# Final Exam

DECEMBER 11, 2015
SUBHADIP CHOWDHURY

Please show **all** your work! Answers without supporting work will not be given credit. No electronic equipment is allowed with you in the exam hall. If you use more than one green book, repeat your name and question numbers in each of them.
**You can score a maximum of 110 points!**

**Time Limit:** 2 Hours

| Question | Points |
|---|---|
| 1 | 10 |
| 2 | 15 |
| 3 | 15 |
| 4 | 15 |
| 5 | 15 |
| 6 | 40 |
| 7 | 0 |
| Total: | 110 |

This exam has 7 questions, for a total of 110 points and 5 bonus points. The maximum possible point for each problem is given on the right side of the problem.

1. Let $f(x) = x^7 + ax + b$. Show that $f(x)$ has at most three *distinct* real roots. 10

**Solution:** $f'(x) = 7x^6 + a$. If $a > 0$, $f'(x) \neq 0$. If $a < 0$, $f'(x) = 0 \implies x = \pm\sqrt[6]{-a/7}$. So $f'(x)$ has at most 2 distinct real roots $\implies f(x)$ has at most 3 distinct real roots by (corollary to) Rolle's theorem.

2. Suppose $f : (0, \infty) \to \mathbb{R}$ is a continuous and differentiable function defined as

$$f(x) = \int_1^{2^x} \left[\frac{1}{\ln(t)} + \ln\left(\frac{1}{t}\right)\right] dt$$

(a) Find $f'(x)$. 8

**Solution:** $f'(x) = 2^x \ln 2 \left(\frac{1}{\ln(2^x)} + \ln(1/2^x)\right)$

(b) Find The critical point(s) of $f$. 3

**Solution:** $f'(x) = 0 \implies x = \frac{1}{\ln 2}$, since $x \in (0, \infty)$.

(c) Is $f$ an injective function? 2

**Solution:** $f$ has a local maxima. So it cannot be injective.

(d) Is $f$ a surjective function? 2

**Solution:** $f$ has a local maxima at $1/ln2$ which is also the global maxima. So it never obtains any bigger value. So it's not surjective.

3. Express the following expression as an integral and evaluate the limit. 15

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{2n}{k^2 - 4n^2}$$

**Solution:** The given limit $= \lim\limits_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{2}{(k/n)^2 - 4} = \int_0^1 \frac{2}{x^2-4} dx = \int_0^1 \frac{1}{2}\left(\frac{1}{x-2} - \frac{1}{x+2}\right) = \ldots.$ The penultimate step was using partial fractions method.

4. A rod of length $a$ is placed on the $X-$axis from $x = -a$ to $x = a$. Suppose the mass density of the rod at $x$ is given by $\lambda(x) = \sqrt{a^2 - x^2}$.

(a) Find the mass of the rod. 8

**Solution:** Substitute $x = a\sin u$. Then mass $= \int_{-a}^{a} a^2 \cos^2 u du = a^2 \int_{-a}^{a} \frac{\cos(2u)+1}{2} du = \ldots = \pi a^2/2$.

(b) Find the center of mass of the rod. 7

**Solution:** Same substitution. Answer $= 0$.

**You do not have to simplify your answer.**

5. (a) Solve the following differential equation with initial conditions: 10

$$xy' - 2y = x^3 e^x, \quad y(1) = e$$

**Solution:** $P(x) = -\frac{2}{x}$, $Q(x) = x^2 e^x$. So $I.F. = e^{-2\ln x} = \frac{1}{x^2}$. Answer: $y = x^2 e^x$.

(b) You drink a beverage with 120 mg of caffeine. Each hour, the caffeine in your system decreases by 12%. How long until you have 10 mg of caffeine left in your body? **You do not have to simplify your answer.** 5

**Solution:** The amount of caffeine follows the formula $y = 120e^{kt}$. At $t = 1, y = 0.88 \times 120 \implies e^k = 0.88$. So $10 = 120 \times (0.88)^t \implies t = \ln_{0.88} \frac{1}{12}$

6. Find the following integrals.

(a) 10

$$\int (1 + \tan x)(1 - 2\tan x) dx$$

**Solution:** Multiply the two terms and expand. Answer: $x - \ln|\sec x| - 2\tan x + 2x + c$.

(b) 10

$$\int \arctan x dx$$

**Solution:** Integration by parts. Answer: $x \arctan x - \frac{1}{2}\ln|1 + x^2| + c$.

(c) $\boxed{10}$

$$\int \frac{1}{x^2+3x+4}dx$$

**Solution:** $x^2+3x+4$ does not have any linear factors. So it's not a partial fraction problem. $x^2+3x+4=(x+3/2)^2+7/4$. Answer: $\frac{2}{\sqrt{7}}\arctan\frac{x+3/2}{\sqrt{7}/2}+c$.

(d) $\boxed{10}$

$$\int e^{\sin x}\left(\sin(2x)+2\cos x\right)dx$$

**Solution:** $u=\sin x$. The integral $=\int e^{\sin x}(2\sin x\cos x+2\cos x)dx=\int e^{\sin x}(2\sin x+2)(\cos x\,dx)=\int e^u(2u+2)du=e^u(2u)+c=2e^{\sin x}\sin x+c$.

7. Find the derivative of $f(x)=(\sin x)^{\ln(\cos x)}$. $\boxed{5\text{ (bonus)}}$

**Solution:** $\ln(f(x))=\ln(\cos x)\ln(\sin x)\implies\frac{f'(x)}{f(x)}=\cot x\ln(\cos x)-\tan x\ln(\sin x)\implies f'(x)=(\sin x)^{\ln(\cos x)}(\cot x\ln(\cos x)-\tan x\ln(\sin x))$