z with respect to y as (x,y) moves towards increasing y-value, along the line $x = 4$.

(c) What happens to the rate of change of z with respect to x as you move from $(4,5)$ towards increasing x-value along the line $y = 5$.

§9.2 Definition

For a two-variable function $f(x,y)$ we define the **partial derivative** with respect to x by

$$f_x(a,b) = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

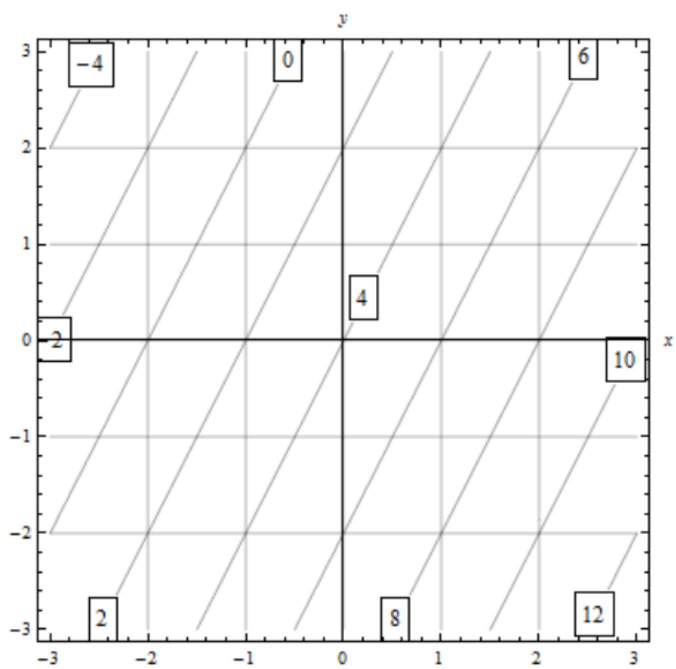
and similarly the **partial derivative** with respect to y by

$$f_y(a,b) = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

In other words, $f_x(a,b)$ is the derivative at a of the single variable function $x \mapsto f(x,b)$ with $y = b$ fixed.

Question 51. ☐

What are the values $f_x(0,0)$ and $f_y(0,0)$ for the linear function $f(x,y)$ whose contour plot looks like



§9.3 Partial derivative function

We can think of partial derivatives as functions of the base point.

- $f_x(x,y)$ is the function giving f's rate of change in the x-direction from any point (x,y).
- $f_y(x,y)$ is the function giving f's rate of change in the y-direction from any point (x,y).

Example 3.7

If $f(x,t)$ is the vertical height of a guitar string at time t and position x units from one end of the guitar,

- $f_x(x,t)$ measures the rate of change of string height as position changes, or the slope of the string "snapshot" at time t.
- $f_t(x,t)$ measures the rate of change of string height as time passes, or the velocity of the string at position x.

Question 52.

For each of the following functions, compute both first partial derivatives f_x and f_y (or f_t):

(a) $f(x,y) = e^x \cos y$

(b) $f(x,y) = x^3 - 3xy^2$

(c) $f(x,t) = e^{-(x+t)^2}$

(d) $f(x,t) = \sin(x-t) + \sin(x+t)$

□

Question 53.

Let $F(M,r) = \frac{GM}{r^2}$ denote the gravitational force experienced by a unit mass at distance r from the center of a planet with mass M. Calculate $\dfrac{\partial F}{\partial M}$ and $\dfrac{\partial F}{\partial r}$.

What does the partial derivative $\frac{\partial F}{\partial M}$ say about how the gravitational force changes when the mass of the planet increases? What does the partial derivative $\frac{\partial F}{\partial r}$ say about how the gravitational force changes when the distance from the planet increases?

□

§9.4 Higher order partials

We can compute higher order derivatives by simply repeating the process. For example,

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

Theorem 4.8: Clairout's Theorem

If f_{xy} and f_{yx} are both continuous, then $f_{xy} = f_{yx}$.

Question 54.

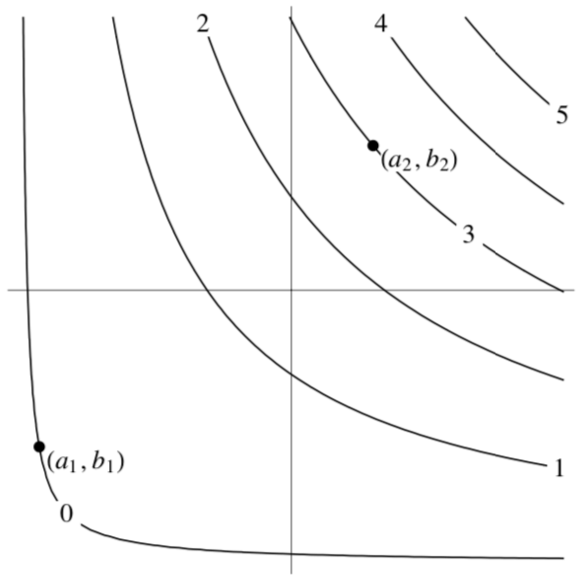
Use Clairaut's Theorem to compute the requested derivatives of the following functions:

(a) f_{xyxyxy} *if* $f(x,y) = x^2\cos(e^y + y^2)$

(b) f_{xxxyy} *if* $f(x,y) = x^3y^2 - \frac{y}{x+\ln(x)}$

Question 55.

Below is the contour plot of a function $f(x,y)$. The value of f on each level set is labeled. Assume that the other level curves are of similar shapes.



Based on the diagram, decide whether each of the statements should be true or false. (For which can you be totally sure, and for which would you need more information to be totally sure?)

(a) $f_x(a_1,b_1) \geq 0$

(b) $f_y(a_2,b_2) \geq 0$

(c) $f_x(a_1,b_1) \geq f_x(a_2,b_2)$

(d) $f_{xx}(a_2,b_2) \geq 0$

(e) $f_{xy}(a_2,b_2) \geq 0$

§9.5 Interpreting second partial derivatives

Question 56.

Go back to our example of a guitar string above. Let $f(x,t)$ be the vertical height of a guitar string at time t and position x units from one end of the guitar.

- *What does f_{xx} represent?*
- *What does f_{xx} represent?*
- *What do f_{xt} and f_{tx} represent?*

Chapter 10 | Directional Derivative and Gradient

Definition 0.9

The **directional derivative** of a function $f(x,y)$ in the direction of a unit vector $\vec{u}$ is the rate of change of the value of f in the direction of $\vec{u}$ and is denoted by $\mathrm{D}_{\vec{u}}f$.

Suppose we start at (a,b) on the XY−plane and move in the direction of the unit vector $\vec{u}$ in the XY−plane. We want to compare the slope of the graph of f above points along the path. To do this, we consider different contractions h of the unit vector $\vec{u} = u_1\hat{i} + u_2\hat{j} = \langle u_1, u_2\rangle$ and find the function value at the end of such a contraction $f(a+hu_1, b+hu_2)$.

The average rate of change of $f(x,y)$ between the two points (a,b) and $(a+hu_1, b+hu_2)$ is given by the difference quotient

$$\frac{\Delta f}{\text{displacement}} = \frac{f(a+hu_1,b+hu_2)-f(a,b)}{\|h\vec{u}\|} = \frac{f(a+hu_1,b+hu_2)-f(a,b)}{h}$$

and so to get the instantaneous rate of change we calculate the limit as $h \to 0$. Thus,

$$\mathrm{D}_{\vec{u}}f(a,b) = \lim_{h\to 0}\frac{f(a+hu_1,b+hu_2)-f(a,b)}{h}$$

Now recall the differential dz or $df = f(x,y)-f(a,b)$ is in fact equal to $f_x(a,b)(x-a)+f_y(a,b)(y-b)$. This is because the instantaneous rate of change in f is the same as the instantaneous rate of change of the linearization $\mathrm{L}(x,y)$ of f at the point P (why?). So

$$\begin{aligned}\mathrm{D}_{\vec{u}}f(a,b) &= \lim_{h\to 0}\frac{f_x(a,b)hu_1 + f_y(a,b)hu_2}{h}\\ &= f_x(a,b)u_1 + f_y(a,b)u_2\\ &= \langle f_x(a,b), f_y(a,b)\rangle\cdot\vec{u}\end{aligned}$$

The **two-dimensional** vector $\langle f_x(a,b), f_y(a,b)\rangle$ is special for its own reasons (as we will see next), and deserves its own name: we call it the **gradient** of f at (a,b). In general ∇f is also denoted by ∇f and is equal to

$$\nabla f = \langle f_x, f_y\rangle$$

Theorem 0.10

The directional derivative of f at (a,b) in the direction of unit vector $\vec{u}$ is given by

$$\mathrm{D}_{\vec{u}}f(a,b) = \nabla f(a,b)\cdot\vec{u}$$

■ Question 57.

□

Find the directional derivative of the function $f(x,y) = x^2+2y^2$ in the direction of the vector $\langle 1,1\rangle$.

§10.1 Significance of the gradient

We observe that

$$\mathrm{D}_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u} = \|\nabla f(a,b)\| \cos(\theta)$$

where θ is the angle between ∇f and $\vec{u}$, since $\|\vec{u}\| = 1$. Hence starting at the point (a,b), the rate of change of $f(x,y)$ i.e. $\mathrm{D}_{\vec{u}} f(a,b)$ is maximized when $\theta = 0$ i.e. when $\vec{u}$ and ∇f are parallel.

In other words,

Theorem 1.11

$\mathrm{D}_{\vec{u}} f(a,b)$ is maximized in the direction of ∇f. Moreover, the maximum value of $\mathrm{D}_{\vec{u}} f(a,b)$ is equal to $\|\nabla f\|$.

Question 58. □

Suppose we drop some water on the surface $z = x^3 y + y^2 x$ at the point $(1,1,2)$. Which way does the water flow?

Question 59. □

True or False: If we know the value of the directional derivative $\mathrm{D}_{\vec{u}} f(a,b)$ in the direction of any two nonparllel different unit vectors, then we can determine $\nabla f(a,b)$.

§10.2 Gradient and Level Curves

What happens to $f(x,y)$ along a level curve of f? If $\vec{u}$ is the direction of the tangent at the point $\mathrm{P} = (a,b)$ to the level curve passing through P, then the rate of change of f in the direction of $\vec{u}$ is zero.

Hence for this choice of $\vec{u}$, we have

$$\mathrm{D}_{\vec{u}} f(a,b) = 0 \implies \theta = \frac{\pi}{2}$$

where θ is the angle between $\vec{u}$ and $\nabla f(a,b)$. In other words,

Theorem 2.12

The gradient vector at a point is perpendicular to the level curve through that point.

Chapter 11 | Three Dimensional Gradient and the Tangent Plane

The directional derivative formula and the gradient have the obvious extensions for functions of three-variables $f(x,y,z)$:

$$\begin{aligned} D_{\vec{u}} f(a,b,c) &= \nabla f(a,b,c) \cdot \vec{u} \\ &= \langle f_x(a,b,c), f_y(a,b,c), f_z(a,b,c) \rangle \cdot \vec{u} \\ &= \|\nabla f(a,b,c)\| \cos(\theta) \end{aligned}$$

so the gradient again points in the direction of greatest increase of f from (a,b,c) and has magnitude equal to that rate of change.

Question 60. ☐

A fly is flying around a room in which the temperature is given by $T(x,y,z) = x^2 + y^4 + 2z^2$. *The fly is at the point* $(1,1,1)$ *and realizes that he's cold. In what direction should he fly to warm up most quickly?*

Question 61. ☐

How should the gradient $\nabla f(a,b,c)$ *be related to the level-surface of* f *through* (a,b,c)*?*

Question 62. ☐

For each of the following functions $F(x,y,z)$*, do the following*

(a) *Compute the gradient* ∇F.

(b) *Identify the level surface* $F(x,y,z) = $ *constant through the point* $(x,y,z) = (1,1,1)$.

(c) *Find the tangent plane to the level surface from part (b) at* $(1,1,1)$.

$$\text{(I) } F(x,y,z) = 3x + 2y + z \qquad \text{(II) } F(x,y,z) = x^2 + y^2 - z^2$$

§11.1 3D Gradient as the Normal vector

Since the gradient $\nabla f(a,b,c)$ is perpendicular to the level surface of f containing (a,b,c), it is the normal vector to the tangent plane of the level surface at (a,b,c). So even if the surface is not the graph of a function, we can use the three dimensional gradient to find tangent planes.

Question 63. ☐

Find the equation of tangent plane to the surface $xy + yz + zx = 5$ *at the point* $(1,1,2)$.

Question 64.

Alice the "A" student is debating Chuck the "C" student. Alice says that the direction of greatest ascent on the graph $z = f(x,y)$ is in the direction ∇f. Chuck says that instead we should look at the level surface $F(x,y,z) = z - f(x,y) = 0$ and go in direction ∇F. Who is right? How would you explain this to the student that is wrong?

□

Question 65.

You're hiking a mountain which is the graph of $f(x,y) = 15 - x^2 - 2xy - 3y^2$. You're standing at $(1,1,9)$. You wish to head in a direction which will maintain your elevation (so you want the instantaneous change in your elevation to be 0). How many possible directions are there for you to head? What are they?

□

Question 66.

Show that every plane that is tangent to the cone $x^2 + y^2 = z^2$ passes through the origin.

□

Question 67.

Where does the normal line to the paraboloid $z = x^2 + y^2$ at the point $(1,1,2)$ intersect the paraboloid a second time?

□

Question 68.

Find the tangent plane to the surface of the solid described by inequalities $0 \le x \le 6$ and $0 \le y^2 + z^2 \le 4$ at the point $(6,1,1)$.

□

Question 69.

The ellipsoid $x^2 + 4y^2 + 9z^2 = 36$ and the surface $z = \sin[\pi(x-y)]$ intersect in a curve, call it $\mathscr{C}$. Find the line tangent to $\mathscr{C}$ at the point $(6,0,0)$. (Please give a parametric vector equation for the line.)

□

Question 70.

Consider the surface

$$\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}.$$

(a) *Find the equation of the tangent plane to this surface at a point (p,q,r).*

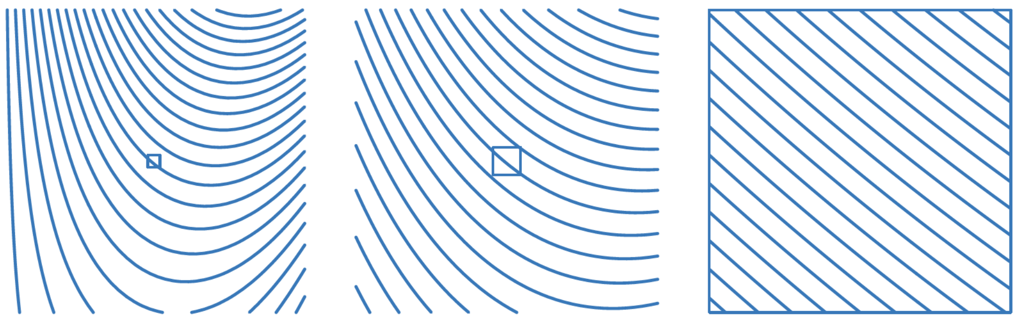
(b) *Show that the sum of X-intercept, Y-intercept and Z-intercept of the above tangent plane does not depend on p, q, and r.*

□

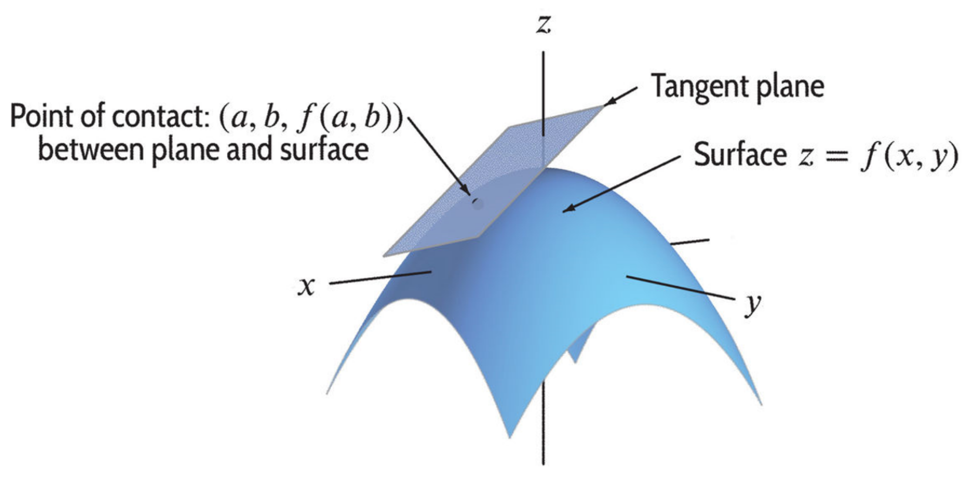
Chapter 12 | Tangent Plane and Linear Approximation

§12.1 Tangent Plane

We have seen earlier that we can view the graph of a function $z = f(x,y)$ as a surface in 3D. If we zoom in on the graph, at most points it seems to flatten out and become planar. Similarly, if we zoom in at the contour plot of $f(x,y)$, the contours look more like equally spaced parallel lines, which are the contours of a linear function. (As we zoom in, we have to add more contours.)



Seeing a plane when we zoom in at a point tells us (provided the plane is not vertical) that $f(x,y)$ is closely approximated near that point by a **linear** function $L(x,y)$. Graphically, the graph of this linear function is a plane, which we call the **tangent plane** at that point.



■ **Question 71.** □

We are going to find the formula of the tangent plane at the point $(a,b,f(a,b))$ on the graph $z = f(x,y)$.

(a) *Write this graph as a level curve of the function of three variables $G(x,y,z) = f(x,y) - z$. What is the 3D gradient of G?*

(b) *Use this to write down the formula for the tangent plane . This is the plane through $(a,b,f(a,b))$ and perpendicular to ∇G.*

Question 72.

Find the equation of the tangent plane to the elliptic paraboloid $z = x^2 + 2y^2$ at the point $(x,y,z) = (1,1,3)$.

§12.2 Linear Approximation

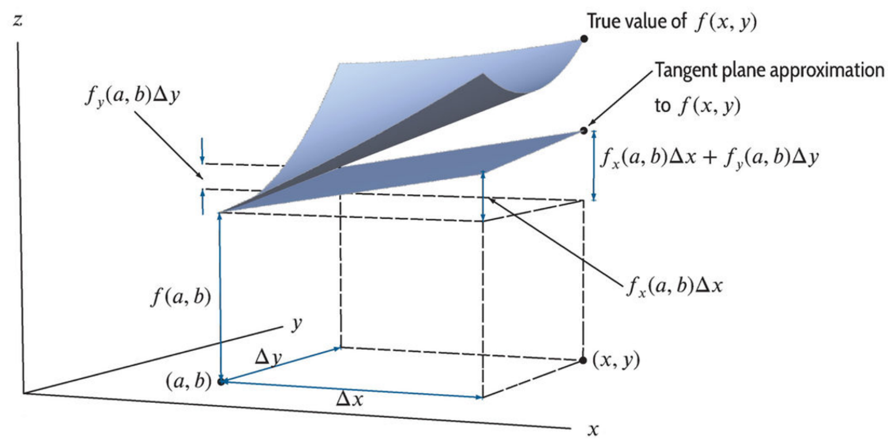
Since the tangent plane function matches f (and its partial derivatives) at (a,b) and is a linear function, it is called a **local linearization** of f near (a,b). We can use the local linearization to approximate a differentiable function at a point.

Theorem 2.13

The local linearization of a differentiable function $f(x,y)$ at a point (a,b) is given by

$$\mathrm{L}(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Here is picture that explains above theorem graphically.



Question 73.

- *Approximate the values of $z = f(x,y) = x^2 + 2y^2$ at the points $(x,y) = (0.9, 1.1)$ and $(x,y) = (0.95, 0.95)$ using the linearization from the previous exercise.*
- *Calculate the actual values of $f(0.9, 1.1)$ and $f(0.95, 0.95)$. Compare them to your answer from last part. Could you have predicted whether your estimate was an over-estimate or an under-estimate before calculating the actual values?*

Question 74.

Approximate the value of $f(x,y) = ye^{xy}$ at the point $(x,y) = (0.90, 0.01)$. Which point should you chose to find the local linearization?

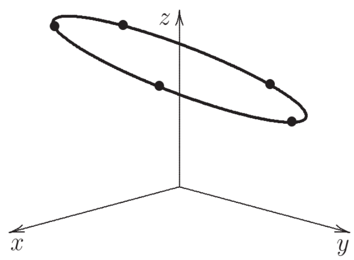
Chapter 13 | Vector Valued Functions and Space Curves

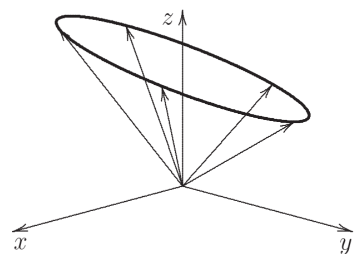
Any curve in 3–space can be described as either a collection of points

$$(x(t), y(t), z(t))$$

for t in some interval (possibly infinite) or as the trace of the heads of the position vectors (which start at the origin)

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$





Definition 0.14

$\vec{r}(t)$ is an example of a **vector valued function** of a real variable t. In general a vector valued function $\vec{\mathbf{F}}(t)$ can be written as

$$\vec{\mathbf{F}}(t) = \langle f(t), g(t), h(t) \rangle$$

We will come back to general vector-valued functions in the next chapter.

Example 0.15: Circle

A circle in the xy-plane can be described as the set of pairs $(x(t), y(t))$ generated by the **parameterization**

$$x(t) = \cos(t) \text{ and } y(t) = \sin(t) \quad \text{for } 0 \le t \le 2\pi$$

associated with the position vectors

$$\vec{r}(t) = \cos(t)\hat{i} + \sin(t)\hat{j} \quad \text{for } 0 \le t \le 2\pi$$

where the parameter t is the angle from the x–axis.

When curves are generated in this manner, they are called **parametrized curves**. Notice that **parameterization** carries more information than simply the final curve shape in the xy-plane. In particular, the example above indicates that the curve shape (circle) in the xy-plane is traced exactly once around, and that this curve shape is traced counterclockwise as t increases through its range.

§13.1 Different ways of parametrizing the same curve

We can parametrize the same curve in different ways and interpret each parametrization as the motion of a particle with the parameter t being time.

Question 75.

Explain why all of the parametrized curves below should look like a circle. In each of the parametrization, determine how many times around, and in which direction (clockwise or counterclockwise) the curve is traced.

- $x(t) = \cos(2t)$ *and* $y(t) = \sin(2t)$ *for* $0 \le t \le 2\pi$
- $x(t) = \cos(t^2)$ *and* $y(t) = \sin(t^2)$ *for* $0 \le t \le 2\pi$
- $x(t) = \cos(t)$ *and* $y(t) = -\sin(t)$ *for* $0 \le t \le 2\pi$
- $x(t) = -\cos(t)$ *and* $y(t) = \sin(t)$ *for* $0 \le t \le 2\pi$
- $x(t) = -\cos(t)$ *and* $y(t) = -\sin(t)$ *for* $0 \le t \le 2\pi$

Question 76.

Recall that $\vec{r}(t) = \vec{r}_0 + t\vec{v}$ (for $t \in \mathbb{R}$) gives the parametrization of a straight line passing through $\vec{r}_0$ and parallel to $\vec{v}$. What kind of curve are the following:

- $\vec{r}(t) = \vec{r}_0 + t^2\vec{v}$ *for* $t \in \mathbb{R}$
- $\vec{r}(t) = \vec{r}_0 + t\vec{v}$ *for* $a \le t \le b$

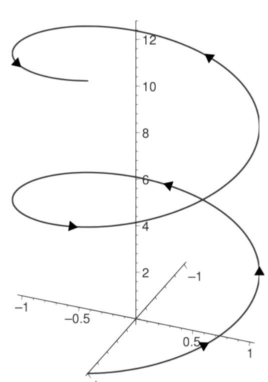
§13.2 3D Curves

13.2.1 Helix

The parametric equation of a helix is given by

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$$

When looked on from above (from the positive z direction), this curve is simply a circle in the xy-plane. The $z = t$ component lifts the circle into the helix spinning above the circle in the plane. If we visualize this as a particle at the tip of the position vector $\vec{r}(t)$, then from above it looks like the particle is simply spinning in a circle. But we also know that $z = t$, so the particle is rising at a constant rate. Hence we get the picture below



Question 77.

Match each vector-valued function to the curve it parametrizes.

(a) $\vec{r}(t) = \langle t\cos t, t\sin t, t\rangle$

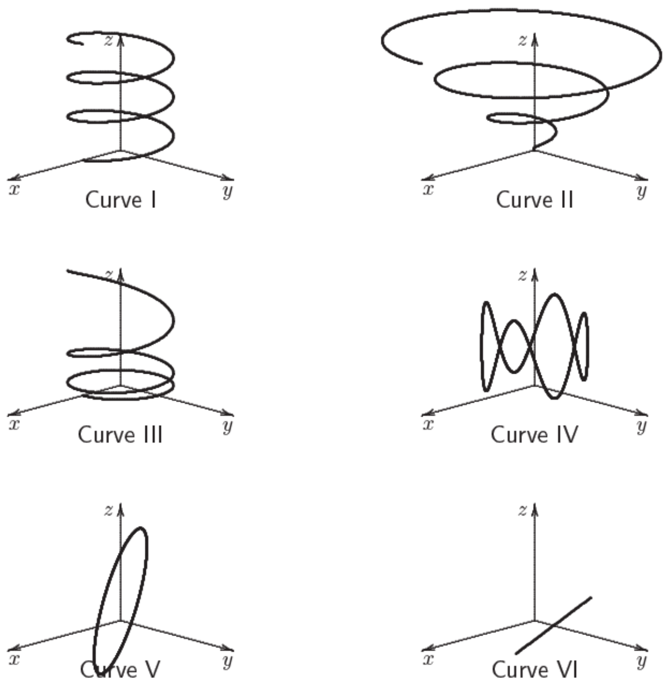
(b) $\vec{r}(t) = \left\langle \cos t, \sin t, t^3\right\rangle$

(c) $\vec{r}(t) = \langle \cos(t^3), \sin(t^3), t^3\rangle$

(d) $\vec{r}(u) = \langle \cos u, \sin u, 1+\sin 4u\rangle$

(e) $\vec{r}(u) = \langle \cos u, \sin u, 1+4\sin u\rangle$

(f) $\vec{r}(t) = \langle 2\cos t, 1+4\cos t, 3\cos t\rangle$



§13.3 Parametrization from equation of curve

Example 3.16

A parametrization of the parabola $x = 1 - y^2$ in xy- plane can be given by

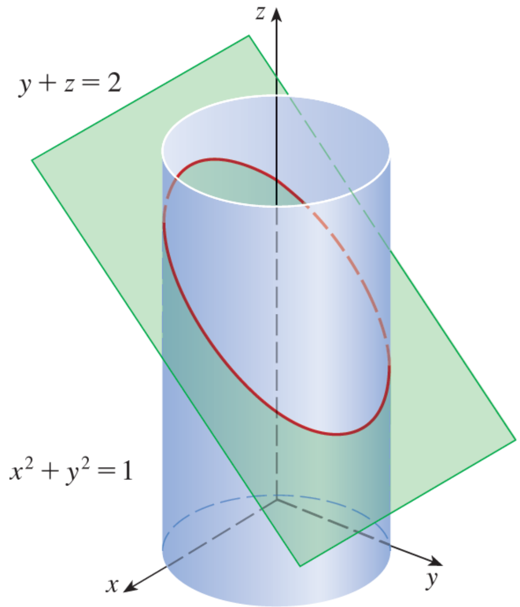
$$y(t) = t, \quad x(t) = 1 - t^2, \quad t \in \mathbb{R}$$

Question 78.

The surfaces $z = \sin(x-y)$ and $y = 2x$ intersect in a curve. Find a parameterization of the curve.

Question 79.

The cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$ intersect in a curve. Find a parameterization of the curve.



z
$y + z = 2$
$x^2 + y^2 = 1$
x
y

Chapter 14 | Motion, Velocity, Speed and Distance

Think of a parametrized curve as the trajectory of a point that is moving on the curve. At time t, its position vector is given by $\vec{r}(t)$. Then considering the appropriate vector difference quotients, we can build a concept of velocity vector of a parametrized curve.

■ Question 80. □

A particle travels along the line $x = 1 + t, y = 5 + 2t, z = -7 + t$. ***When*** *and* ***where*** *does the particle hit the plane $x + y + z = 1$?*

§14.1 Velocity and Acceleration

The idea is to consider the difference vector $\vec{r}(t + \Delta t) - \vec{r}(t)$ between the position vector $\vec{r}(t + \Delta t)$, a little time Δt beyond t, and the position vector $\vec{r}(t)$ at t. By taking $\Delta t \to 0$, the instantaneous **velocity vector** at time t is given by

$$\begin{aligned}\vec{r}\,'(t) &= \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \left(\frac{x(t + \Delta t) - x(t)}{\Delta t} \right) \hat{i} + \lim_{\Delta t \to 0} \left(\frac{y(t + \Delta t) - y(t)}{\Delta t} \right) \hat{j} + \lim_{\Delta t \to 0} \left(\frac{z(t + \Delta t) - z(t)}{\Delta t} \right) \hat{j} \\ &= x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}\end{aligned}$$

Similarly **acceleration vector** is given by

$$\vec{r}\,''(t) = x''(t)\hat{i} + y''(t)\hat{j} + z''(t)\hat{k}$$

■ Question 81. □

Let $f(x, y) = 3x^2 + 4y^2$.

(a) Find the parametric equation $\vec{r}(t)$ of the level curve of f that passes through the point $(1, 1)$.

(b) Find the velocity $\vec{r}\,'(t)$ at $(1, 1)$.

(c) Verify that $\vec{r}\,'(t)$ at $(1, 1)$ is perpendicular to $\nabla f(1, 1)$.

■ Question 82. □

A fly is sitting on the wall at the point $(0, 1, 3)$. At time $t = 0$, he starts flying; his velocity at time t is given by $\vec{v}(t) = \left\langle \cos 2t, e^t, \sin t \right\rangle$. Find the fly's location at time t.

■ Question 83. □

Find the angle of intersection between the curve given by its parametric equation $\vec{r}(t) = \langle t, 2t^2 \rangle$, and the parabola $y = x^2 + 4$.

