Math 1600 Handout 6
Practice Problems for the Final Exam

Subhadip Chowdhury

§1. Related Rates

To solve Related Rates problems, you should first make a (mental) list of the following steps.

(a) What quantities are given in the problem?

(b) What is the unknown?

(c) If possible, draw a picture of the situation for any time t.

(d) Write an equation that relates the quantities.

(e) Take an implicit derivative of both sides of the equation with respect to t.

(f) Finish solving the problem.

■ Exercise 1.1.

If the minute hand of a clock has length r (in centimeters), find the rate at which it sweeps out area as a function of r.

■ Exercise 1.2.

A particle is moving along a hyperbola $xy = 8$. As it reaches the point $(4, 2)$, the y-coordinate is decreasing at a rate of 3 cm/s. How fast is the x-coordinate of the point changing at that instant?

■ Exercise 1.3.

If a spherical snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

[Hint: The formula for the surface area of a sphere is $4\pi r^2$ where r is the radius.]

■ Exercise 1.4.

At noon, ship A is 100 km west of ship B. Ship A is sailing north at 25 km/h while ship B remains still. How fast is the distance between the ships changing at 4:00 pm?

■ Exercise 1.5.

The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?

■ Exercise 1.6.

(Optional) A street light is mounted at the top of a 15 ft tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole?

■ **Exercise 1.7.**

A television camera is positioned 4000 ft from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. Also, the mechanism for focusing the camera has to take into account the increasing distance from the camera to the rising rocket. Lets assume the rocket rises vertically and its speed is 600 ft/s when it has risen 3000 ft.

(a) How fast is the distance from the television camera to the rocket changing at that moment?

(b) If the television camera is always kept aimed at the rocket, how fast is the cameras angle of elevation changing at that same moment?

§2. Optimization

Critical Point: A critical point of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Theorem: If $f(x)$ has a local maximum or minimum at c, then c is a critical point of $f(x)$.

To find local maximum/minimum of a function $f(x)$ do the following steps:

1. Find the critical points of $f(x)$.
2. Find $f''(x)$ and for every critical point p of $f(x)$, check the sign of $f''(p)$.
   (a) If $f''(p) > 0$, then p is a local minimum
   (b) If $f''(p) < 0$, then p is a local maximum
   (c) If $f''(p) = 0$, then check the sign of $f'(x)$ around p.
      i. Choose a small number ϵ e.g. 0.01 so that $(p - \epsilon, p + \epsilon)$ doesn't contain any other critical point. This is to make sure we are checking sign 'locally' i.e. in a small neighborhood of the critical point p.
      ii. If $f'(x)$ changes sign of negative to positive as x passes from left to right of p, then p is a local minimum. In other words, if $f'(p - \epsilon) < 0$ and $f'(p + \epsilon) > 0$, then p is a local min.
      iii. If $f'(x)$ changes sign of positive to negative as x passes from left to right of p, then p is a local minimum. In other words, if $f'(p - \epsilon) > 0$ and $f'(p + \epsilon) < 0$, then p is a local max.
      iv. If $f'(x)$ does not change sign as x passes from left to right of p, then p is neither a local minimum nor a local maximum.

To find global maximum/minimum of a function $f(x)$ **on an interval** $[a, b]$ do the following steps:

1. Find the critical points $f(x)$ that lie inside the interval (a, b). you do not need to check if these are local max/min.
2. Find the value of the function $f(p)$ for every critical point p above.
3. Find the values of $f(x)$ at the endpoints of the interval, i.e. find $f(a)$ and $f(b)$.
4. The largest of the values from Steps 2 and 3 is the global maximum value; the smallest of these values is the global minimum value.
5. You can skip step 3 if you need to find the global max/min over all real numbers $(-\infty, \infty)$.

■ Exercise 2.1.

Problem **4.2.(47,48)**.

■ Exercise 2.2.

Find the global maximum and global minimum values of f on the given interval.

(a) $f(x) = 2x^3 - 3x^2 - 12x + 1, \quad [-2, 3]$

(b) $f(x) = x + \frac{1}{x}, \quad [0.2, 4]$

■ Exercise 2.3.

Prove that the function

$$f(x) = x^{101} + x^{51} + x + 1$$

has neither a local maximum nor a local minimum.

■ Exercise 2.4.

(Problem **4.2.39**) A chemical reaction converts substance A to substance Y. At the start of the reaction, the quantity of A present is a grams. At time t seconds later, the quantity of Y present is y grams. The rate of the reaction, in grams/sec, is given by Rate $= ky(a - y)$, where k is a positive constant.

(a) For what values of y is the rate nonnegative? Graph the rate against y.

(b) For what values of y is the rate a maximum?

■ Exercise 2.5.

(Problem **4.4.34**) Let $f(x) = ax^4 - bx$ for positive constants a and b. Explain why the graph of $f(x)$ is always concave up.

■ Exercise 2.6.

(Problem **4.4.57**) Find a and b if the function $f(x) = y = bxe^{-ax}$ has a local maximum at $(3, 6)$.

■ Exercise 2.7.

(Problem **4.4.60**)

(a) Find all critical points of $f(x) = x^4 + ax^2 + b$.

