# Problem Set 7,8 Solutions

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*The University of Chicago, CAAP 2018: Proof-Based Methods in Calculus (Chowdhury)*

July 18, 2018

## Exercise 1: Fractional Part Function

**Problem 1.1.** *We can define the fractional part of $x$, denoted $\{x\}$, to be $\{x\} = x - \lfloor x \rfloor$. Draw the graph of the fractional part function.*

*Solution.* See Fig. 1. First let's see what happens for $x \in [0,1]$. For $x \in [0,1)$, we have $\lfloor x \rfloor = 0$. So the function grows linearly from the origin with $x$ and approaches but never achieves 1. When $x = 1$, we have $x = \lfloor x \rfloor = 1$. So $\{1\} = 0$. This is in fact true for every integer $x$, since at any whole number, the floor of the whole number equals itself. Next observe that for any other value of $x$, we always have $0 \le x - \lfloor x \rfloor < 1$. Thus, the range of the fractional part function is $[0,1)$. Also since

$$\{x+1\} = x+1-\lfloor x+1 \rfloor = x+1-\lfloor x \rfloor - 1 = x - \lfloor x \rfloor = \{x\},$$

we see that the function is periodic with period 1. Try choosing several points, calculating the fractional part, and plotting.

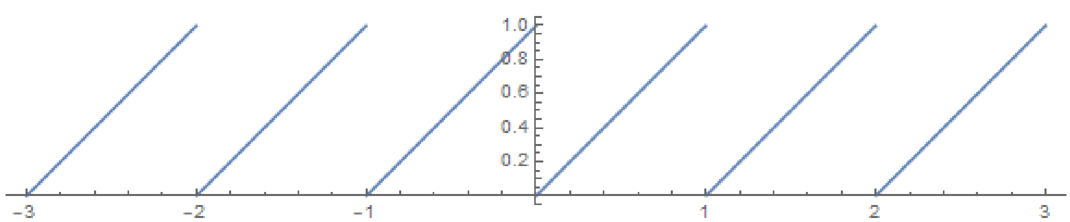


Figure 1: Fractional part function

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## Exercise 2: Ceiling Function

**Problem 2.1.** *The ceiling function, denoted $\lceil x \rceil$, is defined to be the least integer greater than or equal to $x$. Draw the graph of the ceiling function.*

*Solution.* See Fig. 2. The picture is obtained by observing that $\lceil x \rceil = b$ exactly when $x$ is a real number in the interval $(b-1, b]$

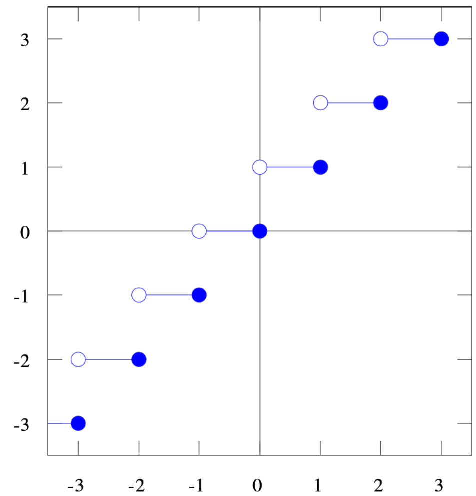


Figure 2: Ceiling function graph, courtesy of wikipedia.org.

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## Exercise 3: Reciprocal & Greatest Integer Function

**Problem 3.1.** *Sketch rough graphs of the functions*

$$f(x) = \left\lfloor \frac{1}{x} \right\rfloor$$
$$g(x) = \frac{1}{\lfloor x \rfloor}$$

*Solution.*

- *Graph of $f(x)$:* Note that $f$ is undefined for $x = 0$. For any value of $x$ bigger than 1, $\frac{1}{x}$ is a positive fraction between 0 and 1, which means the value of $f(x)$ is constant 0. Similarly, for any value of $x$ smaller than $-1$, $\frac{1}{x}$ is a negative fraction between 0 and $-1$, which means the value of $f(x)$ is constant $-1$. So all it remains is to analyze what happens when $x \in [-1, 1]$.

