Teaching Portfolio

Subhadip Chowdhury

Chapter 1

Teaching Philosophy and Methodology

I consider teaching and communicating math an integral part of my life. It always fills me with great joy to witness a classroom full of students make the transition from being doubtful and confused to confident and knowledgeable on a topic, or to help a colleague gain an appreciation for a new and abstract concept. The 2019-2020 academic year marks my sixth year as the instructor of record for a range of pure and applied/interdisciplinary undergraduate Mathematics courses at Bowdoin College and at the University of Chicago, both at introductory and advanced levels, all with great feedback. I have also mentored a diverse group of high school and undergraduate students through *independent study* sessions and REUs, trained students for *Olympiads* and *Putnam*, and have participated in various outreach activities specifically geared towards first generation college students and minority groups.

§1. Active and Inclusive Classroom

From the first day of lecture, I constantly monitor the classroom dynamic to ensure that no one feels isolated and that I pay equal attention and provide honest helpful guidance to both the capable and the less experienced students. By familiarizing myself with their backgrounds and interests through an introduction form, I carefully form small groups that complement each others' strengths and weaknesses. I start planning early to craft rigorous in-class *worksheets* that encourage students to make necessary efforts to discuss, analyze, and discover the relevant results themselves; often this involves working in groups, parallelly on their assigned blackboard. This allows me to engage with every students, understand their needs and modify the instructions on the spot accordingly. Students have found this an integral part of their learning process and always responded very enthusiastically in their feedback: *"[he] was engaging, excited, and patient. He sought class participation from everyone and helped people reach an answer and understanding as necessary."*

§2. Developing Applicable Skills

To make sure that my students are able to demonstrate skills for applying the textbook knowledge to solve complex real life problems in a diverse team environment, I have carefully designed several relevant and practical group projects that act as an extension to their assignments. For example, my students learned the geometry of rotary engines and rocket propulsion in multivariable calculus. In linear algebra, they applied the theory behind Google's page rank algorithm to network flows and learned basics of cryptography. In ordinary differential equation, students analyzed the hysteresis of Spruce Budworm population and found the best kick-flow regimen of caffeine consumption. Some students have used words like *'quest'* and *'adventure'* to describe these projects in their feedback. They liked that it "*...slightly strays away from class - keeps things interesting*" and found the experience satisfying and enjoyable.

§3. Commitment to Diversity

It is important to me that each student is treated as an individual, that multiple perspectives, experiences, and identities are valued and promoted. To adapt the style and content of my teaching that reflect their diversity, I have taken steps like promoting *visual learning* for non-native English speakers, spending extra times with struggling or special-needs students, and in general, create more heterogeneous contents using real life examples from all facets of our society. Students have found my colorful mathematical drawings and physical cues one of my biggest strengths during lectures.

I am fortunate to posses the unique experience of teaching a group of academically talented incoming first-year students at UChicago through the *Chicago Academic Achievement Program* (CAAP), many of whom were first-generation college students or from low-income communities. More recently, I have participated in the *Bowdoin Science Experience*, an orientation program designed towards students from groups underrepresented in the sciences and currently help organize a weekly *Study Group* for students of color in Math, CS and Physics. More details on these and my other outreach efforts can be found in my *Diversity Statement*.

§4. Teaching Beyond the Classroom

I firmly believe that the process of teaching is not confined to the classroom and it is important to keep curious minds busy by engaging them in intellectual reflections and allowing them to find connections to their own interests. I have done independent study courses with Economics students interested in game theory and about optimization techniques for asset revenues, with Computer Science students interested in machine learning, and have mentored several summer *REU* participants. In each case, I also helped them learn mathematical writing and coached them on their presentation skills. A detailed list of these are available in my CV.

I have experience in training high school students for the *Indian National Mathematical Olympiad* and am currently co-organizing a *Problem Solving Seminar* where we work on Math 'puzzles' in an effort to teach important mathematical strategies in a fun setting and train more enthusiastic students for the *Putnam Competition*. Several times over the last years, I have given invited talks in student seminars, participated in panel discussion with undergrads regarding grad school and higher studies, and given colloquium style talks in front of my peers. Through these, I have tried to expose the student community to interesting nonstandard mathematical ideas in an effort to destigmatize math education. Students have regularly praised my availability, enthusiasm, and willingness to help them both in and outside the classroom: *"[he] goes out of his way to explain things, and even beyond his role as an instructor, he cares about his students."*

§5. Formative Assessment Policies

I believe, as a professor, my main focus is not to deliver content, but rather focus on the process and help the students realize their own potential. As such, I have always prioritized rewarding academic growth of a student throughout the course over raw examination scores. Although there are weekly quizzes and two midterms, students get opportunities to make up partial grades, by taking follow-up tests or redoing particular questions to show that they correctly identified their mistakes. In higher-level courses, I sometimes give take-home exams and occasionally require a final Presentation to encourage students to learn effective communication of Math ideas. I have found that the average student performance has gone up when I allow them to demonstrate their abilities in a less constrained environment. Students have often praised my fair grading policy despite moderately hard exams.

§6. Integration of Technology

With the abundance of freely available personalized online content in the 21st century, one of my goals as a teacher is to prepare students to become self-regulated learners. By *curating a list of course-specific online contents* as part of the lecture and assignments, and by making them accountable to their use of resources in form of a cover sheet, I believe to have improved their ability to discern useful information from fraudulent or irrelevant ones.

Incorporating several Mathematica and Octave labs into my curriculum have helped students visualize abstract ideas more easily, observe simulated experiments, understand how to implement the theoretical algorithms in a practical time-efficient way and explore the limitations of computing technologies at the same time.

Finally, *by maintaining an online forum via Blackboard*, I have helped them keep track of their own performance and engage in productive debates more confidently. I have found that the seamless integration of digital and computational tools have significantly lowered the barrier of intimidation and allowed students more control over their learning paces, leading to more time for direct interaction during lecture.

§7. Creating a Supporting Environment

As a way to improve my students' experience and ensure proactive participation, I always strive to create an environment where they feel comfortable to take risks and learn from mistakes. This includes affirming constructive criticisms and recasting negative ones, which empowers them to see criticism as opportunities for advancement, not instances of personal failing. One of my students wrote: *''[he] is just fantastic at instilling a love for math...His good sense of humor always made lectures more fun.''*

In a complimentary direction, I conduct *anonymous student surveys* every three-to-five weeks and try to continually improve myself based on their suggestion. Through weekly teaching seminars at Bowdoin College, the *MAA mentoring network*, and *Bowdoin Teaching Triangle* program, and other pedagogical workshops, I have gained new insights about best teaching practices by following advice from more experienced colleagues.

§8. Future Goals

I have been and always will be very passionate about teaching. Building upon my current experience, I have many specific ideas that I wish to incorporate into my role as a future instructor. Some of them include: developing AR/VR or role-playing-games into the classroom, similar to RTTP, to innovate and transform STEM instruction with new forms of representations and teaching using discovery learning methods and experiment with bolder pedagogical ideas. Finally, by bringing in a wide variety of perspectives, I hope to impact and get support from my peers in designing approaches towards broader, more widely applicable, and more memorable learning.

Chapter 2

Teaching Experience and Responsibilities

§1. Bowdoin College

Over the 2018-2020 academic years, I have been fortunate to posses a liberal arts teaching experience at Bowdoin College. As a Visiting Assistant Professor, I was responsible for designing my own course curriculum, planning lectures and worksheets, designing and grading exams, holding office hours, and assigning individual homeworks and group projects. I also coordinated and mentored several graders, teaching assistant and study group leaders. Brief description of each of the courses I have taught are listed below.

1.1. **Math 2208, Ordinary Differential Equations**

- **Meeting Schedule** - Fall 2019, Spring 2020 - 3 hours per week
- **Class Size** - 20

This course, intended for Junior and Senior Math majors, is designed as a gateway course for students interested in Applied Math. I taught this course using a hybrid discovery method. This was the first applied course I had taught at Bowdoin college. I spent a significant amount of time carefully designing lecture worksheets that consisted of leading questions which helped students figure out the content for themselves. Although a part of the lecture was spent by me presenting on board, a lot of it was spent doing group work on blackboards and experimenting with software. Besides classical methods for solving differential equations, the main emphasis was on modern, qualitative techniques for studying the behavior of solutions to differential equations. We also worked on applications of ODEs in catastrophe theory, flow-kick regimen, resonance, market economy and auto-catalytic biochemical oscillations through multiple lab sessions and projects. Some of the projects and worksheets can be found in the appendix.

1.2. Math 2000, Linear Algebra

- **Meeting Schedule** - Spring 2019 - 3 hours per week
- **Class Size** - 11

This course is intended for Sophomores and Juniors and is designed as a gateway course for Mathematics and interdisciplinary majors. The students taking this course were not expected to have experience with writing proofs. As such, in an effort to make the course less dry, we spent a significant amount of time looking at various applications drawn from flight networks, cryptography, error correcting codes, population dynamics, Markov chains and Google page-rank algorithm, computer graphics, and optimization techniques using least-squares approximations. The students were assessed with multi-part in-class and take-home final exams. The goal was to make sure they are proficient in conceptual and numerical techniques as well as are able to apply their knowledge to practical applications. Some of the projects and exams can be found in the appendix.

1.3. Math 1800, Multivariable Calculus

- **Meeting Schedule** - Fall 2018, Spring & Fall 2019, Spring 2020 - 4.5 hours per week
- **Class Size** - 12 on average

This course is one of my most favorite course to teach. It's aimed towards mathematically inclined students, mostly Freshmen and Sophomores, who have learned differential and integral calculus, and would like to broaden their horizon. I have taught it several times over the last few years and usually the class size is smaller compared to other sections because of less favorable meeting times (three times a week). However the smaller class size allows me incorporate a lot of group discussion style techniques fairly regularly. I can easily keep track of every students' performance and struggles, and could create individualized work for them to catch up with the rest of the class. I have created and refined lecture notes, worksheets, labs, and group projects for this class over the year all of which create an ecosystem where the students learn higher dimensional abstract concepts with relative ease as they get to approach it from numerous viewpoints. Students learned applications of regression techniques in data science, an introduction to the Gradient descent method of optimization techniques used in machine learning, practical modelling of climate change evidences, the Normal probability distribution, and mathematics behind rotary engines and rocket propulsion. Students heavily relied on demonstrations using *Mathematica* and *Desmos*, both to visualize three dimensional pictures of surfaces, vector fields etc. as well as to learn numerical approximation techniques.

1.4. Independent Study

In 2019-2020 academic year, I have and will be mentoring two undergraduate student projects and independent studies. The topics are: (i) Asset revenue modelling using differential equations and machine learning topological dynamics and (ii) non-euclidean geometry and tiling.

1.5. Math 1600, Differential Calculus

- **Meeting Schedule** - Fall 2018 - 4.5 hours per week
- **Class Size** - 32

This course was aimed towards Freshmen and Sophomores from various backgrounds as one of the introductory Mathematics courses offered at Bowdoin College. For a lot of the students, this was the first college Math course and I wanted to make sure they learn the proper way to think about Math from the very beginning. Building on the traditional course structure, one of my main focus in teaching this course was to make sure students are able to interpret and describe symbolic equations using words and conversely be able to transform practical examples and word problems into mathematical models. Over the semester, I created a number of lab sessions which also helped solidify the abstract ideas by doing numerical estimations through Mathematica, and by describing how to implement various algorithms e.g. the Newton-Raphson method.

§2. University of Chicago

Besides my liberal arts teaching experience, I was also fortunate to have the opportunity of being an instructor during and after my PhD at the University of Chicago.

2.1. Proof-Based Methods

- **Meeting Schedule** - Summer 2018 - 6 per week for six weeks
- **Class Size** - 16

After finishing my PhD in summer 2018, I had the unique experience of teaching an *Introduction to Proof* style class to a group of academically talented incoming first-year students at UChicago through the *Chicago Academic Achievement Program* Summer academy, conducted by the *Center for College Student Success*. This class was designed to expose the students to the academic rigor expected of them as they enroll into introductory Math courses at the college, as well as provide a support framework to help them navigate through the new social and cultural norms.

As a class designed essentially to develop Math reasoning, we covered ideas and problem solving strategies from a broad area of topics such as Number Theory, Combinatorics, Graph Theory, Sequences, and limit Calculus. Besides the final exam, the students also were required to give a presentation in front of their peers which I believe helped them with their Mathematical writing and interaction skills. I tried to keep the atmosphere of the class as casual as possible so that they do not get overloaded with too much expectation. The syllabus for this class is available in the appendix.

