Final Report
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Homology of CW-complexes

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1 Introduction

I, Subhadip Chowdhury was selected in SRFP-2011 to work under Prof. Shiva Shankar at Chennai Mathematical Institute. Throughout the summer project in form of a summer course, I have been studying Homology Theory as an introduction to Algebraic Topology. I have used Allen Hatcher's Algebraic Topology and James W. Vick's Homology Theory as my reference. The goal of the project was to be able to calculate Homology groups of various *CW*-complexes.

CW complexes have proved over time to be the most natural class of spaces for Algebraic Topology. This is because more or less every 'simple looking' mathematical object is a *CW* complex made up of 'cells'. Our goal is to mathematically distinguish between the objects so as to categorize them as being 'different'. The fundamental group $\pi_1(X)$ is especially useful when studying spaces of low dimension, as one would expect from its definition which involves only maps from low-dimensional spaces into X , namely loops $I \to X$ and homotopies of loops, maps $I \times I \to X$. This limitation to low dimensions can be removed by considering the natural higher-dimensional analogs of $\pi_1(X)$, the homotopy groups $\pi_n(X)$.However, the higher-dimensional homotopy groups have the serious drawback that they are extremely difficult to compute in general. Even for simple spaces like spheres, the calculation of $\pi_i(S^n)$ for $i > n$ turns out to be a huge problem. Here Homology groups come into picture. The homology group $H_n(X)$ for a *CW* complex X depends only on the $(n+1)$ skeleton. For spheres, the homology groups $H_i(S^n)$ are isomorphic to the homotopy groups $\pi_i(S^n)$ in the range $1 \leq i \leq n$, but homology groups have the advantage that $H_i(S^n) = 0$ for $i > n$.
In what follows, we try to develop the general theory of Singular Homology and applications toward developing techniques for calculating homology groups efficiently. The maximally efficient method is known as cellular homology, whose power comes perhaps from the fact that it is 'homology squared' - homology defined in terms of homology.

2 Definitions

- An *n-cell* is a space homeomorphic to the open n-disk $\text{int}(D_n)$. A cell is a space which is an n-cell for some $n \geq 0$.
- A *CW-Complex* or a *Cell Complex* is a (hausdorff) space X which has a partition of open cells such that the following hold:
  1. For each cell n-cell, usually denoted e_i^n which is read 'the i-th n-cell', we have a continuous map ϕ from the closed n-disk D^n into X such that restricting ϕ to the interior of the domain n-disk gives a homeomorphism onto our cell e_i^n (as in, it takes the interior of the domain disk onto the cell, which should be homeomorphic to that disk anyway) and the image of the boundary of the domain n-disk under ϕ is contained in a union of finitely many cells with dimension less than n.
  2. A subset of X is closed if and only if the (potentially empty) intersection with each cell is closed.

  Note that '*CW*' stands for "Closure-finite" + "Weak topology" . And the second condition in the definition is the same thing as saying we give the *CW*-complex X the weak topology. An n-cell will be said to have dimension n.
- A cell-decomposition of a space X is a family $E = \{e_\alpha | \alpha \in I\}$ of subspaces of X such that each e_α is a cell and $X = \coprod_{\alpha \in I} e_\alpha$. The n-skeleton of X is the subspace

$$X^n = \coprod_{\alpha \in I : dim(e_\alpha) \leq n} e_\alpha$$

  .
- If $X = X^n$ for some n , then X is said to be finite-dimensional, and the smallest such n is the dimension of X , the maximum dimension of cells of X .

2.1 Examples of CW-complexes

1. A 1 dimensional cell complex $X = X^1$ consists of vertices (the 0 cells) to which edges (the 1 cells) are attached. The two ends of an edge can be attached to the same vertex.
2. The sphere S^n has the structure of a cell complex with just two cells, e^0 and e^n , the n cell being attached by the constant map $S^{n-1} \to e^0$. This is equivalent to regarding S^n as the quotient space $D^n/\partial D^n$.
3. $\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$ by attaching an n cell, with the quotient projection $S^{n-1} \to \mathbb{R}P^{n-1}$ as the attaching map. It follows by induction on n that $\mathbb{R}P^n$ has a cell complex structure $e^0 \cup e^1 \cup \cdots \cup e^n$ with one cell e^i in each dimension $i \leq n$.
4. Similarly we obtain a cell structure $\mathbb{C}P^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$ with cells only in even dimensions.

