Practice Problems and review notes

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- The following are a set of practice/review problems for final. Note that they are only **from the syllabus covered after second midterm!** Over this week, I will be going over these problems in office hour, problem session, class etc. Some of these problems will be on the final exam itself.
- Make sure you can solve all of the problems listed below and ask me via email or in person if you have questions.

Problem 1

Determine whether the following series converge or diverge:

(a) $\frac{2}{3} + \frac{2.4}{3.7} + \frac{2.4.6}{3.7.11} + \dots$

(b) $\sum \frac{(2k+1)^{2k}}{(5k^2+1)^k}$

(c) $\sum \frac{1}{k} \left(\frac{1}{\ln k}\right)^{3/2}$

(d) $\sum \frac{(k!)^2}{(pk)!}, p \geq 2, p \in \mathbb{Z}$

(e) $\sum (-1)^k \frac{\cos(\pi k)}{k}$

Problem 2*

Suppose $\sum a_k$ is absolutely convergent. Prove that $\sum a_k^2$ is convergent.

Problem 3

Look up the four boxed formulae in page 607. Memorize those.

Problem 4*

Let

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(a) Use L'Hôpital's rule to show that for all $n \in \mathbb{N}$,

$$\lim_{x \to 0} \frac{e^{-1/x^2}}{x^n} = 0$$

(b) Prove by induction that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

(c) What is the Taylor series of f at 0?

(d) For what values of x does the Taylor series of f actually converge to $f(x)$?

Above question can be also formulated in following forms:

- When is the expansion of $f(x)$ into its Taylor series valid?
- What is the radius of convergence of the Taylor series?

Problem 5

Prove that

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$$

[Hint: $\ln(a/b) = \ln a - \ln b$]

Problem 7

Find $P_{n,a}(x)$ for $\sinh(4+2x)$ when $n = 16, a = -2$. Write it using a $\sum$ notation.

Problem 8

(a) Expand $\ln(a+x)$ in powers of x. When is the expansion valid?

(b) Set $x = y - a$ in above part. What is the expansion of $\ln(x)$ in terms of $x - a$? When is the expansion valid?

Problem 9

Let $n \in \mathbb{N}$. Expand $(x-1)^n$ in powers of x.

Problem 10

Let $n \in \mathbb{N}$. Expand $(1-x)^{-n}$ in powers of x. Expand $(1-2x)^{-4}$ in powers of x. Expand $(1+2x)^{-4}$ in powers of x.

Problem 11

Expand $\sin(\pi x/2)$ in powers of $(x-1)$.

Problem 12

Find the interval of convergence for

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 3^k}(x+2)^k$$

Problem 12

Suppose $\sum_{k=0}^{\infty} a_k(x+2)^k$ converges at $x=4$. At what other values of x must the series converge? Does it necessarily converge at $x=-8$?

Problem 13

Consider the series $\sum a_n x^n$ and assume that the limit

$$l = \lim_{n\to\infty} (a_n)^{1/n}$$

is finite. Prove that the radius of convergence of this series is $1/l$.

Problem 14

Suppose s_k is the $k-$th partial sum of $\sum_{i=1}^{\infty} \frac{1}{i}$. Find the interval of convergence of the series $\sum s_k x^k$.

Problem 15*

Suppose $\sum_{k=0}^{\infty} a_k x^k$ has the property that $a_{k+10} = a_k$ for all $k \geq 0$. What is the radius of convergence?
