Math 2208: Ordinary Differential Equations

Lecture 11 Worksheet

Fall 2019 **Subhadip Chowdhury** **Oct 21**

Title: Phase Portraits of Linear Systems

Summary: We'll explore the various scenarios that occur with linear systems of ODEs that possess real eigenvalues.

§A. Two Real Eigenvalues

Consider a system of linear ODEs with associated matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The *characteristic polynomial* of A is given by

$$p_A(\lambda) = \det(A - \lambda I) = (a - \lambda)(d - \lambda) - bc = \lambda^2 - \lambda(a + d) + (ad - bc) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A)$$

Question 1.

What condition the characteristic polynomial $p_A(\lambda)$ must satisfy in order to produce real eigenvalues?

§B. Classifying Equilibrium Points

Suppose a linear system has two real, nonzero, distinct eigenvalues λ_1 and λ_2. Recall that λ_1 and λ_2 are the solutions to the characteristic polynomial. In what follows, we are going to classify λ_1 and λ_2 into a number of different cases depending on the qualities the eigenvalues possess. In addition, we will also classify the equilibrium at the origin, and a sketch a typical phase portrait for each case.

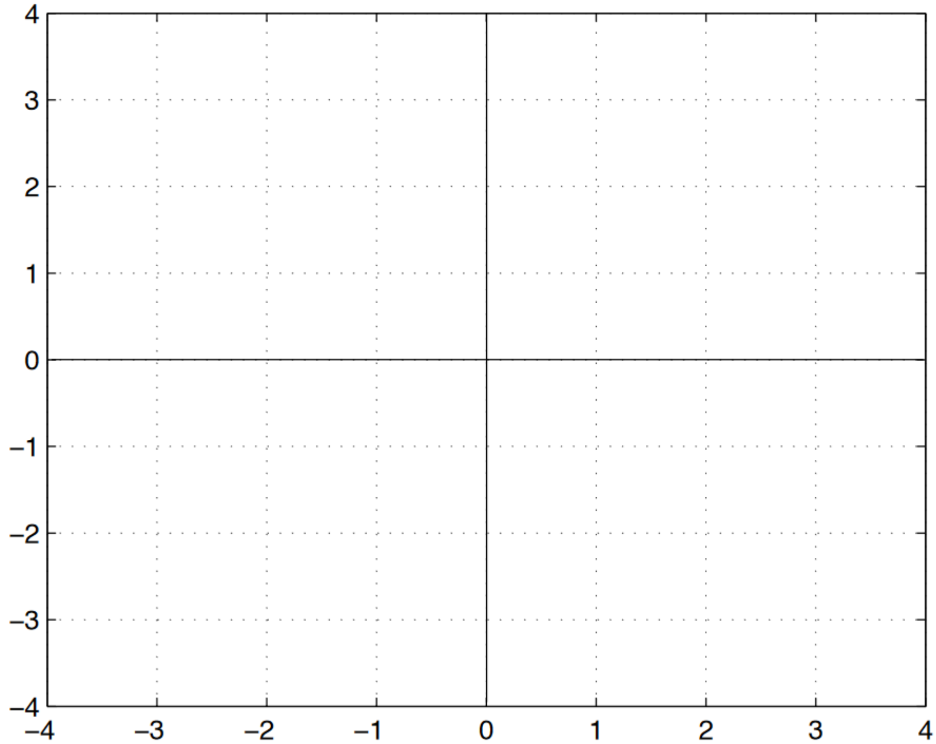
Group Work

On the next few pages, you will find three specific cases, each with an example of an ODE that satisfies the case and a classification of the origin. For each case:

- Check that the given matrix will definitely produce real eigenvalues.
- Then find the eigenvalues and eigenvectors (by hand) in order to write down the straight line solutions and the general solution of the given system.
- What happens to $\vec{r}(t)$ as $t \to \infty$ or $t \to -\infty$? Note that your answer will depend on the initial condition. Find all possible scenarios. In each case, also find out what happens to the ratio $\frac{x(t)}{y(t)}$ as $t \to \pm\infty$.
- Use PPLANE to help you sketch the phase portrait on the given axes. Go to `Solution` menu and sketch the nullclines.
- Write down a few sentences describing your observations of the phase portrait. Are your solution curves consistent with the end behavior you found above?