3 Singular Homology

3.1 Definitions

- The n dimensional analog of the triangle is the n simplex. $\Delta^n = \{(t_0, \cdots, t_n) \in \mathbb{R}^{n+1} | \sum_i t_i = 1$ and $t \geq 0 \forall i\}$.
- A singular n simplex in a space X is by definition just a map $\sigma : \Delta^n \to X$.
- The free abelian group with basis the set of singular n simplices in a space X is called $C_n(X)$ and elements of $C_n(X)$ are called n chains.
- A boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ is defined by:

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0, \cdots, \hat{v_i}, \cdots, v_n]}$$

- The n-th **singular homology group** of X is defined as $H_n(X) = Ker\partial_n / Im\partial_{n+1}$.

3.2 Some useful properties

An interesting feature of homology is that it is the basic properties of homology that are used most often, and not the actual definition itself. So some general properties of Singular Homology is covered before going into Applications.

- The composition $\partial \circ \partial$ is zero.
- If X is a nonempty path-connected space, then

$$H_0(X) \approx \mathbb{Z}$$

- If X is a space and $X_\alpha : \alpha \in A$ are the path components of X, then

$$H_k(X) \approx \sum_{\alpha \in A} H_k(X_\alpha)$$

- Homotopy equivalent spaces have isomorphic homology groups. This can be done by showing that a map $f : X \to Y$ induces a homomorphism $f_* : H_n(X) \to H_n(Y)$ for each n . If two maps $f, g : X \to Y$ are homotopic, then they induce the same homomorphism $f_* = g_* : H_n(X) \to H_n(Y)$, and so f_* is an isomorphism if f is a homotopy equivalence.

3.3 Exact Sequence

- A sequence of homomorphisms

$$\cdots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \to \cdots$$

is said to be **exact** if $Ker\alpha_n = Im\alpha_{n+1}$ for each n .

- Observe that $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact iff α is injective, β is surjective, and $Ker\beta = Im\alpha$, so β induces an isomorphism $C \approx B/Im\alpha$. This can be written $C \approx B/A$ if we think of α as an inclusion of A as a subgroup of B. An exact sequence $0 \to C \to D \to E \to 0$ as above is called a **short exact sequence.**

- If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence of chain complexes and degree zero chain maps, then the long exact sequence

$$\cdots \xrightarrow{f_*} H_n(D) \xrightarrow{g_*} H_n(E) \xrightarrow{\partial} H_{n-1}(C) \xrightarrow{f_*} H_{n-1}(D) \xrightarrow{g_*} \cdots$$

is exact.

- If X is a space and A is a nonempty closed subspace that is a deformation retract of some neighborhood in X ,then the pair of spaces (X, A) will be called good pair.

- Given a space X and a subspace $A \subset X$, let $C_n(X, A)$ be the quotient group $C_n(X)/C_n(A)$. Thus chains in A are trivial in $C_n(X, A)$. Since the boundary map $\partial : C_n(X) \to C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$, it induces a quotient boundary map $\partial : C_n(X, A) \to C_{n-1}(X, A)$. Letting n vary, we have a sequence of boundary maps

$$\cdots \to C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \to \cdots$$

The relation $\partial^2 = 0$ holds for these boundary maps since it holds before passing to quotient groups. So we have a chain complex, and the homology groups $Ker\partial/Im\partial$ of this chain complex are by definition the **relative homology groups** $H_n(X, A)$.

- The relative homology groups $H_n(X, A)$ for any pair (X, A) fit into a long exact sequence

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to H_{n-1}(X) \to$$

$$\cdots \to H_0(X, A) \to 0$$

A collection $\mathfrak{U}$ of subsets of X is a covering of X if $X \subseteq \bigcup_{U \in \mathfrak{U}} U$. For $\mathfrak{U}$ any covering of X, denote by $S_n^{\mathfrak{U}}(X)$ the subgroup of $S_n(X)$ generated by the singular n-simplices $\phi : \sigma_n \to X$ for which $\phi(\sigma_n)$ is contained in some $U \in \mathfrak{U}$.
If $\mathfrak{U}$ is a family of subsets of X such that Int $\mathfrak{U}$ is a a covering of X, then

$$i_* : H_n(S_n^{\mathfrak{U}}(X)) \to H_n(X)$$

is an isomorphism for each n.

