Math 2208: Ordinary Differential Equations

Lecture 14 Worksheet

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Title: Harmonic Oscillators

Summary: We will examine the standard second order constant-coefficient ODE $y'' + py' + qy = 0$ more closely now that we have completed the analysis of the first order system of 2 linear ODEs.

§A. Quick Recap of 2nd Order Linear ODEs

You found in your midterm that the 2nd order constant coefficient ODE $y'' + py' + qy = 0$, where p and q are real-numbered constants, can be written as the linear system

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -pv - qy$$

with coefficient matrix $A = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix}$ and characteristic polynomial $\lambda^2 + p\lambda + q = 0$ which implied the eigenvalues are

Also, if we know the eigenvalues λ_1 and λ_2 , then we know that the eigenvectors of matrix A are $\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$. So we can write the general solution to the system of linear ODEs and get a formula for

$$\vec{r}(t) = \begin{bmatrix} y(t) \\ v(t) \end{bmatrix} =$$

Question 1.

What is the general (real-valued) solution to the second order linear ODE?

§B. Harmonic Motion

The equation

$$my'' + by' + ky = 0 \iff y'' + \frac{b}{m}y' + \frac{k}{m}y = 0$$

represents the displacement $y(t)$ of a mass-spring system where m is the mass, b is the damping constant and k is the spring constant. This type of motion is known as harmonic motion.

Question 2.

First consider the case $b = 0$. This is called a *Simple Harmonic Motion*, also known as the *Undamped* Harmonic Motion.

1. Check that the eigenvalues of associated linear system are complex numbers whose real parts are equal to 0.
2. Write down the general formula for $y(t)$.
3. Check that it is a periodic function of t. What is the period?

Question 3 (Making a clock using Mass-Spring System).

Suppose we wish to make a clock using a mass and a spring sliding on a table. We arrange for the clock to "tick" whenever the mass crosses $y = 0$. We use a spring with spring constant $k = 2$. If we assume there is no friction or damping ($b = 0$), then what mass m must be attached to the spring so that its natural period is one time unit?

Question 4.

If $b \neq 0$, we get a *Damped* Harmonic Motion. Then depending on different values of b, k, and m we will have different behavior for the solution curves as the determinant of the characteristic polynomial changes.

1. Find the determinant D of the characteristic polynomial of the associated system of linear ODEs in terms of m, b, and k.
2. Fill out the following table. We are considering three cases. You can use `pplane` or `Octave`.

	D < 0	D = 0	D > 0
Conditions on m, b, k (Note that $m > 0$)			
Eigenvalues are Real/Complex?			
Number of Eigenvalues			
Does the solution curve y vs. t oscillate? If yes, what's the period?			
Kind of Damping	*Underdamped*	*Critically damped*	*Overdamped*
Equilibrium Type of Phase Portrait in (y, y')-phase plane			

§C. Nonlinear Pendulum

Consider a pendulum consisting of a mass attached to a rigid rod. When the amplitude of motion of the ball is small enough, we make the approximation $\sin(\theta) \approx \theta$ and Hooke's law says the restoring force is proportional to the displacement. This results in the equation

$$y'' = -y$$

But when the amplitudes get bigger, the physics always becomes nonlinear.

Consider an idealized point mass moving in a circle at the end of a rigid weightless bar. The corresponding ODE is

$$y'' = -\sin(y)$$

where $y(t)$ represents the angle from the vertical in radians at time t and constants have been normalized to 1. We are going to try to understand the motion using phase portrait in (y, y')-plane. We are also going to assume the initial condition $y(0) = 0$ i.e the pendulum is vertical at the beginning.

1. Write down the associated system of two first order ODEs.
2. Find the nullclines and the equilibrium points. The equilibrium points are where the nullclines intersect.
3. Use `pplane` to draw some sample solution curves. What would be the best description of the kind of qualitative behavior you observe locally around the equilibrium points?
4. How does the long term behavior depend on $y'(0)$?
5. How would you physically interpret the curves?
6. Justify the following statement using the phase portrait: "In the absence of damping, a pendulum that swings over once swings over infinitely many times."
7. Does there exist a initial value of $y'(0)$ (where $y(0) = 0$) such that the pendulum doesn't exhibit a periodic behavior over time? How would you physically interpret this?