2.2. Math 195, Mathematical Methods for Social Science

- **Meeting Schedule** - Fall 2017, Fall 2018 - 3 hours per week
- **Class Size** - 18 on average

The course consists of topics that are important for students who are planning to become majors in Economics, Political Science, Mathematical Linguistics etc. As such we covered vectors and multivariable calculus up to optimization, but instead of talking about Green's theorem, we covered linear programming next and finally sequences and series with the goal of learning Taylor approximations.

2.3. Independent Study

During Fall 2018, I mentored two interested talented students in my Math 195 course via an independent study of Game Theory and a project on *Least Unique Bid Auction*, where we discussed the usage of Lagrange Multipliers for finding complicated Probability estimates.

2.4. Math 196, Linear Algebra

- **Meeting Schedule** - Summer 2017 - 6 hours per week for 5 weeks
- **Class Size** - 18 on average

This course was offered through the *Graham School of Continuing Liberal and Professional Studies* for computational linear algebra, intended primarily for students in the social sciences who have completed single and multivariable calculus sequence. However, the students weren't expected to have much experience with writing proofs and as such, we spent a lot of time working on examples from many disciplines, in particular ones that relate to their primary fields of interest.

2.5. Math 150's, Standard Calculus Sequence

- **Meeting Schedule** - 3 hours per week
- **Class Size** - 15-30

As a graduate student college instructor, I taught independent section of Calculus courses in 2014-2017. The yearlong rigorous one-variable and multi-variable *standard Calculus* sequence (taught thrice) is designed for science, economics and Math majors. As the instructor of record, I was responsible for designing my own course curriculum, planning lectures, designing and grading exams, holding office hours, and assigning homework. I also mentored teaching assistants, and coordinated junior tutors.

2.6. Math 133, Elementary Functions and Calculus

- **Meeting Schedule** - 3 hours per week
- **Class Size** - 6

I taught a quarter long course on Vector calculus titled 'Elementary Functions and Calculus' to non-science (mostly History, English, and Theater) majors. Teaching students with very little technical background was an unparalleled learning experience.

§3. Other Academic Service

3.1. Math competitions and Problem Solving sessions

I coorganize a weekly Problem Solving Session at Bowdoin College where we work on Math 'puzzles'. The goal is on one hand to teach important mathematical strategies in a fun setting and on the other, train more enthusiastic students for the *Putnam Competition*.

3.2. Undergraduate Mentoring - DRP and REU

At the University of Chicago, I mentored eight undergraduate students (during 2014-2017) through the *Directed Reading Program* (DRP) and the summer *Research Experience for Undergraduates* (REU) on a wide array of topics from geometry, linear algebra, topology, dynamics of group action etc. We usually met twice a week for about 10 weeks, where the students would discuss a paper they have read and any original work they have done, followed by me outlining the next possible direction of approach and available useful literature. In both cases, I also helped them learn mathematical writing and coached them for an end-of-quarter presentation or written paper. The list of students and their papers are available in my CV.

3.3. As a Teaching Assistant and Grader

In 2013-2014 academic year, I worked with professor Eugenia Cheng as a teaching assistant for a year-long *Honors Calculus* sequence, and later worked as a grader for graduate courses on Algebraic Topology, Differential Topology, Differential Geometry, and Riemannian Geometry. Details on these are listed in my CV.

Chapter 3

Teaching Effectiveness - Student Evaluations Numerical Summary

§1. Bowdoin College

A numerical average of all student responses for all the courses I taught at Bowdoin College last two semesters (Spring 2019 and Fall 2018) is attached below. All of the evaluations in full are available on request.

How much did this course contribute to your education?

Name	Resp	Mean
Overall	21	4.33
Linear Algebra	11	4.36
Multivariate Calculus	10	4.30

How effectively did the instructor make use of class sessions to advance your learning?

Name	Resp	Mean
Overall	21	4.20
Linear Algebra	11	4.00
Multivariate Calculus	10	4.40

How helpful to your learning was the feedback you received in the course?

Name	Resp	Mean
Overall	21	4.38
Linear Algebra	11	4.45
Multivariate Calculus	10	4.30

Did you communicate with the instructor outside of class?

Name	% Yes	% No
Overall	85.71%	14.29%
Linear Algebra	90.91%	9.09%
Multivariate Calculus	80.00%	20.00%

How helpful to your learning were these communications with the instructor outside of class?

Name	Resp	Mean
Overall	18	4.84
Linear Algebra	10	4.80
Multivariate Calculus	8	4.88

How accessible was the instructor outside of class?

Name	Resp	Mean
Overall	21	4.71
Linear Algebra	11	4.82
Multivariate Calculus	10	4.60

What is your overall rating of the instructor as a teacher?

Name	Resp	Mean
Overall	20	4.05
Linear Algebra	10	4.20
Multivariate Calculus	10	3.90

How much did this course contribute to your education?

Name	Resp	Mean
Overall	38	3.77
Differential Calculus	30	3.53
Multivariate Calculus	8	4.00

How effectively did the instructor make use of class sessions to advance your learning?

Name	Resp	Mean
Overall	38	3.67
Differential Calculus	30	3.33
Multivariate Calculus	8	4.00

How helpful to your learning was the feedback you received in the course?

Name	Resp	Mean
Overall	37	3.64
Differential Calculus	29	3.41
Multivariate Calculus	8	3.88

Did you communicate with the instructor outside of class?

Name	% Yes	% No
Overall	81.58 %	18.42 %
Differential Calculus	83.33 %	16.67 %
Multivariate Calculus	75.00 %	25.00 %

How helpful to your learning were these communications with the instructor outside of class?

Name	Resp	Mean
Overall	31	4.29
Differential Calculus	25	4.08
Multivariate Calculus	6	4.50

How accessible was the instructor outside of class?

Name	Resp	Mean
Overall	38	4.41
Differential Calculus	30	4.20
Multivariate Calculus	8	4.63

What is your overall rating of the instructor as a teacher?

Name	Resp	Mean
Overall	38	3.60
Differential Calculus	30	3.33
Multivariate Calculus	8	3.88

§2. University of Chicago

A numerical average of all student responses for all the courses I taught at UChicago is attached below. All of the evaluations in full are available on request.

Chowdhury, Subhadip

Course Evaluations

Key of Values

5 - strongly agree
4 - agree
3 - neutral
2 - disagree
1 - strongly disagree

#1: Instructor was organized
#2: Lectures were clear and understandable
#3: Lectures were interesting
#4: instructor exhibited a positive attitude towards the students
#5: Instructor was accessible outside of class
#6: I would recommend this instructor to others

Quarter	Course Number	Section	Instructor (Last)	Instructor (First)	#Eval	#Stud	#1	#2	#3	#4	#5	#6	Overall
Winter 2018	19520	59	Chowdhury	Subhadip	9	10	4.38	4.25	4.00	4.75	4.43	4.38	4.36
Autumn 2017	19520	41	Chowdhury	Subhadip	20	27	3.90	4.10	3.63	4.50	4.56	4.05	4.12
Winter 2017	15300	45	Chowdhury	Subhadip	18	20	3.61	3.56	3.44	3.67	4.28	3.28	3.64
Autumn 2016	15200	45	Chowdhury	Subhadip	28	28	3.79	3.75	3.57	4.00	4.56	3.57	3.87
Spring 2016	13300	22	Chowdhury	Subhadip	6	6	4.17	4.33	3.67	4.50	4.67	4.17	4.25
Winter 2016	15300	48	Chowdhury	Subhaddip	7	8	4.43	4.43	4.43	5.00	5.00	4.71	4.67
Autumn 2015	15200	45	Chowdhury	Subhadip	14	16	3.50	3.64	4.21	4.36	4.50	4.00	4.04
Spring 2015	15300	41	Chowdhury	Subhadip	7	7	4.43	4.29	4.29	5.00	4.86	4.57	4.57
Winter 2015	15200	41	Chowdhury	Subhadip	23	27	3.70	3.61	3.35	4.00	4.50	3.43	3.76
Autumn 2014	15100	41	Chowdhury	Subhadip	22	23	3.32	3.32	3.32	4.18	4.67	3.55	3.72

Chapter 4

Teaching Improvement Activities

§1. Teaching Triangle

Over the academic years 2018-2020, I have been part of the *Teaching Triangle* program at Bowdoin College which involves professors from different departments visiting each others' classes in an effort to gain new insights into teaching and students' learning. Although the process was not evaluative, it facilitated a reflective conversation with my colleagues regarding evidence of student learning and new techniques of student engagement.

§2. Pedagogy Conference

I have attended several pedagogy conferences and workshops organized by *Chicago Center for Teaching* and *Center for Leraning and Teaching* at Bowdoin and Bates College. They taught me about the ever changing role of Math professors in the face of modern technological advances and how the teaching process has evolved to be relevant with the modern day and age. I also learned about new assessment techniques and ways to encourage proactive student involvement.

§3. Teaching Seminar

The Math department at Bowdoin college organizes weekly teaching seminars where we share our methods, coordinate reciprocal classroom visits, and get advice on handling unexpected issues in classroom from more experienced faculty members. Last year I also participated and once led a discussion about the *MAA Instructional Practices Guide*. These meetings have been invaluable for my professional development.

Chapter 5

Future Teaching Goals

Building upon my current experience, I have many specific ideas that I plan to incorporate into my role as a future instructor.

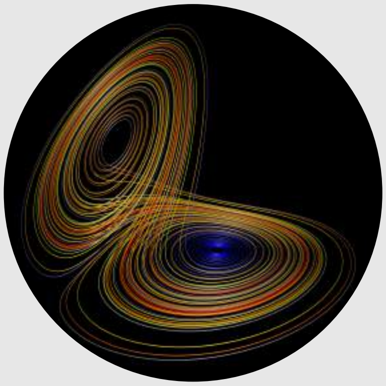
- I intend to develop and introduce augmented reality or role-playing-games into the classroom, similar to RTTP, to innovate and transform STEM instruction with new forms of representation. In a parallel direction I advocate new forms of assignments (e.g. videos or experimental) consistent with modern advancement in technology and welcome any opportunity of employing digital and computational tools to enrich my teaching.
- I would like to teach using discovery learning method and experiment with bolder pedagogical ideas.
- My plans also include summer programs for incoming freshmen for a successful transition to college life, outreach programs such as Math Circles for K-12 students, and professional networking between different communities etc.
- I plan to apply for the Project NExT fellowship and hope to gain guidance from professors in the MAA Mentoring Network to help me navigate through potential difficulties that may arise, develop professionally and learn how to maintain a work-life balance while actively engaging in research and scholarship.

Finally, by bringing in a wide variety of perspectives, I hope to impact and get support from my peers in designing approaches towards broader, more widely applicable, and more memorable learning.

Appendices

Appendix A

Sample Course Syllabi



Ordinary Differential Equations

Math 2208

Instructor Info

Subhadip Chowdhury

Office Hrs: TBD

Searles 104

subhadipchowdhury.github.io

schowdhu@bowdoin.edu

Course Info

Prereq: Math 2000

Mon & Wed

1:15p-2:40p

Searles 215

Lab Info

Wed

1:15p-2:40p

Searles 216

Course Goals

Math 2208 Specific Goals: Learn how to use differential equations (DEs) to model real world phenomena. There are three main categories of tools we focus on to analyze such DE models.

1. Know how to solve a variety of DEs (analytical techniques) "by hand".
2. Know how to analyze and say something about DEs without explicitly solving them (qualitative techniques).
3. Know how to approximate solutions using algorithms implemented on a computer (numerical techniques).

While we will learn several analytical techniques, understanding their limitations should be one of main takeaways of the course. This is why, when compared to more traditional courses on the subject, more emphasis is placed on qualitative and numerical techniques and the use of computer software.

Putting it all together: Given a real world phenomena, learn how to you derive a model, pick the right tool to analyze it (not all tools work on every model), and then interpret that result in the context of the real world phenomena.

Life Long Skills: Your abilities of general problem solving skills, self learning, self evaluation and how to formulate ideas and solutions will be refined throughout the course. This means problems appearing on homework or exam will not "be just like problems" you have seen before. You might be asked to explore new topics in homework before I cover them in class. Finally, how you present your solutions will also be evaluated.

Required Course Materials

Textbook

Differential Equations, 4^{th} edition, by Blanchard, Devaney and Hall

Software

GNU Octave, for your own computer. All computers in Searles 216 have Octave installed.

Others

DFIELD, and *PPLANE*, some Java applets, available from Blackboard.

