Research Statement

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Introduction

I work in low dimensional dynamics, especially the theory of nonabelian group actions on the circle. I have also made contributions to the theory of formal languages, using topological methods.

Extremal actions of free groups on the circle (§1) . Let $\text{Homeo}^{\sim}_{+}(S^1)$ denote the universal central extension of $\text{Homeo}_{+}(S^1)$, the group of orientation-preserving homeomorphisms of the circle. The Poincaré rotation number, denoted $\text{rot}^{\sim}$, is a real-valued semi-conjugacy invariant function on $\text{Homeo}^{\sim}_{+}(S^1)$. If $F := \langle a.b \rangle$ is a free group and $w \in F$, given real numbers r, s, we would like to understand $R(w; r, s) := \sup_{\rho} \text{rot}^{\sim}(\rho(w))$, taken over all $\rho : F \to \text{Homeo}^{\sim}_{+}(S^1)$ for which $\text{rot}^{\sim}(\rho(a)) = r$ and $\text{rot}^{\sim}(\rho(b)) = s$.

If w is in the semigroup generated by a and b, plotting the graph of R against r and s gives a staircase like structure [3], called a Ziggurat. When $r = p/q$ is rational, it turns out that $R(w; p/q, t) = h_a(w)p/q + h_b(w)$ on an entire interval $t \in [1 - \text{fr}_w(p/q), 1)$, where h_a and h_b count the number a's and b's in w, and $\text{fr}_w(p/q)$ is a rational number, a priori depending on p/q in a complicated way. In [6], I proved that $\text{fr}_w(p/q) = \frac{1}{\sigma_w(g) \cdot q}$, where $g = gcd(q, h_b(w))$ and $\sigma_w(g) \cdot g \in \mathbb{Z}$; i.e. fr_w satisfies a *power law*; and the maximal intervals are analogues of Arnol'd tongues in this context (see §1.1). This formula also explains the periodic nature of the ziggurat near the slopes.

As my ongoing and future work in this project, I have plan to prove further rigidity and stability properties of this graph (e.g. the slippery conjecture [3]). Additionally, I have proved and conjectured similar rationality results in the general case of semi-positive and arbitrary words (see §1.2) using ideas from one dimensional dynamics and theory of Diophantine approximations.

A new proof of Salvati's theorem in formal language theory (§2) . A finitely generated group $G = \langle \Sigma \mid R \rangle$ is called context-free (CF) if its word problem $\{w \in \Sigma^\star \mid w =_R 1\}$ has a presentation where a set of replacement rules produce all possible strings in it via recursion. Such hierarchical properties in mathematical linguistics often relate to combinatorial and algebraic properties of groups and their Cayley graph in significant ways. Building on works of Salvati [17] and Nederhof [16], my results in this area give a new proof that the language $\mathbf{O}_2$ (and hence the rationally equivalent $\mathbf{MIX}_3 = \{w \in \{a, b, c\}^* \mid |w|_a = |w|_b = |w|_c\}$) associated to the group $\mathbb{Z}^2$ is generated by a 2-multiple context free grammar, a generalization of CFG. After recasting the statement in topological terms, I give a homological proof, where the existence of a simplification (by which the language may inductively be proved to be mcf) follows by a degree argument for a map on a certain configuration space (see §2.1). This technique opens up the possibility of proving the result in higher dimension with sharper bounds on the arity as conjectured by Nederhof [15], and consequently sheds light on the word problem in $\mathbb{Z}^n$.

Opportunities for undergraduate research. As my research interests include various open but potentially tractable topics, I believe that there are numerous opportunities for undergraduate researchers to meaningfully contribute in this area. My projects use tools and technique that are easily accessible to a student who has taken a course in linear algebra, real analysis, group theory, and topology. In particular, I am currently engaged with a talented senior student at my current position to pursue an independent study course in dynamics, which I hope will lead to some fruitful original work. In the future, I would be thrilled to have the chance to advise undergraduate theses in geometry, topology, and dynamics, as well as guide scholars in any mathematics-related area.

