Math 2208: Ordinary Differential Equations

Lecture 9 Worksheet

Fall 2019 | **Subhadip Chowdhury** | **Oct 16**

Title: System of First Order ODEs

Summary: We will learn how about models that involve a system of first order differential equations. The most famous of these is the Predator-Prey model of Lotka and Volterra. Then we will use direction fields and phase portraits to do qualitative and quantitative analysis.

§A. General Form

The general form for a two-dimensional system of first order ODEs is

$$\frac{dx}{dt} = f(x,y,t)$$
$$\frac{dy}{dt} = g(x,y,t)$$

If f and g are linear in both x and y, the system is called *linear*, otherwise the system is called *nonlinear*.

Note that a second-order ODE can be converted to a two-dimensional system of first-order ODEs via some change in variables. For example, if

$$y'' = f(t, y(t)), y'(t))$$

is a second-order ODE, we can make the substitution $u_2(t) = y'(t)$ and $u_1(t) = y(t)$ to convert the ODE into a system of 2 first-order ODEs

$$\frac{du_1}{dt} = u_2(t)$$
$$\frac{du_2}{dt} = f(t, u_1(t), u_2(t))$$

Question 1.

Show that the third-order linear ODE

$$y''' + 3y'' + 2y' - 5y = \sin(t)$$

can be written as a linear system of three first-order ODEs.

§B. Lotka-Volterra Model

Probably the most famous system of ordinary differential equations of all time is the Lotka-Volterra predator-prey model. We will study a very special case of interaction between exactly two species, one of which -- the predators -- eats the other -- the prey. Such pairs exist throughout nature: lions and gazelles, birds and insects, pandas and eucalyptus trees, foxes and rabbits. To keep our model simple, we will make some assumptions:

- the predator species is totally dependent on a single prey species as its only food supply,
- the prey species has an unlimited food supply, and
- there is no threat to the prey other than the specific predator.

Let $x(t)$ denote the population of prey (rabbits) and let $y(t)$ denote the population of their predators (foxes). Let a, b, c and d be nonnegative parameters. One system of differential equations that might govern the changes in the population of these two species is

$$\frac{dx}{dt} = ax - bxy$$
$$\frac{dy}{dt} = cxy - dy$$

Question 2.

1. What do the constants a, b, c and d represent physically?
2. What is the significance of the xy terms in the model? Is the significance same for rabbits and foxes?
3. What happens to $x(t)$ if $b = c = d = 0$? How does this correspond to our assumptions?
4. What happens to $y(t)$ if $a = b = c = 0$? How does this correspond to our assumptions?
5. What does the model predict will happen if at any time one of the populations of the rabbits or the foxes becomes zero?

Question 3.

Are there any fixed points for the system of equations? In other words, what are the equilibrium solutions for the system?

§C. Phase Portrait

One of the ways to graph the solution of the system is to form the pair $(x(t), y(t))$ and think of it as a point in the xy-plane. In other words, the coordinates of the point are the values of the two populations at time t. As t varies, the pair $(x(t), y(t))$ sweeps out a parametric curve in the xy-plane. This curve is called the *solution curve*.

The xy-plane is called the *phase plane*, and it is analogous to the phase line for an autonomous first-order differential equation.

Observe that the solution curves that correspond to equilibrium solutions are really just points, and we refer to them as *equilibrium points*.

Definition 3.1: Nullclines

For a system

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

the $x-$*nullcline* is the set of points

$$\left\{(x,y)\middle|\frac{dx}{dt} = 0\right\}$$

i.e. the level curve in the phase plane where $f(x,y) = 0$. The $y-$nullcline is the level curve $g(x,y) = 0$.

At an equilibrium point, both $f(x,y)$ and $g(x,y)$ must be zero, hence

Theorem 3.1

The intersection of the nullclines are the equilibrium points.

A *Phase Portrait* of a system consists of the following information on the phase plane.

- the nullclines,
- the equilibrium points
- several solution curves corresponding to different initial conditions.

§D. Direction Field as a (normalized) Vector Field

Instead of thinking of $(x(t), y(t))$ as simply a combination of the two scalar-valued functions $x(t)$ and $y(t)$, we can consider the pair $(x(t), y(t))$ as a vector-valued function in the xy- plane. For each t, let $\vec{\mathbf{R}}(t)$ denote the column vector

$$\vec{\mathbf{R}}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

which corresponds to the position vector of the point $(x(t), y(t))$.

Then using this notation,

$$\frac{d\vec{\mathbf{R}}}{dt} = \begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \vec{\mathbf{F}}(\vec{\mathbf{R}})$$

where $\vec{\mathbf{F}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ is the vector field

$$\vec{\mathbf{F}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$$

Then the solution curves on a phase plane are essentially the *flow lines* of this vector field. If we draw the field after normalizing each vector to some small magnitude and only care about the direction, it gives us a diagram called the *direction field*.

Along the x-nullcline, the x-component of the vector field is zero, and consequently the arrows in the direction field are vertical. They point either straight up or straight down. Similarly, on the y-nullcline, the y-component of the vector field is zero, so the vector field is horizontal. Arrows points either left or right.

Observe that the x–nullcline naturally divides the plane into regions where $f(x,y) > 0$ and $f(x,y) < 0$. Since $x' = f(x,y)$, $f(x;y) > 0$ means x is increasing which in turn means the arrows point rightward in the plane. Similarly, the y–nullcline shows us where y is increasing or decreasing. The following table will help you fill out the direction field once the nullclines are found.

