

TEACHING PORTFOLIO

Subhadip Chowdhury

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Chapter 1

Teaching Experience and Responsibilities

§1. BOWDOIN COLLEGE

Over the 2018-2020 academic years, I have been fortunate to possess a liberal arts teaching experience at Bowdoin College. As a Visiting Assistant Professor, I was responsible for designing my own course curriculum, planning lectures and worksheets, designing and grading exams, holding office hours, and assigning individual homeworks and group projects. I also coordinated and mentored several graders, teaching assistant and study group leaders. Brief description of each of the courses I have taught are listed below.

1.1. Math 2208, Ordinary Differential Equations

- **Meeting Schedule** - Fall 2019, Spring 2020 - 3 hours per week
- **Class Size** - 20

This course, intended for Junior and Senior Math majors, is designed as a gateway course for students interested in Applied Math. I taught this course using a hybrid discovery method. This was the first applied course I had taught at Bowdoin college. I spent a significant amount of time carefully designing lecture worksheets that consisted of leading questions which helped students figure out the content for themselves. Although a part of the lecture was spent by me presenting on board, a lot of it was spent doing group work on blackboards and experimenting with software. Besides classical methods for solving differential equations, the main emphasis was on modern, qualitative techniques for studying the behavior of solutions to differential equations. We also worked on applications of ODEs in catastrophe theory, flow-kick regimen, resonance, market economy and auto-catalytic biochemical oscillations through multiple lab sessions and projects. Some of the projects and worksheets can be found in my website.

1.2. Math 2000, Linear Algebra

- **Meeting Schedule** - Spring 2019 - 3 hours per week
- **Class Size** - 11

This course is intended for Sophomores and Juniors and is designed as a gateway course for Mathematics and interdisciplinary majors. The students taking this course were not expected to have experience with writing proofs. As such, in an effort to make the course less dry, we spent a significant amount of time looking at various applications drawn from flight networks, cryptography, error correcting codes, population dynamics, Markov chains and Google page-rank algorithm, computer graphics, and optimization techniques using least-squares approximations. The students were assessed with multi-part in-class and take-home final exams. The goal was to make sure they are proficient in conceptual and numerical techniques as well as are able to apply their knowledge to practical applications. Some of the projects and exams can be found in my website.

1.3. Math 1800, Multivariable Calculus

- **Meeting Schedule** - Fall 2018, Spring & Fall 2019, Spring 2020 - 4.5 hours per week
- **Class Size** - 12 on average

This course is one of my most favorite course to teach. It's aimed towards mathematically inclined students, mostly Freshmen and Sophomores, who have learned differential and integral calculus, and would like to broaden their horizon. I have taught it several times over the last few years and usually the class size is smaller compared to other sections because of less favorable meeting times (three times a week). However the smaller class size allows me incorporate a lot of group discussion style techniques fairly regularly. I can easily keep track of every students' performance and struggles, and could create individualized work for them to catch up with the rest of the class. I have created and refined lecture notes, worksheets, labs, and group projects for this class over the year all of which create an ecosystem where the students learn higher dimensional abstract concepts with relative ease as they get to approach it from numerous viewpoints. Students learned applications of regression techniques in data science, an introduction to the Gradient descent method of optimization techniques used in machine learning, practical modelling of climate change evidences, the Normal probability distribution, and mathematics behind rotary engines and rocket propulsion. Students heavily relied on demonstrations using *Mathematica* and *Desmos*, both to visualize three dimensional pictures of surfaces, vector fields etc. as well as to learn numerical approximation techniques.

1.4. Independent Study

In 2019-2020 academic year, I have and will be mentoring two undergraduate student projects and independent studies. The topics are: (i) Asset revenue modelling using differential equations and machine learning and (ii) non-euclidean geometry and tiling.

1.5. Math 1600, Differential Calculus

- **Meeting Schedule** - Fall 2018 - 4.5 hours per week
- **Class Size** - 32

This course was aimed towards Freshmen and Sophomores from various backgrounds as one of the introductory Mathematics courses offered at Bowdoin College. For a lot of the students, this was the first college Math course and I wanted to make sure they learn the proper way to think about Math from the very beginning. Building on the traditional course structure, one of my main focus in teaching this course was to make sure students are able to interpret and describe symbolic equations using words and conversely be able to transform practical examples and word problems into mathematical models. Over the semester, I created a number of lab sessions which also helped solidify the abstract ideas by doing numerical estimations through Mathematica, and by describing how to implement various algorithms e.g. the Newton-Raphson method.

§2. UNIVERSITY OF CHICAGO

Besides my liberal arts teaching experience, I was also fortunate to have the opportunity of being an instructor during and after my PhD at the University of Chicago.

2.1. Proof-Based Methods

- **Meeting Schedule** - Summer 2018 - 6 per week for six weeks
- **Class Size** - 16

After finishing my PhD in summer 2018, I had the unique experience of teaching an *Introduction to Proof* style class to a group of academically talented incoming first-year students at UChicago through the *Chicago Academic Achievement Program* Summer academy, conducted by the *Center for College Student Success*. This class was designed to expose the students to the academic rigor expected of them as they enroll into introductory Math courses at the college, as well as provide a support framework to help them navigate through the new social and cultural norms.

As a class designed essentially to develop Math reasoning, we covered ideas and problem solving strategies from a broad area of topics such as Number Theory, Combinatorics, Graph Theory, Sequences, and limit Calculus. Besides the final exam, the students also were required to give a presentation in front of their peers which I believe helped them with their Mathematical writing and interaction skills. I tried to keep the atmosphere of the class as casual as possible so that they do not get overloaded with too much expectation. The syllabus for this class is available in my website.

2.2. Math 195, Mathematical Methods for Social Science

- **Meeting Schedule** - Fall 2017, Fall 2018 - 3 hours per week
- **Class Size** - 18 on average

The course consists of topics that are important for students who are planning to become majors in Economics, Political Science, Mathematical Linguistics etc. As such we covered vectors and multivariable calculus up to optimization, but instead of talking about Green's theorem, we covered linear programming next and finally sequences and series with the goal of learning Taylor approximations.

2.3. Independent Study

During Fall 2018, I mentored two interested talented students in my Math 195 course via an independent study of Game Theory and a project on *Least Unique Bid Auction*, where we discussed the usage of Lagrange Multipliers for finding complicated Probability estimates.

2.4. Math 196, Linear Algebra

- **Meeting Schedule** - Summer 2017 - 6 hours per week for 5 weeks
- **Class Size** - 18 on average

This course was offered through the *Graham School of Continuing Liberal and Professional Studies* for computational linear algebra, intended primarily for students in the social sciences who have completed single and multivariable calculus sequence. However, the students weren't expected to have much experience with writing proofs and as such, we spent a lot of time working on examples from many disciplines, in particular ones that relate to their primary fields of interest.

2.5. Math 150's, Standard Calculus Sequence

- **Meeting Schedule** - 3 hours per week
- **Class Size** - 15-30

As a graduate student college instructor, I taught independent section of Calculus courses in 2014-2017. The yearlong rigorous one-variable and multi-variable *standard Calculus* sequence (taught thrice) is designed for science, economics and Math majors. As the instructor of record, I was responsible for designing my own course curriculum, planning lectures, designing and grading exams, holding office hours, and assigning homework. I also mentored teaching assistants, and coordinated junior tutors.

2.6. Math 133, Elementary Functions and Calculus

- **Meeting Schedule** - 3 hours per week
- **Class Size** - 6

I taught a quarter long course on Vector calculus titled 'Elementary Functions and Calculus' to non-science (mostly History, English, and Theater) majors. Teaching students with very little technical background was an unparalleled learning experience.

§3. OTHER ACADEMIC SERVICE

3.1. Math competitions and Problem Solving sessions

I coorganize a weekly Problem Solving Session at Bowdoin College where we work on Math 'puzzles'. The goal is on one hand to teach important mathematical strategies in a fun setting and on the other, train more enthusiastic students for the *Putnam Competition*.

3.2. Undergraduate Mentoring - DRP and REU

At the University of Chicago, I mentored eight undergraduate students (during 2014-2017) through the *Directed Reading Program* (DRP) and the summer *Research Experience for Undergraduates* (REU) on a wide array of topics from geometry, linear algebra, topology, dynamics of group action etc. We usually met twice a week for about 10 weeks, where the students would discuss a paper they have read and any original work they have done, followed by me outlining the next possible direction of approach and available useful literature. In both cases, I also helped them learn mathematical writing and coached them for an end-of-quarter presentation or written paper. *The list of students and their papers are available in my CV.*

3.3. As a Teaching Assistant and Grader

In 2013-2014 academic year, I worked with professor Eugenia Cheng as a teaching assistant for a year-long *Honors Calculus* sequence, and later worked as a grader for graduate courses on Algebraic Topology, Differential Topology, Differential Geometry, and Riemannian Geometry. Details on these are listed in my CV.

Chapter 2

Teaching Effectiveness - Student Evaluations Numerical Summary

§1. BOWDOIN COLLEGE

Numerical averages of all student responses for all the courses I taught at Bowdoin College last three semesters (Fall 2019, Spring 2019 and Fall 2018) are attached below. All of the evaluations in full are available on request.

How much did this course contribute to your education?

Name	Resp	Mean
Overall	34	4.48
Multivariate Calculus	14	4.71
Ordinary Differ Equations	20	4.25

How effectively did the instructor make use of class sessions to advance your learning?

Name	Resp	Mean
Overall	34	4.34
Multivariate Calculus	14	4.79
Ordinary Differ Equations	20	3.90

How helpful to your learning was the feedback you received in the course?

Name	Resp	Mean
Overall	34	4.06
Multivariate Calculus	14	4.07
Ordinary Differ Equations	20	4.05

Did you communicate with the instructor outside of class?

Name	% Yes	% No
Overall	88.24%	11.76%
Multivariate Calculus	78.57%	21.43%
Ordinary Differ Equations	95.00%	5.00%

How helpful to your learning were these communications with the instructor outside of class?

Name	Resp	Mean
Overall	30	4.74
Multivariate Calculus	11	5.00
Ordinary Differ Equations	19	4.47

How accessible was the instructor outside of class?

Name	Resp	Mean
Overall	34	4.95
Multivariate Calculus	14	5.00
Ordinary Differ Equations	20	4.90

What is your overall rating of the instructor as a teacher?

Name	Resp	Mean
Overall	34	4.33
Multivariate Calculus	14	4.71
Ordinary Differ Equations	20	3.95

How much did this course contribute to your education?

Name	Resp	Mean
Overall	21	4.33
Linear Algebra	11	4.36
Multivariate Calculus	10	4.30

How effectively did the instructor make use of class sessions to advance your learning?

Name	Resp	Mean
Overall	21	4.20
Linear Algebra	11	4.00
Multivariate Calculus	10	4.40

How helpful to your learning was the feedback you received in the course?

Name	Resp	Mean
Overall	21	4.38
Linear Algebra	11	4.45
Multivariate Calculus	10	4.30

Did you communicate with the instructor outside of class?

Name	% Yes	% No
Overall	85.71%	14.29%
Linear Algebra	90.91%	9.09%
Multivariate Calculus	80.00%	20.00%

How helpful to your learning were these communications with the instructor outside of class?

Name	Resp	Mean
Overall	18	4.84
Linear Algebra	10	4.80
Multivariate Calculus	8	4.88

How accessible was the instructor outside of class?

Name	Resp	Mean
Overall	21	4.71
Linear Algebra	11	4.82
Multivariate Calculus	10	4.60

What is your overall rating of the instructor as a teacher?

Name	Resp	Mean
Overall	20	4.05
Linear Algebra	10	4.20
Multivariate Calculus	10	3.90

How much did this course contribute to your education?

Name	Resp	Mean
Overall	38	3.77
Differential Calculus	30	3.53
Multivariate Calculus	8	4.00

How effectively did the instructor make use of class sessions to advance your learning?

Name	Resp	Mean
Overall	38	3.67
Differential Calculus	30	3.33
Multivariate Calculus	8	4.00

How helpful to your learning was the feedback you received in the course?

Name	Resp	Mean
Overall	37	3.64
Differential Calculus	29	3.41
Multivariate Calculus	8	3.88

Did you communicate with the instructor outside of class?

Name	% Yes	% No
Overall	81.58 %	18.42 %
Differential Calculus	83.33 %	16.67 %
Multivariate Calculus	75.00 %	25.00 %

How helpful to your learning were these communications with the instructor outside of class?

Name	Resp	Mean
Overall	31	4.29
Differential Calculus	25	4.08
Multivariate Calculus	6	4.50

How accessible was the instructor outside of class?

Name	Resp	Mean
Overall	38	4.41
Differential Calculus	30	4.20
Multivariate Calculus	8	4.63

What is your overall rating of the instructor as a teacher?

Name	Resp	Mean
Overall	38	3.60
Differential Calculus	30	3.33
Multivariate Calculus	8	3.88

§2. UNIVERSITY OF CHICAGO

A numerical average of all student responses for all the courses I taught at UChicago is attached below. All of the evaluations in full are available on request.

Chowdhury, Subhadip

Course Evaluations

- #1: Instructor was organized
- #2: Lectures were clear and understandable
- #3: Lectures were interesting
- #4: instructor exhibited a positive attitude towards the students
- #5: Instructor was accessible outside of class
- #6: I would recommend this instructor to others

Key of Values

- 5 - strongly agree
- 4 - agree
- 3 - neutral
- 2 - disagree
- 1 - strongly disagree

Quarter	Course Number	Section	Instructor (Last)	Instructor (First)	#Eval	#Stud	#1	#2	#3	#4	#5	#6	Overall
Winter 2018	19520	59	Chowdhury	Subhadip	9	10	4.38	4.25	4.00	4.75	4.43	4.38	4.36
Autumn 2017	19520	41	Chowdhury	Subhadip	20	27	3.90	4.10	3.63	4.50	4.56	4.05	4.12
Winter 2017	15300	45	Chowdhury	Subhadip	18	20	3.61	3.56	3.44	3.67	4.28	3.28	3.64
Autumn 2016	15200	45	Chowdhury	Subhadip	28	28	3.79	3.75	3.57	4.00	4.56	3.57	3.87
Spring 2016	13300	22	Chowdhury	Subhadip	6	6	4.17	4.33	3.67	4.50	4.67	4.17	4.25
Winter 2016	15300	48	Chowdhury	Subhaddip	7	8	4.43	4.43	4.43	5.00	5.00	4.71	4.67
Autumn 2015	15200	45	Chowdhury	Subhadip	14	16	3.50	3.64	4.21	4.36	4.50	4.00	4.04
Spring 2015	15300	41	Chowdhury	Subhadip	7	7	4.43	4.29	4.29	5.00	4.86	4.57	4.57
Winter 2015	15200	41	Chowdhury	Subhadip	23	27	3.70	3.61	3.35	4.00	4.50	3.43	3.76
Autumn 2014	15100	41	Chowdhury	Subhadip	22	23	3.32	3.32	3.32	4.18	4.67	3.55	3.72

Chapter 3

Teaching Improvement Activities

§1. TEACHING TRIANGLE

Over the academic years 2018-2020, I have been part of the *Teaching Triangle* program at Bowdoin College which involves professors from different departments visiting each others' classes in an effort to gain new insights into teaching and students' learning. Although the process was not evaluative, it facilitated a reflective conversation with my colleagues regarding evidence of student learning and new techniques of student engagement.

§2. PEDAGOGY CONFERENCE

I have attended several pedagogy conferences and workshops organized by *Chicago Center for Teaching* and *Center for Learning and Teaching* at Bowdoin and Bates College. They taught me about the ever changing role of Math professors in the face of modern technological advances and how the teaching process has evolved to be relevant with the modern day and age. I also learned about new assessment techniques and ways to encourage proactive student involvement.

§3. TEACHING SEMINAR

The Math department at Bowdoin college organizes weekly teaching seminars where we share our methods, coordinate reciprocal classroom visits, and get advice on handling unexpected issues in classroom from more experienced faculty members. Last year I also participated and once led a discussion about the *MAA Instructional Practices Guide*. These meetings have been invaluable for my professional development.

Chapter 4

Future Teaching Goals

Building upon my current experience, I have many specific ideas that I plan to incorporate into my role as a future instructor.

- I intend to develop and introduce augmented reality or role-playing-games into the classroom, similar to RTTP, to innovate and transform STEM instruction with new forms of representation. In a parallel direction I advocate new forms of assignments (e.g. videos or experimental) consistent with modern advancement in technology and welcome any opportunity of employing digital and computational tools to enrich my teaching.
- I would like to teach using discovery learning method and experiment with bolder pedagogical ideas.
- My plans also include summer programs for incoming freshmen for a successful transition to college life, outreach programs such as Math Circles for K-12 students, and professional networking between different communities etc.
- I plan to apply for the Project NExT fellowship and hope to gain guidance from professors in the MAA Mentoring Network to help me navigate through potential difficulties that may arise, develop professionally and learn how to maintain a work-life balance while actively engaging in research and scholarship.

Finally, by bringing in a wide variety of perspectives, I hope to impact and get support from my peers in designing approaches towards broader, more widely applicable, and more memorable learning.

Chapter 5

Commitment to Diversity, Equity, and Inclusion

A diverse student population have different needs, face distinct challenges, and confront different fears in their journey to academic success. In my statement, I describe my motivation and efforts in mitigating these challenges, as well as my future plans dedicated to improving recruitment and retention of diverse students in all stages of the academic pipeline.

§1. MY BACKGROUND IN INDIA

I come from an average middle-class family from rural India. Of my closest relatives, very few went to college, no one has a degree in STEM field, and no one has a doctorate degree. Growing up, there was a distinct lack of higher academic resources; I remember taking public transportation to the nearest university two hours away to get photocopies of library books. I was lucky to have parents who tried their best to provide me with the most decent schooling they could afford and I attribute my academic success to their encouragement and support; as well as to the guidance of the mentors I found along the way through various math competitions and summer camps. My personal experience plays a crucial role in my familiarity with the problems faced by low-income communities and students with disadvantaged backgrounds, and has helped me build a toolkit for mentoring others who face similar challenges.

§2. SUPPORTING DIVERSITY IN CLASSROOM

Both as a visiting assistant professor at Bowdoin College and as a graduate student at University of Chicago, I was fortunate to find a supportive environment that encouraged diversity among the student demographic. Having found myself in front of a large student body with distinct backgrounds and expectations, it was important for me to set an inclusive tone from the first day of class. By providing introduction index cards to the

students and by building rapport among them through collaborative projects, I have tried to build small communities that appreciate each others strengths, are receptive of new perspectives, and are confident in contributing to discussions. Throughout the semester, I made sure to invite students to move around and work with different partners to avoid stagnant social dynamics, and encouraged them to take advantage of my office hours so that I could engage with every one of them on a personal basis.

Every person learns at a different pace, so it is also important to me that students are allowed academic freedom to pursue the study material more equitably. This has meant creating different ways of delivering content -- visually via slide presentations, graphically via math software and web applets, and practically via online video examples that highlight application of the abstract ideas. It has also included spending extra times with students with disabilities, creating class notes and worksheets that are more heterogeneous in nature, that relate to the personal experiences of students, and using course management systems such as Blackboard to organize and give concrete structure to courses. Finally, it has meant being mindful of my language and rhetoric so that it portrays my spirit of open-mindedness and goodwill.

§3. EFFORTS IN OUTREACH AND MENTORSHIP

My first experience with a diverse student body was as a member of the cultural committee and as an editor of a monthly art magazine at Indian Statistical Institute, my undergraduate institution. Catering to the needs of people from all the different parts of India, with different languages, cultures, and customs, was a demanding but heavily rewarding experience. At the same time, I taught in high school seminars and trained students in the *Indian National Mathematics Olympiad* training camps. Through these, I felt satisfied to have contributed productively to the younger generation of mathematicians, opening fresh minds to new world full of exciting ideas.

Later as a grad student and as a professor, I have continued to regularly give invited talks in student seminars and currently co-organize a *Problem Solving Seminar* at Bowdoin college, where we explore nonstandard mathematical strategies in a more 'fun' setting. I also had a unique experience of supervising the Career Exploration Program of a high school student. I believe my efforts have helped demystify what math education entails and helped encourage a portion of student body to pursue higher level math courses.

