Project Report - Summer 2010
An Introduction to General Topology
The Fundamental Group

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Abstract

As in any branch of Mathematics the 'big problem'of Topology is classification of the Topological Spaces. Showing that two spaces are homeomorphic is a geometrical problem, involving the construction of a specific homeomorphism between them. Attempting to prove that two spaces are not homeomorphic to one another is a problem of an entirely different nature, as we cannot possibly examine each function between the spaces individually. Instead we look for 'topological invariant'of spaces:T_1, T_2, $T_{3\frac{1}{2}}$ etc. properties, seperability, second-countability, connectedness, locally connectedness, compactness, locally compactness etc. As an further topological property, we consider a construction due to Poincaré where the idea is to assign a group to each topological space in such a way that *homeomorphic spaces have isomorphic groups*. Thus problem of distinguishing between two spaces comes down to solving the problem algebraically. Of course, we might be unlucky and end up with isomorphic groups for non-homeomorphic spaces, in which case we must look for a more delicate invariant to seperate the two spaces.

1 Introduction

In topology, there are several restrictions that one often makes on the kinds of topological spaces that one wishes to consider. The kind of restrictions can be classified into three types:

1. **Countability Axioms-** Seperability, Second Countability etc.
2. **Covering Axioms-** Compactness, Lindelöf property etc.
3. **Seperation Axioms-** $T_1, T_2, T_3, T_{3\frac{1}{2}}, T_4$ etc.

2 Countability Axioms

2.1 Definitions

- A space, X, is said to be **first-countable** if each point has a countable neighbourhood basis (local base).
- A space is said to be **second-countable** if its topology has a countable base.
- A topological space is called **separable** if it contains a countable dense subset; that is, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of the space such that every nonempty open subset of the space contains at least one element of the sequence.

2.2 Some theorems

- **Lindelöf's Theorem-**Let X be a second countable space. If a non-empty open set G in X is represented as the union of a class $\{G_i\}$ of open sets, then G can be represented as a countable union of G_i's.
- Let X be a second countable space. Then any open base for X has a countable subclass which is also an open base.
- Every seperable metric space is second countable.

3 Product Topology

3.1 Weak Topology generated by a family of functions

Given a set X and an indexed family $\{Y_i\}_{i \in I}$ of topological spaces with functions

$$f_i : X \to Y_i$$

the weak topology τ on X is the coarsest topology on X such that each

$$f_i : (X, \tau) \to Y_i$$

is continuous. Explicitly, the weak topology may be described as the topology generated by sets of the form $f_i^{-1}(U)$, where U is an open set in Y_i.

3.2 Product Space

A *product space* is the cartesian product of a family of topological spaces equipped with a natural topology called the **product topology**. Given X such that

$$X := \prod_{i \in I} X_i,$$

or the (possibly infinite) Cartesian product of the topological spaces X_i, indexed by $i \in I$, and the canonical projections $\Pi_i : XX_i$, the product topology on X is defined to be the weak topology on X induced by $\{\Pi_i\}_{i \in I}$.

■ **Note:** a map $f : Y \to X$ is continuous if and only if $f_i = \Pi_i o f$ is continuous for all i in I.

4 Covering Axioms

4.1 Definitions

- A **Lindelf** space is a topological space in which every open cover has a countable subcover.
- A **Compact** space is a topological space in which every open cover has a finite subcover.
- A T_2 space is called **Locally Compact** if open sets with closure compact, forms a base.
- A topological space is **sequentially compact** if every sequence has a convergent subsequence.

4.2 Some Theorems

- Let $A \subseteq \mathbb{R}$ be bounded such that $\forall x \in \mathbb{R}, \exists$ open $U \ni x$ such that $U \cap X$ is finite.Then A is finite.
- Let $A \subseteq \mathbb{R}$ be such that $\forall x \in \mathbb{R}, \exists$ open $U \ni x$ such that $U \cap X$ is countable. Then A is countable.
- For a metric space X, the following are equivalent:
  1. X is compact.
  2. X is sequentially compact.
- For a metric space X, the following are equivalent:
  1. X is seperable.
  2. X is second countable.
  3. X is Lindelf.
- Let X be locally compact, T_2. Let Y be T_2. Let $f : X \to Y$ be continuous, open and onto. Then Y is locally compact.

- Let (X, d) be a metric space such that every continuous $f : X \to \mathbb{R}$ is bounded. Then X is compact.

- Let $\{X_i\}_{i \in I}$ be a family of T_2 spaces and $X = \prod_{i \in I} X_i$ with product topology. Then X is locally compact iff each X_i is locally compact and all but finitely many X_i's are compact.