  Observe that $f(x)$ is always an integer. Now

  $$f(x) = n \implies \left\lfloor \frac{1}{x} \right\rfloor = n \implies n \leq \frac{1}{x} < n+1 \implies \frac{1}{n} \geq x > \frac{1}{n+1}$$

  Thus, for $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$ where $n \in \mathbb{Z}$, the function $f(x)$ remains at a constant value of $n$. The length of these intervals on the x-axis is so small compared to the values on the y-axis that it $f$ looks continuous, though it is not. The graph and a zoomed in picture of the same graph is pictured in figure 3.

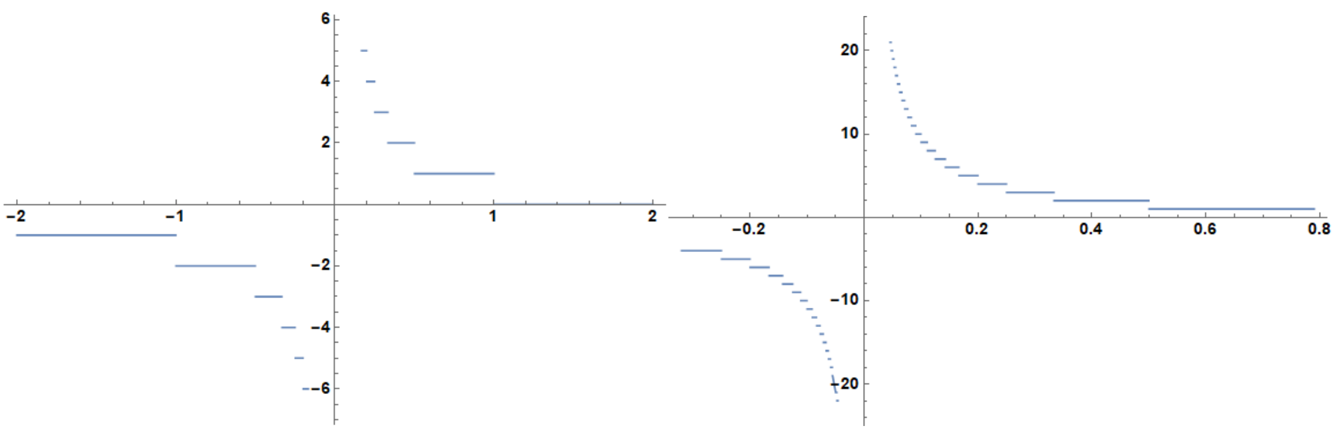


Figure 3: Plot of $f(x)$

- *Graph of $g(x)$:* We will approach this similarly as above. Observe that $g$ is now undefined for $x \in [0, 1)$. This is because for $x \in [0, 1)$, we have $\lfloor x \rfloor = 0$. Otherwise if $x \in [n, n+1)$ for some integer $n \neq 0$, we get

  $$\lfloor x \rfloor = n \implies g(x) = \frac{1}{n}$$

  The graph is pictured in figure 3.