	$f(x,y) < 0$	$f(x,y) = 0$	$f(x,y) > 0$
$g(x,y) < 0$	$\swarrow$	$\downarrow$	$\searrow$
$g(x,y) = 0$	$\leftarrow$	$\cdot$	$\rightarrow$
$g(x,y) > 0$	$\nwarrow$	$\uparrow$	$\nearrow$

§E. Examples

Consider the following Lotka-Volterra model:

$$\frac{dx}{dt} = 2x - 1.2xy$$
$$\frac{dy}{dt} = 0.9xy - y$$

We will use pplane to draw the direction field on the phase plane. In the following picture of direction field, x–axis is prey and y–axis is predator.

Question 4.

Draw the x– and y–nullclines of above system and show the direction of the direction field along the nullclines (up, down, left, right). Find the equilibrium point(s), if any.

Question 5.

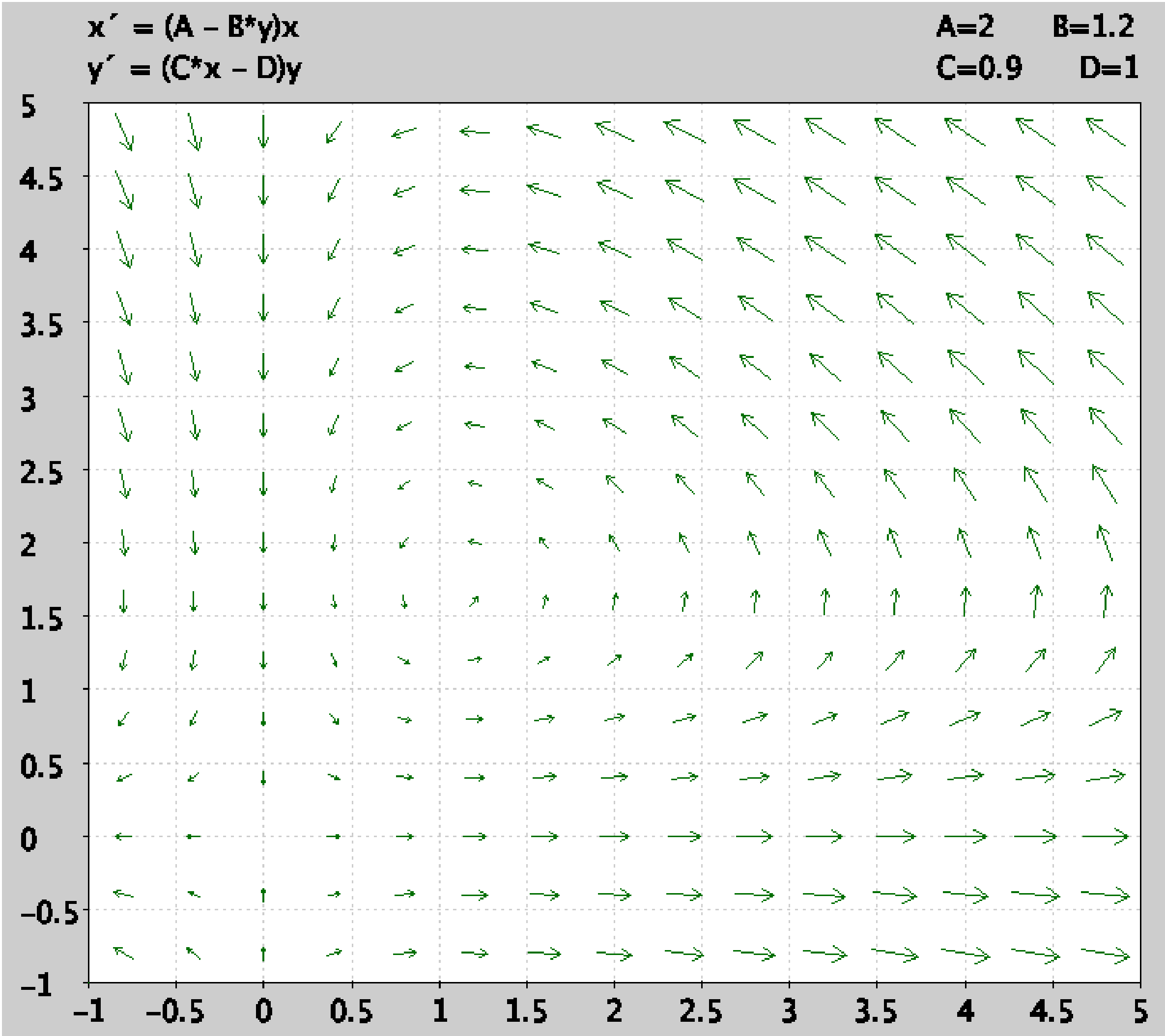
Draw the solution curve that starts at the initial condition $(x(0), y(0)) = (1, 0.5)$. Draw some more solution curves on your paper.

Question 6.

What happens when the initial condition approaches $(10/9, 5/3)$?

Question 7.

Pick one of the solution curves $(x(t), y(t))$. How do the graphs of $x(t)$ and $y(t)$ against t look like? Do you believe they would be periodic based on the behavior of the solution curves in the phase plane?



Question 8.

Can you explain why the two graphs should have the same period? Why does it makes sense that the increase in prey population lag behind the increase in predator population?

Question 9.

Find the equation of the nullclines of the following system, draw them in XY-plane, and find the equilibrium points.

$$\frac{dx}{dt} = x(2-x) - xy$$
$$\frac{dy}{dt} = xy - 1$$

Try drawing a rough diagram of the direction field on the phase plane $0 \le x, y \le 2$, using the table from section D. Here are some instructions: find an easily identifiable point in the plane which is not on one of the nullclines. Then evaluate (f, g) at this point and use this to draw a direction at that specific point. Then the rest of the arrows are easy to fill in using alternate choices.

Can you draw some sample solution curves?