Grading Scheme

15%	Weekly assignments
15%	Projects
15%	Quizzes
15%	Midterm 1
15%	Midterm 2
25%	Final Exam

Scores will NOT be curved. However, the cutoff percentage for letter grades will be set at my discretion. The weights are tentative and subject to change on an individual basis.

Important Dates

Midterm # 1 .... Wednesday, Feb 26, 2020
Midterm # 2 .... Wednesday, Apr 8, 2020
Final Exam .... TBA

Please let me know immediately of any problems with these dates. Please note that the date of the final exam is set by the Registrar's office and cannot be altered. Individual changes in final exam dates are allowed only for particularly serious situations such as three exams in a two-day period.

FAQs

Where can I find Octave?

You can download a free copy for your own use from gnu.org/software/octave.

I can't run DFIELD or PPLANE.

You may need Java Runtime environment to run these on your computer.

What is the late submission and make-up exam policy?

In general, late submission (even 15 mins late) of homework assignments will NOT be accepted. You may turn in *up to two* homeworks late, with no questions asked, so long as you notify me before the time the homework is due. Please see me in case of other extenuating circumstances.
You can make up an exam if certain unavoidable reasons prevent you from taking it and if you inform me in advance. Contact me as soon as possible if you are going to miss an exam. Missed exams can only be made up at my discretion, and are subject to a lost fraction of the grade.

Do I need to attend every class?

Although attendance is not directly part of your grade, it is very easy in a math class to fall behind after skipping even one class. You cannot be an effective and involved member of the class unless you are present! If you miss multiple classes in a row, you may expect a comment card.

Diversity and Inclusion Statement

I consider this classroom to be a place where you will be treated with respect, and I welcome individuals of all ages, backgrounds, beliefs, ethnicities, genders, gender identities, gender expressions, national origins, religious affiliations, sexual orientations, ability - and other visible and non-visible differences. All members of this class are expected to contribute to a respectful, welcoming and inclusive environment for every other member of the class.

No student is required to take an examination or fulfill other scheduled course requirements on recognized religious holidays. Please declare your intention to observe these holidays at the beginning of the semester.

Student Accommodations

- If you are a student with with learning needs that require special accommodation, please see Lesley Levy in the Office of Student Accessibility as soon as possible to make an appointment to discuss your special needs and to obtain an accommodations letter. Please email me as soon as possible in order to set up a time to discuss your learning preferences, challenges you may face learning this semester, and how we can create an effective learning experience for you. *In particular, I understand that the quizzes at the beginning of class can present a challenge, and I'm eager to discuss options with you.*
- As a student, you may experience a range of issues that can cause barriers to learning, such as strained relationships, increased anxiety, alcohol/drug problems, feeling down, difficulty concentrating and/or lack of motivation. These mental health concerns or stressful events may lead to diminished academic performance or reduced ability to participate in daily activities. Bowdoin College is committed to advancing the mental health and well-being of its students. If you or someone you know is feeling overwhelmed, depressed, and/or in need of support, services are available. You can learn more about the broad range of confidential mental health services available on campus at: `https://www.bowdoin.edu/counseling/`

Title IX

As a faculty member I am considered a Responsible Employee, per the Student Sexual Misconduct and Gender Based Violence Policy. While my goal is for you to be able to share information related to your life experiences through discussion and written work, I want to be make sure you understand that as a Responsible Employee I am required to report disclosures of sexual misconduct, dating violence, stalking, and/or sexual and gender-based harassment to the University's Title IX Coordinator, Benje Douglas. My reporting to Benje does NOT mean that any actions will be taken beyond him reaching out to you and trying to schedule a time to talk to see what assistance you might need to be successful as a student here at Bowdoin.

Academic Integrity

I support and adhere to the principles of The Bowdoin College Academic Honor Code. Your work should never be directly copied from another student and I will expect that *you are not reading solution manuals* for this textbook. In particular, I will assume all members of the class are trustworthy in their dealings with me as well as their fellow classmates. However, should a violation of this trust be discovered, it will be reported to the Judiciary Board. The goal is not vengeance against those who violate the Code but fairness for those who adhere to it. If you have any questions about the appropriateness of a particular situation, please communicate with me.

Components of the Course

- You will need to read the textbook. Several homework and suggested review problems will come directly from the book, and possibly quiz or exam problems. Some in class examples will be similar or identical to the book, but many will be different. The overall topic choice and course philosophy will be from the book.
- Weekly assignments will contain questions based on the textbook readings and class work. These assignments with their due dates will be regularly posted on Blackboard.
- You are encouraged to work on the weekly assignments with others, but you must write your final solution in your own words and you must complete and attach an Assignment Cover Sheet with every submission. This sheet can be downloaded from Blackboard.
- As is typical for ODE courses in the Mathematics Department, homework will generally be corrected by student graders who work under my supervision; this is done to ensure that you regularly receive graded assignments in a timely manner. *Please inform me immediately if you find any mistake in graded homeworks.*
- There will be three longer projects built around more challenging questions from the exercises, to showcase interesting applications of the study materials. These will require you to do programming in Octave or use the Java applets.
- You are allowed to work in groups of size at most 2 (two) to work on the projects. In your report you should include pictures and graphs of data and of solutions of your models *as appropriate.* Remember that one carefully chosen picture can be worth a thousand words, but a thousand pictures aren't worth anything. Final submissions must include a Project Report Cover Sheet (downloadable from Blackboard) on which the signatures of all participants must appear along with *brief but substantive* discussions of any issues confronted at your meetings. If any group member did not participate in an important aspect of the assignment, this must be stated in the Report. *One submission for your entire group will suffice.*
- Homeworks are extremely important, as it is the best way for you to engage with the material on a regular basis. I expect that in case you need extra practice with a certain concept, you will seek *extra, unassigned problems from the textbook to work out*; I am always happy to discuss how to locate good practice problems in your book.
- The point of the homeworks is for you to work out what you do and don't understand. When your graded homework has been handed back to you, you should go through it and see if you understand what has been written on it by the grader. If you don't, you should come to office hours and ask.
- As you are solving problems in this course, remember that getting the "answer" is only one of the steps. Don't think of what you write as just showing your instructor that you have done the homework. Write as if you were explaining what you are doing to one of your classmates who missed that day of class. Think of writing as part of the process of learning. The more carefully and clearly you write your mathematics, the more likely it is to be correct, and the more likely you will be to remember it. *Correct answers without explanation will not reap full credit, but clear explanations with an incorrect answer can certainly earn partial credit.*
- We will semi-regularly go into the computer lab during the lecture period for either demonstrations or for you to do your own programming. Homework problems throughout the semester will require you to use Octave/Matlab or other applets.
- Student participation and collaboration is an integral part of this class and is highly valued. Everyone is expected to make thoughtful contributions in the form of questions (even if unprompted), statements, and reasoned arguments. You might be also occasionally invited to present something on the board. Whenever possible, there will be opportunities for you to work through practice problems in small groups during our class meetings. Paper copy of worksheets will be provided and an electronic copy will be available on Blackboard.
- Additionally, there will be occasional quizzes and two midterms given during the semester, as well as a final examination at the end of the semester. The final exam will be according to the Registrar's office schedule. All exams will emphasize the concepts of the course.

General Policies

- Be courteous when using mobile devices. Make sure your cell phone is turned fully off, or silent. If you must make or receive a call, please go outside the classroom.
- Use of laptops or tablets is permitted for note-taking and labs. Please turn off your Wi-fi and sound.
- For any private communication regarding this course, please email me from your bowdoin.edu email address. This is mainly for identity verification purposes.

Class Schedule

The following is a preliminary outline of the topics that we hope to cover. This is an idealized plan, and it *may be adjusted as the semester progresses.* But it should give some indication of the major topics to be covered in this class.

Week No.	Monday	Wednesday
1		22-Jan Syllabus Overview + 1.1 (Modelling via Differential Equations)
2	27-Jan 1.2-1.3 (Separtaion of Variable, Slope Field) + **Using DFIELD**	29-Jan **Intro to Octave - Basic Plotting + Euler's Method**
3	3-Feb 1.5 (Existence and Uniqueness Theorem)	5-Feb **Quiz 1 + ODE45**
4	10-Feb 1.6 (Equilibria and Phase Line)	12-Feb 1.9 (Integrating Factor)
5	17-Feb 1.7 (Bifurcation)	19-Feb ***Project 1 (The Spruce Budworm - Hysteresis and Cusp Catastrophe)***
6	24-Feb Bifurcation contd., Change of Variable techniques + **Quiz 2**	26-Feb **Review + Project 1 due**
	Midterm 1	
7	2-Mar 2.1 (Predator-Prey Model) + 3.1 (Linear System)	4-Mar 3.1 (Linear Systems contd.) + 3.2 (Straight line solutions) + **using PPLANE**
	Spring Break	
8	23-Mar 3.3 (Phase Portraits)	25-Mar 3.4 (Complex Eigenvalues)
9	30-Mar 3.5 (Equal and Zero Eigenvalues) + **Quiz 3**	1-Apr **Trace-Determinant Plane, Defective and Degenerate cases, Bifurcation**
10	6-Apr ***Project 2 (Romeo & Juliet - Bifurcation Diagrams of Love Affairs)***	8-Apr **Review**
	Midterm 2	
11	13-Apr **Second Order Linear ODEs, Simple Harmonic Oscillators**	15-Apr Forced Harmonic Oscillation, Method of Undetermined Coefficients + **Project 2 due**
12	20-Apr Undamped Forcing, Resonance + **Quiz 4**	22-Apr 5.1-5.2 (Equilibrium Point Analysis, Jacobian)
13	27-Apr Almost Linear Systems, Consequences of Poincaré–Bendixson theorem	29-Apr ***Project 3 (Glycolytic Oscillations - Hopf Bifurcation)***
14	4-May **Lorenz Equations**	6-May **Review + Project 3 due**

Multivariable Calculus

Mathematics 1800-C

Fall 2019

Instructor:	Subhadip Chowdhury	**Email:**	schowdhu@bowdoin.edu
Office Location:	Searles 104	**Office Phone:**	(207) 725-3572
Class Sessions:	MWF 10:40–11:35	**Classroom:**	Searles 113
Lab Sessions:	R 1:15–2:40	**Lab:**	Searles 117
Study Group Leader:	Mustafa Aydogdu	**Study Group Session:**	T 8-9PM

Course Webpage

All regular announcements, instructor office hours, daily individual homeworks, group projects, handouts, lab assignments and individual grades will be posted on Blackboard

`http://blackboard.bowdoin.edu`

Check this site on a regular basis to track your progress. General course policies, syllabus, tentative schedule and outline of the course will be also available as pdf files on Blackboard.

Textbooks and Supplies

- *Calculus: Multivariable*, 7^{th} edition, by Hughes-Hallet, Gleason, McCallum et al.

  A scanned copy of chapter 13 is available on Blackboard in case your book hasn't arrived in mail yet.

- *Mathematica*, for your own computer.

  Bowdoin has a license allowing students to download the program onto their personal computers. To learn how to download Mathematica from the Bowdoin network, follow the steps at

  `https://bowdoin.teamdynamix.com/TDClient/KB/ArticleDet?ID=25361`

- *A scientific calculator*

  Though Mathematica will be our most commonly used technology tool, you should also have a scientific calculator. The use of calculators is NOT permitted for most in-class exams. But you may certainly use them when completing homework assignments, and occasionally this may be required.

Prerequisites

In order to be considered for admission into Math 1800 you must either have

1. completed Bowdoin's Math 1700 or Math 1750, or
2. been given a mathematics placement of Math 1800 when you entered Bowdoin.

If you do not satisfy at least one of these two conditions you will need the permission of the Chair of the Mathematics Department in order to register for Math 1800. No prior experience with Mathematica is required but a familiarity with mathematical computing softwares is encouraged.

The MCSR Distribution Requirement

Math 1800 can be used to satisfy Bowdoin's Mathematical, Computational, or Statistical Reasoning (MCSR) distribution requirement. Courses in this category enable students to use mathematics and quantitative models and techniques to understand the world around them either by learning the general tools of mathematics and statistics or by applying them in a subject area.

In Math 1800 you will learn how to apply the tools of calculus to perform fundamental computations and solve fundamental problems in two- and three-dimensions. We live in a three-dimensional world, enough of a reason to require expanding calculus techniques to functions of more than one variable. But dimensionality refers to more than physical dimensions. From this point-of-view (especially in an era of "big data") we often confront problems with literally thousands of dimensions. Math 1800 provides the first steps into how calculus is applied in these multi-dimensional situations.

