Problem Set 11,12 Solutions

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Exercise 1: Coprime Divisors for GCD

Problem 1.1. *Let a, b, d, m, n be positive integers such that $a = md$, $b = nd$ and $(m, n) = 1$. Show that $(a, b) = d$.*

I will give two proofs. The first will rely on prime factorizations.

Proof. Suppose the integers a and b have the following prime factorization:

$$a = p_1 p_2 p_3 \dots p_k, \qquad b = q_1 q_2 q_3 \dots q_l$$

for some natural numbers k and l. Here the p_is (and respectively q_js) are not necessarily distinct since a (respectively b) could have a power of a prime as a divisor. (e.g $60 = 2 \times 2 \times 3 \times 5$, so here $k = 4$, $p_1 = p_2 = 2, p_3 = 3, p_4 = 5$.)

We observe that the greatest common divisor of a and b is, by definition, exactly equal to the product of the common prime factors between the collection of p_is and q_js. This is because we can list all divisors of a as product of some of the p_is and similarly for b. So the greatest common divisor has all the common entries between the collection of p_is and q_js.

Now $(m, n) = 1 \implies (a/d, b/d) = 1 \implies a/d$ and b/d have no factor in common. In other words, d is all that's common between the collections p_is and q_js. So d must be the gcd of a and b. □

This second proof will be by contradiction.

Proof. Suppose, for the sake of contradiction, that $(a, b) = e$ where e is an integer, and $e \neq d$.

Then we first claim that

Claim. *if $d \mid a$ and $d \mid b$, then $d|e$.*

Proof of Claim. Suppose by contradiction that d does not divide e. Also d is smaller than e, since e is the *greatest* common divisor. So we can find at least one prime factor p of d such that p divides a and b but not e. If x, y are the integers such that $ex = a$ and $ey = b$, then $p|x$ and $p|y$ in order for p to still divide a and b without dividing e. This means that there are some integers L, M such that $dL = x$ and $dM = y$. We can construct a larger GCD, ep, which divides a and b since $eLp = a$ and $eMp = b$. Note $ep > e$ since $p > 1$. This contradicts e being the GCD, and we conclude that in fact $d|e$. □

Now we know $\exists x, y \in \mathbb{Z}$ such that $ex = a$ and $ey = b$. Substituting, we know $md = ex$ and $nd = ey$. By the claim above in which we found that $d|e$, and also by the fact that $d > 0$, we conclude that $\frac{e}{d}$ is an integer. In fact, it is greater than 1 since $e > d$. But $m = x\frac{e}{d}$ and $n = y\frac{e}{d}$ implies that $(m, n) \geq \frac{e}{d} > 1$, which is a contradiction.

□

Exercise 2: GCD Coefficient Identity

Problem 2.1. *Suppose a, b, and c are positive integers such that $(ac, bc) = d$. Show that $(a, b) = \frac{d}{c}$.*

Before giving the proof let us recall the following result that was proved in class.

Theorem 2.2. *Let a, b, m, n, d be positive integers such that $(a, b) = d$ and $a = md, b = nd$. Then $(m, n) = 1$.*

Proof. Suppose, for the sake of contradiction, that m and n have a gcd e greater than 1. Then $\exists i, j \in \mathbb{N}$ such that $ei = m$ and $ej = n$. Then, $a = eid$ and $b = ejd$. But then ed is a common divisor of a and b which is greater than d since $e > 1$. But d was supposed to be the GCD of ac and bc, so this is a contradiction. Hence $(m, n) = 1$. □

observe that above theorem is exactly the converse of exercise 1 of this assignment. Now let's do exercise 2.

Proof. Let's take the positive integers m, n to be such that $ac = md$ and $bc = nd$. Then using theorem 2.2, we get that $m = \frac{ac}{d}$ and $n = \frac{bc}{d}$ are coprime.

Let $D = \frac{d}{c}$. We are going to prove $(a, b) = D$. First we rewrite above equalities as

$$m = \frac{ac}{d} = \frac{a}{D}$$

and

$$n = \frac{bc}{d} = \frac{b}{D}$$

Then we have $D \mid a$, $D \mid b$ and $(a/D, b/D) = (m, n) = 1$. By using exercise 1.1, we conclude that $(a, b) = D = \frac{d}{c}$.