- **Lebesgue Number Lemma:** If the metric space (X, d) is compact and an open cover of X is given, then there exists a number $\delta > 0$ such that every subset of X having diameter less than δ is contained in some member of the cover. The number δ is called a **Lebesgue number** of this cover.

4.3 Tychonoff's Theorem

■ **Statement:** *The product of any non-empty class of compact spaces is compact.*
■ **Sketch of the proof:** A **filter** on a set X is a collection of nonempty subsets of X that is closed under finite intersection and under superset. An **ultrafilter** is a maximal filter. The ultrafilter lemma states that every filter on a set X is a subset of some ultrafilter on X (a maximal filter of nonempty subsets of X), which follows from **Zorn's Lemma**.
Thus X is compact
$\Leftrightarrow$ Every family of closed sets having finite intersection property(f.i.p.) has non-empty intersection.
$\Leftrightarrow$ For every family $\{A_i\}_{i \in I}$ of subsets of X with f.i.p. , $\bigcap_{i \in I} \overline{A_i} \neq \phi$.
$\Leftrightarrow$ Foe every filter $\mathcal{F}$ on X,

$$\bigcap \{\overline{F} : F \in \mathcal{F}\} \neq \phi.$$

$\Leftrightarrow$ For every ultrafilter $\mathcal{U}$ on X,

$$\bigcap \{\overline{F} : F \in \mathcal{U}\} \neq \phi.$$

Now let $X := \prod_{i \in I} X_i$ where X_i's are compact, and $\mathcal{U}$ is an ultrafilter on X. Let $\Pi_i : X \to X_i$ be the natural projection map.Then we have to show that

$$\bigcap \{\overline{F} : F \in \mathcal{U}\} \neq \phi.$$

We find that $\exists x_i \in \bigcap_{F \in \mathcal{U}} \overline{\Pi_i(F)}$. Then one can show that $x = (x_i) \in \overline{F}, \forall F \in \mathcal{U}$.

■ **Note:** Let $\mathcal{F}$ be a filter on X. TFAE

1. $\mathcal{F}$ is an ultrafilter.

2. $\forall A \notin \mathcal{F}, \exists C \in \mathcal{F}$ such that $A \cap C = \phi$

3. $\forall A \subseteq X [A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}]$

4. $\forall A \subseteq X, [(C \cap A \neq \phi), \forall C \in \mathcal{F} \Rightarrow A \in \mathcal{F}]$

5 Complete Metric Space

A metric space M is said to be **complete** (or Cauchy) if every Cauchy sequence of points in M has a limit that is also in M or alternatively if every Cauchy sequence in M converges in M.

■ **Cantor Intersection Theorem:** Let X be a complete metric space, and let $\{F_n\}$ be a decreasing sequence of non-empty closed subsets of X such that $d(F_n) \to 0$. Then $F = \bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

■ **Baire Category Theorem:** Let (X, d) be complete metric space and let $\{U_i\}$ be a family of dense open sets in X. Then $\bigcap_{i=1}^{\infty} U_i \neq \phi$.

6 Seperation Axioms

6.1 Definitions

- A T_1 space is a topological space in which, given any pair of distinct points, each has a neighbourhood which does not contain the other.
- A **Hausdorff** or T_2 space is a topological space in which each any pair of distinct points can be sepaerated by open sets, i.e. have disjoint neighbourhoods.
- X is a **regular** space if, given any point x and closed set F in X, if x does not belong to F, then they are separated by neighbourhoods. (In fact, in a regular space, any such x and F will also be separated by closed neighbourhoods.)
- X is a **regular Hausdorff** space, or T_3, if it is both T_0 and regular.
- A **completely regular** space is a T_1-space X with the property that if x is any point in X and F is any closed subspace of X which does not contain x, then there exists a function f in $\mathcal{C}(X, \mathbb{R})$, all of whose values lie in the closed unit interval, such that $f(x) = 0$ and $f(F) = 1$.
- X is a **Tychonoff** space, or $T_{3\frac{1}{2}}$, if it is both T_0 and completely regular.
- X is a **normal** space if any two disjoint closed subsets of X are separated by neighbourhoods.
- X is **normal Hausdorff**, or T_4, if it is both T_1 and normal.

6.2 Some Theorems

- Every compact Hausdorff space is Normal.
- **Urysohn's Lemma-** *Let X be a normal space, and let A and B be disjoint closed subspaces of X. Then there exists a continuous real function f defined on X, all of whose values lie in the closed unit interval* $[0, 1]$, *such that* $f(A) = 0$ *and* $f(B) = 1$.

  Sketch of the proof- For every dyadic fraction $r \in (0, 1)$, we construct an open subset U_r of X such that:

1. $r < s \Rightarrow \overline{U_r} \subseteq U_s$
2. $A \subseteq U_r, \forall r$
3. $B \cap U_r = \phi, \forall r.$

Then define $f(x) = inf\{r : x \in U_r\}$.

