

MATH 1800 HANDOUT 7: APPLICATIONS AND EXTENSIONS OF GREEN'S THEOREM

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Careful Statement of Green's Theorem

Green's Theorem is a statement about only the 2D plane, not 3D space. Here's the setup: Let C be a piecewise smooth simple closed curve that is the boundary of a simply-connected region D in the plane. Let $\vec{F} = P\vec{i} + Q\vec{j}$ be a smooth vector field defined on all of D and C .

Notes on terminology: To say a curve is *simple* means that it doesn't intersect itself, and to say a curve is *closed* means that it is a closed loop, that it starts and ends at the same point. A region is called *simply-connected* if it is just one piece (connected) and doesn't have any holes in it. A vector field is said to be *smooth* if it has continuous first partials.

Since C is the boundary of D , we write $C = \partial D$ and stop referring to C explicitly. When we consider the boundary curve of a simply-connected region, we always orient the curve so that the region is on the left as we follow the curve. (Note: this is harder to specify when one considers 3D surfaces with boundaries.)

Green's Theorem says

$$\oint_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D \text{curl}(F) dA = \iint_D (Q_x - P_y) dA.$$

You may also see this written the Leibniz notation for partial derivatives.

$$\oint_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Using Green's Theorem to Evaluate Difficult Line Integrals

■ If you're asked to find the line integral of an ugly vector field over a closed curve, you should look to see if Q_x and P_y are drastically more simple. If so, use Green's Theorem!

Exercise 1

Evaluate

$$\int_C (2y + \sqrt{9 + x^3}) dx + (5x + e^{\tan^{-1} y}) dy,$$

where C is the circle $x^2 + y^2 = 4$ in the plane oriented counterclockwise.

■ If you're asked to find the line integral over a closed curve that is clearly the boundary of a nice region, it's often a good idea to use Green's Theorem to switch to the double integral over the interior.

Exercise 2

Integrate $\vec{F} = xy\vec{i} + e^x\vec{j}$ over the boundary of the rectangle determined by $0 \leq x \leq 2$, $0 \leq y \leq 3$, oriented clockwise around the boundary.

Exercise 3

Find the line integral of $\vec{F} = 3xy\vec{i} + 2x^2\vec{j}$ over the curve C defined as follows: follow the curve $y = x^2 - 2x$ from $(0, 0)$ to $(3, 3)$, then follow the line $y = x$ from $(3, 3)$ back to $(0, 0)$.

Exercise 4

Evaluate

$$\oint_C (3y - e^{\sin x}) dx + (7x + \sqrt{y^4 + 1}) dy$$

where C is the circle $x^2 + y^2 = 9$, oriented clockwise.

■ Consider the following problem.

Find the line integral of

$$\vec{G}(x, y) = (x + y)\vec{i} + \left(2x + y \ln(\csc \sqrt{1 - y^5})\right)\vec{j}$$

over C_1 , the upper half of the unit circle from $(1, 0)$ to $(-1, 0)$.

The problem is that the vector field is ugly, so parametrizing C_1 is just going to lead to an impossible integral. So we would like to use Green's Theorem, but this isn't a closed curve!

Here's how to fix that issue. Let C_2 be the straight line segment from $(-1, 0)$ to $(1, 0)$. Now $C_1 + C_2$ is a closed loop.

- (a) Let R be the region enclosed by $C_1 + C_2$. Use Green's Theorem to compute $\oint_{C_1+C_2} \vec{G} \cdot d\vec{r}$.
- (b) Parametrize C_2 and directly calculate $\int_{C_2} \vec{G} \cdot d\vec{r}$. (Note that $y = 0$ everywhere on C_2 , which is helpful.)
- (c) Write $\oint_{C_1+C_2} \vec{G} \cdot d\vec{r} = \int_{C_1} \vec{G} \cdot d\vec{r} + \int_{C_2} \vec{G} \cdot d\vec{r}$ and use your answers to part (a) and (b) to finish off the problem and find the line integral of \vec{G} along C_1 .

Exercise 5

Evaluate the integral.

Calculating Area with Green's Theorem

Consider the following vector fields:

$$\vec{F}_1 = x \vec{j}, \quad \vec{F}_2 = -y \vec{i}, \quad \vec{F}_3 = -\frac{1}{2}y \vec{i} + \frac{1}{2}x \vec{j}.$$

What is $Q_x - P_y$ for each of these fields? Applying Green's Theorem to a region D , we get that

$$\oint_{\partial R} \vec{F}_1 \cdot d\vec{r} = \oint_{\partial R} \vec{F}_2 \cdot d\vec{r} = \oint_{\partial R} \vec{F}_3 \cdot d\vec{r} = \iint_D 1 \, dA = \text{Area of } D. (!)$$

Exercise 6

An ellipse with semi-major axis a and semi-minor axis b is parametrized by $x = a \cos t$, $y = b \sin t$ for $0 \leq t \leq 2\pi$. Use \vec{F}_3 to find the area inside this ellipse.