I am fortunate to possess the unique experience of teaching a group of academically talented incoming first-year students at UChicago through the *Chicago Academic Achievement Program* (CAAP), many of whom were first-generation college students or from low-income communities. Apart from the regular coursework, I helped them integrate more actively with the larger college culture, and explore ways of utilizing various campus resources. I also helped them learn mathematical writing and coached them in improving their presentation skills in front of their peers.

I had been an active member of the *Association for Women in Mathematics* for four years and co-organized colloquia, study sessions, and social events specifically geared towards

participation from women and minorities. Over the last year, I have been part of a weekly *Study Group for students of color* in Math, CS and Physics that welcomes underrepresented students in an informal setting and helps them build a support network in college.

§4. SELF IMPROVEMENT

Seeking opportunities to learn from others with different viewpoints, I have attended weekly teaching seminars at my current department, several pedagogical seminars (at *Bowdoin College*) and workshops (organized by *Chicago Center for Teaching*), and have participated in the *Teaching Triangle* programs that involved reciprocal classroom visits with faculty from other departments. In a complimentary direction, I conduct *anonymous student surveys* every three-to-five weeks and try to continually improve myself based on their suggestion.

§5. PLANS FOR THE FUTURE

Under-representation of groups such as women, ethnic minorities, LGBTQ communities, and people with disabilities in Mathematical Sciences is one of the most challenging and critical problems facing us as a field. Building upon my current experience, I have many specific ideas that I wish to incorporate to help fight this issue. For example, these include creating summer programs for incoming freshmen for successful transition to the college life, outreach programs such as Math Circles for K-12 students, professional networking support for student communities with disadvantaged backgrounds etc. I hope to get opportunities where I can actively support the diverse environment of STEM fields as I continue and expand my inclusion efforts, both in my personal and professional life.

Appendices

Appendix A






Sample Course Syllabi







Ordinary Differential Equations

Math 2208




Instructor Info —

-  Subhadip Chowdhury
-  Office Hrs: See Blackboard
-  Searles 104
-  [subhadipchowdhury.github.io](https://github.com/subhadipchowdhury)
-  schowdhu@bowdoin.edu

Course Info —

-  Prereq: Math 2000
-  Mon & Wed
-  1:15p-2:40p
-  Searles 215

Lab Info —

-  Wed
-  1:15p-2:40p
-  Searles 216

Course Goals

Math 2208 Specific Goals: Learn how to use differential equations (DEs) to model real world phenomena. There are three main categories of tools we focus on to analyze such DE models.

1. Know how to solve a variety of DEs (analytical techniques) “by hand”.
2. Know how to analyze and say something about DEs without explicitly solving them (qualitative techniques).
3. Know how to approximate solutions using algorithms implemented on a computer (numerical techniques).

While we will learn several analytical techniques, understanding their limitations should be one of main takeaways of the course. This is why, when compared to more traditional courses on the subject, more emphasis is placed on qualitative and numerical techniques and the use of computer software.

Putting it all together: Given a real world phenomena, learn how to you derive a model, pick the right tool to analyze it (not all tools work on every model), and then interpret that result in the context of the real world phenomena.

Life Long Skills: Your abilities of general problem solving skills, self learning, self evaluation and how to formulate ideas and solutions will be refined throughout the course. This means problems appearing on homework or exam will not “be just like problems” you have seen before. You might be asked to explore new topics in homework before I cover them in class. Finally, how you present your solutions will also be evaluated.

Required Course Materials

Textbook

Differential Equations, 4th edition, by Blanchard, Devaney and Hall

Software

GNU Octave, for your own computer. All computers in Searles 216 have Octave installed.

Others

DFIELD, and PPLANE, some Java applets, will be available from Blackboard.

Grading Scheme

Weekly assignments	20%
Projects	15%
Quizzes	20%
Midterm 1	10%
Midterm 2	15%
Final exam	20%

Scores will NOT be curved. However, the cutoff percentage for letter grades will be set at my discretion. The weights are tentative and subject to change on an individual basis.

Important Dates

Quiz # 1	Wednesday, Feb 5
Quiz # 2	Wednesday, Feb 26
Quiz # 3	Wednesday, Apr 8
Quiz # 4	Wednesday, Apr 22
Midterm # 1	due Monday, Mar 2, 1:15PM
Midterm # 2	due Monday, Apr 13, 1:15PM
Final Exam	due Tuesday, May 12, 4:30PM

Both Midterms and the Final exams will be take-home. The quizzes will serve as in-class tests.

FAQs

? Where can I find Octave?

! You can download a free copy for your own use from gnu.org/software/octave.

? I can't run DFIELD or PPLANE.

! You may need [Java Runtime environment](#) to run these on your computer.

? What is the late submission and make-up exam policy?

! In general, late submission (even 15 mins late) of homework assignments will NOT be accepted. You may turn in *up to two* homeworks late, with no questions asked, so long as you notify me before the time the homework is due. Please see me in case of other extenuating circumstances.

You can make up an exam if certain unavoidable reasons prevent you from taking it and if you inform me in advance. Contact me as soon as possible if you are going to miss an exam. Missed exams can only be made up at my discretion, and are subject to a lost fraction of the grade.

? Do I need to attend every class?

! Although attendance is not directly part of your grade, it is very easy in a math class to fall behind after skipping even one class. You cannot be an effective and involved member of the class unless you are present! If you miss multiple classes in a row, you may expect a comment card.

Diversity and Inclusion Statement

I consider this classroom to be a place where you will be treated with respect, and I welcome individuals of all ages, backgrounds, beliefs, ethnicities, genders, gender identities, gender expressions, national origins, religious affiliations, sexual orientations, ability - and other visible and non-visible differences. All members of this class are expected to contribute to a respectful, welcoming and inclusive environment for every other member of the class.

No student is required to take an examination or fulfill other scheduled course requirements on recognized **religious holidays**. Please declare your intention to observe these holidays at the beginning of the semester.

Student Accommodations

- If you are a student with with learning needs that require special accommodation, please see Lesley Levy in the [Office of Student Accessibility](#) as soon as possible to make an appointment to discuss your special needs and to obtain an accommodations letter. Please email me as soon as possible in order to set up a time to discuss your learning preferences, challenges you may face learning this semester, and how we can create an effective learning experience for you. *In particular, I understand that the quizzes at the beginning of class can present a challenge, and I'm eager to discuss options with you.*
- As a student, you may experience a range of issues that can cause barriers to learning, such as strained relationships, increased anxiety, alcohol/drug problems, feeling down, difficulty concentrating and/or lack of motivation. These **mental health concerns** or stressful events may lead to diminished academic performance or reduced ability to participate in daily activities. Bowdoin College is committed to advancing the mental health and well-being of its students. If you or someone you know is feeling overwhelmed, depressed, and/or in need of support, services are available. You can learn more about the broad range of confidential mental health services available on campus at: <https://www.bowdoin.edu/counseling/>

Title IX

As a faculty member I am considered a **Responsible Employee**, per the [Student Sexual Misconduct and Gender Based Violence Policy](#). While my goal is for you to be able to share information related to your life experiences through discussion and written work, I want to be make sure you understand that as a Responsible Employee I am required to report disclosures of sexual misconduct, dating violence, stalking, and/or sexual and gender-based harassment to the University's Title IX Coordinator, Benje Douglas. My reporting to Benje does NOT mean that any actions will be taken beyond him reaching out to you and trying to schedule a time to talk to see what assistance you might need to be successful as a student here at Bowdoin.

Academic Integrity

I support and adhere to the principles of [The Bowdoin College Academic Honor Code](#). Your work should never be directly copied from another student and I will expect that *you are not reading solution manuals* for this textbook. In particular, I will assume all members of the class are trustworthy in their dealings with me as well as their fellow classmates. However, should a violation of this trust be discovered, it will be reported to the Judiciary Board. The goal is not vengeance against those who violate the Code but fairness for those who adhere to it. If you have any questions about the appropriateness of a particular situation, please communicate with me.

Components of the Course

- You will need to **read the textbook**. Several homework and suggested review problems will come directly from the book, and possibly quiz or exam problems. Some in class examples will be similar or identical to the book, but many will be different. The overall topic choice and course philosophy will be from the book.
- **Weekly assignments** will contain questions based on the textbook readings and class work. These assignments with their due dates will be regularly posted on Blackboard.
- You are encouraged to work on the weekly assignments with others, but you must write your final solution in your own words and you must complete and attach an **Assignment Cover Sheet** with every submission. This sheet can be downloaded from Blackboard.
- As is typical for ODE courses in the Mathematics Department, homework will generally be corrected by student graders who work under my supervision; this is done to ensure that you regularly receive graded assignments in a timely manner. *Please inform me immediately if you find any mistake in graded homeworks.*
- There will be three longer **projects** built around more challenging questions from the exercises, to showcase interesting applications of the study materials. These will require you to do programming in Octave or use the Java applets.
- You are allowed to work in groups of size **at most 2 (two)** to work on the projects. In your report you should include pictures and graphs of data and of solutions of your models *as appropriate*. Remember that one carefully chosen picture can be worth a thousand words, but a thousand pictures aren't worth anything. Final submissions must include a **Project Report Cover Sheet** (downloadable from Blackboard) on which the signatures of all participants must appear along with *brief but substantive* discussions of any issues confronted at your meetings. If any group member did not participate in an important aspect of the assignment, this must be stated in the Report. *One submission for your entire group will suffice.*
- Homeworks are extremely important, as it is the best way for you to engage with the material on a regular basis. I expect that in case you need extra practice with a certain concept, you will seek *extra, unassigned problems from the textbook to work out*; I am always happy to discuss how to locate good practice problems in your book.
- The point of the homeworks is for you to work out what you do and don't understand. When your graded homework has been handed back to you, you should go through it and see if you understand what has been written on it by the grader. If you don't, you should come to office hours and ask.
- As you are solving problems in this course, remember that getting the "answer" is only one of the steps. Don't think of what you write as just showing your instructor that you have done the homework. Write as if you were explaining what you are doing to one of your classmates who missed that day of class. Think of writing as part of the process of learning. The more carefully and clearly you write your mathematics, the more likely it is to be correct, and the more likely you will be to remember it. *Correct answers without explanation will not reap full credit, but clear explanations with an incorrect answer can certainly earn partial credit.*
- We will semi-regularly go into the **computer lab** during the lecture period for either demonstrations or for you to do your own programming. Homework problems throughout the semester will require you to use Octave/Matlab or other applets.
- **Student participation and collaboration** is an integral part of this class and is highly valued. Everyone is expected to make thoughtful contributions in the form of questions (even if unprompted), statements, and reasoned arguments. You might be also occasionally invited to present something on the board. Whenever possible, there will be opportunities for you to work through practice problems in small groups during our class meetings. Paper copy of **worksheets** will be provided and an electronic copy will be available on Blackboard.
- Additionally, there will be four **quizzes** and **two midterms** given during the semester, as well as a **final examination** at the end of the semester. All tests will emphasize the concepts of the course.

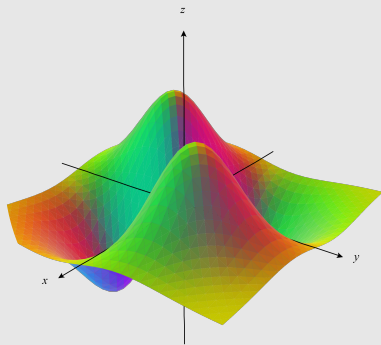
General Policies

- Be courteous when using mobile devices. Make sure your cell phone is turned fully off, or silent. If you must make or receive a call, please go outside the classroom.
- Use of laptops or tablets is permitted for note-taking and labs. Please turn off your Wi-fi and sound.
- For any private communication regarding this course, please email me from your **bowdoin.edu email address**. This is mainly for identity verification purposes.

Class Schedule

The following is a preliminary outline of the topics that we hope to cover. This is an idealized plan, and it *may be adjusted as the semester progresses*. But it should give some indication of the major topics to be covered in this class.






Week No.	Monday	Wednesday
1		22-Jan Syllabus Overview + 1.1 (Modelling via Differential Equations)
2	27-Jan 1.2-1.3 (Separation of Variable, Slope Field) + Using DFIELD	29-Jan Intro to Octave - Basic Plotting + Euler's Method
3	3-Feb 1.5 (Existence and Uniqueness Theorem)	5-Feb Quiz 1 + ODE45
4	10-Feb 1.6 (Equilibria and Phase Line)	12-Feb 1.9 (Integrating Factor)
5	17-Feb 1.7 (Bifurcation)	19-Feb Project 1 (The Spruce Budworm - Hysteresis and Cusp Catastrophe)
6	24-Feb Bifurcation contd., Change of Variable techniques	26-Feb Quiz 2 + Review
	Midterm 1	
7	2-Mar 2.1 (Predator-Prey Model) + 3.1 (Linear System)	4-Mar 3.1 (Linear Systems contd.) + 3.2 (Straight line solutions) + using PPLANE
	Spring Break + Bonus Project (Recipe for an All-nighter - Flow-kick systems)	
8	23-Mar 3.3 (Phase Portraits)	25-Mar 3.4 (Complex Eigenvalues)
9	30-Mar 3.5 (Equal and Zero Eigenvalues)	1-Apr Trace-Determinant Plane, Defective and Degenerate cases, Bifurcation
10	6-Apr Second Order Linear ODEs, Simple Harmonic Oscillators	8-Apr Quiz 3 + Review
	Midterm 2	
11	13-Apr Project 2 (A double spring-mass system - Linear Systems in Higher Dimensions)	15-Apr Forced Harmonic Oscillation, Method of Undetermined Coefficients
12	20-Apr Resonance and Practical Resonance	22-Apr 5.1-5.2 (Equilibrium Point Analysis, Jacobian) + Quiz 4
13	27-Apr Almost Linear Systems, Consequences of Poincaré-Bendixson theorem	29-Apr Project 3 (Glycolytic Oscillations - Hopf Bifurcation)
14	4-May Lorenz Equations	6-May Quiz 5 + Review






Multivariable Calculus

Math 1800-B




Instructor Info —

-  Subhadip Chowdhury
-  Office Hrs: TBD
-  Searles 104
-  [subhadipchowdhury.github.io](https://github.com/subhadipchowdhury)
-  schowdhu@bowdoin.edu

Course Info —

-  Mon, Wed, & Fri
-  10:40a - 11:35a
-  Searles TBA

Lab Info —

-  Thurs
-  2:50p-4:15p
-  Searles TBA

Course Description

The emphasis of the course will be on developing an understanding of the calculus of functions of two and three variables, as well as the geometry of associated curves and surfaces in two and three dimensions. Multivariable calculus is a fundamental pillar for many other things:

- It *extends single variable calculus to higher dimensions*. You will see that the structures are much richer than in single variable and that the fundamental theorem of calculus generalizes to higher dimensions.
- It *provides vocabulary* for understanding fundamental processes and phenomena. Examples are planetary motion, economics, waves, heat, finance, epidemiology, quantum mechanics or optimization.
- It *teaches important background* needed in social sciences, life sciences and economics. But it is rigorous enough that it is also suited for students in core sciences like physics, mathematics or computer science.
- It *builds tools for describing geometrical objects* like curves, surfaces, solids and intuition which is needed in other fields like linear algebra or data analysis. Geometry is currently an extremely popular topic: tomography methods in medicine, computer games, Google earth, social network analysis all use geometry.
- It *relates to culture and history*. The quest for answering questions like "where do we come from", "what will future bring us", "how can we optimize our time in between" all use calculus. The history of calculus contains fascinating stories, starting from Archimedes, 2300 years ago up to the modern times, where new branches of multivariable calculus are developed to understand the structure of nature.
- It *develops problem solving methods*. Examples are optimization problems with and without constraints (which is the bread and butter for economics), geometric problems, computations with scalar and vector fields, area and volume computations.
- It *makes you acquainted with a powerful computer algebra system* which allows you to see the mathematics from a different perspective. Such systems are more and more needed for visualization, experimentation and to build laboratories for your own research.
- It *prepares you for further study in other fields*. Not only in mathematics and its applications, but also in seemingly unrelated fields like game theory, probability theory, discrete mathematics, sociology, or number theory, where similar structures and problems appear, even in a discrete setting. Without geometric intuition and paradigms learned in calculus, it is rather hard to work in those fields.
- It *improves thinking skills*, problem solving skills, visualization skills as well as computing skills. You will see the power of logical thinking and deduction and why mathematics is timeless.

Required Course Materials

Textbook

Calculus: Multivariable, 7th edition, by Hughes-Hallet, Gleason, McCallum et al.

Software

Mathematica, for your own computer.

Prerequisites

In order to be considered for admission into Math 1800 you must either have

1. completed Bowdoin's Math 1700 or Math 1750, or
2. been given a mathematics placement of Math 1800 when you entered Bowdoin.

If you do not satisfy at least one of these two conditions you will need the permission of the Chair of the Mathematics Department in order to register for Math 1800. No prior experience with Mathematica is required but a familiarity with mathematical computing softwares is encouraged.

FAQs

? Where can I find Mathematica?

! Bowdoin has a license allowing students to download the program onto their personal computers for free! To learn how to download Mathematica from the Bowdoin network, follow the steps at [this link](#).

? What is the late submission and make-up exam policy?

! In general, late submission (even 15 mins late) of homework assignments will NOT be accepted. You may turn in *up to two* homeworks late, with no questions asked, so long as you notify me before the time the homework is due. Please see me in case of other extenuating circumstances. You can make up an exam if certain unavoidable reasons prevent you from taking it and if you inform me in advance. Contact me as soon as possible if you are going to miss an exam. Missed exams can only be made up at my discretion, and are subject to a lost fraction of the grade.

? Do I need to attend every class?

! Although attendance is not directly part of your grade, it is very easy in a math class to fall behind after skipping even one class. You cannot be an effective and involved member of the class unless you are present! If you miss multiple classes in a row, you may expect a comment card.

Grading Scheme

Weekly assignments + Labs	20%
Projects	10%
Quizzes	15%
Midterms	15% each
Final exam	25%

Scores will NOT be curved. However, the cutoff percentage for letter grades will be set at my discretion. The weights are tentative and subject to change on an individual basis.

Important Dates

Quiz # 1	Thursday, Feb 6
Midterm # 1	Thursday, Feb 20
Quiz # 2	Wednesday, Mar 25
Midterm # 2	Thursday, Apr 16
Final Exam	Saturday, May 16, 8:30-11:30am

Please let me know immediately of any problems with these dates. Please note that the date of the final exam is set by the Registrar's office and cannot be altered. Individual changes in final exam dates are allowed only for particularly serious situations such as three exams in a two-day period.

Student Accommodations

- If you are a student with learning needs that require special accommodation, please see Lesley Levy in the [Office of Student Accessibility](#) as soon as possible to make an appointment to discuss your special needs and to obtain an accommodations letter. Please email me as soon as possible in order to set up a time to discuss your learning preferences, challenges you may face learning this semester, and how we can create an effective learning experience for you. *In particular, I understand that the quizzes at the beginning of class can present a challenge, and I'm eager to discuss options with you.*
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Diversity and Inclusion Statement

I consider this classroom to be a place where you will be treated with respect, and I welcome individuals of all ages, backgrounds, beliefs, ethnicities, genders, gender identities, gender expressions, national origins, religious affiliations, sexual orientations, ability - and other visible and non-visible differences. All members of this class are expected to contribute to a respectful, welcoming and inclusive environment for every other member of the class.

No student is required to take an examination or fulfill other scheduled course requirements on recognized [religious holidays](#). Please declare your intention to observe these holidays at the beginning of the semester.