§14.2 Distance - Length of the curve

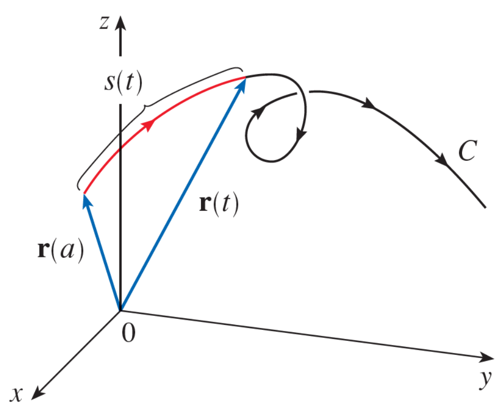
The magnitude $\|\vec{r}\,'(t)\|$ of the velocity vector is the **speed**. Hence the value $\|\vec{r}\,'(t)\|$ approximates the distance traveled from t to $t+\Delta t$ along the curve parameterized by $\vec{r}(t)$. Adding such approximations from $t=a$ to $t=b$ approximates the length of the curve

$$\sum_{t=a}^{t=b} \left\|\vec{r}\,'(t)\right\| \Delta t \approx \text{ Length of curve } \vec{r}(t) \text{ from } t=a \text{ to } t=b$$

Taking $\Delta t \to 0$ gives a perfect approximation in terms of the following integral:

$$s(t) = \int_{t=a}^{t=b} \left\|\vec{r}\,'(t)\right\| dt = \text{ Length of curve } \vec{r}(t) \text{ from } t=a \text{ to } t=b$$

$s(t)$ is then the arc length of the curve - physically, the distance travelled along the curve.



Question 84. □

(a) *What is the speed of an object on the circle parameterized by $\vec{r}(t) = \cos(t)\hat{\imath} + \sin(t)\hat{\jmath}$?*

(b) *Compute the length of this $\vec{r}(t)$ from $0 \le t \le 2\pi$.*

(c) *Use a dot product to find the orientation of the circle's velocity vectors $\vec{r}\,'(t)$ relative to its position vectors $\vec{r}(t) = \cos(t)\hat{\imath} + \sin(t)\hat{\jmath}$.*

(d) *Use a dot product to find the orientation of the circle's acceleration vectors $\vec{r}\,''(t)$ relative to its velocity vectors.*

§14.3 Equation of motion

14.3.1 Projectile motion

Suppose x measures horizontal distance in meters, and y measures distance above the ground in meters. At time $t=0$ in seconds, a projectile starts from a point h meters above the origin with speed v meters/sec at an angle θ to the horizontal. Its path is given by

$$x = (v\cos\theta)t, \quad y = h + (v\sin\theta)t - \frac{1}{2}gt^2$$

Question 85.

Suppose a ball thrown off the top of a cliff travels along the path

$$x = 20t, \quad y = 2 + 25t - 4.9t^2$$

(a) *When and where does the ball hit the ground?*

(b) *At what height above the ground does the ball start?*

(c) *What is the value of g, the acceleration due to gravity?*

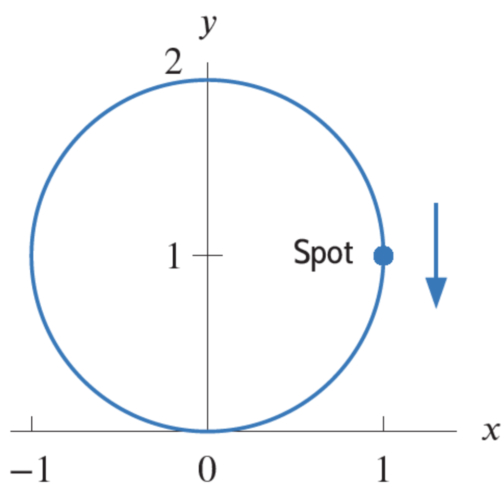
(d) *What are the values of v and θ?*

14.3.2 Cycloid

A **cycloid** is the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slipping.

Question 86.

A wheel of radius 1 meter rests on the x-axis with its center on the y-axis. There is a spot on the rim at the point $(1,1)$. At time $t = 0$ the wheel starts rolling on the x-axis in the direction shown at a rate of 1 radian per second.



(a) *Find parametric equations describing the motion of the center of the wheel.*

(b) *Find parametric equations describing the motion of the spot on the rim. Plot its path.*

§14.4 Calculus with Parametric Curves

Question 87.

A curve C is defined by the parametric equations $x = t^2, y = t^3 - 3t$.

(a) *Show that C has two tangents at the point $(3,0)$ and find their equations.*

(b) *Find the points on C where the tangent is horizontal or vertical.*

(c) *Determine where the curve is concave upward or downward.*

Question 88.

☐

Recall the parametric equation of the cycloid from the last section.

(a) *Find the tangent to the cycloid at the point where $\theta = \pi/3$.*

(b) *Find the length of one arch of the cycloid.*

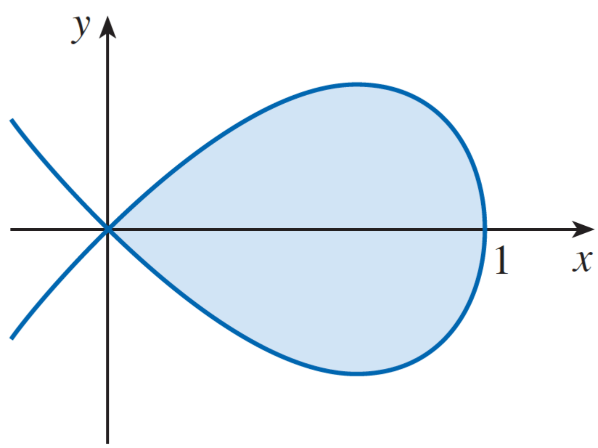
(c) *Find the area under one arch of the cycloid.*

Question 89.

☐

Find the area of the region enclosed by the loop of the curve

$$x = 1 - t^2, \quad y = t - t^3$$



Chapter 15 | Curvature and the TNB Basis

§15.1 Calculus of Vector-valued Functions

Consider a vector-valued function $\vec{\mathbf{F}}(t) = \langle x(t), y(t), z(t)\rangle$. In the last chapter, we defined $\vec{\mathbf{F}}'(t)$ and $\vec{\mathbf{F}}''(t)$. What do you think are the formula for the following? Here $p(t)$ is a one-variable dfifferentiable scalar-valued (real) function of t. .

(a) $\displaystyle\int \vec{\mathbf{F}}(t)\,\mathrm{d}t$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}t}[c\vec{\mathbf{F}}(t)]$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}t}[p(t)\vec{\mathbf{F}}(t)]$ (scalar product rule)

(d) $\dfrac{\mathrm{d}}{\mathrm{d}t}[\vec{\mathbf{F}}(t)\cdot\vec{\mathbf{G}}(t)]$ (dot product rule)

(e) $\dfrac{\mathrm{d}}{\mathrm{d}t}[\vec{\mathbf{F}}(t)\times\vec{\mathbf{G}}(t)]$ (cross product rule)

(f) $\dfrac{\mathrm{d}}{\mathrm{d}t}[\vec{\mathbf{F}}(p(t))$ (chain rule)

Question 90. □

If $\|\vec{r}(t)\| = c$*, then show that* $\vec{r}(t)$ *is perpendicular to* $\vec{r}'(t)$.

§15.2 Definitions of Curvature

A parametrized curve $\vec{r}(t)$ is called **smooth** if $\vec{r}'(t)$ is continuous and $\vec{r}'(t) \neq \vec{\mathbf{0}}$.

Let C be a smooth curve defined by the vector function $\vec{r}(t)$, the unit tangent vector to the curve $\vec{\mathbf{T}}(t)$ is given by

$$\vec{\mathbf{T}}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}.$$

Recall that te arc length of C is given by

$$s(t) = \int \left\|\vec{r}'(t)\right\| dt$$

Definition 2.17

The curvature of C at a given point κ is a measure of how quickly the curve changes direction at that point. Specifically, we define it to be the magnitude of the rate of change of the unit tangent vector with respect to arc length.

$$\kappa = \left\| \frac{\mathrm{d}\vec{\mathbf{T}}}{\mathrm{d}s} \right\|$$

This definition is not very helpful though. So we are going to do some manipulation using chain rule. Note that if we differentiate both sides of the arc length formula, then by using the Fundamental Theorem of Calculus, we obtain

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \|\vec{r}'(t)\|.$$

Hence, by chain rule,

$$\frac{\mathrm{d}\vec{\mathbf{T}}}{\mathrm{d}s} = \frac{\mathrm{d}\vec{\mathbf{T}}}{\mathrm{d}t}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\frac{\mathrm{d}\vec{\mathbf{T}}}{\mathrm{d}t}}{\frac{\mathrm{d}s}{\mathrm{d}t}} \implies \kappa = \frac{\left\|\vec{\mathbf{T}}'(t)\right\|}{\|\vec{r}'(t)\|}.$$

We can further manipulate the above equation by using the definition of $\vec{\mathbf{T}}$ to get

Theorem 2.18

The curvature of the curve given by the vector function $\vec{r}(t)$ is

$$\kappa = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}.$$

When $\vec{r}(t) = \langle f(t), g(t)\rangle$, we can simplify above formula to

$$\kappa = \frac{\|f'g'' - g'f''\|}{\left\|(f')^2 + (g')^2\right\|^{3/2}}.$$

■ **Question 91.** □

Find the curvature $\kappa(t)$ of $\vec{r}(t) = \langle t, \ln(\cos(t))\rangle$.

§15.3 The TNB frame

At a given point on a smooth space curve $\vec{r}(t)$, there are many vectors that are orthogonal to the unit tangent vector $\vec{\mathbf{T}}(t)$. We single out one as follows.

Observe that, $\|\vec{\mathbf{T}}(t)\| = 1$ for all t, hence we have $\vec{\mathbf{T}}(t)\cdot\vec{\mathbf{T}}'(t) = 0$. So $\vec{\mathbf{T}}'(t)$ is orthogonal to $\vec{\mathbf{T}}(t)$.

Now $\vec{\mathbf{T}}'(t)$ is not necessarily a unit vector. But at any point where $\kappa \neq 0$ we can define the principal **unit normal vector** $\vec{\mathbf{N}}(t)$ as

$$\vec{\mathbf{N}}(t) = \frac{\vec{\mathbf{T}}'(t)}{\left|\vec{\mathbf{T}}'(t)\right|}$$

We can think of the unit normal vector as indicating the direction in which the curve is turning at each point.

One other important unit vector is the **binormal vector** which is orthogonal to both $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$. It's given by

$$\vec{\mathbf{B}}(t) = \vec{\mathbf{T}}(t) \times \vec{\mathbf{N}}(t).$$

In general, the vectors $\vec{\mathbf{T}}, \vec{\mathbf{N}}$, and $\vec{\mathbf{B}}$, starting at the various points on a curve, form a set of orthogonal vectors, called the TNB frame, that moves along the curve as t varies. This TNB frame plays an important role in the branch of mathematics known as differential geometry and in its applications to the motion of spacecraft.

Question 92.

Find the unit normal and binormal vectors of $\vec{r}(t) = \langle t, \ln(\cos(t))\rangle$. *What does it tell you about binormal vectors of planar curves?*

Digression

Thus the binormal vector tells us how the curve is "twisting" out of the plane containing its tangent and normal vectors (called the *osculating plane*), and can be used to calculate a quantity called *torsion*.

Question 93.

Find $\vec{\mathbf{T}}, \vec{\mathbf{N}}$, *and* $\vec{\mathbf{B}}$ *for the curve* $\vec{r}(t) = \langle 2\cos t, 2\sin t, 3t\rangle$ *at* $t = \dfrac{3\pi}{2}$.

Question 94.

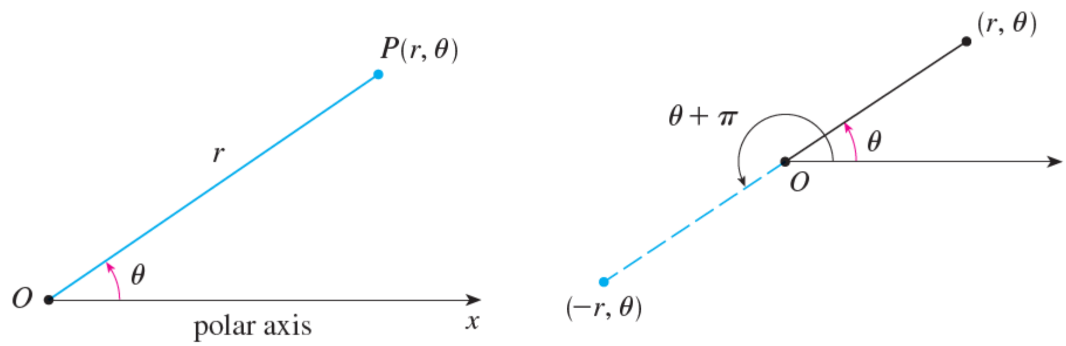
Find the unit tangent, unit normal, and binormal vectors and the curvature for the curve $\vec{r}(t) = \langle t, \sqrt{2}\ln t, 1/t\rangle$ *at the point* $(1, 0, 1)$.

Chapter 16 | Polar Coordinate System and Polar Curves

§16.1 Definition

A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates. Usually we use Cartesian coordinates, which are directed distances from two perpendicular axes. In this chapter, we look at a coordinate system introduced by Newton, called the **polar coordinate system**, which is more convenient for many purposes.

We choose a point in the plane that is called the pole (or origin) and is labeled O. Then we draw a ray (half-line) starting at O called the polar axis. This axis is usually drawn horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates.



If P is any other point in the plane, let r be the distance from O to P and let θ be the angle (usually measured in radians) between the polar axis and the line OP as in the figure above. Then the point P is represented by the ordered pair (r, θ) and r, θ are called polar coordinates of P. We use the convention that an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction. If $P = O$, then $r = 0$ and we agree that $(0, \theta)$ represents the pole for any value of θ.

16.1.1 When r is negative

The convention is to regard r as a 'signed' radius. That means if for some value of θ, we are provided a negative value of r, we go across origin and take the diametrically opposite point. In the second figure above, the points $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance $|r|$ from O, but on opposite sides of O.

Notice that $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$.

Question 95. □

Plot the points $(2, -2\pi/3)$ and $(-3, 3\pi/4)$. Find the Cartesian coordinates of the points above.

Question 96. □

In polar coordinates, what shapes are described by $r = k$ and $\theta = k$, where k is a constant?

(a) Draw $r = 0, r = \frac{2\pi}{3}, r = \frac{4\pi}{3}, r = 2\pi, \theta = 0, \theta = \frac{2\pi}{3}$, and $\theta = \frac{4\pi}{3}$.

(b) Find the equations of above curves in Cartesian coordinates.

§16.2 Converting between Polar and Cartesian Coordinates

Using simple trigonometry, we can find that the polar point (r,θ) corresponds to the Cartesian point $(r\cos\theta, r\sin\theta)$. Equivalently, starting from a Cartesian coordinate (x,y) we can make the change of variables

$$r=\sqrt{x^2+y^2},\quad \theta=\arctan\frac{y}{x}$$

to get (one possible) polar coordinate (r,θ).

Question 97. □

Express the point with polar coordinates $P(3,\frac{3\pi}{4})$ *in Cartesian coordinates. Express the point with Cartesian coordinates* $Q(3,3)$ *in Polar coordinates.*

Question 98. □

Find the polar equation of the straight line $x=5$.

> **Note:** Often you might find it easier to manipulate a given Cartesian equation **using trigonometry** to isolate $r\cos\theta$ and replace it with x and similarly replace $r\sin\theta$ with y.

Question 99. □

Convert the following polar equations into Cartesian:

(a) $r^2=5$.

(b) $r\cos\theta=5$

(c) $r^2\sin(2\theta)=1$

(d) $r=\cos\theta$

§16.3 Polar Curves

The graph of a polar equation $r=f(\theta)$, or more generally $F(r,\theta)=0$, consists of all points P that have at least one polar representation (r,θ) whose coordinates satisfy the equation.

So how do we plot a curve whose equation is of the form $r=f(\theta)$?

One possible method is to change it into Cartesian coordinates and plot that instead.

Question 100. □

Sketch the curve with polar equation $r=2\cos\theta$.

This method is not always easy if the equation is complicated. In such cases, the following procedure is advised.

- **Step 1.** Instead of plotting points on the XY-plane, we first sketch the graph of $r=f(\theta)$ in Cartesian coordinates where the horizontal axis is θ and the vertical axis is r.
- **Step 2.** Use the Cartesian graph in Step 1 as a guide to sketch the values of r that correspond to increasing values of θ. Remember than θ can only go up to 2π.

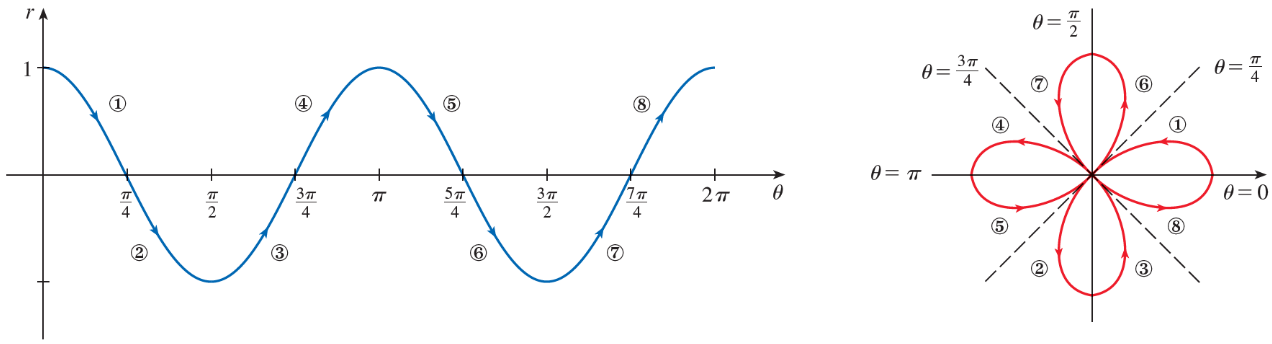
We will demonstrate using an example.

Example 3.19

Suppose we wish to sketch the curve $r=\cos(2\theta)$.

We first sketch $r = \cos 2\theta$ for $0 \le \theta \le 2\pi$, in Cartesian coordinates. As θ increases from 0 to $\pi/4$, the value of r decreases from 1 to 0 and so we draw the corresponding portion of the polar curve (indicated by ①).

As θ increases from $\pi/4$ to $\pi/2$, the value of r decreases from 0 to -1, this means that the distance from O increases from 0 to 1, but instead of being in the first quadrant this portion of the polar curve (indicated by ②) lies on the opposite side of the pole in the third quadrant. The remainder of the curve is drawn in a similar fashion, with the arrows and numbers indicating the order in which the portions are traced out. The resulting curve has four loops and is called a **four-leaved rose**.



■ Question 101.

Which part of the four-leaved rose correspond to $0 \le \theta \le \pi/2$?

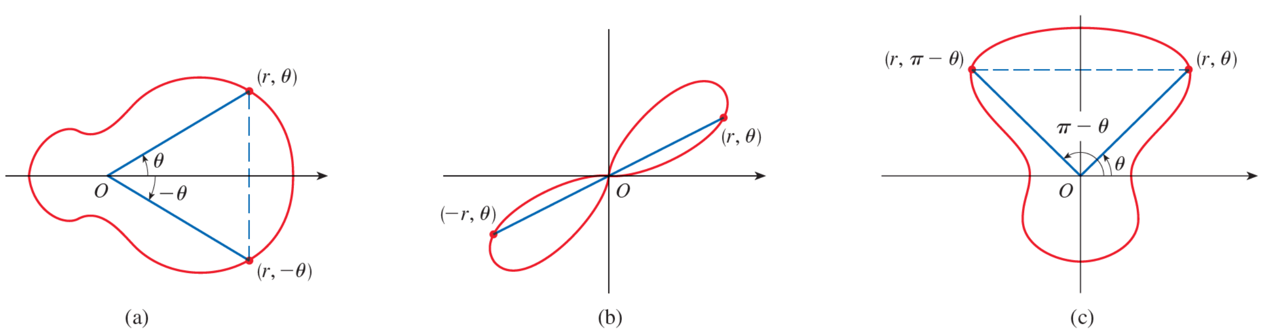
16.3.1 Symmetry

When we sketch polar curves it is sometimes helpful to take advantage of symmetry.

(a) If a polar equation is unchanged when θ is replaced by $-\theta$, the curve is symmetric about the polar axis.

(b) If the equation is unchanged when r is replaced by $-r$, or when θ is replaced by $\theta + \pi$, the curve is symmetric about the pole. (This means that the curve remains unchanged if we rotate it through 180° about the origin.)

(c) If the equation is unchanged when θ is replaced by $\pi - \theta$, the curve is symmetric about the vertical line $\theta = \pi/2$.



Chapter 17 | Chain Rule

Recall from one-variable calculus the chain rule

$$\frac{d}{dt}f(g(t)) = f'(g(t)) \cdot g'(t)$$

which is stated as "the derivative of the outside function (evaluated at inside function), times the derivative of the inside function".

Now think of a particle is moving in a parameterized curve $\vec{r}(t) = \langle x(t), y(t)\rangle$ and we want to understand how some function $f(x, y)$ changes along the curve. For example, we might ask, what is the rate of change of height along a particular racetrack. For any composite function of the form $f(x(t), y(t)) = f(\vec{r}(t))$, the multivariable chain rule is

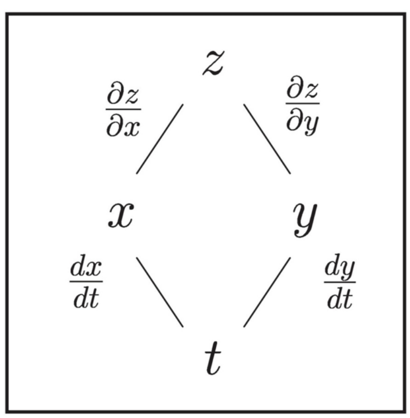
$$\frac{dz}{dt} = \frac{d}{dt}f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle x'(t), y'(t)\rangle$$

where the gradient ∇f acts like the derivative of the outside function f and the velocity $\vec{r}\,'(t)$ acts like the derivative of the inside vector-function $\vec{r}(t) = \langle x(t), y(t)\rangle$.

Warning: Notice that this just like the formula for the directional derivative, but now with $\vec{r}\,'(t)$ playing the role of the direction $\vec{u}$. Thus we can think of $\frac{df}{dt}$ as (a multiple of) the rate of change of f in the tangent direction of the parameterized motion.

§17.1 Chain Rule diagram

A chain of dependence diagram is a convenient way to represent the chain rule for $z = f(x(t), y(t))$.



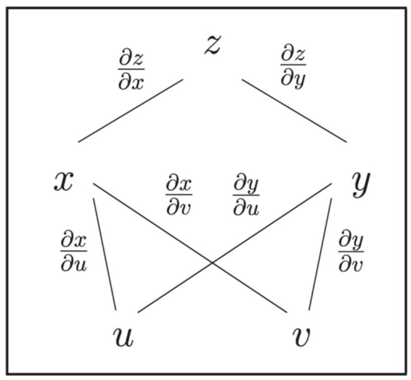
Then, the derivative $\dfrac{dz}{dt}$ is obtained by adding the products of the derivatives on each path of the diagram leading from z to t:

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}x'(t) + \frac{\partial z}{\partial y}y'(t)\\ &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\end{aligned}$$

where the terms $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are both evaluated at $(x(t), y(t))$.

§17.2 More complicated diagram

Curves in the xy-plane can also sometimes be described by even more parameters, like $x(u,v)$ and $y(u,v)$. Then the chain rule for differentiating $z = f(x(u,v), y(u,v))$ is even more complicated; but a dependence diagram can help.



Now the partial derivative $\frac{\partial z}{\partial u}$ is obtained by adding the products of the derivatives on each path from z to u in the diagram:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

and the partial derivative $\frac{\partial z}{\partial v}$ is obtained by adding the products of the derivatives on each path from z to v in the diagram:

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

■ Question 102. □

Let $w = x^2e^y, x = 4u$, and $y = 3u^2 - 2v$. Compute $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ using the chain rule.

■ Question 103. □

Let $z = f(x,y) = x^2 + y^3$, and $x = x(s,t)$ and $y = y(s,t)$; i.e., x and y are functions of s and t. Suppose that when $(s,t) = (0,1)$, we have:

$$x(0,1) = -1, \quad x_s(0,1) = -4, \quad x_t(0,1) = -7, \quad y(0,1) = 2, \quad y_s(0,1) = 10, \quad y_t(0,1) = 5.$$

Compute $\dfrac{\partial z}{\partial t}$ at $(s,t) = (0,1)$.

■ Question 104. □

The length of a side of a triangle is increasing at a rate of 3 in/s, the length of another side is decreasing at a rate of 2 in/s, and the contained angle θ is increasing at a rate of 0.05 radian/s. How fast is the area of the triangle changing when $x = 40$ in, $y = 50$ in, and $\theta = \pi/6$?

■ Question 105. □

Let $p = g(u,v)$ be a differentiable function of two variables. Let $u = \frac{x}{y}$ and $v = \frac{y}{z}$. Show that

$$x\frac{\partial p}{\partial x} + y\frac{\partial p}{\partial y} + z\frac{\partial p}{\partial z} = 0$$

■ Question 106. □

(a) *Use the chain rule to compute $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ for $z = f(x(r,\theta), y(r,\theta))$ where*

$$x(r,\theta) = r\cos(\theta) \quad y(r,\theta) = r\sin(\theta)$$

(b) *Show that*

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

Chapter 18 | Limit, Continuity, and Differentiability

§18.1 Limit and Continuity

The definition of a limit for a function of two variables is pretty much the same as for a single variable function.

Definition 1.20

We write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if $f(x,y)$ approaches L as (x,y) approaches (a,b).

The key question here is figuring out what "approaches" means. In one-dimension, x can approach a number c either from left or right. But in two-dimension we can have lots of different paths to "approach" a point (a,b).

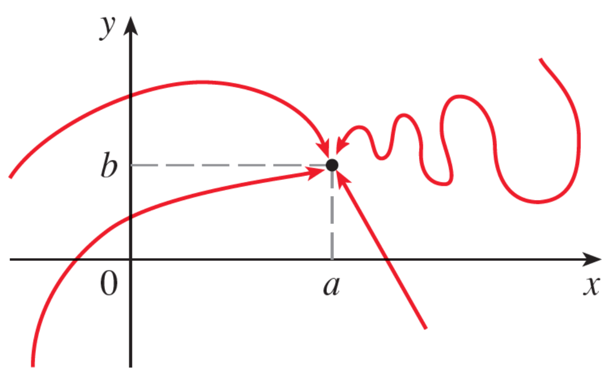


Figure 18.1: Different paths approaching (a,b)

In particular, if $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Warning: We can only follow paths that are in the domain of our function and that go to the limit point!

In general, it's hard to show algebraically that a limit exists since we must consider infinitely many possible paths to approach (a,b). If the function is of certain specific types, we can simplify the limit into a one-varibale limit and calculate it that way.

Sometimes you can separate the variables to turn the multivariable limit into a single variable limit which means we can use L'Hopital.

Example 1.21

Consider $L = \lim_{(x,y)\to(0,3)} \frac{1-\cos x}{xy}$.

We can write the limit as the product of two different limits ________________________.

Now calculating each separately, we get L = ________________________________.

Sometimes limits at a point like $(0,0)$ may be easier to evaluate by converting to polar coordinates. Remember that the same limit must be obtained as $r \to 0$ regardless of θ.

Example 1.22

Consider $L = \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{5x^2+5y^2}$.

We can rewrite the limit in polar coordinates as ________________________________.

This is a one-variable limit. Hence, L = ________________________________.

For a general function, we will need to use the $\varepsilon-\delta$ definition. It is comparatively easier to show that a limit does not exist.

Example 1.23

We will show that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

First let's approach $(0,0)$ along the x-axis. Along this path $y =$ ________________________.

So the function becomes $f(x,y) =$ ______________, and hence the limit is ______________.

Next, approach $(0,0)$ along the y-axis. Along this path $x =$ ________________________.

So the function becomes $f(x,y) =$ ______________, and hence the limit is ______________.

Since f has two different limits as (x,y) approaches $(0,0)$ along two different curves, the given limit does not exist.

Question 107.

Show that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ *does not exist by consider approaches along* $x = 0$ *and* $x = y$.

□

Question 108.

Show that $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ *does not exist by consider approaches along the line* $y = x$ *and the parabola* $x = y^2$.

□

§18.2 Continuity

We define continuity for functions of two variables in the same way as for functions of one variable:

Definition 2.24

We say that a function f is **continuous at the point** (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

A function is **continuous on a region** R in the xy-plane if it is continuous at each point in R.

Theorem 2.25

If $f(x,y)$ is a polynomial or a rational function $\dfrac{p(x,y)}{q(x,y)}$ where $q(x,y) \neq 0$, then f is continuous.

Example 2.26

he function $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ is clearly discontinous at the origin (since it's not even defined there). But as a rational function, f is continuous everywhere else.

Just as for functions of one variable, composition of two continuous functions is continuous.

Question 109. □

Where is the function $h(x,y) = e^{-(x^2+y^2)}$ continuous?

§18.3 Differentiability

First of all, recall that the in the case of one-variable functions, the derivative is equal to slope of the tangent line. Analogously, we say a function of two-variables is differentiable at a point if the function is "well-approximated" by the tangent plane at the point.

From theorem 13, the tangent plane is given by the local linearization of $f(x,y)$ at (a,b)

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

So we will focus on the precise meaning of the phrase "well approximated." Let's define the error function $E(x,y) = f(x,y) - L(x,y)$. We will define differentiability at a point in terms of this error and the distance from nearby points.

Consider a nearby point $(a+h, b+k)$. The distance between this point and (a,b) is $\sqrt{h^2+k^2}$.

Definition 3.27

We say that the function $f(x,y)$ is differentiable at the point (a,b) if

$$\lim_{(h,k)\to(0,0)} \frac{\mathrm{E}(a+h,b+k)}{\sqrt{h^2+k^2}} = 0.$$

The function f is differentiable on a region R if it is differentiable at each point of R.

Question 110.

Consider the function $f(x,y)=\sqrt{x^2+y^2}$. Show that f is not differntiable at the origin by showing that above limit does not exist.

HINT: What happens if we approch along the path $y=x$.

Theorem 3.28

If $f_x(x,y)$ and $f_y(x,y)$ both exist and are both continuous on a small disk R centered around the point (a,b), then $f(x,y)$ is differentiable at (a,b).

Note: Just the existence of both partial derivatives at a point does not guarantee differentiability of the function there. For example, consider the unction

$$f(x,y)=\begin{cases} \dfrac{xy}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$

We have already seen that the function is not continuous at the origin. However, we can verify that both partial derivatives exist at $(0,0)$.

Question 111.

Show that $f(x,y)=xe^{xy}$ is differentiable at $(1,0)$.