- **the Tietze Extension Theorem-**
  1. Let X be a normal space, F a closed subspace, and f a continuous real function defined on F whose values lie in the losed interval $[a, b]$. Then f has a continuous extension $f^{'}$ defined on all of X whose values also lie in $[a, b]$.
  2. In above statement, $[a, b]$ can be replaced by $\mathbb{R}$.

- **the Urysohn Imbedding Theorem-** Let X be a topological space, $\mathcal{F}$ a family of continuous functions $f : X \to [0, 1]$ such that:
  1. $\mathcal{F}$ seperates points:
  $$\forall x \neq y, \exists f \in \mathcal{F} \text{ such that } f(x) \neq f(y)$$
  .
  2. $\mathcal{F}$ seperates points and closed sets:
  $$\forall x \text{ and } \forall \text{ closed } F \subseteq X, [x \notin F \Rightarrow \exists f \in \mathcal{F} \text{ such that } [f(x) \notin \overline{f(F)}]]$$

  Then the evaluation map
  $$e : X \to [0, 1]^{\mathcal{F}}$$
  is an embedding.
  Cor. 1- Every $T_{3\frac{1}{2}}$ space can be embedded into a compact, T_2 space.
  Cor. 2- If countable $\mathcal{F}$ satisfying (1) and (2) can be found, then X can be embedded into a compact metric space.
  Remark: If X is T_4 and second countable then countable $\mathcal{F}$ can be found.

- Let X be locally compact,T_2, second countable. Then X can be written as $\bigcup_n K_n$, where each K_n is compact and $K_n \subseteq K^{\circ}_{n+1}$.

- Let Y be locally compact, dense subspace of a Hausdorff space X.Then Y is open in X.

7 Compactification

7.1 One Point Compactification

Let X be a locally compact, non-compact T_2 space. Take $\infty \notin X$. Set
$$\hat{X} := X \cup \{\infty\}$$
. Define a topology τ on $\hat{X}$ as follows:
$U \subseteq \hat{X}$ is open iff either [$U \subseteq X$ and U open in X], or [$\infty \in U$ and $\hat{X} \setminus U$ is compact]. Then

1. $\hat{X}$ is compact,T_2 and X is dense in $\hat{X}$

2. Let Y be a compact,T_2 space and $h : X \to Y$ be an embedding such that $|Y \setminus h(X)| = 1$. Then Y is homeomorphic to $\hat{X}$.

7.2 StoneČech compactification

For each topological space X let $F(X)$ be the family of all continuous functions on X to the closed unit interval Q. The cube $Q^{F(X)}$(the product of the unit interval Q taken $F(X)$ times) is compact by Tychonoff theorem. The evaluation map e carries a member x of X into the member $e(x)$ of $Q^{F(X)}$ whose $f-$th coordinate is $f(x)$ for each $f \in F(X)$. Evaluation is a continuous map of X into the cube $Q^{F(X)}$, and if X is a Tychonoff space, then e is a homeomorphism of X onto a subspace of $Q^{F(X)}$. The Stone-Čech compactification is the pair $(e, \beta(X))$ where $\beta(X)$ is the closure of $e[X]$ in the cube $Q^{F(X)}$.
■ **Theorem:** Let X be a Tychonoff space and f is a continuous function on X to a compact Hausdorff space Y. if $(e, \beta(X))$ is the Stone-Čech compactification, then $f \circ e^{-1}$ can be extended to a continuous function on $\beta(X)$ to Y.
■ **Sketch of Proof:** Given f define $f^* : F(Y) \to F(X)$ by $f^*(a) = a \circ f$. Define $f^{**} : Q^{F(X)} \to Q^F(Y)$ by $f^{**}(q) = q \circ f^*$. Then consider the following diagram:

$$\begin{tikzcd} \beta(X)\subseteq Q^F(X) \arrow[r, "f^{**}"] & Q^F(Y) \supseteq\beta(Y) \\ X \arrow[u, "e"] \arrow[r, "\mathrm{f}"] & Y \arrow[u, "g"'] \end{tikzcd}$$

Then it can be shown that $f^{**} \circ e = g \circ f$.

8 Connectedness

8.1 Definitions

- A **connected** space is a topological space which cannot be represented as the union of two or more disjoint nonempty open subsets.

- A topological space is said to be **locally connected** at a point x if every neighbourhood of x contains a connected open neighbourhood.

- A **path** from a point x to a point y in a topological space X is a continuous function f from the unit interval $[0, 1]$ to X with $f(0) = x$ and $f(1) = y$.

