Assignment 2 (6/29)

Subhadip Chowdhury

- *This homework is due at the beginning of class on* ***Tuesday*** 7/3. *You are encouraged to work together on these problems, but you must write up your solutions independently.*
- *Exercise* 7 *is extra credit and optional.*

More on Pigeonhole Principle

- We talked about a more general version of PHP and how it can be applied to problems. The general version reads as follows:

  If $(nk+1)$ pearls are placed into n boxes, then there is at least one box contains more than k pearls.

- An application of this is Exercise 4 from assignment 1.

More on Graph Theory

- We can draw the same graph in different ways. Convince yourself that the following two graphs are actually 'same'. The corresponding vertices have been given the same color.

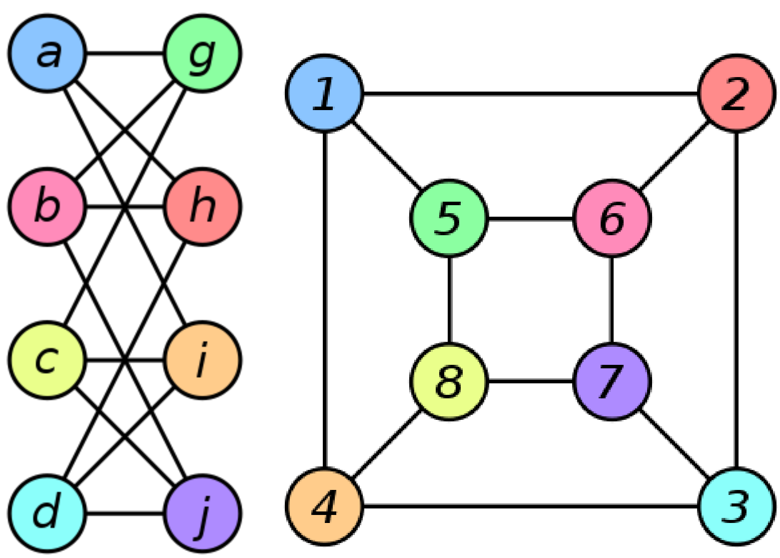

Figure 1: Two Graphs, Source: Wikipedia

- We discussed degree of a vertex last time. This time, we proved the following theorem.

Theorem: *A graph cannot have exactly one vertex of odd degree.*

Proof: Consider the sum of all the degrees of all the vertices. We will show that this is an even number.

Observe that each edge in the graph contributes 2 to the total degree of all vertices. Since counting all the edges also counts all the degrees of the vertices, it follows that the total number of edges is half the total degrees. Hence the total degree is twice the number of edges, i.e. an even number.

On the other hand, if the graph had exactly one vertex of odd degree, the total degree would be an odd number, which is not the case.

Hence, the theorem is proved. □

- Next we used PHP to prove an interesting property of graphs.

Problem: Show that there exist at least two vertices of same degree in a graph.

- The main idea behind the proof is to show that in graph of n vertices, you cannot have one vertex of degree 0 and another of degree $n-1$ at the same time.
- This problem is essentially equivalent to Exercise 3 from assignment 1 and hence we will not write any more details of the proof here.
- Below are some more problems you can solve using PHP.

Exercise 5

In the sequence $1, 1, 2, 3, 5, 8, 3, 1, 4, \ldots$ each term starting with the third is the sum of the two preceding terms, but we only write the unit's digit. Thus 5 and 8 is followed by the unit's digit of 13, which is 3. Prove that the sequence is periodic i.e. it will start repeating after some time.
[HINT: Any two consecutive terms of the sequence determine all succeeding terms. Thus the sequence will become periodic if any pair (a, b) of successive terms repeats.]

Exercise 6

Consider any five points P_1, P_2, P_3, P_4, and P_5 in the interior of a square S of side length 1. Show that we can always find two of them at distance at most $1/\sqrt{2}$ apart.
[HINT: Subdivide the square into four smaller squares of equal size.]

Exercise 7 (Extra Credit)

Show that any positive integer n has a multiple that looks like $11\ldots100\ldots0$, i.e. some number of ones followed by some (possibly none) number of zeroes.
[HINT: Consider the list of numbers $1, 11, 111, 1111, \ldots, 111\cdots1$, where the last entry has $(n+1)$ ones. What can you say about the remainders when they are divided by n?]