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Components of the Course

- You will need to [read the textbook](#). Several homework and suggested review problems will come directly from the book, and possibly quiz or exam problems. Some in class examples will be similar or identical to the book, but many will be different. The overall topic choice and course philosophy will be from the book.
- [Individual assignments](#) will contain questions based on the textbook readings and class work. These assignments with their due dates will be regularly posted on Blackboard. The typical due date pattern is:
 - Monday's homework is due Friday same week,
 - Wednesday and Friday's homeworks are due Wednesday next week.
- You are encouraged to work on the weekly assignments with others, but you must write your final solution in your own words and you must complete and attach an [Assignment Cover Sheet](#) with every submission.
- As is typical for multivariable calculus courses in the Mathematics Department, homework will generally be corrected by student graders who work under my supervision; this is done to ensure that you regularly receive graded assignments in a timely manner. *Please inform me immediately if you find any mistake in graded homeworks.*
- Around six longer [collaborative projects](#) will be built around more challenging questions. Electronic copies of the assignment details will be available on Blackboard. These will be due typically within seven to ten days. The teams for the projects ([of size 3-4](#)) will be decided in second week and will change several times over the semester.
- All members of the group must not only participate in the analysis of the project but should discuss the specific phrasing and organization of the final submission. Final submissions must include a [Project Report Cover Sheet](#) that contains the signatures of all participants along with *brief but substantive* discussions of any issue confronted at your meetings. If any group member did not participate in an important aspect of the assignment, this must be stated in the Report. *A single submission for your entire group will suffice.*
- In the [computer lab](#) sessions you will work on Mathematica projects designed to deepen your understanding of the primary course concepts. Depending on your familiarity with Mathematica, you may find that you complete labs during the lab period, or you may find that you need some more time to complete them as homework. Either way is fine.
- [Student participation and collaboration](#) is an integral part of this class and is highly valued. Everyone will be expected to make thoughtful contributions in the form of questions (even if unprompted), statements, and reasoned arguments. You might be also occasionally invited to present something on the board. Whenever possible, there will be opportunities for you to work through practice problems in small groups during our class meetings. Paper copy of [worksheets](#) will be provided and an electronic copy will be available on Blackboard.
- Additionally, there will be [two quizzes](#) and [two Midterms](#) given during the semester as well as a [Final Examination](#) at the end of the semester. The quizzes are not cumulative but the midterms are semi-cumulative. The final will be all-inclusive. I will explain the syllabus for reach exam in more details in class.

Electronics Policies

- Be courteous when using mobile devices. Make sure your cell phone is turned fully off, or silent. If you must make or receive a call, please go outside the classroom.
- Use of laptops or tablets is permitted for note-taking and labs. Please turn off your Wi-fi and sound.

Class Schedule

The following is a preliminary outline of the topics that we hope to cover. This is an idealized plan, and it *may be adjusted as the semester progresses*. But it should give some indication of the major topics to be covered in this class.

Monday	Wednesday	Thursday	Friday
	22-Jan	23-Jan	24-Jan
	Syllabus Overview + 3D Coordinate Geometry + 13.1 (Vectors in 3D)	Lab 0 (Intro to Mathematica + Vectors)	13.3 (Dot Product, Angle, Projection)
27-Jan	29-Jan	30-Jan	31-Jan
13.4 (Cross Product, Area, Volume)	Lines and Planes	Lab 1 (Lines and Planes) + Distances	Practice Problems
3-Feb	5-Feb	6-Feb	7-Feb
12.1-12.2 (Functions of several variables)	12.3, 12.5 (Contour Plots) + Conic Sections and Quadric Surfaces	Quiz 1	Lab 2 (3D Graphing and Contour Plots, Linear Functions)
10-Feb	12-Feb	13-Feb	14-Feb
17.1 (Parametrized Curves - Straight line, Circle, Helixes)	17.2 (Motion, Velocity, Speed and Distance, Cycloid)	Lab 3 (Parametric Plotting)	Epicycloid and the Rotary Engine
17-Feb	19-Feb	20-Feb	21-Feb
Practice Problems	Review	Midterm 1	Polar Coordinate System and Some Interesting Parametric Curves
24-Feb	26-Feb	27-Feb	28-Feb
14.1-14.2 (Partial Derivatives)	14.3 (Tangent Plane and Local Linearity)	Practice Problems	14.4 (Gradients and Directional Derivatives)
2-Mar	4-Mar	5-Mar	6-Mar
14.5 (Three dimensional Gradient and Tangent Plane)	Practice Problems	Lab 4 (Gradient Vector, Tangent Plane, and Directional Derivative)	14.6 (Chain Rule)
Mar 7 - Mar 22			
Spring Break			
23-Mar	25-Mar	26-Mar	27-Mar
Practice Problems	Quiz 2	15.1 (Local Optimization)	Taylor Approximation and Second Derivative Test
30-Mar	1-Apr	2-Apr	3-Apr
15.3 (Constrained Optimization) + Rocket Science	15.2 (Global Optimization)	Lab 5 (Linear Regression - Ordinary Least Square vs Gradient Descent)	16.1-16.2 (Definite Integral of Functions of Two Variables)
6-Apr	8-Apr	9-Apr	10-Apr
16.2-16.3 (Type I/II regions, Triple Integrals)	16.4 (Double Integral in Polar Coordinates) + Normal Probability Distribution	Lab 6 (Volume Integration)	Polar Volume Integration (Cylindrical Coordinates)
13-Apr	15-Apr	16-Apr	17-Apr
Practice Problems	Review	Midterm 2	17.3 (Vector Fields)
20-Apr	22-Apr	23-Apr	24-Apr
17.4 (Flow of a Vector Field)	18.1-18.2 (Line Integrals on Parameterized Curves)	Practice Problems	18.3 (Gradient Fields - Path-Independent)
1-Apr	29-Apr	30-Apr	1-May
18.4 (Path-dependent fields, Circulation, Curl)	18.4 (Path-Dependent Fields and Green's Theorem)	Practice Problems	Applications and Generalizations of Green's Theorem
4-May	6-May		
Practice Problems	Review	Reading Period	Reading Period

LINEAR ALGEBRA

MATHEMATICS 2000-B

Spring 2019

Instructor:	Subhadip Chowdhury	Email:	schowdhu@bowdoin.edu
Office Location:	Searles 103	Office Phone:	(207) 725-3572
Class Sessions:	MWF 11:45–12:40	Classroom:	Searles 213

Course Webpage

All regular announcements, instructor office hours, daily homeworks, group projects, handouts, and individual grades will be posted on Blackboard

<http://blackboard.bowdoin.edu>

Check this site on a regular basis to track your progress. General course policies, syllabus, tentative schedule and outline of the course will be also available as pdf files on Blackboard.

Office Hours

- TBA. These time slots are common for all the courses I am teaching this semester.
- If you can't make it to any of the weekly office hours, you can email me to schedule appointments with me. These will depend on my availability.
- I am usually in the office every weekday about 10-6PM. *If my door is open*, you are welcome to knock on my door and come in with quick questions.
- Any and all questions are welcome in class or in my office, but be aware that I will not simply "give you the answer" to any problem. Big-picture questions beyond "How do I solve this problem?" are highly encouraged.
- I also welcome questions through email. Though I strive to answer all email questions as clearly as possible, please realize that certain questions are best answered in a face-to-face discussion.

Prerequisites

In order to be considered for admission into Math 2000 you must either have

1. completed Bowdoin's Math 1800, or
2. been given a mathematics placement of Math 2000 when you entered Bowdoin.

If you do not satisfy at least one of these two conditions you will need the permission of the Chair of the Mathematics Department in order to register for Math 2000.

Textbooks and Supplies

- *Linear Algebra and Its Applications*, 5th edition, by David C. Lay.
- *A scientific calculator*

The use of calculators is NOT permitted for all in-class exams. But you may certainly use them when completing homework assignments, and occasionally this may be required.

Course Description

At the heart of Linear Algebra is an interplay between linear equations, matrix algebra, and geometry. It has a broad range of applications within the sciences and other areas of mathematics, from geometry and computer graphics to differential equations and the dynamics of populations.

Linear Algebra begins with the study of systems of linear equations and their solutions. These systems can be viewed geometrically as vector equations, and alternatively as matrix equations. A major goal of the course will be to understand long-term behavior of linear systems over time, through the techniques of eigenvectors and eigenvalues of matrices. We will also learn how inner products allow us to find the closest approximate solution to equations for which no exact solution exists.

The MCSR Distribution Requirement

Math 2000 can be used to satisfy Bowdoin's Mathematical, Computational, or Statistical Reasoning (MCSR) distribution requirement, through the development of the mathematical tools of linear equations, vectors, and matrices, and also through the application of such techniques to linear models in science and engineering.

The Components of the Course

- You will need to **read the textbook**. In particular, the designated sections of the text should be read prior to the class sessions for which they are assigned. This will get updated in the 'Prep Assignment' section of the Blackboard menu. You do not need to submit the solutions for the practice problems in the prep assignment, but you should try to work them out yourself to solidify your understanding. We will explain the material and work out harder examples from the section in class.
- **Daily Assignments** will contain questions based on the textbook readings and class work. These assignments with their due dates will be regularly posted on Blackboard. The typical due date pattern is:
 - Monday's homework is due Friday same week,
 - Wednesday and Friday's homeworks are due Wednesday next week.

As is typical for linear algebra courses in the Mathematics Department, homework will generally be corrected by student graders who work under my supervision; this is done to ensure that you regularly receive graded assignments in a timely manner.

- There will be five or six longer **Extra Credit Collaborative Projects** built around more challenging questions from the exercises, to showcase interesting applications of the study materials. Electronic copies of the assignment details will be available on Blackboard. These will be due typically within seven to ten days.
- we will use some of the class sessions to work on practice problems. Paper copy of **Handouts** will be provided and an electronic copy will be available on Blackboard. Depending on how much we are able to cover during class period, part of it might get assigned as homework.
- Additionally, there will be occasional quizzes and **two Midterms** given during the semester, as well as a **Final Examination** at the end of the semester. The midterms will be during Friday class times. The final exam will be according to the Registrar's office schedule. All exams will emphasize the concepts of the course.

Grading Policy

- Grades will be given for each daily assignments, quizzes, and exams. In addition, each lab will include a short assignment that will be collected and graded. Both your score and how it ranks relative to the other scores in the class will determine your final grade.
- *You can get an additional 10% score by completing the extra credit collaborative projects.*
- *Scores will NOT be curved. However, the cutoff percentage for letter grades will be set at my discretion.*

The individual weights are as follows:

Assignments	20%
Quizzes, class work and class participation	15%
Midterm 1	15%
Midterm 2	20%
Final exam	30%
Extra Credit Projects	10%

Important Dates

Midterm # 1	Friday, February 22, 2019
Midterm # 2	Friday, April 12, 2019
Final Exam	Friday, May 17, 2019, 8:30 AM - 11:30 AM

Please let me know immediately of any problems with these dates. Please note that the date of the final exam is set by the Registrar's office and cannot be altered. Individual changes in final exam dates are allowed only for particularly serious situations such as three exams in a two-day period.

Assignment and Projects Policies

- *Often there will be no example in the text or in class work that exactly mirrors an assigned problem or project. This is by design.* To learn how to apply the principles discussed in the text and the class sessions, you cannot merely copy procedures you see laid out in examples.

- Homeworks are extremely important, as it is the best way for you to engage with the material on a regular basis. The problems assigned will be carefully chosen to highlight essential concepts. I also expect that in case you need extra practice with a certain concept, you will seek *extra, unassigned problems from the textbook to work out*; I am always happy to discuss how to locate good practice problems in your book.
- You are encouraged to work on the **daily assignments** with others, but you must write your final solution in your own words and you must complete and attach an **Assignment Cover Sheet** with every submission. This sheet can be downloaded from Blackboard. Assignments will need to be submitted to me personally at the beginning of the classes.
- You are allowed to work in groups of size **at most 3 (three)** to work on the **extra credit collaborative projects**. All members of the group must not only participate in the analysis of the project but should discuss the specific phrasing and organization of the final submission. Final submissions must include a **Collaboration Report** (downloadable from Blackboard) on which the signatures of all participants must appear along with *brief but substantive* discussions of the issues confronted at your meetings. If any group member did not participate in an important aspect of the assignment, this must be stated in the Report. *One submission for your entire group will suffice.*
- The point of the homeworks is for you to work out what you do and don't understand. You should help each other to understand the materials and come and ask me if all of you get stuck together. When your graded homework has been handed back to you, you should go through it and see if you understand what has been written on it by the grader. If you don't, you should come to office hours and ask.
- As you are solving problems in this course, remember that getting the "answer" is only one of the steps. Don't think of what you write as just showing your instructor that you have done the homework. Write as if you were explaining what you are doing to one of your classmates who missed that day of class. Think of writing as part of the process of learning. The more carefully and clearly you write your mathematics, the more likely it is to be correct, and the more likely you will be to remember it. Correct answers without explanation will not reap full credit, but clear explanations with an incorrect answer can certainly earn partial credit.

Advice on Collaborative Learning

Collaborative learning teams are said to attain higher level thinking and preserve information for longer times than students working individually. You are strongly encouraged to work together, both in-class and on homeworks, throughout this course. As a member of a group you are responsible not only for your own learning but also for the learning of the other members of your group. This means that when the work is completed and submitted, every member of the group should be able to explain how to solve all the problems. Here are some ideas that past students have come up with to help your group function at its full potential.

- **Schedule enough meetings**, well in advance, and make sure to attend every one of them.
- **Be prepared**. Prior to meeting do the readings and think about the problems.
- **Contribute** to the assignment solutions. Make sure that everyone is equally involved.
- **Listen carefully** and with respect to each other. Don't interrupt and don't tune out.

- **Ask for help** when you need it.
- **Give help** when it is requested.
- **Criticize ideas, not people.** Be tolerant, respectful, and caring.
- **Never agree to something you don't understand.** Don't rush to the finish before others.

Low scores and late submission policies

- You can **replace up to three quiz grades** by going to a mathematics or related talk, and turning in a 1–2 page summary of the talk. Talks from other departments with a math flavor to them can also count. (eg: biology, chemistry, computer science, digital and computational studies, earth and oceanographic science, economics, education, environmental studies, neuroscience and physics are all good places to look) For talk announcements, check out the posters around Searles, Druckenmiller, Kanbar, Adams, VAC and elsewhere. Also check the Bowdoin events calendar, dept. websites, the digest, e-mail announcements, and the ES newsletter.
- In general, late submission (even 15 mins late) of homework assignments will **NOT** be accepted. In extenuating circumstances, with proper prior notice, I will try to provide extensions to individuals. If I am not present to receive your submission, you can put it in the Math 2000-B homework box located at the South end of Searles' first floor hallway.
- I will drop two of your lowest *daily assignment* scores, no-questions-asked.
- If you think you are going to miss any quiz or exam for unavoidable reasons, please notify me beforehand. Missed exams can only be made up at my discretion, and are subject to a lost fraction of the grade.

Class Participation

Student participation is an integral part of this class and is highly valued. Everyone is expected to make thoughtful contributions in the form of questions, statements, and reasoned arguments. You might be also occasionally invited to present something on the board. Whenever possible, there will be opportunities for you to work through practice problems in small groups during our class meetings. This is a chance for you to learn in an active way, collaborating with your peers. Please express yourself within the bounds of courtesy and respect. Please share your thoughts and be willing to listen attentively to perspectives that may differ from your own.

Class Attendance

You cannot be an effective and involved member of the class unless you are present! Please try to be punctual as well. If you are late to the occasional quizzes that are administered at the beginning of the class, you will *not* be allowed any extra time.

General Policy

- Be courteous when using mobile devices. Make sure your cell phone is turned fully off, or silent. If you must make or receive a call, please go outside the classroom.
- Use of laptops or tablets is permitted for note-taking but only with prior permission. Please turn off your Wi-fi and sound.
- The final exam is based on all material covered in class. If you have to miss a lecture, then I strongly recommend you study the material you missed before you return to class. I recommend doing the following steps:
 - Look at the tentative course schedule from Blackboard.
 - Read the relevant sections from the textbooks, class note, internet etc.
 - Find someone who was in class and make a copy of their notes,

Once you have done these steps, and you still need more clarification on lectures you missed, email me to schedule an appointment.

- *For any communication regarding this course, please email me from your bowdoin.edu email address. This is mainly for identity verification purposes.*

Miscellaneous Items of Interest

- It is my intent that students from all backgrounds and perspectives receive **equitable access and opportunity** in this course, that students' learning needs be addressed both in and out of class, and that the diversity students bring to this class be viewed as a resource, strength and benefit. It is my intent to employ materials and engage in activities and dialogue that are respectful of: gender identity, sexuality, disability, age, socioeconomic status, ethnicity, race, nationality, religion, and culture. Please share your preferences for your name and pronouns.
- No student is required to take an examination or fulfill other scheduled course requirements on recognized **religious holidays**. Students are expected to declare their intention to observe these holidays at the beginning of the semester.
- Students with **documented accommodations** have a right to have these met. I encourage you to see me in the first 2 week of class to discuss how your accommodations may support your learning process in this course. I highly encourage all students to meet with me in the first few weeks of class (or as soon as you become aware of your needs) to discuss your learning preferences, challenges you may face learning this semester, and how we can create an effective learning experience for you. *In particular, I understand that the quizzes at the beginning of class can present a challenge, and I'm eager to discuss options with you.* If you are interested in learning more about accommodations please see Lesley Levy in the Office of Student Accessibility

<https://www.bowdoin.edu/accessibility/student-accessibility-office/index.html>

- As a student, you may experience a range of issues that can cause barriers to learning, such as strained relationships, increased anxiety, alcohol/drug problems, feeling down, difficulty concentrating and/or

lack of motivation. These **mental health concerns** or stressful events may lead to diminished academic performance or reduced ability to participate in daily activities. Bowdoin College is committed to advancing the mental health and well-being of its students. If you or someone you know is feeling overwhelmed, depressed, and/or in need of support, services are available. You can learn more about the broad range of confidential mental health services available on campus at:

<https://www.bowdoin.edu/counseling/>

- As a faculty member I am considered a **Responsible Employee**, per the [Student Sexual Misconduct and Gender Based Violence Policy](#). While my goal is for you to be able to share information related to your life experiences through discussion and written work, I want to be make sure you understand that as a Responsible Employee I am required to report disclosures of sexual misconduct, dating violence, stalking, and/or sexual and gender-based harassment to the University's Title IX Coordinator, Benje Douglas. My reporting to Benje does NOT mean that any actions will be taken beyond him reaching out to you and trying to schedule a time to talk to see what assistance you might need to be successful as a student here at Bowdoin. For more information please check out:

www.bowdoin.edu/title-ix

The Honor Code

I support and adhere to the principles of [The Bowdoin College Academic Honor Code](#). Your work should never be directly copied from another student and I will expect that *you are not reading solution manuals* for this textbook. In particular, I will assume all members of the class are trustworthy in their dealings with me as well as their fellow classmates. However, should a violation of this trust be discovered, it will be reported to the Judiciary Board. The goal is not vengeance against those who violate the Code but fairness for those who adhere to it. If you have any questions about the appropriateness of a particular situation, please communicate with me.

Tentative Course outline and Schedule

The following is a preliminary outline of the topics that we hope to cover. This is an idealized plan, and it *may be adjusted as the semester progresses*. But it should give some indication of the major topics to be covered in this class.

