Optional Group Project: Vectors and Euclidean Geometry

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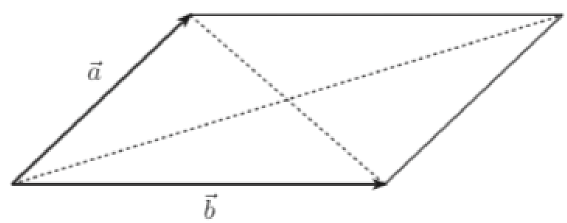
Recall your high school course in geometry. In such a course you would have constructed proofs, many of which were quite difficult. However, it is often possible to use vectors to find simple proofs for geometric theorems. Below are four propositions from geometry, some of which you may have seen in high school geometry. Using the properties of vector algebra we've developed in our course, give short proofs of each of these propositions.

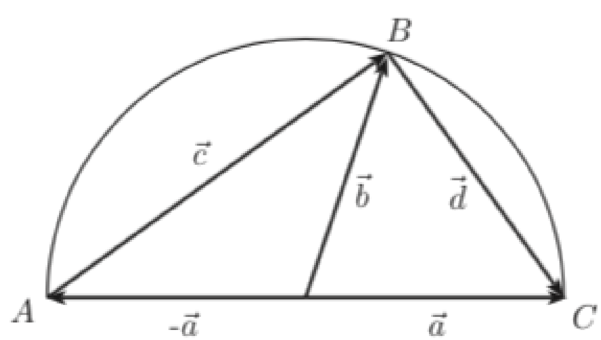
Proposition 1

If diagonals of a paralelogram are perpendicular to each other, then the parallelogram is a rhombus.



Proposition 2

Every angle inscribed in a semicircle is a right angle.



[Hint: In above picture show that $\vec{c}$ and $\vec{d}$ are perpendicular for any choice of $\vec{b}$]

Proposition 3

If a triangle has side lengths a, b, c with opposite angles A, B, C respectively, then

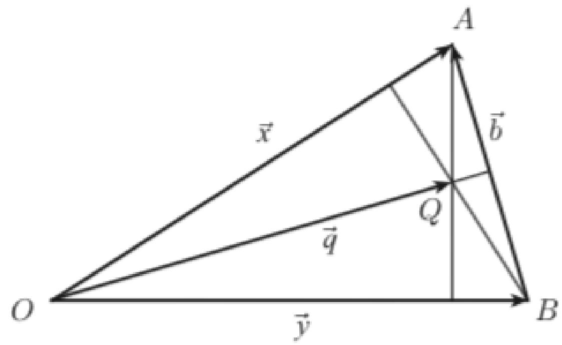
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

*The idea and problems in this project are due to Professor William Barker.

[HINTS: Consider the three vectors $\vec{a}, \vec{b}$, and $\vec{c}$ as usual. What is $(\vec{a}+\vec{b}+\vec{c})$? What can you say about $\vec{a} \times (\vec{a}+\vec{b}+\vec{c})$? Show that $\vec{a} \times \vec{b} = \vec{c} \times \vec{a}$. Then use the geometric definition of cross products.]

Proposition 4

The three altitudes of a triangle $\triangle OAB$ meet at one point.



[HINTS: Suppose the altitudes from A and B intersect at Q. Then to show the third altitude from O also contains Q you only need to show the vector $\vec{q} = \overrightarrow{OQ}$ is perpendicular to the vector $\vec{b} = \overrightarrow{BA}$.

The vector $\overrightarrow{QA}$ is perpendicular to the vector $\vec{y} = \overrightarrow{OB}$, and the $\overrightarrow{QB}$ is perpendicular to the vector $\vec{x} = \overrightarrow{OA}$. (These facts follow from the definition of an altitude of a triangle.) Convert these facts into statements about dot products and use them to prove $\vec{q} \perp \vec{b}$.]