1. Nonabelian group actions on a circle and extremal rotation number

The main goal of this research project is to study nonlinear actions of finitely generated groups on manifolds, in particular a circle, using the proper analog for character varieties. Since the real-valued rotation number $\text{rot}^\sim : \text{Homeo}_+^\sim(S^1) \to \mathbb{R}$ defined as

$$\text{rot}^\sim(g) = \lim_{n\to\infty} \frac{g^n(0)}{n}$$

is a semi-conjugacy class invariant function (see e.g. Ghys [8] or Bucher-Frigerio-Hartnick [2]), it is reasonable to ask how $\text{rot}^\sim$ is constrained on the image of a finite subset of Γ under a homomorphism to $\text{Homeo}_+^\sim(S^1)$. Given $F = \langle a, b\rangle$, any word $w \in F$ and some representation $\rho : F \to \text{Homeo}_+^\sim(S^1)$, our main motivating question is as follows:

Research Program 1.1. For fixed values of $\text{rot}^\sim(\rho(a)) =: r$ and $\text{rot}^\sim(\rho(b)) =: s$, what is the set $X(w; r, s)$ of possible rotation number of w?

It is easy to prove that $X(w; r, s)$ is a compact interval whose extrema is achieved for some ρ and if we define$R(w; r, s) = \max\{X(w; r, s)\}$ then $\min\{X(w; r, s)\} = -R(w; -r, -s)$. So all the information about the program can be recovered by studying the function R, which we undertake in this project.

1.1. **Positive words and Calegari-Walker ziggurats.**

Theorem 1.2 (Calegari-Walker [3, Theorem 3.4,3.7])**.** *Suppose w is positive i.e. w is in the semigroup generated by a and b (and not a power of a or b), and suppose r and s are rational. Then*

1. *$R(w; r, s)$ is rational with denominator no bigger than the smaller of the denominators of r and s; and*
2. *there is some $\epsilon(r, s) > 0$ so that $R(w; \cdot, \cdot)$ is constant on $[r, r + \epsilon) \times [s, s \times \epsilon)$.*

Furthermore, when r and s are rational and w is positive, Calegari-Walker give an explicit combinatorial algorithm to compute $R(w; r, s)$; it is the existence and properties of this algorithm that proves Theorem 1.2. Computer implementation of this algorithm allows one to draw pictures of the graph of R (restricted to $[0, 1) \times [0, 1)$) for certain short words w, producing a stairstep structure dubbed a *Ziggurat*; see Figure 1a.

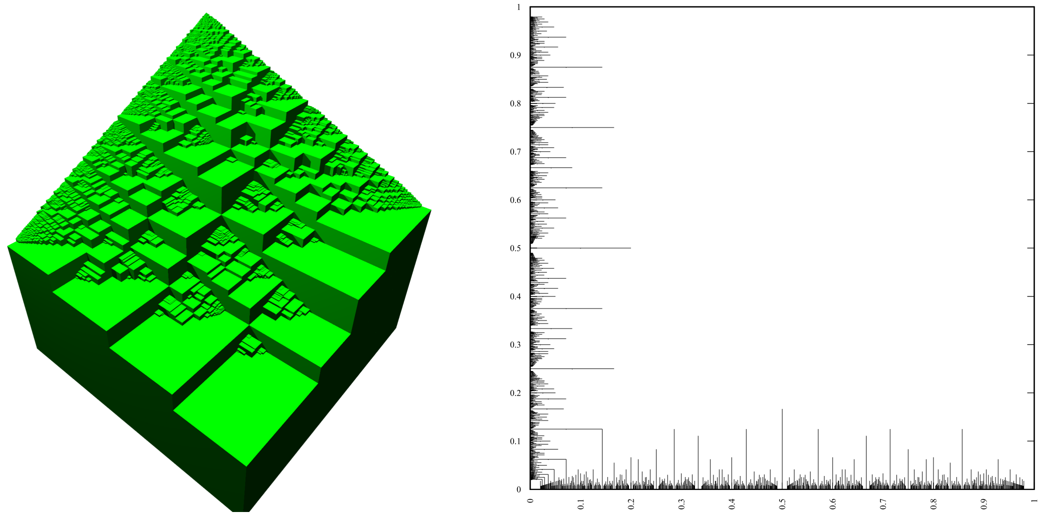
In the special case of the word $w = ab$, a complete analysis can be made, and an explicit formula obtained for $R(ab; \cdot, \cdot)$ (this case arose earlier in the context of the classification of taut foliations of Seifert fibered spaces, where the formula was conjectured by Jankins-Neumann [11] and proved by Naimi [14]). But in *no other case* is any explicit formula known or even conjectured, and even the computation of $R(w; r, s)$ takes time exponential in the denominators of r and s.