□

Exercise 3: GCD of One Less Than Consecutive Powers of Two

Problem 3.1. *Show that*

$$(2^m - 1, 2^{m+1} - 1) = 1$$

for any positive integer m.

Proof. Apply the identity $(a + bc, b) = (a, b)$ with $a = 1$, $b = 2^m - 1$, $c = 2$:

$$(2^m - 1, [2(2^m - 1) + 1]) = (1, 2^m - 1)$$

1 has only itself as a positive factor, so the GCD of any integer with 1 is 1. □

Exercise 4: GCD Multiplicative Property

Problem 4.1. *Suppose a, b, x and d are positive integers such that $(a,x) = d$ and $(b,x) = 1$. Show that $(ab,x) = d$.*

Proof. Since $(a,x) = d$, there are integers i, j such that $di = a$ and $dj = x$. Consider $(ab,x) = (dib,dj)$. Hence d is a common divisor of ab and x.

We know that $(i, j) = 1$ by theorem 2.2. Now, we know by assumption that $x = dj$ and b share no common factor greater than 1. Then j and b must be coprime as well.[1] In other words j doesn't share any common prime factor with either i or b. Hence $(ib, j) = 1$. Then by exercise 1.1, we get that $(ab,x) = d$.

□

Exercise 5: Coprime Divisibility (Extra Credit)

Problem 5.1. *Suppose a, b, and c are positive integers such that $c|ab$ and $(b,c) = 1$. Then show that $c|a$.*

The first proof uses the result of exercise 4 directly.

Proof. Suppose, for the sake of contradiction, that c does not divide a. Then let $(a,c) = d \neq c$. We also have $(b,c) = 1$. then exercise 4 with x replaced with c gives us $(ab,c) = d$. This is a contradiction since $c \mid ab \implies (ab,c) = c$ and $d \neq c$. □

Here's another proof using prime factorization.

Proof. Suppose, for the sake of contradiction, that c does not divide a. Then there must be at least one prime factor of c that is not a prime factor of a. Call this p. (To be more exact, there must be at least one prime p whose multiplicity is greater in the prime factorization of c than in the prime factorization[2] of a.) Now, since $c|ab$, we have $p|ab$, but p does not divide a, so $p|b$. But then $p|c$ and $p|b$, which means that $p > 1$ is a common divisor of c and b. But this contradicts $(b,c) = 1$. □

Exercise 6: Floor Identities for Integers

Problem 6.1. *Find whether the following statements are True or False. If False, give a counterexample. If True, give a proof.*

$\forall x, y \in \mathbb{Z}$,

(a) $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$

(b) $\lfloor 2x \rfloor = 2\lfloor x \rfloor$

(c) $\lfloor x^2 \rfloor = \lfloor x \rfloor^2$

(d) $\lfloor x + 0.5 \rfloor = \lfloor x - 0.5 \rfloor$

(e) $\lfloor x \rfloor + \lfloor -x \rfloor = 0$

1. for if there were, by contradiction, some prime factor p_j that did, then (b,x) would be at least p_j, since we know $p_j|j$ and $j|x$. But this contradicts $(b,x) = 1$.

2. From now on, we will assume that we mean this, i.e. that prime factors are not necessarily distinct.

Solution. (a) True.

Proof. For every integer z, we calculate $\lfloor z \rfloor = z$, as an integer is its own floor. We know $x + y$ is an integer since $x, y \in ⅽ$. So $\lfloor x + y \rfloor = x + y$. Moreover, $\lfloor x \rfloor = x$ and $\lfloor y \rfloor = y$, so the right side is also $x + y$. ■

(b) True.

Proof. $2x \in \mathbb{Z}$, so $\lfloor 2x \rfloor = 2x$. The right side is $2\lfloor x \rfloor = 2x$, as $x \in \mathbb{Z}$. ■

(c) True.

Proof. $x^2 \in \mathbb{Z}$, so $\lfloor x^2 \rfloor = x^2$. The right side is $\lfloor x \rfloor^2 = (x)^2$, since x is an integer. ■

(d) False.

Proof. Choose $x = 1$. Then $\lfloor x + 0.5 \rfloor = 1$ and $\lfloor x - 0.5 \rfloor = 0$. ■

(e) True.

