Assignment 9 (1/29)

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- *This homework is due at the beginning of class on* ***Friday*** 2/2. *You may cite results from class as appropriate. Unless otherwise stated, you must provide a complete explanation for your solutions, not simply an answer. You are encouraged to work together on these problems, but you must write up your solutions independently.*
- *Hand in the exercises only, not the reading material part. You are encouraged to think about the exercises marked with a* ($\star$) *or* (†) *if you have time, but you don't need to hand them in. If you correctly solve a* (†)*-marked problem, you will get a candy!*
- *Remember that you can always use the result of the previous assignment problems without proof to solve the new assignment problems.*
- *We are currently covering Chapter* 14 *from Stewart.*

Important Points and Reading Materials

- Lagrange Multipliers with multiple constraints
  - When optimizing a function $F(x, y, z)$ subject to constraints $g(x, y, z) = k_1$ and $h(x, y, z) = k_2$, we solve the equation
  $$\nabla F = \lambda \nabla g + \mu \nabla h$$
  where λ and μ are called the Lagrange multipliers.
  - This equation arises from the fact that at the optimal point, ∇F is perpendicular to both the tangent planes to the surfaces $g = k_1$ and $h = k_2$ and hence is coplanar with ∇g and ∇h. Earlier we learned that coplanarity can be determined by the scalar triple product. Here we are giving a new characterization for coplanarity. Three vectors are coplanar iff one of them can be written as a linear combination of the two other vectors.

- Finding Max/Min of functions
  - If f has a local max or min at (a, b), then $\nabla f(a, b) = 0$.
  - A point (a, b) is called critical point of $f(x, y)$ if either $f_x(a, b) = 0$, or $f_y(a, b) = 0$, or if one of these partial derivatives do not exist.
  - A local mac/min for f is a critical point of f.
  - Not all critical point of f is a local max or min.

Problems

Exercise 1

Find the maximum and minimum volume of a rectangular box whose surface area is 1500 cm^2 and whose total edge length is 200 cm.

Exercise 2

1. Maximize $\sum_{i=1}^{n} x_i y_i$ subject to the constraints $\sum_{i=1}^{n} x_i^2 = 1$ and $\sum_{i=1}^{n} y_i^2 = 1$, where x_i, y_i's are real numbers.
2. ($\star$) Using above result, prove that for any real numbers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$, we have the following (famous!) inequality, known as *Cauchy-Schwarz Inequality*
$$\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right)$$
3. (†) Can you prove Cauchy-Schwarz Inequality using dot product of vectors?

Exercise 3

Find the critical points of the following functions:

1. $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 1$
2. $f(x, y) = e^x \cos y$

Exercise 4⋆

Find the points on the hyperboloid $x^2 + 4y^2 - z^2 = 4$ where the tangent plane is parallel to $2x + 2y + z = 5$.

Exercise 5

If $z = y + f(x^2 - y^2)$, where f is differentiable, show that

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = x$$

Exercise 6

Find the linear approximation to the function $f(x, y) = 2 - \sin(-x - 3y)$ at the point $P = (0, \pi)$, and then use your answer to estimate $f(0.001, \pi)$.

Exercise 7

Let $f(x, y) = x \cos x \cos y$ and $P = (0, \pi)$. Show that f is differentiable at P.