1.1.1. *Results.*

The *fringe* associated to w and a rational number $0 \leq p/q < 1$ is the interval $[s, 1)$ for which there is a homomorphism from F to $\text{Homeo}_+^\sim(S^1)$ with $r = p/q$ and

$$\text{rot}^\sim(\rho(w)) = \lim_{t\uparrow 1} R(w; p/q, t) = h_a(w)p/q + h_b(w).$$

Essentially, this is the length of the step starting inwards from the boundary at $(p/q, 1)$ before we drop down. The *fringe length* (see figure 1b), denoted $\text{fr}_w(p/q)$, is equal to $1 - s$. My main result gives an explicit formula for fr_w for any positive word w and establishes a (partial) integral projective self-similarity for fringes, thus giving a theoretical basis for the experimental observations of Calegari-Walker and Gordenko [9].



(a) The *abbbabaaaabbabb* ziggurat (b) Fringe of *abbbabaaaabbab* ziggurat

Figure 1

Fringe Formula 1.3 (Chowdhury, [6]). *If w is positive, and p/q is a reduced fraction, then*

$$\mathrm{fr}_w(p/q) = \frac{1}{\sigma_w(g)\cdot q}$$

where $\sigma_w(g)$ depends only on the word w and $g := \gcd(q, h_a(w))$; and $g\cdot\sigma(g)$ is an integer.

The proof uses the stairstep algorithm by Calegari-Walker to transform the dynamical question to a linear programming problem, and using a theorem by Kaplan [12] and some combinatorial number theory we get the result.

As $t \to 1$, the dynamics of F on S^1 is approximated better and better by a linear model. For t close to 1, the non-linearity can be characterized by a perturbative model; fringes are the maximal regions where this perturbative model is valid. My theorem says that the size of this region of stability follows a power law. This is a new example of (topological) nonlinear phase locking in 1-dimensional dynamics giving rise to a power law, of which the most famous example is the phenomenon of Arnol'd Tongues [7].

The function $\sigma_w(g)$ depends on w and on q in a complicated but easily calculable way, and there are some special cases which are very easy to understand. In particular, I prove the following inequality:

σ-inequality 1.4 (Chowdhury, [6]). *Suppose $w = a^{\alpha_1}b^{\beta_1}a^{\alpha_2}b^{\beta_2}\dots a^{\alpha_n}b^{\beta_n}$. Then the function $\sigma_w(g)$ satisfies the inequality*

$$\frac{h_b(w)}{h_a(w)} \le \sigma_w(g) \le \max\beta_i$$

Moreover, $h_b(w)/h_a(w) = \sigma_w(g)$ when h_a divides q, and $\sigma_w(g) = \max\beta_i$ when q and $h_a(w)$ are coprime.

Projective self-similiarity. The Fringe Formula explains the fact that $\mathrm{fr}_w(p/q)$ is independent of p (for $\gcd(p,q) = 1$) and implies a periodicity of fr_w on infinitely many scales. In particualr, I also show that the unit interval can be decomposed into some finite number of intervals Δ_i such that there exist a further decomposition of each Δ_i into a disjoint union of subintervals $I_{i,j}$ such that the graph of $\mathrm{fr}(x)$ on each of $I_{i,j}$ is similar to that on some $\Delta_{k(i,j)}$ under projective linear transformations.