■

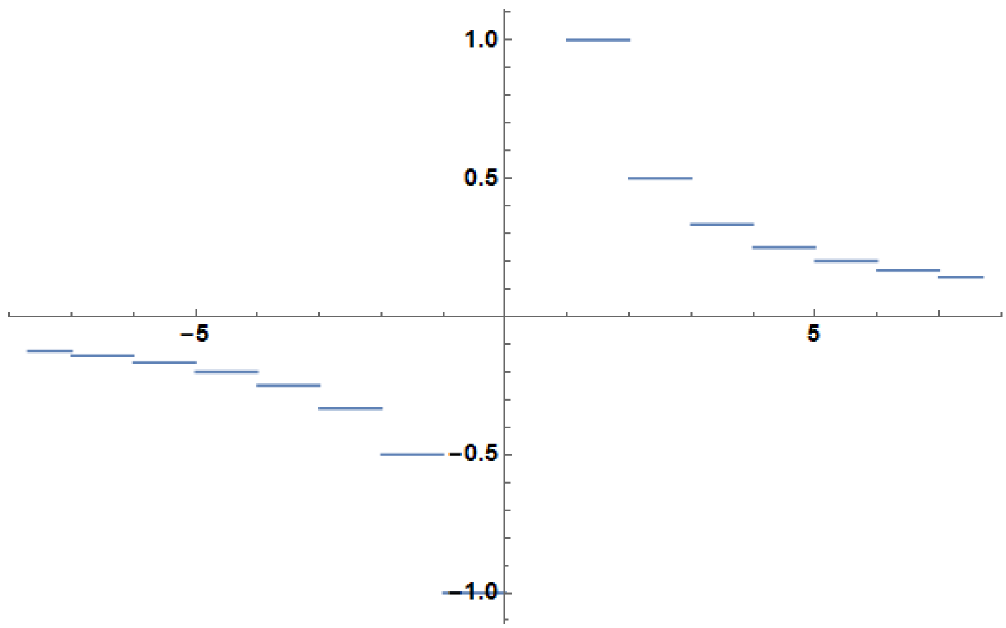


Figure 4: Plot of $g(x)$

## Exercise 4: Greatest Integer Function Equation

**Problem 4.1.** *Solve the following equation*

$$\lfloor 3.2 + \lfloor 2.5x - 7.9 \rfloor \rfloor = -5.$$

*Solution.* $x \in [-0.04, 0.36)$.

Let $z = \lfloor 2.5x - 7.9 \rfloor$, then

$$\lfloor 3.2 + z \rfloor = -5 \implies 3.2 + z \in [-5, -4) \implies z \in [-8.2, -7.2)$$

Now note that the Floor function takes only integer values, so the only way $z \in [-8.2, -7.2)$ is when $z = -8$.

Now $z = -8 \implies \lfloor 2.5x - 7.9 \rfloor = -8$. Hence

$$-8 \leq 2.5x - 7.9 < -7 \implies -0.1 \leq 2.5x < 0.9 \implies -\frac{0.1}{2.5} \leq x \leq \frac{0.9}{2.5}$$

Simplifying we get $x \in [-0.04, 0.36)$.

■

# Exercise 5: Floor Identity

**Problem 5.1.** *Prove that* $\lfloor 2x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor$ *for all real numbers* $x$.

*Proof.* We are going to divide the proof in to two cases. Observe that any real number $x$ can be always written as $\lfloor x \rfloor + \{x\}$ where $\{\cdot\}$ is the fractional part of $x$. We observed above that $0 \leq \{x\} < 1$. The proof of above identity is going to be different when $0 \leq \{x\} < \frac{1}{2}$ and when $\frac{1}{2} \leq \{x\} < 1$.

Let $\lfloor x \rfloor = n$.

**Case 1:** Assume $n \leq x < n + \frac{1}{2}$, so that $0 \leq \{x\} < \frac{1}{2}$.

In this case, observe that $2n \leq 2x < 2n + 1$. Hence $\lfloor 2x \rfloor = 2n$. On the other hand,

$$n \leq x < n + \frac{1}{2} \implies n + \frac{1}{2} \leq x + \frac{1}{2} < n + 1 \implies n < x + 1/2 < n + 1 \implies \lfloor x + 1/2 \rfloor = n$$

Thus

$$\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + n = 2n = \lfloor 2x \rfloor$$

**Case 2:** Assume $n + \frac{1}{2} \leq x < n + 1$, so that $\frac{1}{2} \leq \{x\} < 1$.

In this case, observe that $2n + 1 \leq 2x < 2n + 2$. Hence $\lfloor 2x \rfloor = 2n + 1$. On the other hand,

$$n + \frac{1}{2} \leq x < n + 1 \implies n + 1 \leq x + \frac{1}{2} < n + \frac{3}{2} \implies n + 1 \leq x + 1/2 < n + 2 \implies \lfloor x + 1/2 \rfloor = n + 1$$

Thus

$$\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = n + (n + 1) = 2n + 1 = \lfloor 2x \rfloor$$

□

# Exercise 6: Irrational Element

**Problem 6.1.** $\sqrt{7}$ *is irrational.*

*Proof.* Suppose, for the sake of contradiction, that $\sqrt{7} \in \mathbb{Q}$. Then, $\exists p, q$ such that $\frac{p}{q} = \sqrt{7}$, and we can choose $p, q$ such that they are relatively prime.[1] Then, $\frac{p^2}{q^2} = 7$, and in fact $p^2 = 7q^2$. This means that $7 | p^2$, as $q^2$ is an integer. But we know that $7 | p^2$ implies that $7^2 | p^2$, as we have proven before. So $\exists k \in \mathbb{Z}$ such that $p^2 = 7^2 k$. Substituting in our previous value of $p^2$, we find $7q^2 = 7^2 k \implies q^2 = 7kp^2$, and since $k$ is an integer, we know that $7 | q$. But then 7 divides both $p$ and $q$. This contradicts the fact that $p, q$ are coprime. □

1. If $p, q$ are not relatively prime, they have a GCD $k$ where $k > 1$. Then we have integers $p', q'$ where $p = p'k$ and $q = q'k$, and take those two instead.

