Midterm 1

Please show **all** your work! Answers without supporting work will not be given credit. Answer the questions in the spaces provided on the question sheets.
If you run out of room for an answer, continue on the back of the page.
Please note that use of calculator is not allowed.

Full Name: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	15	
5	20	
Total:	65	

This exam has 5 questions, for a total of 65 points.
The maximum possible point for each problem is given on the right side of the problem.

1. **CIRCLE** the correct option (only one) in each of the following. Here 'lub' stands for least upper bound and 'glb' stands for greatest lower bound. 10

(a) Let $M = lub$ of a nonempty bounded subset S of $\mathbb{R}$. Then M is also equal to

A. the glb of the set of all lower bounds (in $\mathbb{R}$) of S.

B. the lub of the set of all lower bounds (in $\mathbb{R}$) of S.

C. the glb of the set of all upper bounds (in $\mathbb{R}$) of S.

D. the lub of the set of all upper bounds (in $\mathbb{R}$) of S.

(b) Consider the following four intervals where $a < b$ are both fixed real numbers:

$$(a,b),\ [a,b],\ [a,b),\ \text{and } (a,b]$$

Which of the following is true about the upper and lower bounds of these intervals?

A. The glb for the intervals $[a,b)$ and $[a,b]$, while same to each other, differs from the glb of the intervals $(a,b]$ and (a,b), which in turn are the same. Similarly, The lub for the intervals $(a,b]$ and $[a,b]$, while same to each other, differs from the lub of the intervals $[a,b)$ and (a,b), which in turn are same.

B. The intervals $[a,b]$ and $[a,b)$ have a glb and the intervals $(a,b]$ and (a,b) do not. Further, the intervals $[a,b]$ and $(a,b]$ have a lub, and the intervals $[a,b)$ and (a,b) do not.

C. None of the intervals has a greatest lower bound or a least upper bound.

D. $[a,b]$ is the only interval among the four intervals that has a glb and a lub.

E. All of them have the same lub and glb.

(c) Which of the following is true?

A. Any bounded monotonic sequence converges to its glb.

B. Any bounded monotonic sequence converges to its lub.

C. Any bounded monotonic sequence is convergent. It converges to its glb if it is non-increasing and to its lub if it is non-decreasing.

D. Any bounded monotonic sequence is convergent. It converges to its lub if it is non-increasing and to its glb if it is non-decreasing.

(d) Which of the following is true?

A. A sequence $\{a_n\}_{n\in\mathbb{N}}$ is decreasing (i.e. $a_{n+1} \le a_n$) if and only if its first term equals its lub.

B. If a sequence is decreasing, then its first term is its lub, but the converse is not true in general.

C. If the first term of a sequence is its lub, then the sequence is decreasing, but the converse is not true in general.

(e) Suppose S is a nonempty bounded subset of $\mathbb{R}$. Denote by $-S$ the set $\{-s \mid s \in S\}$. Which of the following is true about S?

A. $lub(-S) = lub(S)$ and $glb(-S) = glb(S)$

B. $lub(-S) = glb(S)$ and $glb(-S) = lub(S)$

C. $lub(-S) = -lub(S)$ and $glb(-S) = -glb(S)$

D. $lub(-S) = -glb(S)$ and $glb(-S) = -lub(S)$

2. Suppose that a sequence $a_n \to L$. Show that if $a_n \leq M$ for all n, then $L \leq M$. 10

   [Note that L is **NOT** necessarily the *lub* of the sequence.]

3. Define a sequence $\{a_n\}_{n\in\mathbb{N}}$ by $a_1 = 0$, $a_2 = 1$, and $\boxed{10}$

$$a_n = a_{n-1} + 2a_{n-2} \text{ for } n \geq 3$$

Let $b_n = \frac{a_n}{a_{n-1}}$ for $n \geq 2$. Assuming that the sequence b_n converges, find its limit.
[Note that the sequence $\{a_n\}$ diverges i.e. $\lim_{n\to\infty} a_n$ does not exist.]

4. (a) Find $\boxed{5}$

$$\lim_{n\to\infty} \frac{n^2 \ln n}{e^n}$$

(b) Let S_n = Sum of the first n terms of the arithmetic progression

$$1, 4, 7, 10, 13, \ldots$$

i. Find S_n. $\boxed{4}$

ii. Find $\boxed{6}$

$$\lim_{n\to\infty} \frac{S_n}{n^2}$$

5. Find whether the following sequences converge or diverge. You **MUST** specify what test(s) or theorem(s) you are using.

   (a) 5

$$\sum_{k=1}^{\infty} \frac{k+\cos k}{k^2+1}$$

   (b) 7

$$\sum_{k=1}^{\infty} \frac{1}{k}\left(2+\frac{1}{k}\right)^k$$

(c) 8

$$\sum_{k=1}^{\infty} \frac{k!}{k^k}$$