Problem Set 9,10 Solutions

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Exercise 2: GP Containing an AP

Problem 1.1. *The first, second and seventh term of a certain AP are the first three terms of a GP. If their sum is* 93*, then find the numbers.*

Solution. $31, 31, 31$ or $3, 15, 75$.

Let's write $a_1 = a$, so that $a_2 = a + d$ and $a_7 = a + 6d$. Now, the ratio of the first to second term is common with the ration of the second to seventh term, as they form a GP. So, equating these common ratios,

$$\frac{a}{a+d} = \frac{a}{a+6d}$$

Moving all terms to one side, we have

$$a^2 + 6ad - (a^2 + 2ad + d^2) = 4ad - d^2$$

$$= d(4a - d)$$

which equals zero, which implies that either $d = 0$ or $d = 4a$. In the first case, a common difference of zero corresponds to a constant sequence, in which our terms are $31, 31$, and 31. In the second case, consider the sum:

$$a_1 + a_2 + a_3 = 3a + 7d$$

$$= 3a + 28a$$

which equals 93, giving $a = 3$, and $d = 12$. The terms of interest in this sequence are $3, 15$, and 75. ∎

Exercise 4: Triangle GP

Problem 2.1. *One side of an equilateral triangle is* 24 *cm. The mid-points of its sides are joined to form a triangle. Again another triangle is formed by joining the mid-points of the sides of this triangle and this process is continued. Determine the sum of the area of the triangles after you have repeated the process* 100 *times. (To clarify, we have drawn a total of* 100 *triangles, including the starting one with side length* 24 *cm.) Your answer should be in closed-form, but you do not have to simplify the powers.*

Solution. $\frac{1}{3} \times 24^2 \times \sqrt{3}(1-(\frac{1}{4})^{100})\ cm^2 \approx 333\ cm^2$.

The original triangle has area $A = \frac{1}{2} \times 24 \times 12\sqrt{3}\ cm^2$. The second triangle covers an area that is one-fourth of the first triangle area; and, furthermore, the $(k+1)^{th}$ triangle is $\frac{1}{4}$ the area of the k^{th} triangle. So our common ratio $r = \frac{1}{4}$ yields a sum of the first 100 terms in this GP[1] as

$$A_{100} = \frac{A(1-r^{100})}{1-r}$$

which yields approximately $\frac{1}{3}24^2 \times \sqrt{3}\ cm^2 \approx 333\ cm^2$. ■

Exercise 5: Two-Digit GP (Extra Credit)

Problem 3.1. *Five distinct* 2*-digit positive integers are in a geometric progression. Find the middle term.*

Solution. 36

First, note that every five-term solution of the form a, ar, ar^2, ar^3, ar^4 is accompanied by a solution of the form ar^4, ar^3, ar^2, ar, a, i.e. we can follow any solution GP backwards by using the reciprocal of the common ratio as the new common ratio. Thus, for every solution of common ratio $r > 1$, the common ratio $r' = \frac{1}{r}$ is also a common ratio of a solution, where $0 \le r' < 1$ (note that r cannot be negative since we require that all terms be positive. Moreover, $r \neq 1$ since the terms would not be distinct).

So let us choose to look for the solutions where $r > 1$, since when we find those solutions, we know how to find all the solutions.

Moreover, we need each a_i, for $i \in \{1,2,3,4,5\}$, to satisfy $10 \le a_i \le 99$, for the sequence to be of only two digit integers. In particular, given our choice of $r > 1$, we need $a_1 \ge 10$ and $a_5 \le 99$, which we can write as $a \ge 10$ and $ar^4 \le 99$. Dividing these two inequalities, we find $r^4 \le \frac{99}{10}$, which means approximately that $r < 1.77$. But it is sufficient enough to note $r < 2$.

