Assignment 16 (7/24)

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- *This homework is due at the beginning of class on **Tuesday** 7/31. You are encouraged to work together on these problems, but you must write up your solutions independently.*

Least Upper Bound

Theorem 1.1. *Let M be the least upper bound of a set S. Then for all $\epsilon > 0$, there exists a $x \in S$ such that $M - \epsilon < x \leq M$.*

Before beginning the proof, let's try to understand the second sentence above. We can break it up into 3 parts as follows;

For all $\epsilon > 0$,

there exists a $x \in S$ such that,

$M - \epsilon < x \leq M$.

We would like to do a proof by contradiction. So we will start by finding out what is the *negation* of above sentence.

The opposite of saying that "for all $\epsilon > 0$, something happens" is to say that "we can find **some** $\epsilon > 0$ for which that something does not happen". ...①

The "something" here is the part "there exists a $x \in S$ such that $M - \epsilon < x \leq M$".

So next we need to understand how to write the part: "something does not happen."

The opposite of saying "there exists a $x \in S$ such that blah is true" is to say "there does **not** exist **any** $x \in S$ such that blah is true". This is equivalent to saying "blah is false **for all** $x \in S$".

Now the "something" part reads as "there exists a $x \in S$ such that blah inequality is true".

Thus to write that the "something does not happen", is to say "blah inequality is false **for all** $x \in S$". ...②

Combining ① and ②, we have the following so far:

we can find **some** $\epsilon > 0$ for which blah inequality is false **for all** $x \in S$

$\ldots ③$

So all that remains is to understand what it means to say "blah inequality is false".

Note that blah inequality is $M-\epsilon < x \leq M$, there are two inequalities here. We can interpret the chain of inequalities as "$M-\epsilon < x$ **and** $x \leq M$". So the "chain of inequalities is false" means

$$\text{either } M-\epsilon \geq x \text{ or } x > M.$$

$\ldots ④$

Combining ③ and ④, we have the complete negation statement:

we can find some $\epsilon > 0$ for which either $M-\epsilon \geq x$ or $x > M$ for all $x \in S$.

Rewriting above sentence in more common mathematical language, we get

There exists some $\epsilon > 0$ such that either $M-\epsilon \geq x$ or $x > M$ for all $x \in S$.

Now let's begin the proof.

Proof. Assume, for the sake of contradiction, that there exists some $\epsilon > 0$ such that either $M-\epsilon \geq x$ or $x > M$ for all $x \in S$.

Since M is an upper bound of S, the option $x > M$ is not possible. Hence we must have $M-\epsilon \geq x$ for all $x \in S$.

Then, by definition, $(M-\epsilon)$ is an upper bound of S. But $(M-\epsilon)$ is less that M, which is supposed to be the *least* i.e. the smallest upper bound. Contradiction! □

Also look up theorem 11.1.2 from the book for another way of writing the same proof.

Problems

Exercise 5. *Write the negation of the following statements.*

(a) *For all $x \in P$, there exists $y \in Q$ such that, R is true for all $z \in S$.*

(b) *For all $\epsilon > 0$, there exists a natural number N such that, $\frac{1}{n} < \epsilon$ for all $n > N$.*

(c) *For all $\epsilon > 0$, there exists a natural number N such that, if $n > N$ then $\frac{1}{n} < \epsilon$.*

(d) *For all $\epsilon > 0$, there exists a natural number N such that, $|a_n - l| < \epsilon$ for all $n > N$.*

(e) *For all $\epsilon > 0$, there exists a $\delta > 0$ such that, if $|x-c| < \delta$ then $|f(x)-l| \leq \epsilon$.*

Exercise 6. *Do exercise* 11.1.(32*a*).