Exercise 7

Let C be the curve parametrized by $\vec{r}(t) = (t^2 - 3)\vec{i} + (t^3 - 4t + 1)\vec{j}$, $-2 \leq t \leq 2$. This is a closed loop. Use Green's Theorem and \vec{F}_1 to find the area inside this loop.

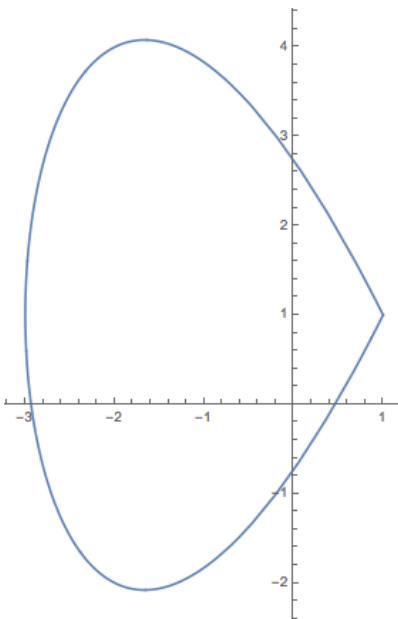


Figure 1: Exercise 7

Extended Versions of Green's Theorem

Although we have proved Green's Theorem only for the case where D is simple, we can now extend it to the case where D is a *finite union of simple regions*. For example, if D is the region shown in Figure 2, then we can write $D = D_1 \cup D_2$, where D_1 and D_2 are both simple. The boundary of D_1 is $C_1 + C_3$ and the boundary of D_2 is $C_2 + (-C_3)$. so, applying Green's Theorem to D_1 and D_2 separately, we get

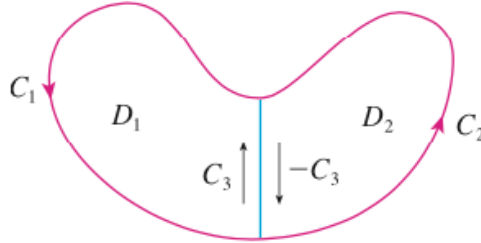


Figure 2

$$\oint_{C_1+C_3} Pdx + Qdy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\oint_{C_2+(-C_3)} Pdx + Qdy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Adding the two integrals above, we get,

$$\oint_{C_1+C_2} Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Note that we can similarly extend the theorem to regions that are finite *intersections* of simple regions.

Exercise 8

Evaluate

$$\oint_C y^2 dx + 3xy dy$$

where C is the boundary of the semiannular region D in the upper half plane between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Green's Theorem can be also extended to apply to regions with holes, that is, regions that are not simply-connected. Observe that the boundary C of the region D in Figure 3 consists of two simple closed curves C_1 and C_2 . We assume that these boundary curves are oriented so that the region D is always on the left as the curve C is traversed. Thus the positive direction is counterclockwise for the outer curve C_1 but clockwise for the inner curve C_2 . If we divide D into two regions D' and D'' by means of the lines shown in Figure 3 and then apply Green's Theorem to each of D' and D'' , we get

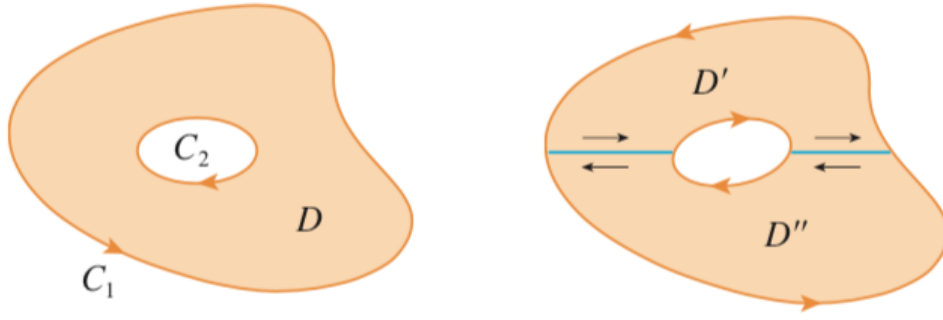


Figure 3

$$\begin{aligned}\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA &= \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\ &= \oint_{\partial D'} (Pdx + Qdy) + \oint_{\partial D''} (Pdx + Qdy)\end{aligned}$$

Since the line integrals along the common boundary lines are in opposite directions, they cancel each other out and we get

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_{C_1} (Pdx + Qdy) + \oint_{C_2} (Pdx + Qdy) = \oint_{C_1+C_2} Pdx + Qdy$$

Exercise 9

If $\vec{F}(x, y) = \frac{-y\vec{i} + x\vec{j}}{x^2 + y^2}$, show that $\oint_C \vec{F} \cdot d\vec{r} = 2\pi$ for every positively oriented closed path C that encloses the origin.

Exercise 10

Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ where

$$\vec{F}(x, y) = \frac{2xy\vec{i} + (y^2 - x^2)\vec{j}}{(x^2 + y^2)^2}$$

and C is a positively oriented closed path that encloses the origin.