Math 2208: Ordinary Differential Equations

Lecture 7 Worksheet

Fall 2019 **Subhadip Chowdhury** **Sep 25**

Title: Bifurcation Theory

Summary: We will learn about a modern analytical technique which allows one to characterize how solutions of differential equations which contain parameters changes when the parameter values vary.

Over the past couple of weeks we have developed the *logistic model* of population growth in a system with limited resources. For small values of the population, growth is proportional to the size of the population. But as the population gets larger, resource constraints and overcrowding force growth to slow down, even become negative, leading to the idea of a carrying capacity. The DE model is given by

$$\frac{dp}{dt} = kp\left(1 - \frac{p}{N}\right)$$

where N is the carrying capacity and k is the proportionality constant.

Over the course of next few lectures, we would like to investigate several models of such a system in a scenario where the population gets 'harvested' over time. For example, this could be a commercially valuable species, e.g. fish in a pond, that gets repeatedly harvested. We will try to analyze how the equilibrium points change as the parameter values vary.

§1. Parameter Sensitivity

Consider a fish population in a lake with the following harvesting scheme. A certain number of fishing licenses are issued that specify how many fish can be caught per year. This would cause a direct decrease in the population, modeled by

$$\frac{dp}{dt} = kp\left(1 - \frac{p}{N}\right) - h \tag{1}$$

where h is the number of fish harvested each year via fishing and $p(t)$ is the number of fish at time t in years. If no fish are caught $(h = 0)$, then the model is the standard logistic population model. Notice that our model is an autonomous ODE. For the remainder of this worksheet, assume $k = N = 6$ so that the ODE is

$$p' = p(6 - p) - h \tag{2}$$

The goal is to determine how the fish population is effected as the value of h increases away from 0.

Question 1.

a. In the space below, draw phase lines for the critical points when the value of h equals 0, 4, 8, and 12. Identify and classify any and all critical points for each value of h. What do you notice?

$\boxed{\text{b}}$ Is there a particular value of h for which the nature of the solution changes? If so, find it.

Bifurcation. When a small change in the value of a parameter leads to a drastic change in the qualitative nature of the phase portrait or long-term behavior of the solution of a differential equation, the phenomenon is called a bifurcation of the ODE. The value of the parameter at which such changes occur is known as a bifurcation value of the ODE.

Another possible way of saying this is as follows. A point (μ_0, y_0) is a bifurcation point for the ODE $y' = f_\mu(y)$, if the number of solutions of the equation $f_\mu(y) = 0$ for y in a neighborhood of (y_0, μ_0) is *not* a constant independent of μ.

§2. Analysis of Bifurcations

Bifurcation Diagram. A bifurcation diagram is a picture of the phase lines near a bifurcation value. It appears as a curve in the plane with the autonomous variable y on the vertical axis, and the bifurcation parameter on the horizontal axis. Generally a dotted line is used to indicate unstable sections of the curve (i.e. sources) and a solid line is used to indicate stable sections (i.e. sinks).

The most important bifurcations in one dimension are described locally by the following ODEs:

$$y' = \mu - y^2 \qquad \text{(saddle-node)}$$
$$y' = \mu y - y^2 \qquad \text{(transcritical)}$$
$$y' = \mu y - y^3 \qquad \text{(supercritical pitchfork)}$$
$$y' = \mu y + y^3 \qquad \text{(subcritical pitchfork)}$$

■ **Question 2.**

(a) Go back to the ODE in the first page. Draw the bifurcation diagram with p on the vertical axis and h on the horizontal axis.

This bifurcation is called a *saddle-node* bifurcation. This is probably the most typical kind of bifurcation to arise. In it, a pair of equilibria, one stable and one unstable, coalesce at the bifurcation point, annihilate each other and disappear. Note that up to a change of variable $y = p-3$ and $\mu = 9-h$, the ODE in equation (2) transforms to $y' = \mu - y^2$.

(b) Based on your bifurcation diagram above, explain why it might be a bad idea to harvest at a rate of h less than but very very close to 9?

Theorem 2.1

Given a one-parameter family of differential equations $y' = f_\mu(y)$, a point (μ_0, y_0) is a bifurcation point iff both $f_\mu(y_0) = 0$ and $f'_{\mu_0}(y_0) = 0$.

In the previous harvesting scheme, when $0 < h < 9$, you should have observed that with low enough initial population, the model predicts $p(t) \to -\infty$, which is obviously something we don't want to happen. This suggests that the model needs serious revision to account for lower populations, as it is clear that one cannot keep harvesting a fixed limit h when the population becomes small. To modify the model, we propose a second harvesting scheme. We will harvest a fixed *proportion* h of the fishes per year. If the population is $p(t)$ at time t in years, the *harvest rate* is hp, and the equation now becomes

$$p' = p(6-p) - hp \tag{3}$$

Question 3.

(a) First let's make a change of variable $\mu = 6 - h$. So our ODE becomes

$$p' = \mu p - p^2 \tag{4}$$

Draw phase lines for the critical points when the value of μ equals -1, 0 and 1. Identify and classify any and all critical points for each value of μ.

(b) Draw the bifurcation diagram with p on the vertical axis and μ on the horizontal axis.

This *transcritical* bifurcation arises in systems where there is some basic "trivial" solution branch, corresponding here to $p = 0$, that exists for all values of the parameter μ. (This differs from the case of a saddle-node bifurcation, where the solution branches exist locally on only one side of the bifurcation point.). There is a second solution branch $p = \mu$ that crosses the first one at the bifurcation point $(p_0, \mu_0) = (0, 0)$. When the branches cross one solution goes from stable to unstable while the other goes from stable to unstable. This phenomenon is referred to as an "exchange of stability.".

(c) For what values of h (recall that $h = 6 - \mu$), is it possible for the fishes to go extinct with this harvesting scheme?

(d) If you harvest at a per-capita rate $h < 6$, what will be the long-term population size? What will be the long term harvest rate?

(e) What value of h maximizes the harvest rate hp in the long term?

Question 4.

Draw the bifurcation diagram for each of the following ODEs. Here the parameter μ is a real number.

(a) $y' = \mu y - y^3$

This type of bifurcation is called a pitchfork (more precisely *supercritical pitchfork*) bifurcation and the reason for the nomenclature will be clear once you draw it. In this, a stable solution branch bifurcates into two new stable branches (as well as an unstable branch) as the parameter μ is increased.

(b) $y' = \mu y + y^3$

This is called a *subcritical pitchfork*. A supercritical pitchfork bifurcation leads to a "soft" loss of stability, in which the system can go to nearby stable equilibria when the equilibrium $y = 0$ loses stability as μ passes through zero. On the other hand, a subcritical pitchfork bifurcation leads to a "hard" loss of stability, in which there are no nearby equilibria and the system goes to some far-off dynamics (or perhaps to infinity) when the equilibrium $y = 0$ loses stability.