3.4 Excision

Theorem: If $(X, A)$is a pair of spaces and U is a subset of A with $\bar{U}$ contained in the interior of A, then the inclusion map

$$i : (X - U, A - U) \to (X, A)$$

induces an isomorphism on relative homology groups

$$i_* : H_*(X - U, A - U) \to H_*(X, A).$$

That is, such a set U may be excised without altering the relative homology groups. Equivalently, for subspaces $A, B \subset X$ whose interiors cover X , the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_n(B, A \cap B) \to H_n(X, A)$ for all n .

Proposition: For good pairs (X, A) , the quotient map $q : (X, A) \to (X/A, A/A)$ induces isomorphisms $q_* : H_n(X, A) \to H_{n-1}(X/A, A/A)$

As an application we get the following : If nonempty open sets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$.

4 Applications: Tools to calculate Homology Groups

4.1 Mayer-Vietoris Sequence

Our first task is the development of a technique for studying the homology of a space X in terms of the homology of the components of a covering $\mathfrak{U}$ of X. In the simplest nontrivial case the covering consists of two subsets U and V for which $\text{Int}U \cup \text{Int}V = X$.
Setting A to be the set of all singular n-simplices in U and B to be the set of all singular n-simplices in V, we get

$$S_n(U) = F(A), S_n(V) = F(B)$$

$$S_n(U \cap V) = F(A \cap B), S_n^{\mathfrak{U}}(X) = F(A \cup B)$$

where $F(Z)$ is the free group generated by Z. Note that there is a natural onto homomorphism

$$h : F(A) \oplus F(B) \to F(A \cup B)$$

given by

$$h(a_i, b_j) = a_i + b_j$$

Similarly there is a natural $1 - 1$ homomorphism

$$G : F(A \cap B) \to F(A) \oplus F(B)$$

given by

$$g(b_i) = (b_i, -b_i)$$

These maps induce the following short exact sequence

$$0 \to S_n(U \cap V) \to S_n(U) \oplus S_n(V) \to S_n^{\mathfrak{U}}(X) \to 0$$

which gives a long exact sequence which can be transformed to following:

$$\cdots \xrightarrow{\Delta} H_n(U \cap V) \xrightarrow{g_*} H_n(U) \oplus H_n(V) \xrightarrow{h_*} H_n(X) \xrightarrow{\Delta} H_{n-1}(U \cap V) \to \cdots$$

Above sequence is called the **Mayer-Vietoris Sequence**.

Example: Take $X = S^n$ with A and B the northern and southern hemispheres, so that $A \cap B = S^{n-1}$. Then in the Mayer-Vietoris sequence the terms $H_i(U) \oplus H_i(V)$ are zero, so we obtain isomorphisms $H_i(S^n) \approx H_{i-1}(S^{n-1})$. This gives a way of calculating the homology groups of S^n by induction. Thus we get for any integer $n \geq 0, H_*(S^n)$ is a free abelian is a free abelian group with two generators, one in dimension zero and one in dimension n.

Application: ∂D^n is not a retract of D^n . Hence every map $f : D^n \to D^n$ has a fixed point. **[Brouwer fixed-point theorem]**

4.2 Degree

For a map $f : S^n \to S^n$ with $n > 0$, the induced map $f_* : H_n(S^n) \to H_n(S^n)$ is a homomorphism from an infinite cyclic group to itself and so must be of the form $f_*(\alpha) = d\alpha$ for some integer d depending only on f . This integer is called the degree of f , with the notation deg f . Here are some basic properties of degree:

1. deg $\mathbb{1} = 1$, since $\mathbb{1}_* = 1$

2. deg $f = 0$ if f is not surjective. For if we choose a point $x_0 \in S^n - f(S^n)$ then f can be factored as a composition $S^n \to S^n - \{x_0\} \hookrightarrow S^n$ and $H_n(S^n - \{x_0\}) = 0$ since $S^n - \{x_0\}$ is contractible. Hence $f_* = 0$.