Chapter 19 | Local Optimization

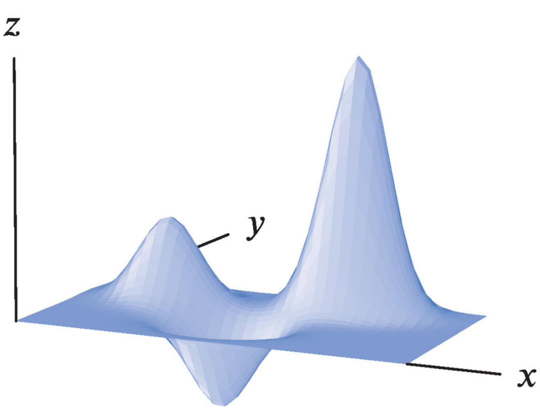
Definition 0.29

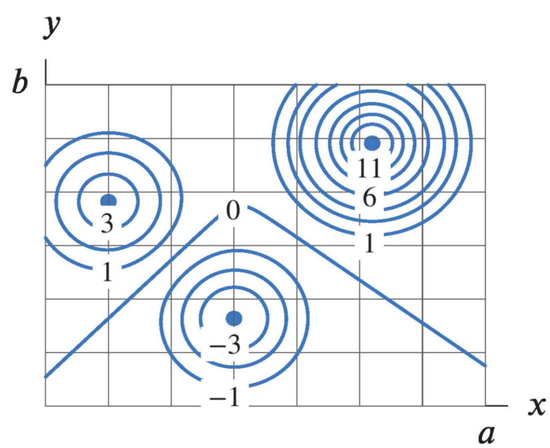
Suppose f is a continuous function on its domain D. Let P_0 be a point in D. Then we say f has

- a local maximum at the point P_0 if $f(P_0) \geq f(P)$ for all points P near P_0.
- a local minimum at the point P_0 if $f(P_0) \leq f(P)$ for all points P near P_0.
- a global (absolute) maximum at the point P_0 if $f(P_0) \geq f(P)$ for all points P in the domain of f.
- a global (absolute) minimum at the point P_0 if $f(P_0) \leq f(P)$ for all points P in the domain of f.

§19.1 How to detect a local extremum from the Contour Plot?

Consider the graph and contour plot of a function $z = f(x, y)$ as follows:





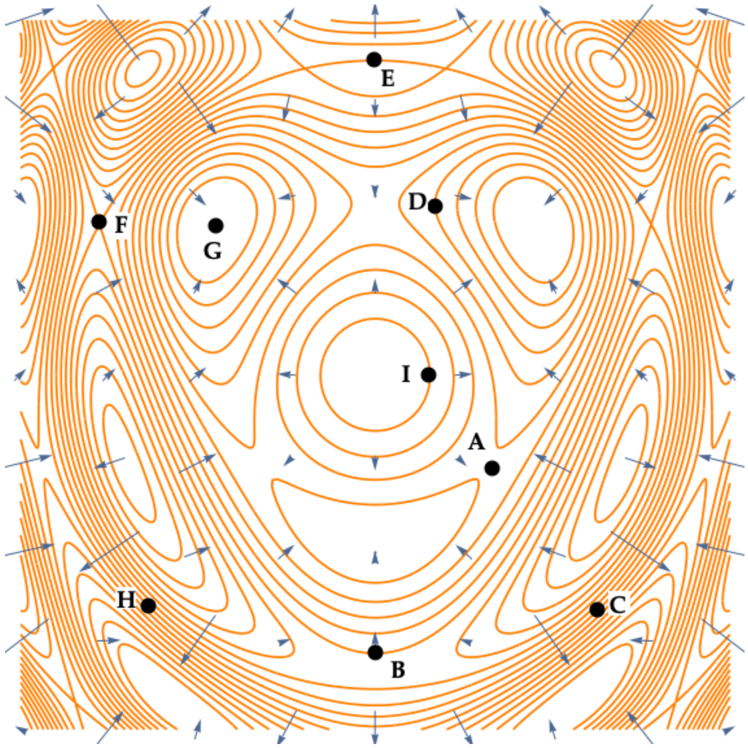
Question 112. □

What does the contour plot look like around a local extremum? What can you say about the partial derivatives and the gradient at the local extremum?

Use similar logic to answer the following question.

Question 113. □

A function $f(x, y)$ of two variables has level curves as shown in the picture. Some of the gradient vectors are also given for your convenience. Fill in the table below with all appropriate choices of points.



Enter all possible choices from A-I	*is/are point(s) where*
	$f_x(x,y) = 0$ *and* $f_y(x,y) \neq 0$
	$f_y(x,y) = 0$ *and* $f_x(x,y) \neq 0$
	$f_y(x,y) = 0$ *and* $f_x(x,y) = 0$
	$f(x,y)$ has either a local maxima or a local minima.

§19.2 Critical Points

Definition 2.30

A point (a,b) is called a **critical point** for the function $z = f(x,y)$ if it is one of the following two types.

- **Stationary Point**: an interior point at which f is differentiable and $f_x(a,b) = 0$ and $f_y(a,b) = 0$.
- **Singular Point**: an interior point at which f is not differentiable.

A point (a,b) is called a **stationary point** for the function $z = f(x,y)$ iff That is,

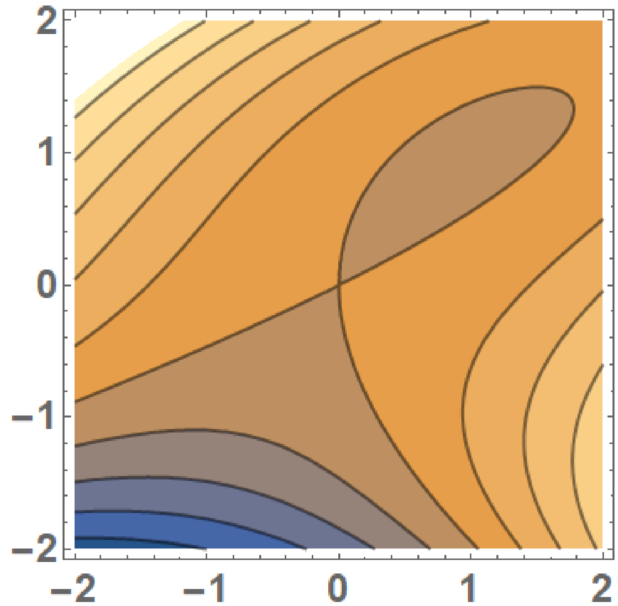
From the table above, it should be clear that a local maxima or local minima is a critical point, but not all critical points are local extremum.

Note: Stationary points are the points where $\nabla f = \langle f_x, f_y \rangle = 0$. If f is differentiable everywhere, critical points and sationary points are the same things.

Question 114.

Find the critical points of the function $f(x,y) = 8y^3 + 12x^2 - 24xy$. Are all of them local extrema?

Here is a contour plot to help you.



Definition 2.31

A critical point which is neither a local maximum nor a local minimum is called a saddle point.

§19.3 Classifying Stationary Points - The Second-Derivative test

The **Hessian** of a function $f(x,y)$ is defined to be the matrix $\mathrm{H} = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$. Suppose (a,b) is a point where $\nabla f(a,b) = \vec{\mathbf{0}}$ and let

$$\mathrm{D} = \det(\mathrm{H}(a,b)) = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2$$

The second-derivative test says

Theorem 3.32: The Second Derivative test

Suppose the second partial derivatives of f are continuous for all points in an open disk around $(a,b$ where $\nabla f(a,b) = \vec{\mathbf{0}}$.

- *If $\mathrm{D} > 0$ and $f_{xx}(a,b) > 0$, then f has a local minimum at (a,b).*
- *If $\mathrm{D} > 0$ and $f_{xx}(a,b) < 0$, then f has a local maximum at (a,b).*
- *If $\mathrm{D} < 0$, then f has a saddle point at (a,b).*
- *If $\mathrm{D} = 0$, anything can happen: f can have a local maximum, or a local minimum, or a saddle point, or none of these, at (a,b).*

Question 115. □

How does the contour plot look like near a saddle point? Identify the saddle points in the plots above.

Question 116. □

Find and classify all stationary points of the following functions:

(a) $f(x,y) = x^2 + 2xy + 2y^2 - 8y + 12$

(b) $f(x,y) = x^2 - 2y^2 + xy - 4$

(c) $f(x,y) = 8 - x^2 - xy - y^2$

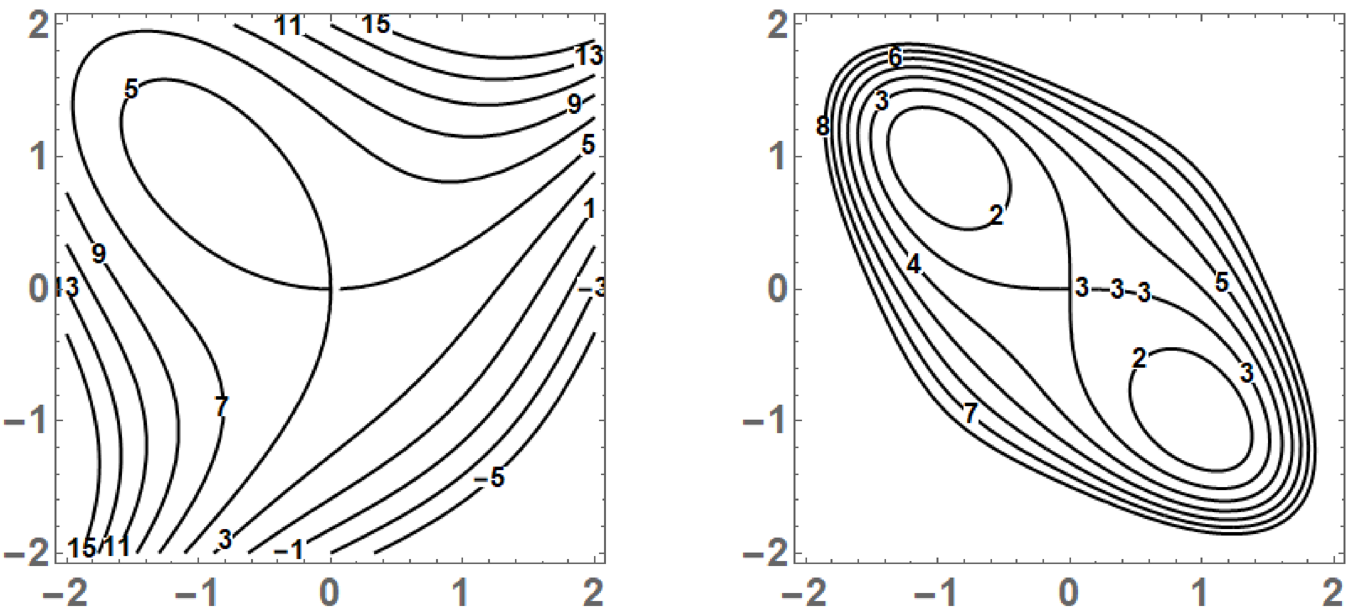
(d) $f(x,y) = x^4 + y^4 + 4xy + 3$

(e) $f(x,y) = y^2 + 2xy - x^2 + 2x - 2y + 36$

(f) $f(x,y) = 5 - x^3 + y^3 + 3xy$

Question 117.

Two of the above functions are shown as contour plots below. For each plot, predict the location of the stationary points and classify them as maxima, minima, or saddle points. Can you identify which functions are plotted here?

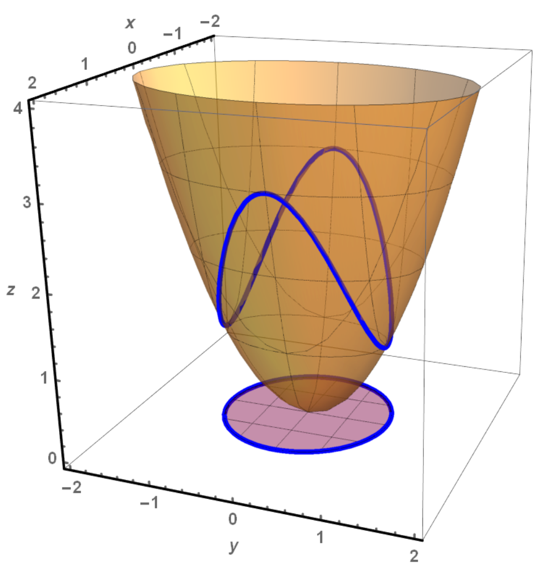


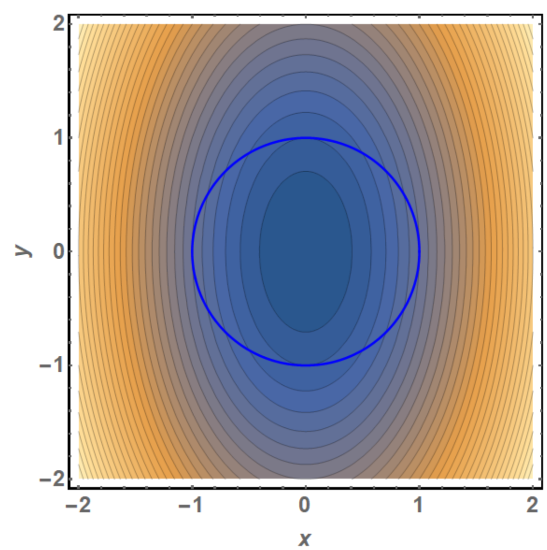
Chapter 20 | Constrained Optimization

§20.1 Motivation

A racehorse lives in a valley which happens to be the graph of $f(x,y) = 3x^2 + y^2$. He is doomed to wander his racetrack, which is the set of points in the valley where $x^2 + y^2 = 1$. The racehorse secretly wishes to be a mountain climber, and his fantasy is to escape from his racetrack and take the steepest path up the mountain. At what point should he make his escape, and in what direction should he run?

Below we have drawn 3-D depiction of the valley and the racetrack in the first picture, and the contour plot of the function $f(x,y)$ along with the projection of the racetrack in the second picture.



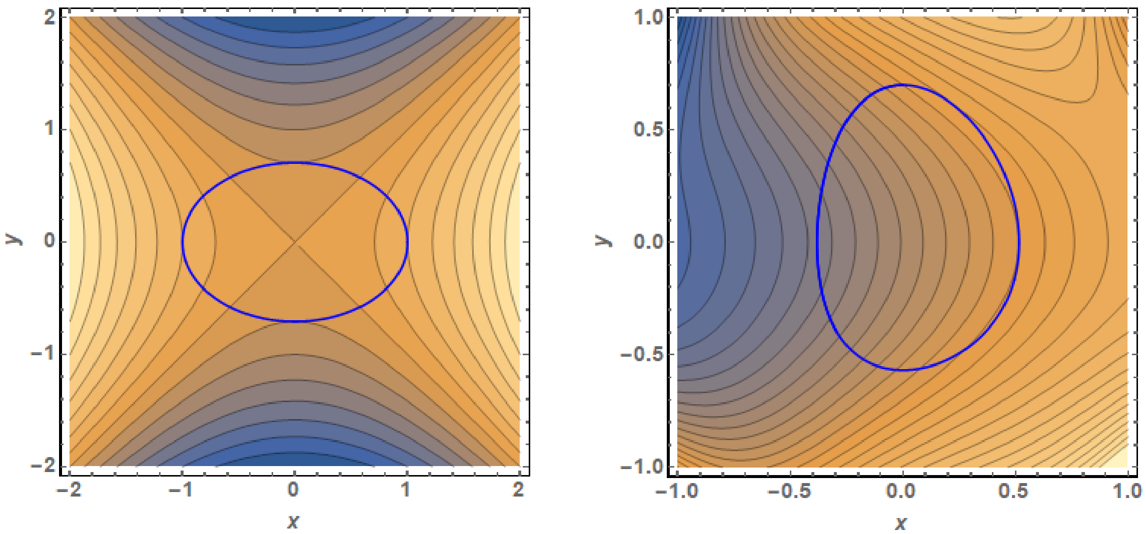


It makes sense that the horse will make his escape when $f(x,y)$ is largest along the racetrack. Can you see how to locate those points on the contour plot?

§20.2 Graphical Reasoning

■ **Question 118.** □

Consider in general, the contours of a differentiable function $f(x,y)$ overlaid with a curve $g(x,y) = c$.



(a) *For each plot, locate the local and global maximizer(s) and minimizer(s) of $f(x,y)$ on the curve $g(x,y) = c$.*

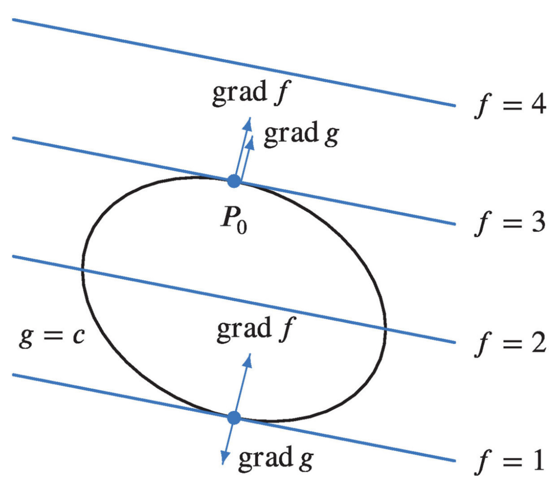
(b) *How are the tangents to the level curve of f and the curve $g(x,y) = c$ related at the extreme points?*

(c) *The curve $g(x,y) = c$ is evidently a level curve for the function $g(x,y)$. What does this imply about the relationship between ∇f and ∇g at the extreme points of f on the curve $g(x,y) = c$?*

[Hint: How is the gradient of any function related to the level curves of that function?]

(d) *Find the other points on the curve $g(x,y) = c$ which satisfy the gradient relationship. What is the behavior of f relative to the curve $g(x,y) = c$ at these points?*

§20.3 Algebraic Solution



From the last exercise, we observe that ∇f must be parallel to ∇g at the extreme points (a,b) of f on the curve $g(x,y) = c$. So we should be able to find some scalar λ such that

$$\nabla f(a,b) = \lambda \nabla g(a,b)$$

This is called the **Lagrange multiplier rule** in honor of Joseph-Louis Lagrange, and the particular scalar λ that lines up the two vectors exactly is called the **Lagrange multiplier**.

In the picture above ∇f and ∇g lie on the same line at the extreme points.

20.3.1 Algorithm

Step 1. To find critical points of a function $f(x,y)$ on a constraint curve $g(x,y)=c$, first solve the following system of simultaneous equations for x,y, and λ:

$$f_x(x,y)=\lambda g_x(x,y)$$
$$f_y(x,y)=\lambda g_y(x,y)$$
$$g(x,y)=c$$

Step 2. Once you have found all the critical points (a,b), evaluate $f(a,b)$ at each of them. The critical points where f is greatest are maxima and the critical points where f is smallest are minima.

Warning: In case of functions of three variables, Lagrange Multiplier problems with one constraint equation means solving a system of four simultaneous equations.

20.3.2 Tips for Solving the System of Equations

Solving the system of equations can be hard! Here are some possible approaches for solving these systems of equations:

- Solve for x,y,z in terms of λ. Then you can plug these back into the constraint,

  find the value of λ, which will give you x,y,z.
- Or, eliminate λ to solve for one variable in terms of the others. Substitute them in to $g(x,y)=c$ to solve for x and y.
- Remember that whenever you take a square root, you must consider both the positive and the negative square roots.
- Remember that whenever you divide an equation by an expression, you must be sure that the expression is not 0. It may help to split the problem into two cases:

  first solve the equations assuming that a variable is 0, and then solve the equations assuming that it is not 0.

Question 119. □

For each of the following, find the maxima and minima of f, on the constraint curve g.

(a) $f(x,y)=xy, \quad g(x,y)=3x^2+y^2=6.$

(b) $f(x,y)=\dfrac{1}{x}+\dfrac{1}{y}, \quad g(x,y)=\dfrac{1}{x^2}+\dfrac{1}{y^2}$

(c) $f(x,y)=x^2+4xy+y^2, \quad g(x,y)=x-y-6$

§20.4 Three-variable Functions

Question 120. □

Let $\mathrm{T}(x,y,z)=x-2y+5z+64$ *give the temperature, in degrees Celsius, at the point* (x,y,z)*, where* x,y*, and are measured in meters. Find the maximum and minimum temperature on the sphere* $x^2+y^2+z^2=120$.

Suppose we want to find extreme values of $f(x,y,z)$ and we have more than one constraint, say two constraints $g(x,y,z)=k$ and $h(x,y,z)=c$. So we are looking for the max and min values of f which lie on the intersection of $g(x,y,z) = k$ and $h(x,y,z) = c$. In this case our ∇f is determined by both ∇g and ∇h. So we now have Lagrange multipliers for each constraint function. So in this example, we have

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

Question 121.

Set up, but do not solve, the equations you would solve to find the extreme values of $f(x,y,z) = z$ subject to $x^2 + y^2 = z^2$ and $x + y + z - 24 = 0$

§20.5 Interpretation of λ as a rate of change

The value of λ can be interpreted as the rate of change of the optimum value of f as c increases (where $g(x,y) = c$). Suppose the optimum value of f is obtained at the point (a,b), where both a and b depend on the choice of c, and suppose the optimum value is given by

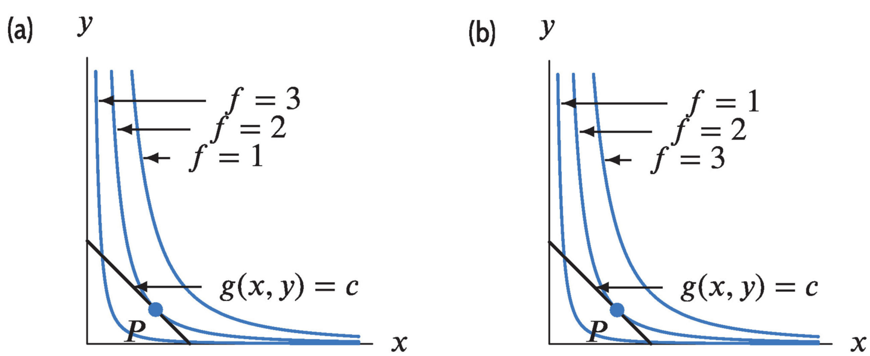
$$f_{min}(c) = f(a(c), b(c))$$

as a function of c. Then

$$\lambda = \frac{d}{dc} f_{min}(c)$$

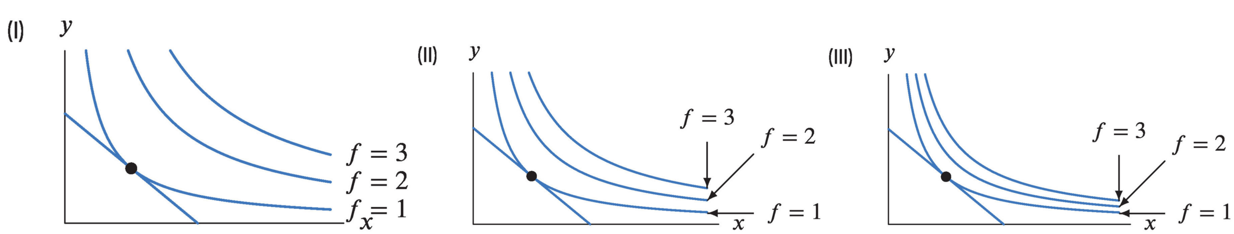
Question 122.

The image below shows the contours of a function f. For the graphs (a) and (b), Is P a maximizer or minimizer of f subject to $g(x,y) = x + y = c$ with $x, y \geq 0$? What is the sign of λ? Where could the other extreme be located?



Question 123.

The images below show the optimal point (marked with a dot) in three optimization problems with the same constraint. Arrange the corresponding values of λ in increasing order. (Assume λ is positive.)



Chapter 21 | Global Optimization

§21.1 Definitions

- f has a global maximum on a two-dimensional region R at the point P_0 if $f(P_0) \geq f(P)$ for all points $P \in R$.
- f has a global minimum on a two-dimensional region R at the point P_0 if $f(P_0) \leq f(P)$ for all points $P \in R$.

Warning: If the region R is not stated explicitly, we take it to be the whole xy-plane unless the context of the problem suggests otherwise.

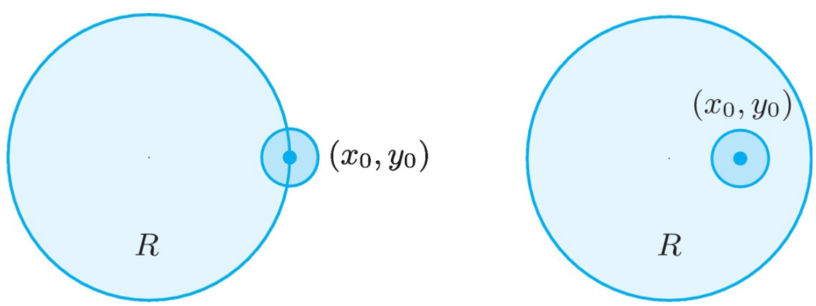
§21.2 How do we know whether a function has a global maximum or minimum?

- A **closed** region is one which contains its boundary.
- A **bounded** region is one which does not stretch to infinity in any direction.

Theorem 2.33: Extreme Value Theorem

If f is a continuous function on a closed and bounded region R, then f achieves its global maximum and minimum at some points in R.

The global extrema could be on the boundary or in the interior of R.



(a) A point on the boundary (b) A point in the interior

Question 124. □

For the function $f(x,y) = (x-2)^2 + y^2$, find a region R such that

(a) f attains a global maximum value of 4 and a global minimum value of 0 over R.

(b) f attains a global maximum value of 3 and a global minimum value of 1 over R.

(c) f attains a global maximum value of 9 but has no global minimum over R.

§21.3 Algorithm for Global Optimization

■ Question 125. □

Consider the function

$$F(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 1$$

defined on the disk D *of radius* $\sqrt{2}$ *centered at the origin i.e.*

$$D = \{(x,y) \mid x^2 + y^2 \leq 2\}.$$

Follow the steps below to find the global maximum and minimum of f *on* D.

(a) *Find all the stationary critical points of* F. *[Hint: There are 4 such points.]*

(b) *In the list of critical points from part (a), identify the ones lying* ***inside*** D *(excluding the boundary).*

(c) *Evaluate* F *at the critical point(s) from part (b).*

(d) *Use Lagrange multiplier to find the maximum and minimum of* $F(x,y)$ *subject to the constraint* $x^2 + y^2 = 2$. *Note that this gives the extreme values of* F *on the boundary circle of* D.

(e) *Evaluate* F *at the critical point(s) from part (d).*

(f) *Compare the extreme values of* F *from part (c), and the extreme values of* F *from part (e), to find the global maximum and minimum of* $F(x,y)$ *on* D.

§21.4 Practice Problems

■ Question 126. □

For which of the regions D *described below is it true that every continuous function* $f(x,y)$ *must attain an global maximum value and global minimum value on* D*? (There may be more than one.)*

(i) D *is the set of points* (x,y) *such that* $|x| \leq 4$ *and* $|y| < 2$

(ii) D *is the set of points* (x,y) *such that* $|x+y| \leq 1$

(iii) D *is the set of points* (x,y) *such that* $x^2 + 4y^2 \leq 1$

(iv) D *is the set of points* (x,y) *such that* $x^2 + 4y \leq 1$

(v) D *is the set of points* (x,y) *such that* $-x \leq y \leq x$

Then do the following:

(a) *For one of the regions* D *that you picked, find the global minimum and global maximum value of* $f(x,y) = x^2 - 4x + y^2$ *on the region.*

(b) *For one region you didn't pick, find a function* $f(x,y)$ *which has either no maximum or no minimum on the region* D.

■ Question 127. □

Find all global extrema of the function

$$f(x,y) = 2x^3 + 2y^3 - 3x^2 - 3y^2 + 6$$

on the disc $D = \{(x,y) \mid x^2 + y^2 \leq 4\}$.

Chapter 22 | Definite Integration in two variables

§22.1 Introduction

In one-variable calculus, we learned that the definite integral $\int_a^b f(x)\,dx$ gave the area under the graph of $f(x)$ over the interval $[a, b]$ on the x -axis. For two-variable functions, the input space is the xy-plane, so we compute volume integrals over two-dimensional regions R of the plane:

$$\int_{\mathrm{R}} f(x, y)\,\mathrm{dA}$$

which measure the amount of volume under the graph of $f(x, y)$ and over the region R. Note that R is NOT necessarily a rectangle. The dA indicates that the integration is taken over a region R with (A)rea in the xy-plane.

§22.2 Construction

Recall that we approximate definite integrals in one-variable calculus By chopping the interval $[a, b]$ into subintervals and summing the areas of rectangles (usually) based on each subinterval.

For volume integrals, we can chop R into subregions and sum the volumes of rectangular (usually) prisms based on each subregion. As with one-variable integrals, using smaller and smaller subregions of R leads to better and better approximations.

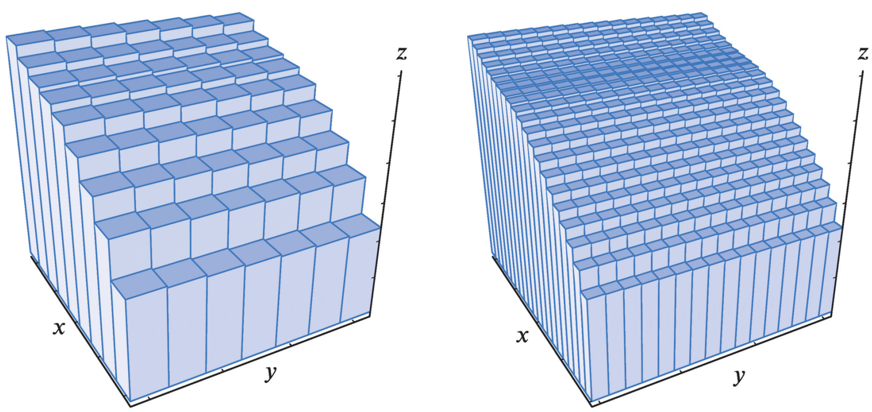


Figure 22.1: Approximating volume under a graph with finer and finer Riemann sums

Digression

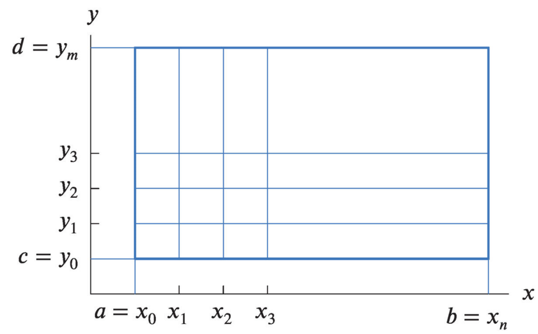
The Riemann Sum definition: If R is a rectangle, we can express this approximation procedure more formally using Riemann sums and limits as follows.