1.1.2. *Ongoing and Future Work.* Although my formula gives an essentially complete characterization for the ziggurats near the boundaries, much is still unknown regarding the interior. For example, the following result that claims that nonlinearity of the extremal action corresponds to rigidity of the rotation number, remains unproven in general:

Conjecture 1.5 (Calegari-Walker)**.** *Suppose w is a positive word of the form $w = a^{\alpha_1}b^{\beta_1}a^{\alpha_2}b^{\beta_2}\ldots a^{\alpha_n}b^{\beta_n}$. If $R(w;r,s) = c/q$ where c/q is reduced, then*

$$|c/q - h_a(w)r - h_b(w)s| \leq n/q$$

It seems plausible that an approach using the stairstep algorithm and linear programming will lead to a possible resolution, and that is one of my future goals to pursue. Indeed using this technique, I have improved previously known bounds in a specific case:

Proposition 1.6 (Chowdhury)**.** *For any positive word w of the form $w - a^{\alpha_1}b^{\beta_1}\ldots a^{\alpha_n}b^{\beta_n}$, if $R(w;p/q,s) = c/q$ where p and c are coprime to q, we have the inequality*

$$|c/q - h_a(w)p/q - h_b(w)s| \leq n/q$$

The general case seems to be quite within reach. I have also been able to give bounds on the size of the stability region in the interior in specific cases such as follows:

Proposition 1.7. *If $(3,q) = 1$, then*

$$\mathbb{R}abaabt1 - \frac{1}{q} \textit{ is constant } \quad \forall\, t \in \left[\frac{p}{q}, \frac{p}{q} + \frac{1}{3q^2}\right).$$

If $(3,q) \neq 1$, then

$$\mathbb{R}abaabt1 - \frac{3}{2q} \textit{ is constant } \quad \forall\, t \in \left[\frac{p}{q}, \frac{p}{q} + \frac{1}{2q^2}\right).$$

The techniques used can be easily generalized to the case of prime q.

1.2. **Generalization to arbitrary words in a free group and interval game.** Moving to the case of arbitrary words in free group, the first part of theorem 1.2 does hold for semi-positive words but in general, we have the conjecture

Conjecture 1.8 ([3])**.** *Let $w \in F$ be arbitrary and let $r,s \in \mathbb{Q}$. Then $R(w;r,s) \in \mathbb{Q}$.*

Since the monotonicity and stability results of ziggurats are not satisfied anymore, we use a different dynamical problem to analyze this case, called the *interval game*. Existence of a *winning* interval in this game ensures that conjecture 1.8 is true.

Goal 1.9. Given a finite collection of homeomorphisms $\varphi_1, \varphi_2, \ldots, \varphi_m$, called enemies, and a homeomophism ψ with irrational rotation number θ, we want find an interval I such that

- $\exists n \in \mathbb{N}$ such that $\psi^n(I^+) \in I$ and
- $\psi^i(I)$ is dijoint from $\varphi_j(I)$ for all $0 \leq i \leq n, q \leq j \leq m$.

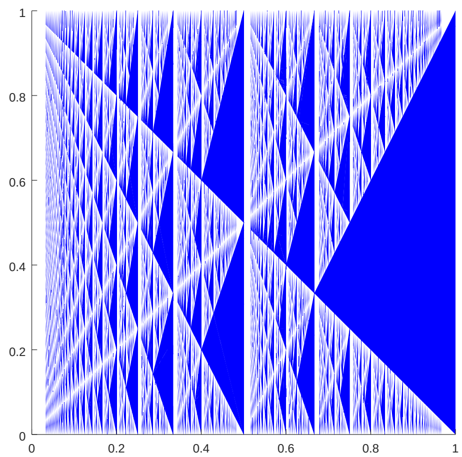


Figure 2: Recursive structure of the dense set of points where interval game can be won (§1.2.1)

1.2.1. *Results.* Calegari and Walker show that a winning interval exists when φ_i's are generic C^1 diffeomorphisms and for all cases of a single homeomorphism φ except when φ is a rigid rotation. The case when both ψ and φ are rigid rotations by θ and α, the set of points (θ, α) for which the game can be won, forms an open, dense subset of the unit square. In this context, the authors' description of the dense set and the proof had some minor mistakes , that I have fixed and the new dense set is pictured in figure 2.

