Assignment 1 (6/28)

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- *This homework is due at the beginning of class on* ***Tuesday*** 7/3. *You are encouraged to work together on these problems, but you must write up your solutions independently.*
- *Submit exercise* 1 *and* 2 *to me at the beginning of class on Monday. Submit exercise* 3 *and* 4 *on Tuesday along with problems from assignment* 2.

Königsberg Bridge Problem

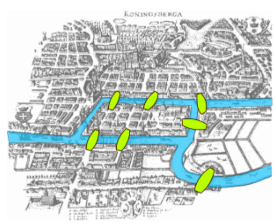

Figure 1: City of Konisberg, Source: Wikipedia

- The original problem was to devise a walk through the city of Konisberg (figure 1) that would cross each of those bridges exactly once.
- Euler pointed out that the choice of route inside each land mass is irrelevant. The only important feature of a route is the sequence of bridges crossed. This allowed him to reformulate the problem in abstract terms (laying the foundations of graph theory), eliminating all features except the list of land masses and the bridges connecting them. [Source: Wikipedia]
- We are going to replace each land mass with an abstract **vertex** or node, and each bridge with an abstract connection, an **edge**, which only serves to record which pair of vertices (land masses) is connected by that bridge. The resulting mathematical structure is called a **graph** (see figure 2). It has 4 vertices and 7 edges.

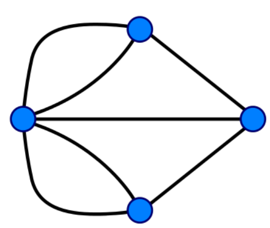

Figure 2: Graph of Konisberg Problem, Source: Wikipedia

- A **graph** is thus defined as a collection of vertices and edges. Don't worry about the proper set theoretic definition. Think of it as an abstract way to visualize a collection of (data)points (vertices) that are related (connected) in some way (via edges). Check out the wiki for lots more examples. We will use *Graph Theory* to formulate many different kinds of problems in upcoming lectures.
- We define **degree** of a vertex to be the number of edges connected to it. What are the degrees of each of the vertices in figure 2?
- An **Eulerian Path** in a graph is a trail that visits each edge exactly once.

- When reformulated in terms of graph theory, the Konisberg problem asks us to find an Eulerian path in the graph in figure 2.
- We will show that such a path can not be found in exercise 1 below.

Exercise 1

> Suppose we have a graph with an Eulerian Path on it. Prove that if a vertex has odd degree, then it must be either the starting or the ending point in the Eulerian path.

Exercise 2

> Using Exercise 1, prove that if a graph has more than two vertices of odd degree, then it has no Eulerian Path.

- In particular, the solution to above exercises will show that if a graph has exactly two vertices of odd degree, then one of those vertices must be the starting and the other the ending point of the Eulerian path.
- We also conclude that the Konisberg Bridge Problem has a negative solution i.e. an Eulerian Path doesn't exist.

I want you to write out whatever you think a 'proof' is for the exercises above. These will not be graded, I just want to find out how you would approach such a question. The best way would be just write out an explanation of the solution using proper English sentences. Don't worry about having a correct 'format' or page limit. Explain as much as you want, but don't be repetitive. Make sure your solution is generic, i.e. it works for any graph, not one example in particular.

The strategy we used to solve this problem was to identify an **invariant**. This principle is applicable to problems that ask us to show the existence (or nonexistence) of some kind of algorithm. We try to identify what remains unchanged over the steps of the algorithm. In the Konisberg Bridge problem, or more generally to find an Eulerian path on a graph, we see that the *parity* (i.e. odd vs even-ness) of the degree of a vertex plays the role of this invariant. We will discuss more examples of the Invariance Principle in next class.

The Pigeonhole Principle

We explain the principle by considering the following problem. (You don't need to write a proof for this)

> There are 20 students present in a classroom. Every student knows at least one person other than herself/himself (we assume every student knows themselves). Prove that among them there are two students who know the same *number* of people in the room.

- First of all, note that the problem does not ask you find which two students have same no. of acquaintances. It doesn't ask you to find that number of acquaintances either. It only asks you show that such two people can be found if we wanted to.
- The argument here is roughly as follows. We find that a student can have at least 2 and at most 20 acquaintances; so there are 19 options. But there are 20 students. So some option must get chosen more than once. We explain the strategy more formally in terms of the following theorem.

The Pigeonhole Principle: If $(n+1)$ pigeons are put into n pigeon-holes, then at least one hole has more than one pigeon.

- We use the Pigeonhole Principle (abbreviated as PHP) to solve existence problems. It gives no help in identifying the multiply occupied hole, it only tells us that such a hole exists. The main difficulty to apply PHP in a problem is to figure out the holes and the pigeons correctly.
- In above classroom problem, we say a student (pigeon) goes into hole number i if she has i acquaintances. There are 19 holes but 20 students (i.e. n is equal to 19 here). So at least two students must 'share a hole'.
- Convince yourself that the number 20 is nothing special. It could be any number n and the assertion would still be valid.

The problem now gets slightly more complicated if we remove the assumption of knowing someone other than yourself and add a different assumption instead. The new problem reads:

Exercise 3

> There are 20 students present in a classroom. Assume that if student A is friends with student B, then student B is friends with student A. Prove that there are two students who have the same number of acquaintances in the room.

I claim that it can be still solved using PHP. Can you find a proof? The main problem is to identify the number of 'holes's and show that it's less than 20. As above, don't worry too much about the formal structure of the proof. Write in your own words what you consider to be a proof.

A different way to formulate a PHP problem is as follows.

> There are n students present in a classroom. Find the least value of n to ensure that at least two students will have birthdays on the same day of the week.

The answer is 8. Do you see why?

Here are another exercise of the same format. You should try to provide a complete explanation for your solution, not simply a numerical answer.

Exercise 4

> There are n students present in a conference. Find the least value of n to ensure that at least three students will have birthdays on the same day of the year (the birth-year doesn't have to same, only the month and day have to be same). Assume all years have 365 days.