Math 1800-C Handout 7: Applications of Green's Theorem

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Careful Statement of Green's Theorem

Green's Theorem is a statement about only the 2D plane, not 3D space. Here's the setup: Let C be a piecewise smooth simple closed curve that is the boundary of a simply-connected region R in the plane. Let $\vec{\mathbf{F}} = P\,\vec{\mathbf{i}} + Q\,\vec{\mathbf{j}}$ be a smooth vector field defined on all of R and C.

Notes on terminology: To say a curve is *simple* means that it doesn't intersect itself, and to say a curve is *closed* means that it is a closed loop, that it starts and ends at the same point. A region is called *simply-connected* if it is just one piece (connected) and doesn't have any holes in it. A vector field is said to be *smooth* if it has continuous first partials.

Since C is the boundary of R, we write $C = \partial R$ and stop referring to C explicitly. When we consider the boundary curve of a simply-connected region, we always orient the curve so that the region is on the left as we follow the curve. (Note: this is harder to specify when one considers 3D surfaces with boundaries.)

Green's Theorem says

$$\oint_{\partial R} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_R (Q_x - P_y)\, dA.$$

You may also see this written the Leibniz notation for partial derivatives.

$$\oint_{\partial R} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Using Green's Theorem to Evaluate Difficult Line Integrals

If you're asked to find the line integral of an ugly vector field over a closed curve, you should look to see if Q_x and P_y are drastically more simple. If so, use Green's Theorem!

Exercise 1

Evaulate

$$\int_C (2y + \sqrt{9 + x^3})\, dx + (5x + e^{\tan^{-1} y})\, dy,$$

where C is the circle $x^2 + y^2 = 4$ in the plane oriented counterclockwise.

If you're asked to find the line integral over a closed curve that is clearly the boundary of a nice region, it's often a good idea to use Green's Theorem to switch to the double integral over the inside.

Exercise 2

Integrate $\vec{\mathbf{F}} = xy\,\vec{\mathbf{i}} + e^x\,\vec{\mathbf{j}}$ over the boundary of the rectangle determined by $0 \le x \le 2$, $0 \le y \le 3$, oriented clockwise around the boundary.

Exercise 3

Find the line integral of $\vec{\mathbf{F}} = 3xy\,\vec{\mathbf{i}} + 2x^2\,\vec{\mathbf{j}}$ over the curve C defined as follows: follow the curve $y = x^2 - 2x$ from $(0,0)$ to $(3,3)$, then follow the line $y = x$ from $(3,3)$ back to $(0,0)$.

Exercise 4

Evaluate

$$\oint_C \left(3y - e^{\sin x}\right) dx + \left(7x + \sqrt{y^4+1}\right) dy$$

where C is the circle $x^2 + y^2 = 9$, oriented clockwise.

Exercise 5

We would like to find

$$\vec{\mathbf{G}}(x,y) \;=\; (x+y)\,\vec{\mathbf{i}} + \left(2x + y\ln\!\left(\csc\sqrt{1-y^5}\right)\right)\vec{\mathbf{j}}$$

over C_1, the upper half of the unit circle from $(1,0)$ to $(-1,0)$.

The problem is that the vector field is ugly, so parametrizing C_1 is just going to lead to an impossible integral. So we would like to use Green's Theorem, but this isn't a closed curve!

Here's how to fix that issue. Let C_2 be the straight line segment from $(-1,0)$ to $(1,0)$. Now $C_1 + C_2$ is a closed loop.

(a) Let R be the region enclosed by $C_1 + C_2$. Use Green's Theorem to compute $\displaystyle\oint_{C_1+C_2} \vec{\mathbf{G}} \cdot d\vec{\mathbf{r}}$.

(b) Parametrize C_2 and directly calculate $\displaystyle\int_{C_2} \vec{\mathbf{G}} \cdot d\vec{\mathbf{r}}$. (Note that $y = 0$ everywhere on C_2, which is helpful.)

(c) Write $\displaystyle\oint_{C_1+C_2} \vec{\mathbf{G}} \cdot d\vec{\mathbf{r}} = \int_{C_1} \vec{\mathbf{G}} \cdot d\vec{\mathbf{r}} + \int_{C_2} \vec{\mathbf{G}} \cdot d\vec{\mathbf{r}}$ and use your answers to part (a) and (b) to finish off the problem and find the line integral of $\vec{\mathbf{G}}$ along C_1.

Calculating Area with Green's Theorem

Consider the following vector fields:

$$\vec{\mathbf{F}}_1 = x\,\vec{\mathbf{j}}, \qquad \vec{\mathbf{F}}_2 = -y\,\vec{\mathbf{i}}, \qquad \vec{\mathbf{F}}_3 = -\tfrac{1}{2}y\,\vec{\mathbf{i}} + \tfrac{1}{2}x\,\vec{\mathbf{j}}.$$

What is $Q_x - P_y$ for each of these fields? Applying Green's Theorem to a region R, we get that

$$\oint_{\partial R} \vec{\mathbf{F}}_1 \cdot d\vec{\mathbf{r}} \;=\; \oint_{\partial R} \vec{\mathbf{F}}_2 \cdot d\vec{\mathbf{r}} \;=\; \oint_{\partial R} \vec{\mathbf{F}}_3 \cdot d\vec{\mathbf{r}} \;=\; \iint_R 1\,dA \;=\; \text{Area of } R. \text{ (!)}$$

Exercise 6

An ellipse with semi-major axis a and semi-minor axis b is parametrized by $x = a\cos t,\ y = b\sin t$ for $0 \le t \le 2\pi$. Use $\vec{\mathbf{F}}_3$ to find the area inside this ellipse.