- A path-component of X is an equivalence class of X under the equivalence relation defined by x is equivalent to y if there is a path from x to y. The space X is said to be **path-connected** if there is only one path-component, i.e. if there is a path joining any two points in X.

- Let X be a set and a, b in X. A chain from a to b is a finite sequence $U_1, U_2, U_3, \cdots, U_n$ of subsets of X such that $a \in U_1, b \in U_n$ and $U_i \cap U_{i+1} \neq \phi$ for every $1 \leq i < n$.

8.2 Some Theorems

- Every connected, locally path connected space is path-connected.

- Every connected, locally connected complete metric space is path-connected. **Sketch of the proof:** The proof uses the following proposition: Let X be a connected space and $\mathcal{U}$ an open cover. For every a, b in X, there is a chain $U_1, U_2, \cdots, U_n$ in $\mathcal{U}$ from a to b. Thus a chain may be regarded as a sort of approximation for a path. By joining two points by finer and finer chains,we should come closer and closer to a path. This is the main idea behind the proof. Another fact used is that any uniformly continuous funtion on a dense subset D of a metric space X, to a complete metric space Y can be extended to X.

- **Hahn-Mazurkiewicz Theorem:** A hausdorff space is a continuous image of [0,1] iff it is compact,connected,locally connected and metrizable.

- Every connected, T_4 space X with $|X| > 1$ is of cardinality at least c.

- Every locally constant function on a connected space is constant.

9 Quotient Space

9.1 definition

- Suppose X is a topological space and $\sim$ is an equivalence relation on X. We define a topology on the quotient set $X/\sim$ (the set consisting of all equivalence classes of $\sim$) as follows: a set of equivalence classes in $X/\sim$ is open if and only if their union is open in X . This is the quotient topology on the quotient set $X/\sim$.

- Given a surjective map $f : XY$ from a topological space X to a set Y we can define the quotient topology on Y as the finest topology for which f is continuous. The map f induces an equivalence relation on X by saying $x_1 \sim_f x_2$ if and only if $f(x_1) = f(x_2)$.

9.2 Some Theorems

- Let X be compact and Y be a T_2 space.Let $f : X \to Y$be continuous and onto. Then Y is homeomorphic to $X/\sim_f$.

- If in the above statement, X and Y are any two topological spaces and f is continuous, onto and open(or closed) map then Y is homeomorphic to $X/\sim_f$.

- Let $\sim$ be an equivalence relation on X such that each equivalence class is connected and $X/\sim$ is connected. Then X is connected.

9.3 n-Manifold

We all know that

$$\forall x \in S^2, \exists \text{ open } U \ni x \text{ such that } [U \text{ is homeomorphic to } \mathbb{R}^2]$$

. We can also show that

$$\forall x \in \mathbb{T}^n, \exists \text{ open } U \ni x \text{ such that } [U \text{ is homeomorphic to } \mathbb{R}^n]$$

. All such topological spaces(like $\mathbb{T}^n$)for which above property is true is called a **n-manifold**.

- Any connected, n-manifold is path connected.
- Klein's Bottle and $\mathbb{RP}^2$ are 2-manifold.

9.4 Some Examples

- S^1 is homeomorphic to $\mathbb{R}/\sim$ where $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$.
- $\mathbb{T}^2 = S^1 \times S^1$ is homeomorphic to $\mathbb{R}^2/\sim$ where $(x, y) \sim (u, v) \Leftrightarrow x - u, y - v \in \mathbb{Z}$.
- $\mathbb{T}^n = \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{\text{n times}}$ is homeomorphic to $\mathbb{R}^n/\sim$ where $(x_1, ..., x_n) \sim (y_1, ..., y_n) \Leftrightarrow x_i - y_i \in \mathbb{Z}, \forall i$. $\mathbb{T}^n$ is called the **n dimensional Torus**.
- $[0,1] \times [0,1]/\sim$ where $(0, x) \sim (1, 1 - x)$ is called a **Möbius Strip**.
- $[0,1] \times [0,1]/\sim$ where $(0, x) \sim (1, 1 - x)$ and $(x, 0) \sim (x, 1)$ is called a **Klein Bottle**.
- $[0,1] \times [0,1]/\sim$ where $(0, x) \sim (1, 1 - x)$ and $(x, 0) \sim (1 - x, 1)$ is homeomorphic to $S^2/\sim_a$ where $z \sim_a -z$, and also homeomorphic to $(\mathbb{R}^3 \setminus \{0\})/\sim_b$ where $p \sim_b q \Leftrightarrow p, q$ lie on the same straight line through the origin. The quotient space is called the **Real Projective Plane** , written as $\mathbb{RP}^2$.
- $\mathbb{RP}^n$ can obtained from (i)S^n by identifying the antipodal points, (ii) S^n_+ by identifying antipodal points on the boundary, (iii) 1-dimensional subspaces of $\mathbb{R}^{n+1}$.
- $D/\sim$ is homeomorphic to S^2 where $z \sim Z' \Leftrightarrow |z| = |z'| = 1$.
- $(S^1 \times I)/\sim = D$ where $(x, 1) \sim (x', 1)$.