Proof. Both x and $-x$ are integers, so we have $\lfloor x \rfloor = x$ and $\lfloor -x \rfloor = -x$. The left side is thus $x + (-x) = 0$. ■

■

Exercise 7: Floor Equation

Solve the following equation for x.

$$\lfloor 5 - \lfloor x \rfloor \rfloor = 15$$

Solution. $x \in [-10, -9)$

If $y = 5 - \lfloor x \rfloor$, then $y \in [15, 16)$. This implies that

$$10 \leq -\lfloor x \rfloor < 11$$

Viewing the inequalities in terms of the multiplicative inverse of the three quantities, we have

$$-10 \geq \lfloor x \rfloor > -11$$

Now $\lfloor x \rfloor$ must be an integer, and the only integer in this interval is -10. So

$$\lfloor x \rfloor = -10$$

which means that the solution for x is $[-10, -9)$. ■

Exercise 8: Minimum Power of Ten (Extra Credit)

Problem 8.1. *Assume k is an integer and n is a natural number such that $\lfloor \frac{10^n}{k} \rfloor = 2018$. What is the minimum possible value of n?*

Solution. $n = 7$

For $\frac{10^n}{k}$ to have a floor of 2018, it must lie in the interval $[2018, 2019)$. With $k \neq 0$, we have $10^n \in [2018k, 2019k)$. Now, if k were negative, we would be unable to find any n such that 10^n is negative. So we must start with positive k. The length of the interval $[2018k, 2019k)$ is $2019k - 2018k = k$, so we must find an integer k such that $0 < 10^n - 2018k \leq k$. For every value of n, the closest approach of k to test is $k = \lfloor \frac{10^n}{2018} \rfloor$. So we look for when the inequality is satisfied by $2018\lfloor \frac{10^n}{2018} \rfloor + \lfloor \frac{10^n}{2018} \rfloor > 10^n$, or when $\lfloor \frac{10^n}{2018} \rfloor > \frac{10^n}{2019}$. The two sides of this inequality are equal until $n = 7$ (Can you say why? Observe the fractional part of $\frac{10^n}{2018}$), which produces $k = \lfloor \frac{10^n}{2018} \rfloor = 4955$, and the inequality now holds true. Observe that the conditions required by the problem are satisfied by these values. ■

Exercise 9: Fractional and Floor GP

Problem 9.1. *If $\{x\} = x - \lfloor x \rfloor$, and x is a real number such that $\{x\}, \lfloor x \rfloor, and x$ form a GP, find x.*

Solution. $x = \frac{1}{2} + \frac{\sqrt{5}}{2}$ or approximately 0.618, 1, 1.618

Let a be a real number and b be an integer, such that $a = \{x\}$ and $b = \lfloor x \rfloor$. So $x = a + b$. Note that $0 \leq a < 1$. Now, the ratio of the first term to the second term is the same as the ratio of the second to the third, so

$$\frac{a}{b} = \frac{b}{a+b}$$

Note that b cannot be zero. Since b is the floor of x, x cannot be in $[0, 1)$. Arranging the terms, we find

$$a^2 + ab - b^2 = 0$$

Since b is an integer, there are only a few possibilities. Observe that:

if $b = 1$, a has real roots $-\frac{1}{2} \pm \frac{\sqrt{5}}{2}$. Of these two, $a = -\frac{1}{2} + \frac{\sqrt{5}}{2}$ lies in $[0, 1)$.

if $b = -1$, a has real roots $\frac{1}{2} \pm \frac{\sqrt{5}}{2}$. Neither of these is within $[0, 1)$.

if $b > 1$ or $b < -1$, the parabola only widens (Can you say why?), and the roots lie increasingly far from $[0, 1)$. So the positive root for $b = 1$ is the only solution. ■

Exercise 1: Inequality Plots

Problem 10.1. *Find and sketch the solution set of the following equations/inequalities. If the equation has one variable, the answer is an interval. If it has two variables, then it is a region on the plane. You must explain how you found your answer.*

(a) $|x| + |y| < 1$
(b) $\frac{x+3y}{3x+y} < 1$
(c) $|x - 1| + |x - 2| < 5$
(d) $x^2 - |x + 2| + x > 0$

Solution. (a) We split into four cases at which the absolute-valued expressions change signs: $x < 0$ and $y < 0$; $x < 0$ and $y > 0$; $x > 0$ and $y < 0$; and $x > 0$ and $y > 0$. This yields four inequalities, respectively:

$(-x) + (-y) < 1$
$(-x) + y < 1$
$x + (-y) < 1$
$x + y < 1$

For each of the four inequalities, plot the corresponding equation as a dotted line and use the test point $(0, 0)$ to determine the side which is the solution. You will notice that all four regions intersection as shown in Fig. 10.