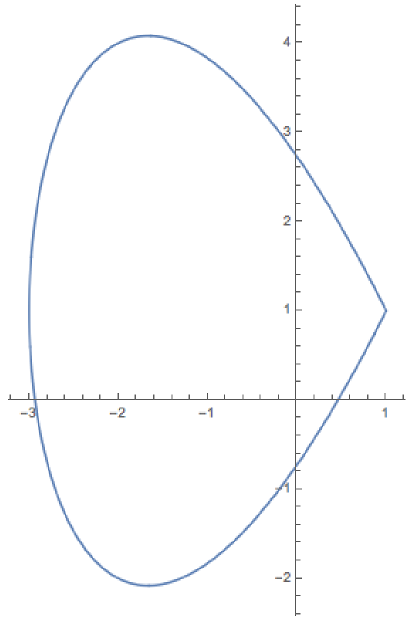

Figure 1: Exercise 7

Exercise 7

Let C be the curve parametrized by $\vec{\mathbf{r}}(t) = (t^2 - 3)\,\vec{\mathbf{i}} + (t^3 - 4t + 1)\,\vec{\mathbf{j}}$, $-2 \leq t \leq 2$. This is a closed loop. Use Green's Theorem and $\vec{\mathbf{F}}_1$ to find the area inside this loop.

Extended Versions of Greens Theorem

Although we have proved Greens Theorem only for the case where D is simple, we can now extend it to the case where D is a finite union of simple regions. For example, if D is the region shown in Figure 2, then we can write $D = D_1 \cup D_2$, where D_1 and D_2 are both simple. The boundary of D_1 is $C_1 \cup C_3$ and the boundary of D_2 is $C_2 \cup \overline{C_3}$. so, applying Greens Theorem to D_1 and D_2 separately, we get

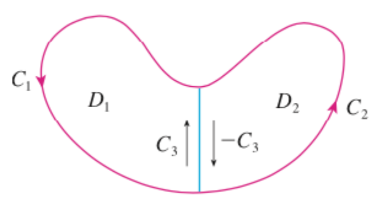

Figure 2

$$\oint_{C_1 \cup C_3} Pdx + Qdy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\oint_{C_2 \cup \overline{C_3}} Pdx + Qdy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Adding the two integrals above, we get,

$$\oint_{C_1 \cup C_2} Pdx + Qdy = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Exercise 8

Evaluate

$$\oint_C y^2 dx + 3xydy$$

where C is the boundary of the semiannular region D in the upper half plane between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

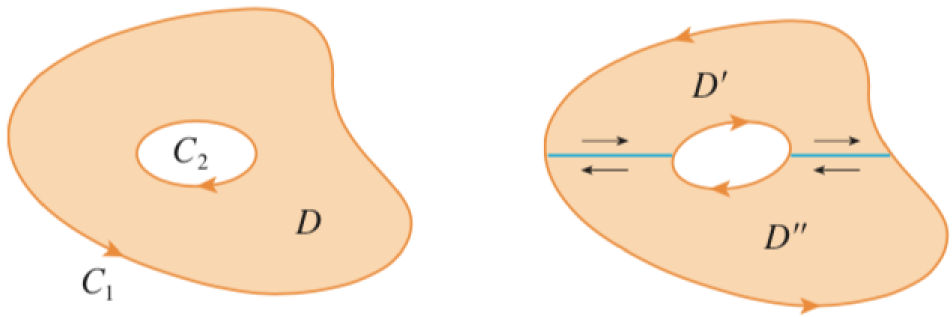

Figure 3

Greens Theorem can be extended to apply to regions with holes, that is, regions that are not simply-connected. Observe that the boundary C of the region D in Figure 3 consists of two simple closed curves C_1 and C_2. We assume that these boundary curves are oriented so that the region D is always on the left as the curve C is traversed. *Thus the positive direction is counterclockwise for the outer curve C_1 but clockwise for the inner curve C_2.* If we divide D into two regions D' and D'' by means of the lines shown in Figure 3 and then apply Greens Theorem to each of D' and D'', we get

$$\begin{aligned} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA &= \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\ &= \oint_{\partial D'} (Pdx + Qdy) + \oint_{\partial D''} (Pdx + Qdy) \end{aligned}$$

Since the line integrals along the common boundary lines are in opposite directions, they cancel and we get

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_{C_1} (Pdx + Qdy) + \oint_{C_2} (Pdx + Qdy) = \oint_C Pdx + Qdy$$

Exercise 9

If $\vec{\mathbf{F}}(x, y) = (-y\vec{\mathbf{i}} + x\vec{\mathbf{j}}) / (x^2 + y^2)$, show that $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 2\pi$ for every positively oriented closed path C that encloses the origin.

Exercise 10

Evaluate $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 2\pi$ where

$$\vec{\mathbf{F}}(x, y) = \frac{2xy\vec{\mathbf{i}} + (y^2 - x^2)\vec{\mathbf{j}}}{(x^2 + y^2)^2}$$

and C is a positively oriented closed path that encloses the origin.