I am currently working on extending the possible cases of winning scenarios to more than one homeomophism φ. In particular, the goal is to allow a wider variety, and eventually generic homeomorphisms that will have winning intervals. Currently I have shown that

Proposition 1.10. *Consider the interval game with two or more enemies φ_i and suppose rotation number of ψ is irrational. Let μ be an invariant probability measure for ψ. Suppose the following conditions are met:*

1. *there is a point $r \in S^1$ such that φ_i's are either weakly locally contarcting or expanding to the right of r.*
2. *the enemies that are weakly locally expanding to the right are locally Lipschitz to the right of r.*

Then a winning interval exits.

Here by weakly locally contracting (resp. expanding) we mean that the graph of φ_i lies strictly below (resp. above) the the slope 1 line passing through $(r, \varphi_i(r))$.

1.2.2. *Future Work.* The immediate goal in future is to get around the Lipshitz condition in proposition 1.10 and generalize it further. I also would like to analyze the particular case of the word $aabAB$ in order to gain some insights about its extremal rotation numbers. Several geometric group theoretic tools including quasimorphism and stable commutator length are some of the most useful tools in this regard.

2. Multiple Context Free Languages and Salvati's theorem

Given a finitely presented group G with finite generating set $\Sigma = \Sigma^{-1}$, the *word problem* asks whether a word in $\Sigma^* = \{a_1 a_2 \ldots a_n \mid a_i \in \Sigma, n \in \mathbb{N}\}$ is equal to the identity of G. The subset of Σ^* that satisfy the word problem is called the group language $L(G)$. Different presentations of the group give rationally equivalent group languages. A broad research program started by Anisimov [1] asks the following:

Research Program 2.1. Relate algebraic properties of finitely presented and finitely generated groups with language-theoretic properties of group languages.

There are several theorems in this area relating Chomsky hierarchies [4] of $L(G)$ to different class of groups, e.g. G is finite iff $L(G)$ is regular [1] and an infinite group G has more than one end if $L(G)$ is context-free [13]. In fact Muller and Schupp prove the stronger theorem that

Theorem 2.2 ([13])**.** *Let G be a finitely generated group. Then the following are equivalent.*

1. *G is virtually free.*
2. *$L(G)$ is context-free.*
3. *There exists a constant K such that every closed path in the Cayley graph $\Gamma(G)$ can be K-triangulated.*

A multiple context free grammar [18] deals with tuples of strings, with rewriting rules of the form

$$f[A_1, A_2, \ldots, A_n] \leftarrow A_0$$

where f is a function with tuples of strings as argument that satisfy

1. Each component of the value of f is a concatenation of some constant string and some component of its argument.
2. Each component is not allowed to appear in the value of f more than once.

Salvati [17] recently showed that the group language

$$\mathbf{O}_2 = \{w \in \{a, b, A, B\}^* \mid |w|_a = |w|_A \wedge |w|_b = |w|_B\}$$

and hence the rationally equivalent language

$$\mathbf{MIX}_3 = \{w \in \{a, b, c\}^* \mid |w|_a = |w|_b = |w|_c\}$$

are $2-$multiple context free i.e. they are generated by a 2-MCFG, which shows that they fall into the class of mildly context sensitive, the class that arguably best captures natural languages. The proof however has the disadvantage of using geometric ideas specific to two dimension e.g. Jordan curve theorem and complex exponential function, and as such no clear way of generalizing to higher dimensional variants. I give a new topological proof of this theorem using Homology theory of CW complexes, ideas not specific to 2-dimension, that raise possibility of generalization to higher dimensions.

2.1. A topological proof.

Salvati proves that the following grammar generates $\mathbf{O}_2$.

$$G = (\Omega, \mathcal{A}, R, S)$$

where $\Omega = (\{S; \mathrm{Inv}\}, \rho)$ with $\rho(S) = 1$ and $\rho(\mathrm{Inv}) = 2$; $\mathcal{A} = \{a, A, b, B\}$ and R consists of the following rules:

1. $S(x_1x_2) \leftarrow \mathrm{Inv}(x_1, x_2)$
2. $\mathrm{Inv}(t_1, t_2) \leftarrow \mathrm{Inv}(x_1, x_2)$ where $t_1t_2 \in \mathrm{perm}(x_1x_2aA) \cup \mathrm{perm}(x_1x_2bB)$

3. $\mathrm{Inv}(t_1, t_2) \leftarrow \mathrm{Inv}(x_1, x_2), \mathrm{Inv}(y_1, y_2)$ where $t_1 t_2 \in \mathrm{perm}(x_1 x_2 y_1 y_2)$
4. $\mathrm{Inv}(\epsilon, \epsilon)$

Since a word in O_2 corresponds uniquely to a closed path in the integer lattice $\mathbb{Z}^2$ (a and b correspond to $\rightarrow$ and $\uparrow$, and A, B to $\leftarrow$ and $\downarrow$ respectively), we prove a purely topological result about smooth closed curves in the plane that as a corollary gives Salvati's theorem.