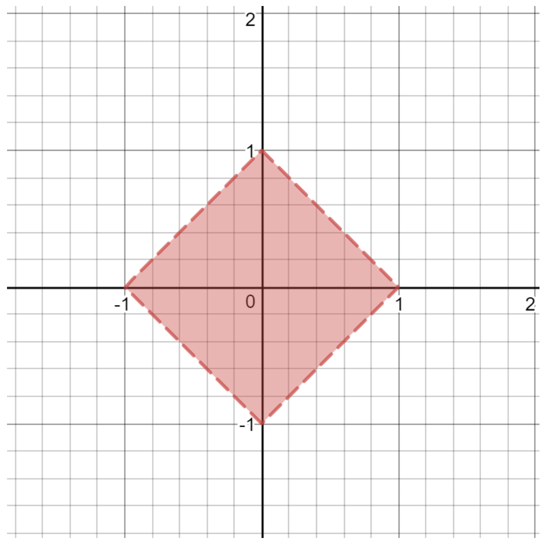

Figure 1: Plot of $|x| + |y| < 1$, created with Desmos

(b) Note that we must exclude the points x, y such that $3x + y = 0$, for otherwise we would divide by zero. Now, split the rest of the possible denominator values into two cases. The solution will be either the first case OR the second case, i.e. the *union* of the two solution regions.

If $3x + y < 0$, then multiplying both sides by $3x + y$ will result in the inequality sign changing. So in this case, solutions will obey $x + 3y > 3x + y$, or $y > x$. Plot both analogous equations as dotted lines, and test a point. This is the region to the left in Fig. 10.

If $3x + y > 0$, then we have $y < x$. Plot both analogous equations as dotted lines, and test a point. This is the region to the right in Fig. 10

(c) $x \in (-1, 4)$

Note that the signs of the expressions inside absolute values change at $x = 1$ and at $x = 2$. We can write separate inequalities for each of the three regions:

If $x \leq 1$, then we have $-(x-1) + -(x-2) < 5$ which is also $-2x - 2 < 0$, or $x > -1$. So this region has solutions only in the interval where $-1 < x \leq 1$.

If $1 < x < 2$, then we have $(x-1) + -(x-2) < 5$, which is satisfied when $-1 + 2 < 5$, i.e. true for all x in this region. So x is also a solution when $1 < x < 2$.

If $x \geq 2$, then we have $(x-1) + (x-2) < 5$, which is true when $2x < 8$. So the solutions in this region are x where $2 \leq x < 4$.

Taking the union of all these solution intervals, we have $\{x \in \mathbb{R} | -1 < x < 4\}$.

(d) $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

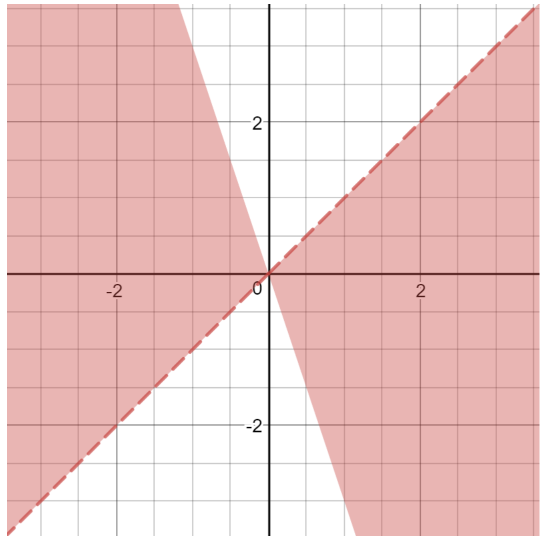

Figure 2: Plot of $\frac{x+3y}{3x+y} < 1$, created with Desmos

The sign of $x+2$ changes at $x=-2$, so let us consider two cases.

