# Math 2208: Ordinary Differential Equations

## Lecture 19 Worksheet

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Title: Lorenz Equations

Summary: We will see an example of *chaos* in three dimensional ODE systems.

Write answers on this sheet, as this worksheet will count as a homework grade. Turn in before lecture concludes. You are allowed to work with others, but turn your own worksheet. I will walk around and answer your questions.

**Note:** Systems in three dimensions (the topic of this worksheet) will NOT be on the final.

## Preliminaries

### Stability of equilibrium solutions

**Definition 1.1**

If every solution that is close to an equilibrium stays close to that equilibrium for all time, then the equilibrium solution is called *stable*. Otherwise, it is called *unstable*.

**Question 1.**

(a) Circle the types of equilibrium solutions that are stable according to the above definition:

Source, spiral source, saddle, sink spiral sink, center

(b) Fill in the blanks with either the word "*stable*" or "*unstable*":

If an equilibrium solution has eigenvalues with real parts that are non-positive, then the equilibrium solution is ________________.

If an equilibrium solution has at least one eigenvalue with a positive real part, then the equilibrium solution is ________________.

(c) It turns out, the conclusion you made in problem (b) holds even for systems with more than two dependent variables. For example, if we have a 3x3 matrix for a linear system, we can compute the eigenvlaues. By inspecting the real parts of the eigenvalues only, we can determine the stability. Does this make sense, yes or no? ________________.

If yes, please continue. If not, call me over.

## Three Dimensional Systems

We want to analyze the so-called Lorenz equations, which is a famous system of differential equations derived by Edward Lorenz when studying convection rolls in the atmosphere,

$$\frac{dx}{dt} = \alpha(y-x) \quad (1)$$

$$\frac{dy}{dt} = x(\rho-z)-y \quad (2)$$

$$\frac{dz}{dt} = -\beta z+xy \quad (3)$$

where $\alpha, \beta$ and $\rho$ are real parameter values. Note that $(x,y,z)=(0,0,0)$ is an equilibrium solution to this ODE for all parameter values.

$$(x,y,z)=\left(\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho-1\right)$$

and

$$(x,y,z)=\left(-\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}, \rho-1\right)$$

are equilibrium solutions if $\rho>1$. There are no other equilibria.

### Linearization in three dimensions

**Question 2.**

Write down the Jacobian matrix of the Lorenz system (the Jacobian matrix for a 3 dimensional system is a natural extension of the one you know for two dimensions. So if you think about it for a moment, you can probably deduce the correct matrix). You can write your answer in terms of $x,y,z$ (you don't have to evaluate the Jacobian at an equilibrium solution)

**Question 3.**

The formula for eigenvalues of 3x3 matrices is quite nasty. We will use Octave to do that for us. Download the file `EigenvaluesLorenz.m` from Blackboard and then open it in Octave. The code assumes first that $\alpha=10$, $\rho=5$ and $\beta=8/3$. Click the "Run" button and inspect the output. Classify each of the three equilibrium solutions as stable or unstable. Use notation like

$$(x,y,z): \text{stable}$$

where you fill in numerical values for $x, y, z$.

### Question 4.

Consider a solution with the initial value $x(0) = 1$, $y(0) = 0, z(0) = 0$ and $\alpha = 10$, $\rho = 5$ and $\beta = 8/3$. Based on what you know about the stability of the three equilibrium solutions, try to draw a very rough sketch of the solution in the $(x, y, z)$ plane. Try to focus on the long term and qualitative behavior. Also, indicate with a point each of the three equilibrium solutions on your graph. This won't be the easiest to draw, but try your best. Don't spend more than two to three minutes on this problem. We will draw the solution using computer soon.

## Using numerical methods to compute the solutions of the Lorenz equations

### Question 5.

Download the file `solve_lorenz.m`. This file computes a solution to the Lorenz equations, see Eq. (1) - (3), with $\alpha = 10$, $\beta = 8/3$ and $\rho = 5$ and $x(0) = 1$, $y(0) = 0, z(0) = 0$ for $t$ from 0 to 100. A plot of the solution in the three dimensional phase space $(x, y, z)$ will appear. The green dot is the initial value, and the red dot is the solution at $t = 100$.

Click the left-most icon on the toolbar labelled "Rotate". Now you can change the angle how you view the three dimensional plot. Note, the value of $x$,$y$, and $z$ is printed on the command line at the final time $t = 100$. Is the result consistent with your analysis in problem (4)? What was the value of the solution at time $t = 100$?

$(x(100), y(100), z(100)) =$

### Question 6.

Consider the same problem but with an initial value just slightly different, e.g. $x(0) = 1.001$, $y(0) = 0$, $z(0) = 0$. How would you expect the solution curve to look like compared to problem 5? In particular, how would you expect the solution at time $t = 100$ in this scenario when compared to the one in problem 5?

### Question 7.

Re-run the program with with the new value of $x$. What was the value of the solution at time $t = 100$?

$(x(100), y(100), z(100)) =$

## Something strange in the Lorenz equations

### Question 8.

Repeat questions 3 and 4, but change the parameters to $\alpha = 10$, $\rho = 28$ and $\beta = 8/3$.

### Question 9.

Run the `solve_lorenz.m` program once again, but change the value of $\rho$ to $\rho = 28$ (see line 19, the variable is called rho), and change the initial value back to $x(0) = 1$, $y(0) = 0, z(0) = 0$.

Inspect the graph. Is the result consistent with your analysis in problem 8? What is different?

### Question 10.

Consider problem 8, but with an initial value to be just slightly different, like $x(0) = 1.001$, $y(0) = 0, z(0) = 0$. How would you expect the solutions at time $t = 100$ to compare in this scenario when compared to the one in problem 8?

### ■ Question 11.

Re-run `solve_lorenz.m` with $x(0) = 1.001$, $y(0) = 0, z(0) = 0$ and $\alpha = 10$, $\beta = 8/3$ and $\rho = 28$. How does this compare with your expectation in 8?

### ■ Question 12.

Re-run the program with $x(0) = 1.000001$, $y(0) = 0, z(0) = 0$ and $\alpha = 10$, $\beta = 8/3$ and $\rho = 28$. How does this compare with your expectation in 11? Why is this surprising?

### ■ Question 13.

What you just observed is exactly what Edward Lorenz did in the 60s. It was something never seen, or even thought of before. The equations are completely deterministic (no statistical variation), yet the solutions can change drastically, even if the initial conditions are changed only slightly. All solutions live on that funny looking butterfly surface, which is now called a strange attractor. This is an example of Chaos, which you can learn more about in *Math 3208: Advanced Dynamics*. Weather systems have this same chaotic property (infinite sensitivity to initial conditions). What does this tell you about long term weather forecasts?