9.5 Cone and Suspension

9.5.1 Definitions

- Given any topological space X, let $\sim$ be an equivalence relation defined on $X \times I$ as follows, for $x, x' \in X, (x, 1) \sim (x', 1)$. Then $(X \times I)/\sim$ is called CX, the **Cone on X**.
- Similarly if $\sim'$ is another realtion defined as: $(x, 0) \sim' (x', 0); (x, 1) \sim' (x', 1)$, then $(X \times I)/\sim'$ is called SX, the **Suspension of X**.
- Let $A \subseteq X$. Define $x, x' \in A \Rightarrow x \sim x'$.Then $X/\sim$ is written as X/A.

9.5.2 Facts

- CX is path-connected.
- $SX = CX/(X \times \{0\})$
- $SS^n = S^{n+1}$

9.6 Some special Quotient Space

9.6.1

If X and Y are pointed spaces (i.e. topological spaces with distinguished base-points x_0 and y_0) the wedge sum of X and Y is the quotient of the disjoint union of X and Y by the identification $x_0 \sim y_0$:

$$X \vee Y = (X \amalg Y) \ / \ \{x_0 \sim y_0\}$$

.

9.6.2

Let $A \subseteq X$. Let f be a continuous function from A into Y. We define $\sim$ on $X \oplus Y$ as $a \sim f(a)$. Then $(X \oplus Y)/\sim$ is written as $X \bigcup_f Y$.

■**Examples**

- $S^1 \subseteq D$. Let us take two identical copies of D and name then D_1 and D_2. Let us say that $\sim$ on $D_1 \bigcup D_2$ identifies the corresponding points on the boundary(i.e. S^1) of D_1 and D_2. Then $(D_1 \bigcup D_2)/\sim$ is homeomorphic to S^2.
- Let $D^n = \{(x_1, x_2, \cdots, x_{n+1}) | \sum_i x_i^2 \leq 1\}$ We know that $\partial D^n = S^n$. Thus $S^n \subseteq D^n$ and let q be the natural quotient map from S^n to $\mathbb{RP}^n$. Then $D_n \bigcup_q \mathbb{RP}^n = \mathbb{RP}^{n+1}$.
- Let $M = I^2/\sim$ be the Möbius Strip, where $(x, 0) \sim (1-x, 1)$. Then $\partial M = S^1$ Also there is a natural inclusion map $\imath$ from S^1 to D. Then $M \bigcup_i D = \mathbb{RP}^2$.
- Let M_1 and M_2 be two Möbius Strips. Let S_1^1, S_2^1 denote their respective boundaries. Suppose any two corresponding points on the two identical copies of S^1 which are S_1^1 and S_2^1 are identified. Then the resulting quotient space is a Klein Bottle.

10 Homotopy

10.1 Definitions

- let $f, g : X \to Y$ be two continuous functions. We say that f and g are **homotopic** if $\exists$ a continuous $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. We say that $\underline{H : f \simeq g}$.

- A **loop** at $x_0 \in X$ is a continuous function $\gamma : [0,1] \to X$ such that $\gamma(0) = \gamma(1) = x_0$.

- **Relative Homotopy:** Let $A \subseteq X \overset{f}{\underset{g}{\to}} Y$ be such that f, g are continuous and $f|_A \equiv g|_A$. We say that $f \simeq g$(rel. A) if $\exists$ a continuous $H : X \times I \to Y$ such that $H(x,0) = f(x)$;$H(x,1) = g(x)$ and $H(x,t) = f(x) = g(x), \forall 0 \leq t \leq 1, x \in A$.

- We say that X and Y are **homotopically equivalent** if $\exists f : X \to Y$ and $\exists g : Y \to X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$. We say that f, g are **homotopy equivalence**.

10.2 Retract

- A subset A of X is called a **retract** of X if $\exists$ a continuous $\gamma : X \to A$ such that $\gamma|_A = id_A$. γ is called a **retraction**.
  $\blacksquare S^n$ is a retract of $\mathbb{R}^{n+1}_{\times} = \{\tilde{x} \in \mathbb{R}^{n+1} | \tilde{x} \neq \tilde{0}\}$.$[\gamma : x \mapsto \frac{x}{|x|}]$

- A is called a **strong deformation retract** of X if $id_x \simeq$a retraction(rel. A) i.e. $\exists$ a continuous $H : X \times I \to X$ such that (i)$H(x,0) = x$;
  (ii)$H(x,1) \in A$ and
  (iii)$H(x,t) = x, \forall 0 \leq t \leq 1, x \in A$.