## Exercise 7: Diophantine Equation

**Problem 7.1.** *Show that $x^2 - 4y = 2$ does not have any integer solution. In other words, show that if $x$ and $y$ are integers then $x^2 - 4y$ can not be equal to 2. (Use proposition 1.6).*

*Proof.* We will divide the proof into two cases, when $x$ is odd and when $x$ is even.

Suppose $x$ is an odd integer. Note that an odd number times an odd number is odd, so $x^2$ is odd. Now, an even number added to an odd number is odd. So $x^2 + 4y$ is odd for any odd $x$, and therefore cannot equal 2.

Next suppose $x$ is an even integer. We know $2|x \implies 2^2|x^2$. Also, $4|4y$, and so $4|(x^2 + 4y)$. But 4 does not divide 2, and so we conclude that $(x^2 + 4y) \neq 2$. $\square$

Here is another way of proving this (by Subhadip).

*Proof.* Suppose, for the sake of contradiction, that the equation is satisfied for some integer values of $x$ and $y$. Then $x^2 = 4y + 2$, is an even integer. Hence $2 \mid x^2 \implies 2 \mid x$ i.e. $x = 2k$ for some integer $k$. Then

$$x^2 - 4y = 2 \implies 4k^2 - 4y = 2 \implies 2k^2 - 2y = 1$$

which is a contradiction since the LHS is even and the RHS is odd. $\square$

## Exercise 8: Factorial Power (Extra Credit)

**Problem 8.1.** *If $n \in \mathbb{N}$, then $4|\frac{(2n)!}{(n!)^2} \implies n$ is not a power of 2. (Use Legendre's Formula, by contradiction.)*

*Proof.* For the sake of contradiction, suppose $\exists k \in \mathbb{Z}$ such that $n = 2^k$.
Let $P = (2n)!, Q = (n!)$ and $N = \frac{P}{Q^2}$. Then,

$$P = (2^{k+1})!$$
$$Q = (2^k)!$$
$$N = \frac{(2^{k+1})!}{(2^k!)^2}$$

Let's calculate the highest power of 2 that divides the numerator and denominator of $N$ with Legendre's formula. The formula says that the exponent of the largest power of 2 that divides $J!$ is given by the sum

$$\nu(J) = \left\lfloor \frac{J}{2^1} \right\rfloor + \left\lfloor \frac{J}{2^2} \right\rfloor + \left\lfloor \frac{J}{2^3} \right\rfloor + \ldots + \left\lfloor \frac{J}{2^i} \right\rfloor + \ldots$$

Now observe that if $P = (2^{k+1})!$, then

$$\nu(P) = 2^k + 2^{k-1} + \ldots + 1$$

.

On the other hand, if $Q = (2^k)!$, then

$$\nu(Q) = 2^{k-1} + 2^{k-2} + \ldots + 1$$

Note that when we square a number, the highest power of 2 that divides it, also gets squared, and so the exponent gets multiplied by 2. In other words, the exponent of the highest power of 2 that divides $Q^2$ is

$$2 \times (2^{k-1} + 2^{k-2} + \ldots + 1) = 2^k + 2^{k-1} + \ldots + 2$$

Next observe that the powers of 2 dividing $P$ and $Q^2$ cancel out when we divide $P$ by $Q^2$. Hence the exponent of the largest power of 2 that divides $N$ is $\nu(P) - 2\,\nu(Q)$, which simplifies to

$$(2^k + 2^{k-1} + \ldots + 2 + 1) - (2^k + 2^{k-1} + \ldots + 2) = 1$$

Hence $4 = 2^2$ does not divide $N$. Contradiction.

