Problem Set 4 Solutions

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Exercise 7: Negation

Problem 1.1. *What is the negation of the following sentence? "There exists some natural number N such that all elements of set A are less than N."*

Solution. $\forall N \in \mathbb{N}$, $\exists x \in \mathbb{A}$ such that $x \geq N$.
The above statement reads: For all natural numbers N, there exists an element x in $\mathbb{A}$ that is greater than or equal to N. ■

Exercise 8: Sylvester Problem

Problem 2.1. *A finite set S of points in the plane has the property that any line through two of them passes through a third. Show that all the points lie on a line.*

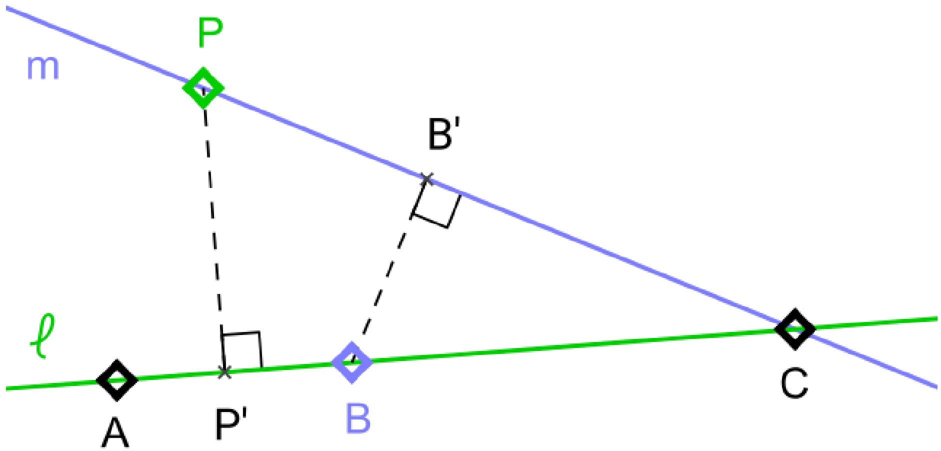

Figure 1: A drawing of an example geometry for the Sylvester problem.

Proof. For the sake of contradiction, suppose that not all the points in S are collinear. Consider the set of lines L which contains all the lines between any two points in S. For every line in L, there must exist some point in S that does *not* lie on the line, i.e. the distance d between them is positive.[1] Now, since the set L is finite (since S is finite), there must exist some line l and some point P not on it that achieve the minimum such distance d, which we shall call d_{min}. Refer to figure 1 for an example of such a situation. In the figure the square points are elements of S. P' is the foot of perpendicular from P to l and $d_{min} = |PP'|$.

Recall that any line through two points of S passes through a third. Hence, there are at least three points A, B, and C on l. Now, notice that the point P' divides l into two regions. By the Pigeonhole Principle, at least two of the three points A, B, and C lie in the same region; let's say B and C lie to the right of P' (in that order).[2]

We construct the line m through the points P and C. This still in L. We are going to consider the perpendicular distance from B to m, call it d'. This is the length BB' in above figure.

We are going to show that $d' < d_{min}$. There are many ways to show this. One way is to see that $\triangle BB'C$ and $\triangle PP'C$ are similar and hence the ratios of similar sides are equal. Since $|CP'| > |CB| > |CB'|$, we get that $\frac{|BB'|}{|PP'|} = \frac{|CB'|}{|CP'|} < 1$. Hence $d' < d_{min}$.

Another way is via a simple construction. Assume the stratight line through B that is parallel to PP', intersects m at D. Then $\triangle BB'D$ is a right angled triangle, implying that $|BB'| < |BD|$. But clearly $|BD| < |PP'|$, since B is between C and P'. Hence $d' = |BB'| < |PP'| = d_{min}$.

Thus we have shown $d' < d_{min}$. But this is a contradiction, as d_{min} was supposed to be the minimum such distance for all lines in the set L. □

1. Here, we define distance of a point to a line as the length of the perpendicular from the point to the line.
2. The rest of the proof still holds if one of the points is P'.

Exercise 9: Proof By Induction

Problem 3.1. *Prove that* $\forall n \in \mathbb{N}$,

$$1 \times 2 + 2 \times 3 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3} \tag{$\star$}$$

Proof. We will prove the identity by inducting on n. We will begin by proving that the above identity holds for $n = 1$. Observe that the right-hand side can be written $\frac{1(1+1)(1+2)}{3} = 2$ which is equal to the left-hand side, $1 \times 2 = 2$.

Now, let us assume that the identity holds for some $k \in \mathbb{N}$. Thus we have,

$$1 \times 2 + 2 \times 3 + \ldots + k(k+1) = \frac{(k)(k+1)(k+2)}{3}$$

Adding the next term, $(k+1)(k+1+1)$ to both sides, we have

$$\begin{aligned} 1 \times 2 + 2 \times 3 + \ldots + (k)(k+1) + (k+1)(k+2) &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\ &= (k+1)(k+2)\left(\frac{k}{3} + 1\right) \\ &= \frac{(k+1)(k+2)(k+3)}{3} \end{aligned}$$

Thus the identity holds for $n = k+1$ whenever the identity holds for $n = k$. Hence, by the principle of mathematical induction, the identity ($\star$) holds $\forall n \in \mathbb{N}$.

$\square$

$\boxed{\textbf{Exercise 10: Induction on Odds}}$

Problem 4.1. *Guess a formula for the sum of the first n odd natural numbers and then prove it by induction.*

We claim that the formula for sum of the first n odd natural numbers is n^2.

Claim. $\forall n \in \mathbb{N}$,

$$1+3+5+\ldots+2n-1=n^2 \tag{†}$$

Proof. We will prove the identity by inducting on n. We observe that the above identity holds for $n=1$: the right side is 1^2 which is equal to the sum up to the first odd natural number, i.e. 1.

Now, let us make the induction assumption that there is some $k \in \mathbb{N}$ for which this identity holds:

$$1+3+5+\ldots+2(k-1)=k^2$$

where we have summed up to the k^{th} odd natural number. Adding the $(k+1)^{th}$ odd number, $(2k+1)$, to both sides, we have as the left side

$$1+3+5+\ldots+(2k-1)+(2k+1)=1+3+5+\ldots+(2k+1)$$

Moreover, the right side is

$$(k)^2+(2k+1)=k^2+2k.1+1^2=(k+1)^2$$

Thus the identity holds for $n=k+1$ whenever the identity holds for $n=k$. Hence, by the principle of mathematical induction, the identity (†) holds $\forall n \in \mathbb{N}$. □

Exercise 11: Induction Step Fallacy (Extra Credit)

Problem 5.1. *Explain what is wrong with it the proof by induction of the following: All real numbers are equal. (see assignment for the incorrect proof).*

Solution. The induction step contains the fallacy. To be precise, consider the logic given in the induction step applied to the case $k = 1$. The base case is trivially true. So we are good so far, there is no issue in there. Next, as we attempt to make the inductive step, the proof reads: Applying the induction hypothesis to the first $k(= 1)$ numbers, we get

$$a_1$$

and

$$a_2.$$

Each of the chain of equalities in the fake proof in fact contains only one term. Since there is no common terms in the two 'equations', the conclusion $a_1 = a_2$ is false. So the argument doesn't work when $k = 1$, and our induction process never gets started. ∎