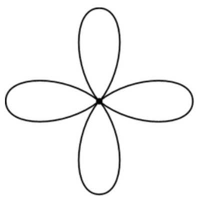
- if $\exists x \in X$ such that $\{x\}$ is a s.d.f. of X , then X is called **contractible.**

- A is called a **deformation retract** of X if $\exists$ a continuous $H : X \times I \to X$ such that (i)$H(x,0) = x$;
  (ii)$H(x,1) \in A$ and
  (iii)$H(x,1) = x, \forall x \in A$.
  i.e. $H(\cdot, 1) : X \to A$ is a retraction.

10.3 Theorems and Examples

- Every convex $X \subseteq \mathbb{R}^n$ is contractible.

- $A \subseteq X$ is a retract of X and X is T_2. $\Rightarrow$ A is closed in X.

-  is a s.d.r. of $\mathbb{R}^2 \setminus \{4 \text{ points}\}$.

- Every cone CX is contarctible.

- Every contractible space i s path-connected.

- $f : X \to Y$ is homotopic to a constant map iff $\exists g : CX \to Y$ such that g is continuous.

- If Y is contractible,then every continuous map $f : X \to Y$ is homotopic to a constant map.

- S^1 is a s.d.r. of the Möbius Strip.
- $S^1 \vee S^1$ is a s.d.r. of the punctured torus.
- $\mathbb{RP}^{n-1}$ is a s.d.r. of punctured $\mathbb{RP}^n$.

11 The Fundamental Group

11.1 Definition

We observe the following facts:

- "$\simeq$(rel. A)"on the set of all continuous functions from X to Y which agree with some given map on A, is an equivalence relation. The equivalence classes are called **homotopy class**.
- Suppose we have maps $X \underset{g}{\overset{f}{\to}} Y \xrightarrow{h} Z$. Then if $H : f \simeq g$(rel. A) , then $h \circ H : h \circ f \simeq h \circ g$(rel A).
- The product $f * g$ of two loops f and g based at the point $x_o \in X$ is defined by setting $(f * g)(t) := f(2t)$ if $0 \leq t \leq 1/2$ and $(f * g)(t) := g(2t1)$ if $1/2 \leq t \leq 1$. Thus the loop $f * g$ first follows the loop f with "twice the speed" and then follows g with twice the speed. The product of two homotopy classes of loops $< f >$ and $< g >$ is then defined as $< f * g >$, and it can be shown that this product does not depend on the choice of representatives.
- The set of homotopy classes of loops in X based at a certain point p forms a group under the multiplication $< f > . < g > = < f * g >$. The identity element is the homotopy class of the constant loop e at p defined by $e(s) = 1$ for $0 \leq s \leq 1$.

The group constructed is called **the fundamental group of** X **based at p**, written as $\Pi_1(X, p)$.Although the fundamental group in general depends on the choice of base point, it turns out that, up to isomorphism, this choice makes no difference so long as the space X is path-connected. For path-connected spaces, therefore, we can write $\Pi_1(X)$ instead of $\Pi_1(X, p)$ without ambiguity whenever we care about the isomorphism class only.
■ We can define a function

$$f_* : \Pi_1(X, p) \to \Pi_1(Y, q)$$

by $f_*(< \alpha >) = < f \circ \alpha >$. We can clearly see that f_* is a homomorphism. We say that f_* is induced by f.
■ Note: $(g \circ f)_* = g_* \circ f_*$.

11.2 Theorems and Examples

- $X \subseteq \mathbb{R}$ is convex $\Rightarrow \Pi_1(X, x_0)$ is trivial.

- If $x_0 \in A \subseteq X$ and A is a s.d.r. of X , then $\Pi_1(X, x_0) = \Pi_1(A, x_0)$ upto isomorphism.
- $\mathbb{R}^2 \setminus 0$ and S^1 have the same fundamental group.
- $\Pi_1(X, x_0) \times \Pi_1(Y, y_0) = \Pi_1(X \times Y, (x_0, y_0))$.

Now we turn our attention to calculate the fundamental group of S^1 and S^n for $n \geq 2$.

11.3 Fundamental group of $S^n, (n \geq 2)$

First observe that if $\gamma : [0, 1] \to S^n$ is a loop at $x_0 \in S^n$, and if γ is **not** onto, then it is homotopic to the constant map. It is obvious since if $y \notin$Range(γ), then by projecting S^n from y, we get $\mathbb{R}^n$ which is convex subset of itself.
But the main problem occurs when γ is **onto**, which is possible by **Hahn-Mazurkiewicz theorem** mentioned earlier.