□

## Exercise 9: Sequence Induction

**Problem 9.1.** *By mathematical induction, prove that if $a_1 = 1$ and for $n \geq 1$*

$$a_{(n+1)} = 2a_n + 1$$

*then $a_n = 2^n - 1$ $\forall n \in \mathbb{N}$.*

*Proof.* We will prove the formula by inducting on $n \in \mathbb{N}$. The base case, $n = 1$, holds because if $a_1 = 1$ and $a_2 = 2a_1 + 1$, then $a_2 = 3$ which agrees with the closed form formula that $a_2 = 2^2 - 1$.

Now let us make the induction assumption that $\exists k \in \mathbb{N}$ such that $a_k = 2^k - 1$. We want to prove that $a_{k+1} = 2^{k+1} - 1$.

We know that $a_{k+1} = 2a_k + 1$, so

$$\begin{aligned} a_{k+1} &= 2(2^k - 1) + 1 \\ &= 2^{k+1} - 2 + 1 \\ &= 2^{k+1} - 1 \end{aligned}$$

This completes our induction step. Hence, the identity holds for $n = k+1$ whenever it holds for $n = k$. By the principle of mathematical induction, it holds $\forall n \in \mathbb{N}$, where $n \geq 1$.

□

## Exercise 10: Closed Forms of Recursive Sequences

**Problem 10.1.** *Give the first 6 terms of the following sequences and then guess a formula for the $n^{th}$ term, for $n \geq 2$. You don't need to provide a proof.*

*(a)* $a_1 = 1$, $a_2 = 3$, $a_{n+1} = 2a_n - a_{n-1}$

*(b)* $a_1 = 1$, $a_2 = 3$, $a_{n+1} = 3a_n - 2a_{n-1}$

*Solution.* (a) $1, 3, 5, 7, 9, 11$

$$a_n = 2n - 1$$

(b)$1, 3, 7, 15, 31, 63$

$$a_n = 2^n - 1$$

■

# Exercise 11: Square Root Recursion Induction

**Problem 11.1.** *If* $\{a_i\}_{i\in\mathbb{N}}$ *is defined as* $a_1 = \sqrt{2}$*,* $a_{n+1} = \sqrt{2+a_n}$ *for* $n \geq 1$*, show by induction that* $a_n < 2$ $\forall n \in \mathbb{N}$.

*Proof.* We will prove the inequality by inducting on $n$. Note that the first term (i.e. when $n = 1$) is $a_1 = \sqrt{2}$, which is clearly less than 2. Hence the base case is true.

Now, make the induction assumption that for some $k \geq 1$, we have $a_k < 2$. We want to prove that $a_{k+1} < 2$.

By definition, $a_{k+1} = \sqrt{2+a_k}$. Since $a_k < 2$, we get that

$$a_{k+1} < \sqrt{2+2} = 2$$

Thus the inequality holds for $n = k+1$ whenever it holds for $n = k$. Hence, by the principle of mathematical induction, it holds $\forall n \in \mathbb{N}$.

□

## Exercise 12: AP Terms

**Problem 12.1.** *The fifth term of an AP is* 1 *and the* $31^{st}$ *is* $-77$*. Find the* $20^{th}$ *term.*

*Solution.* Subtracting $a_5 = a+4d$ from $a_{31} = a+30d$, we find a difference of $(30-4)d$, which we can set equal to $a_{31}-a_5 = (-77-1)$. This gives a common difference of $-3$. From the $5^{th}$ term to the $20^{th}$, we add the common difference 15 times, so $a_{20} = a+19d = a_5+15d$. This gives $a_{20} = -44$. ■

## Exercise 13: AP Proofs

**Problem 13.1.** *(a) The* $m^{th}$ *term of an AP is* $n$ *and the* $n^{th}$ *term is* $m$*. Show that the* $(m+n)^{th}$ *term is zero.*

*(b) The sum of first* $m$ *terms of an AP is* $n$ *and the sum of first* $n$ *terms is* $m$*. Show that the sum of the first* $(m+n)$ *terms is* $(m+n)$*.*

*Proof.* (a) Let's assume that the AP looks like

$$a, a+d, a+2d, a+3d, \ldots$$

where the $i$th term is given by $a_i = a+(i-1)d$. We are given $a_m = n$ and $a_n = m$. Using the formula, we get

$$a+(m-1)d = n$$
$$a+(n-1)d = m$$

Subtract the two equations to eliminate $a$, and we get that $d(m-n) = (n-m) \implies d = -1$. Substituting this value of $d$ into either of the two equation above we get that $a = m+n-1$. Hence $a_{m+n} = a+(m+n-1)d = m+n-1-(m+n-1) = 0$.