Suppose the function $f(x, y)$ is continuous on R, the rectangle $a \le x \le b, c \le y \le d$. Consider the partitions

$$\mathrm{P} : \{a = x_0 < x_1 < \cdots < x_n = b\}$$

and

$$Q : \{c = y_0 < y_1 < \cdots < y_m = d\}$$



For a choice of points $\left(x^*_{ij}, y^*_{ij}\right)$ in the (i, j)th subrectangle $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$,we define the Riemann Sum of f over R as

$$S^*_{(P,Q)} = \sum_{i=1}^{n}\sum_{j=1}^{m} f\left(x^*_{ij}, y^*_{ij}\right)\Delta x_i \Delta y_j = \sum_{i=1}^{n}\sum_{j=1}^{m} f\left(x^*_{ij}, y^*_{ij}\right)(x_i - x_{i-1})\left(y_j - y_{j-1}\right)$$

Also let $\|(P,Q)\|$ denote the maximum area of a subrectangle i.e.

$$\|(P,Q)\| = \max_{i,j}(x_i - x_{i-1})\left(y_j - y_{j-1}\right)$$

Then the definite integral of f over R is given by

$$\int_R f\, dA = \lim_{\|(P,Q)\|\to 0} S^*_{(P,Q)}$$

In the case the partitions are uniform, above limit can be rewritten as

$$\int_R f\, dA = \lim_{n,m\to\infty} S^*_{(P,Q)}$$

■ Question 128. □

If R is any region in the plane, what does the double integral $\iint_R 1\, dA$ *represent? Why?*

■ Question 129. □

Suppose the shape of a flat plate is described as a region R in the plane, and $f(x,y)$ gives the density (mass of unit area) of the plate at the point (x,y) in kilograms per square meter. What does the double integral $\iint_R f(x,y)\, dA$ *represent? Why?*

■ Question 130. □

What does the double integral $\dfrac{\displaystyle\iint_R f(x,y)\,dA}{\displaystyle\iint_R 1\,dA}$ *represent? Why?*

■ Question 131. □

Let R *be the rectangle* $-1 \le x \le 1, -1 \le y \le 1$, *and* T *is the top half* $-1 \le x \le 1, 0 \le y \le 1$, *and* L *is the left half* $-1 \le x \le 0\ -1 \le y \le 1$.

Without evaluating any of the integrals, decide which of them are positive, which are negative, and which are zero.

(i) $\displaystyle\int_R x\,dA$ (ii) $\displaystyle\int_R y\,dA$ (iii) $\displaystyle\int_T y\,dA$ (iv) $\displaystyle\int_R \left(x - x^2\right)dA$

(v) $\displaystyle\int_T \left(y - y^2\right)dA$ (vi) $\displaystyle\int_L \left(x^2 - x\right)dA$ (vii) $\displaystyle\int_L \left(y + y^3\right)dA$ (viii) $\displaystyle\int_R (2x + 3y)\,dA$

■ Question 132. □

For the same set of integrals above, arrange $(a), (c), (e)$ *and* (f) *in increasing order.*

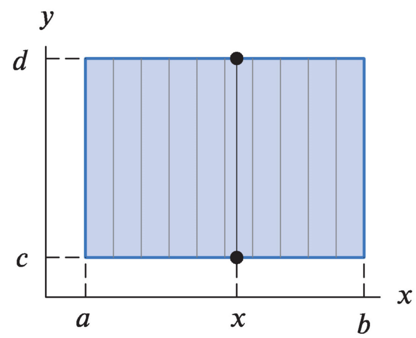
■ Question 133. □

Evaluate $\iint_R (f(x,y)\,dA$ *for the function below*

$$f(x,y) = \begin{cases} 2, & 0 \le x \le 5, 0 \le y \le 1 \\ 3, & 0 \le x \le 5, 1 \le y \le 3 \end{cases}$$

Chapter 23 | Iterated Integrals

Consider the volume integral of a function f over a rectangular region R. Instead of dicing R into a grid to approximate the integral, we can cut it in "vertical strips" (in the y-direction) in the xy-plane. Then, we build solid slices whose heights are determined by the value of $f(x,y)$ along one edges of each strip.



Volume of such a solid slice is given by

$$\underbrace{\left(\int_c^d f(x_{i-1},y)dy\right)}_{\text{Area of a vertical cross-section parallel to YZ-plane}} \quad \times \quad \underbrace{(x_i - x_{i-1})}_{\text{width in X-direction}}$$

Now if we sum up the volumes of these solids and take a limit, we should get back the volume integral

$$\int_R f\,dA = \lim_{\Delta x_i \to 0} \sum_{i=1}^n \left(\int_c^d f(x_{i-1},y)dy\right)(x_i - x_{i-1}) = \int_a^b \int_c^d f(x,y)dydx$$

This is called an **iterated integral**.

§23.1 Type I and Type II regions

A region R is called **Type I** if it can be written in the following way:

$$R = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

We can then compute a double integral as

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

A region R is called **Type II** if it can be written in the following way:

$$R = \{(x,y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$$

We can then compute a double integral as

$$\iint\limits_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Note: For a quick reference, the rules on the limits of an iterated integral are as follows:

- The limits on the outer integral must be constants.
- The limits on the inner integral can involve only the variable in the outer integral. For example, if the inner integral is with respect to x, its limits can be functions of y.

Question 134.

Match the integral with the appropriate region of integration.

(a) R_1 : *The triangle with vertices* $(0,0),(2,0),(0,1)$

(b) R_2 : *The triangle with vertices* $(0,0),(0,2),(1,0)$

(c) R_3 : *The triangle with vertices* $(0,0),(2,0),(2,1)$

(d) R_4 : *The triangle with vertices* $(0,0),(1,0),(1,2)$

(i) $\displaystyle\int_0^1 \int_0^{2-2x} f(x,y)\,dy\,dx$

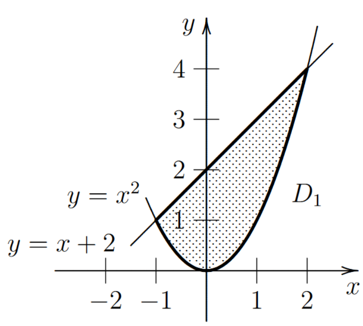
(ii) $\displaystyle\int_0^1 \int_0^{2-2y} f(x,y)\,dx\,dy$

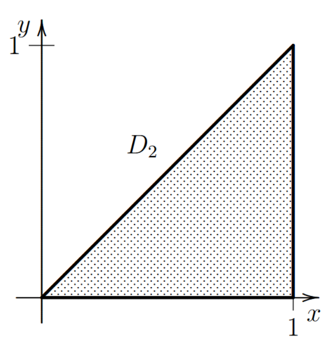
(iii) $\displaystyle\int_0^1 \int_0^{2x} f(x,y)\,dy\,dx$

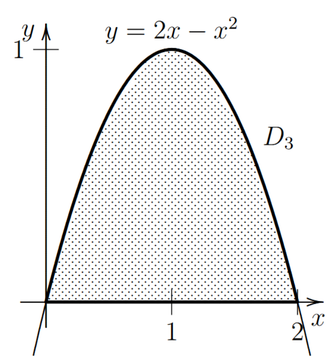
(iv) $\displaystyle\int_0^1 \int_{2y}^{2} f(x,y)\,dx\,dy$

Question 135.

Here are some type I regions. For each of them set up the given double integral as an iterated integral and evaluate it.







(i) $\displaystyle\iint\limits_{D_1} xy\,dA$ *Ans:* 45/8

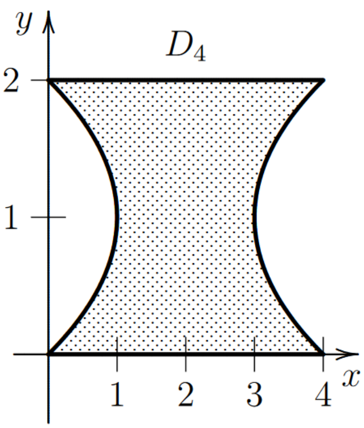
(ii) $\displaystyle\iint\limits_{D_2} e^{x^2}\,dA$ *Ans:* $(e-1)/2$

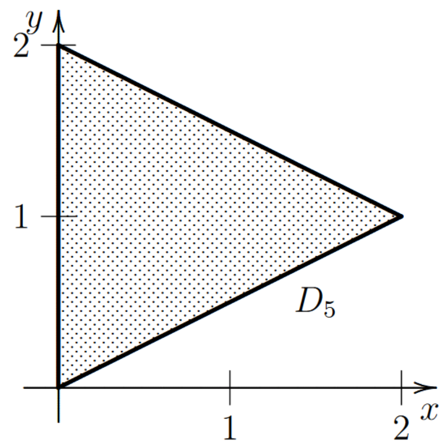
(iii) $\displaystyle\iint\limits_{D_3} (x-1)y^2\,dA$ *Ans:* 0

(iv) $\displaystyle\iint\limits_{D_1} 1\,dA$ *Ans:* 4.5

Question 136.

Here are some type II regions. For each of them set up the given double integral as an iterated integral and evaluate it. The curves in D_4 are $x = 2y - y^2$ and $x = y^2 - 2y + 4$.





(i) $\displaystyle\iint_{D_2} (1-y)^3 \, dA$ *Ans:* 1/5

(ii) $\displaystyle\iint_{D_4} (y-1)x^2 \, dA$ *Ans:* 0

(iii) $\displaystyle\iint_{D_2} cos(x^2) \, dA$ *Ans:* $\frac{\sin(1)}{2}$

(iv) $\displaystyle\iint_{D_5} (x-1) \, dA$ *Ans:* $-2/3$

§23.2 Switching the Order of Integration

If a region is both type I and type II, you may find that one order of integration will be simpler to deal with than the other. Sometimes, when converting a double integral to an iterated integral, we decide the order of integration based on the integrand, rather than the shape of the region - some integrands are easy to integrat ewith respect to one variable and much harder (or even impossible) to integrate with respect to the other.

Question 137.

For each of the following integrals, draw the region in question, write down an integral with the reverse order of integration, then finally integrate.

(i) $\displaystyle\int_0^2 \int_x^2 (x+y) \, dy \, dx$ *Ans:* 4

(ii) $\displaystyle\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1} \, dy \, dx$ *Ans:* 26

(iii) $\displaystyle\int_0^1 \int_y^1 e^{x^2} \, dx \, dy$ *Ans:* $\frac{e-1}{2}$

(iv) $\displaystyle\int_0^{\pi/2} \int_y^{\pi/2} \frac{\sin(x)}{x} \, dx \, dy$ *Ans:* 1

(v) $\displaystyle\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} (3x-x^3)^{10} \, dx \, dy$ *Ans:* $\frac{2^{12}}{33}$

(vi) $\displaystyle\int_1^{e^3} \int_{\ln(y)}^3 (e^x - x)^5 \, dx \, dy$ *Ans:* $\frac{e^3-4}{6}$

Chapter 24 | Double Integral in Polar Coordinates

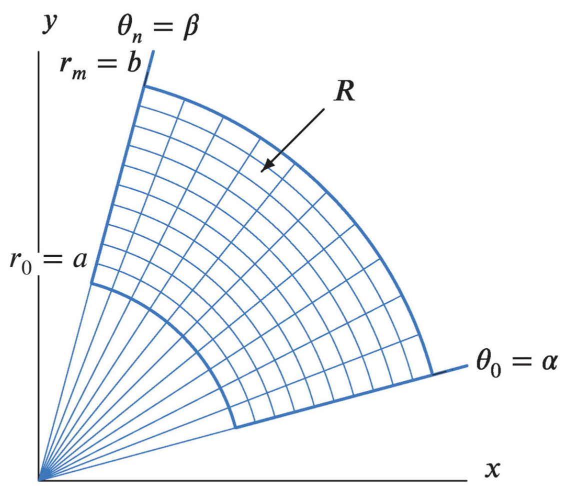
§24.1 When to Use it

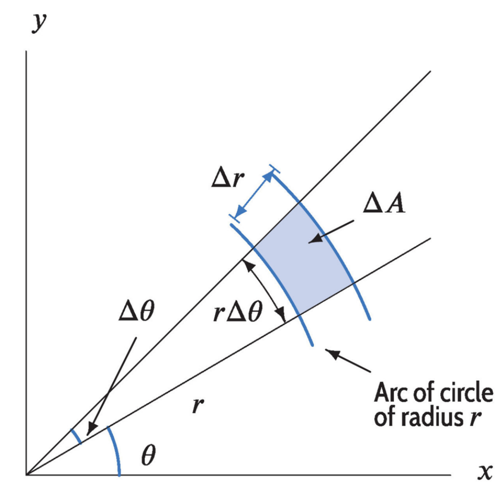
We will use Polar coordinates when the region of integration is easier to express in polar coordinates. This might happen mostly two cases:

- The region is interior of a polar curve of the form $r = f(\theta)$.
- The region is circularly symmetric and either the bounds or the integrand has terms involving $x^2 + y^2$.

§24.2 What is dA in Polar Coordinates?

Consider the way a Riemann Sum is constructed on a region with polar coordinates. Subdividing a region of the form $\mathrm{R} = \{(r,\theta) \mid a \le r \le b, \alpha \le \theta \le \beta\}$ into smaller partitions gives us the following pictures:





Accordingly, ΔA is approximated as $r\Delta r\Delta\theta$ and after taking a limit of the Riemann sum we get the following identity

$$\iint_{\mathrm{R}} f\,\mathrm{dA} = \iint_{\mathrm{R}} f(r\cos(\theta), r\sin(\theta))\,r\,\mathrm{d}r\,\mathrm{d}\theta$$

Warning: Note that extra r in the second integral. This shows up due to the fact that the area infinitesimal dA does not look like a small square in polar coordinates. A more detailed explanation will be given after we learn about Jacobians and change of variables.

Digression

When doing integrals in polar coordinates, you often need to integrate trigonometric functions. The double-angle formulas are very useful for this.

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta$$

These two identities make it easy to integrate $\sin^2\theta$ and $\cos^2\theta$.

Question 138.

For the following problems

- *Sketch the region of integration.*
- *Try to describe the region in polar coordinates and decide whether you should use polar coordinates or cartesian coordinates.*
- *Evaluate the integral.*

(a) $\iint\limits_R \sqrt{x^2+y^2}\,dA$, *where* R *is the region* $x^2+y^2 \leq 1$. *Ans:* $2\pi/3$

(b) $\iint\limits_R x\,dA$, *where* R *is the region* $x^2+y^2 \leq 1, x \geq 0$. *Ans:* $2/3$

(c) $\iint\limits_R (x+y)^2\,dA$, *where* R *is the region* $1 \leq x^2+y^2 \leq 9, x \geq 0, y \geq 0$. *Ans:* $10\pi + 20$

(d) $\iint\limits_R \sqrt{x^2+y^2}\,dA$, *where* R *is the region* $0 \leq x \leq 1, 0 \leq y \leq 1$ *Ans:* $2/3$

(e) $\displaystyle\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} e^{x^2+y^2}\,dy\,dx$ *Ans:* $\pi\frac{e^4-1}{2}$

(f) $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sqrt{1-x^2-y^2}\,dx\,dy$ *Ans:* $2\pi/3$

(g) $\displaystyle\int_{\pi/4}^{\pi/2}\int_{1/\sin\theta}^{4/\sin\theta} r\,dr\,d\theta$ *Ans:* 7.5

Question 139.

(a) *Find the area of the region* R *lying between the curves* $r = 2 + sin(3\theta)$ *and* $r = 4 - \cos(3\theta)$.

(b) *The region* R *lying inside the curve* $r = 2 + \sin 3\theta$ *and outside the curve* $r = 3 - \sin(3\theta)$ *consists of three pieces. Find the area of one of those pieces.*

(c) *Find the area of the region which lies inside the circle* $x^2 + (y-1)^2 = 1$ *but outside the circle* $x^2 + y^2 = 1$.

Chapter 25 | Triple Integrals

Note: We will learn this topic using a Mathematica notebook. I have only compiled the main results and some exercises here.

§25.1 Volume and Mass as an Iterated Integral

The volume of a three-dimensional solid W is given by $\iiint_W dV$. If the density at point (x, y, z) is given by $f(x, y, z)$, then it's mass is given by $\iiint_W f\, dV$.

For suitable three dimensional regions W, the triple integral can be represented as an iterated integral of the form:

$$\int_W f\, dV = \int_a^b \left(\int_{\psi_1(z)}^{\psi_2(z)} \left(\int_{\varphi_1(y,z)}^{\varphi_2(y,z)} f(x, y, z)\, dx \right) dy \right) dz$$

where y and z are treated as constants in the innermost (dx) integral, and z is treated as a constant in the middle (dy) integral.

Other orders of integration are possible and sometimes necessary to make the integration feasible. In general, the rules on the limits on a triple integral are as follows:

- The limits for the outer integral are constants.
- The limits for the middle integral can involve only one variable (that in the outer integral).
- The limits for the inner integral can involve two variables (those on the two outer integrals).

§25.2 Concept Test

Question 140. □

What does the integral $\int_0^1 \int_0^1 \int_0^1 2dzdydx$ *represent?*

(a) *Twice the volume of a cube of side* 1.

(b) *The volume of a cube of side length* 1.

(c) *Twice the volume of a sphere of radius* 1.

(d) *The volume under the plane* $z = 2$ *and over a square of side length 1 in the xy-plane.*

Question 141. □

For each integral (a) - (d) that makes sense, match it with its region of integration, I or II.

(i) $\displaystyle\int_1^3\int_{y-1}^2\int_0^y f(x,y,z)\,dz\,dx\,dy$ *(ii)* $\displaystyle\int_1^3\int_0^y\int_2^{y-1} f(x,y,z)\,dx\,dy\,dz$

(iii) $\displaystyle\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$ *(iv)* $\displaystyle\int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dx\,dy$

I. *The region below the plane $z = y$ and above the triangle with vertices $(0,1),(2,1),(2,3)$ in the xy-plane.*

II. *The region between the upper hemisphere of $x^2+y^2+z^2 = 1$ and the xy-plane.*

■ Question 142. □

Find a solid W such that the integral

$$I = \iiint_W (x^2 - x)\,dV$$

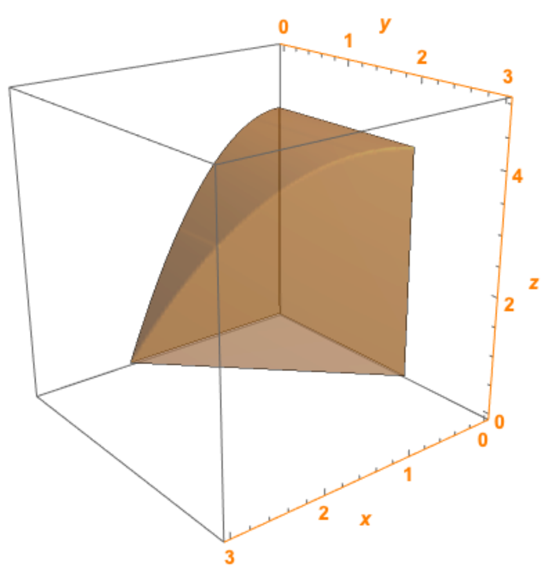
(i) *is positive.* *(ii)* *is negative* *(iii)* *is zero.*

§25.3 Practice Problems

■ Question 143. □

Do the same for the region T in the first octant bounded by the plane $x + y = 2$ and the parabolic cylinder $z = 4 - x^2$.

(i) $dz\,dy\,dx$ *(ii)* $dy\,dz\,dx$ *(iii)* $dy\,dx\,dz$

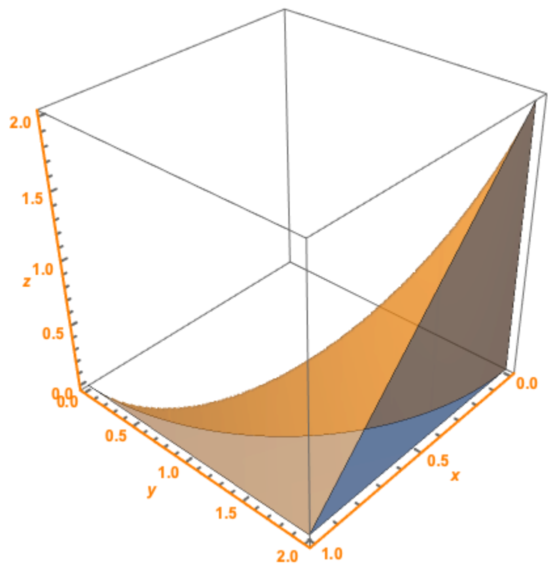


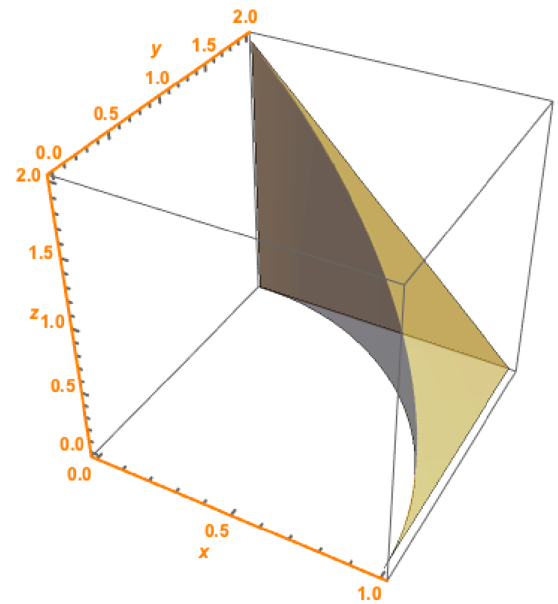
■ Question 144. □

Consider the region T in the first octant bounded by the elliptical cylinder $4x^2 + y^2 = 4$, the plane $2x + z = 2$ and the plane $y = 2$. For the given order of integration, write an iterated integral equivalent to the triple integral $\displaystyle\iiint_T 1\,dV$.

(i) $dx\,dy\,dz$ *(ii)* $dy\,dx\,dz$ *(iii)* $dz\,dx\,dy$

Here are two pictures of the region T *from two different viewpoints.*



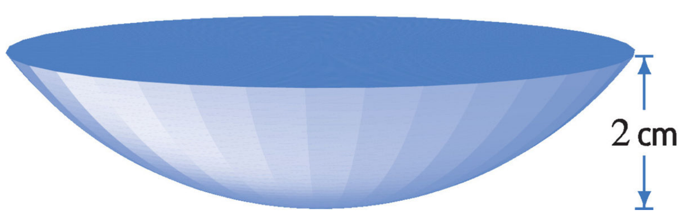


§25.4 Practice Problems with Polar Coordinates

The following exercises require you to set up the base of integration in polar coordinates. The coordinate system (r,θ,z) is called a cylindrical coordinate system. Don't forget the extra r when setting up polar integrals.

■ Question 145. □

A spherical ball of radius 5 *cm is sliced off by a plane that leaves the following solid. Write an iterated triple integral in* $dz\,dr\,d\theta$ *which represents the volume of this region.*

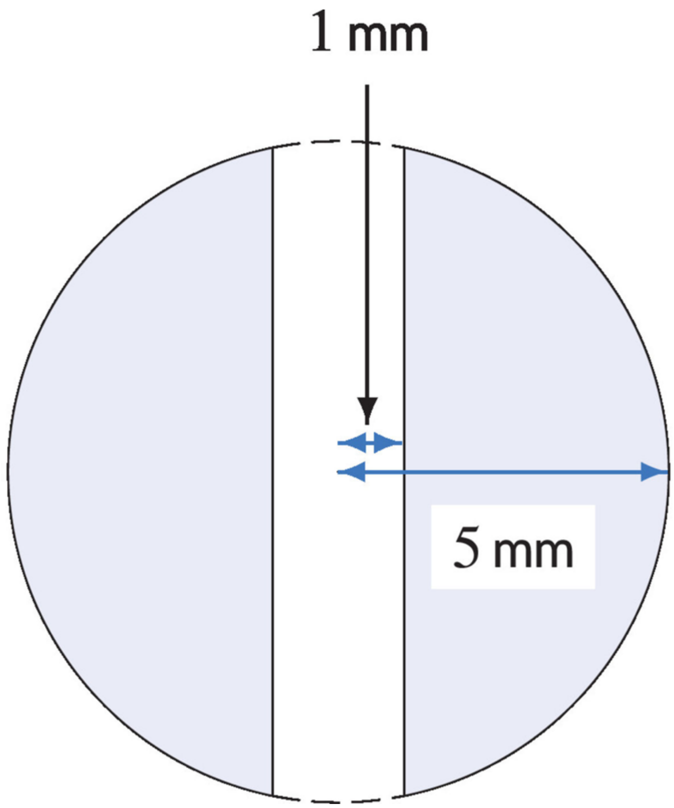


■ Question 146. □

A bead is made by drilling a cylindrical hole of radius 1 *mm through a sphere of radius* 5 *mm.*

(a) *Set up a triple integral in* $dz\,dr\,d\theta$ *that represents the volume of the bead.*

(b) *Evaluate the integral.*


1 mm
5 mm

Chapter 26 | Change of Variables in Multiple Integrals

In one-dimensional calculus, we often need to use a change of variable to evaluate integrals. This is known as the technique of u-substitution. The formula goes as follows:

$$\int_a^b f(x)\,\mathrm{d}x = \int_c^d f(g(u))g'(u)\,\mathrm{d}u$$

where $x = g(u)$, $a = g(c)$, and $b = g(d)$. Note that we have reversed the roles of x and u, so f is a function of x and x is a function of u. This chapter deals with the question of how to deal with change of variables in double or triple integrals.

Example 0.34

We have already seen an example of a change of variables for double integrals: conversion to polar coordinates. The new variables r and θ are related to the old variables x and y by the equations

$$x = x(r,\theta) = r\cos\theta, \quad y = y(r,\theta) = r\sin\theta$$

and the change of variables formula can be written as

$$\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_S \underbrace{f(r\cos(\theta), r\sin(\theta))}_{\text{analogue of } f(g(u))} \quad \underbrace{r}_{\text{analogue of } g'(u)} \quad \mathrm{d}r\,\mathrm{d}\theta$$

where S is the region in the $r\theta$-plane that corresponds to the region R in the xy-plane.

§26.1 One-to-one Transformation

More generally, we are going to consider a change of variables that is given by a transformation T from the uv-plane to the xy-plane:

$$T(u,v) = (x,y)$$

where

$$x = x(u,v), \quad y = y(u,v)$$

Question 147. □

If we write x and y as functions of new variables u and v so if $x = 2u + 3v$ and $y = u - v$, what are the image of the following points in the $u - v$ in the xy plane?

(a) $(0,0)$

(b) $(3,0)$

(c) $(3,1)$

(d) $(0,1)$

We usually assume that T is a C^1 transformation, which means that x and y have continuous first-order partial derivatives with respect to u and v. Additionally, we will assume that the function T is a bijection, so that T^{-1} exists.

Question 148.

Find the transformation from the xy plane into the uv plane for $u = 2x - 3y$ and $v = y - x$. In other words, find equations for x and y in terms of u and v.

Question 149.

A transformation is defined by the equations

$$x = u^2 - v^2, \quad y = 2uv.$$

Find the image of the square $S = \{(u,v) \mid 0 \le u \le 1, 0 \le v \le 1\}$.

Hint: First find the images of the boundary curves.

§26.2 The Jacobian

Now let's see how a change of variables affects a double integral. The first observation is that image of a rectangle in the uv-plane is not necessarily a rectangle any more. So we can't simply replace $\mathrm{d}u\,\mathrm{d}v$ with $\mathrm{d}x\,\mathrm{d}y$. But we can approximate the area as a cross product of the boundary curves, which can be in turn approximated using tangent lines.

To write down the correct formula, we first need to define a quantity called the Jacobian.

Definition 2.35: The Jacobian

The Jacobian of the transformation T given by $x = x(u,v)$ and $y = y(u,v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Question 150.

Compute the Jacobians for the transformations in the previous section.

Theorem 2.36

Suppose that T is a one-to-one C^1 transformation whose Jacobian is nonzero on the interior of a region S in uv-plane and that T maps the region S in the uv-plane onto a region R in the xy-plane. If $f(x,y)$ is continuous on R, then

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v$$

So to perform a change of variables in a double integral, we need to complete three steps:

(a) Substitute for x and y in the integrand in terms of u and v.

(b) Use the absolute value of the Jacobian to change the $\mathrm{d}A$ term.

(c) Change the bounds of the integrals accordingly.

■ Question 151.

Check that the Jacobian for the change to polar coordinates is equal to r.

■ Question 152.

Find the absolute value of the Jacobian for the given change of coordinates.

(a) $x = 5s + 2t, y = 3s + t$

(b) $x = s^2 - t^2, y = 2st$

(c) $x = e^s \cos t, y = e^s \sin t$

(d) $x = s^3 - 3st^2, y = 3s^2t - t^3$

■ Question 153.

Find the region R *in the* xy*-plane corresponding to the region* $\mathrm{T} = \{(s,t) \mid 0 \le s \le 3, 0 \le t \le 2\}$ *under the change of coordinates* $x = 2s - 3t, y = s - 2t$. *Check that*

$$\int_{\mathrm{R}} dx dy = \int_{\mathrm{T}} \left| \frac{\partial(x,y)}{\partial(s,t)} \right| ds dt.$$

■ Question 154.

Use the change of coordinates $s = x + y, t = y$ *to find the area of the ellipse* $x^2 + 2xy + 2y^2 \le 1$.

26.2.1 Harder Problems

■ Question 155.

Consider $\iint_{\mathrm{R}} y^2 \, \mathrm{dA}$ *where* R *is the region in the upper half-plane bounded by the parabolas* $x = y^2, x = y^2 - 9, x = 4 - y^2$, *and* $x = 16 - y^2$. *Use change of variables to convert* R *into a rectangular region and set up the new integral.*

■ Question 156.

Use the change of variables $x = u^2 - v^2, y = 2uv$ *to evaluate the integral* $\iint_{\mathrm{R}} y \, \mathrm{dA}$, *where* R *is the region bounded by the x-axis and the parabolas* $y^2 = 4 - 4x$ *and* $y^2 = 4 + 4x$, $y \ge 0$.

■ Question 157.

Evaluate the integral $\iint_{\mathrm{R}} e^{(x+y)/(x-y)} \, \mathrm{dA}$, *where* R *is the trapezoidal region with vertices* $(1,0), (2,0), (0,-2)$, *and* $(0,-1)$.

