# **Assignment 8** (1/22)

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- *This homework is due at the beginning of class on* ***Friday*** 1/26. *You may cite results from class as appropriate. Unless otherwise stated, you must provide a complete explanation for your solutions, not simply an answer. You are encouraged to work together on these problems, but you must write up your solutions independently.*
- *Hand in the exercises only, not the reading material part. You are encouraged to think about the exercises marked with a* ($\star$) *or* (†) *if you have time, but you don't need to hand them in. If you correctly solve a* (†)*-marked problem, you will get a candy!*
- *Remember that you can always use the result of the previous assignment problems without proof to solve the new assignment problems.*
- *We are currently covering Chapter* 14 *from Stewart.*

## Important Points and Reading Materials

- Gradient and Tangent Line/Plane
  - We proved that the gradient vector is perpendicular to level surfaces. Consequently, given a surface of the form $f(x,y,z)=k$, we can use the gradient vector to determine the equation of the normal line at any point. Note that the normal line at a point $P$ on the surface is the straight line perpendicular to the surface passing through $P$.

    For example, given surface $x^2+yz+zx^3=3$, the normal line at $(1,1,1)$ is the straight line that passes through $(1,1,1)$ and is parallel to $\nabla f(1,1,1)=\langle 2x+3zx^2, z, y+x^3\rangle\,|_{(1,1,1)}=\langle 5,1,2\rangle$. So equation of the straight line is
    $$\frac{x-1}{5}=\frac{y-1}{1}=\frac{z-1}{2}$$
    And equation of the tangent plane at $(1,1,1)$ to the surface is the plane whose normal vector is $\langle 5,1,2\rangle$. Thus its equation is
    $$5(x-1)+1(y-1)+2(z-1)=0$$
  - The same holds for a function of 2 variables where the gradient vector and the tangent line to the level curve are perpendicular.
- Lagrange Multipliers
  - Understand why a point $(x_0,y_0,z_0)$ on $g(x,y,z)=k$ cannot be optimal for $f(x,y,z)$ if $\nabla f$ and $\nabla g$ are not parallel.
  - Observe that if $\nabla g=0$, the two vectors $\nabla f$ and $\nabla g$ are always parallel; so Lagrange multiplier method is useless.
  - Understand that finding min/max for a function of three variable $f(x,y,z)$ with a constraint $g(x,y,x)=k$ using Lagrange multiplier will involve solving a system of 4 equations. $\nabla f=\lambda\nabla g$ is a system of 3 equations and $g(x,y,z)=k$ is the fourth.
  - Make sure you know how to solve these equations.
    - $*$ Do NOT neglect this fact when studying for the exam. It is extremely easy to look over a few examples and convince yourself that you know how to do this, and then realize on the test that you have no clue what to do after writing down the equations.
    - $*$ These equations are non-linear, so this is not simply a matter of memorizing a few steps and applying that to any problem you encounter. You need to think of each one as its own problem, and need to come up with a new approach each time.
    - $*$ Remember basic tricks for solving equations. Try combining some equations to eliminate a variable, or otherwise simplify things. Try solving for some variables in terms of another variable (say, express $x$ and $y$ in terms of $\lambda$) and then substitute that into another equation.

* Remember that you can only divide by things that are nonzero. That means whenever you are dividing equations, you need to check that some variables are nonzero. In some cases, there max be maxima/minima at points where a variable is zero, which you would miss if you just divided without checking this. Similarly make sure to take $\pm$ when taking square roots.
* For example if you have an equation of the form $xy = \lambda x^2$, that means either $x = 0$ or $y = \lambda x$. You need to consider both cases as potential solutions.
* The ONLY way to get good at this is by doing lots of examples. This assignment cannot possibly cover all the different kind of exercises one can come up with regarding Lagrange multipliers. So do some more exercises from the book yourself.

# Problems

Please note that for this assignment you should work out the star problems on you own as practice for the midterm, even if you don't submit them.

## Exercise 1

If $g(x,y) = x^2 + y^2 - 4x$, find the gradient vector $\nabla g(1,2)$ and use it to find the equation of the tangent line to the level curve $g(x,y) = 1$ at the point $(1,2)$. Sketch the level curve, the tangent line and the gradient vector.

## Exercise 2

Where does the normal line to the paraboloid $z = x^2 + y^2$ at the point $(1,1,2)$ intersect the paraboloid a second time?

## Exercise 3

Find the equation of the tangent plane and the normal line to the surface $x + y + z = e^{xyz}$ at $(0,0,1)$.

## Exercise 4

Find the parametric equation for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$ at the point $(-1,1,2)$.

## Exercise 5

Suppose the curve given by $\vec{r}(t) = \langle \cos(\pi t), \sin(\pi t), t \rangle$ intersects the paraboloid $z = x^2 + y^2$ at a point $P = (x_0, y_0, z_0)$.

1. Find the coordinates of $P$.
2. ($\star$) Find equation of the tangent plane to the paraboloid at $P$.
3. ($\star$) What is the equation of the tangent line to the curve $\vec{r}(t)$ at $P$?
4. What is the angle of intersection between the curve and the paraboloid? This is the angle between the tangent line in part (3) and the plane in part (2). Leave the answer in $\arccos(\theta)$ form, don't use a calculator.

## Exercise 6

Find the **angle** at which the curve

$$\vec{r}(t) = \cos(\pi t)\,\hat{\imath} + \sin(\pi t)\,\hat{\jmath} + 2t\,\hat{k}, \quad t \in \mathbb{R}$$

intersects the surface

$$2x^2 + z^2 - xy + 2xz - 1 = 0$$

at the point $(0,1,1)$. Leave the answer in $\arccos(\theta)$ form, don't use a calculator.

**Exercise 7**

Use Lagrange multipliers to find the extreme values of the following functions subject to the given constraints.

1. ($\star$) $F(x,y,z) = e^{xyz}$ subject to $2x^2 + y^2 + z^2 = 24$
2. ($\star$) $F(x,y,z) = x^2 + y^2 + z^2$ subject to $x^4 + y^4 + z^4 = 1$
3. $F(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 1$ subject to $x^2 + y^2 = 2$
4. $F(x,y) = 2y + x$ subject to $y^2 + xy - 1 = 0$

[Use Wolfram Alpha to double check your answer, email me if you have questions.]