CAAP 2018

Quiz 3

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Full Name:

1. Determine which of the following statements are true or false. Write T/F in the box accordingly. **If the statement is False, give a counterexample. If it's true, no explanation is necessary.**

   (a) If **P** is the statement $\boxed{2}$

   There exists a natural number n such that π^n is rational,

   then **not P** is the statement

   π^n is irrational for all natural numbers n.

   Solution: True.

   (b) Given natural numbers a and b, if $6 \mid ab$, then either $6 \mid a$ or $6 \mid b$. $\boxed{2}$

   Solution: False. Take $a = 2, b = 3$.

   (c) For all real numbers x, we have $\boxed{2}$

   $$\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$$

   Solution: False. Take $x = 0.3$, $y = 0.8$.

   (d) Every composite number n must have at least two distinct positive factors other than 1 and n. $\boxed{2}$

   Solution: False. Take $n = 9$.

   (e) Given natural numbers a, b, and c, if $a \mid b$ and $a \mid c$, then $(a, b) = (a, c)$. $\boxed{2}$

   Solution: True.

2. Prove by induction that the following identity holds for all natural numbers n. $\boxed{10}$

$$\frac{1}{2\times 5} + \frac{1}{5\times 8} + \frac{1}{8\times 11} + \cdots + \frac{1}{(3n-1)\times(3n+2)} = \frac{n}{2(3n+2)}$$

Solution: We will prove the identity by inducting on n.

Base Case: When $n = 1$, the LHS$= \frac{1}{(3\times1-1)(3\times1+2)} = \frac{1}{2\times5} = \frac{1}{10}$ and the RHS$= \frac{1}{2\times(3\times5+2)} = \frac{1}{10}$. Hence the identity is true for $n = 1$.

Induction Hypothesis: Assume that the identity is true for some natural number k. Then by our induction assumption we have,

$$\frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+\cdots+\frac{1}{(3k-1)\times(3k+2)}=\frac{k}{2(3k+2)} \qquad (\star)$$

Induction Step: The $(k+1)$th term in the LHS of the identity is $\frac{1}{(3(k+1)-1)(3(k+1)+2)} = \frac{1}{(3k+2)(3k+5)}$.

Adding $\frac{1}{(3k+2)(3k+5)}$ to both sides of $(\star)$ we get,

$$\begin{aligned}
&\frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+\cdots \\
&\cdots+\frac{1}{(3k-1)\times(3k+2)}+\frac{1}{(3k+2)\times(3k+5)} = \frac{k}{2(3k+2)}+\frac{1}{(3k+2)(3k+5)} \\
&= \frac{k(3k+5)+2}{2(3k+2)(3k+5)} \\
&= \frac{3k^2+5k+2}{2(3k+2)(3k+5)} \\
&= \frac{(3k+2)(k+1)}{2(3k+2)(3k+5)} \\
&= \frac{k+1}{2(3k+5)} \\
&= \frac{k+1}{2(3(k+1)+2)}
\end{aligned}$$

Thus we have shown that the identity holds true for $n=k+1$ whenever it's true for $n=k$.

Hence by the Induction Principle, the identity is true for all natural number n.

3. Prove that two consecutive odd numbers are always relatively prime to each other. 7

[HINT: How do you write odd numbers? Consecutive odd numbers? You might need to use the identity $(a+bc,b)=(a,b)$.]

Solution: We can write two consecutive odd numbers as $2k-1$ and $2k+1$ for some integer k. Then taking $b=2k-1, c=1, a=2$ in the identity $(a+bc,b)=(a.b)$, we get that

$$(2k-1,2k+1)=(2+(2k-1)\times 1, 2k-1)=(2,2k-1)$$

Again taking $a=-1, b=2, c=2$ in the identity, we get

$$(2,2k-1)=(-1+2k,2)=(-1,2)=(1,2)=1$$

Hence $(2k-1)$ and $(2k+1)$ are relatively prime.

4. Consider the following Arithmetic Progression. $\boxed{8}$

$$24, 21, 18, 15, \ldots$$

If sum of the first n terms of this AP is 105, then find n.

Solution: The sum of first n terms of an AP is given by $S_n = \frac{n}{2}[2p + (n-1)d]$, where p is the first term and d is the common difference. Here $p = 24$, $d = -3$. So we have

$$\begin{aligned}
&\frac{n}{2}[48 + (n-1)(-3)] = 105 \\
\implies &48n - 3(n-1)n = 210 \\
\implies &48n - 3n^2 + 3n = 210 \\
\implies &3n^2 - 51n + 210 = 0 \\
\implies &n^2 - 17n + 70 = 0 \\
\implies &n^2 - 10n - 7n + 70 = 0 \\
\implies &(n-10)(n-7) = 0 \\
\implies &n = 7 \text{ or } 10
\end{aligned}$$

[Extra Credit, 1 pt.] Can you explain why there are two possible values of n?

Solution: There are two possible values of n because when the AP decreases to 0 and the terms become negative, they cancel out some of the positive terms that came before. More precisely, the $8, 9, 10$th term of this AP are $3, 0$, and -3, which add up to 0. Thus sum of first 7 terms is same as sum of first 10 terms.