Consider a closed (oriented) loop K in $\mathbb{R}^2$ given by

$$\varphi : S^1 \rightarrow \mathbb{R}^2, \quad Image(\varphi) =: K$$

It makes no difference to set the domain of φ equal to $[0, 1]$ and assume $\varphi(0) = \varphi(1)$. Let $p = \varphi(0), q = \varphi(1/2)$ and let r and s be two arbitrary points on K. Together the four points $\{p, q, r, s\}$ break up K into 4 (possibly degenerate) arcs K_1, K_2, K_3 and K_4 such that the starting point of K_{i+1} is the same as the ending point of K_i. Denote by $\vec{v}_i$ the vector which is defined as

$$\vec{v}_i = \text{End point of } K_i - \text{ Starting point of } K_i$$

The vectors $\vec{v}_i$ satisfy

$$\vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4 = 0$$

My main technical result is the following.

Proposition 2.3 (Chowdhury, [5]). *Assume that φ is differentiable at p and q and $\varphi'(0)$ is not antiparallel to $\varphi'(1/2)$. Then there exists a pair of r and s on K different from $\{p, q\}$ such that the set of* 4 *vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ can be partitioned into two sets of* 2 *vectors each of which sum up to zero.*

The proof of Proposition 2.3 proceeds in multiple steps as follows. First we construct a 2 dimensional cell complex $\mathcal{X}$ whose points parametrizes choices of $\{r, s\} \neq \{p, q\}$ on K along with the particular choice of partition of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ into two sets of 2 vectors each. Then we define a function $f : \mathcal{X} \rightarrow \mathbb{R}^2$ with the property that a zero for f gives a choice of $\{r, s\}$ satisfying the conclusion of proposition 2.3. Finally using degree of f near link of certain vertices of $\mathcal{X}$, we show that f must have a zero. We subsequently prove Salvati's theorem using an induction argument and the following observation:

Lemma 2.4 (Chowdhury, [5]). *When φ is a closed path the the integer lattice $\mathbb{Z}^2$, we can choose r and s to be points with integer coordinates.*

2.2. **Ongoing and Future work.** Recently in a paper, Meng-Che Ho [10] shows that the language O_n is $(8\lfloor\frac{n+1}{2}\rfloor - 2)$-MCFL. However the sharper bound still remains a conjecture:

Conjecture 2.5 (Nederhof). *O_n is a n-MCFL. In particular, O_3 is a* $3-$*MCFL and is generated by the three dimensional version of the Group language in beginning of section 2 where e.g. $\Omega = (\{\mathrm{Circ}; S\}, \rho)$ with $\rho(\mathrm{Circ}) = 3$ and rule* 3 *is modified to*

$$\mathrm{Circ}(t_1, t_2, t_3) \leftarrow \mathrm{Circ}(x_1, x_2, x_3), \mathrm{Circ}(y_1, y_2, y_3)$$

where $t_1 t_2 t_3 \in \mathrm{perm}(x_1 x_2 x_3 y_1 y_2 y_3)$.

My computer experiments suggest that this is indeed true for words of small length and given $t_1 t_2 t_3 \in O_3$, it is always possible to choose x_1, x_2, x_3, y_1, y_2, and y_3 in such a way that $t_1 t_2 t_3 = x_1 y_1 x_2 y_2 x_3 y_3$ and $x_1 x_2 x_3, y_1 y_2 y_3 \in O_3$; i.e. it is possible to break the word $t_1 t_2 t_3$ into six subwords such that the two sets of three *alternate* subwords together make two words of O_3. My ongoing research in this direction is about finding and working with the correct configuration space in this scenario.

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