Math 1800 Project 4: Running Circles around Circles

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In this project we investigate families of curves, called *hypocycloids* and *epicycloids*, that are generated by the motion of a point on a circle that rolls inside or outside another circle. Note that the answers to questions (c), (d), (e), and (f) are open-ended. Try to give a one or two sentence description of the observations you make. There no *one* correct answer, it may vary from student to student.

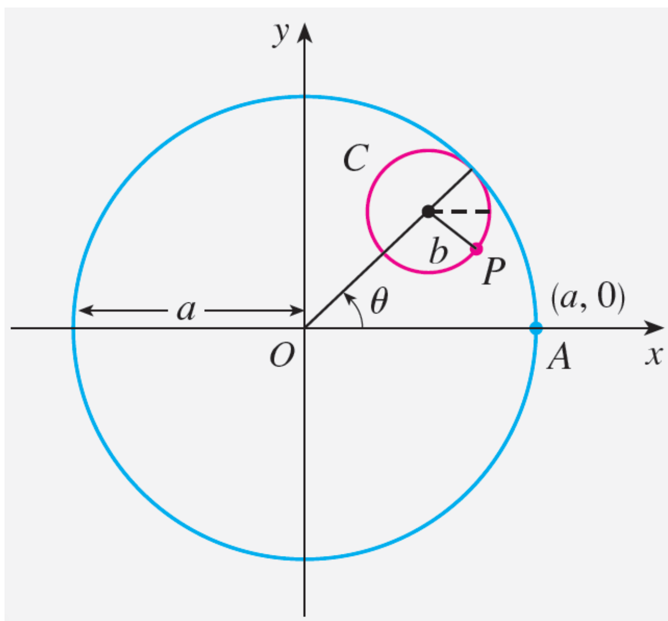

Figure 1

(a) A *hypocycloid* is a curve traced out by a fixed point P on a circle C of radius b as C rolls on the inside of a circle with center O and radius a. Show that if the initial position of P is $(a, 0)$ and the parameter θ is chosen as in the figure 1, then parametric equations of the hypocycloid are

$$x = (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}\theta\right)$$

$$y = (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}\theta\right)$$

[Hint: First find the parametric coordinates for P if the red circle C was centered at the origin. Then find the parametric equation of the path of the center of C. Add these two to get the answer.]

(b) Use the attached Mathematica notebook to draw the graphs of hypocycloids with a a positive integer and $b = 1$. How does the value of a affect the graph?

(c) Look up the formula for $\cos(3\theta)$ and $\sin(3\theta)$ in terms of $\cos\theta$ and $\sin\theta$. Use these to show that if we take $a = 4$ and $b = 1$, then the parametric equations of the hypocycloid reduce to

$$x = 4\cos^3\theta \qquad y = 4\sin^3\theta$$

This curve is called a **hypocycloid of four cusps**, or an **astroid**.

(d) Now try $b = 1$ and $a = n/d$, a fraction where n and d have no common factor.

(i) First let $n = 1$ and try to determine graphically the effect of the denominator d on the shape of the graph. Try $d = 2, 3, 4, 10$. What do you observe?

(ii) Then let n vary while keeping d constant. Let $d = 2$ and $n = 3, 5, 7$. What do you observe?

(iii) What happens when $n = d + 1$?

(iv) As d increases, we must expand the range of θ in order to get the full closed curve. What can you say about the relation between the range and d?

(v) Try the values $a = \frac{3}{2}, \frac{5}{4}, \frac{11}{10}$ etc.

(e) What happens if $b = 1$ and a is irrational? Experiment with an irrational number like $\sqrt{2}$ or $\frac{e}{3}$. There are two distinct possibilities depending on whether a is less than on bigger than 1. Can you see what those are?

(f) Take larger and larger values for θ and speculate on what would happen if we were to graph the hypocycloid for all real values of θ.

(g) If the circle C rolls on the outside of the fixed circle, the curve traced out by P is called an *epicycloid*. Find parametric equations for the epicycloid.