Chapter 27 | Vector Fields

§27.1 Motivation

Given a function $f(x,y)$ of two variables, consider what the gradient function $\vec{\nabla} f(x,y)$ represents. To each point in the plane it associates a vector that represents the magnitude and direction greatest increase of $f(x,y)$ at the at point. We have learned earlier that vector $\vec{\nabla} f(x,y)$ is always perpendicular to the contour plot of f through the point (x,y). Using this knowledge we can draw all these vectors in XY-plane as in the following picture:

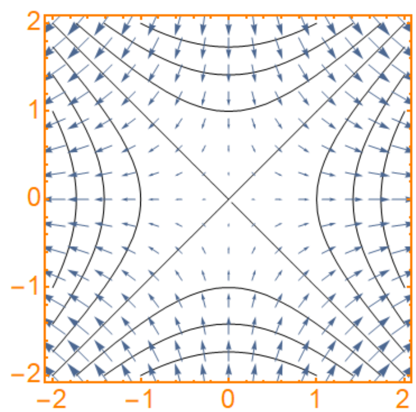


Figure 27.1: Contour Plot and Gradients of $f(x,y) = x^2 - y^2$

Thus $\vec{\nabla} f$ is a function that takes vector inputs (position vector of a point) and gives a vector output. In general, functions of this kind are given a special name: they are called **Vector Fields**.

Definition 1.37

A vector field in 2-space is a function

$$\vec{\mathbf{F}}(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$$

whose value at a point (x,y) is a 2-dimensional vector. Similarly, a vector field in 3-space is a function $\vec{\mathbf{F}}(x,y,z)$ whose values are 3-dimensional vectors.

There are many other examples of vector fields that we come across in real life frequently e.g. Ocean currents, wind flow in your weather app, electromagnetic force field, gravitational force etc. It is not necessary that a vector field $\vec{\mathbf{F}}$ comes from gradient of a function.

§27.2 Basic Examples

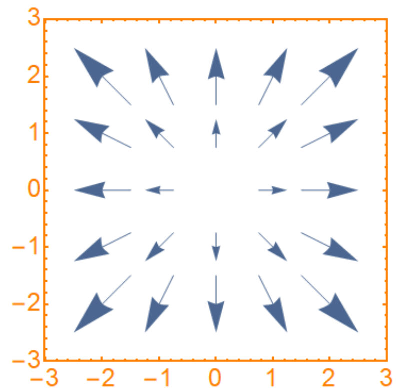
Example 2.38

Consider the vector field $\vec{\mathbf{F}}(x,y) = x\hat{i} + y\hat{j}$. We are interested in what it looks like.

- Along the x-axis, we have $y = 0$, so the arrows are purely horizontal.

- Along the y-axis, we have $x = 0$, so the arrows are purely vertical.
- What about the arrow at other points? According to the formula, the vector at the point (x, y) is equal to its position vector. So the arrow points radially outward and has length equal to the distance of the point from the origin.

With these observations, our vector field looks like:



■ Question 158.

Plot the following vector fields:

(a) $\vec{\mathbf{F}}(x, y) = y\hat{\imath}$

(b) $\vec{\mathbf{F}}(x, y) = (x + y)\hat{\imath} + (x - y)\hat{\jmath}$

■ Question 159.

Give examples of vector fields where:

(a) *all output vectors are parallel to the x-axis, and have the same magnitude if they are located on the same vertical line.*

(b) *all output vectors are unit length, and perpendicular to the position vectors* $x\hat{\imath} + y\hat{\jmath}$.

§27.3 Gradient Vector Field

A vector field $\vec{\mathbf{F}}(x, y)$ is called a gradient vector field if it can be written as the gradient of some underlying function $f(x, y)$ i.e.

$$\vec{\mathbf{F}} = \vec{\nabla} f = \langle f_x, f_y \rangle$$

In that case, the function f is called the potential function for the field $\vec{\mathbf{F}}$ (the terminology comes from Physics and has to do with potential energy of objects).

■ Question 160.

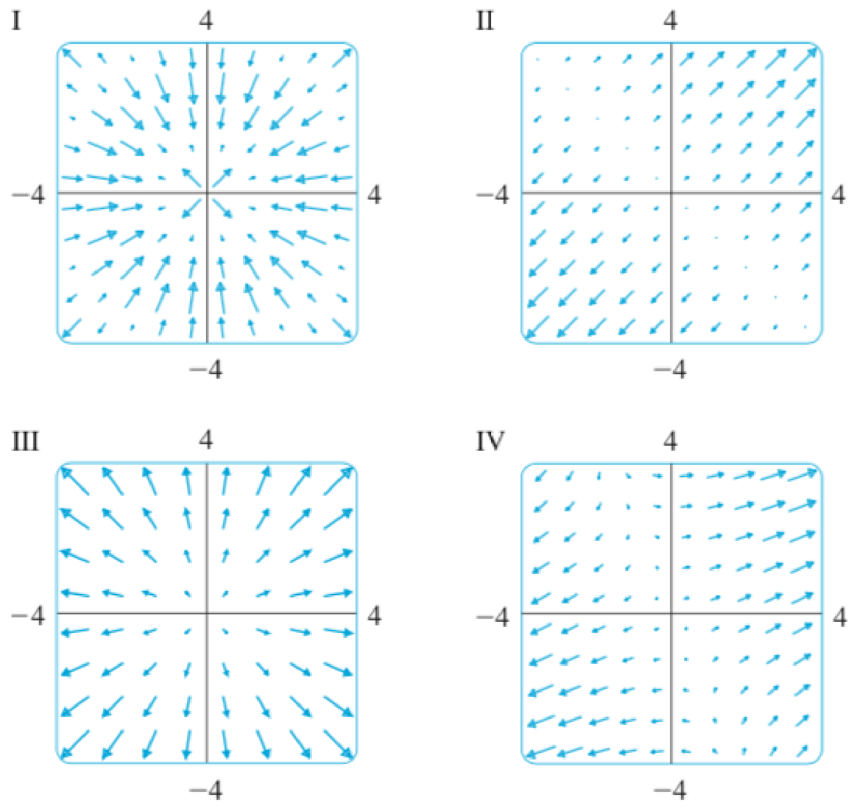
Match the following functions with their gradient vector fields.

(a) $x^2 + y^2$

(b) $x(x + y)$

(c) $(x + y)^2$

(d) $\sin\sqrt{x^2 + y^2}$



Gradient vector fields have some very nice properties (which we'll explore soon), so it will be important for us to identify when a given vector field is a gradient vector field. Recall that Clairaut's theorem says $f_{xy} = f_{yx}$ for a smooth function. That means for a gradient vector field $\vec{\mathbf{F}} = \vec{\nabla} f$, we must have $P_y = Q_x$. As a contrapositive, we have the following test.

Theorem 3.39

If $\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$, *then* $\vec{\mathbf{F}}$ *is not a gradient vector field.*

Warning: The natural follow-up question to ask is whether the equality of the two partials then guarantees a gradient vector field. Unfortunately, we cannot yet answer this question with the material covered so far. We will show that the answer is affirmative in a couple of chapters.

27.3.1 Finding the potential function

Assume that you are told that a given vector field $\vec{\mathbf{F}}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$ is in fact a gradient vector field. How do we then find the exact formula of a function $f(x,y)$ such that $\vec{\mathbf{F}} = \vec{\nabla} f$?

Here is an algorithm. Note that we want to find f such that

$$f_x = P \text{ and } f_y = Q$$

Integrating the first equation with respect to x gives us

$$f(x,y) = \int P(x,y)dx + g(y)$$

The extra $g(y)$ serves as a replacement for the arbitrary constant term '$+c$' because $\dfrac{\partial g(y)}{\partial x} = 0$. With this presumptive model for $f(x,y)$ we must have

$$\frac{\partial}{\partial y}\left(\int \mathrm{P}(x,y)dx + g(y)\right) = \mathrm{Q}$$

From here you can solve for g. We can also start from Q if that seems easier.

It's best to learn the algorithm by doing examples:

Example 3.40

Suppose $\vec{\mathbf{F}}(x,y) = (x+y)\hat{\imath} + (x-y)\hat{\jmath} = \vec{\nabla} f(x,y)$. Then $f_x = x+y$ and $f_y = x-y$. The first equation tells us

$$f(x,y) = \frac{x^2}{2} + xy + g(y)$$

Then

$$x - y = \frac{\partial}{\partial y}\left(\frac{x^2}{2} + xy + g(y)\right) = 0 + x + g'(y) \implies g'(y) = -y \implies g(y) = -\frac{y^2}{2} + c$$

So overall, $f(x,y) = \frac{x^2}{2} + xy - \frac{y^2}{2} + c$.

Question 161.

For each of the following vector fields, find a potential function or explain why one doesn't exists.

(a) $x\hat{\imath}$

(b) $y\hat{\imath}$

(c) $(x^2 - y^2)\hat{\imath} - 2xy\hat{\jmath}$

(d) $(2xy + 5)\hat{\imath} + (x^2 + 8y^3)\hat{\jmath}$

(e) $2xe^{x^2}\sin y\hat{\imath} + e^{x^2}\cos y\hat{\jmath}$

(f) $\left(yze^{xyz} + z^2\cos\left(xz^2\right)\right)\hat{\imath} + xze^{xyz}\hat{\jmath} + \left(xye^{xyz} + 2xz\cos\left(xz^2\right)\right)\hat{k}$

(g) $\dfrac{-y\hat{\imath} + x\hat{\jmath}}{x^2 + y^2}$

(h) $\dfrac{\vec{r}}{\|\vec{r}\|^3} = \dfrac{x}{(x^2+y^2+z^2)^{3/2}}\hat{\imath} + \dfrac{y}{(x^2+y^2+z^2)^{3/2}}\hat{\jmath} + \dfrac{z}{(x^2+y^2+z^2)^{3/2}}\hat{k}$

Question 162.

Match the following vector fields with their drawings in fig. 27.2.

(a) $\langle y, 1\rangle$ $\boxed{3}$

(b) $\langle 0, 2y\rangle$ $\boxed{9}$

(c) $\langle -x, -2y\rangle$ $\boxed{2}$

(d) $\langle -2y, 3x\rangle$ $\boxed{7}$

(e) $\langle 0, x^2 y\rangle$ $\boxed{8}$

(f) $\langle -2y, -x\rangle$ $\boxed{1}$

(g) $\langle x^2 y, 0\rangle$ $\boxed{4}$

(h) $\langle -x, 0\rangle$ $\boxed{11}$

(i) $\langle -2y, 1\rangle$ $\boxed{5}$

(j) $\langle -y - x, x\rangle$ $\boxed{12}$

(k) $\langle -y, x\rangle$ $\boxed{10}$

(l) $\langle x^2, y^2\rangle$ $\boxed{6}$

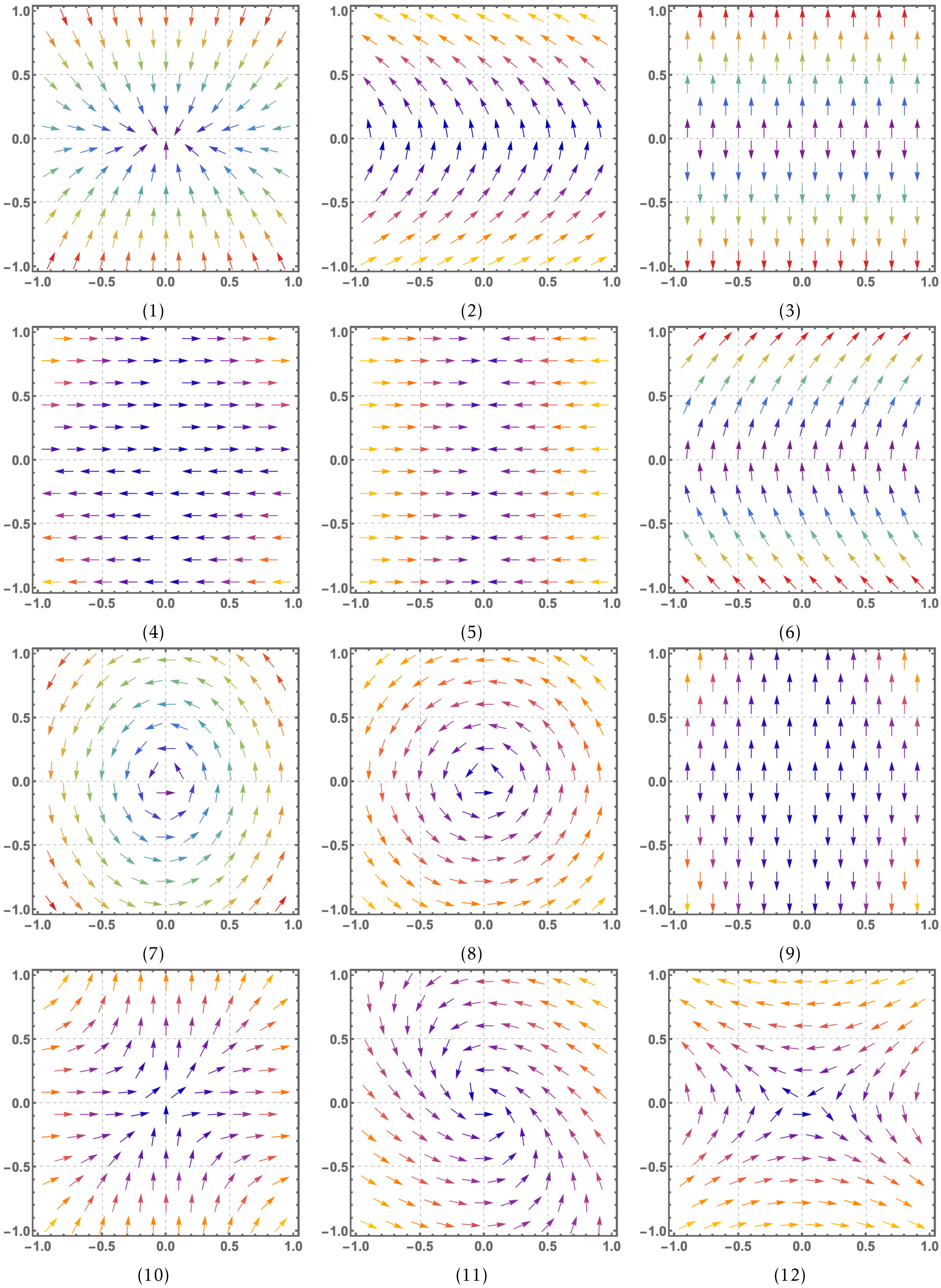


Figure 27.2

Chapter 28 | Flow Lines of a Vector Fields

§28.1 Motivation

In the last lecture we defined vector fields and learned how to graphically represent them on coordinate plane. For a practical application, consider a ocean current map that depicts the water velocity vector field and suppose we are interested in predicting the path of an iceberg that just showed up. The iceberg will obviously try to "go with the flow" and move along a curve such that it follows the arrow directions. In this chapter, we are interested in finding equations of those curves given the vector field. It will be helpful to follow along in the `lec25_flow_line.nb` file.

§28.2 Vector Field along a Curve

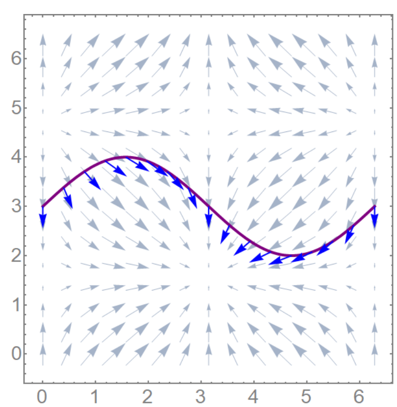
Instead of looking at all of the vectors $\vec{\mathbf{F}}(x,y)$ in the plane, suppose we focus only on those along a given parameterized curve $\vec{r}(t) = \langle x(t), y(t)\rangle$. To express this symbolically, we write $\vec{\mathbf{F}}(\vec{r}(t))$ and interpret the input $\vec{r}(t)$ to the vector field as the position vector of the point $(x(t), y(t))$ coming from the parameterization.

Example 2.41

For the vector field $\vec{\mathbf{F}}(x,y) = \sin(x)\vec{i} + \cos(y)\vec{j}$ and the parameterized curve $\vec{r}(t) = \langle t, \sin(t)+3\rangle$, we get the (blue) vectors

$$\vec{\mathbf{F}}(\vec{r}(t)) = \vec{\mathbf{F}}(x(t),y(t)) = \sin(x(t))\vec{i} + \cos(y(t))\vec{j} = \sin(t)\vec{i} + (\cos(\sin(t)+3))\vec{j}$$

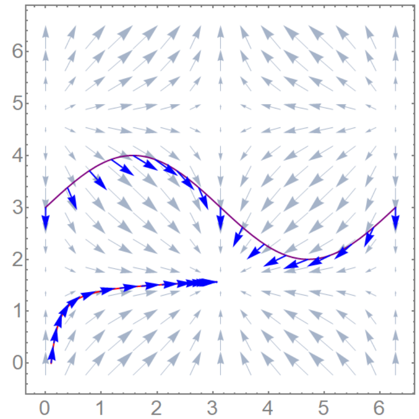
along the (purple) parameterized curve.



See Curve through vector field in `flow_line.nb`

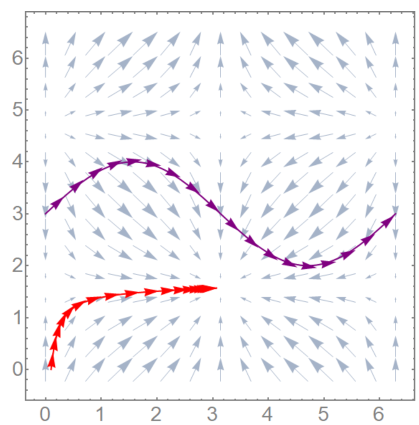
§28.3 Flow Lines

There are special curves which fit the vector field perfectly, so that the arrows from the vector field are exactly the same as the velocity vectors associated with the parameterized curve. These curves are called flow lines because they "go with the flow" of the vector field (they aren't always lines, but that's what we call them).



See Flow line through vector field in `flow_line.nb`

In the image above, the shorter red curve "goes with the flow" of the vector field. To study flow lines carefully, we need to express this idea via an equation. We recall that the velocity vector $\vec{r}'(t)$ represents the tangent direction of the parameterized curve at any t. The image below shows the velocity vectors along our two different curves through the vector field.



See Velocity vectors along curves in `flow_line.nb`

Definition 3.42

A **flow line** of a vector field $\vec{\mathbf{F}}(x, y)$ is a path whose velocity vector equals the vector field along the path. Thus,

$$\vec{r}'(t) = \vec{\mathbf{F}}(\vec{r}(t))$$

The **flow** of a vector field is the family of all of its flow lines.

Obviously, not all parameterized curves $\vec{r}(t)$ satisfy this flow line equation (e.g., the purple curve in the above image), but we would like to identify the ones that do. This is relatively easy to do visually, however coming up with the formulas $\vec{r}(t)$ for the flow lines is not always so easy.

§28.4 Identifying Flow Line formula

To identify the formula for the parameterized curve, we need to solve the flow line equation. If $\vec{\mathbf{F}}(x,y) = \mathrm{P}(x,y)\hat{i} + \mathrm{Q}(x,y)\hat{j}$ then the equation can be rewritten as a system

$$\frac{dx}{dt} = \mathrm{P}(x,y)$$
$$\frac{dy}{dt} = \mathrm{Q}(x,y)$$

This is called a system of Differential Equations. There are no general methods of solving such a system and the solutions can be quite complicated. However, given a solution curve, it is easy to check that the curve indeed satisfies the differential equations.

Example 4.43

Let's take a look at some easy differential equations for which we can guess a solution.

Suppose $\vec{\mathbf{F}}(x,y) = 3\hat{i} + 2\hat{j}$. The corresponding differential equations are

$$x'(t) = 3, \qquad y'(t) = 2$$

So the flow line curves are $\vec{r}(t) = \langle 3t + c_1, 2t + c_2 \rangle$. Specific values of the arbitrary constants c_1 and c_2 in the formula determines which specific flow line we are looking at.

Question 163. □

Determine the flow lines of the following vector fields.

(i) $2\hat{i} + x\hat{j}$ *(ii)* $-y\hat{i} + x\hat{j}$

Warning: The differential equation

$$x'(t) = \sin(x)$$

is not the same as

$$x'(t) = \sin(t)$$

The later has the solution $x(t) = c - \cos(t)$ whereas the former has solution $x(t) = 2\operatorname{arccot}\left(e^{-t-c}\right)$.

§28.5 Practice Problems

Question 164. □

Consider a vector field $\vec{\mathbf{F}} = -h_y\hat{i} + h_x\hat{j}$ *where* $h(x,y)$ *is a smooth function. Explain why*

(a) $\vec{\mathbf{F}}$ *is perpendicular to* $\vec{\nabla}h$.

(b) the flow lines of $\vec{\mathbf{F}}$ *are along the level curves of* h.

Question 165. □

Show that every flow line of the vector field $\vec{\mathbf{F}}(x,y) = y\hat{i} + x\hat{j}$ *lies on a level curve for the function* $f(x,y) = x^2 - y^2$.

Question 166.

☐

(a) *Show that* $h(t) = e^{-2at}\left(x^2 + y^2\right)$ *is constant along any flow line of* $\vec{\mathbf{F}} = (ax - y)\hat{i} + (x + ay)\hat{j}$.

(b) *Show that points moving with the flow that are on the unit circle centered at the origin at time* 0 *are on the circle of radius* e^{at} *centered at the origin at time* t.

§28.6 Conceptual Problems

Question 167.

☐

Fill the boxes with 'certainly', 'possibly', or 'certainly not'.

(a) *The plot of the vector field* $\vec{\mathbf{G}}(x, y) = \vec{\mathbf{F}}(2x, 2y)$ *is* $\boxed{}$ *drawn by doubling the length of all the arrows in the plot of* $\vec{\mathbf{F}}(x, y)$.

(b) *If one paprametrization of a curve is a flow line for a vector field, then all of its paramterizations are* $\boxed{}$ *flow lines for the vector field.*

(c) *If the flow lines for the vector field* $\vec{\mathbf{F}}(x, y)$ *are all concentric circles centered at the origin, then the dot-product* $\vec{\mathbf{F}}(x, y) \cdot (x\hat{i} + y\hat{j})$*is* $\boxed{}$ *equal to zero.*

(d) *If the flow lines for the vector field* $\vec{\mathbf{F}}(x, y)$ *are all straight lines parallel to the constant vector* $\vec{v} = 3\hat{i} + 5\hat{j}$, *then* $\vec{\mathbf{F}}(x, y)$ *is* $\boxed{}$ *equal to* $\vec{v}$.

(e) *The flow lines of the vector field* $\vec{\mathbf{F}}(x, y) = e^x\hat{i} + y\hat{j}$ $\boxed{}$ *cross the* X*-axis.*

Question 168.

☐

Are the following true or false? If $\vec{r}(t)$ *is a flow line for a vector field* $\vec{\mathbf{F}}$, *then*

(a) $\vec{r}_1(t) = \vec{r}(t - 5)$ *is a flow line for the same vector field* $\vec{\mathbf{F}}$.

(b) $\vec{r}_2(t) = \vec{r}(2t)$ *is a flow line for the vector field* $2\vec{\mathbf{F}}$.

(c) $\vec{r}_3(t) = 2\vec{r}(t)$ *is a flow line for the vector field* $2\vec{\mathbf{F}}$.

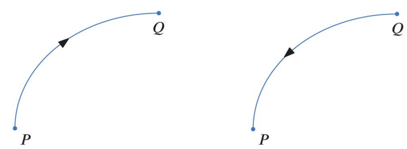
Chapter 29 | Line Integrals on Parameterized Curves

§29.1 Motivation

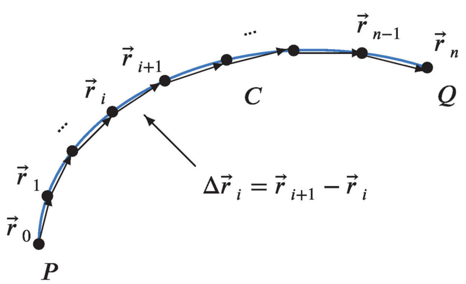
Now that we know about parameterized curves and vector fields, we are ready to introduce a new kind of integral which uses both of these concepts. The line integral measures the extent to which a curve in a vector field is, overall, going with the vector field or against it. For instance, a flow line goes completely with the flow of its vector field, so it should have a comparatively high line integral value.

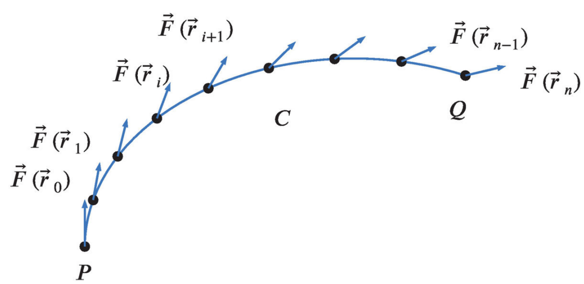
§29.2 Definition

Suppose we are given a vector field $\vec{\mathbf{F}}(x,y)$ and a parametrized curve $\mathrm{C} : \vec{\boldsymbol{r}}(t), a \leq t \leq b$. First we will need to fix an **orientation** of C so that we know the direction of travel along the curve.



We will assume that the orientation is as in the first picture. Next, recall that we can use dot product to measure the extent to which two vectors point in the same or opposing directions. So we will break our parameterized curve C from $\vec{\boldsymbol{r}}(a) = \mathrm{P}$ to $\vec{\boldsymbol{r}}(b) = \mathrm{Q}$ into a linked trail of "secant" vectors $\Delta\vec{\boldsymbol{r}}_i$, connecting points along the curve, and then compare each of these vectors to the corresponding vector from the vector field vector at the same points.





We then add the dot products and create a Riemann sum whose limit gives us the Line Integral.

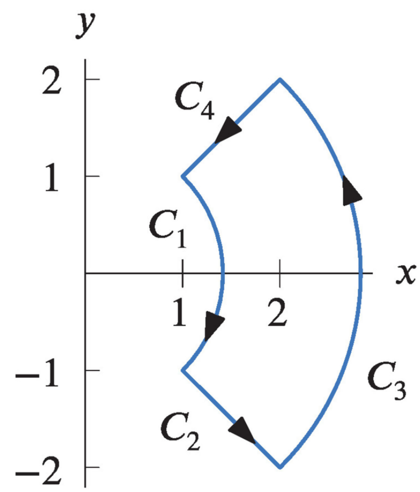
$$\int_{\mathrm{C}} \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{r}} = \lim_{\|\Delta\vec{\boldsymbol{r}}_i\| \to 0} \sum_{i=0}^{n-1} \vec{\mathbf{F}}(\vec{\boldsymbol{r}}_i) \cdot \Delta\vec{\boldsymbol{r}}_i$$

Note that the value of the line integral also defines the work done by the force $\vec{\mathbf{F}}$ as displacement happens along the curve C, which is an important quantity in physics.

Question 169. □

For each curve C_i in the picture and for each of the following vector fields, determine if possible whether $\int_{\mathrm{C}} \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{r}}$ would be positive, negative, or zero.

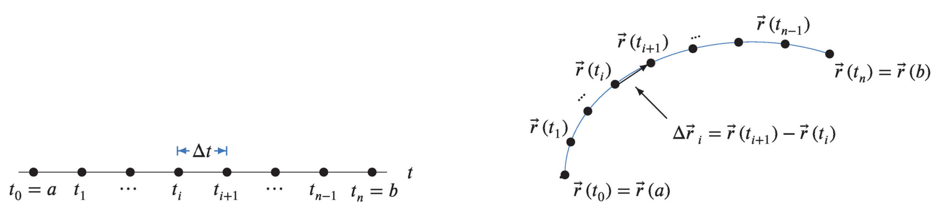
(i) $x\hat{i}$ *(ii)* $y\hat{i} - x\hat{j}$



§29.3 Evaluating a line integral

The key to evaluating line integration is to relate the secant vectors $\Delta\vec{r}_i$ to the velocity vectors $\vec{r}'(t_i)$ as

$$\Delta\vec{r}_i \approx \vec{r}'(t_i)\Delta t$$



As $\Delta t \to 0$, we then find the limit of the Riemann sum as

$$\int_C \vec{\mathbf{F}} \cdot d\vec{r} = \lim_{\Delta t\to 0} \sum_{a\le t\le b} \vec{\mathbf{F}}(\vec{r}(t)) \cdot \vec{r}'(t)\Delta t = \int_a^b \vec{\mathbf{F}}(\vec{r}(t)) \cdot \vec{r}'(t)dt$$

Observe that this is essentially a u−substitution, where we are changing the variable of integration to t.

Example 3.44

Let's compute the line integral

$$\int_C \langle x-y, x\rangle \cdot d\vec{r}$$

along the upper-half C of the unit circle traversed counterclockwise. First we need to parameterize C and find $\vec{r}'(t)$.

$$\vec{r}(t) = \cos t\hat{i} + \sin t\hat{j}, \qquad 0 \le t \le \pi$$

So $\vec{r}'(t) = \langle -\sin t, \cos t\rangle$. Then using the formula for line integrals,

$$\begin{aligned}
\int_C \langle x-y, x\rangle \cdot d\vec{r} &= \int_a^b \vec{\mathbf{F}}(\vec{r}(t)) \cdot \vec{r}'(t)dt \\
&= \int_0^\pi \underbrace{\langle \cos(t)-\sin(t), \cos(t)\rangle}_{\vec{\mathbf{F}}(\vec{r}(t))} \cdot \underbrace{\langle -\sin(t), \cos(t)\rangle}_{\vec{r}'(t)} dt \\
&= \int_0^\pi \Big(-\cos(t)\sin(t) + \sin^2(t) + \cos^2(t)\Big) dt \\
&= \int_0^\pi -\cos(t)\sin(t)\,\mathrm{d}t + \int_0^\pi 1 dt \\
&= \left(\frac{\cos^2(t)}{2} + t\right)\Bigg]_0^\pi = \pi
\end{aligned}$$

■ Question 170. □

Compute the line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{r}$ *for the following pairs of vector fields* $\vec{\mathbf{F}}$ *and curves* C.

(a) $\vec{\mathbf{F}} = \langle y, x\rangle$ *and* C *is the quarter-circle centered at the origin starting at* $(2,0)$ *and proceeding counterclockwise to* $(0,2)$.