■ **Case 1: $\lambda_1 > \lambda_2 > 0$.**

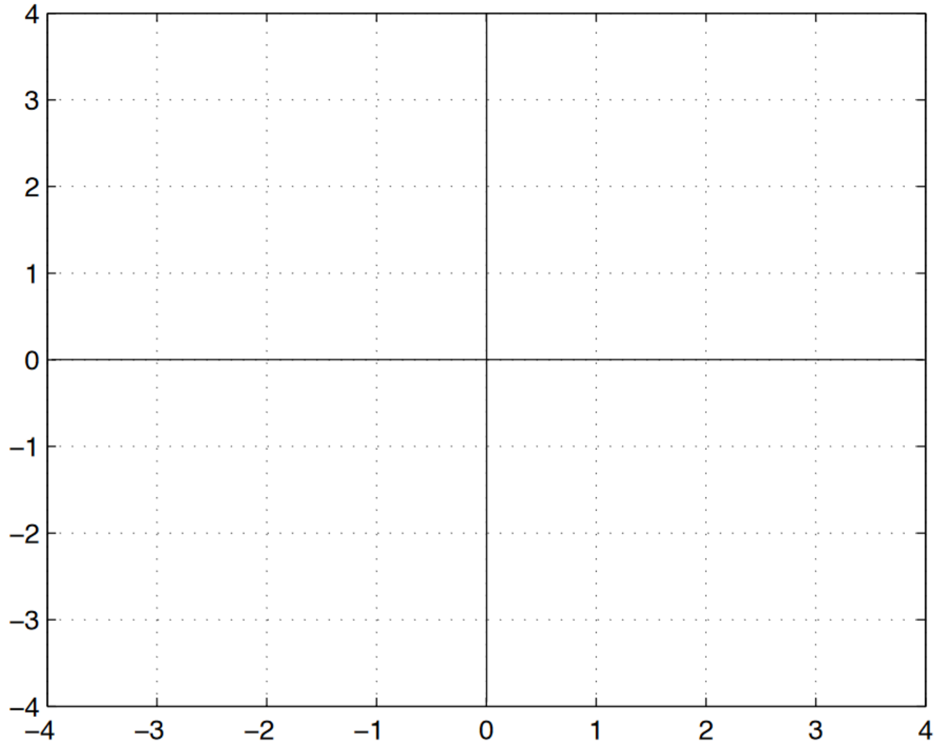
Solve $\dfrac{d\vec{r}}{dt} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \vec{r}$.



In this case the origin is an *unstable source*.

Case 2: $\lambda_1 < \lambda_2 < 0$.

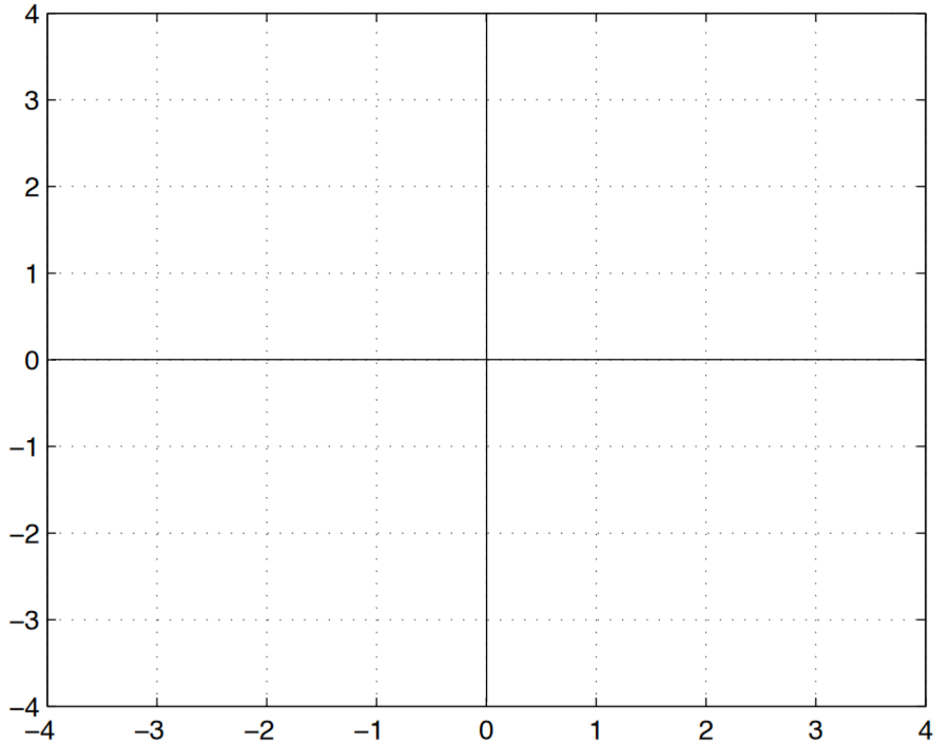
Solve $\dfrac{d\vec{r}}{dt} = \begin{bmatrix} -2 & -1 \\ 2 & -5 \end{bmatrix}\vec{r}$.



In this case the origin is an *stable sink*.

Case 3: $\lambda_1 > 0 > \lambda_2$.

Solve $\dfrac{d\vec{r}}{dt} = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix}\vec{r}$.



In this case the origin is an *unstable saddle*.