Office Hours

- MW 3-4:30PM, T 2-4PM, R 5:45-7PM. These time slots are common for all the courses I am teaching this semester.
- If you can't make it to any of the weekly office hours, you can email me to schedule appointments with me. These will depend on my availability.
- I am usually in the office every weekday about 10-6PM. *If my door is open*, you are welcome to knock on my door and come in with quick questions.
- Any and all questions are welcome in class or in my office, but be aware that I will not simply "give you the answer" to any problem. Big-picture questions beyond "How do I solve this problem?" are highly encouraged.
- I also welcome questions through the discussion forum available on Blackboard. Though I strive to answer all online questions as clearly as possible, please realize that certain questions are best answered in a face-to-face discussion.

Course Description

The emphasis of the course will be on developing an understanding of the calculus of functions of two and three variables, as well as the geometry of associated curves and surfaces in two and three dimensions. Multivariable calculus is a fundamental pillar for many other things:

- It *extends single variable calculus to higher dimensions*. You will see that the structures are much richer than in single variable and that the fundamental theorem of calculus generalizes to higher dimensions.
- It *provides vocabulary* for understanding fundamental processes and phenomena. Examples are planetary motion, economics, waves, heat, finance, epidemiology, quantum mechanics or optimization.
- It *teaches important background* needed in social sciences, life sciences and economics. But it is rigorous enough that it is also suited for students in core sciences like physics, mathematics or computer science.
- It *builds tools for describing geometrical objects* like curves, surfaces, solids and intuition which is needed in other fields like linear algebra or data analysis. Geometry is currently an extremely popular topic: tomography methods in medicine, computer games, Google earth, social network analysis all use geometry.

- It *relates to culture and history*. The quest for answering questions like "where do we come from", "what will future bring us", "how can we optimize our time in between" all use calculus. The history of calculus contains fascinating stories, starting from Archimedes, 2300 years ago up to the modern times, where new branches of multivariable calculus are developed to understand the structure of nature.
- It *develops problem solving methods*. Examples are optimization problems with and without constraints (which is the bread and butter for economics), geometric problems, computations with scalar and vector fields, area and volume computations.
- It *makes you acquainted with a powerful computer algebra system* which allows you to see the mathematics from a different perspective. Such systems are more and more needed for visualization, experimentation and to build laboratories for your own research.
- It *prepares you for further study in other fields*. Not only in mathematics and its applications, but also in seemingly unrelated fields like game theory, probability theory, discrete mathematics, sociology, or number theory, where similar structures and problems appear, even in a discrete setting. Without geometric intuition and paradigms learned in calculus, it is rather hard to work in those fields.
- It *improves thinking skills*, problem solving skills, visualization skills as well as computing skills. You will see the power of logical thinking and deduction and why mathematics is timeless.

The Components of the Course

- You will need to read the textbook. In particular, the designated sections of the text should be read *prior/concurrently to the class sessions for which they are assigned.* This will get updated in the lecture notes and homework. You do not need to submit the solutions for the practice problems in the lecture notes, but you should try to work them out yourself to solidify your understanding. We will explain the material and work out harder examples from the section in class.
- Individual assignments will contain questions based on the textbook readings and class work. These assignments with their due dates will be regularly posted on Blackboard. The typical due date pattern is:
  - Monday's homework is due Friday same week,
  - Wednesday and Friday's homeworks are due Wednesday next week.

  You are encouraged to work on the weekly assignments with others, but you must write your final solution in your own words and you must complete and attach an Assignment Cover Sheet with every submission. This sheet can be downloaded from Blackboard.

  As is typical for multivariable calculus courses in the Mathematics Department, homework will generally be corrected by student graders who work under my supervision; this is done to ensure that you regularly receive graded assignments in a timely manner. *Please inform me immediately if you find any mistake in graded homeworks.*
- Around six longer collaborative projects will be built around more challenging questions. Electronic copies of the assignment details will be available on Blackboard. These will be due typically within seven to ten days. The teams for the projects will be decided in second week and will change several times over the semester.

  The collaborative projects will be completed in your Assignment Group (of size 3-4). All members of the group must not only participate in the analysis of the project but should discuss the specific phrasing and organization of the final submission. Final submissions must include a Project Report Cover Sheet (downloadable from Blackboard) on which the signatures of all participants must appear along with *brief but substantive* discussions of the issues confronted at your meetings. If any group member did not participate in an important aspect of the assignment, this must be stated in the Report. *A single submission for your entire group will suffice.*

- In the computer lab sessions you will work on Mathematica projects designed to deepen your understanding of the primary course concepts. Depending on your familiarity with Mathematica, you may find that you complete labs during the lab period, or you may find that you need some more time to complete them as homework. Either way is fine. I will announce when the Lab Homework is due depending on the workload.
- Research shows that interactions and being active lead to deep learning. Thus, you can expect each class to contain portions where students will work on problems. Paper copy of handouts will be provided and an electronic copy will be available on Blackboard.
- Additionally, there will be occasional quizzes and two Midterms given during the semester as well as a Final Examination at the end of the semester. The midterms will be during Thursday Lab times. The final exam will be according to the Registrar's office schedule. All exams will emphasize the concepts of the course.

Grading Policy

- Grades will be given for each daily assignments, quiz, and exams. In addition, each lab will include short assignments that will be collected and graded. Both your score and how it ranks relative to the other scores in the class will determine your final grade.
- *Scores will* NOT *be curved. However, the cutoff percentage for letter grades will be set at my discretion.*
- The following weights are tentative and subject to change on an individual basis.

The partial weights are as follows:

Individual assignments	20%
Group assignments (projects + handouts)	15%
Quizzes	10%
Midterm 1	15%
Midterm 2	15%
Final exam	25%

Important Dates

Midterm # 1	Thursday, October 10, 2019
Midterm # 2	Thursday, November 14, 2019
Final Exam	TBA

Please let me know immediately of any problems with these dates. Please note that the date of the final exam is set by the Registrar's office and cannot be altered. Individual changes in final exam dates are allowed only for particularly serious situations such as three exams in a two-day period.

Assignment and Projects Policies

- *Often there will be no example in the text or in class work that exactly mirrors an assigned problem or project. This is by design.* To learn how to apply the principles discussed in the text and the class sessions, you cannot merely copy procedures you see laid out in examples.
- Homeworks are extremely important, as it is the best way for you to engage with the material on a regular basis. The problems assigned will be carefully chosen to highlight essential concepts. I also expect that in case you need extra practice with a certain concept, you will seek *extra, unassigned problems from the textbook to work out*; I am always happy to discuss how to locate good practice problems in your book.
- The point of the homework is for you to work out what you do and don't understand. When your graded homework has been handed back to you, you should go through it and see if you understand what has been written on it by the grader. If you don't, you should come to office hours and ask.
- As you are solving problems in this course, remember that getting the "answer" is only one of the steps. Don't think of what you write as just showing your instructor that you have done the homework. Write as if you were explaining what you are doing to one of your classmates who missed that day of class. Think of writing as part of the process of learning. The more carefully and clearly you write your mathematics, the more likely it is to be correct, and the more likely you will be to remember it. *Correct answers without explanation will not reap full credit, but clear explanations with an incorrect answer can certainly earn partial credit.*
- When appropriate you are encouraged to use Mathematica to help with problem solutions.

Late submission policies

- In general, late submission (even 15 mins late) of homework assignments will **NOT** be accepted. You may turn in *up to two* homeworks late, with no questions asked, so long as you notify me before the time the homework is due. If there are extenuating circumstances in your life you may be able to hand in more than two late homework. Please see me in such an event.
- You can make up an exam if certain unavoidable reasons prevent you from taking it and if you inform me in advance. Contact me as soon as possible if you are going to miss an exam. Missed exams can only be made up at my discretion, and are subject to a lost fraction of the grade.

Student Participation and Collaboration

Student participation is an integral part of this class and is highly valued. Everyone is expected to make thoughtful contributions in the form of questions (even if unprompted), statements, and reasoned arguments. You might be also occasionally invited to present something on the board. Whenever possible, there will be opportunities for you to work through practice problems in small groups during our class meetings.

Collaboration is an excellent way to facilitate learning (by formulating questions and answers verbally), and will help prepare you for your future (where you most likely will have to work with others at some point). Plus you may make some new friends! Please express yourself within the bounds of courtesy and respect. Please share your thoughts and be willing to listen attentively to perspectives that may differ from your own. Note that as a member of a group you are responsible not only for your own learning but also for the learning of the other members of your group. This means that when the work is completed and submitted, every member of the group should be able to explain how to solve all the problems.

Class Attendance

Attend every class. Although attendance is not directly part of your grade, it is very easy in a math class to fall behind after skipping even one class. You cannot be an effective and involved member of the class unless you are present!

General Policies

- Be courteous when using mobile devices. Make sure your cell phone is turned fully off, or silent. If you must make or receive a call, please go outside the classroom.
- Use of laptops or tablets is permitted for note-taking but only with prior permission. Please turn off your Wi-fi and sound.
- *There will be no class on Monday of the Thanksgiving week.*
- For any private communication regarding this course, please email me from your bowdoin.edu email address. This is mainly for identity verification purposes.

Miscellaneous Items of Interest

- It is my intent that students from all backgrounds and perspectives receive equitable access and opportunity in this course, that students' learning needs be addressed both in and out of class, and that the diversity students bring to this class be viewed as a resource, strength and benefit. It is my intent to employ materials and engage in activities and dialogue that are respectful of: gender identity, sexuality, disability, age, socioeconomic status, ethnicity, race, nationality, religion, and culture. Please share your preferences for your name and pronouns.
- No student is required to take an examination or fulfill other scheduled course requirements on recognized religious holidays. Students are expected to declare their intention to observe these holidays at the beginning of the semester.
- Students with documented accommodations have a right to have these met. I encourage you to see me in the first 2 week of class to discuss how your accommodations may support your learning process in this course. I highly encourage all students to meet with me in the first few weeks of class (or as soon as you become aware of your needs) to discuss your learning preferences, challenges you may face learning this semester, and how we can create an effective learning experience for you. *In particular, I understand that the quizzes at the beginning of class can present a challenge, and I'm eager to discuss options with you.* If you are interested in learning more about accommodations please see Lesley Levy in the Office of Student Accessibility

  https://www.bowdoin.edu/accessibility/student-accessibility-office/index.html

- As a student, you may experience a range of issues that can cause barriers to learning, such as strained relationships, increased anxiety, alcohol/drug problems, feeling down, difficulty concentrating and/or lack of motivation. These mental health concerns or stressful events may lead to diminished academic performance or reduced ability to participate in daily activities. Bowdoin College is committed to advancing the mental health and well-being of its students. If you or someone you know is feeling overwhelmed, depressed, and/or in need of support, services are available. You can learn more about the broad range of confidential mental health services available on campus at:

https://www.bowdoin.edu/counseling/

- As a faculty member I am considered a Responsible Employee, per the Student Sexual Misconduct and Gender Based Violence Policy. While my goal is for you to be able to share information related to your life experiences through discussion and written work, I want to be make sure you understand that as a Responsible Employee I am required to report disclosures of sexual misconduct, dating violence, stalking, and/or sexual and gender-based harassment to the University's Title IX Coordinator, Benje Douglas. My reporting to Benje does NOT mean that any actions will be taken beyond him reaching out to you and trying to schedule a time to talk to see what assistance you might need to be successful as a student here at Bowdoin. For more information please check out:

  www.bowdoin.edu/title-ix

- I support and adhere to the principles of The Bowdoin College Academic Honor Code. In particular, I will assume all members of the class are trustworthy in their dealings with me as well as their fellow classmates. However, should a violation of this trust be discovered, it will be reported to the Judiciary Board. The goal is not vengeance against those who violate the Code but fairness for those who adhere to it. If you have any questions about the appropriateness of a particular situation, please communicate with me.

Tentative Course outline and Schedule

The following is a preliminary outline of the topics that we hope to cover. This is an idealized plan, and it *may be adjusted as the semester progresses.* But it should give some indication of the major topics to be covered in this class.