11.3.1 van-Kampen Theorem

■**Theorem** A space is called **simply-connected** if it is path-connected and has trivial fundamental group.Let X be a space which can be written as the union of two simply connected open sets U, V in such a way that $U \cap V$ is path-connected. Then X is simply connected.
■ **Proof:**
We show that any loop in X is homotopic to a product of loops each contained in either U or V. This is enough to prove the theorem since both are simply connected.
Choose $p \in U \cap V$ and let $\gamma : I \to X$ be a loop at p. From Lebesgue's Lemma, we can find points $0 = t_0 < t_1 < t_2 < ... < t_n = 1$ in I such that $\gamma[t_{k-1}, t_k]$ is always contained in U or V. Write γ_k for the path $s \mapsto \gamma((t_k - t_{k-1})s + t_{k-1}), 0 \leq s \leq 1$. Join p to each point $\gamma(t_k), 1 \leq k \leq n-1$ by a path α_k which lies in U if $\gamma(t_k) \in U$ and which lies in V if $\gamma(t_k) \in V$. Now by pathconnectedness of $U \cap V$, our γ is homotopic to

$$(\gamma_1.\alpha_1^{-1}).(\alpha_1.\gamma_2.\alpha_2^{-1}).(\alpha_2.\gamma_3.\alpha_3^{-1}). \cdots .(\alpha_{n-1}.\gamma_n)$$

each member of which is a loop contained in U or V.
■ To apply this result to $S^n, (n \geq 2)$, take distinct points x, y(so, $n > 1$ is necessary) and set $U = S^n - x, V = S^n - y$. Then we clearly see that S^n **has trivial fundamental group for** $n \geq 2$.

Now we turn our attention to finding $\Pi_1(S^1)$. But this requires a special treatment. So before finding that let us look into one other important concept:

11.4 Covering Space

11.4.1 Definition

A **covering space** of a space B is a space U together with a map $p : UB$ satisfying the following condition: $\forall b \in B \exists$ open $U \ni b$ such that $p^{-1}(U)$ is a disjoint union of open sets in U , each of which is mapped homeomorphically onto U by p .

11.4.2 Group action on a topological space

A topological group G is said to act as a group of homeomorphisms on a space X if each group element induces a homeomorphism of the space in such a way that:
(i)$hg(x) = h(g(x))$ for all $h, g \in G, x \in X$
(ii) $e(x) = x$ for all $x \in X$ where e is the indentity element of G
(iii)the function $G \times X \to X$ defined by $(g, x) \mapsto g(x)$ is continuous.
The corresponding *Orbit space* is written X/G.

11.4.3 Examples of Covering Space

- If D is a discrete space, $U \times D$ is a covering space of U,called the trivial covering space. Here p is natural projection map π_1.
- S^1 is a covering space of itself with p being the map $z \mapsto z^n, (n \geq 1)$.
- $\mathbb{R}^n$ is a covering space for $\underbrace{S^1 \times S^1 \times S^1 \times \cdots \times S^1}_{ntimes} = \mathbb{R}^n/\mathbb{Z}^n$ with p being the map $(t_1, t_2, ..., t_n) \mapsto (e^{2\pi i t_1}, e^{2\pi i t_2}, ..., e^{2\pi i t_n})$.
- S^n is a covering space for $\mathbb{RP}^n = s^1/\mathbb{Z}^2$ with p being the natural quotient map q.

11.4.4 Some Theorems

- Suppose $p : E \to B$ is a covering map and B is connected. Then $\forall b_1, b_2 \in B, [|p^{-1}(b_1)| = |p^{-1}(b_2)|]$.
- Suppose G is a group acting on X. Then $q : X \to X/G$ is a covering map if $\forall x \in X, \exists$ open $U \ni x$ such that $\forall g \in G$ for which $(U \cap g \cdot U = \phi), g_1 \neq g_2 \Rightarrow g_1 \cdot U \cap g_2 \cdot U \neq \phi$.