(b) $\vec{\mathbf{F}} = \langle y, x\rangle$ *and* C *is the line segment starting at* $(2,0)$ *and proceeding counterclockwise to* $(0,2)$

(c) $\vec{\mathbf{F}} = \langle x, y, -2z\rangle$ *and* C *is the "twisted cubic"* $\vec{r}(t) = \langle t, t^2, t^3\rangle$ *from* $t=0$ *to* $t=1$

(d) $\vec{\mathbf{F}} = \langle x, y, -2z\rangle$ *and* C *is the line segment from* $(0,0,0)$ *to* $(1,1,1)$

■ Question 171. □

True or False? *The line integral of a vector field along a path depends on the parameterization of the curve.*

Chapter 30 | The Fundamental theorem of Line Integrals

§30.1 Introduction

Recall the Fundamental Theorem of (single-variable) Calculus:

$$\int_a^b f'(x)dx = f(b) - f(a)$$

This essentially says that integration is the opposite operation of differentiation (that's the reason we call the indefinite integral the anti-derivative). In this chapter we'll see a multivariable version of the Fundamental Theorem of Calculus, which uses the line integral of a (continuous) gradient $\vec{\mathbf{F}} = \nabla f$. Among other things, this result will allow us to compute some line integrals quickly and without resorting the approach from last time involving parameterization.

§30.2 The Fundamental theorem of Line Integrals

30.2.1 Formulation

The single-variable FTC says that, to evaluate the integral of a function f' over an interval $[a,b]$, we find its antiderivative f, plug in the endpoints and subtract. By analogy, think of the potential function $f(x,y)$ as being an "antiderivative" for the gradient field $\vec{\mathbf{F}} = \nabla f$. And let C be a piece-wise smooth oriented path with starting point A and endpoint B. Then we have the following theorem

Theorem 2.45: The Fundamental Theorem for Line Integral

$$\int_{\mathrm{C}} \vec{\nabla} f \cdot d\vec{r} = f(\mathrm{B}) - f(\mathrm{A})$$

Before we prove this to be true, note how it simplifies the computation of line integrals for gradient fields! Instead of parametrizing C, we just have to identify its endpoints, find the potential function, and plug in the endpoints and subtract.

Proof of theorem 45.

Suppose C is parametrized by $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, $a \le t \le b$. Recall that from last chapter, we have

$$\begin{aligned}\int_{\mathrm{C}} \vec{\nabla} f \cdot d\vec{r} = \int_a^b \vec{\nabla} f \cdot \vec{r}'(t)\,dt &= \int_a^b \langle f_x, f_y\rangle \cdot \langle x'(t), y'(t)\rangle\,dt \\ &= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\right)dt\end{aligned}$$

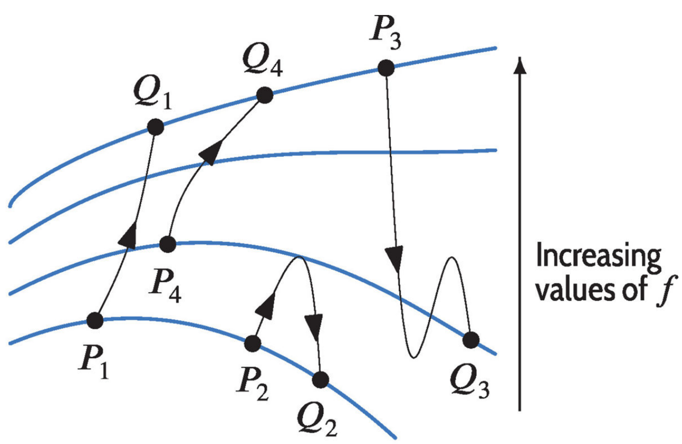
$$= \int_a^b \frac{df}{dt}\,dt$$

$$= \int_A^B df = f(B) - f(A)$$

■

Question 172.

Consider the line integrals, $\int_{C_i} \vec{\mathbf{F}} \cdot d\vec{r}$ *for* $i = 1, 2, 3, 4$, *where* $\vec{\mathbf{F}}(x, y) = \vec{\nabla} f$ *and* C_i *is the path from* P_i *to* Q_i *shown below. Some level curves of f are also shown in the figure.*



(a) *Which of the line integral(s) is (are) zero?*

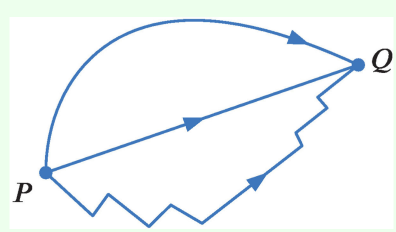
(b) *Arrange the four line integrals in ascending order (from least to greatest).*

(c) *Two of the nonzero line integrals have equal and opposite values. Which are they? Which is negative and which is positive?*

§30.3 Path Independence

Definition 3.46

A vector field is called **path-independent** or **conservative** if the value of its line integral is the same along any paths having the same endpoints.



In other words, the line integrals depend only on the starting and ending points and NOT on the particular path between those two points. Here 'path' means a piecewise smooth curves C that avoid any locations where $\vec{\mathbf{F}}$ is undefined.

How can we tell whether a given vector field is path-independent or path-dependent, without computing a bunch of line integrals to check? Note that with this new nomenclature, the FTLI can be stated as

$$\boxed{\text{Gradient Vector Field} \implies \text{Path independent}}$$

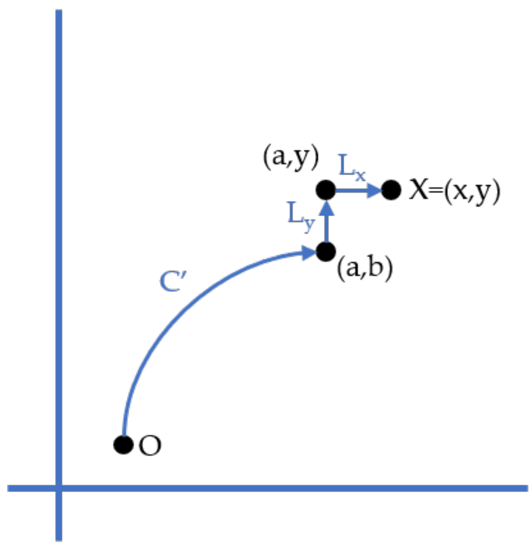
In fact, the converse is true as well. We are going to include the proof below as a digression, but it is not within the scope of the current discussion.

Digression

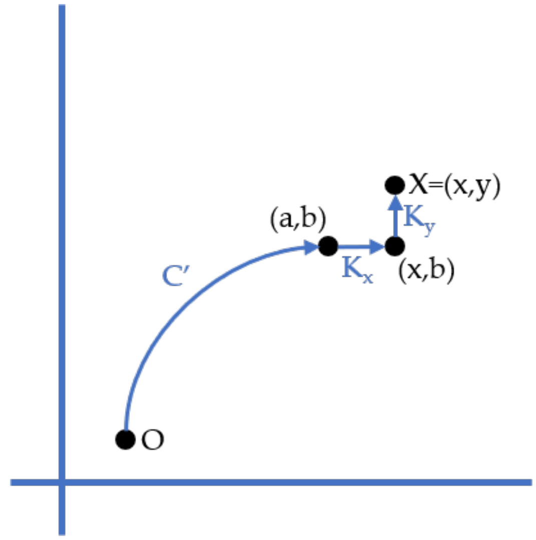
Suppose $\vec{\mathbf{F}} = \mathrm{P}\hat{\imath} + \mathrm{Q}\hat{\jmath}$ is a path-independent vector field. Consider the function f defined as follows:

$$f(x,y) = \int_{\mathrm{C}} \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{r}}$$

where C is a path from a fixed starting point O to a variable point $\mathrm{X} = (x,y)$. This integral has the same value for any path from O to X because $\vec{\mathbf{F}}$ is path-independent. The two figures below then explain why $\vec{\boldsymbol{\nabla}} f = \vec{\mathbf{F}}$.



The path $\mathrm{C}' + \mathrm{L}_y + \mathrm{L}_x$ is used to show $f_x = \mathrm{P}$



The path $C' + K_x + K_y$ is used to show $f_y = Q$

So we can conclude that

$$\boxed{\text{Gradient Vector Field} \iff \text{Path independent}}$$

So it is a natural question to ask if a given vector field is a gradient vector field or not since that would make our life considerably easier! And if it is a gradient vector field, how can find f so that $\vec{F} = \vec{\nabla} f$? Fortunately, we have already answered these questions in a previous chapter.

Warning: Note that we still haven't proved that a vector field is gradient v.f. if $Q_x = P_y$, although it is true in **most** cases.

Question 173.

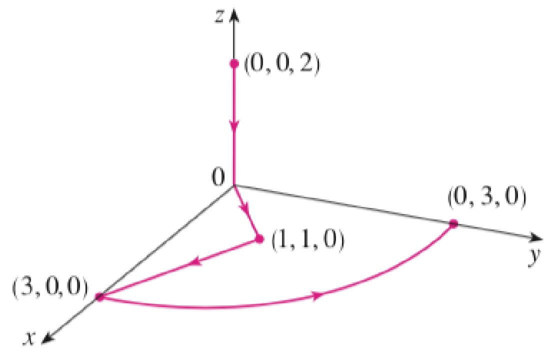
Evaluate the following line integrals. Not all of them can be done using FTLI. Identify the ones that are gradient vector fields.

(a) $\int \vec{F} \cdot d\vec{r}$, *where* $\vec{F}(x,y) = xy\,\vec{i} + x^2\,\vec{j}$ *and* C *is given by* $\vec{r}(t) = \sin t\,\vec{i} + (1+t)\,\vec{j}$, $0 \le t \le \pi$

(b) $\int \vec{F} \cdot d\vec{r}$, *where* $\vec{F}(x,y) = \left(4x^3y^2 - 2xy^3\right)\vec{i} + \left(2x^4y - 3x^2y^2 + 4y^3\right)\vec{j}$ *and* C *is given by* $\vec{r}(t) = (t + \sin \pi t)\,\vec{i} + (2t + \cos \pi t)\,\vec{j}$, $0 \le t \le 1$

(c) $\int \vec{F} \cdot d\vec{r}$, *where* $\vec{F}(x,y,z) = \sin y\,\vec{i} + x\cos y\,\vec{j} - \sin z\,\vec{k}$, *and* C *is the helix* $x = 3\cos t, y = t, z = 3\sin t$ *from* $(3,0,0)$ *to* $(0,\pi/2,3)$

(d) $\int \vec{F} \cdot d\vec{r}$, *where* $\vec{F}(x,y,z) = \left(3x^2yz - 3y\right)\vec{i} + \left(x^3z - 3x\right)\vec{j} + \left(x^3y + 2z\right)\vec{k}$ *and* C *is the curve shown below.*



(e) $\int \vec{\mathbf{F}} \cdot d\vec{r}$, *where*

$$\vec{\mathbf{F}}(x,y,z) = \left\langle 4xe^{2x^2+3y^2+4z^2}, 6ye^{2x^2+3y^2+4z^2}, 8ze^{2x^2+3y^2+4z^2} \right\rangle$$

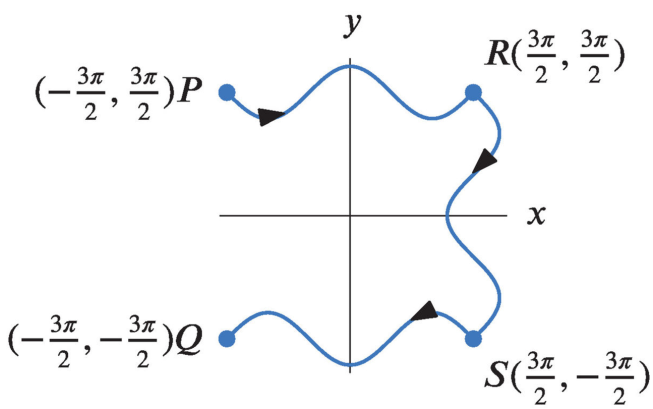
and C *is*

$$\vec{r}(t) = \langle (2+\cos(7t))\cos(t), (2+\cos(7t))\sin(t), \sin(7t) \rangle$$

parameterized by $0 \le t \le \pi$ *starting at* $t = 0$ *and ending at* $t = \pi$.

(f) $\int \vec{\mathbf{F}} \cdot d\vec{r}$, *where* $\vec{\mathbf{F}}(x,y) = xy^2\,\vec{i} + x^2y\,\vec{j}$, *and* C *is* $\vec{r}(t) = \cos t\,\vec{i} + 2\sin t\,\vec{j}$, $\quad 0 \le t \le \pi/2$

(g) $\int \vec{\mathbf{F}} \cdot d\vec{r}$, *where* $\vec{\mathbf{F}}(x,y) = \left(\sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)\right)\hat{i} - \left(\cos\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right)\right)\hat{j}$, *and* C *is as in the picture below:*



§30.4 Circulation Freedom

Definition 4.47

A curve C is called **closed** (or a closed loop) if it starts and ends at the same point.

Line integral on a closed loop is sometimes explicitly denoted by the symbol $\oint$, although it is not necessary.

Definition 4.48

A vector field $\vec{\mathbf{F}}$ is called **circulation-free** if $\oint_C \vec{\mathbf{F}} \cdot d\vec{r} = 0$ for **every** closed path C.

Again, here 'path' means a piecewise smooth curve C that avoids any locations where $\vec{\mathbf{F}}$ is undefined.

One of the immediate observations we could make from the definition is that for a vector field

$$\boxed{\text{Path independent} \implies \text{Circulation-free}}$$

The converse is non-trivial but still easy to prove.

Question 174.

Explain why circulation-free vector fields are path-independent. □

§30.5 Conclusion

Overall, the results of this chapter show the following

$$\boxed{\text{Gradient v.f.} \iff \text{Path independent} \iff \text{Circulation-free}}$$

Chapter 31 | Irrotational Vector Fields

§31.1 Curl and Rotation

Definition 1.49

Given a vector field $\vec{\mathbf{F}} = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$, the curl of the vector field is defined as

$$\operatorname{curl}(\vec{\mathbf{F}}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Of course, this is only defined at points where P and Q are differentiable with respect to y and x respectively.

Definition 1.50

A vector field is called **irrotational** if its curl is zero at every point where curl is defined.

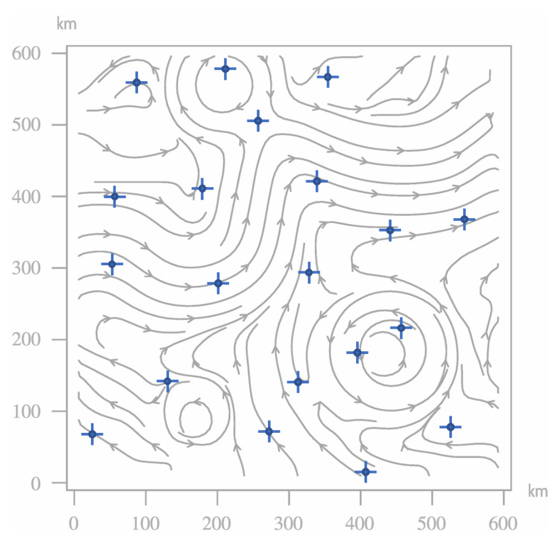
You might recall seeing the terms in $\operatorname{curl}(\vec{\mathbf{F}})$ before. In fact, we have proved the following in an earlier chapter using Clairaut's theorem.

$$\boxed{\text{Gradient vector field} \implies \text{Irrotational}}$$

31.1.1 Understanding Curl

The nomenclature suggests that $\operatorname{curl}(\vec{\mathbf{F}})$ has something to do with rotation in a vector field. There is a way to visually interpret the algebraic quantity that shows how curl corresponds to rotation.

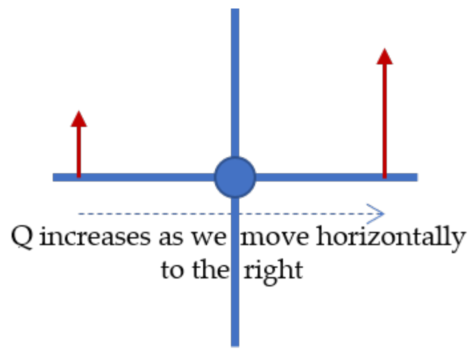
Imagine an infinitesimally small paddle-wheel lying in a vector field $\vec{\mathbf{F}}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$. We will show that as long as $\vec{\mathbf{F}}$ is smooth, the curl measures how much (and which way) it turns at any point.



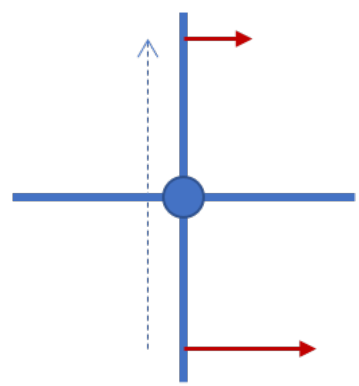
A positive curl means the paddle-wheel will rotate counterclockwise, and a negative curl means the paddle-wheel rotates clockwise. An irrotational vector field has no spinning paddle-wheels anywhere.

In general, the first term $\frac{\partial Q}{\partial x}$ in the curl function measures the effect of the vector field $\vec{\mathbf{F}}$ on the horizontal part of the paddle wheel, and a positive value corresponds to a counter-clockwise contribution to the rotation of the paddle-wheel. This follows because Q is the j-component of the vector field, and its partial derivative with respect to x measures its change when moving in horizontally, keeping y fixed.

The picture below shows a situation where $\frac{\partial Q}{\partial x}$ is **positive** since the (red) vectors representing the j-component $Q(x,y)$ of the vector field increases as we move horizontally. In this case, the paddle-wheel rotates counterclockwise.



Likewise, the second term $\frac{\partial P}{\partial y}$ in the curl function measures the effect of the vector field on the vertical part of the paddle wheel, but now a negative value corresponds to a counterclockwise contribution to the rotation of the paddle-wheel (that's why this term appears in the curl with a negative sign).

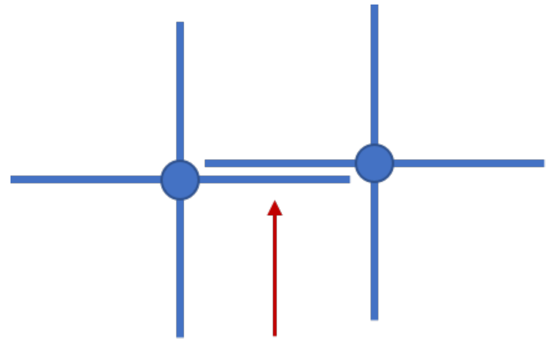


P decreases as we move vertically up

§31.2 Green's Idea: Volume Integrate the Curl

The fact that gradient vector fields are both circulation free and irrotational lead a famous mathematical physicist named George Green to guess that there might be a relationship between circulation integrals to volume integrals of the curl.

To approximate the volume integral of the curl function over a closed bounded region R, we can first break R into smaller rectangular sub-rectangles and then add up the volumes of the rectangular solids with heights equal to the curl at one of the corners of the sub-rectangles.



The key thing to notice here is that the vector field spins "adjacent" paddle-wheels in opposite directions: The above red vector gives counterclockwise spin to the left paddle-wheel and the same amount of clockwise spin to the right paddle-wheel. This means that the curls associated with interior points of the region R will essentially be cancelled out by each other. So, when we take the limit of the volume approximation, all that is left is the spin due to the (counterclockwise) push of the vector field along the boundary of the region. This is measured by the line integral of the vector field around the (counterclockwise oriented) boundary.

This is of course, very much of a heuristic reasoning. An exact algebraic proof can be found in the book if you are interested.

31.2.1 Careful Formulation of Green's Theorem

Green's Theorem is a statement about only the 2D plane, not 3D space. Here's the setup: Let C be a piecewise smooth simple closed curve that is the boundary of a simply-connected region R in the plane. Let $\vec{\mathbf{F}} = P\,\vec{i} + Q\,\vec{j}$ be a smooth vector field defined on all of R and C.

Definition 2.51

To say a curve is **simple** means that it doesn't intersect itself, and to say a curve is **closed** means that it is a closed loop, that it starts and ends at the same point.

Definition 2.52

A region is called **simply-connected** if it is just one piece (connected) and doesn't have any holes in it.

Definition 2.53

A vector field is said to be **smooth** if its first partials are continuous functions.

Since C is the boundary of R, we will denote it as $C = \partial R$ and stop referring to C explicitly. When we consider the boundary curve of a simply-connected region, we always orient the curve so that the region is on the left as we follow the curve.

Theorem 2.54: Green's Theorem

Green's theorem relates the line integral of $\vec{\mathbf{F}}$ *over* ∂R *to the volume integral of* curl(F) *over* R.

$$\oint_{\partial R} \vec{\mathbf{F}} \cdot d\vec{r} = \iint_{D} \operatorname{curl}(F)\, dA = \iint_{R} (Q_x - P_y)\, dA.$$

You may also see this written in the Leibniz notation for partial derivatives.

$$\oint_{\partial R} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Question 175.

Use Green's theorem to explain why smooth irrotational vector fields are guaranteed to be circulation free, and hence gradient vector fields.

§31.3 The Venn Diagram

Consider the following particularly illuminating example.

Example 3.55

We start with the vector field
$$\vec{\mathbf{F}}(x,y) = \frac{y}{x^2+y^2}\hat{i} - \frac{x}{x^2+y^2}\hat{j}$$

(a) Check that the vector field is irrotational.

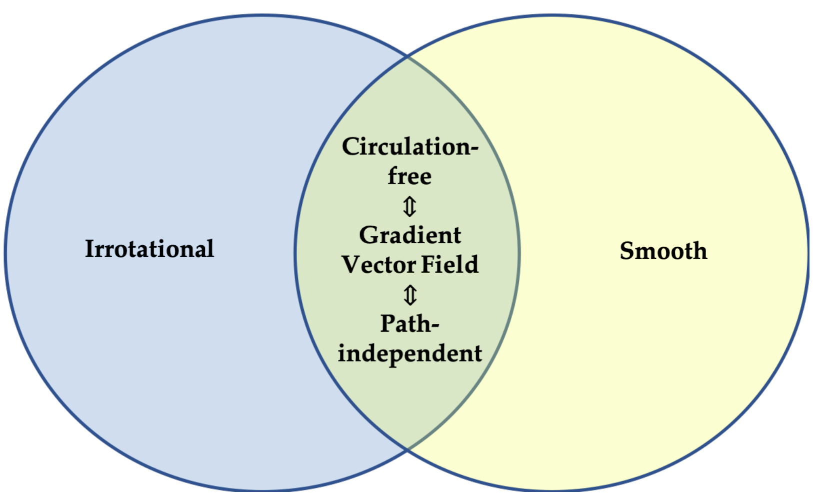
(b) Use a parameterization to compute $\oint_C \vec{\mathbf{F}} \cdot d\vec{r}$ where C is the unit circle traversed once in a counterclockwise direction.

(c) Explain how you can conclude that $\vec{\mathbf{F}}$ is not a gradient vector field.

(d) Let $f(x,y) = \arctan\frac{y}{x}$. Calculate $\vec{\nabla} f$.

(e) How do you explain the apparent contradiction?

This example highlights the requirement of smoothness in Green's theorem. In particular it shows that not all irrotational vector fields are gradient vector fields. Overall, the venn diagram of vector fields looks as follows:



Chapter 32 | Green's Theorem - Applications and Generalizations

§32.1 Applications of Green's Theorem

32.1.1 Evaluating Difficult Line Integrals

Example 1.56

If you're asked to find the line integral of an ugly vector field over a closed curve, you should look to see if Q_x and P_y are drastically more simple. If so, use Green's Theorem!

Question 176. □

Evaulate

$$\oint_C (2y + \sqrt{9+x^3})\,dx + (5x + e^{\tan^{-1} y})\,dy,$$

where C *is the circle* $x^2 + y^2 = 4$ *in the plane oriented counterclockwise.*

Example 1.57

If you're asked to find the line integral over a closed curve that is clearly the boundary of a nice region, it's often a good idea to use Green's Theorem to switch to the double integral over the interior.

Question 177. □

Integrate $\vec{\mathbf{F}} = xy\,\vec{i} + e^x\,\vec{j}$ *over the boundary of the rectangle determined by* $0 \le x \le 2$, $0 \le y \le 3$, *oriented clockwise around the boundary.*

Question 178. □

Find the line integral of $\vec{\mathbf{F}} = 3xy\,\vec{i} + 2x^2\,\vec{j}$ *over the curve* C *defined as follows: follow the curve* $y = x^2 - 2x$ *from* $(0,0)$ *to* $(3,3)$, *then follow the line* $y = x$ *from* $(3,3)$ *back to* $(0,0)$.

Question 179. □

Evaluate

$$\oint_C \left(3y - e^{\sin x}\right)dx + \left(7x + \sqrt{y^4+1}\right)dy$$

where C *is the circle* $x^2 + y^2 = 9$, *oriented clockwise.*

Example 1.58

Consider the following problem.

Find the line integral of

$$\vec{\mathbf{G}}(x,y) = \left(x+y\right)\vec{i} + \left(2x + y\ln\left(\csc\sqrt{1-y^5}\right)\right)\vec{j}$$

over C_1, the upper half of the unit circle from $(1,0)$ **to** $(-1,0)$**.**

The problem is that the vector field is ugly, so parametrizing C_1 is just going to lead to an impossible integral. So we would like to use Green's Theoerem, but this isn't a closed curve!

Here's how to fix that issue. Let C_2 be the straight line segment from $(-1,0)$ to $(1,0)$. Now $C_1 + C_2$ is a closed loop.

(a) Let R be the region enclosed by $C_1 + C_2$. Use Green's Theorem to compute $\displaystyle\oint_{C_1+C_2} \vec{\mathbf{G}} \cdot d\vec{r}$.

(b) Parametrize C_2 and directly calculate $\displaystyle\int_{C_2} \vec{\mathbf{G}} \cdot d\vec{r}$. (Note that $y = 0$ everywhere on C_2, which is helpful.)

(c) Write $\displaystyle\oint_{C_1+C_2} \vec{\mathbf{G}} \cdot d\vec{r} = \int_{C_1} \vec{\mathbf{G}} \cdot d\vec{r} + \int_{C_2} \vec{\mathbf{G}} \cdot d\vec{r}$ and use your answers to part (a) and (b) to finish off the problem and find the line integral of $\vec{\mathbf{G}}$ along C_1.

Question 180.

Evaluate the integral.

Question 181.

Compute the line integral of the vector field

$$\vec{\mathbf{F}}(x,y,z) = \left\langle \cos(x), 2+\cos(y), e^z + x\left(y^2+z^2\right)\right\rangle$$

along the curve

$$\vec{r}(t) = \langle t, \cos(t), \sin(t)\rangle \text{ with } 0 \le t \le 3\pi.$$

[HINT: Write $\vec{\mathbf{F}}$ as $\vec{\mathbf{G}} + \vec{\mathbf{H}}$ where $\vec{\mathbf{G}}$ is a gradient vector field. Then do the two integrals separately.]

32.1.2 Calculating Area

Consider the following vector fields:

$$\vec{\mathbf{F}}_1 = x\,\vec{j}, \qquad \vec{\mathbf{F}}_2 = -y\,\vec{i}, \qquad \vec{\mathbf{F}}_3 = -\tfrac{1}{2}y\,\vec{i} + \tfrac{1}{2}x\,\vec{j}.$$

What is $Q_x - P_y$ for each of these fields? Applying Green's Theorem to a region D, we get that

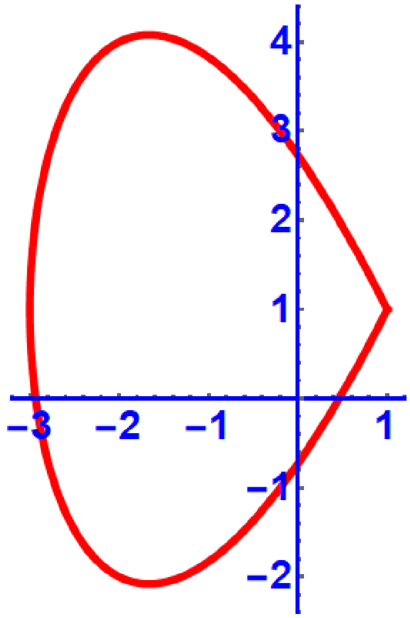
$$\oint_{\partial D} \vec{\mathbf{F}}_1 \cdot d\vec{r} = \oint_{\partial D} \vec{\mathbf{F}}_2 \cdot d\vec{r} = \oint_{\partial D} \vec{\mathbf{F}}_3 \cdot d\vec{r} = \iint_D 1\,dA = \text{Area of D. (!)}$$

Question 182.

An ellipse with semi-major axis a and semi-minor axis b is parametrized by $x = a\cos t$, $y = b\sin t$ for $0 \le t \le 2\pi$. Use $\vec{\mathbf{F}}_3$ to find the area inside this ellipse.

Question 183.

Let C be the curve parametrized by $\vec{r}(t) = (t^2 - 3)\,\vec{i} + (t^3 - 4t + 1)\,\vec{j}$, $-2 \le t \le 2$. This is a closed loop. Use Green's Theorem and $\vec{\mathbf{F}}_1$ to find the area inside this loop.



§32.2 Extended Versions of Green's Theorem

32.2.1 Finite union of simple regions

Although we have proved Green's Theorem only for the case where D is simple, we can now extend it to the case where D is a **finite union of simple regions**. For example, if D is the region shown in Figure 32.1, then we can write $D = D_1 \cup D_2$, where D_1 and D_2 are both simple. The boundary of D_1 is $C_1 + C_3$ and the boundary of D_2 is $C_2 + (-C_3)$. so, applying Green's Theorem to D_1 and D_2 separately, we get

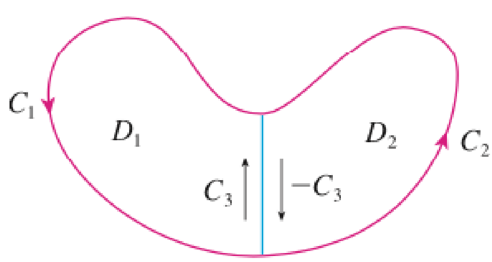


Figure 32.1

$$\oint_{C_1+C_3} \vec{\mathbf{F}} \cdot d\vec{r} = \iint_{D_1} \mathrm{curl}(F)\, dA$$

$$\oint_{C_2+(-C_3)} \vec{\mathbf{F}} \cdot d\vec{r} = \iint_{D_2} \mathrm{curl}(F)\, dA$$

Adding the two integrals above, we get,

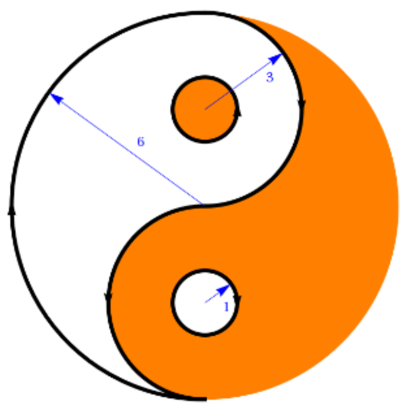
$$\oint_{C_1+C_2} \vec{\mathbf{F}} \cdot d\vec{r} = \iint_{D} \operatorname{curl}(F)\, dA$$

Question 184.