Monday	Wednesday	Thursday	Friday
	4-Sep	5-Sep	6-Sep
	Syllabus Overview + 3D Coordinate Geometry + 13.1 (Vectors in 3D)	**Lab 0 (Intro to Mathematica)** + Vectors	13.3 (Dot Product, Angle, Projection)
9-Sep	11-Sep	12-Sep	13-Sep
13.4 (Cross Product, Area, Volume)	Lines and Planes	**Lab 1 (Lines and Planes)** + **Distances**	**Quiz 1** + **Handout 1**
16-Sep	18-Sep	19-Sep	20-Sep
12.1-12.2(Functions of several variables)	12.3, 12.5 (Contour Plots) + **Conic Sections and Quadric Surfaces**	**Lab 2 (3D Graphing)**	**Quiz 2** + **Handout 2**
23-Sep	25-Sep	26-Sep	27-Sep
12.4 (Linear Functions)	17.1 (Parametrized Curves - Straight line, Circle, Helixes)	**Handout 3 (Cycloid and Hypocycloid)** + **Epicycloid and the Rotary Engine**	17.2 (Arc length and Curvature)
30-Sep	2-Oct	3-Oct	4-Oct
Lab 3 (Parametric Plotting)	**Handout 4 (Lab contd. + Angles)**	**Quiz 3** + 14.1-14.2 (Partial Derivatives)	14.3 (Tangent Plane and Local Linearity)
7-Oct	9-Oct	10-Oct	11-Oct
Handout 5 (Review)	**Review**	**Midterm 1**	14.6 (Chain Rule)
14-Oct	16-Oct	17-Oct	18-Oct
Fall Vacation	14.4 (Gradients and Directional Derivatives)	**Lab 4 (Gradients and Contour Plots)**	14.5 (Three dimensional Gradient and Tangent Plane)
21-Oct	23-Oct	24-Oct	25-Oct
Quiz 4 + **Handout 6**	15.1 (Stationary Points) + **Mathematica Project**	**Lab 5 (Ordinary Linear Regression)**	14.6 (Clairaut's Theorem and Hessian)

Monday	Wednesday	Thursday	Friday
28-Oct	30-Oct	31-Oct	1-Nov
15.3 (Lagrange Multipliers) + **Rocket Science**	15.2 (Unconstrained Optimization)	**Lab 6 (Max/Min Problems) + Quiz 5**	16.1-16.2 (Definite Integral of Functions of Two Variables)
4-Nov	6-Nov	7-Nov	8-Nov
16.2-16.3 (Type I/II regions, Triple Integrals)	16.4 (Double Integral in Polar Coordinates) + **Normal Probability Distribution**	**Lab 7 (Volume Integration)**	Polar Volume Integration (Cylindrical Coordinates)
11-Nov	13-Nov	14-Nov	15-Nov
Handout 8 (Curves in Polar Coordinates)	**Review**	**Midterm 2**	Spherical Coordinates
18-Nov	20-Nov	21-Nov	22-Nov
17.3 (Vector Fields)	17.4 (Flow of a Vector Field)	**Handout 9**	18.1-18.2 (Line Intergrals on Paramterized Curves)
25-Nov	27-Nov	28-Nov	29-Nov
Thanksgiving Break			
2-Dec	4-Dec	5-Dec	6-Dec
18.3 (Gradient Fields - Path-Independent)	18.4 (Path-dependent fields, Circulation, Curl)	**Lab 8 (Vector Fields)**	18.4 (Path-Dependent Fields and Green's Theorem)
9-Dec	11-Dec	12-Dec	13-Dec
Applications and Generalizations of Green's Theorem	**Handout 10**	**Reading Period**	**Reading Period**

Proof-Based Methods in Calculus

CAAP Summer Academy 2018

Instructor:	Subhadip Chowdhury	**Email:**	subhadip@math.uchicago.edu
Teaching Assistant:	John Judge	**Email:**	jjudge@uchicago.edu
Time:	MTRF 11–12:30	**Classroom:**	RY 277

Course Webpages

1. For announcement and grades, check Canvas (http://canvas.uchicago.edu/)
2. For assignment and course policies, check
   https://subhadipchowdhury.github.io/teaching/Summer2018.CAAP

Office Hours

TR 7-9PM or by appointment, or email me your questions.

Textbooks:

We will be using materials mainly from the following two textbooks. I will also provide a summary of topics covered in each class, with appropriate references, at the beginning of assignments.

- *Number, Shape, & Symmetry*, 1^{st} edition, by Herrmann and Sally, Jr.
- *Calculus - One Variable*, 3^{rd} edition, by Salas, Hille, Etgen

Course Objectives

This course intends to acclimate incoming students to the sophistication of UChicago math courses, and to give them a head start on the harder topics in first-year calculus.

Homework Policy

- Homework is to be turned in on the given date at the **BEGINNING** of the class. Usually an assignment will be set in every class. Homework sets will be due **TWICE** a week.
- **Thursday and Friday's assignments are due on Tuesday next week. Monday and Tuesday's assignments are due on Friday (same week).**
- The point of the homework is for you to work out what you do and don't understand. You should help each other to understand things and come and ask me if all of you get stuck together. When your graded homework has been handed back to you, you should go through it and see if you understand what has been written on it by the TA. If you don't, you should come to office hours and ask.
- I encourage you to work together on homework, but you should make sure you understand what you have written down. If you write up what was done in a group without understanding it, that counts as cheating.

Participation

Student participation is an integral part of this class and is highly valued. Everyone is expected to make thoughtful contributions in the form of questions, statements, and reasoned arguments. You might be also occasionally invited to present something on the board. Please express yourself within the bounds of courtesy and respect. Please share your thoughts and be willing to listen attentively to perspectives that may differ from your own.

Class Project

In the second half of this course, you will be working a class project. This project is open-ended, and you can choose any topic that interests you. I will post a list of project ideas sometime around the middle of July. Possible projects include parsing the solution strategy of an interesting puzzle, applying mathematical ideas to model a real life example, or understanding the multistep proof of a complicated result etc. You are also encouraged to come up with ideas on your own. I think it is a very good idea to work in groups of two, but you may choose not to do so. The project has two components - an in-class Presentation and a written Final Paper. More guidelines and details on these will be posted soon.

Grading Policy

Apart from homework, and the class project, there will be three in-class quizzes and a final exam. The individual weights are as follows:

Participation	5%
Homework assignments	15%
Quizzes	15%
Class Project	25%
Final exam	40%

Important Dates

Quiz # 1	Friday, July 6, 2018
Quiz # 2	Friday, July 13, 2018
Quiz # 3	Thursday, July 19, 2018
Project Sign-up	Friday, July 20, 2018
Project Outline	Sunday, July 22, 2018
Progress Meetings	Thursday & Friday, July 26-27, 2018
In class Presentations	Thursday & Friday, August 2-3, 2018
Project Report Due	Sunday, August 5, 2018
Final Exam	Tuesday, August 7, 2015

Class Policy

- Regular attendance is essential and expected.
- Be courteous when using mobile devices. Make sure your cell phone is turned fully off, or silent. If you must make or receive a call, please go outside the classroom.
- Use of computers is permitted for note-taking (with prior permission). Please turn off your Wi-fi and sound.
- The final exam is based on all material covered in class. If you have to miss a lecture, then I strongly recommend you study the material you missed before you return to class. I recommend doing the following steps:
  - Look at the course schedule written below.
  - Find someone who was in class and make a copy of their notes,
  - Read the relevant sections from the textbooks, class note, Wikipedia, etc.

  Once you have done these steps, and you need more clarification on lectures you missed, email me to schedule an appointment to review the materials.
- The class on Friday July 20 will be spent discussing possible projects for presentation at the end of the course.
- For any communication regarding this course, please email me from your uchicago email address. This is mainly for identity verification purposes.

Miscellaneous Items of Interest

- This course is open to all students who meet the academic requirements for participation. Any student who has documented a need for accommodation should contact Student Disability Services and the faculty member or instructor privately to discuss the specific situation as soon as possible. Student Disability Services can be reached at (773) 702-6000 or disabilities@uchicago.edu. SDS staff will coordinate accommodations for students.
- At times, you may experience some academic challenges, and you may find it helpful to utilize some of the services provided by the Student Services of The College. Various professional staff and advanced students are committed to helping you address your academic challenges in a variety of ways. Visit their webpage to learn about tutoring and mentoring options (low-cost, individual, free group, and drop-in), guidance on study skills and time management, and one-on-one assistance to ensure that you are using the best possible strategies for success in your course work.

Tentative Course Outline

We will discuss the 'Main Topics' in class using the 'Illustrative Examples'. The 'Workout' problems will be covered if time permits and will be assigned as homework or reading topics otherwise.

Time	Description
June 28,29 [RF]	WARM-UP PROBLEMS **Illustrative Examples:** • The Pigeonhole Principle - Birthday Problem • The Invariance Principle - Königsberg Bridge Problem **Weekend Workout:** Friends and Strangers: $R(3,3) = 6$
July 2,3 [MT]	SET THEORY AND LOGIC **Main Topics:** • Number Sets - $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$. • Logical Comparisons using Examples - Implication, Inverse, Converse, Contrapositive, and Negation. Necessary vs. Sufficient. • Proof techniques and strategies **Illustrative Examples:** • Proof by Contradiction: The Extremal Principle - set of midpoints (biggest), $n\sqrt{2}$ is not an integer (smallest). • The Induction Principle - $1+2+\cdots+n$, $\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}$ **Wednesday Workout:** True/False logic quiz, Knights and knaves puzzle, NYT confirmation bias problem, The $0=1$ fallacy
July 5,6 [RF]	DIVISIBILTY AND PRIMES **Main Topics:** • Definition and fundamental properties of the Divisibility Relation • Greatest Common Divisor • Prime and Composite numbers **Illustrative Examples:** • The infinitude of primes • Set of triple primes • The greatest integer function $\lfloor . \rfloor$ - multiplicity of a prime divisor **Weekend Workout:** Sophie Germain Identity, Divisibility tests
July 9,10 [MT]	SEQUENCES

Main Topics:
- Definition - Explicit vs. Recursive
- Arithmetic and Geometric Progression
- Solving Linear Recurrence
- Idea of Convergence - Monotone Convergence Theorem?
- Calculating Limits

Illustrative Examples:
- Binet's Formula

Wednesday Workout: Periodic Sequences, Finding limit of Convergent Sequences

July 12,13 [RF] BASIC ENUMERATIVE COMBINATORICS

Main Topics:
- Counting!
- Binomial Theorem
- Riemann Sum?

Illustrative Examples:
- Pascal's Triangle and Binomial coefficients

Weekend Workout: Erdős-Szekeres theorem

July 16-19 [MTR] PRECALCULUS REVIEW

Main Topics:
- Fractions and Decimals
- Absolute value and Inequalities
- Factorization of quadratic (and higher degree) polynomials
- Trigonometry

Illustrative Examples:
- Decimal Expansion
- Triangle Inequality
- Sketching Region of Solution
- Triangle Problems

Wednesday Workout: Different Bases, Coordinate Geometry

July 20 [F] DISCUSSING CLASS PROJECT IDEAS

July 23,24 [MT] FUNCTIONS AND GRAPHS

Main Topics:
- Functions as Mappings
- Bijective Functions and Inverses
- Graphs of Functions

Illustrative Examples:
- Linear, Quadratic, Rational, Trigonometric, Exponential, and Logarithmic Functions.
- Dirichlet function and Thomae's function

Wednesday Workout: Properties of Transcendental functions

July 26,27 [RF] INTRODUCTION TO LIMITS

Main Topics:
- $\epsilon - \delta$ definition
- Arithmetic of Limits
- One-sided Limit

Illustrative Examples:
- Finding Limit from Graphs
- Linear and Quadratic functions

July 30,31 [MT] MORE ON LIMITS AND CONTINUITY

Main Topics:
- Definition of Continuity
- Types of Discontinuity

August 2,3 [RF] PROJECT PRESENTATIONS

August 6 [M] REVIEW

August 7 [T] FINAL EXAM

Appendix B

Sample Projects and Labs

Math 2208: Ordinary Differential Equations

Project 3: An Application from Chemistry - The Brusselator and Hopf Bifurcation

Fall 2019 **Subhadip Chowdhury** **Due: Dec 11**

The *Brusselator model* is a famous model of chemical reactions with oscillations. The dynamics and chemistry of oscillating reactions has been the subject of study for the last almost 60 years, starting with the work of Boris Belousov. While studying the Kreb's cycle, he witnessed a mixture of citric acid, bromate and cerium catalyst in a sulphuric acid solution undergoing periodic colour changes. These changes indicated the cycle formation and depletion of differently oxidized Ce(III) and Ce(IV) ions. This was the first reaction, where biochemical oscillations was observed in 1950. In 1961, Zhabotinsky reproduced Belousov's work and showed further oscillating reactions. The reaction mechanism is an example of an autocatalytic, oscillating chemical reaction. Autocatalytic reactions are chemical reactions in which at least one of the reactants is also a product and vice versa. The rate equations for autocatalytic reactions are fundamentally nonlinear.