□

*Proof.* (b) We use the formula for the sum of first $i$ terms in an AP $\{a_k\}_{k\in\mathbb{N}}$ to get

$$S_m = \frac{m(2a_1+a_m)}{2} = \frac{m}{2}(2a+(m-1)d) = n \tag{1}$$
$$S_n = \frac{n(2a_1+a_n)}{2} = \frac{n}{2}(2a+(n-1)d) = m \tag{2}$$

Expanding the LHS of equation (1) and (2), we get

$$\frac{m(m-1)d}{2} + am = n \tag{3}$$
$$\frac{n(n-1)d}{2} + an = m \tag{4}$$

Now adding (3) and (4) we get

$$\frac{d}{2}(m^2-m+n^2-n) + a(m+n) = m+n \tag{5}$$

By the formula for sum, we have

$$S_{m+n} = a(m+n) + \frac{d}{2}(m+n)(m+n-1)$$

Substitute $a(m+n)$ from (5) in above equation and simplify to get

$$S_{m+n} = \frac{d}{2}(m+n)(m+n-1) + (m+n) - \frac{d}{2}(m^2 - m + n^2 - n) = m + n - mnd \tag{6}$$

Next multiply both sides of Eq. 3 with $n$ and both sides of Eq. 4 with $m$ and subtract them. This eliminates $a$. We get

$$\frac{dmn}{2}((m-1)-(n-1)) = n^2 - m^2 = (n-m)(n+m) \implies \frac{dmn}{2} = -(m+n)$$

Thus $dmn = -2(m+n)$, and substituting this value in to equation 6, we get

$$S_{m+n} = (m+n) - 2(m+n) = -(m+n)$$

□

## Exercise 14: Student Loans

**Problem 14.1.** *A student decides to pay off her student loan of* \$36000 *in* 40 *annual installments which form an arithmetic progression. When* 30 *of the installments are paid, she gives up and flees the country, leaving one-third of the debt unpaid. Find the value of the first installment.*

*Solution.* Since the installments are in an AP, let's assume that the amount paid on day $n$ of 40 days is given by

$$a_n = a + (n-1)d \tag{7}$$

where $d$ is the common difference between payments and $a$ is the first installment.

The plan is to pay all debt in 40 years, so the sum of first 40 terms is 36000. Thus,

$$S_{40} = \frac{40(2a+39d)}{2} = 36000 \implies 2a + 39d = 1800$$

We are also given that $S_{30} = \frac{2}{3}S_{40}$, so

$$S_{30} = \frac{30(2a+29d)}{2} = 24000 \implies 2a + 29d = 1600$$

Use method of elimination to solve for $a$ and $d$. You should get $d = 20$ and $a = 510$.

■

## Exercise 15: Wall Builders (Extra Credit)

**Problem 15.1.** 150 *workers were engaged to build a wall in a certain number of days. Due to some reason, four workers dropped on the second day, four more on the third day and so on. It took* 8 *more days than initially planned to finish the wall. Assuming all workers work at the same rate, find the number of days in which the work was completed. (HINT: Use AP and unitary method!)*

*Solution.* The number of workers on the $n$th day is $n$th term of an AP with starting term 150 and common difference −4. Let's assume that the project was actually finished in $d$ days. Then the total number of workers that worked on the project over all days is the sum of the first $d$ terms of the AP, which is

$$S_d = \frac{d}{2}[2 \times 150 + (d-1)(-4)] = d(150 - 2(d-1)) = d(152 - 2d)$$

The problem says that the 150 original workers were scheduled to complete the project in $d - 8$ days. Assuming all workers work at the same rate, we get that each worker builds $\frac{1}{150(d-8)}$ of a wall every day[2].

Then $S_d$ workers built a total of $\frac{S_d}{150(d-8)} = 1$ wall. So

$$\frac{d(152-2d)}{150(d-8)} = 1 \implies d(152-2d) = 150(d-8) \implies d^2 - d - 600 = 0 \implies (d-25)(d+24) = 0$$

hence the wall was completed in $d = 25$ days. ■

2. This is what is meant by the *unitary* method. We find the work rate for a *single* worker in a single day.