Assignment 9 (1/30)

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For all of the following problems, sufficient reasoning/proof is needed.

Problem 1

1. Consider the sequence $a_n = 2a_{n-1} - \alpha$, with $a_1 = \beta$, for α, β real numbers. What can we say about this sequence for sure?

   (A) $\{a_n\}$ is eventually increasing for all values of α, β.

   (B) $\{a_n\}$ is eventually decreasing for all values of α, β.

   (C) $\{a_n\}$ is eventually constant for all values of α, β.

   (D) $\{a_n\}$ is either increasing or decreasing, and which case occurs depends on the values of α and β.

   (E) $\{a_n\}$ is eventually constant, increasing or decreasing, and which case occurs depends on the values of α and β.

2. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\lim\limits_{x\to 0} \frac{g(x)}{x} = L$ for some $L \neq 0$. What is $\lim\limits_{x\to 0} \frac{g(g(x))}{x} =$

   (A) 0

   (B) L

   (C) L^2

   (D) $g(L)$

   (E) $g(L)/L$

3. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\lim\limits_{x\to 0} \frac{g(x)}{x^2} = L$ for some $L \neq 0$. What is $\lim\limits_{x\to 0} \frac{g(g(x))}{x^4} =$

   (A) L

   (B) L^2

   (C) L^3

   (D) $L^2 g(L)$

   (E) $g(L)/L^2$

4. Suppose a function $y(t)$ satisfies the differential equation $y' = f(y)$ for all t, where f is a continuous function. Further suppose $\lim_{t\to\infty} y = L$, for some finite L. What can we conclude about L?

   (A) $f(L) = L$
   (B) $f(L) = 0$
   (C) $f'(L) = L$
   (D) $f'(L) = 0$
   (E) $f''(L) = 0$

Problem 2

1. Which of the following series converge(s)?

   (A) $\sum \frac{1}{\ln(\ln k)}$
   (B) $\sum \frac{1}{\ln k}$
   (C) $\sum \frac{1}{(\ln(\ln k))^2}$
   (D) $\sum \frac{1}{k(\ln k)(\ln(\ln k))}$
   (E) $\sum \frac{1}{k(\ln k)(\ln(\ln k))^2}$
   (F) All of the above

2. Which of the following series converge(s)?

   (A) $\sum \frac{1}{k\ln(k^2+1)}$
   (B) $\sum \frac{1}{k(\ln(k^2)+1)}$
   (C) $\sum \frac{1}{k\ln(k^2)+1}$
   (D) $\sum \frac{1}{k((\ln k)^2+1)}$
   (E) All of the above.

3. What can you say about the series $\sum \frac{1}{2^{2^k}}$

   (a) It converges and the sum is strictly between 0 and 1.
   (b) It converges and the sum is strictly between 1 and 2.
   (c) It converges and the sum is equal to 1.

(d) It converges and the sum is equal to 2.

(e) It diverges.

4. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^p}$, where x is a positive real number and p is a natural number. For what values of x and p does it converge?

Problem 3 (Optional Bonus Problem)

Suppose $p > 1$ and let $\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$.
ζ is pronounced as zeta.

1. What is the value of $\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^p}$? *Hint: sum of the odd-numbered terms is the total minus the sum of even numbered terms.*

   (A) $2\zeta(p) - 1$

   (B) $\zeta(p)/3$

   (C) $(2^p - 1)\zeta(p)$

   (D) $(1 - 2^{-p})\zeta(p)$

   (E) $\zeta(p)/(2^p + 1)$

2. Suppose $p > 1$. What is the value of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$?

   (A) $-\zeta(p)$

   (B) $\zeta(p)/3$

   (C) $\zeta(p)/2$

   (D) $(2^{p-1} - 1)\zeta(p)$

   (E) $(2^{1-p} - 1)\zeta(p)$

   There is a result of calculus which states that, *under suitable conditions*, if $f_1, f_2, \ldots, f_n, \ldots$ are all functions, and we define $f(x) := \sum_{n=1}^{\infty} f_n(x)$, then $f^{(r)}(x) = \sum_{n=1}^{\infty} f_n^{(r)}(x)$ for any positive integer r. In other words, under suitable assumptions, we can repeatedly differentiate a sum of countably many functions by repeatedly differentiating each of them and adding up the derivatives.

3. Suppose $p > 1$. Assume that the required assumptions as above are valid for this summation, so that $\zeta''(p)$ is the sum of the second derivatives of each of the summands (w.r.t. p). Then $\zeta''(p)$ is equal is

   (A) $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln p)^2}{n^p}$

   (B) $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln p)(\ln n)}{n^p}$

   (C) $\displaystyle\sum_{n=1}^{\infty} \frac{-(\ln p)(\ln n)}{n^p}$

(D) $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^p}$

(E) $\displaystyle\sum_{n=1}^{\infty} \frac{-(\ln n)^2}{n^p}$

4. What can you say about the nature of ζ on $(1, \infty)$?

   (A) Increasing and concave up.

   (B) Decreasing and concave up.

   (C) Increasing and concave down.

   (D) Decreasing and concave down.

   (E) Decreasing, initially concave down, then concave up.

Problem 4 (Optional Bonus Problem)

If an infinite series with *positive* summands converges, then you are allowed to rearrange the summands (i.e. the underlying sequence) and the new series still converges to the same limit.

5. Show that $\displaystyle\sum_{p=2}^{\infty} (\zeta(p) - 1) = 1$.

   - Start by writing each of the $\zeta(p)$ as a sum as above and then change the order of summation i.e. first sum over p and then over n.
   - Use formula for geometric series to simplify the first sum and then a telescoping sum technique for the second one.
   - Note that p starts at 2.