Consider a Brusselator whose equations are given as follows[1]:

$$\mathbf{A} \xrightarrow{k_1} \mathbf{X}$$
$$\mathbf{B} + \mathbf{X} \xrightarrow{k_2} \mathbf{Y} + \mathbf{D}$$
$$2\mathbf{X} + \mathbf{Y} \xrightarrow{k_3} 3\mathbf{X}$$
$$\mathbf{X} \xrightarrow{k_4} \mathbf{E}$$

where **A** and **B** are reactants (substrates), **D** and **E** are products and **X** and **Y** are the autocatalytic reactants of the set of reactions. Also k_1, k_2, k_3 and k_4 are the rate of reactions for each component reaction.

The governing equations of the Brusselator are obtained using law of mass action (the rate of a chemical reaction is directly proportional to the product of the concentration of reactant) and the set of equations for the change in concentrations of **X** and **Y** are found to be (after some scaling):

$$\frac{dx}{dt} = a - (1+b)x + yx^2$$
$$\frac{dy}{dt} = bx - yx^2$$

where x and y represent the concentrations of the autocatalytic reactants and $a, b > 0$. Because of obvious reasons, we are only going to be interested in the case where x and y are positive.

We will be using `Desmos` and `PPlane` for this project.

Equilibrium Point Analysis

Read the above text and watch the linked video

https://www.youtube.com/watch?v=uWh8reiXq58

before you begin! The video uses different compounds than the original experiment by Belousov but the theory is still the same.

[1] https://en.wikipedia.org/wiki/Brusselator

Question 1.

Find the equation of the nullclines and the equilibrium point in terms of a and b.

Question 2.

Find the Jacobian of the system at the equilibrium point. Show that its trace is

$$T = b - 1 - a^2$$

and determinant is

$$D = a^2$$

Bifurcation

Question 3.

Suppose we keep a fixed and vary b. Write down the type of stability that should occur at the equilibrium in the following three cases:

a) $a = 1$ and $b = 1$. b) $a = 1$ and $b = 3$. c) $a = 1$ and $b = 5$.

Question 4.

We conclude that there are at least two bifurcation values of b when a is fixed at 1.

(a) Check that D is always positive, so the equilibrium is source or sink based on values of T.

(b) Draw the curves $T = 0$ and $T^2 = 4D$ in the (a, b)-parameter plane.

(c) Find the bifurcation values of b when $a = 1$.

Question 5.

Find all possible behaviour of the system as a function of parameters a and b. In particular, draw the regions in (a, b)-plane (remember that $a, b > 0$) that correspond to stable/unstable node or spirals.

Nullclines and Direction Field

Question 6.

Use Desmos to draw the nullclines. Set sliders for a and b.

(a) Move the slider to check that the nullclines intersect only once for all values of a and b. How does changing a and b change the x and y coordinates of the point of intersection? Is this consistent with the formula of equilibrium point you obtained above?

(b) Set $a = 1, b = 2.5$ and copy the resulting graphs of the nullclines on to paper. Draw the direction field along the nullclines.

Question 7.

Find whether x' and y' are positive or negative in other regions (4 of them) and use this information to draw the direction field arrows in those regions (on your paper).

Phase Portrait

■ **Question 8.**

Something very interesting happens in the phase portrait as the equilibrium becomes unstable. Let's use `pplane` to draw the phase portrait in each of the three cases from question (3). You may need to change the window width in each case for a comfortable viewing size.

Include a printout of the phase portrait with some sample solution curves in each case. You can collect all three on one page using e.g. MS Word to save paper.

Existence of a Limit Cycle

Could we have predicted that we would be getting a limit cycle when the equilibrium becomes unstable? Recall the four possibilities for a trajectory of a solution curve for a nonlinear system as a consequence of *Poincaré–Bendixson* theorem. Assuming the requirements hold, the only possibilities are either the solution curves that start near $(0,0)$ become unbounded or they spiral towards a limit cycle. How do we know they are not going to infinity? In the following steps we will work this out.

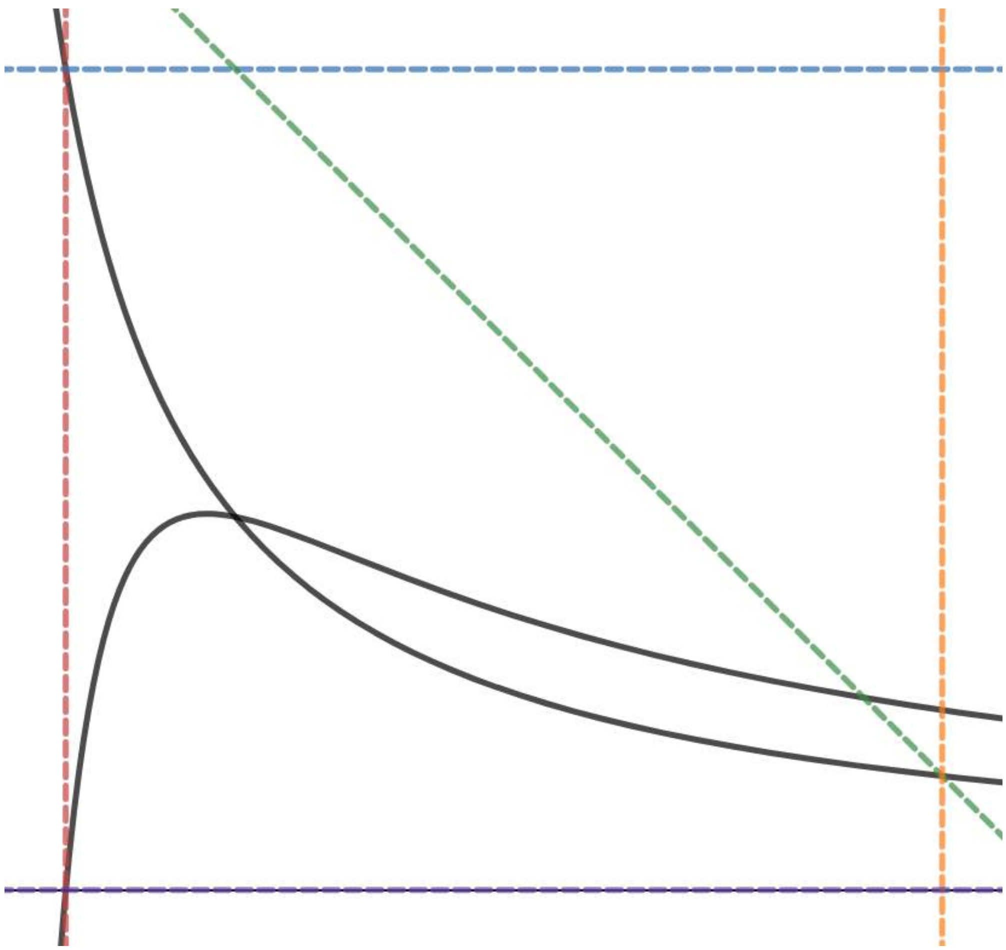


Figure 1: Trapping Region

Consider the five dotted lines that we have drawn along with the nullclines in figure 1. We will describe and find equations for them in the next problem.

■ Question 9.

(a) The red dotted line is vertical and it passes through the point of intersection of the x-nullcline and the x-axis. Find it's equation in terms of a and b.

(b) The blue dotted line is horizontal and it passes through the point of intersection of the red line and the y-nullcline. Find it's equation in terms of a and b.

(c) The green dotted line has slope -1 and passes through the point $\left(a, \frac{b(b+1)}{a}\right)$. Find it's equation in terms of a and b.

(d) The orange dotted line is vertical and passes through the intersection of the green line and the y-nullcline. Find it's equation in terms of a and b.

(e) The purple dotted line is just the $\mathbf{X}-$axis. Mark the region bounded by the five dotted straight lines.

■ Question 10.

We are going to find the direction field along the boundary of this region. Arrows along the horizontal and vertical segments are easy to find. So we only need to check the diagonal segment.

(a) Show that $-y' > x'$ if $x > a$. [Hint: Calculate $x' + y'$.]

(b) Explain how this inequality implies that the vector field points inward on the diagonal line curve 3.

[Hint: What is dy/dx for $x > a$?]

■ Question 11.

Draw the arrows along the five boundary (dotted) line segments. What can you say about the long-term behavior of a solution curve that crosses into the region? Can it ever leave the region?

The region we have constructed above is called a *Trapping Region*. It 'traps' the trajectories inside as all the arrows are pointing inward! On the other hand, when the equilibrium point is a spiral source, all trajectories that start near the point are going outwards. Since they can't escape the trapping region, they are doomed to converge towards a limit cycle! Note that the requirement of being a repelling equilibrium point is crucial, since otherwise all trajectories could have converged to a point.

Hopf Bifurcation

The kind of bifurcation you observed above is called a *Hopf Bifurcation*. The uniqueness of such bifurcations lies in two aspects: unlike other common types of bifurcations (e.g. pitchfork, saddle-node or transcritical) Hopf bifurcation cannot occur in one dimension. The minimum dimensionality has to be two. The other aspect is that Hopf bifurcation deals with birth or death of a periodic solution or limit cycle as and when it emanates from or shrinks onto a fixed point, the equilibrium. Recently, Hopf bifurcations of some famous chaotic systems have been investigated and it is becoming one of the most active topics in the field of chaotic systems.

Math 2000 Project 5: Markov Chains, the Perron-Frobenius theorem and Google's PageRank Algorithm*

Subhadip Chowdhury

- **Purpose:** To analyze Markov chains and investigate steady state vectors.
- **Prerequisite:** Eigenvalues and eigenvectors.
- **Resources:** Use Mathematica as needed. You might also want to take a look at `http://setosa.io/ev/markov-chains/` and `http://setosa.io/markov/index.html`

Web Surfing

Definition 1. A **Stochastic matrix** (aka Markov Matrix) is a square matrix, all of whose entries are between 0 and 1 (inclusive), and such that the entries in each column add up to 1.

We can think of the matrix entries as probabilities of different events happening. For example, consider the matrix

$$A = \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.4 & 0.2 \\ 0.1 & 0.5 & 0.6 \end{bmatrix}.$$

(Note that the entries in each column add up to 1.) We can use this matrix to make a *very very* simple model of the internet, as follows.

Pretend there are only 3 domains on the internet: reddit.com, google.com, and instagram.com (Hereafter referred to as Domains 1, 2, and 3, respectively.) In every five-minute period, say, the people surfing this web have some probability of switching to another domain. For example, let's say that, in a given five-minute period, out of all the people clicking around on Domain 1, 70% will remain on Domain 1, 20% will end up clicking on a link to Domain 2, and 10% will end up on Domain 3. Notice that these are precisely the numbers in the first column of A. I.e., the first column encodes what happens to the people surfing Domain 1. Similarly, the second and third columns tells the probabilities of what happens to the people on Domain 2 and Domain 3, respectively.

Another way to say the same thing: if we identify matrix entries in the standard way, where a_{ij} represents the entry in row i and column j, then a_{ij} here is the probability that someone surfing Domain j will end up on Domain i five minutes from now.

*Most of this project is made using or copied from Lay's Linear Algebra book and Interactive Linear Algebra by Dan Margalit, Joseph Rabinoff.

Definition 2. The matrix A is called the *Transition Matrix* of this system and the columns of A are called the *Transition Probability Vectors*.

By our definition, the transition matrix is a stochastic matrix. We will see more examples of stochastic matrices later.

Exercise 1

Draw a graph illustrating this situation: make a vertex(node) for each of Domain 1, 2, and 3, and draw arrows between the nodes, each labelled with the probability of moving from one bubble to the next. You might want to check out the links above for some pretty neat animations.

Suppose initially, at time $t = 0$, **50%** of the surfers are on Domain **1**, **30%** are on Domain **2**, and **20%** are on Domain **3**. Encode this by the vector

$$\vec{\mathbf{x}}_0 = \begin{bmatrix} 0.5 \\ 0.3 \\ 0.2 \end{bmatrix}.$$

Then $\vec{\mathbf{x}}_1 = A\vec{\mathbf{x}}_0$ tells us what proportion of the surfers are on each domain after one time increment. And $\vec{\mathbf{x}}_2 = A\vec{\mathbf{x}}_1$ tells us where the surfers are after two time increments. And so on.