Monday	Wednesday	Friday
	23-Jan	25-Jan
	Syllabus Overview + System of Linear Equation (1.1)	Matrices, Elem Row Operation, Reduced Echelon Form (1.1-1.2)
28-Jan	30-Jan	1-Feb
Consistent Matrices, Homogeneous Matrices (1.2, 1.5)	Matrix Operations (2.1) + Adjacency Matrix of a Graph	Properties of Matrix Operations (2.1 cont., 1.4)
4-Feb	6-Feb	8-Feb
Handout 1	Scalar Product, Norm, Linear Independence (1.3,1.7)	More on Linear Independence (1.7 cont.)
11-Feb	13-Feb	15-Feb
Handout 2	Matrix Inverses (2.2) + Cryptography	Inverse Matrix cont. and Determinants (2.3,3.1)
18-Feb	20-Feb	22-Feb
Handout 3	Review	Midterm 1
25-Feb	27-Feb	1-Mar
Elementary Operations and Determinants (3.2)	Vector Space (4.1)	Null Space and Range (4.2) + Subspaces
4-Mar	6-Mar	8-Mar
Computing $N(A)$ and $R(A)$ (4.2 cont.)	Spanning Subset and Bases (4.3)	Handout 4
11-Mar	13-Mar	15-Mar
Spring Vacation		
18-Mar	20-Mar	22-Mar
Spring Vacation		
25-Mar	27-Mar	29-Mar
Bases cont. and Dimension (4.5)	Rank and Orthogonal Bases (4.6)	Orthogonal and Orthonormal Sets, Gram-Schmidt Process (6.1-6.4)
1-Apr	3-Apr	5-Apr
Handout 5 + Orthogonal Projection and Least Squares	Linear Transformations (1.8,4.4) + Lin. Transf. of the Plane	Linear Transformation cont. (4.7)
8-Apr	10-Apr	12-Apr
Handout 6	Review	Midterm 2
15-Apr	17-Apr	19-Apr
Eigenvalues and Eigenvectors (5.1)	Characteristic Polynomials (5.2)	Complex Eigenvalues and Similar Matrices (5.5)
22-Apr	24-Apr	26-Apr
Handout 7	Diagonalization (5.3)	Eigenvectors and Linear Transformations (5.4)
29-Apr	1-May	3-May
Discrete Dynamical System (5.6) + Spotted Owl Population	Handout 8	Diagonalization of Symmetric Matrices (7.1)
6-May	8-May	9-May
Handout 9	Review	Reading Period

Appendix B

Sample Projects and Labs

MATH 2208: ORDINARY DIFFERENTIAL EQUATIONS

PROJECT 3: AN APPLICATION FROM CHEMISTRY - THE BRUSSELATOR AND HOPF BIFURCATION

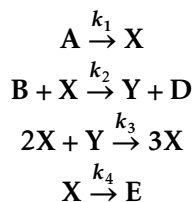
Fall 2019

Subhadip Chowdhury

Due: Dec 11

The *Brusselator model* is a famous model of chemical reactions with oscillations. The dynamics and chemistry of oscillating reactions has been the subject of study for the last almost 60 years, starting with the work of Boris Belousov. While studying the Krebs's cycle, he witnessed a mixture of citric acid, bromate and cerium catalyst in a sulphuric acid solution undergoing periodic colour changes. These changes indicated the cycle formation and depletion of differently oxidized Ce(III) and Ce(IV) ions. This was the first reaction, where biochemical oscillations was observed in 1950. In 1961, Zhabotinsky reproduced Belousov's work and showed further oscillating reactions. The reaction mechanism is an example of an autocatalytic, oscillating chemical reaction. Autocatalytic reactions are chemical reactions in which at least one of the reactants is also a product and vice versa. The rate equations for autocatalytic reactions are fundamentally nonlinear.

Consider a Brusselator whose equations are given as follows¹:



where **A** and **B** are reactants (substrates), **D** and **E** are products and **X** and **Y** are the autocatalytic reactants of the set of reactions. Also k_1, k_2, k_3 and k_4 are the rate of reactions for each component reaction.

The governing equations of the Brusselator are obtained using law of mass action (the rate of a chemical reaction is directly proportional to the product of the concentration of reactant) and the set of equations for the change in concentrations of **X** and **Y** are found to be (after some scaling):

$$\begin{aligned}\frac{dx}{dt} &= a - (1 + b)x + yx^2 \\ \frac{dy}{dt} &= bx - yx^2\end{aligned}$$

where x and y represent the concentrations of the autocatalytic reactants and $a, b > 0$. Because of obvious reasons, we are only going to be interested in the case where x and y are positive.

We will be using Desmos and PP1ane for this project.

Equilibrium Point Analysis

Read the above text and watch the linked video

<https://www.youtube.com/watch?v=uWh8reiXq58>

before you begin! The video uses different compounds than the original experiment by Belousov but the theory is still the same.

¹<https://en.wikipedia.org/wiki/Brusselator>

■ Question 1.

Find the equation of the nullclines and the equilibrium point in terms of a and b .

■ Question 2.

Find the Jacobian of the system at the equilibrium point. Show that its trace is

$$T = b - 1 - a^2$$

and determinant is

$$D = a^2$$

Bifurcation

■ Question 3.

Suppose we keep a fixed and vary b . Write down the type of stability that should occur at the equilibrium in the following three cases:

a) $a = 1$ and $b = 1$.

b) $a = 1$ and $b = 3$.

c) $a = 1$ and $b = 5$.

■ Question 4.

We conclude that there are at least two bifurcation values of b when a is fixed at 1.

- (a) Check that D is always positive, so the equilibrium is source or sink based on values of T .
- (b) Draw the curves $T = 0$ and $T^2 = 4D$ in the (a, b) -parameter plane.
- (c) Find the bifurcation values of b when $a = 1$.

■ Question 5.

Find all possible behaviour of the system as a function of parameters a and b . In particular, draw the regions in (a, b) -plane (remember that $a, b > 0$) that correspond to stable/unstable node or spirals.

Nullclines and Direction Field

■ Question 6.

Use Desmos to draw the nullclines. Set sliders for a and b .

- (a) Move the slider to check that the nullclines intersect only once for all values of a and b . How does changing a and b change the x and y coordinates of the point of intersection? Is this consistent with the formula of equilibrium point you obtained above?
- (b) Set $a = 1, b = 2.5$ and copy the resulting graphs of the nullclines on to paper. Draw the direction field along the nullclines.

■ Question 7.

Find whether x' and y' are positive or negative in other regions (4 of them) and use this information to draw the direction field arrows in those regions (on your paper).

Phase Portrait

■ Question 8.

Something very interesting happens in the phase portrait as the equilibrium becomes unstable. Let's use pp1ane to draw the phase portrait in each of the three cases from question (3). You may need to change the window width in each case for a comfortable viewing size.

Include a printout of the phase portrait with some sample solution curves in each case. You can collect all three on one page using e.g. MS Word to save paper.

Existence of a Limit Cycle

Could we have predicted that we would be getting a limit cycle when the equilibrium becomes unstable? Recall the four possibilities for a trajectory of a solution curve for a nonlinear system as a consequence of *Poincaré–Bendixson* theorem. Assuming the requirements hold, the only possibilities are either the solution curves that start near $(0, 0)$ become unbounded or they spiral towards a limit cycle. How do we know they are not going to infinity? In the following steps we will work this out.

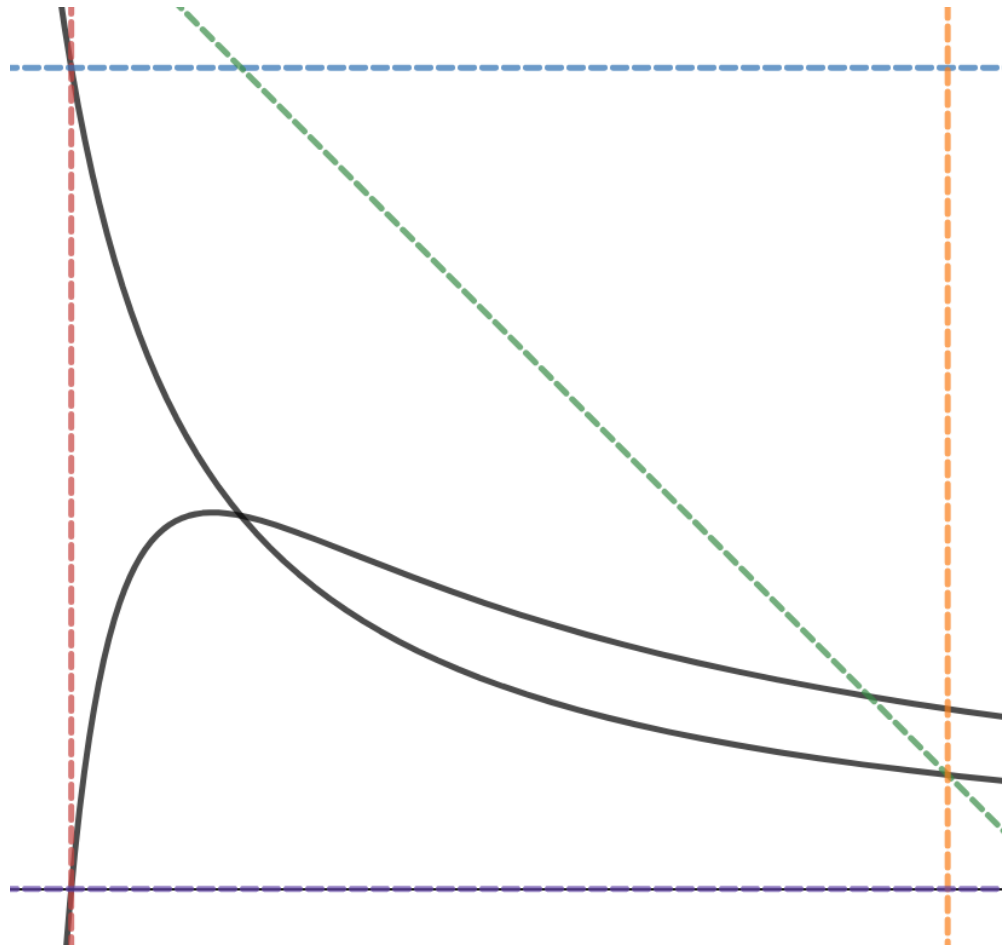


Figure 1: Trapping Region

Consider the five dotted lines that we have drawn along with the nullclines in figure 1. We will describe and find equations for them in the next problem.

■ Question 9.

- (a) The **red** dotted line is vertical and it passes through the point of intersection of the x -nullcline and the x -axis. Find its equation in terms of a and b .
- (b) The **blue** dotted line is horizontal and it passes through the point of intersection of the **red** line and the y -nullcline. Find its equation in terms of a and b .
- (c) The **green** dotted line has slope -1 and passes through the point $\left(a, \frac{b(b+1)}{a}\right)$. Find its equation in terms of a and b .
- (d) The **orange** dotted line is vertical and passes through the intersection of the **green** line and the y -nullcline. Find its equation in terms of a and b .
- (e) The **purple** dotted line is just the X -axis. Mark the region bounded by the five dotted straight lines.

■ Question 10.

We are going to find the direction field along the boundary of this region. Arrows along the horizontal and vertical segments are easy to find. So we only need to check the diagonal segment.

- (a) Show that $-y' > x'$ if $x > a$. [HINT: Calculate $x' + y'$.]
- (b) Explain how this inequality implies that the vector field points inward on the diagonal line curve 3. [HINT: What is dy/dx for $x > a$?]

■ Question 11.

Draw the arrows along the five boundary (dotted) line segments. What can you say about the long-term behavior of a solution curve that crosses into the region? Can it ever leave the region?

The region we have constructed above is called a *Trapping Region*. It 'traps' the trajectories inside as all the arrows are pointing inward! On the other hand, when the equilibrium point is a spiral source, all trajectories that start near the point are going outwards. Since they can't escape the trapping region, they are doomed to converge towards a limit cycle! Note that the requirement of being a repelling equilibrium point is crucial, since otherwise all trajectories could have converged to a point.

Hopf Bifurcation

The kind of bifurcation you observed above is called a *Hopf Bifurcation*. The uniqueness of such bifurcations lies in two aspects: unlike other common types of bifurcations (e.g. pitchfork, saddle-node or transcritical) Hopf bifurcation cannot occur in one dimension. The minimum dimensionality has to be two. The other aspect is that Hopf bifurcation deals with birth or death of a periodic solution or limit cycle as and when it emanates from or shrinks onto a fixed point, the equilibrium. Recently, Hopf bifurcations of some famous chaotic systems have been investigated and it is becoming one of the most active topics in the field of chaotic systems.

MATH 2000 PROJECT 5: MARKOV CHAINS, THE PERRON-FROBENIUS THEOREM AND GOOGLE'S PAGERANK ALGORITHM*

Subhadip Chowdhury

- **Purpose:** To analyze Markov chains and investigate steady state vectors.
- **Prerequisite:** Eigenvalues and eigenvectors.
- **Resources:** Use Mathematica as needed. You might also want to take a look at <http://setosa.io/ev/markov-chains/> and <http://setosa.io/markov/index.html>

Web Surfing

Definition 1. A **Stochastic matrix** (aka Markov Matrix) is a square matrix, all of whose entries are between 0 and 1 (inclusive), and such that the entries in each column add up to 1.

We can think of the matrix entries as probabilities of different events happening. For example, consider the matrix

$$A = \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.4 & 0.2 \\ 0.1 & 0.5 & 0.6 \end{bmatrix}.$$

(Note that the entries in each column add up to 1.) We can use this matrix to make a *very very* simple model of the internet, as follows.

Pretend there are only 3 domains on the internet: reddit.com, google.com, and instagram.com (Hereafter referred to as Domains 1, 2, and 3, respectively.) In every five-minute period, say, the people surfing this web have some probability of switching to another domain. For example, let's say that, in a given five-minute period, out of all the people clicking around on Domain 1, 70% will remain on Domain 1, 20% will end up clicking on a link to Domain 2, and 10% will end up on Domain 3. Notice that these are precisely the numbers in the first column of A . I.e., the first column encodes what happens to the people surfing Domain 1. Similarly, the second and third columns tell the probabilities of what happens to the people on Domain 2 and Domain 3, respectively.

Another way to say the same thing: if we identify matrix entries in the standard way, where a_{ij} represents the entry in row i and column j , then a_{ij} here is the probability that someone surfing Domain j will end up on Domain i five minutes from now.

*Most of this project is made using or copied from Lay's Linear Algebra book and Interactive Linear Algebra by Dan Margalit, Joseph Rabinoff.

Definition 2. The matrix A is called the *Transition Matrix* of this system and the columns of A are called the *Transition Probability Vectors*.

By our definition, the transition matrix is a stochastic matrix. We will see more examples of stochastic matrices later.

Exercise 1

Draw a graph illustrating this situation: make a vertex(node) for each of Domain 1, 2, and 3, and draw arrows between the nodes, each labelled with the probability of moving from one bubble to the next. You might want to check out the links above for some pretty neat animations.

Suppose initially, at time $t = 0$, 50% of the surfers are on Domain 1, 30% are on Domain 2, and 20% are on Domain 3. Encode this by the vector

$$\vec{x}_0 = \begin{bmatrix} 0.5 \\ 0.3 \\ 0.2 \end{bmatrix}.$$

Then $\vec{x}_1 = A\vec{x}_0$ tells us what proportion of the surfers are on each domain after one time increment. And $\vec{x}_2 = A\vec{x}_1$ tells us where the surfers are after two time increments. And so on.

Exercise 2

Compute \vec{x}_1 and \vec{x}_2 .

Note that for each of \vec{x}_0 , \vec{x}_1 , and \vec{x}_2 , the entries add up to 1. This makes sense for \vec{x}_0 , since its entries are probabilities covering all the cases. But it's not so obvious for \vec{x}_1 and \vec{x}_2 .

Exercise 3

Show in general that, given a Markov matrix M and a vector \vec{v} whose entries add up to 1, the entries of $M\vec{v}$ also add up to 1.

Definition 3. The sequence of vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3, \dots, \vec{x}_n, \dots$ is called a *Markov chain*.

It lets us track the evolution of this system, seeing where the populations end up over time. However, in order to find something like \vec{x}_{100} , we would need to compute A^{100} . In other words, we need eigenstuff and diagonalization!

The first claim is that A (or any Markov matrix) always has an eigenvalue of 1. It's not trivial to find a \vec{v} such that $A\vec{v} = \vec{v}$ in general, but...

Exercise 4

Find an easy nonzero \vec{v} such that $A^T\vec{v} = \vec{v}$. This shows that 1 is an eigenvalue of A^T .
HINT: use the fact that, since we've transposed, the entries in each row of A^T add up to 1.

Recall that A and A^T have the same eigenvalues. Thus $\mathbf{1}$ is an eigenvalue of A as well.

Exercise 5

- (a) Find the characteristic polynomial of A , and use the fact that we already know $(\lambda - 1)$ will appear in the factorization to find the other eigenvalue(s) of A .
- (b) Find the eigenspace for each eigenvalue of A , and write down a nice eigenbasis for \mathbb{R}^3 .
- (c) Write down the diagonalization $A = BDB^{-1}$.
- (d) Compute A^{100} and \vec{x}_{100} .
- (e) Find $\lim_{n \rightarrow \infty} \vec{x}_n$.
- (f) In the long-term, what percentage of surfers end up on reddit.com, what percentage end up on google.com, and what percentage end up on instagram.com?
- (g) Did it matter here what our particular initial distribution \vec{x}_0 was? If 100% started at reddit.com, would we still end up with the same percentages on each domain over the long term?

Steady State and the Perron-Frobenius Theorem

The eigenvalues of stochastic matrices have very special properties.

Proposition 1. *Let A be a stochastic matrix. Then:*

- (a) $\mathbf{1}$ is an eigenvalue of A .
- (b) If λ is a (real or complex) eigenvalue of A , then $|\lambda| \leq 1$.

As we observed in the last section in exercise 4, we can prove that $\mathbf{1}$ is always an eigenvalue of a stochastic matrix. Let's prove the second part. We will restrict to the case of real eigenvalues for the sake of this project.

Exercise 6

1. Let λ be any real eigenvalue of A . Explain why we can always find a vector \vec{x} such that $A^T \vec{x} = \lambda \vec{x}$.
2. Let $\vec{x} = [x_1 \ x_2 \ \dots \ x_n]^T$. Choose x_j with the largest absolute value, so that $|x_i| \leq |x_j|$ for all i . Explain the steps in the following chain of inequality

$$|\lambda| \cdot |x_j| = \left| \sum_{i=1}^n a_{ij} x_i \right| \leq \sum_{i=1}^n (a_{ij} \cdot |x_i|) \leq \left(\sum_{i=1}^n a_{ij} \right) \cdot |x_j| = 1 \cdot |x_j|$$

Hence we can conclude that $|\lambda| \leq 1$.

Definition 4. We say that a matrix A is *positive* if all of its entries are positive numbers.

For a *positive* stochastic matrix A , one can show that if $\lambda \neq 1$ is a (real or complex) eigenvalue of A , then $|\lambda| < 1$. The $\mathbf{1}$ -eigenspace E_1 of a stochastic matrix is very important.

Definition 5. If A is a stochastic matrix, then a *steady-state vector* (or equilibrium vector) for A is a probability vector \vec{q} such that

$$A\vec{q} = \vec{q}$$

In other words, it is an eigenvector \vec{q} of A with eigenvalue $\mathbf{1}$, such that the entries are positive and sum to $\mathbf{1}$.

The Perron-Frobenius theorem describes the long-term behavior of such a process represented by a stochastic matrix. Its proof is complicated and is beyond the scope of this project.

Theorem 2 (Perron-Frobenius Theorem). *Let A be a *positive* stochastic matrix. Then A admits a *unique* steady state vector \vec{q} , which spans the $\mathbf{1}$ -eigenspace E_1 . Further, if \vec{x}_0 is any initial state and $\vec{x}_{k+1} = A\vec{x}_k$ then the Markov chain $\{\vec{x}_k\}$ converges¹ to \vec{q} as $k \rightarrow \infty$.*

Why is this nontrivial? For two reasons:

- Apriori, we did not know whether all the entries of the eigenvector corresponding to the eigenvalue $\mathbf{1}$ are positive. We also did not know about the geometric multiplicity of the eigenvalue $\mathbf{1}$. P-F theorem tells us that in fact, $\dim(E_1) = 1$ and we can find a vector $\vec{q} \in E_1$ such that all entries of \vec{q} are positive and sum to $\mathbf{1}$!
- If a Markov process has a positive transition matrix, the process will converge to *the* steady state \vec{q} regardless of the initial state.

¹We say that a sequence of vectors $\{\vec{x}_k\}$ converges to a vector \vec{q} as $k \rightarrow \infty$ if the entries in \vec{x}_k can be made as close as desired to the corresponding entries in \vec{q} by taking k sufficiently large.

Exercise 7

Let $A = \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$ be a stochastic matrix.

- Find the eigenvalues and corresponding eigenvectors of A .
- Using the eigenvector corresponding to the eigenvalue 1 , find the steady-state vector \vec{q} of A .

