Assignment 10 (7/13)

Subhadip Chowdhury

- *This homework is due at the beginning of class on **Thursday** 7/19. You are encouraged to work together on these problems, but you must write up your solutions independently.*

Binet's Formula

Recall that the Fibonacci sequence is defined as

$$\begin{aligned} &F_1 = F_2 = 1 \\ &F_n = F_{n-1} + F_{n-2} \text{ for } n > 2 \end{aligned} \tag{$\star$}$$

We are going to find an explicit formula for the nth term F_n. This will be done in two steps. First we will find a general format of sequence that satisfy the recurrence relation. Then we will customize that format to fit the starting conditions.

Let's start by analyzing the equation ($\star$). Suppose there is some GP that has property ($\star$) i.e. each term (starting with the third) is sum of the previous two terms. We can write the terms of the GP as

$$a, ar, ar^2, ar^3, \ldots, ar^{n-1}, \ldots$$

Equation ($\star$) says

$$\begin{aligned} ar^{n-1} &= ar^{n-2} + ar^{n-3} \\ \implies r^{n-1} &= r^{n-2} + r^{n-3} \\ \implies r^2 &= r + 1 \\ \implies r^2 - r - 1 &= 0 \\ \implies r &= \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

The equation $r^2 = r + 1$ is called the *characteristic equation* of ($\star$). We will use α, β to denote the two roots. The constant α is also known as the *Golden Ratio*.

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}$$

What have we proved? We showed that the numbers α and β satisfy

$$\alpha^{n-1} = \alpha^{n-2} + \alpha^{n-3} \text{ and } \beta^{n-1} = \beta^{n-2} + \beta^{n-3}$$

for all natural numbers $n \geq 4$. Thus is particular, the sequences

$$1, \alpha, \alpha^2, \alpha^3, \alpha^4, \ldots$$

and

$$1, \beta, \beta^2, \beta^3, \beta^4, \ldots$$

satisfy the equation ($\star$) and each term (starting with the third) is sum of the previous two terms.

Note that a cancelled out from both sides while we were solving for r. That tells us, if a sequence satisfies ($\star$), and each term is multiplied by the same non-zero quantity, then the resulting sequence also satisfies ($\star$). To conclude, we have shown,

Theorem 1.1. *Suppose we have a Geometric Progression $\{a_i\}_{i\in\mathbb{N}}$ that satisfies*

$$a_i = a_{i-1} + a_{i-2} \quad \textit{for } i > 2$$

Then the sequence must be in one of the following two formats:

$$c, c\alpha, c\alpha^2, c\alpha^3, c\alpha^4, \ldots \quad \textit{for some constant } c$$

or

$$d, d\beta, d\beta^2, d\beta^3, d\beta^4, \ldots \quad \textit{for some constant } d.$$

At this point you might ask, why did we start by considering GP? The answer is because they are interesting. That is of course not a satisfactory reason, but it is in fact an example of a theme we follow to solve complicated Math problems. The theme can be summarized as follows:

Suppose we need to find a function that satisfies a complicated system of equation. We will start by the method of trial and error. Take some of the easiest functions you know and see if it fits the equations. This is pure guesswork and you just hope that you will get lucky. Maybe, it fits only some of the equations. Then make some more complicated function out of the easy functions, e.g. by adding them, multiplying them etc., and see if the new function still fits those equations. If it does, maybe you could make a family of functions, that have the same pattern, up to some difference in constants, that satisfy the equations. Now choose the constants in a way so that the rest of the equations are satisfied.

This method is used for solving recurrence relations, differential equations, functional equations etc. Here, in this example, we started with GPs because they are nice, and we hoped that we could fit them in ($\star$) i.e. there will be some GP that would satisfy part of the requirements. Once we found such candidates, now it's time to see if we can combine them in some way to make a family of sequences that satisfy ($\star$).

Claim. *A sequence $\{z_i\}_{i\in\mathbb{N}}$ defined as*

$$z_i = c\alpha^{i-1} + d\beta^{i-1}$$

for some constants c and d, has the property that each term (starting with the third) is sum of the previous two terms.

Proof. Since $\{c\alpha^i\}_{i\in\mathbb{N}}$ and $\{d\beta^i\}_{i\in\mathbb{N}}$ satisfy ($\star$), we get that

$$\begin{aligned} z_i &= c\alpha^{i-1} + d\beta^{i-1} \\ &= c\alpha^{i-2} + c\alpha^{i-3} + d\beta^{i-2} + d\beta^{i-3} \\ &= c\alpha^{i-2} + d\beta^{i-2} + c\alpha^{i-3} + d\beta^{i-3} \\ &= z_{i-1} + z_{i-2} \end{aligned}$$

□

So we might choose $F_i = z_i$ for all $i > 2$. At this point we will try to find the member of the family that also satisfies $F_1 = F_2 = 1$. Why now? Let's go back and see if could have tried that earlier.