3. If $f \simeq g$ then deg f = deg g since $f_* = g_*$.

4. deg fg = deg f deg g , since $(fg)_* = f_* g_*$. As a consequence, deg $f = \pm 1$ if f is a homotopy equivalence since $fg \simeq 1$ implies deg f deg g = deg $\mathbb{1} = 1$.

5. deg $f = -1$ if f is a reflection of S^n , fixing the points in a subsphere S^{n-1} and interchanging the two complementary hemispheres.

6. The antipodal map $-\mathbb{1} : S^n \to S^n$, has degree $(-1)^{n+1}$ since it is the composition of $n+1$ reflections, each changing the sign of one coordinate in $\mathbb{R}^{n+1}$.

7. deg Sf = deg f , where $Sf : S^{n+1} \to S^{n+1}$ is the suspension of the map $f : S^n \to S^n$.

8. If f : $S^n \to S^n$ has no fixed points then deg $f = (-1)^{n+1}$. For if $f(x) \neq x$ then the line segment from $f(x)$ to $-x$, defined by $t \mapsto (1-t)f(x) - tx$ for $0 \leq t \leq 1$,does not pass through the origin. Hence if f has no fixed points, the formula $f_t(x) = \frac{(1-t)f(x)-tx}{||(1-t)f(x)-tx||}$ defines a homotopy from f to the antipodal map. Note that the antipodal map has no fixed points, so the fact that maps without fixed points are homotopic to the antipodal map is a sort of converse statement.

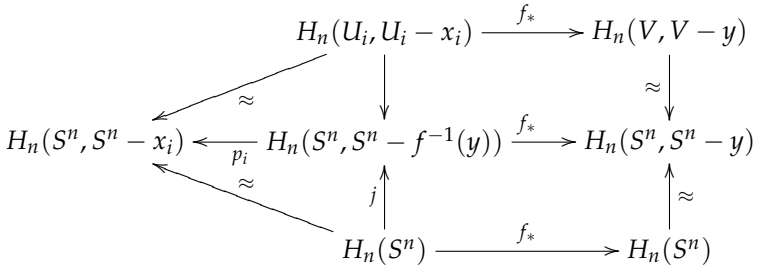
   Corollary: If $f : S^{2n} \to S^{2n}$ is a map, then there exists an x in S^{2n} with $f(x) = x$ or there exists a y in S^{2n} with$f(y) = -y$.

   Corolllary: There is no continuous map $f : S^{2n} \to S^{2n}$ such that x and $f(x)$ are orthogonal for all x.

   Corolllary: There exists no nonzero vector field on S^{2n}.

4.2.1 Local Degree

Definition : Suppose $f : S^n \to S^n, n > 0$, has the property that for some point $y \in S^n$, the preimage $f^{-1}(y)$ consists of only finitely many points, say $x_1, \cdots, x_m$. Let $U_1, \cdots, U_m$ be disjoint neighborhoods of these points, mapped by f into a neighborhood V of y . Then $f(U_i - x_i) \subset V - y$ for each i , and we have a commutative diagram



where all the maps are the obvious ones, in particular k_i and p_i are induced by inclusions. The two isomorphisms in the upper half of the diagram come from excision, while the lower two isomorphisms come from exact sequences of pairs. Via these four isomorphisms, the top two groups in the diagram can be identified with $H_n(S^n) \approx Z$, and the top homomorphism f_* becomes multiplication by an integer called the **local degree** of f at x_i , written deg $f|x_i$.

Use: The most important use of local degree is in computing degree of a map using the following theorem: deg $f = \sum_i$ deg $f|x_i$ for any map $f : S^n \to S^n$.