Finally, we need to impose the requirement that the terms a_i be integers. While we know a must be an integer right away, r must be a rational between 1 and 2. We know $\exists p, q \in \mathbb{Q}$ such that $\frac{p}{q} = r$ and $p > 0$ and $q > 0$. Moreover, these terms must be integers: $\frac{ap}{q}, \frac{ap^2}{q^2}, \frac{ap^3}{q^3}, \frac{ap^4}{q^4}$. This requires that a be divisible by the first four powers of q. In particular, we need $q^4 | a$.

Since we chose to look for $r > 1$, we know $p > q$. There are only a few possibilities for q, so let us look at them individually. If $q = 1$, then p must be 2 or any greater integer. But then $r = p$ is greater than or equal to 2, and we already have the requirement that $r < 2$. Let's continue and look at $q = 2$. In this case, the only value p can take is 3, since otherwise r would not be between 1 and 2. We get

1. This summation is up to the term Ar^{99}. A way to check you are using the correct n as exponent is to check that the polynomial degrees are equal on both sides of your application of the formula.

$r = \frac{3}{2}$, and need some a such that $2^4|a$ yet $ar^4 < 100$. In fact, the smallest[2] possibility, $a = 2^4$, works, and the next largest possibility, which is $a = 32$, is too big (calculate that $32 * (\frac{3}{2}^4) > 99$). So we have one solution, $r = \frac{3}{2}$. Are there any more? If $q = 3$, then the smallest that a can be is 81. The smallest r can be is $\frac{4}{3}$. But then $a_2 = 4^4$ which is already a three-digit integer. And for any $q \geq 4$, the minimum that a can be is q^4, which is three digits for any $q > 3$.

So we have only one pair of solutions:

$$16, 24, 36, 54, 81$$

or

$$81, 54, 36, 24, 16$$

and for both, the middle term is 36.

■

Exercise 6: Closed GP Forms for Linear Recursions

Problem 4.1. *Suppose a Geometric Progression $\{x_i\}_{i\in\mathbb{N}}$ satisfies*

$$x_{n+1} = 7x_n - 12x_{n-1} \tag{1}$$

for $n > 2$. Find all such sequences.

Solution. Let us find all sequences of the form $x_n = ar^n$. Substituting this closed form into Eq. 1, we find

$$ar^{n+1} = 7ar^n - 12ar^{n-1}$$

Dividing by ar^{n-1} and moving terms to one side, we find a quadratic equation:

$$0 = r^2 - 7r + 12$$

which has roots $r = 3$ and $r = 4$. Let these sequences be called $a_n = A(3^{n-1})$ and $b_n = B(3^{n-1})$. We did not specify any conditions for initial terms, so A and B can each be any real number. Hence all geometric progressions that satisfy Eq. 1 are in the form[3]

$$x_n = A(3^{n-1})$$

or

$$x_n = B(4^{n-1})$$

where $A, B \in \mathbb{R}$, and $n > 2$.

■

2. $a \geq q^4$. Can you tell why?

3. While a linear combination of these two GP's would indeed be a solution to Eq. 1, we were told to find only sequences that are GP's. In general, a linear combination of GP's is not a GP.

Exercise 7: Closed GP Form for Linear Recursion

Problem 5.1. *Solve the recurrence relation for $n > 2$:*
$a_1 = 1,\ \ a_2 = 2,\ \ a_n = 5a_{n-1} - 6a_{n-2}$

Solution. We follow the same method as in Exercise 6 (Sec. 4) to solve the quadratic equation $r^2 - 5r + 6 = 0$. Observe that, as proved in class, any linear combination of GP solutions to a recurrence relation is yet another solution. Hence $a_n = A(2^{n-1}) + B(3^{n-1})$. We use the values of a_1 and a_2 to determine A and B. By the equation for a_1, we have

$$1 = A + B \tag{2}$$

And by the equation for a_2, we have

$$2 = A(2) + B(3)$$

Solving, we get $A = 1$ and $B = 0$. So the recurrence relation has the closed form

$$a_n = 2^{n-1}$$

for $n > 2$. ∎