When $x<-2$, we have $x^2 - -(x+2) + x > 0$, which is true when $x^2+2x-2>0$. The roots of this quadratic, when set equal to 0, are $-1 \pm \sqrt{3}$. Testing the point $x=-1$ in the original inequality, we find that this holds for $x < -1-\sqrt{3}$ or $x > -1+\sqrt{3}$. Intersecting these intervals with $x<-2$, we have that $x < -1-\sqrt{3}$ is the interval containing solutions for this case.

When $x \geq -2$, we have $x^2-(x+2)+x>0$, which is true when $x^2>2$. These x are the intervals such that $x>\sqrt{2}$ or $x<-\sqrt{2}$. The intersection with $x \geq -2$ is x such that $-2 \geq x < -\sqrt{2}$ or $x > \sqrt{2}$.

Now the union of the two solution intervals found in the two cases above is x such that $x < -\sqrt{2}$ or $x > \sqrt{2}$.

■

Exercise 2: Quadratic Inequality Conditions

Problem 11.1. *If $x^2 - ax + 3 > 0$ for all real number x, then find all possible values of a.*

Solution. $-2\sqrt{3} < a < 2\sqrt{3}$

Note that as x tends towards large positive or negative values, the right side of the inequality is positive, as the x^2 dominates the other terms. Thus, we now only need to guarantee that the minimum occurs above the x-axis. Hence, the inequality $x^2-ax+3>0$ holds when $x^2-ax+3=0$ has no real roots. This is true when the discriminant is negative, i.e. when

$$(-a)^2 - 4(1)(3) < 0$$

Which is true when $-2\sqrt{3} < a < 2\sqrt{3}$. ■

Exercise 3:

Problem 12.1. *Suppose λ is a real number such that the two roots of the equation*

$$(\lambda-1)(x^2+x+1)-(\lambda+1)(x^2-x+1)=0$$

are real and distinct. Prove that $\lambda < -2$ or $\lambda > 2$.

Proof. Write the quadratic equation in the form $ax^2 + bx + c = 0$, where $a = (\lambda - 1) - (\lambda + 1) = -2$, $b = (\lambda - 1) - -(\lambda + 1) = 2\lambda$, and $c = (\lambda - 1) - (\lambda + 1) = -2$. The discriminant must be positive for there to be two distinct real roots, so

$$(2\lambda)^2 - 4(-2)(-2) > 0$$

So then

$$\lambda^2 > 4$$

This inequality has solutions $\lambda < -2$ or $\lambda > 2$. □

Exercise 4:

Problem 13.1. *If the two roots of of the equation $(b-c)x^2 + (c-a)x + (a-b) = 0$ are equal to each other, then show that a, b, and c are in an AP.*

Proof. The two roots of the quadratic equation are equal when the discriminant D is equal to zero. Hence,

$$D = (c-a)^2 - 4(b-c)(a-b) = 0$$

Now, we want to show that there is a common difference between a, b, and c. We want to prove that $b - a = c - b$, or that $c - a - 2b = 0$. So we should try to show that $c - a - 2b$ is a factor (and the only factor) of $(c-a)^2 - 4(b-c)(a-b)$.

Now expanding the discriminant D, we get

$$(c-a)^2 - 4(b-c)(a-b) = (c^2 - 2ac + a^2) - 4(b^2 - bc - ba + ac) = a^2 + 4ba + 2ac + c^2 - 4bc - 4b^2$$

Dividing by $(c - a - 2b)$, we see that

$$a^2 + 4ba + 2ac + c^2 - 4bc - 4b^2 = (c-a-2b)(c-a-2b) = (c-a-2b)^2$$

Thus $D = 0 \implies c + a - 2b = 0$, and we are done.

□

Exercise 5:

Problem 14.1. *Suppose a, b, and c are three real numbers in a GP and x is a real number such that $a + b + c = bx$. Find all possible values of x.*

Solution. $\{x | x \leq -1 \text{ or } x \geq 3\}$

Let us write the equation in the form

$$a + ar + ar^2 = x(ar)$$

Dividing by a, we can write this as a quadratic equation:

$$r^2 + (1-x)r + 1 = 0$$

For a, b, and c to indeed form a GP of real numbers, the common ratio r must be real. Thus the equation above must have real roots for r. This requires that the discriminant be non-negative, i.e.

$$(1-x)^2 - 4(1)(1) \geq 0$$

which is satisfied when $x \leq -1$ or $x \geq 3$. ■