Let's try to give a visual interpretation of the linear transformation defined by the matrix above. This matrix A is diagonalizable; we have $A = CDC^{-1}$ for

$$C = \begin{pmatrix} 7 & -1 & 1 \\ 6 & 0 & -3 \\ 5 & 1 & 2 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -.2 & 0 \\ 0 & 0 & .1 \end{pmatrix}$$

The matrix D leaves the x -coordinate unchanged, scales the y -coordinate by $-1/5$, and scales the z -coordinate by $1/10$. Repeated multiplication by D makes the y - and z -coordinates very small, so it *sucks all vectors into the x -axis*.

The matrix A does the same thing as D , but with respect to the coordinate system defined by the columns $\vec{u}_1, \vec{u}_2, \vec{u}_3$ of C . This means that A *sucks all vectors into the 1 -eigenspace*, without changing the sum of the entries of the vectors.

Google's PageRank Algorithm

In 1996, Larry Page and Sergey Brin invented a way to rank pages by importance. They founded Google based on their algorithm. Here is how it works (Roughly). Each web page has an associated importance, or *rank*. This is a positive number. If a page P links to n other pages Q_1, Q_2, \dots, Q_n , then each page Q_i inherits $\frac{1}{n}$ of P 's importance.

Definition 6. Consider an Internet with n pages. The *Rank matrix* is the $n \times n$ matrix A whose i, j -entry is the importance that page j passes to page i .

Observe that the rank matrix is a stochastic matrix, assuming every page contains a link: if page i has m links, $m \leq n$, then the i th column contains the number $\frac{1}{m}$, a total of m times, and the number zero in the other entries.

The goal is to find the steady-state rank vector of this Rank matrix. We would like to use the Perron-Frobenius theorem to find the rank vector. Unfortunately, the Rank Matrix is not always a *positive* stochastic matrix.

Here is Page and Brin's solution. First we fix the rank matrix by replacing each zero column with a column of $\frac{1}{n}$ s, where n is the number of pages.

So for example,

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{becomes} \quad A' = \begin{pmatrix} 0 & 0 & 1/3 \\ 0 & 0 & 1/3 \\ 1 & 1 & 1/3 \end{pmatrix}$$

The **modified Rank Matrix** A' is always stochastic.

Now we choose a number p in $(0, 1)$, called the damping factor. (A typical value is $p = 0.15$.)

Definition 7 (The Google Matrix). Let A be the Rank Matrix for an Internet with n pages, and let A' be the modified Rank Matrix. The **Google Matrix** is the matrix

$$G = (1 - p) \cdot A' + p \cdot B \quad \text{where} \quad B = \frac{1}{n} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

Exercise 8

Show that G is a positive stochastic matrix.

If we declare that the ranks of all of the pages must sum to one, then we find:

Definition 8 (The 25 Billion Dollar Eigenvector). The PageRank vector is the steady state of the Google Matrix.

This exists and has positive entries by the Perron-Frobenius theorem. The hard part is calculating it: in real life, the Google Matrix has zillions of rows.

MATH 1800 LAB 5: TAYLOR APPROXIMATION AND SECOND DERIVATIVE TEST

Subhadip Chowdhury

WHY DOES THE SECOND DERIVATIVE TEST WORK?

It combines 2 ideas:

- (a) It's not too hard to classify critical points for *quadratic* polynomials in two variables, i.e. polynomials where no term is bigger than x^2 , y^2 , or xy .
- (b) We can use Taylor series to approximate any (smooth) function by a quadratic polynomial.

Let's begin with the first point.

UNDERSTANDING QUADRATICS IN TWO VARIABLES

1. Give a simple explanation why $f(x, y) = x^2 + y^2$ has a minimum at $(0, 0)$.
HINT: what is the minimum possible value this function could have?
2. For the simple function $f(x) = ax^2$, can you tell whether it's concave up or down at $x = 0$ just by the sign of a ? When a is positive, is it concave up or down? When a is negative?

Even though the example $x^2 + y^2$ is easy, let's think about it in a different way. A function $z = f(x, y)$ has a local minimum at a point if it's "concave up in every direction" at that point. To be more precise, when we take any vertical slice through this point, the resulting curve should be concave up.

Now let's convince ourselves that $f(x, y) = x^2 + y^2$ is concave up on every vertical slice going through $(0, 0, 0)$.

3. Show it's concave up for the two standard cross-sections $x = 0$ and $y = 0$.
4. Think about intersecting $z = x^2 + y^2$ with the vertical plane $y = x$ (you could call this the "45-degree vertical plane" I suppose if you looked down on it from above). Note that this plane does go through the origin. Justify that the slice of the surface lying on this plane is concave up. (Just plug $y = x$ into the equation!)
5. Now do the same thing with slice from the vertical plane $y = 2x$. And $y = -3x$. And $y = 0.02x$. It should be concave up on each of these slices.
6. Every vertical slice is found by intersecting the surface with the plane $y = mx$ for some m . (Or $x = 0$ if you want to think about the slice where m goes to infinity.) Plug $y = mx$ into $z = x^2 + y^2$ and get an equation of the form $z = ax^2$. What is a in terms of m ? Argue that a is always positive.

This last part shows that the surface is concave up in every direction, thus we really do have a local minimum at the origin!

7. Repeat the same analysis as above for the quadratic function $z = g(x, y) = x^2 - y^2$, at the critical point $(0, 0)$ and show that it is concave up in some directions and concave down in others. This explains why g has a saddle point at the origin.
8. Repeat the same analysis as above for the quadratic function $z = h(x, y) = x^2 + 3xy + y^2$ at the critical point $(0, 0)$. Is it concave up in every direction, or are there some directions where it is concave down? Then say whether this critical point is a local max or min or saddle point.
9. One more specific example: try $z = f(x, y) = 2x^2 - 4xy + 3y^2$. When you plug in $y = mx$, you get $z = p(m) \cdot x^2$ where $p(m)$ is a quadratic in m . How can you tell if $p(m)$ takes on both positive and negative values?
HINT: quadratic equation, discriminant. Justify that the critical point at $(0, 0)$ is a local min.

THE GENERAL QUADRATIC

10. Show that the quadratic $z = q(x, y) = Ax^2 + Bxy + Cy^2$ has a critical point at $(0, 0)$.
11. Intersect the surface $z = q(x, y)$ with the vertical plane $y = mx$. As before, we get $z = p(m)x^2$ where $p(m)$ is a quadratic in m . Write down $p(m)$.
12. Convince yourself that if $p(m)$ takes on only positive values (for any choice of m), then the critical point is a local minimum. And that if $p(m)$ takes on only negative values, then the critical point is a local max. And that if $p(m)$ can be either positive or negative depending on the choice of m , then the critical point is a saddle point.
13. Show that the sign of the expression $B^2 - 4AC$ determines whether $p(m)$ takes on only positive (or only negative) values or both positive and negative values.

MORE GENERAL QUADRATIC?

The most general formula of a quadratic function is

$$Q(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + G$$

We can rewrite this by completing some squares as follows:

$$Q(x, y) = A\left(x + \frac{D}{2A}\right)^2 + C\left(y + \frac{E}{2C}\right)^2 + Bxy + \underbrace{\left(G - \frac{D^2}{4A} - \frac{E^2}{4C}\right)}_{\text{constant}}$$

For the following three questions, try experimenting in [Fall12019_1800_Lab5.nb](#).

14. What is the effect on the graph of Q if we change G ?
15. What is the effect on the graph of Q if we change E ?
16. What is the effect on the graph of Q if we change D ?

We should be able to conclude that the *shape* of Q is unaffected by different choices of D, E and G . So we can assume $D = E = G = 0$. Hence the shape of a general quadratic can be fully understood by analyzing $q(x, y)$ from above.

THE TAYLOR QUADRATIC

Recall that the linear approximation of the function $f(x, y)$ at $(0, 0)$ corresponds to the tangent plane at $(0, 0)$:

$$T(x, y) = f_x(0, 0)x + f_y(0, 0)y + f(0, 0)$$

Note that a linear approximation essentially finds a linear function such that the slope of the function matches with the linear function in x - and y - direction. But clearly it can't distinguish between local max/min/saddle point.

So we want to find a better approximation of f by matching more derivatives. For that we'll need something more complicated than linear functions, a quadratic function.

Suppose $Q(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + G$ such that

$$Q(0, 0) = f(0, 0)$$

$$Q_x(0, 0) = f_x(0, 0)$$

$$Q_y(0, 0) = f_y(0, 0)$$

$$Q_{xx}(0, 0) = f_{xx}(0, 0)$$

$$Q_{yy}(0, 0) = f_{yy}(0, 0)$$

$$Q_{xy}(0, 0) = f_{xy}(0, 0)$$

$$Q_{yx}(0, 0) = f_{yx}(0, 0)$$

17. Find A, B, C, D, E, G .

These choices lead us to the *Taylor Quadratic* for $f(x, y)$ at $(0, 0)$.

$$TQ(x, y) = \frac{f_{xx}(0, 0)}{2}x^2 + f_{xy}(0, 0)xy + \frac{f_{yy}(0, 0)}{2}y^2 + f_x(0, 0)x + f_y(0, 0)y + f(0, 0)$$

Or more generally the Taylor Quadratic approximation at a point (a, b) is given by,

$$TQ(x, y) = \frac{f_{xx}(a, b)}{2}(x - a)^2 + f_{xy}(a, b)(x - a)(y - b) + \frac{f_{yy}(a, b)}{2}(y - b)^2 + f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

Using the same argument as last section, we can then conclude that the shape of this quadratic is entirely determined by

$$4AC - B^2 = 4 \frac{f_{xx}(a, b)}{2} \frac{f_{yy}(a, b)}{2} - (f_{xy}(a, b))^2 = \det[\text{Hessian}]$$

18. Apply the $4AC - B^2$ rule to determine the graph shapes of the Taylor quadratics:

(a) $-2xy - 4y^2 - 3x$

(b) $-3(x + 1)^2 - 2(x + 1)(y - 1) - 4(y - 1)^2 - 2(x + 1) - 6(y - 1)$

19. We can also rewrite $q(x, y)$ as

$$q(x, y) = Ax^2 + Bxy + Cy^2 = A\left(x + \frac{B}{2A}y\right)^2 + \left(C - \frac{B^2}{4A}\right)y^2$$

So when $B^2 - 4AC = 0$, the y^2 term vanishes. What is the shape of the surface

$$z = A\left(x + \frac{B}{2A}y\right)^2 ?$$

20. What can you conclude when $B^2 - 4AC = 0$? Is it a local max/min/saddle point? Try experimenting in [Fall2019_1800_Lab5.nb](#).

Appendix C

Sample Worksheets and Handouts

MATH 2208: ORDINARY DIFFERENTIAL EQUATIONS

LECTURE 9 WORKSHEET

Fall 2019

Subhadip Chowdhury

Oct 16

TITLE: System of First Order ODEs

SUMMARY: We will learn how about models that involve a system of first order differential equations. The most famous of these is the Predator-Prey model of Lotka and Volterra. Then we will use direction fields and phase portraits to do qualitative and quantitative analysis.

§A. General Form

The general form for a two-dimensional system of first order ODEs is

$$\begin{aligned}\frac{dx}{dt} &= f(x, y, t) \\ \frac{dy}{dt} &= g(x, y, t)\end{aligned}$$

If f and g are linear in both x and y , the system is called *linear*, otherwise the system is called *nonlinear*.

Note that a second-order ODE can be converted to a two-dimensional system of first-order ODEs via some change in variables. For example, if

$$y'' = f(t, y(t), y'(t))$$

is a second-order ODE, we can make the substitution $u_2(t) = y'(t)$ and $u_1(t) = y(t)$ to convert the ODE into a system of 2 first-order ODEs

$$\begin{aligned}\frac{du_1}{dt} &= u_2(t) \\ \frac{du_2}{dt} &= f(t, u_1(t), u_2(t))\end{aligned}$$

■ Question 1.

Show that the third-order linear ODE

$$y''' + 3y'' + 2y' - 5y = \sin(t)$$

can be written as a linear system of three first-order ODEs.

§B. Lotka-Volterra Model

Probably the most famous system of ordinary differential equations of all time is the Lotka-Volterra predator-prey model. We will study a very special case of interaction between exactly two species, one of which -- the predators -- eats the other -- the prey. Such pairs exist throughout nature: lions and gazelles, birds and insects, pandas and eucalyptus trees, foxes and rabbits. To keep our model simple, we will make some assumptions:

- the predator species is totally dependent on a single prey species as its only food supply,
- the prey species has an unlimited food supply, and
- there is no threat to the prey other than the specific predator.

Let $x(t)$ denote the population of prey (rabbits) and let $y(t)$ denote the population of their predators (foxes). Let a , b , c and d be nonnegative parameters. One system of differential equations that might govern the changes in the population of these two species is

$$\begin{aligned}\frac{dx}{dt} &= ax - bxy \\ \frac{dy}{dt} &= cxy - dy\end{aligned}$$

■ Question 2.

1. What do the constants a, b, c and d represent physically?
2. What is the significance of the xy terms in the model? Is the significance same for rabbits and foxes?
3. What happens to $x(t)$ if $b = c = d = 0$? How does this correspond to our assumptions?
4. What happens to $y(t)$ if $a = b = c = 0$? How does this correspond to our assumptions?
5. What does the model predict will happen if at any time one of the populations of the rabbits or the foxes becomes zero?

■ Question 3.

Are there any fixed points for the system of equations? In other words, what are the equilibrium solutions for the system?

§C. Phase Portrait

One of the ways to graph the solution of the system is to form the pair $(x(t), y(t))$ and think of it as a point in the xy -plane. In other words, the coordinates of the point are the values of the two populations at time t . As t varies, the pair $(x(t), y(t))$ sweeps out a parametric curve in the xy -plane. This curve is called the *solution curve*.

The xy -plane is called the *phase plane*, and it is analogous to the phase line for an autonomous first-order differential equation.

Observe that the solution curves that correspond to equilibrium solutions are really just points, and we refer to them as *equilibrium points*.

Definition 3.1: Nullclines

For a system

$$\begin{aligned}\frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y)\end{aligned}$$

the x -*nullcline* is the set of points

$$\left\{ (x, y) \mid \frac{dx}{dt} = 0 \right\}$$

i.e. the level curve in the phase plane where $f(x, y) = 0$. The y -nullcline is the level curve $g(x, y) = 0$.

At an equilibrium point, both $f(x, y)$ and $g(x, y)$ must be zero, hence

Theorem 3.1

The intersection of the nullclines are the equilibrium points.

A *Phase Portrait* of a system consists of the following information on the phase plane.

- the nullclines,
- the equilibrium points
- several solution curves corresponding to different initial conditions.

§D. Direction Field as a (normalized) Vector Field

Instead of thinking of $(x(t), y(t))$ as simply a combination of the two scalar-valued functions $x(t)$ and $y(t)$, we can consider the pair $(x(t), y(t))$ as a vector-valued function in the xy - plane. For each t , let $\vec{\mathbf{R}}(t)$ denote the column vector

$$\vec{\mathbf{R}}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

which corresponds to the position vector of the point $(x(t), y(t))$.

Then using this notation,

$$\frac{d\vec{\mathbf{R}}}{dt} = \begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix} = \vec{\mathbf{F}}(\vec{\mathbf{R}})$$

where $\vec{\mathbf{F}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ is the vector field

$$\vec{\mathbf{F}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$$

Then the solution curves on a phase plane are essentially the *flow lines* of this vector field. If we draw the field after normalizing each vector to some small magnitude and only care about the direction, it gives us a diagram called the *direction field*.

Along the x -nullcline, the x -component of the vector field is zero, and consequently the arrows in the direction field are vertical. They point either straight up or straight down. Similarly, on the y -nullcline, the y -component of the vector field is zero, so the vector field is horizontal. Arrows point either left or right.

Observe that the x -nullcline naturally divides the plane into regions where $f(x, y) > 0$ and $f(x, y) < 0$. Since $x' = f(x, y)$, $f(x, y) > 0$ means x is increasing which in turn means the arrows point rightward in the plane. Similarly, the y -nullcline shows us where y is increasing or decreasing. The following table will help you fill out the direction field once the nullclines are found.

	$f(x, y) < 0$	$f(x, y) = 0$	$f(x, y) > 0$
$g(x, y) < 0$	↙	↓	↘
$g(x, y) = 0$	←	·	→
$g(x, y) > 0$	↖	↑	↗

§E. Examples

Consider the following Lotka-Volterra model:

$$\begin{aligned}\frac{dx}{dt} &= 2x - 1.2xy \\ \frac{dy}{dt} &= 0.9xy - y\end{aligned}$$

We will use **pp1ane** to draw the direction field on the phase plane. In the following picture of direction field, x -axis is prey and y -axis is predator.

■ Question 4.

Draw the x - and y -nullclines of above system and show the direction of the direction field along the nullclines (up, down, left, right). Find the equilibrium point(s), if any.

■ Question 5.

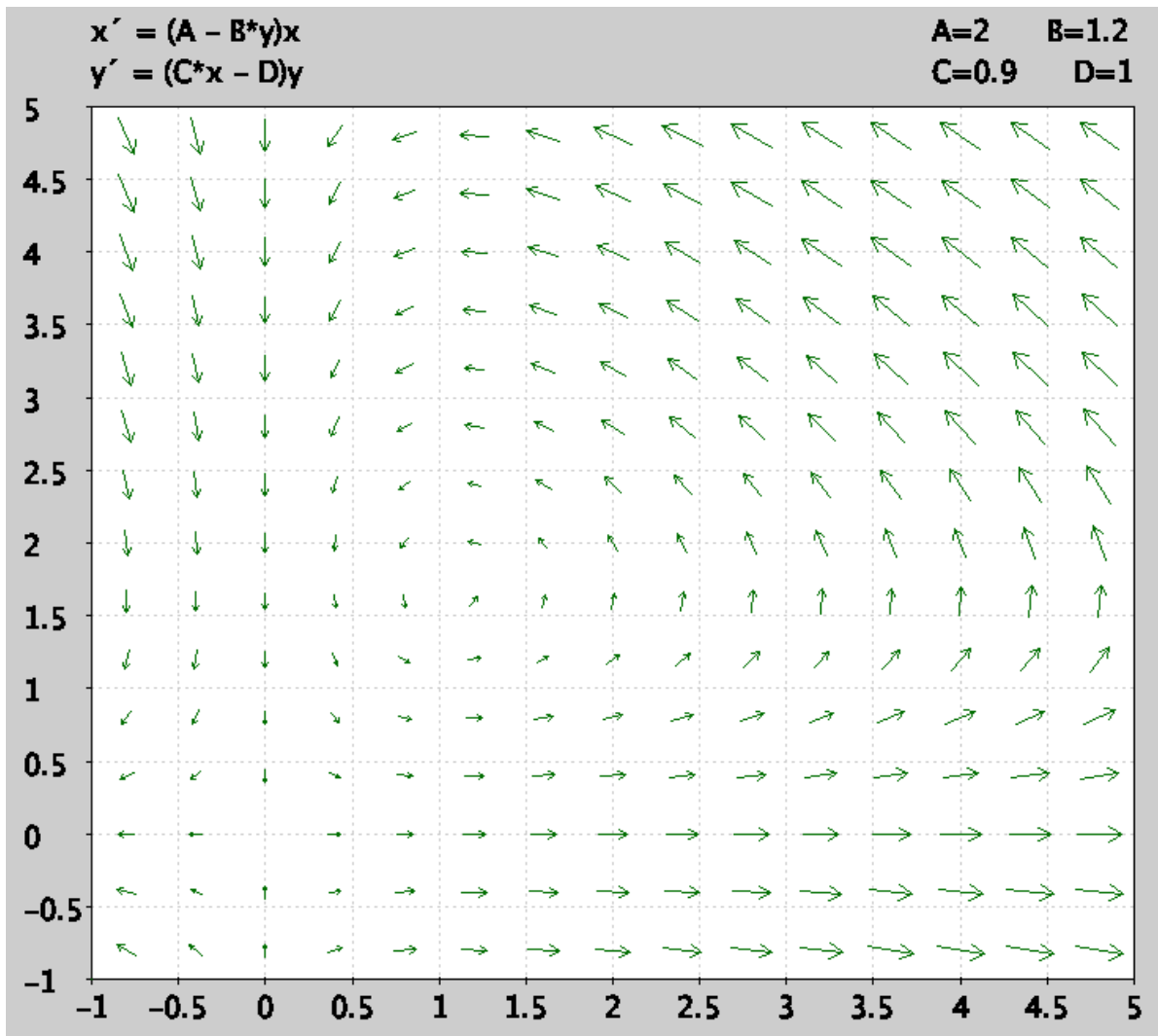
Draw the solution curve that starts at the initial condition $(x(0), y(0)) = (1, 0.5)$. Draw some more solution curves on your paper.