Let C be the boundary curve of the white Yang part of the Ying-Yang symbol in the disc of radius 6. You can see in the picture that the curve C has three parts, and that the orientation of each part is given. Find the line integral of the vector field

$$\vec{\mathbf{F}}(x,y) = \langle -y + \sin(e^x), x\rangle$$

around C. Notice that the Ying and the Yang have the same area.



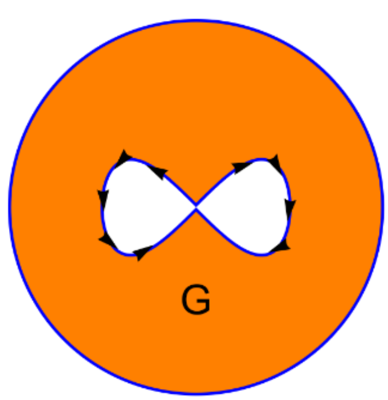
Question 185.

*Look at the shaded region G bounded by a circle of radius 2 and an inner **figure eight lemniscate** with parametric equation*

$$\vec{r}(t) = \langle \sin(t), \sin(t)\cos(t)\rangle$$

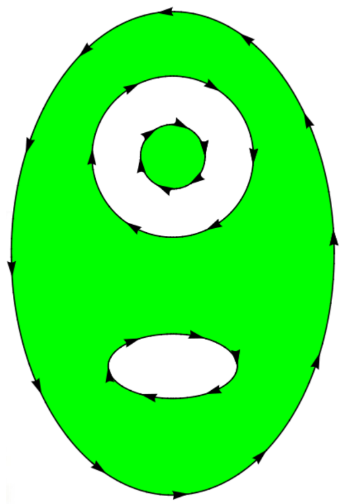
with $0 \le t \le 2\pi$. The picture shows the curve and the arrows indicate some of the velocity vectors of the curve. Find the area of this region G.

[HINT: Use Green's theorem and the vector field $x\hat{j}$.]



Question 186.

Compute the line integral of $\vec{\mathbf{F}}(x,y) = \langle 5y+3y^2, 6xy+y^5\rangle$ along the boundary of the green ***Cyclops region*** *given in the figure below. There are four boundary curves,* ***oriented as shown in the picture****: a large ellipse of area 16, two circles of area 2 (the eyeball) and 1 (the iris) as well as a small ellipse (the mouth) of area 3.*



32.2.2 Finite intersection of simple regions

We can similarly extend the theorem to regions that are finite **intersections** of simple regions.

Question 187.

Evaluate

$$\oint_C y^2 dx + 3xy dy$$

where C is the boundary of the semiannular region D in the upper half plane between $x^2+y^2=1$ and $x^2+y^2=4$.

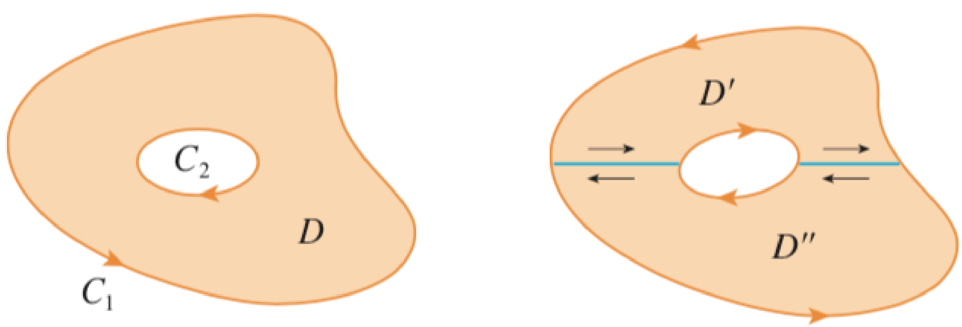


Figure 32.2

32.2.3 Not simply-connected regions

Green's Theorem can be also extended to apply to regions with holes, that is, regions that are not simply-connected. Observe that the boundary C of the region D in Figure 32.2 consists of two simple closed curves C_1 and C_2. We assume that these boundary curves are oriented so that the region D is always on the left as the curve C is traversed. **Thus the positive direction is counterclockwise for the outer curve C_1 but clockwise for the inner curve C_2.** If we divide D into two regions D' and D'' by means of the lines shown in Figure 32.2 and then apply Green's Theorem to each of D' and D'', we get

$$\iint_{D} \operatorname{curl}(F)\,dA = \iint_{D'} \operatorname{curl}(F)\,dA + \iint_{D''} \operatorname{curl}(F)\,dA$$
$$= \oint_{\partial D'} \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{r}} + \oint_{\partial D''} \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{r}}$$

Since the line integrals along the common boundary lines are in opposite directions, they cancel each other out and we get

$$\iint_{D} \operatorname{curl}(F)\,dA = \oint_{C_1} \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{r}} + \oint_{C_2} \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{r}} = \oint_{C_1+C_2} \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{r}}$$

Question 188.

If $\vec{\mathbf{F}}(x,y) = \dfrac{-y\,\vec{\boldsymbol{i}} + x\,\vec{\boldsymbol{j}}}{x^2+y^2}$, *show that* $\oint_{C} \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{r}} = 2\pi$ *for every positively oriented closed path* C *that encloses the origin.*

Question 189.

Evaluate $\oint_{C} \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{r}}$ *where*

$$\vec{\mathbf{F}}(x,y) = \frac{2xy\,\vec{\boldsymbol{i}} + \left(y^2 - x^2\right)\vec{\boldsymbol{j}}}{(x^2+y^2)^2}$$

and C *is a positively oriented closed path that encloses the origin.*

§32.3 More Practice Problems

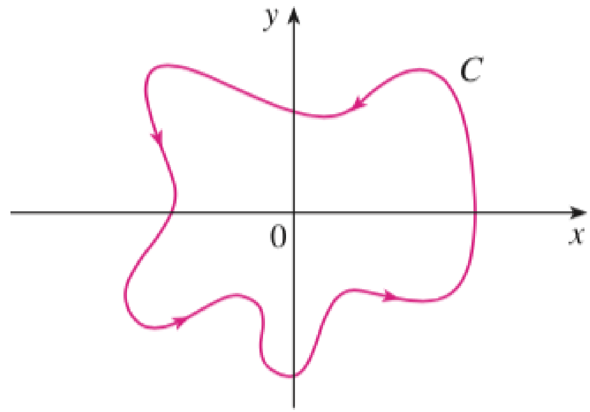
Question 190.

Evaluate the following line integrals.

(a) $\oint_{C} y\,dx + \left(x + y^2\right)dy$ *where* C *is the ellipse* $4x^2 + 9y^2 = 36$ *with counterclockwise orientation.*

(b) $\oint_{C} \sqrt{1+x^3}\,dx + 2xy\,dy$ *where* C *is the triangle with vertices* $(0,0), (1,0)$, *and* $(1,3)$

(c) $\oint_{C} \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{r}}$, *where* $\vec{\mathbf{F}}(x,y) = \frac{\left(2x^3+2xy^2-2y\right)\vec{\boldsymbol{i}} + \left(2y^3+2x^2y+2x\right)\vec{\boldsymbol{j}}}{x^2+y^2}$ *and* C *is the curve shown below.*

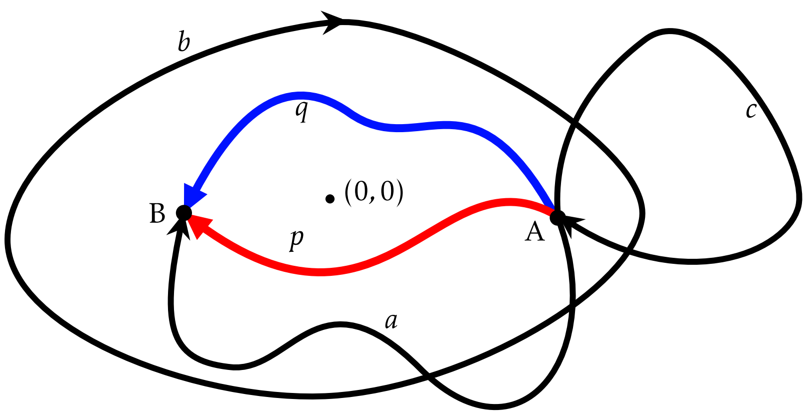


(d) $\oint_C \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{r}}$, *where* $\vec{\mathbf{F}}(x,y,z) = \langle y\cos x - xy\sin x, xy + x\cos x\rangle$, *and* C *is the triangle from* $(0,0)$ *to* $(0,4)$ *to* $(2,0)$ *to* $(0,0)$.

■ Question 191.

□

Suppose $\vec{\mathbf{F}}$ *is an irrotational vector field in the plane (that is, its curl is everywhere zero) that is not defined at the origin* $O = (0,0)$. *Suppose the line integral of* $\vec{\mathbf{F}}$ *along the path* p *from* A *to* B *is* 5 *and the line integral of* $\vec{\mathbf{F}}$ *along the path* q *from* A *to* B *is* -4. *Find the line integral of* F *along the paths* a, b *and* c.



Projects

§A Distances

In this project, we will use Dot product and Cross product of vectors to derive formula for calculating distances between points, lines and planes. We will use the notation $d(\cdot,\cdot)$ to denote distance.

A.1 Distance between two points

To begin with, the distance between two points P and Q with position vectors $\vec{\mathbf{P}}$ and $\vec{\mathbf{Q}}$ is simply given by

$$d(\mathrm{P},\mathrm{Q}) = \|\vec{\mathbf{Q}} - \vec{\mathbf{P}}\| = \|\overrightarrow{\mathbf{PQ}}\|$$

where $\|\cdot\|$ denotes the magnitude of a vector.

Question 1001. □

Find the distance between $(-5,2,4)$ *and* $(-2,2,0)$.

A.2 Distance from a point to a plane

The distance of a point P from a plane Σ is defined as the length of the perpendicular from P to Σ. Suppose the plane Σ passes through a point Q and has normal vector $\vec{n}$.

Question 1002. □

Explain using a picture why the distance from P *to* Σ *is the length of the projection of* $\overrightarrow{\mathbf{PQ}}$ *onto* $\vec{n}$. *Then*

$$d(\mathrm{P},\Sigma) = \frac{|\overrightarrow{\mathbf{PQ}} \cdot \vec{n}|}{\|\vec{n}\|}$$

Question 1003. □

Find the distance of the point $(7,1,4)$ *from the plane* $2x + 4y + 5z = 9$.

Question 1004. □

Without the absolute sign in the numerator of the distance formula, your answer in question (2) would have been negative. What does the negative sign signify here?

A.3 Distance from a point to a line

The distance of a point P from a line $\mathscr{L}$ is defined as the length of the perpendicular from P to $\mathscr{L}$. Suppose the line $\mathscr{L}$ passes through a point Q and is parallel to a vector $\vec{u}$ (i.e. its parametric equation looks like $\vec{r}(t) = \vec{\mathbf{Q}} + t\vec{u}$).

Question 1005. □

Use the definition of cross product to derive the following formula:

$$d(\mathrm{P},\mathscr{L}) = \frac{\|\overrightarrow{\mathbf{PQ}} \times \vec{u}\|}{\|\vec{u}\|}$$

Question 1006. □

Find the distance of the point $(2,3,1)$ *from the straight line* $\vec{r}(t) = (1,1,2) + t\langle 5,0,1\rangle$.

■ Question 1007.

What is the equation of the plane which contains the point P *and the line* $\mathscr{L}$*?*

A.4 Distance between two straight lines

Suppose the two straight lines $\mathscr{L}_1$ and $\mathscr{L}_2$ are given by

$$\vec{r}_1(t) = \vec{\mathbf{P}} + t\vec{u} \quad \text{and} \quad \vec{r}_2(t) = \vec{\mathbf{Q}} + t\vec{v}$$

i.e. the straight lines pass through P (and Q respectively) and is parallel to $\vec{u}$ (and $\vec{v}$ respectively).

■ Question 1008.

Draw a picture and explain using geometry why the distance between the two straight lines is given by

$$d(\mathscr{L}_1, \mathscr{L}_2) = \frac{|\overrightarrow{\mathbf{PQ}} \cdot (\vec{u} \times \vec{v})|}{\|\vec{u} \times \vec{v}\|}$$

■ Question 1009.

Find the distance between the lines $\vec{r}_1(t) = (2,1,4) + t\langle -1,1,0\rangle$ *and* $\vec{r}_2(t) = (-1,0,2) + t\langle 5,1,2\rangle$.

A.5 Distance between two planes

Before deriving the formula, observe that the distance between two planes is non-zero iff the two planes are parallel to each other, in which case they have the same normal vector $\vec{n} = \langle a,b,c\rangle$. Suppose the two planes Σ_1 and Σ_2 are given by

$$ax + by + cz = d \quad \text{and} \quad ax + by + cz = e$$

■ Question 1010.

Show that the distance formula is given by

$$d(\Sigma_1, \Sigma_2) = \frac{|d-e|}{\|\vec{n}\|}$$

■ Question 1011.

Find the distance between the planes $5x + 4y + 3z = 8$ *and* $5x + 4y + 3z = 1$.

■ Question 1012.

Find the distance between the planes $x + 3y - 2z = 2$ *and* $5x + 15y - 10z = 30$.

§B Conic Sections and Quadric Surfaces

B.1 Conic Sections

A **conic section** (or simply conic) is a curve obtained as the intersection of the surface of a cone with a plane. The three types of conic sections are the **hyperbola**, the **parabola**, and the **ellipse**. The circle is type of ellipse, and is sometimes considered to be a fourth type of conic section.

A cone has two identically shaped parts called **nappes**. One nappe is what most people mean by ‘‘cone", and has the shape of a dunce hat. It can be thought of as the surface of revolution of a straight line around an axis.

- If the intersecting plane is parallel to the axis of revolution of the cone, then the conic section is a hyperbola.
- If the plane is parallel to the generating line, the conic section is a parabola.
- If the plane is perpendicular to the axis of revolution, the conic section is a circle.
- If the plane intersects one nappe at an angle to the axis (other than 90°), then the conic section is an ellipse.

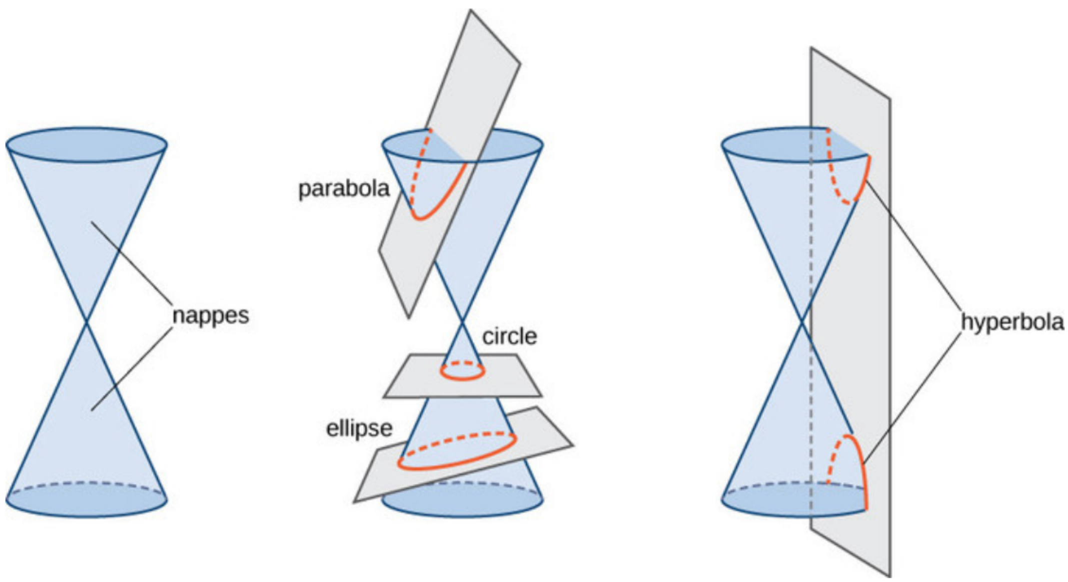


Figure 3: Conic Sections

Observe that when the intersecting plane is parallel to the axis of revolution and passes through the vertex of the cone, the conic section becomes a pair of straight lines (also known as a degenerate hyperbola).

Standard Equations in Cartesian Coordinates

The **Major Axis** is the chord between the two vertices: the longest chord of an ellipse, the shortest chord between the branches of a hyperbola. The **Minor Axis** is the shortest chord of an ellipse.

- In each of the above cases the center of the conic is at the origin. If the curve is translated h units horizontally and k units vertically, its new equation is obtained by replacing x with $(x-h)$ and y with $(y-k)$.

Conic Type	Standard Equation	Major Axis	Minor Axis
Circle	$x^2 + y^2 = r^2, \quad r \geq 0$	$2r$	$2r$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a, b > 0$	$2a$	$2b$
Parabola	$y^2 = 4ax$	N/A	N/A
	$x^2 = 4ay$	N/A	N/A
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad a, b > 0$	$2a$	N/A
	$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1, \quad a, b > 0$	$2b$	N/A
Pair of Straight Lines	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0, \quad a, b > 0$	N/A	N/A
	$\iff y = \pm\frac{b}{a}x$	N/A	N/A

Question 1013.

Use your precalculus memory or your favorite computer graphics software (e.g. DESMOS*) to draw a picture of each of the above conic sections. Clearly denote the center, radius, major axis, minor axis etc. and specify their lengths in terms of a, b etc. as applicable.*

Question 1014.

For each of the following curves, find out what kind of conic section it is. See if you can answer without using a computer first.

(a) $(x-3)^2 + (y-4)^2 = y^2$

(b) $(x-3)^2 + (y-4)^2 = 2y^2$

(c) $(x-3)^2 + (y-4)^2 = \frac{y^2}{2}$

(d) $(x-3)^2 + (y-4)^2 = 1$

Question 1015.

Find the range of value(s) of p for which the curve $\dfrac{x^2}{9-p} + \dfrac{y^2}{p-3} = 1$ *looks like a*

(a) *Circle.*

(b) *Ellipse.*

(c) *Hyperbola.*

B.2 Quadric Surface

Equations of surfaces in three dimension are of the form $f(x,y,z) = c$. One special case of interest is when $f(x,y,z)$ is a polynomial which is not linear and quadratic at most, containing terms involving x, y, z, x^2, y^2, and z^2 only. This kind of surface is called a **Quadric Surface**. Quadric surfaces are often used as example surfaces since they are relatively simple. There are nine different basic quadric surfaces listed below. A catalog of the equations and pictures of the quadric surfaces is available on section 2.6 of your openstax textbook.

- **Cylinders:** A cylinder basically has no control over one of the variables. Take some sort of a curve in the plane, and draw a family of parallel lines so that each of the lines intersects the curve in a point.

For example, a cylinder over the line $(0,t,t)$ would be all points of the form (s,t,t) for any values of s and t. This is a plane (a plane is a cylinder over a line!), and has the equation $y = z$.

The most common picture you think of when you hear the term cylinder, is that of a cylinder over a circle. Some basic variations are

- **elliptical cylinder:** A cylinder over an ellipse
- **parabolic cylinder:** A cylinder over a parabola
- **hyperbolic cylinder:** A cylinder over a hyperbola

- **Ellipsoid**, the three-dimensional analogue of the ellipse. A sphere is an uniform ellipsoid.
- **Elliptic paraboloid**, a sort of cup or a bowl
- **Hyperbolic paraboloid**, looks like a horse-saddle or a pringle
- **Cone**, take a straight line intersecting the $z-$axis and consider its surface of revolution around the $z-$axis
- **Hyperboloids:** In three dimensions there are two different analogs of hyperbolas. The word "sheet" is used in an antique, specialized sense with surfaces: it means one connected "piece" of a surface. So a hyperboloid with one sheet is a surface with one (connected) piece, and a hyperboloid with two sheets is a surface with two (connected) pieces.
  - **hyperboloid of one sheet**, obtained by revolving a hyperbola around its minor axis. The surface is connected but there is a hole in it.
  - **hyperboloid of two sheet**, obtained by revolving a hyperbola around its major axis. This surface has two pieces.

Warning: In each case, note the direction of the axes relative to the surfaces and how the corresponding variables show up in the equation. For example, $z = 2y^2 - x^2$ is a hyperbolic paraboloid that goes downward in the $x-$axis direction and upwards in $y-$axis direction. The equation $y = 2x^2 - z^2$ is also a hyperbolic paraboloid that goes downward in the $z-$axis direction and upwards in $x-$axis direction. Similarly, $y = x^2 + z^2$ is an elliptical paraboloid that opens in the $y-$axis direction.

Question 1016.

For each of the following quadric surfaces, use the catalog to pick the term from the list above which seems to most accurately describe the surface.

*Then describe what kind of conic sections are obtained by taking its cross-sections parallel to the YZ−plane, XZ−plane and XY−plane. These are called **traces**.*

(a) $\frac{x^2}{9} - \frac{y^2}{16} = z.$

- *Intersection with $x = k$:*
- *Intersection with $y = k$:*
- *Intersection with $z = k$:*

(b) $\frac{x^2}{4} + \frac{y^2}{25} + \frac{z^2}{9} = 1.$

- *Intersection with $x = k$:*
- *Intersection with $y = k$:*
- *Intersection with $z = k$:*

(c) $\frac{x^2}{4} + \frac{y^2}{9} = \frac{z}{2}$.

- *Intersection with $x = k$:*
- *Intersection with $y = k$:*
- *Intersection with $z = k$:*

(d) $\frac{z^2}{4} - x^2 - \frac{y^2}{4} = 1$.

- *Intersection with $x = k$:*
- *Intersection with $y = k$:*
- *Intersection with $z = k$:*

(e) $x^2 + \frac{y^2}{9} = \frac{z^2}{16}$.

- *Intersection with $x = k$:*
- *Intersection with $y = k$:*
- *Intersection with $z = k$:*

(f) $\frac{x^2}{9} + y^2 - \frac{z^2}{16} = 1$.

- *Intersection with $x = k$:*
- *Intersection with $y = k$:*
- *Intersection with $z = k$:*

Now try identifying the following surfaces without the help of the catalog to see if you remember the names. Try to do this **without** graphing them first.

■ Question 1017. □

Identify the following surfaces.

(a) $9y^2 + 4z^2 = 36$

(b) $y^2 + 2y + z^2 = x^2$

(c) $4x^2 - y^2 + z^2 + 9 = 0$

■ Question 1018. □

(A non-basic Quadric Surface) *Google* `Geogebra 3D calculator`. *Use the website to plot the surface*

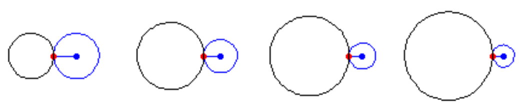
$$x^2 - 17z^2 - 2y^2 - 2xz - 12yz - 1 = 0$$

Which of the nine basic quadric surface does this resemble most closely? Can you explain how the equation of the surface might tell us what kind of surface it is, without using a graphing software?

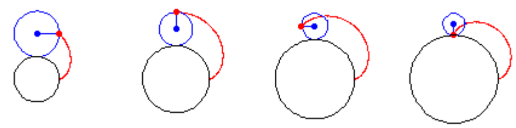
[HINT: Complete the squares.]

§C Epicycloids

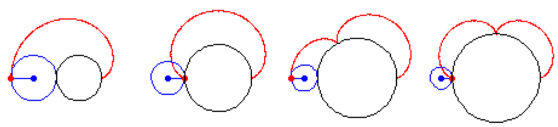
Consider a (black) circle of radius R with its center at the origin O. A (blue) circle of radius r rolls around the **outside** of the circle of radius R. See figure 4 for diagrams of different values of r. A (red) point P is located on the circumference of the rolling circle. The path traced out by P is called an **epicycloid**.



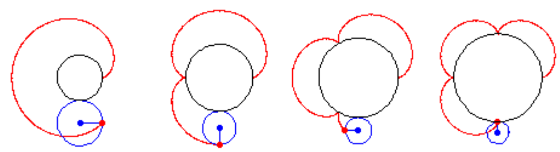
(a) $t = 0$



(b) $t = \pi/2$



(c) $t = \pi$



(d) $t = 3\pi/2$

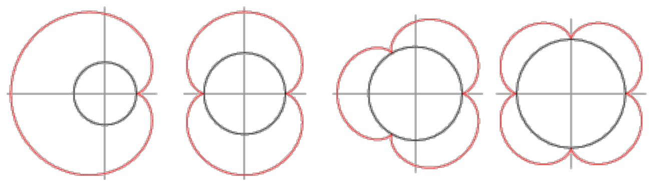


Figure 4: Epicycloids for $R/r = 1, 2, 3, 4$

Assume that initially at time $t = 0$, the rolling circle sits to the right of the fixed circle and the point P is located at $(R, 0)$. After the rolling circle has moved a bit, draw a line from the center O of the large circle to the point of contact with the rolling circle and let t be the angle the line makes with the positive X–axis. If the location of P at this moment is given by $(x(t), y(t))$ (see figure 5), then the parametric equation of the **epicycloid** is given by

$$x(t) = (R + r)\cos t - r\cos\frac{(R + r)t}{r}$$

$$y(t) = (R + r)\sin t - r\sin\frac{(R + r)t}{r}$$

Warning: If the initial location of P and the rolling circle is chosen differently, you will get the same shape with a different orientation and the form of the parametric equations will change slightly. For example, if the sin and cos functions are interchanged ($\sin \leftrightarrow \cos$), we get a vertically oriented epicycloid. If we replace t with $t + \varphi$ we get an epicycloid that has been rotated by angle φ.

■ Question 1019. □

Use the diagram above to analyze the geometry of the epicycloid. Write the detailed steps for the derivation of the parametric form of the epicycloid.

[**Hints:** The length of the blue arc and the green arc are equal, why? Use this to derive the angle $\angle PCA$ in terms of t. The x–coordinate of P is given by BD which is equal to $BC - CD$. The y–coordinate is equal to $OB - PD$.]

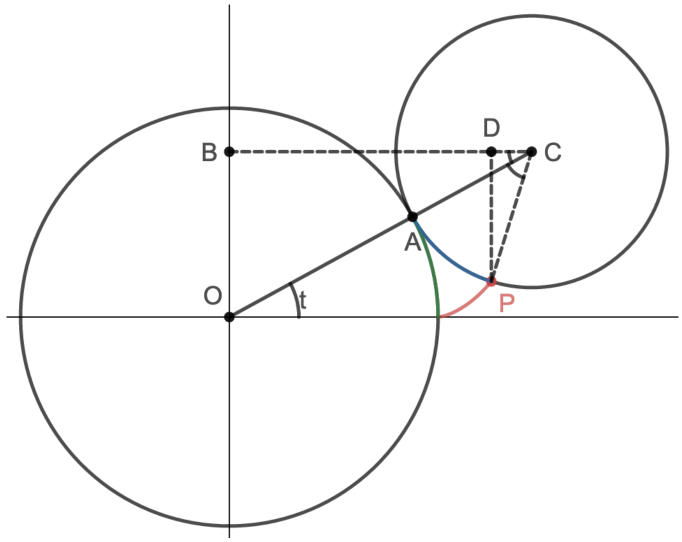


Figure 5

■ Question 1020. ☐

Suppose $\frac{R}{r} = m$, a positive integer. Show that the number of times the point P touches the fixed circle is exactly m.

[**Hints:** When the point P touches the big fixed circle, $x(t) = R\cos t$ and $y(t) = R\sin t$ (why?).]

■ Question 1021. ☐

Now we will use `Mathematica` *to draw a couple of different epicycloids when $\frac{R}{r}$ is a rational number $\frac{p}{q}$ (in its lowest terms) for different values of p and q.*

- *First set $r = 2$, and let $R = 3$. What do you observe? Does the epicycloid complete a full revolution (i.e. does the point P completes a full cycle) for $t \in [0, 2\pi]$?*
- *Next set $r = 2$, and let $R = 4, 5, 6$. What do you observe? What are the values of p and q in each case? What does the quantity p correspond to in the pictures?*
- *Next set $R = 5$, and let $r = 2, 3, 4, 5$. Note that R/r has to be at least than 1 for an epicycloid to form. What do you observe? What does the quantity q correspond to in the pictures?*

■ Question 1022. ☐

Try plotting the curve with $R = e, r = 1$. How long does it take to complete a full revolution? Make a conjecture about what happens in general if $\frac{R}{r}$ is an irrational number.

■ Question 1023. ☐

Find the length of the arc of the epicycloid between two points where it touches the fixed circle for a general value of $m = \frac{R}{r}$. For example, when $R/r = 3$, there are three such equal length arcs.

§D Stationary Points with Mathematica

A stationary point for a function of two variables is a point where both first partial derivatives equal zero. In this project you will investigate the stationary points for the following functions:

$$f(x,y) = x^3 + 3xy + y^3$$
$$g(x,y) = x^2 + 6xy + y^2 + 14x + 10y$$
$$h(x,y) = 16x^2 + 8xy + y^2$$

D.1 Computing Stationary Points

Not only will **Mathematica** calculate the partial derivatives for you, but it also has a built-in `NSolve` command that you can use to find the points where the derivatives are both zero.

(a) Define the first function by executing the command

```
f[x_,y_] := x^3 + 3*x*y + y^3
```

Similarly define $g(x,y)$ and $h(x,y)$ next.

(b) Solve for the stationary points by executing the command

```
Solve[Grad[f[x,y],{x,y}]=={0,0}, {x, y}, Reals]
```

(c) We can define a routine `StatPts` in Mathematica that will take f, g or h as input and produce the list of stationary points directly as follows. Type and execute

```
StatPts[func_] := Solve[Grad[func[x,y], {x, y}] == {0, 0}, {x, y}, Reals]
```

Check that `StatPts[f]` produces the same list of points as part (2). The advantages of doing this step are as follows.

- First, to find the stationary points of `g` and `h`, we can skip writing a long command as in part (2). Instead, we can directly get a list by using `StatPts[g]` and `StatPts[h]`.
- Secondly, we have given the list of stationary points a name, that we can refer to later.

(d) Find and record the stationary points of f, g, and h for future reference in the table below.

> **Note:** Note that the third function $h(x,y)$ has an entire line of stationary points, and you should choose any **one** of these for your investigation.

■ **Question 1024.** □

Draw the 3D plots and the contour plots of the functions f, g and h and try to visually classify your critical points as local maximum, local minimum, or saddle point. ***Choose a big enough domain so that it contains all the stationary points you are looking at.*** *Recall that the command for 3D Plot looks like*

```
Plot3D[f[x,y],{x,a,b}, {y,c,d}]
```

and the command for Contour Plot looks like

```
ContourPlot[f[x,y],{x,a,b}, {y,c,d}]
```

Note that you will need to replace a, b, c and d with appropriate numbers to see the full pictures.