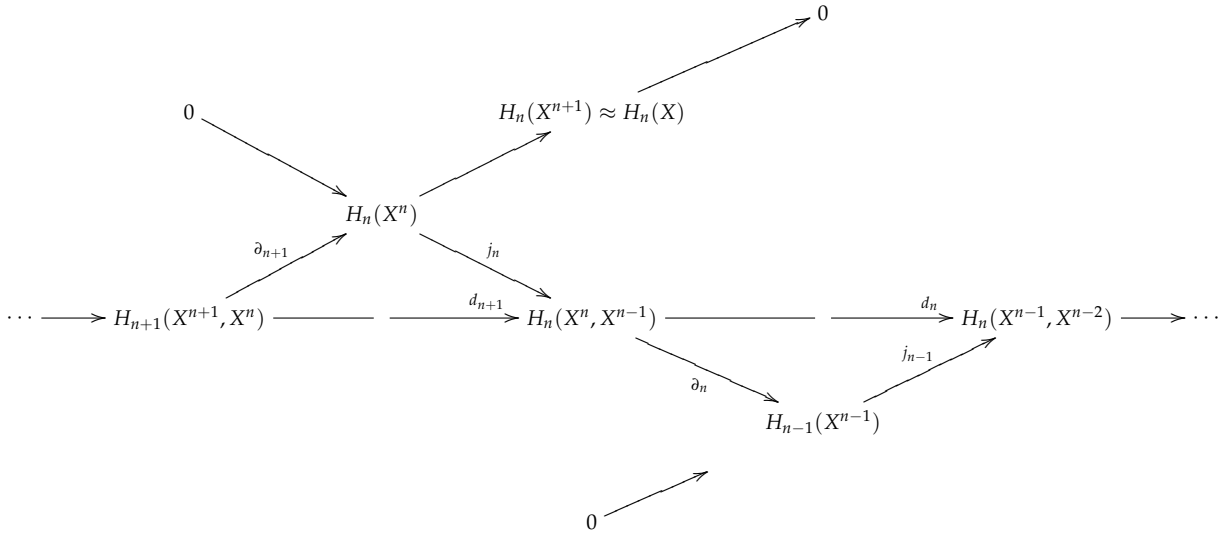
5 Cellular Homology

Now we come to the main task of calculating Homology group of CW-complexes. We use degree calculation to find the boundary maps and relative homology of the cell structures to find out the required homology group for a CW-complex.
we first establish a few preliminary facts:

Lemma: If X is a CW complex, then :

1. $H_k(X^n, X^{n-1})$ is zero for $k \neq n$ and is free abelian for $k = n$, with a basis in one-to-one correspondence with the n cells of X .
2. $H_k(X^n) = 0$ for $k > n$. In particular, if X is finite-dimensional then $H_k(X) = 0$ for $k >$ dim X.
3. The inclusion $i : X^n \hookrightarrow X$ induces an isomorphism $i_* : H(X^n) \to H(X)$ if $k < n$.

Definition: Let X be a CW complex. Using above lemma, portions of the long exact sequences for the pairs (X^{n+1}, X^n), (X^n, X^{n-1}), and (X^{n-1}, X^{n-2}) fit into a diagram



where d_{n+1} and d_n are defined as the compositions $j_n\partial_{n+1}$ and $j_{n-1}\partial_n$, which are just 'relativizations' of the boundary maps ∂_{n+1} and ∂_n . The composition d_nd_{n+1} includes two successive maps in one of the exact sequences, hence is zero. Thus the horizontal row in the diagram is a chain complex, called the **cellular chain complex** of X since $H_n(X^n, X^{n-1})$ is free with basis in one-to-one correspondence with the n cells of X , so one can think of elements of $H_n(X^n, X^{n-1})$ as linear combinations of n cells of X . The homology groups of this cellular chain complex are called the ***cellular homology groups*** of X . Temporarily we denote them $H_n^{CW}(X)$.

Theorem: $\mathbf{H_n^{CW}(X) \approx H_n(X)}$.

Proof: From the diagram above, $H_n(X)$ can be identified with $H_n(X^n)/$ Im ∂_{n+1}.Since j_n is injective, it maps Im ∂_{n+1} isomorphically onto Im$(j_n\partial_{n+1}) =$ Im d_{n+1} and $H_n(X^n)$ isomorphically onto Im j_n = Ker ∂_n . Since j_{n-1} is injective, Ker $\partial_n =$ Ker d_n. Thus j_n induces an isomorphism of the quotient $H_n(X^n)/$ Im ∂_{n+1} onto Ker $d_n/$ Im d_{n+1}.