■ Question 6.

What happens when the initial condition approaches $(10/9, 5/3)$?

■ Question 7.

Pick one of the solution curves $(x(t), y(t))$. How do the graphs of $x(t)$ and $y(t)$ against t look like? Do you believe they would be periodic based on the behavior of the solution curves in the phase plane?



■ Question 8.

Can you explain why the two graphs should have the same period? Why does it make sense that the increase in predator population lag behind (in time) the increase in prey population?

■ Question 9.

Find the equation of the nullclines of the following system, draw them in XY-plane, and find the equilibrium points.

$$\frac{dx}{dt} = x(2 - x) - xy$$

$$\frac{dy}{dt} = xy - 1$$

Try drawing a rough diagram of the direction field on the phase plane $0 \leq x, y \leq 2$, using the table from section D. Here are some instructions: find an easily identifiable point in the plane which is not on one of the nullclines. Then evaluate (f, g) at this point and use this to draw a direction at that specific point. Then the rest of the arrows are easy to fill in using alternate choices.

Can you draw some sample solution curves?

MATH 2000-B HANDOUT 7: GEOMETRY OF LINEAR TRANSFORMATIONS

Subhadip Chowdhury

As shown in Theorem 10 in Section 1.9, when a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ given, it can be identified with a matrix, and there is an easy way to get a formula for the function as follows. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ a linear transformation and let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ denote the columns of the $n \times n$ identity matrix.

Figure out what each $T(\mathbf{e}_i)$ should be and write each $T(\mathbf{e}_i)$ as a column vector. If you then define the matrix $A = [T(\mathbf{e}_1) \ T(\mathbf{e}_2) \ \dots \ T(\mathbf{e}_n)]$, then it will be true that $T(\mathbf{x}) = A\mathbf{x}$ for all \mathbf{x} , and $A\mathbf{x}$ gives a formula for the function. In other words, given a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ if you know its values at just the n independent vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ then its value at every point \mathbf{x} is determined!

Example 1. The 2×2 linear transformation that maps $\vec{\mathbf{e}}_1$ to $\vec{\mathbf{e}}_1 + \vec{\mathbf{e}}_2$ and $\vec{\mathbf{e}}_2$ to $\vec{\mathbf{e}}_1 - \vec{\mathbf{e}}_2$ is $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Example 2. The 3×3 matrix transformation that maps $\vec{\mathbf{e}}_1$ to $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$, $\vec{\mathbf{e}}_2$ to $\begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix}$, and $\vec{\mathbf{e}}_3$ to $\begin{bmatrix} 5 \\ 4 \\ -1 \end{bmatrix}$ is $\begin{bmatrix} 3 & 6 & 5 \\ -2 & 0 & 4 \\ 1 & 7 & -1 \end{bmatrix}$.

Exercise 1. (a) The function that reflects \mathbb{R}^2 across the line $\mathbf{y} = -\mathbf{x}$ is a linear transformation. What are the images of $\vec{\mathbf{e}}_1$ and $\vec{\mathbf{e}}_2$ under this map? What is the matrix of this linear transformation?

(b) Write a 2×2 matrix that reflects \mathbb{R}^2 across the line $\mathbf{y} = \mathbf{x}$.

A matrix transformation always maps a line onto a line or a point, and maps parallel lines onto parallel lines or onto points. (See exercises 25-28 in Section 1.8.) In the following exercises, you will verify these things for a particular matrix.

Exercise 2. Let $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

(a) Explain why the function $T(\mathbf{x}) = M\mathbf{x}$ maps the x -axis onto the line $\mathbf{y} = \mathbf{x}$, and why it maps the line $\mathbf{y} = 2$ onto the line $\mathbf{y} = \mathbf{x} + 2$.

[HINT: Use the parametric equation form of a straight line.]

(b) Sketch and label the lines and their images on a graph.

Because a matrix transformation maps parallel lines to parallel lines, it will map any parallelogram to another parallelogram. (The parallelogram could be degenerate-one line segment or a single point.) When a linear transformation and parallelogram are given, the easy way to draw the image of the parallelogram is to plot the images of its four vertices and connect those points to make a parallelogram.

Define the *standard unit square* to be the square in \mathbb{R}^2 whose vertices are $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$. When you want to visualize what a 2×2 matrix transformation does geometrically, it is particularly useful to sketch the image of this standard square. Seeing how this square gets moved or distorted shows what the transformation does to the x -axis and y -axis and thus gives a good idea what the transformation does geometrically to the whole plane. Recall that any linear transformation maps the origin to itself (why?), so you only need to figure out where the transformation maps the other three vertices.

Example 3. The image of the standard unit square under the matrix from exercise 2 looks like figure 1.

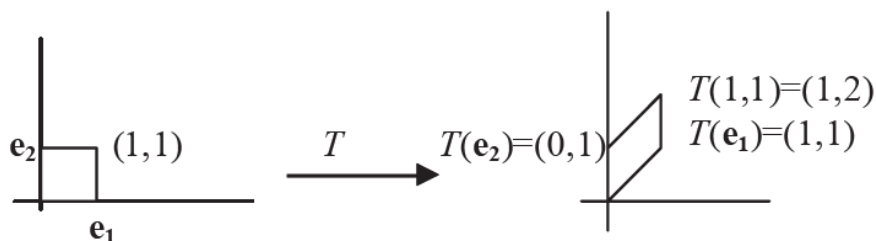


Figure 1

So you can see that the y -axis stays fixed and the x -axis is mapped onto the line $y = x$, causing a vertical *shear* of the plane. See more examples like this in Table 3 in Section 1.9.

Example 4. Find a matrix M which maps the standard unit square to the parallelogram with vertices $(0,0)$, $(3,1)$, $(2,2)$, $(-1,1)$. To do this, sketch the parallelogram and recognize that $M\vec{e}_1$ and $M\vec{e}_2$ must be $(3,1)$ and $(-1,1)$, or vice versa. (Why?) So either of the matrices $\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ or $\begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$ will work.

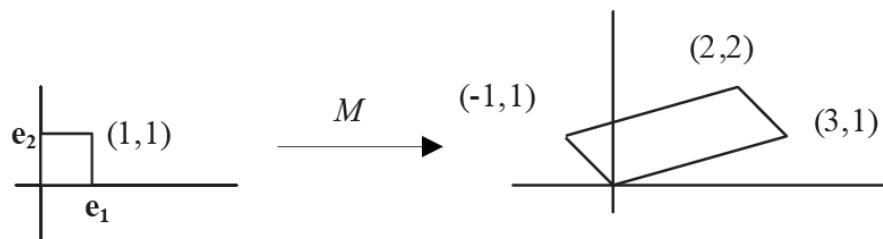


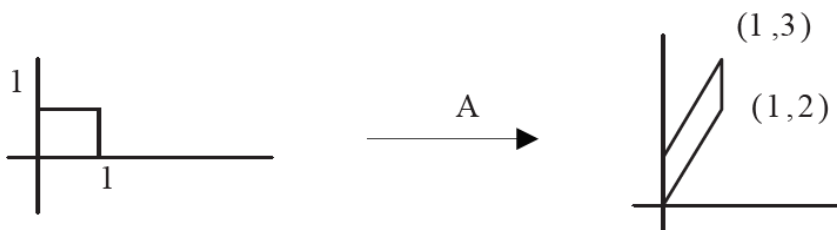
Figure 2

Calculate the image of $(1,1)$ under these matrices and verify that it's $(2,2)$.

Exercise 3. Each of the following seven matrices is one of the special, simple types described in Sections 1.8 and 1.9. Each determines a linear transformation of \mathbb{R}^2 . For each, sketch the image of the standard unit square, label the vertices of the image, and describe how the matrix is transforming the plane. To get you started, answers are given for the first matrix.

(A) $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

Description: A vertical shear. It leaves the y -axis fixed and increases the slope of all other lines through the origin.



(B) $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

(C) $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(D) $D = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

(E) $E = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(F) $F = \begin{bmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{bmatrix}$

(G) $G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Exercise 4. (a) Draw the image of the standard unit square under the transformations EFA and FCA . Here A, C, E, F are from the last exercise.

Exercise 5. Sketch the parallelogram with vertices $(0, 0), (4, 2), (0, -4), (4, -2)$ and write two different 2×2 matrices X and Y which would transform the standard unit square into this parallelogram.

Exercise 6. Sketch the parallelogram with vertices $(1, 1), (1, 2), (3, 1), (3, 2)$. Explain why no 2×2 matrix transformation could map the standard unit square onto this figure.

Exercise 7. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ a linear transformation. Prove that T is completely determined by its values on any n linearly independent vectors.

HINT: Here is an outline of the proof to get you started. Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ be linearly independent vectors in \mathbb{R}^n , and suppose $T(\vec{v}_1), T(\vec{v}_2), \dots, T(\vec{v}_n)$ are known. Let \vec{x} be any element of \mathbb{R}^n . Explain why $\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = \mathbb{R}^n$, so \vec{x} equals some linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. Then show $T(\vec{x})$ can be calculated using $T(\vec{v}_1), T(\vec{v}_2), \dots, T(\vec{v}_n)$.

MATH 1800 HANDOUT 7: APPLICATIONS AND EXTENSIONS OF GREEN'S THEOREM

Subhadip Chowdhury

Careful Statement of Green's Theorem

Green's Theorem is a statement about only the 2D plane, not 3D space. Here's the setup: Let C be a piecewise smooth simple closed curve that is the boundary of a simply-connected region D in the plane. Let $\vec{F} = P\vec{i} + Q\vec{j}$ be a smooth vector field defined on all of D and C .

Notes on terminology: To say a curve is *simple* means that it doesn't intersect itself, and to say a curve is *closed* means that it is a closed loop, that it starts and ends at the same point. A region is called *simply-connected* if it is just one piece (connected) and doesn't have any holes in it. A vector field is said to be *smooth* if it has continuous first partials.

Since C is the boundary of D , we write $C = \partial D$ and stop referring to C explicitly. When we consider the boundary curve of a simply-connected region, we always orient the curve so that the region is on the left as we follow the curve. (Note: this is harder to specify when one considers 3D surfaces with boundaries.)

Green's Theorem says

$$\oint_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D \text{curl}(F) dA = \iint_D (Q_x - P_y) dA.$$

You may also see this written the Leibniz notation for partial derivatives.

$$\oint_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Using Green's Theorem to Evaluate Difficult Line Integrals

■ If you're asked to find the line integral of an ugly vector field over a closed curve, you should look to see if Q_x and P_y are drastically more simple. If so, use Green's Theorem!

Exercise 1

Evaluate

$$\int_C (2y + \sqrt{9 + x^3}) dx + (5x + e^{\tan^{-1} y}) dy,$$

where C is the circle $x^2 + y^2 = 4$ in the plane oriented counterclockwise.

■ If you're asked to find the line integral over a closed curve that is clearly the boundary of a nice region, it's often a good idea to use Green's Theorem to switch to the double integral over the interior.

Exercise 2

Integrate $\vec{F} = xy\vec{i} + e^x\vec{j}$ over the boundary of the rectangle determined by $0 \leq x \leq 2$, $0 \leq y \leq 3$, oriented clockwise around the boundary.

Exercise 3

Find the line integral of $\vec{F} = 3xy\vec{i} + 2x^2\vec{j}$ over the curve C defined as follows: follow the curve $y = x^2 - 2x$ from $(0, 0)$ to $(3, 3)$, then follow the line $y = x$ from $(3, 3)$ back to $(0, 0)$.

Exercise 4

Evaluate

$$\oint_C (3y - e^{\sin x}) dx + (7x + \sqrt{y^4 + 1}) dy$$

where C is the circle $x^2 + y^2 = 9$, oriented clockwise.

■ Consider the following problem.

Find the line integral of

$$\vec{G}(x, y) = (x + y)\vec{i} + \left(2x + y \ln(\csc \sqrt{1 - y^5})\right)\vec{j}$$

over C_1 , the upper half of the unit circle from $(1, 0)$ to $(-1, 0)$.

The problem is that the vector field is ugly, so parametrizing C_1 is just going to lead to an impossible integral. So we would like to use Green's Theorem, but this isn't a closed curve!

Here's how to fix that issue. Let C_2 be the straight line segment from $(-1, 0)$ to $(1, 0)$. Now $C_1 + C_2$ is a closed loop.

- (a) Let R be the region enclosed by $C_1 + C_2$. Use Green's Theorem to compute $\oint_{C_1+C_2} \vec{G} \cdot d\vec{r}$.
- (b) Parametrize C_2 and directly calculate $\int_{C_2} \vec{G} \cdot d\vec{r}$. (Note that $y = 0$ everywhere on C_2 , which is helpful.)
- (c) Write $\oint_{C_1+C_2} \vec{G} \cdot d\vec{r} = \int_{C_1} \vec{G} \cdot d\vec{r} + \int_{C_2} \vec{G} \cdot d\vec{r}$ and use your answers to part (a) and (b) to finish off the problem and find the line integral of \vec{G} along C_1 .

Exercise 5

Evaluate the integral.

Calculating Area with Green's Theorem

Consider the following vector fields:

$$\vec{F}_1 = x \vec{j}, \quad \vec{F}_2 = -y \vec{i}, \quad \vec{F}_3 = -\frac{1}{2}y \vec{i} + \frac{1}{2}x \vec{j}.$$

What is $Q_x - P_y$ for each of these fields? Applying Green's Theorem to a region D , we get that

$$\oint_{\partial D} \vec{F}_1 \cdot d\vec{r} = \oint_{\partial D} \vec{F}_2 \cdot d\vec{r} = \oint_{\partial D} \vec{F}_3 \cdot d\vec{r} = \iint_D 1 \, dA = \text{Area of } D. (!)$$

Exercise 6

An ellipse with semi-major axis a and semi-minor axis b is parametrized by $x = a \cos t$, $y = b \sin t$ for $0 \leq t \leq 2\pi$. Use \vec{F}_3 to find the area inside this ellipse.

Exercise 7

Let C be the curve parametrized by $\vec{r}(t) = (t^2 - 3)\vec{i} + (t^3 - 4t + 1)\vec{j}$, $-2 \leq t \leq 2$. This is a closed loop. Use Green's Theorem and \vec{F}_1 to find the area inside this loop.

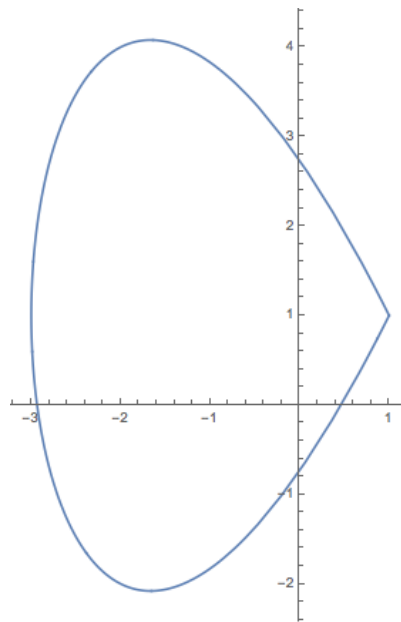


Figure 1: Exercise 7

Extended Versions of Green's Theorem

Although we have proved Green's Theorem only for the case where D is simple, we can now extend it to the case where D is a *finite union of simple regions*. For example, if D is the region shown in Figure 2, then we can write $D = D_1 \cup D_2$, where D_1 and D_2 are both simple. The boundary of D_1 is $C_1 + C_3$ and the boundary of D_2 is $C_2 + (-C_3)$. so, applying Green's Theorem to D_1 and D_2 separately, we get

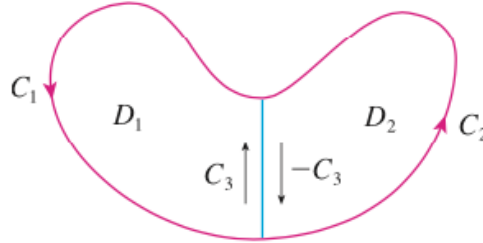


Figure 2

$$\oint_{C_1+C_3} Pdx + Qdy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\oint_{C_2+(-C_3)} Pdx + Qdy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Adding the two integrals above, we get,

$$\oint_{C_1+C_2} Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Note that we can similarly extend the theorem to regions that are finite *intersections* of simple regions.

Exercise 8

Evaluate

$$\oint_C y^2 dx + 3xy dy$$

where C is the boundary of the semiannular region D in the upper half plane between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Green's Theorem can be also extended to apply to regions with holes, that is, regions that are not simply-connected. Observe that the boundary C of the region D in Figure 3 consists of two simple closed curves C_1 and C_2 . We assume that these boundary curves are oriented so that the region D is always on the left as the curve C is traversed. Thus the positive direction is counterclockwise for the outer curve C_1 but clockwise for the inner curve C_2 . If we divide D into two regions D' and D'' by means of the lines shown in Figure 3 and then apply Green's Theorem to each of D' and D'' , we get

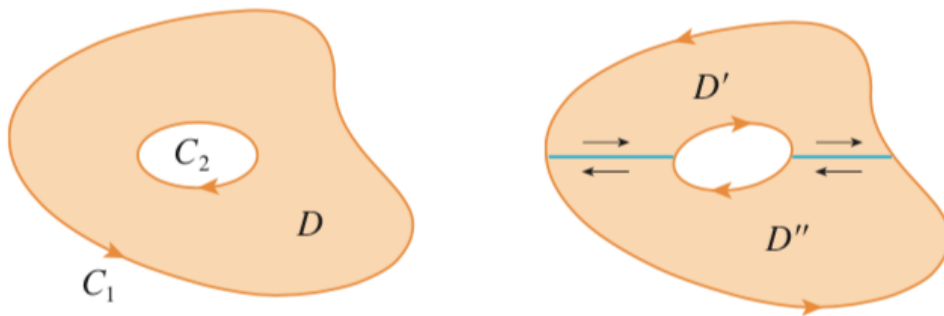


Figure 3

$$\begin{aligned} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA &= \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\ &= \oint_{\partial D'} (Pdx + Qdy) + \oint_{\partial D''} (Pdx + Qdy) \end{aligned}$$

Since the line integrals along the common boundary lines are in opposite directions, they cancel each other out and we get

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_{C_1} (Pdx + Qdy) + \oint_{C_2} (Pdx + Qdy) = \oint_{C_1+C_2} Pdx + Qdy$$

Exercise 9

If $\vec{F}(x, y) = \frac{-y\vec{i} + x\vec{j}}{x^2 + y^2}$, show that $\oint_C \vec{F} \cdot d\vec{r} = 2\pi$ for every positively oriented closed path C that encloses the origin.

Exercise 10

Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ where

$$\vec{F}(x, y) = \frac{2xy\vec{i} + (y^2 - x^2)\vec{j}}{(x^2 + y^2)^2}$$

and C is a positively oriented closed path that encloses the origin.

Appendix D

Sample Exams

Final Exam

ORDINARY DIFFERENTIAL EQUATIONS -- MATH 2208

SUBHADIP CHOWDHURY

Due: Noon, Dec 16, 2019

Instructions:

- This is a take-home exam. As such, your written arguments will be held to a higher standard than on a sit-down in-class exam. Please submit clear and carefully composed solutions, and explain the concepts you are using and the connections among them. As always, points may be deducted for any unjustified steps, and generous partial credit will be given if you explain your thought process to me.
- You may consult and use our course materials while taking this exam, including the textbook, class notes, your problem sets, and any of the worksheets on Blackboard. You are not allowed to use the web links (for eigenvalues or TD plane etc.) on worksheets.

You may **NOT** use a calculator or any graphing tool.

You may **NOT** use `dfield` or `pplane` or `Octave`.

You may **NOT** consult the internet or discuss problem specifics with other people.

You may email me to ask questions.

If you are not sure if some resource is allowed, please ask!

