Practice Problems and review notes

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- The following are a set of practice/review problems for final. Over this week, I will be going over these problems in office hour, problem session, class etc. Some of these problems will be on the final exam itself.
- Make sure you can solve all of the problems listed below and ask me via email or in person if you have questions.
- **You may skip problem 4 and $5(d)$.**
- **Problem 5 gives solution to the last problem in Midterm 2.**
- Problem 13 and 14 has some exercises from the book based on topic covered today. I will also assign problems from exercise 7.6 , exponential growth and decay in next class. You do NOT have to submit solution to these exercises, but you should solve them on your own and ask me if you have questions.
- Problem 15 gives a formal write-up of the proof we did in class today.
- I will try to make another such practice set for injectiveness/surjectiveness if I get time. In any case, I will be going over more examples of those in class/office hour/problem session.
- Apart from these problems, you should also go through the extra problems (i.e. those outside the book) that I had assigned in homeworks over the quarter. Ask me if you need clarification with any of those.

Problem 1

$$\int_{1/e}^{e^2} \left|\frac{\ln(x)}{x}\right| dx$$

[Hint: Be careful about any sign change which affects the absolute($|.|$) function.]

Problem 2

$$\int \frac{1-x^2}{x(2-x^2)} dx$$

[Hint: Multiply Num and Den by x. Then take $u = 1 - x^2$.]

Problem 3

Prove the addition formula for $\tan$ using the corresponding formulae for $\sin$ and $\cos$.

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Problem 4

Recall the following property of definite integrals:

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx.$$

Use this to evaluate

$$\int_0^{\pi/4} \ln(1+\tan(\theta))d\theta.$$

Problem 5

Note that

$$\begin{aligned}\int_0^{\pi/2} \frac{1}{1+\tan x}dx &= \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}dx \\ &= \int_0^{\pi/2} \frac{\cos(\pi/2-x)}{\cos(\pi/2-x)+\sin(\pi/2-x)}dx \\ &= \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}dx\end{aligned}$$

Hence

$$\begin{aligned}2\int_0^{\pi/2} \frac{1}{1+\tan x}dx &= \int_0^{\pi/2} \frac{\cos x + \sin x}{\cos x + \sin x}dx \\ &= \int_0^{\pi/2} 1dx \\ &= \pi/2 \\ \implies \int_0^{\pi/2} \frac{1}{1+\tan x}dx &= \pi/4\end{aligned}$$

Use the same kind of logic to evaluate

(a)

$$\int_0^{\pi/2} \frac{a+b\tan x}{1+\tan x}dx$$

(b)

$$\int_0^{\pi/2} \frac{\sin^6 x - \cos^6 x}{\sin^7 x + \cos^7 x}dx$$

(c)

$$\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}} dx$$

[Ans: $\pi/12$]

(d)

$$\int_0^{\pi} \frac{x \sin x}{1+\sin x} dx$$

[Ans: $\frac{\pi}{2}(\pi - 2)$]

Problem 6

$$\int \frac{\tan x}{\sec x + \tan x} dx$$

Problem 7

(a)

$$\int \frac{1}{1+\cos(2x)+\sin(2x)} dx$$

[Hint: Use double angle formulae. Then divide both Num and Den by $\cos^2 x$.]

(b) Find

$$\int \frac{1+\sin(2x)}{1+\cos(2x)} dx$$

[Hint: Separate into two integrals and then use double angle formulae.]

(c)

$$\int \frac{1+\cos x}{\sin(2x)} dx$$

Problem 8

$$\int \frac{1+x^2}{x^4+1-2x^2} dx$$

[Hint: Divide Num and Den by x^2.]

Problem 9

(a)

$$\int \frac{dx}{\sin x + \sqrt{3}\cos x}$$

(b)

$$\int \frac{dx}{\sin x + \cos x}$$

[Hint: Use the addition formula for $\sin$ to write the denominators as $\sin(x+\theta)$ for some suitable angle θ.]