Applications :

- $H_n(X) = 0$ if X is a CW complex with no n cells.
- More generally, if X is a CW complex with k n cells, then $H_n(X)$ is generated by at most k elements. For since $H_n(X^n, X^{n-1})$ is free abelian on k generators, the subgroup Ker d_n must be generated by at most k elements, hence also the quotient Ker d_n / Im d_{n+1} .
- If X is a CW complex having no two of its cells in adjacent dimensions, then $H_n(X)$ is free abelian with basis in one-to-one correspondence with the n cells of X . This is because the cellular boundary maps d_n are automatically zero in this case. Thus

$$H_i(\mathbb{C}P^n) \approx \begin{cases} \mathbb{Z} & \text{for } i = 0, 2, 4, \cdots, 2n \\ 0 & \text{otherwise} \end{cases}$$

Cellular Boundary Formula: Next we describe how the cellular boundary maps d_n can be computed. In case X is connected and has only one 0 cell, then d_1 must be 0 , otherwise $H_0(X)$ would not be

$\mathbb{Z}$ When $n > 1$ we can compute d_n in terms of degrees:
$d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$ where $d_{\alpha\beta}$ is the degree of the map $S^{n-1}_\alpha \to X^{n-1} \to S^{n-1}_\beta$ that is the composition of the attaching map of e^n_α with the quotient map collapsing $X^{n-1} - e^{n-1}_\beta$ to a point.

6 Examples

1. For the n-dimensional sphere we have the cell complex $S^n = e^0 \cup e^n$. So for $C_n(S^n)$ we have one n-dimensional cell, thus the elements in $C_n(S^n)$ are $a_n < e^n >$, is isomorphic to $\mathbb{Z}$, that is $C_n(S^n) \approx Z$. Now for $C_{n-1}(S^n)$ through $C_1(S^n)$ we have $\{0\}$. However since we have one 0-dimensional cell we have $C_0(S^n) \approx \mathbb{Z}$. Thus we have the following chain complex

$$\cdots \to \{0\} \to C_n(S^n) \to \{0\} \to \cdots \to \{0\} \to C_0(S^n) \to \{0\}.$$

Now for our boundary ∂_n we are mapping $\partial_n : C_n(S^n) \approx \mathbb{Z} \to \{0\}$ we have that $\partial_n = 0$. Now for the kernel we have that $\ker(\partial_n) \approx \mathbb{Z}$ and for the image we have $\mathrm{im}(\partial_{n+1}) \approx \{0\}$.

$$H_n(S^n) \approx ker(\partial_n)/im(\partial_{n+1}) \approx \mathbb{Z}/\{0\} \approx \mathbb{Z}.$$

Now for $H_0(S^n)$ we have the same thing as $H_n(S^n)$ and thus $H_0(S^n) \approx \mathbb{Z}$. All the other homology groups of S^n are $\{0\}$. So for S^n we have the following:

$$H_p(S^n) \approx \begin{cases} \mathbb{Z} & p = m \text{ or } p = 0 \\ \{0\} & \text{otherwise} \end{cases}$$

2. Let M_g be the closed orientable surface of genus g with its usual CW structure consisting of one 0-cell, $2g$ 1-cells, and one 2-cell attached by the product of commutators $[a_1, b_1] \cdots [a_g, b_g]$. The associated cellular chain complex is

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \to 0$$

As observed above, d_1 must be 0 since there is only one 0-cell. Also, d_2 is 0 because each a_i or b_i appears with its inverse in $[a_1, b_1] \cdots [a_g, b_g]$, so the maps $\Delta_{\alpha\beta}$ are homotopic to constant maps. Since d_1 and d_2 are both zero, the homology groups of M_g are the same as the cellular chain groups, namely, $\mathbb{Z}$ in dimensions 0 and 2 , and $\mathbb{Z}^{2g}$ in dimension 1 .

3. The closed nonorientable surface N_g of genus g has a cell structure with one 0-cell, g 1-cells, and one 2-cell attached by the word $a_1^2 a_2^2 \cdots a_g^2$. Again $d_1 = 0$, and $d_2 : \mathbb{Z} \to \mathbb{Z}^g$ is specified by the equation $d_2(1) = (2, \cdots, 2)$ since each a_i appears in the attaching word of the 2-cell with total exponent 2 , which means that each $\Delta_{\alpha\beta}$ is homotopic to the map $z \mapsto z^2$, of degree 2 . Since $d_2(1) = (2, \cdots, 2)$, we have d_2 injective and hence $H_2(N_g) = 0$. If we change the basis for $\mathbb{Z}^g$ by replacing the last standard basis element $(0, \cdots, 0, 1)$ by $(1, \cdots, 1)$, we see that $H_1(N_g) \approx \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$.

