# MATH 2208: ORDINARY DIFFERENTIAL EQUATIONS

## LECTURE 15 WORKSHEET

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TITLE: Forced Harmonic Oscillators

SUMMARY: We will examine the second order constant-coefficient linear, nonautonomous ODE $y'' + by' + ky = f(t)$ and explore the idea of Resonance.

## §A. Forced Harmonic Oscillation

If we apply an external force to the harmonic oscillator system, the differential equation governing the motion becomes

$$y'' + py' + qy = f(t)$$

where $p = \frac{b}{m} > 0$ and $q = \frac{k}{m} > 0$ and $f(t)$ measures the external force. As a system, the forced harmonic oscillator equation becomes

$$\frac{d\vec{R}(t)}{dt} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix}\vec{R}(t) + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

where $\vec{R}(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$. In your midterm, you showed that the general solution of this system is given by

$$\vec{R}(t) = \vec{R}_0(t) + \vec{R}_h(t)$$

where $\vec{R}_0(t)$ is one particular solution of this equation and $\vec{R}_h(t)$ is the general solution to the *associated homogeneous system*

$$\frac{d\vec{R}(t)}{dt} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix}\vec{R}(t)$$

Therefore, since we know all solutions of the homogeneous equation, we can find the general solution to the non-homogeneous equation, provided that we can find *just one particular solution* to the non-homogeneous equation. Often one gets such a solution by simply guessing that solution. The guessing method is usually called the *method of undetermined coefficients*.

## §B. Method of Undetermined Coefficient

If $f(t)$ is sufficiently simple, we can make some intelligent guess for the general form of a solution $y_0(t)$ for the system

$$y'' + by' + ky = f(t) \tag{$\star$}$$

For example, if $f(t)$ is sinusoidal function

$$f(t) = m\sin(\omega t) + n\cos(\omega t)$$

then it is reasonable to expect a particular solution

$$y_0(t) = \mathrm{M}\sin(\omega t) + \mathrm{N}\cos(\omega t)$$

which is of the same form but with as yet *undetermined coefficients* M and N. The reason is that any derivative of such a linear combination of $\cos(\omega t)$ and $\sin(\omega t)$ has the same form.

To find M and N, we substitute this form of $y_0$ in Eq. ($\star$), and then -- by equating coefficients of $\cos(\omega t)$ and $\sin(\omega t)$ on both sides of the resulting equation -- attempt to determine the coefficients M and N so that $y_0$ will, indeed, be a particular solution.

### Question 1.

Find a particular solution of $3y'' + y' - 2y = 2\cos(t)$.

Similarly note that any derivative of $e^{\omega t}$ is a constant multiple of $e^{\omega t}$. So if

$$f(t) = me^{\omega t}$$

we make a reasonable guess that the particular solution $y_0(t)$ is of the form

$$y_0(t) = \mathrm{M}e^{\omega t}$$

### Question 2.

Find a particular solution of $y'' - 4y = 2e^{3t}$.

## §C. Undamped Forcing

Certain systems have one or more natural frequencies at which they oscillate. If you force such a system by an input that oscillates at close to a natural frequency, the response may be very large in amplitude. In this section we explore this phenomenon.

Consider a mass-spring apparatus lying on a table and moving back and forth periodically. Now we force the table to tilt periodically as well with some constant frequency. We get the following equation of motion (up to some scaling)

$$y'' + \nu^2 y = \cos(\omega t)$$

### Question 3.

Write down the general solution to the *associated homogeneous equation*.

The constant $\nu$ is called the *resonant frequency* and $\omega$ is called the *forcing frequency*.

### Question 4.

Find a particular solution to the nonhomogeneous system using the method of undetermined coefficients.

We conclude that the general solution looks like

$$y(t) = \frac{\cos(\omega t)}{\nu^2 - \omega^2} + k_1\sin(\nu t) + k_2\cos(\nu t)$$

**Qualitative Analysis.** Fix $\nu = 1$, then the solution depends on $\omega$ as a parameter. Suppose we have the initial condition $y(0) = y'(0) = 0$. Solving for $k_1$ and $k_2$, we get that the solution to this IVP is

$$y(t) = \frac{\cos(\omega t) - \omega\cos(t)}{1 - \omega^2}$$

Open `https://mathlets.org/mathlets/forced-damped-vibration/` and set $b = 0, A = 1$. Check the `Show Trajectory` option.

### Question 5.

Set $\omega = 0.1$. What does the solution curve look like? You may have to increase the time range to see the full picture.

Can you explain the behavior physically? Is the solution periodic for all values of $\omega$?

### Question 6.

What happens when $\omega = 0.5$ or $\omega = 2$? What's special about these values of $\omega$?

### Question 6.

Now set $\omega = 0.9$, a value very close to $\nu$. You may have to increase the plot range to see the full picture. What do you observe?

### Question 7.

As $\omega \to \nu$, how does the amplitude of $y(t)$ change? Does it become more 'regular'? The phenomenon you observed above is called *Beating* which occurs when the resonant frequency and forcing response have approximately the same value. You can hear the phenomenon of beating when listening to a piano or a guitar that is slightly out of tune.

## §D. Resonance

What happens when $\omega = \nu$? Above algebraic calculations don't make sense in that case.

### Question 8.

Check that

$$y(t) = \frac{t\sin t}{2}$$

is the solution to the IVP

$$y'' + y = \cos(t), \qquad y(0) = y'(0) = 0$$

What does the solution curve look like?