Exercise 2

Compute $\vec{\mathbf{x}}_1$ and $\vec{\mathbf{x}}_2$.

Note that for each of $\vec{\mathbf{x}}_0$, $\vec{\mathbf{x}}_1$, and $\vec{\mathbf{x}}_2$, the entries add up to 1. This makes sense for $\vec{\mathbf{x}}_0$, since its entries are probabilities covering all the cases. But it's not so obvious for $\vec{\mathbf{x}}_1$ and $\vec{\mathbf{x}}_2$.

Exercise 3

Show in general that, given a Markov matrix M and a vector $\vec{\mathbf{v}}$ whose entries add up to 1, the entries of $M\vec{\mathbf{v}}$ also add up to 1.

Definition 3. The sequence of vectors $\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \vec{\mathbf{x}}_3, \ldots, \vec{\mathbf{x}}_n, \ldots$ is called a **Markov chain**.

It lets us track the evolution of this system, seeing where the populations end up over time. However, in order to find something like $\vec{\mathbf{x}}_{100}$, we would need to compute A^{100}. In other words, we need eigenstuff and diagonalization!

The first claim is that A (or any Markov matrix) always has an eigenvalue of $\mathbf{1}$. It's not trivial to find a $\vec{\mathbf{v}}$ such that $A\vec{\mathbf{v}} = \vec{\mathbf{v}}$ in general, but…

Exercise 4

Find an easy nonzero $\vec{\mathbf{v}}$ such that $A^T\vec{\mathbf{v}} = \vec{\mathbf{v}}$. This shows that $\mathbf{1}$ is an eigenvalue of A^T.
Hint: use the fact that, since we've transposed, the entries in each *row* of A^T add up to 1.

Recall that A and A^T have the same eigenvalues. Thus 1 is an eigenvalue of A as well.

Exercise 5

(a) Find the characteristic polynomial of A, and use the fact that we already know $(\lambda - 1)$ will appear in the factorization to find the other eigenvalue(s) of A.

(b) Find the eigenspace for each eigenvalue of A, and write down a nice eigenbasis for $\mathbb{R}^3$.

(c) Write down the diagonalization $A = BDB^{-1}$.

(d) Compute A^{100} and $\vec{\mathbf{x}}_{100}$.

(e) Find $\lim_{n\to\infty} \vec{\mathbf{x}}_n$.

(f) In the long-term, what percentage of surfers end up on reddit.com, what percentage end up on google.com, and what percentage end up on instagram.com?

(g) Did it matter here what our particular initial distribution $\vec{\mathbf{x}}_0$ was? If 100% started at reddit.com, would we still end up with the same percentages on each domain over the long term?

Steady State and the Perron-Frobenius Theorem

The eigenvalues of stochastic matrices have very special properties.

Proposition 1. *Let A be a stochastic matrix. Then:*

(a) 1 is an eigenvalue of A.

(b) If λ is a (real or complex) eigenvalue of A, then $|\lambda| \leq 1$.

As we observed in the last section in exercise 4, we can prove that 1 is always an eigenvalue of a stochastic matrix. Let's prove the second part. We will restrict to the case of real eigenvalues for the sake of this project.

Exercise 6

1. Let λ be any real eigenvalue of A. Explain why we can always find a vector $\vec{\mathbf{x}}$ such that $A^T\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$.

2. Let $\vec{\mathbf{x}} = [x_1 \quad x_2 \quad \dots \quad x_n]^T$. Choose x_j with the largest absolute value, so that $|x_i| \leq |x_j|$ for all i. Explain the steps in the following chain of inequality

$$|\lambda| \cdot |x_j| = \left|\sum_{i=1}^{n} a_{ij}x_i\right| \leq \sum_{i=1}^{n} (a_{ij} \cdot |x_i|) \leq \left(\sum_{i=1}^{n} a_{ij}\right) \cdot |x_j| = 1 \cdot |x_j|$$

Hence we can conclude that $|\lambda| \leq 1$.

Definition 4. We say that a matrix A is *positive* if all of its entries are positive numbers.

For a positive stochastic matrix A, one can show that if $\lambda \neq 1$ is a (real or complex) eigenvalue of A, then $|\lambda| < 1$. The $1-$eigenspace E_1 of a stochastic matrix is very important.

Definition 5. If A is a stochastic matrix, then a *steady-state vector* (or equilibrium vector) for A is a probability vector $\vec{\mathbf{q}}$ such that

$$A\vec{\mathbf{q}} = \vec{\mathbf{q}}$$

In other words, it is an eigenvector $\vec{\mathbf{q}}$ of A with eigenvalue 1, such that the entries are positive and sum to 1.

The Perron-Frobenius theorem describes the long-term behavior of such a process represented by a stochastic matrix. Its proof is complicated and is beyond the scope of this project.

Theorem 2 (Perron-Frobenius Theorem)**.** *Let A be a positive stochastic matrix. Then A admits a unique steady state vector $\vec{\mathbf{q}}$, which spans the $1-$eigenspace E_1. Further, if $\vec{\mathbf{x}}_0$ is any initial state and $\vec{\mathbf{x}}_{k+1} = A\vec{\mathbf{x}}_k$ then the Markov chain $\{\vec{\mathbf{x}}_k\}$ converges*[1] *to $\vec{\mathbf{q}}$ as $k \to \infty$.*

Why is this nontrivial? For two reasons:

- Apriori, we did not know whether all the entries of the eigenvector corresponding to the eigenvalue 1 are positive. We also did not know about the geometric multiplicity of the eigenvalue 1. P-F theorem tells us that in fact, $\mathbf{dim}(E_1) = 1$ and we can find a vector $\vec{\mathbf{q}} \in E_1$ such that all entries of $\vec{\mathbf{q}}$ are positive and sum to 1!

- If a Markov process has a positive transition matrix, the process will converge to *the* steady state $\vec{\mathbf{q}}$ *regardless of the initial state.*

[1]We say that a sequence of vectors $\{\vec{\mathbf{x}}_k\}$ converges to a vector $\vec{\mathbf{q}}$ as $k \to \infty$ if the entries in $\vec{\mathbf{x}}_k$ can be made as close as desired to the corresponding entries in $\vec{\mathbf{q}}$ by taking k sufficiently large.

Exercise 7

Let $A = \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$ be a stochastic matrix.

(a) Find the eigenvalues and corresponding eigenvectors of A.

(b) Using the eigenvector corresponding to the eigenvalue 1, find the steady-state vector $\vec{\mathbf{q}}$ of A.

Let's try to give a visual interpretation of the linear transformation defined by the matrix above. This matrix A is diagonalizable; we have $A = CDC^{-1}$ for

$$C = \begin{pmatrix} 7 & -1 & 1 \\ 6 & 0 & -3 \\ 5 & 1 & 2 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -.2 & 0 \\ 0 & 0 & .1 \end{pmatrix}$$

The matrix D leaves the $x-$coordinate unchanged, scales the $y-$ coordinate by $-1/5$, and scales the $z-$coordinate by $1/10$. Repeated multiplication by D makes the $y-$ and $z-$coordinates very small, so it *sucks all vectors into the $x-$axis.*

The matrix A does the same thing as D, but with respect to the coordinate system defined by the columns $\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \vec{\mathbf{u}}_3$ of C. This means that A *sucks all vectors into the* $\mathbf{1}-$*eigenspace*, without changing the sum of the entries of the vectors.

Google's PageRank Algorithm

In 1996, Larry Page and Sergey Brin invented a way to rank pages by importance. They founded Google based on their algorithm. Here is how it works (Roughly). Each web page has an associated importance, or *rank*. This is a positive number. If a page P links to n other pages $Q_1, Q_2, \ldots, Q_n$, then each page Q_i inherits $\frac{1}{n}$ of P's importance.

Definition 6. Consider an Internet with n pages. The Rank matrix is the $n \times n$ matrix A whose i, j-entry is the importance that page j passes to page i.

Observe that the rank matrix is a stochastic matrix, assuming every page contains a link: if page i has m links, $m \leq n$, then the ith column contains the number $\frac{1}{m}$, a total of m times, and the number zero in the other entries.

The goal is to find the steady-state rank vector of this Rank matrix. We would like to use the Perron-Frobenius theorem to find the rank vector. Unfortunately, the Rank Matrix is not always a positive stochastic matrix.

Here is Page and Brin's solution. First we fix the rank matrix by replacing each zero column with a column of $\frac{1}{n}$s, where n is the number of pages.

So for example,

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{becomes} \quad A' = \begin{pmatrix} 0 & 0 & 1/3 \\ 0 & 0 & 1/3 \\ 1 & 1 & 1/3 \end{pmatrix}$$

The modified Rank Matrix A' is always stochastic.

Now we choose a number p in $(0, 1)$, called the damping factor. (A typical value is $p = 0.15$.)

Definition 7 (The Google Matrix)**.** Let A be the Rank Matrix for an Internet with n pages, and let A' be the modified Rank Matrix. The *Google Matrix* is the matrix

$$G = (1-p)\cdot A' + p\cdot B \quad \text{where} \quad B = \frac{1}{n}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

Exercise 8

Show that G is a positive stochastic matrix.

If we declare that the ranks of all of the pages must sum to one, then we find:

Definition 8 (The 25 Billion Dollar Eigenvector)**.** The PageRank vector is the steady state of the Google Matrix.

This exists and has positive entries by the Perron-Frobenius theorem. The hard part is calculating it: in real life, the Google Matrix has zillions of rows.

Lab 7 : Volume Integration

0. If U is any solid in 3D space, what does the triple integral $\int\int\int_U 1\, dV$ represent?

Response:

1. In this problem, we will use double integrals to calculate volume bounded by surfaces. We will look at some three dimensional pictures of solids, and we will look at the projection of these solids into the three coordinate planes in order to set up the triple integral in Cartesian coordinates.

Consider a space region T bounded below by the surface $z = f_{\text{bottom}}(x, y)$ and above by the surface $z = f_{\text{top}}(x, y)$, and whose 'shadow' (i.e. projection) in the XY-plane is a region R. Then the volume of T is

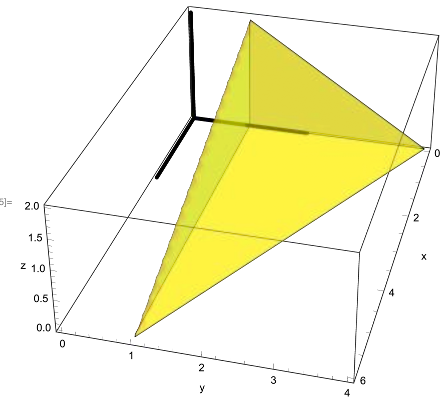
$$\int\int_R \left(f_{\text{top}}(x, y) - f_{\text{bottom}}(x, y)\right) dx\, dy =$$
$$\int\int_R \int_{f_{\text{bottom}}(x,y)}^{f_{\text{top}}(x,y)} 1\, dz\, dx\, dy$$

In general it can be written as $\int\int_R \left(f_{\text{top}} - f_{\text{bottom}}\right) dA = \int\int_R \int_{f_{\text{bottom}}}^{f_{\text{top}}} dV$.

1a. Consider the solid B bounded by $x = 0,\ y = 1,\ z = 0,\ x + 2y + 3z = 8$. Let's look at a picture of it. (Execute the following.)

In[85]:=
```
solid1 = Show[positiveaxes[2], RegionPlot3D[
   x ≥ 0 && y ≥ 1 && z ≥ 0 && x + 2 y + 3 z ≤ 8, {x, 0, 6}, {y, 0, 6}, {z, 0, 6},
   PlotPoints → 60, PlotStyle → Directive[Yellow, Opacity[0.5]], Mesh → None],
  Axes → True, AxesLabel → mylabels, ViewPoint → defaultvp]
```

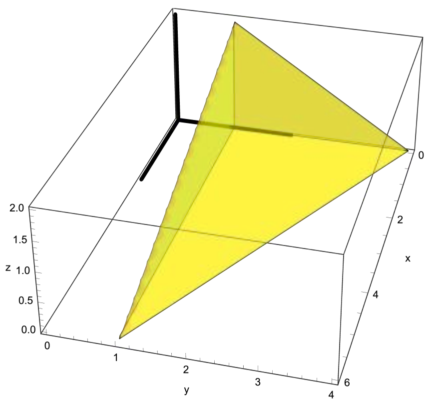
Out[85]=



We should drag-rotate this solid around so we can see it from various viewpoints. Pay attention to the three axes as you rotate the solid, so that you don't forget which axis is which. First, let us look at this solid from the viewpoint along the **z-direction**, projecting onto the **xy-plane**. Rotate the solid until you get a viewpoint in the **z-direction**, from the **top**.