■ **Question 1025.** □

Fill out the following table with information you obtained from above plots.

Function:	$f(x,y)$		$g(x,y)$	$h(x,y)$
Stationary points: $x =$				
$y =$				
Sign of $\frac{\partial^2}{\partial x^2}$:				
Sign of $\frac{\partial^2}{\partial y^2}$:				
Classification:				

D.2 Using Second Derivative test

You may have found that it was hard to visually classify the stationary points of g. We can use the second derivative test to give a definite answer in such cases.

(a) Recall that the Hessian is the matrix $\mathrm{H} = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$. We can think of this as the gradient of the gradient of f. Type and execute

```
H[func_]:=Grad[Grad[func[x,y],{x,y}],{x,y}]
```

to define a routine `H`. Check that `H[f,{x,y}]` produces the Hessian matrix for f.

(b) We can define the determinant of the Hessian `d` as

```
d[func_]:=Det[H[func]]
```

Execute the command above.

(c) Find `d[f]`. Next find the determinant for other two functions by changing f to g and h.

(d) We can evaluate the determinant at each of the stationary point above as follows. Type and execute

```
d[f]/.StatPts[f]
```

to evaluate $d(f,x,y)$ at the stationary points of f. Recall that the '`/.`' operator was used in last lab, it's called `ReplaceAll`.

(e) You can get the top left entry in the Hessian matrix by typing `H[f][[1,1]]`. Find the value of f_{xx} at the stationary points by executing

```
H[f][[1,1]]/.StatPts[f]
```

Write down the values of f_{xx}, g_{xx}, and h_{xx} at corresponding stationary points from the table. You can copy and paste above lines of codes and change f to g or h everywhere to investigate stationary points of g and h.

■ **Question 1026.** □

Classify the stationary points as local maxima, local minima or saddle point using the values you got in the last step. Check that your classification is consistent with the table above.

§E Why does the Second Derivative Test Work?

The proof of second derivative test depends on two ideas:

- It's not too hard to classify critical points for **quadratic** polynomials in two variables, i.e. polynomials where no term is bigger than x^2, y^2, or xy.
- We can use Taylor series to approximate any (smooth) function by a quadratic polynomial.

Let's begin with the first point.

E.1 Understanding quadratics in two variables

Question 1027. ☐

Give a simple explanation (without using any test) why $f(x,y) = x^2 + y^2$ has a minimum at $(0,0)$.

HINT: what is the minimum possible value this function could have?

Question 1028. ☐

For the simple function $f(x) = ax^2$, can you can tell whether it's concave up or down at $x = 0$ just by the sign of a? When a is positive, is it concave up or down? When a is negative?

Even though the example $x^2 + y^2$ is easy, let's think about it in a different way. A function $z = f(x,y)$ has a local minimum at a point if it's "concave up in every direction" at that point. To be more precise, when we take any vertical slice through this point, the resulting curve should be concave up.

Now let's convince ourselves that $f(x,y) = x^2 + y^2$ is concave up on every vertical slice going through $(0,0,0)$.

Question 1029. ☐

Show it's concave up for the two standard cross-sections $x = 0$ and $y = 0$.

Question 1030. ☐

Think about intersecting $z = x^2 + y^2$ with the vertical plane $y = x$ (you could call this the "45-degree vertical plane" I suppose if you looked down on it from above). Note that this plane does go through the origin. Justify that the slice of the surface lying on this plane is concave up. (Just plug $y = x$ into the equation!)

Now do the same thing with slice from the vertical plane $y = 2x$. And $y = -3x$. And $y = 0.02x$. It should be concave up on each of these slices.

Every vertical slice is found by intersecting the surface with the plane $y = mx$ for some m. (Or $x = 0$ if you want to think about the slice where m goes to infinity.) Plug $y = mx$ into $z = x^2 + y^2$ and get an equation of the form $z = ax^2$.

Question 1031. ☐

What is a in terms of m? Argue that a is always positive.

This last part shows that the surface is concave up in every direction, thus we really do have a local minimum at the origin!

■ Question 1032.

Repeat the same analysis as above for the quadratic function $z = g(x, y) = x^2 - y^2$, at the critical point $(0, 0)$ and show that it is concave up in some directions and concave down in others. This explains why g has a saddle point at the origin.

■ Question 1033.

Repeat the same analysis as above for the quadratic function $z = h(x, y) = x^2 + 3xy + y^2$ at the critical point $(0, 0)$. Is it concave up in every direction, or are there some directions where it is concave down? Then say whether this critical point is a local max or min or saddle point.

One more specific example: try $z = f(x, y) = 2x^2 - 4xy + 3y^2$. When you plug in $y = mx$, you get $z = p(m) \cdot x^2$ where $p(m)$ is a quadratic in m.

■ Question 1034.

How can you tell if $p(m)$ takes on both positive and negative values?

HINT: quadratic equation, discriminant. Justify that the critical point at $(0, 0)$ is a local min.

E.2 The General Quadratic

■ Question 1035.

Show that the quadratic $z = q(x, y) = \mathrm{A}x^2 + \mathrm{B}xy + \mathrm{C}y^2$ has a critical point at $(0, 0)$.

Intersect the surface $z = q(x, y)$ with the vertical plane $y = mx$. As before, we get $z = p(m)x^2$ where $p(m)$ is a quadratic in m. Write down $p(m)$.

Convince yourself that if $p(m)$ takes on only positive values (for any choice of m), then the critical point is a local minimum. And that if $p(m)$ takes on only negative values, then the critical point is a local max. And that if $p(m)$ can be either positive or negative depending on the choice of m, then the critical point is a saddle point.

■ Question 1036.

Show that the sign of the expression $\mathrm{B}^2 - 4\mathrm{AC}$ determines whether $p(m)$ takes on only positive (or only negative) values or both positive and negative values.

E.3 More general Quadratic?

The most general formula of a quadratic function is

$$\mathrm{Q}(x, y) = \mathrm{A}x^2 + \mathrm{B}xy + \mathrm{C}y^2 + \mathrm{D}x + \mathrm{E}y + \mathrm{G}$$

We can rewrite this by completing some squares as follows:

$$\mathrm{Q}(x, y) = \mathrm{A}\left(x + \frac{\mathrm{D}}{2\mathrm{A}}\right)^2 + \mathrm{C}\left(y + \frac{\mathrm{E}}{2\mathrm{C}}\right)^2 + \mathrm{B}xy + \underbrace{\left(\mathrm{G} - \frac{\mathrm{D}^2}{4\mathrm{A}} - \frac{\mathrm{E}^2}{4\mathrm{C}}\right)}_{\text{constant}}$$

■ Question 1037. □

Try experimenting in the Mathematica file `SecondDerivativeDerivation.nb` *from Moodle to answer the following questions.*

(a) *What is the effect on the shape of the graph of* Q *if we change* G*?*

(b) *What is the effect on the shape of the graph of* Q *if we change* E*?*

(c) *What is the effect on the shape of the graph of* Q *if we change* D*?*

We should be able to conclude that the **shape** of Q is unaffected by different choices of D, E and G. So we can assume $D = E = G = 0$. Hence the shape of a general quadratic can be fully understood by analyzing $q(x, y)$ from above.

E.4 The Taylor Quadratic

Recall that the linear approximation of the function $f(x, y)$ at $(0, 0)$ corresponds to the tangent plane at $(0, 0)$:

$$T(x, y) = f_x(0, 0)x + f_y(0, 0)y + f(0, 0)$$

Note that a linear approximation essentially finds a linear function such that the slope of the function matches with the linear function in $x-$ and $y-$ direction. But clearly it can't distinguish between local max/min/saddle point.

So we want to find a better approximation of f by matching more derivatives. For that we'll need something more complicated than linear functions, a quadratic function.

Suppose $Q(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + G$ such that

$$\begin{aligned}
Q(0, 0) &= f(0, 0)\\
Q_x(0, 0) &= f_x(0, 0)\\
Q_y(0, 0) &= f_y(0, 0)\\
Q_{xx}(0, 0) &= f_{xx}(0, 0)\\
Q_{yy}(0, 0) &= f_{yy}(0, 0)\\
Q_{xy}(0, 0) &= f_{xy}(0, 0)\\
Q_{yx}(0, 0) &= f_{yx}(0, 0)
\end{aligned}$$

■ Question 1038. □

Find A, B, C, D, E, G.

These choices lead us to the **Taylor Quadratic** for $f(x, y)$ at $(0, 0)$.

$$TQ(x, y) = \frac{f_{xx}(0, 0)}{2}x^2 + f_{xy}(0, 0)xy + \frac{f_{yy}(0, 0)}{2}y^2 + f_x(0, 0)x + f_y(0, 0)y + f(0, 0)$$

Or more generally the Taylor Quadratic approximation at a point (a, b) is given by,

$$\begin{aligned}
TQ(x, y) = &\frac{f_{xx}(a, b)}{2}(x - a)^2 + f_{xy}(a, b)(x - a)(y - b) + \frac{f_{xx}(a, b)}{2}(y - b)^2\\
&+ f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)
\end{aligned}$$

Using the same argument as last section, we can then conclude that the shape of this quadratic is entirely determined by

$$4AC - B^2 = 4\frac{f_{xx}(a, b)}{2}\frac{f_{xx}(a, b)}{2} - (f_{xy}(a, b))^2 = \det[Hessian]$$

Question 1039.

Apply the $4AC - B^2$ *rule to determine the graph shapes of the Taylor quadratics:*

(a) $-2xy - 4y^2 - 3x$

(b) $-3(x+1)^2 - 2(x+1)(y-1) - 4(y-1)^2 - 2(x+1) - 6(y-1)$

Question 1040.

We can also rewrite $q(x,y)$ *as*

$$q(x,y) = Ax^2 + Bxy + Cy^2 = A\left(x + \frac{B}{2A}y\right)^2 + \left(C - \frac{B^2}{4A}\right)y^2$$

So when $B^2 - 4AC = 0$*, the* y^2 *term vanishes. What is the shape of the surface*

$$z = A\left(x + \frac{B}{2A}y\right)^2?$$

Question 1041.

What can you conclude when $B^2 - 4AC = 0$*? Is it a local max/min/saddle point? Try experimenting in* `SecondDerivativeDerivation.nb` .

§F Rocket Science

Many rockets, such as the **Pegasus XL** currently used to launch satellites and the **Saturn V** that first put men on the moon, are designed to use three stages (engines) in their ascent into space. A large first stage initially propels the rocket until its fuel is consumed, at which point the stage is jettisoned to reduce the mass of the rocket. The smaller second and third stages function similarly in order to place the rocket's payload into orbit about the earth. (With this design, at least two stages are required in order to reach the necessary velocities, and using three stages has proven to be a good compromise between cost and performance.) Our goal here is to determine the individual masses of the three stages, which are to be designed to minimize the total mass of the rocket while enabling it to reach a desired velocity.

For a single-stage rocket consuming fuel at a constant rate, the change in velocity resulting from the acceleration of the rocket vehicle has been modeled by

$$\Delta v = -c \ln\left(1 - \frac{(1-S)M_r}{P + M_r}\right)$$

where

- M_r is the mass of the rocket engine including initial fuel,
- P is the mass of the payload,
- S is a structural factor determined by the design of the rocket (specifically, it is the ratio of the mass of the rocket vehicle without fuel to the total mass of the rocket with payload),
- and c is the **(constant)** speed of exhaust relative to the rocket.

Now consider a rocket with **three** stages and a payload of mass A. Assume that outside forces are negligible and that c and S remain constant for each stage. If M_i is the mass of the ith stage, we can initially consider the rocket engine (i.e. the first stage) to have mass M_1 and its payload to have mass $M_2 + M_3 + A$; the second and third stages can be handled similarly.

■ Question 1042. □

Show that the velocity attained after all three stages have been jettisoned is given by

$$v = c\left[\ln\left(\frac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A}\right) + \ln\left(\frac{M_2 + M_3 + A}{SM_2 + M_3 + A}\right) + \ln\left(\frac{M_3 + A}{SM_3 + A}\right)\right]$$

We wish to minimize the total mass $M = M_1 + M_2 + M_3$ of the rocket engine subject to the constraint that the final velocity v from Problem 1 is equal to some desired velocity v_f. The method of Lagrange multipliers is appropriate here, but difficult to implement using the current expressions.

To simplify, we define variables N_i so that the constraint equation may be expressed as

$$v_f = c(\ln N_1 + \ln N_2 + \ln N_3).$$

Thus for example $N_1 = \frac{M_1+M_2+M_3+A}{SM_1+M_2+M_3+A}$.

■ Question 1043. □

Show that

$$\frac{M_1+M_2+M_3+A}{M_2+M_3+A} = \frac{(1-S)N_1}{1-SN_1}$$
$$\frac{M_2+M_3+A}{M_3+A} = \frac{(1-S)N_2}{1-SN_2}$$
$$\frac{M_3+A}{A} = \frac{(1-S)N_3}{1-SN_3}$$

and conclude that

$$\frac{M+A}{A} = \frac{(1-S)^3 N_1 N_2 N_3}{(1-SN_1)(1-SN_2)(1-SN_3)} \tag{$\star$}$$

But now M is difficult to express in terms of the N_i's. So instead of minimizing M, we are going to find N_i's that minimize $\ln\frac{M+A}{A}$.

■ Question 1044. □

To see why this works, show that $\ln\frac{M+A}{A}$ *is an increasing function of* M. *Explain why this implies that* $\ln\frac{M+A}{A}$ *is minimized at the same place where* M *is minimized.*

■ Question 1045. □

Now write $\ln\frac{M+A}{A}$ *as a function of* N_1, N_2, *and* N_3 *by using the result of equation* ($\star$). *Then use Lagrange multipliers to minimize* $\ln\frac{M+A}{A}$ *subject to the constraint* $c(\ln N_1 + \ln N_2 + \ln N_3) = v_f$. *Find expressions for the values of* N_i *in terms of* v_f *where the minimum occurs.*

[HINT: Use properties of logarithms to help simplify the expressions. You should be getting $N_1 = N_2 = N_3$.*]*

■ Question 1046. □

Show that the minimum value of M *as a function of* v_f *is*

$$M = \frac{A(1-S)^3 e^{v_f/c}}{\left[1 - Se^{v_f/(3c)}\right]^3} - A$$

■ Question 1047. □

If we want to put a three-stage rocket into orbit 100 *miles above the earth's surface, a final velocity of approximately* 17,500 *mph is required. Suppose that each stage is built with a structural factor* $S = 0.2$ *and an exhaust speed of* $c = 6000$ *mph.*

(a) *Find the minimum total mass* M *of the rocket engines as a function of* A.

(b) *Find the mass of each individual stage as a function of* A. *(They are not equally sized!)*

§G Ordinary Linear Regression

Suppose that a scientist has reason to believe that two quantities x and y are related linearly, that is, $y = mx + b$, at least approximately, for some values of m and b. The scientist performs an experiment and collects data in the form of points $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$, and then plots these points. The points don't lie exactly on a straight line, so the scientist wants to find constants m and b so that the line $y = mx + b$ "fits" the points as well as possible (see figure 6).

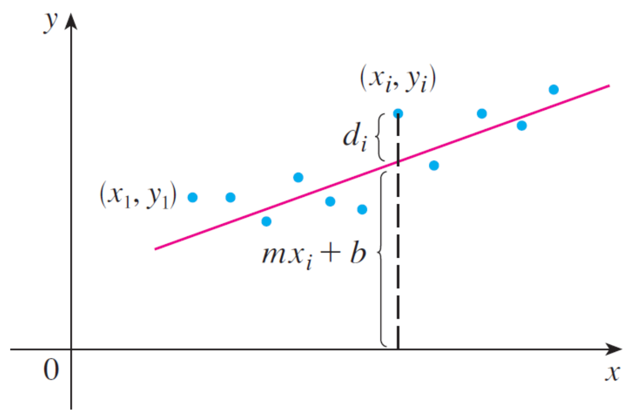


Figure 6

For each data point (x_i, y_i), observe that the corresponding point directly above or below it on the best fit line has y-coordinate $b + mx_i$. Let d_i be the vertical distance between each data point and the corresponding point on the straight line found above (see figure 6). Then for each data point (x_i, y_i), we get

$$d_i = y_i - (b + mx_i)$$

We can think of these d_is as error measurement of each data point. The method of **Ordinary Linear Regression** tries to minimize the **S**um of the **S**quares of the **E**rrors. In Statistics, this is known as SSE or RSS (Residual Sum of Squares). It is given by the function $f(b, m)$ as follows:

$$f(b, m) = \sum_{i=1}^{n} (y_i - (b + mx_i))^2$$

Our goal is to find b and m that minimizes $f(b, m)$.

■ Question 1048. □

Show that the partial derivatives $\frac{\partial f}{\partial b}$ and $\frac{\partial f}{\partial m}$ are given by

$$\frac{\partial f}{\partial b} = -2\sum_{i=1}^{n} (y_i - (b + mx_i))$$

and

$$\frac{\partial f}{\partial m} = -2\sum_{i=1}^{n} (y_i - (b + mx_i)) \cdot x_i$$

■ Question 1049.

☐

Show that the critical point equations $\frac{\partial f}{\partial b} = 0$ *and* $\frac{\partial f}{\partial m} = 0$ *lead to a pair of simultaneous linear equations in* b *and* m*:*

$$nb + \left(\sum_{i=1}^{n} x_i\right) m = \sum_{i=1}^{n} y_i$$
$$\left(\sum_{i=1}^{n} x_i\right) b + \left(\sum_{i=1}^{n} x_i^2\right) m = \sum_{i=1}^{n} x_i y_i$$

■ Question 1050.

☐

Solve the equations above for b *and* m*, and show that*

$$b = \frac{\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i y_i}{n \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} \qquad m = \frac{n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

■ Question 1051.

☐

Use the second derivative test to make sure that the stationary point we obtained above is in fact a local minimum.

■ Question 1052.

☐

Either using the formula or from the construction of $f(b, m)$*, find the line of best fit for the following data points:* $(1, 1), (2, 1)$*, and* $(3, 3)$.

Go to this link: `https://mathlets.org/mathlets/linear-regression/` to experiment with best fit lines yourself.

§H Normal Probability Distribution

In statistical applications it is important to know the exact value of the area under the bell–shaped curve $y = e^{-t^2/2}$, i.e., we need to evaluate the improper integral

$$k = \int_{-\infty}^{\infty} e^{-t^2/2} dt$$

Using integral k, the standard normal density function is given by

$$f(t) = \frac{1}{k} e^{-t^2/2}.$$

In this project, as a byproduct of our ability to evaluate double integrals using polar coordinates, we will obtain an exact value for k.

Notice that, by symmetry, we have

$$k = \int_{-\infty}^{\infty} e^{-t^2/2} dt = 2\int_{0}^{\infty} e^{-t^2/2} dt$$

In addition, by definition of the improper integral from 0 to ∞, we also have

$$\int_{0}^{\infty} e^{-t^2/2} dt = \lim_{a\to\infty} \int_{0}^{a} e^{-t^2/2} dt$$

■ **Question 1053.** □

(a) *Let a be positive, and let D_a be the square domain $[0,a]\times[0,a]$. Show that*

$$\iint_{D_a} e^{-(x^2+y^2)/2} dA = \left(\int_{0}^{a} e^{-t^2/2} dt\right)^2$$

(b) *Use part (a) to show that*

$$\lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)/2} dA = \frac{1}{4}k^2$$

(c) *Now designate by S_a the quarter-disk of radius a consisting of points with polar coordinates satisfying $0 \le \theta \le \pi/2$ and $0 \le r \le a$, and designate by T_a the quarter-disk of radius $\sqrt{2}a$ consisting of points with polar coordinates satisfying $0 \le \theta \le \pi/2$ and $0 \le r \le \sqrt{2}a$. Explain geometrically why*

$$\iint_{S_a} e^{-(x^2+y^2)/2} dA \le \iint_{D_a} e^{-(x^2+y^2)/2} dA \le \iint_{T_a} e^{-(x^2+y^2)/2} dA$$

(d) *Transform*

$$\iint_{S_a} e^{-(x^2+y^2)/2} \, dA$$

into an iterated integral in polar coordinates, and evaluate this integral exactly.

(e) *Transform*

$$\iint_{T_a} e^{-(x^2+y^2)/2} \, dA$$

into an iterated integral in polar coordinates, and evaluate this integral exactly.

(f) *Show that the integrals in parts (d) and (e) both approach the same limiting value as a approaches infinity.*

(g) *Use the results of parts (a) through (f) to find the value of k.*

§I A 2-page Summary of Line Integrals

I.1 The Theorems

We learned the following theorems over the last couple of lectures.

Theorem 9.59: Line Integral on Parameterized Curves

If the curve C *can be parametrized as* $\vec{r}(t)$, $a \le t \le b$, *then*

$$\int_C \vec{\mathbf{F}} \cdot d\vec{r} = \int_a^b \vec{\mathbf{F}}(\vec{r}(t)) \cdot \vec{r}\,'(t)dt$$

Theorem 9.60: Fundamental Theorem of Line Integrals

If the vector field $\vec{\mathbf{F}}$ *is a gradient vector field i.e.* $\vec{\mathbf{F}} = \nabla f$, *and the curve* C *starts at* P *and ends at* Q, *then*

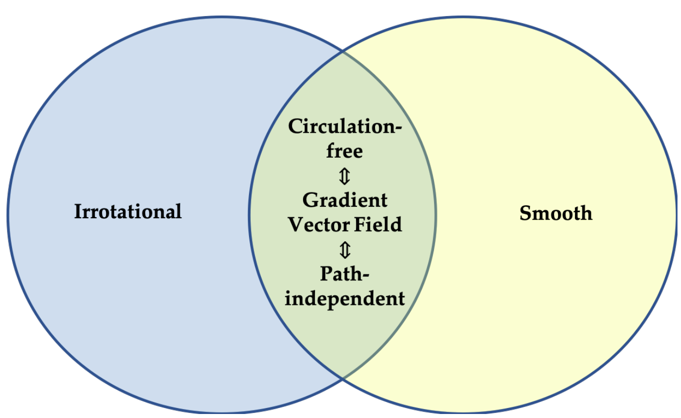
$$\int_C \vec{\mathbf{F}} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\mathrm{Q}) - f(\mathrm{P})$$

Theorem 9.61: Green's Theorem

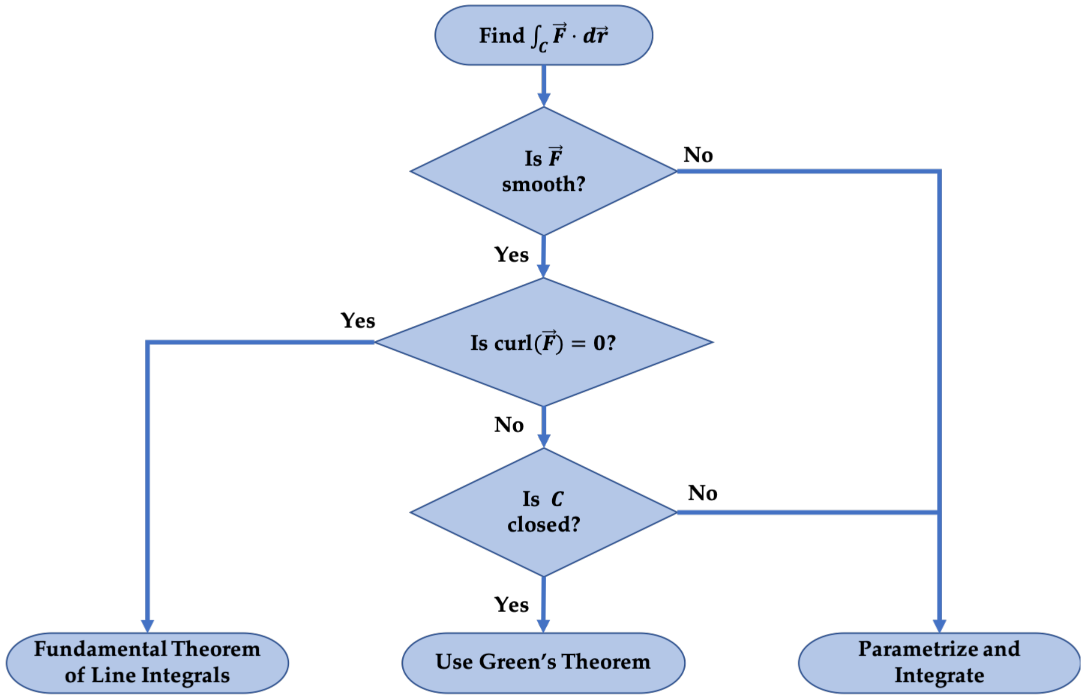
If C *is a* ***simple, closed, oriented*** *curve and the vector field* $\vec{\mathbf{F}}$ *is* ***smooth*** *over the simply-connected region* R *enclosed by* C *(oriented so that* R *is always to the left of* C*), then*

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{r} = \iint_R \operatorname{curl} \vec{\mathbf{F}}\, dA$$

I.2 The Venn Diagram



1.3 The Flowchart



§J (Optional) 3D Curl and Divergence

J.1 3D Curl

Consider the mathematical abstract 'construct' $\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$. This is called a "differential operator". It's an operation you can apply to a function of a Field. We will use it to give a new and more general definition of curl.

Definition 10.62

Given a vector field $\vec{\mathbf{F}} = \mathrm{P}\,\vec{i} + \mathrm{Q}\,\vec{j} + \mathrm{R}\,\vec{k}$, we define the **curl** of $\vec{\mathbf{F}}$ to be the **vector field**

$$\operatorname{curl}\vec{\mathbf{F}} = \vec{\nabla}\times\vec{\mathbf{F}}.$$

You should check that in the case of 2-dimensional vector fields, above definition gives the correct familiar formula for $\nabla\vec{\mathbf{F}}$ and curl $\vec{\mathbf{F}}$.

Warning: Since we have defined the differential operator $\vec{\nabla}$ which looks like a vector, we might naturally ask what is the dot product of the operator and the vector field. Note that the dot product $\vec{\nabla}\cdot\vec{\mathbf{F}}$ is NOT the gradient because gradient of a vector field doesn't make sense. The gradient of a function $g(x,y,z)$ can be written as $\vec{\nabla}g$ without any 'dot' (since it's not a dot product). We will see how to interpret the dot product in the next section.

Question 1054. □

Write out the formula for curl $\vec{\mathbf{F}}$ *for a 3D vector field using* $\hat{\imath},\hat{\jmath},\hat{k}$ *notation.*

Question 1055. □

Now suppose $\vec{\mathbf{F}}$ *is a gradient vector field. I.e.,*

$$\vec{\mathbf{F}} = \mathrm{P}\,\hat{\imath} + \mathrm{Q}\,\hat{\jmath} + \mathrm{R}\,\hat{k} = g_x\,\hat{\imath} + g_y\,\hat{\jmath} + g_z\,\hat{k}$$

for some function $g(x,y,z)$*. Use Clairaut's theorem to prove that* curl $\vec{\mathbf{F}} = 0$.

This statement is sometimes written as

$$\vec{\nabla}\times(\vec{\nabla}g) = 0.$$

Question 1056. □

(a) *Show that*

$$\vec{\mathbf{F}}(x,y,z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$$

is an irrotational vector field.

(b) *Find a function* f *such that* $\mathrm{F} = \vec{\nabla}f$ *and conclude that* $\vec{\mathbf{F}}$ *is a conservative vector field.*

Question 1057.

□

Show that Green's Theorem (in 2D) can be rewritten as

$$\oint_{\partial R} \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{r}} = \iint_{R} \left((\operatorname{curl} \vec{\mathbf{F}}) \cdot \hat{k}\right) dA.$$

This is called the **vector form** of Green's Theorem. It generalizes to 3D situations in the form of **Stokes' Theorem**!

J.2 Divergence

If the curl can be interpreted of as a **vector derivative** of the vector field, we define the **scalar derivative** of the vector field as follows.

Definition 10.63

If $\frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}$, and $\frac{\partial R}{\partial z}$ exist, then the **divergence** of $\vec{\mathbf{F}}$ is defined to be the **scalar** quantity

$$\operatorname{div} \vec{\mathbf{F}} = \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{F}} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Question 1058.

□

Show that If $\vec{\mathbf{F}} = P\,\hat{i} + Q\,\hat{j} + R\,\hat{k}$ *is a vector field defined on* $\mathbb{R}^3$ *and* $P, Q,$ *and* R *have continuous second-order partial derivatives, then*

$$\operatorname{div}\operatorname{curl} \vec{\mathbf{F}} = 0.$$

This is also sometimes written as

$$\vec{\boldsymbol{\nabla}} \cdot (\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{F}}) = 0.$$

Question 1059.

□

Show that the vector field

$$\vec{\mathbf{F}}(x, y, z) = xz\hat{i} + xyz\hat{j} - y^2\hat{k}$$

cannot be written as the curl of another vector field, that is, $\vec{\mathbf{F}} \neq \operatorname{curl} \vec{\mathbf{G}}$.

Question 1060.

□

Show that Green's Theorem can also be written in (yet another vector form) as

$$\oint_{\partial R} \vec{\mathbf{F}} \cdot \vec{\boldsymbol{n}}\, ds = \iint_{R} \operatorname{div} \vec{\mathbf{F}}\, dA$$

where $\vec{\boldsymbol{n}}$ *is the outward unit normal vector to* ∂R.

Above result generalizes to 3D situations in the form of **Divergence Theorem**!

Understanding Divergence

The reason for the name divergence can be understood in the context of fluid flow. If $\vec{\mathbf{F}} = P\,\hat{i} + Q\,\hat{j} + R\,\hat{k}$ the velocity of a fluid (or gas), then $\operatorname{div} \vec{\mathbf{F}}$ represents the net rate of change (with respect to time) of the mass of fluid (or gas) flowing from the point (x, y, z) per unit volume. In other words, $\operatorname{div} \vec{\mathbf{F}}$ measures the tendency of the fluid to diverge from the point (x, y, z). if $\operatorname{div} \vec{\mathbf{F}} = 0$ then the vector field $\vec{\mathbf{F}}$ is said to be **incompressible.**

J.3 Laplace Operator

For the sake of completion we also mention another differential operator that occurs when we compute the divergence of a gradient vector field.

$$\mathrm{div}(\vec{\nabla} f) = \vec{\nabla} \cdot \left(\vec{\nabla} f\right)$$

is abbreviated as $\nabla^2 f$, and the operator ∇^2 is called the **Laplace operator**. $\nabla^2 f$ is also denoted as Δf.

Question 1061. □

Check that

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$