- When submitting your exam, print and staple this first page on top, and sign the “Honor Signature” to indicate that you followed Bowdoin’s Honor Code with respect to this exam.

Full Name: _____

Honor Signature: _____

Section Number	A	B	C	D	E	Total
Available Points	30	25	35	35	20	145
Your Score						

§A. Air Resistance and Terminal Velocity

A paratrooper jumps out of an airplane at a sufficiently high altitude (with initial downward velocity $v(0) = 0$), falls freely for 20 seconds and then opens her parachute. Assume her mass is m .

Her vertical motion is subject to two forces:

- a downward gravitational force $F_G = mg$ and
- a force F_R of air resistance that is proportional to velocity (so that $F_R = kv$) and of course directed opposite to the direction of motion of the body (i.e. upward).

Newton's law of motion says that the net force acting on the paratrooper is equal to her mass times her acceleration.

■ Question 1 (5 points).

Show that at time t , the velocity $v(t)$ of the paratrooper can be found by solving the ODE

$$\frac{dv}{dt} = -\rho v + g$$

where $\rho = \frac{k}{m}$.

■ Question 2 (10 points).

For the first 20 seconds, without the parachute opened, ρ is given by 0.5. Find $v(20)$. Assume $g = 10 \text{ m/s}^2$.

■ Question 3 (10 points).

With the parachute open, ρ increases to 1.5. Find a formula for $v(t)$ for $t > 20$.

■ Question 4 (5 points).

Show that as $t \rightarrow \infty$, the paratrooper's velocity does not increase indefinitely. Instead, it approaches a finite limiting velocity, called the *terminal velocity*. Find the terminal velocity of the paratrooper.

§B. Boundary Value Problems

We know that the solution of a second-order linear differential equation is uniquely determined by two initial conditions. In particular, the solution of the initial value problem

$$y'' + py' + qy = 0, \quad y(a) = 0, y'(a) = 0 \tag{1}$$

has a unique solution $y(t) \equiv 0$. The situation is quite radically different for a problem such as

$$y'' + py' + qy = 0, \quad y(a) = 0, y(b) = 0 \quad (2)$$

The difference between the problems in equations (1) and (2) is that in (2) the two conditions are imposed at two different points a and b with (say) $a < b$. In (2) we are to find a solution of the differential equation on the interval (a, b) that satisfies the conditions $y(a) = 0$ and $y(b) = 0$ at the endpoints of the interval. Such a problem is called an endpoint or *boundary value problem*.

Consider the Boundary Value Problem (BVP)

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0$$

■ Question 1 (5 points).

Let $\lambda = 3$. Show that the only solution to above BVP in this case is the trivial solution $y(t) = 0$ for all t .

■ Question 2 (4 points).

Show that if $\lambda = 0$, then the only solution to the BVP is the trivial solution.

■ Question 3 (6 points).

Show that if $\lambda < 0$, then the only solution to the BVP is the trivial solution.

■ Question 3 (10 points).

Show that if $\lambda > 0$, then the BVP has a non-trivial solution if and only if λ is of the form

$$\lambda = n^2, \quad n = 1, 2, 3, \dots$$

§C. Competing Species

The blue crab is native to the US Atlantic coast, but there is concern that the population is on the decline. The European green crab is an invasive species (recently introduced to the US in the ballast waters of ships), that competes with the blue crab. Assume that the interaction between the blue crab, x , and the green crab, y , is modeled by (up to some scaling):

$$\frac{dx}{dt} = x(100 - x) - 2xy$$

$$\frac{dy}{dt} = y(400 - 6y) - xy$$

■ Question 1 (8 points).

Find equation of the nullclines and the four equilibrium solutions (three of which are non-negative, representing biologically relevant values).

■ Question 2 (7 points).

Assume x and y are non-negative. Draw the direction field in the first quadrant of the phase plane. What does the model predict about the long-term fate of the two species?

■ Question 3 (8 points).

Suppose an intervention effort is launched to preserve the blue crab by harvesting a proportion h of the green crabs, so the equations modeling the system become (this is up to scaling, so h can be any positive number):

$$\begin{aligned}\frac{dx}{dt} &= x(100 - x) - 2xy \\ \frac{dy}{dt} &= y(400 - 6y) - xy - hy\end{aligned}$$

It would be nice if we could find harvesting values h so that the species can co-exist. Calculate the new equilibrium solutions and observe that for some values of h , the fourth equilibrium solution can be found at a point in the first quadrant (it will represent co-existence of the species, since both populations will have positive values). For what range of h values does the fourth equilibrium solution have positive coordinates? Find the exact h values.

■ Question 4 (12 points).

Suppose that you found in the previous problem that for $h_1 < h < h_2$ there is a fourth equilibrium solution in first quadrant of the phase plane. Show that, if $h_1 < h < h_2$ then the equilibrium solution where both blue and green crab co-exist is the **only** stable equilibrium solution among the four possible options. Thus, no matter what the initial value (as long as it is positive), the green crab and blue crab will co-exist. (yay!)

For this problem, you can assume that the Jacobian (associated to linearization) at each equilibrium point always has two real eigenvalues.

[HINT: Try to answer using the (T, D) -plane. Don't calculate the eigenvalues.]

[NOTE: The algebra will be a little messy in this question. So be patient during your calculation.]

§D. Damped Harmonic Oscillator with Sinusoidal Forcing

Consider a damped harmonic oscillator with sinusoidal forcing whose equation is given by

$$\frac{d^2y}{dt^2} + p \frac{dy}{dt} + y = \cos(\omega t) \quad (\star)$$

where

- $y(t)$ denotes the displacement at time t ,
- $p > 0$ is the damping constant,
- the spring constant is 1, and
- $\omega > 0$ is the forcing frequency.

■ Question 1 (9 points).

Find a particular solution $y_0(t)$ to above differential equation (\star) using the *method of undetermined coefficient*. Your solution should be in terms of ω and p .

■ Question 2 (6 points).

Suppose your solution $y_0(t)$ is of the form

$$y_0(t) = a \cos(\omega t) + b \sin(\omega t)$$

- (a) Suppose the polar form of the complex number $z = a + ib$ is given by $re^{i\theta}$. What's the polar form of $a - ib$ in terms of r and θ ?
- (b) Calculate the real part of $(a - ib)e^{i\omega t}$ in two different ways to show that the solution $y_0(t)$ can be also written as

$$y_0(t) = r \cos(\omega t - \theta)$$

■ Question 3 (10 points).

Show that regardless of initial conditions, all general solutions of the differential equation (\star) converge to $y_0(t)$ for large values of t .

[Hint: Show that the two-dimensional system of first order differential equations corresponding to the *associated homogeneous equation* always has a sink of some type at the origin. What do the general solutions of the nonhomogeneous system (\star) look like?]

■ Question 4 (3 points).

We conclude that if damping is present, in the long-term, every solution of (\star) oscillates with frequency ω and amplitude r . Write r in terms of a and b and consequently as a function of ω and p .

■ Question 5 (7 points).

Fix p and let $r(\omega)$ be the amplitude of the particular solution when the system is forced at frequency ω . We say that *Practical resonance* occurs when $r(\omega)$ achieves its maximum as a function of ω .

- (a) Show that if $p < \sqrt{2}$, then practical resonance occurs at $\omega = \sqrt{1 - \frac{p^2}{2}}$.
- (b) Show that if $p > \sqrt{2}$, then no practical resonance occurs.

§E. Existence and Uniqueness

Consider the differential equation $y' = f(t, y)$ where $f(t, y)$ is continuously differentiable in t and y i.e. the partial derivatives are continuous functions. Suppose $f(t, y)$ is a periodic function of t with period T i.e.

$$f(t + T, y) = f(t, y) \quad \text{for all } t \text{ and } y$$

and suppose there are constants p, q with $p < q$ such that

$$f(t, p) > 0, \quad f(t, q) < 0 \quad \text{for all } t$$

■ Question 1 (10 points).

- (a) If $y_p(t)$ is a solution curve such that $y_p(0) = p$, then explain why $y_p(T) > p$.
- (b) If $y_q(t)$ is a solution curve such that $y_q(0) = q$, then explain why $y_q(T) < q$.

[Hint: What does the direction field look like?]

■ Question 2 (10 points).

Show that there is a periodic solution $\tilde{y}(t)$ with period T such that $p < \tilde{y}(0) < q$.

[Hint: Why is it enough to find \tilde{y} such that $\tilde{y}(T) = \tilde{y}(0)$?]

Note: An example of such a function is $f(t, y) = \sin(t) - y$. However, you are not allowed choose a fixed example to prove above question statements. You have to prove that the statements are true for all functions satisfying the given assumptions.

Final Exam Part 1

LINEAR ALGEBRA -- MATH 2000

SUBHADIP CHOWDHURY

Due: May 17, 2019, 8:30 AM

Instructions:

- This is a take-home exam. As such, your written arguments will be held to a higher standard than on a sit-down in-class exam. Please submit clear and carefully composed solutions, and explain the concepts you are using and the connections among them. As always, points may be deducted for any unjustified steps, and generous partial credit will be given if you explain your thought process to me.
- You may consult and use our course materials while taking this exam, including the textbook, class notes, your problem sets, and any of the handouts or Mathematica code on Blackboard. If you use Mathematica, please print and attach your commands and output to your exam. If you are not sure if some resource is allowed, please ask! You may NOT consult the internet or discuss problem specifics with other people. You may email me to ask questions.
- when submitting your exam, staple this packet on top, and sign the "Honor Signature" to indicate that you followed Bowdoin's Honor Code with respect to this exam.

Full Name: _____

Honor Signature: _____

Total Points Available: 125

Section Number	1	2	3	4	5	6	7	Total
Available Points	18	20	12	20	20	8	27	125
Your Score								

§1. The Pell Sequence

The Fibonacci sequence is arguably the most famous integer sequence in Mathematics, making surprise appearances in everything from seashell patterns to the Parthenon. It is defined recursively as follows: each term of the sequence is equal to the sum of the previous two. A generalization of Fibonacci Sequence called c -Fibonacci sequence is defined as follows:

$$\begin{aligned} p_0 &= 1 \\ p_1 &= 1 \\ p_n &= cp_{n-1} + p_{n-2}, \quad n \geq 2 \end{aligned}$$

where c is any natural number. The 2-Fibonacci sequence is also known as the *Pell Sequence*. In this section, we will try to figure out the long term behaviour of the Pell sequence by calculating an explicit formula for p_n . In the following questions, we will assume $c = 2$.

Question 1 (3 points). For $n \geq 1$, define the vector \vec{u}_n as

$$\vec{u}_n = \begin{bmatrix} p_{n-1} \\ p_n \end{bmatrix}$$

Find a 2×2 matrix A such that

$$\vec{u}_{n+1} = A\vec{u}_n \text{ for all } n \geq 1.$$

We can conclude that

$$\begin{bmatrix} p_{n-1} \\ p_n \end{bmatrix} = \vec{u}_n = A^{n-1}\vec{u}_1 = A^{n-1} \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} = A^{n-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So if we can calculate A^{n-1} , we can easily give a formula for p_n .

Question 2 ((3+4+3+3+2) points). (a) Find the eigenvalue(s) of A .

(b) Find an eigenbasis.

(c) Write down the diagonalization $A = SDS^{-1}$ i.e. find S, D , and S^{-1} .

(d) Find A^{n-1} .

(e) Find \vec{u}_n and a formula for p_n .

Question 3 (Bonus 2 points). Show that

$$\lim_{n \rightarrow \infty} \frac{p_{n+1}}{p_n} = 1 + \sqrt{5}$$

The number $1 + \sqrt{5}$ is called the *Silver Ratio*.

§2. Nul space, Col space and Rank

Question 4 ((3+3+4+3) points). For some real matrix A , the following vectors form a basis for its column space and null space:

$$\text{Col}(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

$$\text{Nul}(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

- What is the size $m \times n$ of A ? What is $\text{rank}(A)$?
- What are the dimensions of $\text{Col}(A^T)$, $\text{Nul}(A^T)$, and $\text{Row}(A)$?
- Give a matrix A with this $\text{Col}(A)$ and $\text{Nul}(A)$. There are many possible answers.
- Give a vector \vec{b} for which $A\vec{x} = \vec{b}$ has at least one solution, and give all the solutions \vec{x} for your A from the previous part.

HINT: you should not have to do Row-reduction.

Question 5 ((4+3) points). Let A be a $m \times n$ matrix and let B be a $n \times p$ matrix.

- Explain why $\text{rank}(AB) \leq \text{rank}(A)$.

HINT: What is the relation between the column space of AB and that of A ?

- Explain why $\text{rank}(AB) \leq \text{rank}(B)$.

HINT: What can you say about $\text{rank}((AB)^T)$?

§3. Characteristic Polynomial

Question 6 (7 points). Let A be an 5×5 diagonalizable matrix and let

$$p_A(t) = c_0 + c_1t + c_2t^2 + c_3t^3 + c_4t^4 + t^5$$

be the characteristic polynomial of A . Show that

$$\det(A) = c_0 \quad \text{and} \quad \text{tr}(A) = -c_4.$$

Question 7 (5 points). If A is an invertible and diagonalizable $n \times n$ matrix, show that A^{-1} can be written as a linear combination of the matrices $A, A^2, A^3, \dots, A^{n-1}$.

HINT: Use the Cayley-Hamilton theorem from assignment 22, chapter 5 supplementary exercise 7.

§4. Matrix of a Linear Transformations with respect to a Basis

Question 8 ((1+5+6) points). Consider the vector space \mathbb{V} consisting of 3×3 matrices A such that $A^T = -A$. These type of matrices are called skew-symmetric.

- (a) Show that the diagonal entries of any skew-symmetric matrix are zero.
- (b) Find the dimension of \mathbb{V} and a basis \mathfrak{B} for \mathbb{V} .
- (c) Let $T : \mathbb{V} \rightarrow \mathbb{V}$ be a linear transformation defined by

$$T(M) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} M - M \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

for all $M \in \mathbb{V}$. Find the \mathfrak{B} -matrix of T for your choice of \mathfrak{B} from part (b).

Question 9 ((6+2) points). Consider the linear transformation $T : \mathbb{P}_2 \rightarrow \mathbb{P}_2$ defined as

$$T(f(x)) = f(1) + (x - 1)f'(1)$$

- (a) Consider the basis \mathfrak{B} of \mathbb{P}_2 given by $\mathfrak{B} = \{1, (x - 1), (x - 1)^2\}$. Find the \mathfrak{B} -matrix of T .
- (b) Is T an isomorphism?

§5. Orthogonality

Question 10 ((3+2+5+1) points). Let \vec{v} be a unit vector in \mathbb{R}^3 and let $H = I - 2\vec{v}\vec{v}^T$.

- (a) Show that $H^2 = I$.
- (b) Show that H is an orthogonal matrix.
- (c) Find the eigenvalues of H and describe the eigenspaces geometrically.
- (d) Can you give a geometric description of the linear transformation corresponding to H ?
HINT: What is $H\vec{u}$ for $\vec{u} \perp \vec{v}$?

Question 11 ((6+3) points). Let P be a 3×3 matrix that corresponds to the linear transformation that projects every vector in \mathbb{R}^3 orthogonally onto the plane $x + 2y + 2z = 0$.

- (a) Find an orthogonal basis of \mathbb{R}^3 consisting of three eigenvectors of P .
HINT: Investigate the geometry of the eigenspaces.
- (b) Find the matrix P .

§6. Vector Space and Dimension

Let $B = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ be a diagonal 3×3 matrix, and let V be the vector space defined as

$$V = \{A \in \mathbb{R}^{3 \times 3} \mid AB = BA\}$$

i.e. V consists of all 3×3 matrices A that commute with B .

Question 12 (8 points). Find the possible dimensions of V . In each case, give a basis of V .

HINT: Consider the following cases separately: (1) when $x = y = z$, (2) any two of x, y, z are equal but the third is not, (3) none of them are equal to each other.

§7. The Least-Square Approximation problem

One of the fundamental problems in Machine Learning is to find the best approximate solution to equations of the form $A\vec{x} = \vec{b}$ where A be a $m \times n$ matrix and \vec{b} is a $m \times 1$ vector and the system is possibly inconsistent. Recall that the space $\text{Col}(A)$ is a subspace of \mathbb{R}^m , defined as

$$\text{Col}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$$

So if $\vec{b} \notin \text{Col}(A)$, then $A\vec{x} = \vec{b}$ is not consistent, but we can still try to find an \vec{x} that will minimize the “error”. We define a *least-square solution* \vec{x}^* to be an $\vec{x} \in \mathbb{R}^n$ such that

$$\|A\vec{x}^* - \vec{b}\| \leq \|A\vec{x} - \vec{b}\|$$

for all $\vec{x} \in \mathbb{R}^n$. In other words, \vec{x}^* is such that $\vec{b}^* = A\vec{x}^*$ is the point in $\text{Col}(A)$ that is closest to \vec{b} .

Geometrically speaking, $\vec{b}^* = A\vec{x}^*$ is the orthogonal projection of \vec{b} onto $\text{Col}(A)$ i.e

$$\vec{b}^* = \text{proj}_{\text{Col}(A)} \vec{b}.$$

Question 13 ((2+2) points). (a) Explain why $\vec{b} - \vec{b}^*$ is in $(\text{Col}(A))^\perp$.

(b) Explain why part (a) is equivalent to the equation $A^T \vec{b} = A^T A \vec{x}^*$.

HINT: What is $(\text{Col}(A))^\perp$ equal to?

(c) If $A\vec{x} = \vec{b}$, explain why $\vec{b}^* = \vec{b}$.

The equation obtained in part (b) above is called the *Normal Equation* for $A\vec{x} = \vec{b}$. By definition, the Normal Equation is always consistent, however it may have more than one solution i.e. there may be more than one best approximation. One particular case of interest is when $\text{Nul}(A) = \{0\}$.

Question 14 ((8+3+2) points). Let A be an $m \times n$ matrix.

(a) Show that $\mathbf{Nul}(A) = \mathbf{Nul}(A^T A)$.

HINT: It is easy to show that $\mathbf{Nul}(A) \subseteq \mathbf{Nul}(A^T A)$. To show the opposite containment $\mathbf{Nul}(A^T A) \subseteq \mathbf{Nul}(A)$, use orthogonal complements.

(b) Show that $\mathbf{rank}(A) = \mathbf{rank}(A^T A)$.

(c) Show that if $\mathbf{Nul}(A) = \{\mathbf{0}\}$, then $A^T A$ is an $n \times n$ invertible matrix.

So if $\mathbf{Nul}(A) = \{\mathbf{0}\}$, we get $A^T A$ is invertible and in that case the normal equation has a unique solution

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b}. \quad (\star)$$

Scientists are often interested in fitting a function of a certain type, e.g. a polynomial to experimental data they have gathered. Since physical experiments often have unavoidable errors, there may not be an exact polynomial that satisfies given data-set.

Consider for example an experiment where we know that the quantity V is supposed to be a quadratic polynomial in variable T because of theoretical reasons. Let's write

$$V = a + bT + cT^2 = p(T)$$

During the experiment we note down a list of input versus output (T, V) data-points as follows:

Observation No.	T	V
1	-1	8
2	0	8
3	1	4
4	2	16

Our goal is to find a, b and c that will give the best approximate polynomial with least amount of error. In other words,

Question 15 (10 points). Use equation (\star) to find the least-square solution $\vec{x}^* = \begin{bmatrix} a^* \\ b^* \\ c^* \end{bmatrix}$ to the system of equations

$$V_i = p(T_i) \text{ for } i = 1, 2, 3, 4$$

where (T_i, V_i) is the data-point from observation i .

A picture of $p^*(T) = a^* + b^*T + c^*T^2$ is given below.