11.4.5 Path Lifting

Given any path $\gamma : [0,1] \to B, \exists$ a path $\Gamma : [0,1] \to E$ called a **lift** of γ such that $p \circ \Gamma = \gamma$. Further, if Γ' is another lift of γ, then

$$\{t | \Gamma(t) = \Gamma'(t)\} = \phi \text{ or } [0,1]$$

. Also if $p(e) = \gamma(t)$, whatever be t, $\exists!$ lifting Γ of γ such that $\Gamma(t) = e$.
Homotopy Lifting: If $F : I \times I \to B$ is a map such that $F(0,t) = F(1,t) = 1, \forall t \in [0,1]$, there is a unique map $\tilde{F} : I \times I \to E$ which satisfies

$$p \circ \tilde{F} = F; \text{ and } \tilde{F}(0,t) = 0, \forall t \in [0,1].$$

11.4.6

■ Let γ be a loop at $b_0 \in B$.

$$\begin{tikzcd}
(E, e_0) & (E, e_0) \arrow[d, "p"] \\
([0,1], 0) \arrow[u, "\exists!"', "\Gamma, \Gamma'"] \arrow[r, "\gamma", "\gamma'"'] & (B, b_0)
\end{tikzcd}$$

Thus $p(\Gamma(1)) = b_0 \Rightarrow \Gamma(1) \in p^{-1}(b_0)$, but $\Gamma(1)$ may not be equal to e_0.

Now let $H : \gamma \simeq \gamma'$(rel 0,1). let G be the lift of H with $G(0,t) = e_0, \forall t$. Then

$$\underbrace{G(1 \times [0,1])}_{connected} \subseteq \underbrace{p^{-1}(b_0)}_{discrete}.$$

So $G(1 \times [0,1])$ is constant. $\Rightarrow G(0,\cdot)$ and $G(1,\cdot)$ are homotopic rel.0,1.
■ We know that if E is simply connected, and if γ, γ' are two paths in E with $\gamma(0) = \gamma'(0)$ and $\gamma(1) = \gamma'(1)$ then $\gamma \simeq \gamma'$(rel. 0,1).

11.5 Fundamental Group of S^1

Let γ be a loop at $1 \in S^1$.

$$\begin{array}{ccc} (\mathbb{R},0) & & (\mathbb{R},0) \\ \exists! \uparrow \Gamma & & \downarrow p(t)=e^{2\pi i t} \\ ([0,1],0) & \xrightarrow{\gamma} & (S^1,1) \end{array}$$

Then by above paragraph

$$\Gamma(1) \in \mathbb{Z}.$$

and since Γ is unique, $\Gamma(1)$ is well defined.
Claim:

$$\gamma \simeq \gamma' \text{ (rel. 0,1) } \Leftrightarrow \Gamma(1) = \Gamma'(1).$$

Proof of the claim:
$\Rightarrow$ follows from Homotopy lifting.
$\Leftarrow$ follows from the last result in previous subsection and the fact that $\mathbb{R}$ is simply connected. □

So, we can derive a bijection $h : \Pi_1(S^1,1) \to \mathbb{Z}$. Also observe that if $h(<\gamma>) = m$ and $h(<\gamma'>) = n$, then $h(<\gamma.\gamma'>) = m+n$. Thus h is a homomorphism.
$\Rightarrow h$ is an isomorphism.
$\Rightarrow \underline{\Pi_1(S^1) = \mathbb{Z}}$.

11.6 Other examples of Fundamental Group

■

$$(E, e_0) \xrightarrow{p} (E/G, <e_0>)$$

Folowing a similar argument to all those done in the previous subsection, we can also deduce easily that

$$\Pi_1(E/G) = G$$

, whenever E is simply connected, and G is a topological group acting on E.
Thus we also deduce $\underline{\Pi_1(\mathbb{RP}^n) = \Pi_1(S^n/\mathbb{Z}_2) = \mathbb{Z}_2}$.
■ Also fundamental group of $S^1 \vee S^1$ is free product of $\mathbb{Z}$ and $\mathbb{Z}$.
■ In general if U and V has fundamental groups G and H, then fundamental group of $U \vee V$ is free product of G and H.

11.7 An Application

■**The Brouwer fixed-point theorem:** *A continuous function f from a ball(of any dimension) to itself must leave at least one point fixed.*
■**Proof for dimension 1:**obvious considering the function $g(x) = f(x) - x$.
■**Proof for dimension 1:**Assume a continuous map $f : D \to D$ exists which has no fixed point. Then the map g which sends x to the intersection point of the ray joining $f(x)$ to x and the boundary of $D = S^1$, is continuous. Also $g(x) = x, \forall x \in S^1$.
Now take $p = (1,0)$ as base point for both S^1 and D and denote the inclusion of S^1 in D by $\imath$. Then

$$S^1 \xrightarrow{\imath} D \xrightarrow{g} S^1 \Rightarrow \Pi_1(S^1, p) \xrightarrow{\imath_*} \Pi_1(D, p) \xrightarrow{g_*} \Pi_1(S^1, p).$$

Now $g \circ \imath(x) = x, \forall x \in S^1 \implies g_* \circ \imath_*$ is the identity homomorphism and g_* must be onto. But $\Pi_1(D, p)$ is trivial and $\Pi_1(S^1, p)$ is isomorphic to $\mathbb{Z}$. Hence,Contradiction.