(b) Under what conditions on a and b does this function have exactly one critical point? What is the one critical point, and is it a local maximum, a local minimum, or neither?

(c) Under what conditions on a and b does this function have exactly three critical points? What are they? Which are local maxima and which are local minima?

(d) Is it ever possible for this function to have two critical points? No critical points? More than three critical points? Give an explanation in each case.

■ Exercise 2.8.

(Problem **4.*Review*.(30)**) For a a positive constant, find all critical points of $f(x) = x - a\sqrt{x}$. What value of a gives a critical point at $x = 5$? Does $f(x)$ have a local maximum or a local minimum at this critical point?

■ **Exercise 2.9.**

On the graph of f in Figure 1, indicate the x-values that are critical points of the function f itself. Are they local maxima, local minima, or neither?

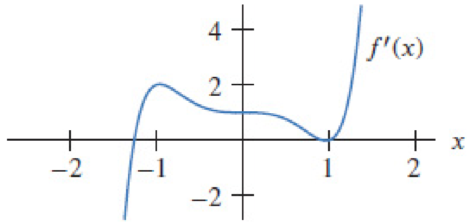

Figure 1

§3. Curve Sketching

The following checklist is intended as a guide to sketching a curve $y = f(x)$ by hand. Not every item is relevant to every function. (For instance, a given curve might not have an asymptote or possess symmetry.) But the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function.

A. **Domain** Its often useful to start by determining the domain D of f , that is, the set of values of x for which $f(x)$ is defined.

B. **Intercepts** The y-intercept is $f(0)$ and this tells us where the curve intersects the y-axis. To find the x-intercepts, we set $y = 0$ and solve for x. (You can omit this step if the equation is difficult to solve.)

C. **Symmetry**

   (i) If $f(-x) = f(x)$ for all x in D, that is, the equation of the curve is unchanged when x is replaced by $-x$, then f is an **even function** and the curve is symmetric about the y-axis. This means that our work is cut in half. If we know what the curve looks like for $x > 0$, then we need only reflect about the y-axis to obtain the complete curve. Here are some examples: $y = x^2, y = x^4, y = |x|$, and $y = \cos x$.

   (ii) If $f(-x) = -f(x)$ for all x in D, then f is an **odd function** and the curve is symmetric about the origin. Again we can obtain the complete curve if we know what it looks like for $x > 0$. [Rotate $180°$ about the origin] Some simple examples of odd functions are $y = x, y = x^3, y = x^5$, and $y = \sin x$.

   (iii) If $f(x + p) = f(x)$ for all x in D, where p is a positive constant, then f is called a periodic function and the smallest such number p is called the period. For instance, $y = \sin x$ has period 2π and $y = \tan x$ has period π. If we know what the graph looks like in an interval of length p, then we can use translation to sketch the entire graph.

D. **Asymptotes**

   (i) A line $y = L$ is a horizontal asymptote of a function $f(x)$ if $f(x) \to L$ as $x \to \infty$ or $x \to -\infty$. Note that a function may have different **horizontal asymptotes** as it goes towards $+\infty$ or $-\infty$. If it turns out that $f(x) \to \infty$ or $-\infty$ as $x \to \infty$, then we do not have an asymptote to the right, but this fact is still useful information for sketching the curve.

   (ii) A line $x = K$ is a vertical asymptote of a function $f(x)$ if $f(x) \to \infty$ or $-\infty$ as $x \to K+$ and $x \to K-$. For rational functions you can locate the **vertical asymptotes** by equating the denominator to 0 after canceling any common factors. But for other functions this method does not apply.

E. **Intervals of Increase or Decrease** Find the intervals on which $f'(x)$ is positive (f is increasing) and the intervals on which $f'(x)$ is negative (f is decreasing).

F. **Local Maximum and Minimum Values** Follow the steps from last section.

G. **Concavity and Points of Inflection** Compute $f''(x)$. The curve is concave upward where $f''(x) \geq 0$ and concave downward where $f''(x) \leq 0$. Inflection points occur where the direction of concavity changes.

H. **Drawing the Graph** Using the information in items A-G, draw the graph. Sketch the asymptotes as dashed lines. Plot the intercepts, maximum and minimum points, and inflection points. Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes. If additional accuracy is desired near any point, you can compute the value of the derivative there. The tangent indicates the direction in which the curve proceeds.

■ Exercise 3.1.

Use the guidelines above to sketch the graph of the rational function

$$y = \frac{2x^2}{x^2 - 1}.$$

■ Exercise 3.2.

(Optional) Draw the graph of $y = x \cos x$.

■ Exercise 3.3.

Use the guidelines above to sketch the graph of the rational function $y = x^4 - 8x^2 + 8$.

■ Exercise 3.4.

(Problem $4.Review.(5, 7)$) For the following functions

(i) Find f' and f''.

(ii) Find the critical points of f.

(iii) Find any inflection points.

(iv) Evaluate f at the critical points and the endpoints. Identify the global maxima and minima of f.

(v) Sketch f. Indicate clearly where f is increasing or decreasing, and its concavity.

(a) $e^{-x} \sin x$, $0 \leq x \leq 2\pi$

(b) $2x^3 - 9x^2 + 12x + 1$

■ Exercise 3.5.

Problems $4.1.(40 - 43, 59, 61)$.