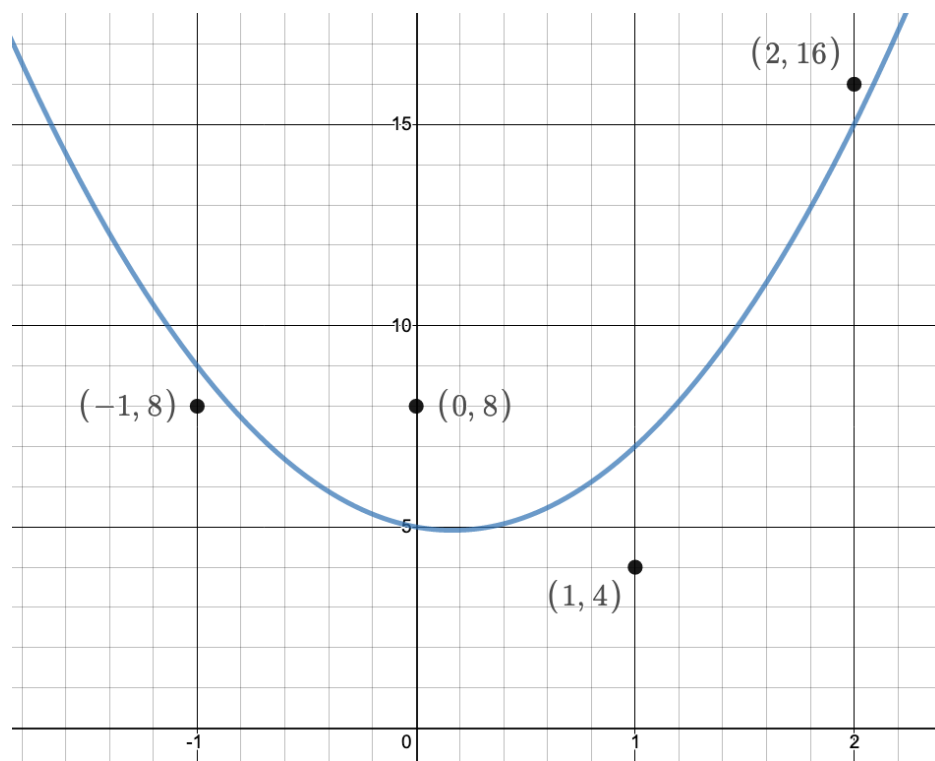


Figure 1

INSTRUCTIONS:

- Please show ALL your work! Answers without supporting justification will not be given credit.
- Answer the questions in the blue books provided.
- Write legibly and clearly mark the answer.
- Please note that use of any books or notes is not allowed. You are allowed to use the two pages of handwritten letter-sized note that you brought. Use of calculators are not allowed.
- If you write down the correct formula/procedure to find an answer, you will get some partial credit regardless of whether you evaluated the exact values or not.
- Unless otherwise specified, you may use any valid method to solve a problem.

Full Name: _____

Question	Points	Score
1	12	
2	12	
3	16	
4	4	
5	4	
6	4	
Total:	52	

This exam has 6 questions, for a total of 52 points.
The maximum possible point for each problem is given on the right side of the problem.

1. For each of the following statements, find out whether it is '**always true**', '**sometimes true**', or '**always false**'. Give a brief explanation for your answer. If you think an answer is 'sometimes true', give an example or criterion when it's false.

(a) If we have **2020** vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{2020} \in \mathbb{R}^{2019}$, then each \vec{v}_i can be written as a linear combination of the other **2019** vectors. 3

(b) If V_k denotes the set consisting of all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in \mathbb{R}^3 such that $xyz = k$ for some real constant k , then V_k is a subspace of \mathbb{R}^3 . 3

(c) If all the diagonal entries of an $n \times n$ matrix A are odd integers and all other entries are even integers, then A is an invertible matrix. 3

(d) If A and B are invertible $n \times n$ matrices then AB and BA are similar matrices. 3

2. Is it possible or impossible to find examples of each of the following? If possible, please provide an example. If impossible, please explain why.

(a) A real number t such that $A = \begin{pmatrix} 2 & t & 0 \\ t & 2 & t \\ 0 & t & 2 \end{pmatrix}$ has eigenvalue 0. 3

(b) A square matrix whose nullspace is $\text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. 3

(c) Two real invertible 3×3 matrices A and S such that $S^T A S = -A$. 3

(d) Two 3×3 nonzero diagonalizable matrices that have same eigenvalues but are not similar. 3

3. Choose the correct answer(s) from the given choices. No explanation is necessary.

There may be more than one correct choice. You will get partial credit for circling correct but incomplete set of choices, full credit for correct and complete set of choices, and zero credit if you circle any incorrect choice.

(a) Consider the following operations on \mathbb{P}_2 . Which of them is/are linear transformation(s)? 4

$\alpha.$ $T(p(x)) = p(x) - p(1)$

$\beta.$ $T(p(x)) = p(x) - 1$

$\gamma.$ $T(p(x)) = x - p(1)$

$\delta.$ $T(p(x)) = x^2 p\left(\frac{1}{x}\right)$

(b) Which of the following statements is/are true for all $n \times n$ matrices A and B ? 4

α . $\text{Col}(A) = \text{Col}(AB)$

β . $\text{Col}(A) = \text{Col}(BA)$

γ . $\text{Nul}(A) = \text{Nul}(AB)$

δ . $\text{Nul}(A) = \text{Nul}(BA)$

(c) Suppose A and B are two 4×4 matrices and let $\text{rank}(A) = 2$, $\text{rank}(B) = 3$. Then which of the following are possible values of $\text{rank}(A + B)$? 4

α . 0

β . 2

γ . 3

δ . 5

(d) Consider the system of linear equations 4

$$\begin{cases} x + y + z = 1 \\ 2x \quad + z = 2 \\ -x + y + az = b \end{cases}$$

Then,

α . If $a \neq 0$ then the system has exactly one solution regardless of the value of b .

β . If $a = 0$ then the system has no solution regardless of the value of b .

γ . If $a = 0$ then the system has no solution for exactly one value of b .

δ . If $a = 0$ then the system has infinitely many solutions for exactly one value of b .

4. If $A = Q^T D Q$ for a diagonal matrix D and an orthogonal matrix Q , then explain why the eigenvectors of A are the rows of Q . 4

5. Suppose A is a 3×3 matrix with eigenvalues $\lambda = 0, 1, -1$ and corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Find a basis of $\text{Col}(A)$ in terms of \vec{v}_1, \vec{v}_2 , and \vec{v}_3 . 4

6. The matrix 4

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

is diagonalizable and $H^2 = 4I$. Find the eigenvalues of H and their algebraic multiplicities.

HINT: You should not need to calculate any determinant.

INSTRUCTIONS:

- Please show ALL your work! Answers without supporting justification will not be given credit.
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- Start every question on a new page.
- Write legibly and clearly mark the answer.
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- If you write down the correct formula for an answer, you will get some partial credit regardless of whether you evaluated the exact values or not.
- Unless otherwise specified, you may use any valid method to solve a problem.

Full Name: _____

Question	Points	Score
1	10	
2	4	
3	4	
4	4	
5	5	
6	10	
7	14	
8	12	
9	8	
10	9	
11	10	
Total:	90	

This exam has 11 questions, for a total of 90 points.
The maximum possible point for each problem is given on the right side of the problem.

1. For the following problems, determine whether each statement is 'True' or 'False'. If it is true, explain your reasoning **briefly**. If it is false, explain why or give a counterexample that disproves the statement.

(a) The scalar triple product $\vec{u} \cdot (\vec{v} \times \vec{w})$ between three vectors \vec{u}, \vec{v} and \vec{w} is zero if and only if two or more of the 3 vectors are parallel. 2

(b) If $\vec{u} \times \vec{v} = 2019\vec{w}$ then $(\text{Proj}_{\vec{v}} \vec{u}) \times (\text{Proj}_{\vec{u}} \vec{v}) = \lambda \vec{w}$ for some real number λ . 2

(c) Every flow line of the vector field $\vec{F}(x, y) = x\hat{i} - y\hat{j}$ lies on a level curve of the function $f(x, y) = xy$. 2

(d) Different parameterizations of the same curve result in identical tangent vectors at a given point on the curve. 2

(e) For a particle moving along the flow line $\vec{r}(t)$ of a gradient vector field $\vec{F} = \nabla f$, the velocity $\vec{r}'(t)$ is zero at a stationary critical point of f . 2

2. Compute the line integral of $\vec{F}(x, y) = \langle 5y + 3y^2, 6xy + y^5 \rangle$ along the boundary of the green *Cyclops* region given in figure 1. There are four boundary curves, *oriented as shown in the picture*: a large ellipse of area 16, two circles of area 2 (the eye) and 1 (the eyeball) as well as a small ellipse (the mouth) of area 3. 4

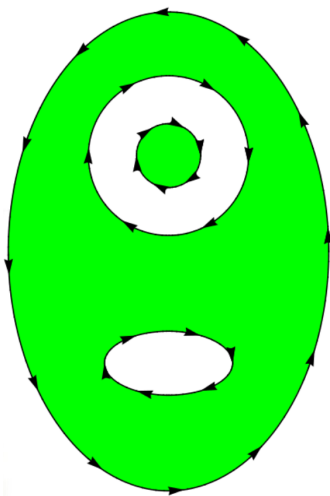


Figure 1

4

3. Match the Vector Fields in figure 2 with their formula. No explanation is necessary.

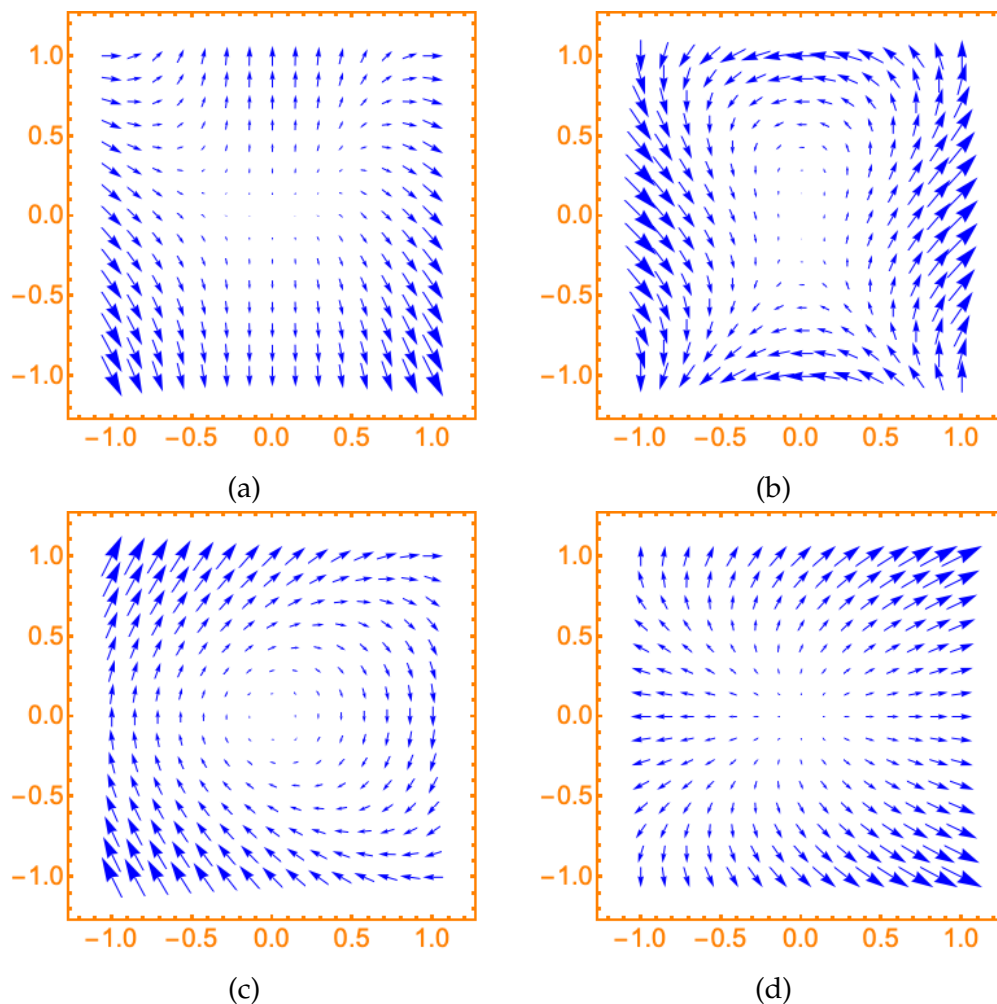


Figure 2

$$(1) \langle y, y^2 - x \rangle, \quad (2) \langle x^2, y - x^2 \rangle, \quad (3) \langle x^2 - y^2, x \rangle, \quad (4) \langle x + y^2, y \rangle,$$

4. Let $L(x, y)$ be the linearization of a function $f(x, y)$ and let $\vec{r}(t)$ be a parameterized curve. Fill in the blanks with following three options. Options may be used more than once. No explanation is necessary.

• $\nabla f(\vec{r}(0))$

• $\vec{r}'(0)$

• $\vec{r}(t) - \vec{r}(0)$

(a) $L(\vec{r}(t)) = f(\vec{r}(0)) + \boxed{} \cdot \boxed{}$

2

(b) $\left. \frac{d}{dt} f(\vec{r}(t)) \right|_{t=0} = \boxed{} \cdot \boxed{}$

2

5. Match each of the following iterated integrals with its domain of integration. No explanation is necessary.

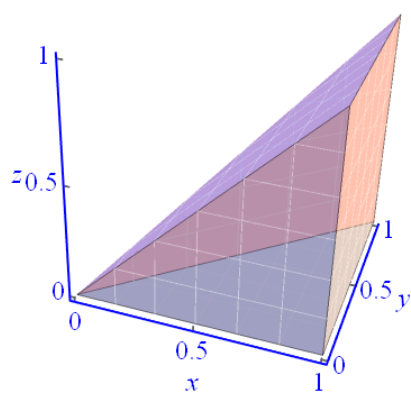
1. $\int_0^1 \int_y^1 \int_0^y f(x, y, z) dz dx dy$

2. $\int_0^1 \int_0^y \int_y^1 f(x, y, z) dz dx dy$

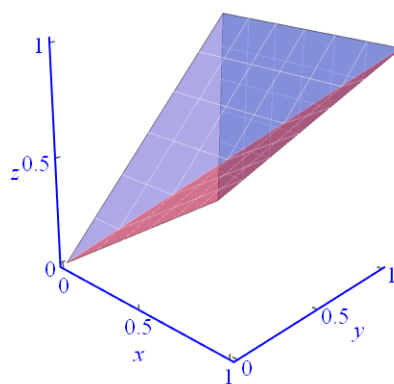
3. $\int_0^1 \int_y^1 \int_0^x f(x, y, z) dz dx dy$

4. $\int_0^1 \int_0^y \int_x^1 f(x, y, z) dz dx dy$

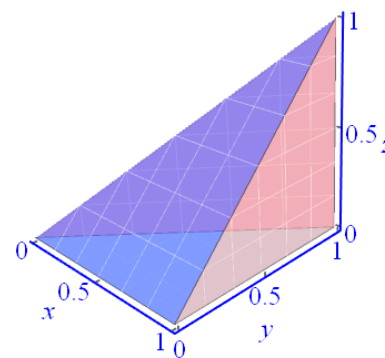
5. $\int_0^1 \int_0^y \int_x^y f(x, y, z) dz dx dy$



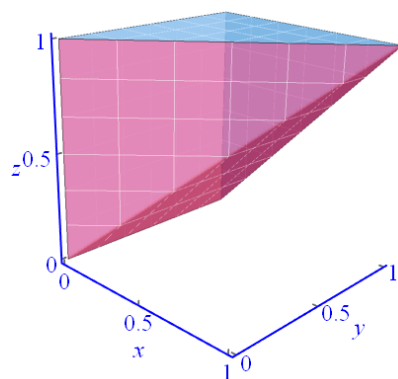
(a)



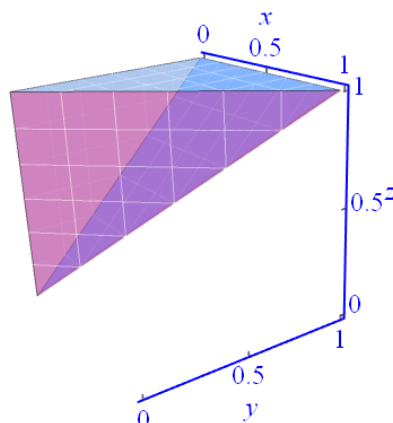
(b)



(c)



(d)



(e)

Figure 3: The five regions

6. Find the line integral of the vector field

10

$$\vec{F} = \langle 45x^4y^2 - 2y^6 + 3, 18x^5y - 12xy^5 + 7 \rangle$$

along the curve

$$r = 2 \sin(\theta) + \sqrt{|5 \cos^2 \theta - 1|}$$

from $\theta = 0$ to $\theta = 3\pi/2$. A polar plot of the curve is given below in figure 4.

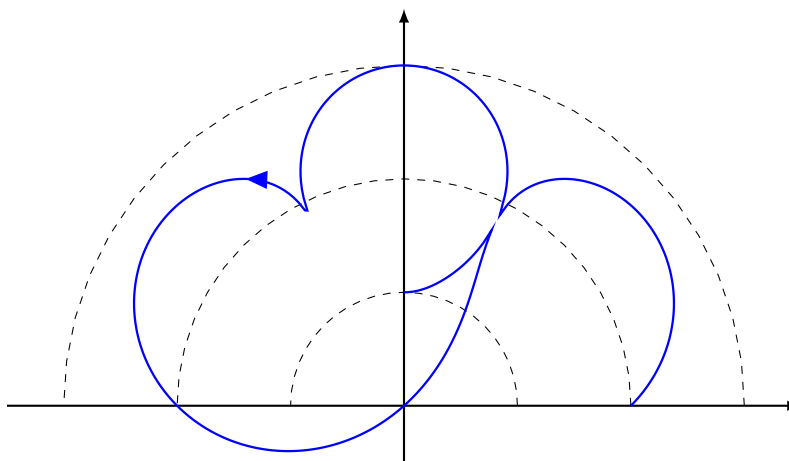


Figure 4

7. Let \vec{F} be the vector field

14

$$\vec{F} = \langle xy^2, -x^2y \rangle$$

Let C be the blue curve given in figure 5. It consists of two circular arcs, one horizontal line segment, and one vertical line segment. Calculate the line integral $\oint_C \vec{F} \cdot d\vec{r}$ using Green's theorem.

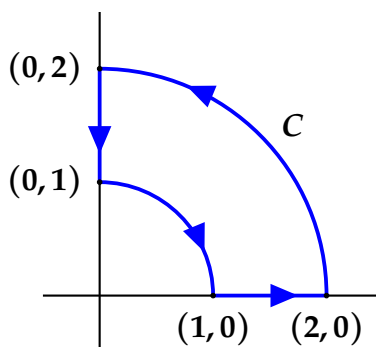


Figure 5

8. (a) Find the equations of the tangent planes at the points $A = (1, 1, 2)$ and $B = (-2, 1, 5)$ to the surface

$$x^2 + y^2 - z = 0.$$

- (b) Check that $(-1/2, 1, -1)$ lies on both of the tangent planes above. Find the parametric equation of the line of intersection of these two tangent planes.

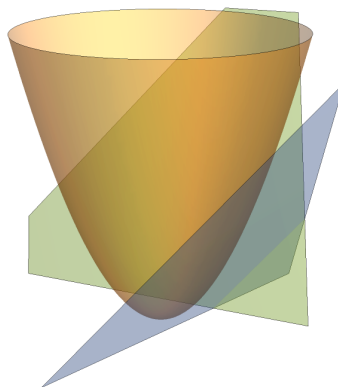


Figure 6

9. Use the Lagrange multipliers to find the points (x, y) where

$$f(x, y) = 4x^2y + y^3$$

achieves its maximum and minimum on the ellipse

$$g(x, y) = 2x^2 + y^2 = 6.$$

10. The function $F(x, y) = x^2y - 4xy + 3x^2 + 12y^2$ has three critical points, at $x = 0, x = 1$, and $x = 5$.

- (a) Find the values of y at these three critical points.
 (b) Classify each critical point as a maximum, minimum, or saddle point.

11. Let

$$\iint_R f(x, y) dA = \int_1^2 \int_{2-x}^{\sqrt{2-x}} \frac{1}{2y^3 - 3y^2 + 5} dy dx$$

- (a) Sketch the region R .
 (b) Rewrite the double integral as an iterated integral with the order interchanged.
 (c) Evaluate the integral.