Before theorem 1.1, we only had specific examples that satisfy ($\star$), and we didn't have any choice regarding the values of α and β. So none of the sequences we had would satisfy $F_1 = F_2 = 1$, henceforth called the initial conditions.

After the theorem 1.1, but before the claim, we had two possible choices. But again, you could only change one parameter (c **or** d) since α and β were fixed. You would have two equations and one variable that way, and the initial conditions might not get satisfied.

Looking at the starting conditions, we see that there are two equations. So we aimed to find a family of functions that satisfy ($\star$), and has two varying parameters (c **and** d) that we could specify ourselves. This led to proving the claim, and now we can try to find specific values ofr c and d that would help solve the initial conditions.

Now, if we want $F_i = z_i$ for all i, and not just $i > 2$, we must also have

$$F_1 = z_1 \text{ and } F_2 = z_2$$

In other words, we get two equations

$$F_1 = 1 = c\alpha^0 + d\beta^0 = z_1$$
$$F_2 = 1 = c\alpha^1 + d\beta^1 = z_2$$

Simplifying, we get,

$$c + d = 1 \quad \text{ and } c\alpha + d\beta = 1$$

Solving this system of linear equations for c and d we find that

$$c = \frac{1-\beta}{\alpha-\beta}, d = \frac{\alpha-1}{\alpha-\beta}$$

Using the exact values for α and β, we find that $1-\alpha = \beta$ and $1-\beta = \alpha$ and $\alpha - \beta = \sqrt{5}$. Thus,

$$c = \frac{\alpha}{\sqrt{5}}, \quad d = \frac{-\beta}{\sqrt{5}}$$

We conclude that for all $i \in \mathbb{N}$, we have

$$F_i = z_i = c\alpha^{i-1} + d\beta^{i-1} = \frac{\alpha}{\sqrt{5}}\alpha^{i-1} + \frac{-\beta}{\sqrt{5}}\beta^{i-1} = \frac{\alpha^i - \beta^i}{\sqrt{5}}$$

Remark 1.2. The formula $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$ for all n, is called *Binet's Formula*. Although we now have an explicit way to define the nth term of Fibonacci sequence, it is still prudent to try to use induction first to solve interesting properties of the sequence.

Example 1.3. Let's try to use the same method as above to solve the recurrence relation

$$a_1 = 0, \quad a_2 = 2, \quad a_n = 3a_{n-1} - 2a_{n-2} \text{ for } n > 2$$

The corresponding *characteristic equation* is $r^2 = 3r - 2$. The roots are $r = 1$ and $r = 2$. Hence the general solution is of the format $a_n = c1^{n-1} + d2^{n-1}$. Using the initial conditions, we get

$$c + d = 0, c + 2d = 2 \implies c = -2, d = 2$$

Hence $a_n = -2 \times 1^{n-1} + 2 \times 2^{n-1} = 2^n - 2$ for all n.

Example 1.4. Let's try one more recurrence relation, now with three initial conditions.

$$a_1 = 3, \quad a_2 = 1, \quad a_3 = 9, \quad a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3} \text{ for } n > 3$$

Note that we need three initial conditions since every term (starting with the fourth) is defined using previous **three** terms.

The corresponding *characteristic equation* is $r^3 = 2r^2 + r - 2$. We can factorize

$$r^3 - 2r^2 - r + 2 = (r-1)(r^2 - r - 2) = (r-1)(r-2)(r+1)$$

Hence, the roots are $r = 1$, $r = 2$, and $r = -1$. Then the general solution to the recurrence is of the format $a_n = c \times 1^{n-1} + d \times 2^{n-1} + e \times (-1)^{n-1}$. Using the initial conditions, we get

$$\left.\begin{aligned} c + d + e &= 3 \\ c + 2d - e &= 1 \\ c + 4d + e &= 9 \end{aligned}\right\} \implies \begin{cases} c &= -1 \\ d &= 2 \\ e &= 2 \end{cases}$$

Hence

$$a_n = (-1) \times 1^{n-1} + 2 \times 2^{n-1} + 2 \times (-1)^{n-1} = 2^n - 1 - 2(-1)^n$$

for all n.

Exercise 7. *Solve the recurrence relation*

$$a_1 = 1, \quad a_2 = 2, \quad a_n = 5a_{n-1} - 6a_{n-2} \quad \textit{for } n > 2$$