Problem 10

Note that for any continuous differentiable fumction f, we have

$$\frac{d}{dx}e^x f(x) = e^x(f(x) + f'(x))$$

Thus,

$$\int e^x(f(x) + f'(x))dx = e^x f(x) + c.$$

Using this fact, find the following integrals:

(a)

$$\int e^{x/2} \sin\left(\frac{x}{2} + \frac{\pi}{4}\right) dx$$

[Hint: Expand using addition formula.]

(b)

$$\int e^x \left(\frac{x-1}{x^2}\right) dx$$

(c)

$$\int e^{2x} \frac{1+\sin(2x)}{1+\cos(2x)} dx$$

[Hint: cf. $7(b)$, Separate into two integrals, use double angle formulae and substitute $u = 2x$.]

Problem 11

$$\int \cot^3 x \csc^4 x \, dx$$

Problem 12

Evaluate

$$\int_0^{25} [\sqrt{x}] dx,$$

where $[\,.\,]$ denotes the greatest integer function, i.e. $[x]$ is the greatest integer $\leq x$. For example, $[1.65] = [1.2] = 1, [0.3] = 0, [-3.2] = -4$ etc.

[Hint: Draw the graph of $[\sqrt{x}]$.
For $x \in [0, 1)$, we have $0 \leq \sqrt{x} < 1$, so that $[\sqrt{x}] = 0$.
For $x \in [1, 4)$, we have $1 \leq \sqrt{x} < 2$, so that $[\sqrt{x}] = 1$. So on...]

$\boxed{\textbf{Problem 13}}$

Recall that $a^x = e^{x\ln(a)}$ for $a > 0$. Using this fact, we can write

$$\begin{aligned}
\frac{d}{dx} f(x)^{g(x)} &= \frac{d}{dx} e^{g(x)\ln(f(x))} \\
&= e^{g(x)\ln(f(x))} . \frac{d}{dx}[g(x)\ln(f(x))] && \text{[By Chain rule]} \\
&= f(x)^{g(x)} . \left(\frac{g(x)}{f(x)} f'(x) + \ln(f(x)) g'(x) \right) && \text{[By Product rule]}
\end{aligned}$$

Now do problem $7.5.(45, 49, 50)$.

$\boxed{\textbf{Problem 14}}$

Problems $7.5.(63, 65, 67)$.

$\boxed{\textbf{Problem 15}}$

Today in class we proved the following theorem:
Theorem. *If*

$$f'(t) = kf(t)$$

for all t in some interval $[a, b]$, then there exists a constant C such that

$$f(t) = Ce^{kt}$$

for all $t \in [a, b]$.

This is also Theorem $7.6.1$ in book. However the proof in book is different from what I did in class. If I ask to prove this in exam, you may do either of them. Just for completeness, here is the proof as I did in class.
Proof.

$$\begin{aligned}
& f'(t) = kf(t) \\
\implies & \frac{df(t)}{dt} = kf(t) \\
\implies & \frac{df(t)}{f(t)} = kdt
\end{aligned}$$

Integrating both sides from $t = a$ to $t = T$ and substituting $u = f(t)$

$$\begin{aligned}
&\implies \int_{f(a)}^{f(T)} \frac{du}{u} = \int_a^T kdt \\
&\implies [\ln|u|]_{f(a)}^{f(T)} = [kt]_a^T \\
&\implies \ln\left|\frac{f(T)}{f(a)}\right| = k(T - a) \\
&\implies f(T) = f(a)e^{kT-ka} \\
&\qquad\qquad = (f(a)e^{-ka})e^{kT} = Ce^{kT} \text{ where } C = f(a)e^{-ka}
\end{aligned}$$

$\square$

In the study of Exponential growth or decay we usually have $a = 0$. Thus the formula becomes

$$f(T) = f(0)e^{kT}$$

for all $T \in [0, b]$.

$\boxed{\textbf{Look in the book for a different proof.}}$

I will assign practice problems from exercise 7.6 on Wednesday.