4. T 3 1 A 3 dimensional torus $T^3 = S^1 \times S^1 \times S^1$ can be constructed from a cube by identifying each pair of opposite square faces.If a slightly different pattern of identifications of opposite faces are done, with the front and back faces now identified via a rotation of the cube around a horizontal left-right axis. The space produced by these identifications is the product $K \times S^1$ of a Klein bottle and a circle. For both T^3 and $K \times S^1$ we have a CW structure with one 3-cell, three 2-cells, three 1-cells, and one 0-cell. The cellular chain complexes thus have the form

$$0 \to \mathbb{Z} \xrightarrow{d_3} \mathbb{Z}^3 \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \to 0$$

In the case of the torus T^3 the cellular boundary map d_2 is zero by the same calculation as for the 2 dimensional torus. We claim that d_3 is zero as well. This amounts to saying that the three maps $\Delta_{\alpha\beta} : S^2 \to S^2$ corresponding to the three 2-cells have degree zero. Each $\Delta_{\alpha\beta}$ maps the interiors of two opposite faces of the cube homeomorphically onto the complement of a point in the target

S^2 and sends the remaining four faces to this point. Computing local degrees at the center points of the two opposite faces, we see that the local degree is +1 at one of these points and -1 at the other, since the restrictions of $\Delta_{\alpha\beta}$ to these two faces differ by a reflection of the boundary of the cube across the plane midway between them, and a reflection has degree -1 . Since the cellular boundary maps are all zero, we deduce that $H_i(T^3)$ is $\mathbb{Z}$ for $i = 0, 3$, $\mathbb{Z}^3$ for $i = 1, 2$, and 0 for $i > 3$.

For $K \times S^1$, when we compute local degrees for the front and back faces we find that the degrees now have the same rather than opposite signs since the map $\Delta_{\alpha\beta}$ on these two faces differs not by a reflection but by a rotation of the boundary of the cube. The local degrees for the other faces are the same as before. Using the letters A, B, C to denote the 2-cells given by the faces orthogonal to the edges a, b, c , respectively, we have the boundary formulas $d_3e^3 = 2C, d_2A = 2b, d_2B = 0$, and $d_2C = 0$. It follows that $H_3(K \times S^1) = 0, H_2(K \times S^1) = \mathbb{Z} \oplus \mathbb{Z}_2$, and $H_1(K \times S^1) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$.

5. **Real Projective Space** $\mathbb{R}P^n$. As we saw earlier $\mathbb{R}P^n$ has a CW-structure with one cell e_k in each dimension $k \leq n$, and the attaching map for e_k is the 2-sheeted covering projection $\phi : S^{k-1} \to \mathbb{R}P^n$. To compute the boundary map d_k we compute the degree of the composition $S^{k-1} \xrightarrow{\phi} \mathbb{R}P^{k-1} \xrightarrow{q} \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} = S^{k-1}$, with q the quotient map. The map $q\phi$ is a homeomorphism when restricted to each component of $S^{k-1} - S^{k-2}$, and these two homeomorphisms are obtained from each other by precomposing with the antipodal map of S^{k-1} , which has degree $(-1)^k$ .Hence deg $q\phi$ = deg $\mathbb{1}$+ deg $(-\mathbb{1}) = 1 + (-1)^k$, and so d_k is either 0 or multiplication by 2 according to whether k is odd or even. Thus the cellular chain complex for $\mathbb{R}P^n$ is

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \text{ if } n \text{ is even}$$

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \text{ if } n \text{ is odd}$$

So it follows that $H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & \text{for } k = 0 \text{ and for } k = n \text{ odd} \\ \mathbb{Z}_2 & \text{for } k \text{ odd}, 0 < k < n \\ 0 & \text{otherwise} \end{cases}$

6. Complex projective space $\mathbb{C}P^n$ has a CW decomposition with one cell in each even dimension $0, 2, \cdots, 2n$. It follows that $H_{2k}(\mathbb{C}P^n) \approx \mathbb{Z}$ for $k = 1, \cdots, n$, and the odd-dimensional homology groups vanish.