In[86]:=
```
Show[solid1] (*Execute this to get another image to play with*)
```

Out[86]=



Does this look like a feasible viewpoint to use? Why or why not? (Think about the top surface and bottom surface, and the type of base domain the projection yields.) [Here, “feasible” just means that we have an elementary region.]

Response:

If the answer is yes, fill in the following:

Equation of the Top surface in the form $z = f_{top}(x, y)$:

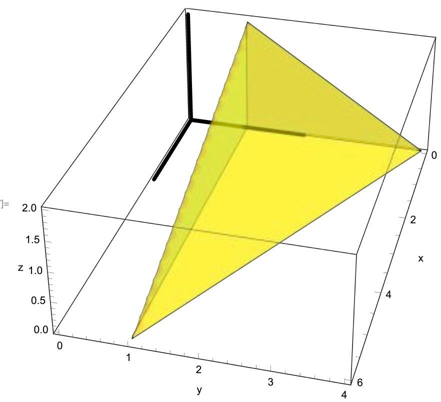
Equation of the Bottom surface in the form $z = f_{top}(x, y)$:

Description of the base domain R:

Next, let us look at this solid from the viewpoint along the **y-direction**, projecting onto the **xz-plane**. Rotate the solid until you get a viewpoint in the **y-direction**, from the **side**.

In[87]:=
```
Show[solid1] (*Execute this to get another image to play with*)
```


Out[87]=



Does this look like a feasible viewpoint to use? Why or why not? (Caution: note that in the y-viewpoint, the x-axis above runs in the opposite direction!!)

Response:

If the answer is yes, fill in the following:

Equation of the Top surface in the form $y = f_{top}(x, z)$:

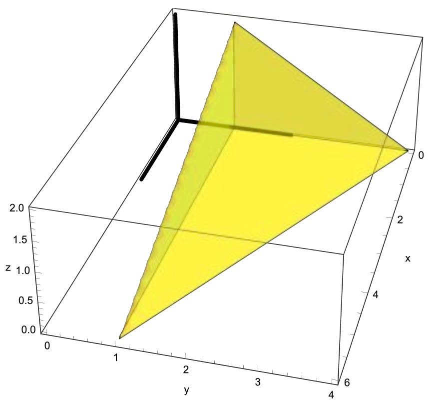
Equation of the Bottom surface in the form $y = f_{top}(x, z)$:

Description of the base domain R:

Finally, let us look at this solid from the viewpoint along the **x-direction**, projecting onto the yz-**plane**. Rotate the solid until you get a viewpoint in the **x-direction**, from the **front**.

In[88]:=
```
Show[solid1] (*Execute this to get another image to play with*)
```

Out[88]=



Does this look like a feasible viewpoint to use? Why or why not?

Response:

If the answer is yes, fill in the following:

Equation of the Top surface in the form $x = f_{top}(y, z)$:

Equation of the Bottom surface in the form $x = f_{top}(y, z)$:

Description of the base domain R:

Now, fill in and execute each feasible parametrization. That is, a feasible view-point is one where you can set up using just one integral.

In[89]:= way1 = $\int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} 1 \times d\square \times d\square \times d\square$

Out[89]= 0

In[90]:= way2 = $\int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} 1 \times d\square \times d\square \times d\square$

Out[90]= 0

In[91]:= way3 = $\int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} 1 \times d\square \times d\square \times d\square$

Out[91]= 0

To verify that these gave the same answer, execute the following.


```
In[92]:= N[way1]
         N[way2]
         N[way3]
```

Out[92]= 0.

Out[93]= 0.

Out[94]= 0.

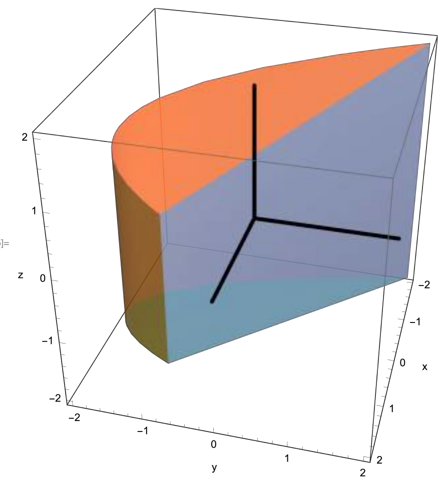
Did you get the same answer?

Response:

1b. Consider the solid B bounded by $y = x^2 - 2,\ x + y = 0,\ z = -2,\ x + 2z = 2.$ Let's look at a picture of it. (Execute the following.)

```
In[95]:= solid2 =
  Show[positiveaxes[2], ParametricPlot3D[{{u, u^2 - 2, -2 v + 1/2 (1 - v) (2 - u)},
      {u, -u, -2 v + 1/2 (1 - v) (2 - u)}, {u, (u^2 - 2) v + (1 - v) (-u), -2},
      {u, (u^2 - 2) v + (1 - v) (-u), (2 - u)/2}}, {u, -2, 1}, {v, 0, 1},
     PlotPoints → 10, PlotRange → {{-2, 2}, {-2, 2}, {-2, 2}},
     PlotStyle → Opacity[0.6], Mesh → None],
   Axes → True, AxesLabel → mylabels, ViewPoint → defaultvp]
```

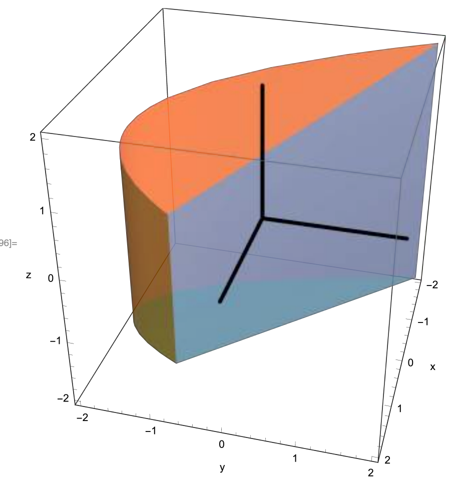
Out[95]=



We should drag-rotate this solid around so we can see it from various viewpoints. Pay attention to the three axes as you rotate the solid, so that you don't forget which axis is which. First, let us look at this solid from the viewpoint along the **z-direction**, projecting onto the **xy-plane**. Rotate the solid until you get a viewpoint in the **z-direction**, from the **top**.

In[96]:=
```
Show[solid2] (*Execute this to get another image to play with*)
```


Out[96]=



A region is called "feasible" if it elementary, of type I or type II. Does this look like a feasible viewpoint to use? Why or why not? (Think about the top surface and bottom surface, and the type of base domain the projection yields.)

Response:

If the answer is yes, fill in the following:

Equation of the Top surface, $f_{top}(x, y)$*:*

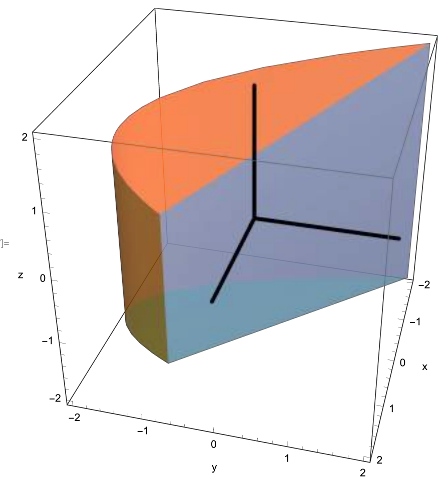
Equation of the Bottom surface, $f_{top}(x, y) =$

Description of the base domain R:

Next, let us look at this solid from the viewpoint along the **y-direction**, projecting onto the **xz-plane**. Rotate the solid until you get a viewpoint in the **y-direction**, from the **side**.

In[97]:=
```
Show[solid2] (*Execute this to get another image to play with*)
```

Out[97]=



Does this look like a feasible viewpoint to use? Why or why not? (Caution: note that in the y-viewpoint, the x-axis above runs in the opposite direction!!)

Response:

If the answer is yes, fill in the following:

Equation of the Top surface, $f_{top}(x, z)$ =

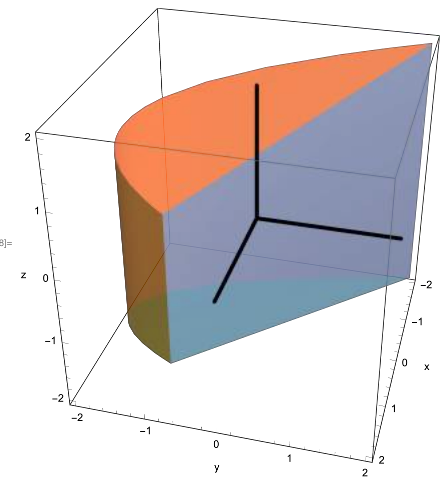
Equation of the Bottom surface, $f_{top}(x, z)$ =

Description of the base domain R:

Finally, let us look at this solid from the viewpoint along the **x-direction**, projecting onto the yz-**plane**. Rotate the solid until you get a viewpoint in the **x-direction**, from the **front**.

```
In[98]:= Show[solid2] (*Execute this to get another image to play with*)
```


Out[98]=



Does this look like a feasible viewpoint to use? Why or why not?

Response:

If the answer is yes, fill in the following:

Equation of the Top surface, $f_{top}(y, z)$ =

Equation of the Bottom surface, $f_{top}(y, z)$ =

Description of the base domain R:

Now, fill in and execute each feasible parametrization (you might not do all three, but hint: at least two are feasible). That is, a feasible viewpoint is one where you can set up using just one integral.

```
In[99]:= bway1 = ∫□^□ ∫□^□ ∫□^□ 1 × d□ × d□ × d□
Out[99]= 0

In[100]:= bway2 = ∫□^□ ∫□^□ ∫□^□ 1 × d□ × d□ × d□
Out[100]= 0
```

In[101]:=
```
bway3 = ∫□^□ ∫□^□ ∫□^□ 1 × ⅆ□ × ⅆ□ × ⅆ□
```
Out[101]= 0

To verify that these gave the same answer, execute the following.

In[102]:=
```
N[bway1]
N[bway2]
N[bway3]
```
Out[102]= 0.

Out[103]= 0.

Out[104]= 0.

Did you get the same answer?

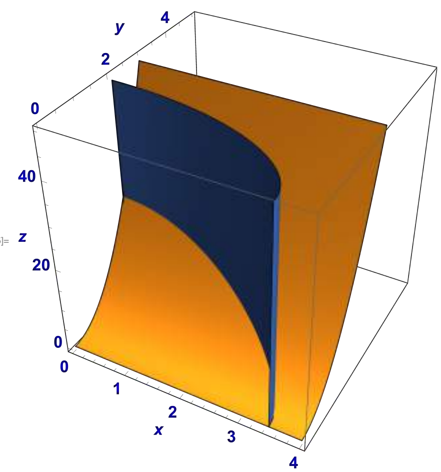
Response:

2. In this exercise we are going to repeat the above process to set up the volume integral in three different ways.

Consider a region E in the *first octant* bounded by the parabolic cylinder $z = 6\,y^2$ and the elliptical cylinder $x^2 + 3\,y^2 = 12$. First let's plot the two surfaces.

In[105]:=
```
ContourPlot3D[{z == 6 y^2, x^2 + 3 y^2 == 12}, {x, -0, 4},
 {y, 0, 5}, {z, 0, 50}, AxesLabel → Automatic, PlotPoints → 100,
 Mesh → None, LabelStyle → Directive[Blue, Bold, Larger]]
```

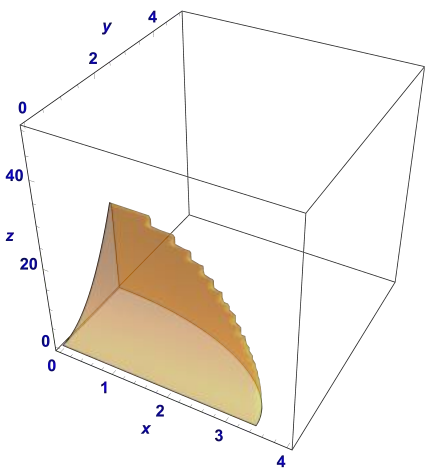

Out[105]=



The region E has four sides. Execute the following command to plot the region E. Note in particular how the region was defined using inequalities. You can increase the **PlotPoints** parameter to make your curves smoother. However, note that it will take exponentially longer time to execute.

In[106]:=
```
RegionPlot3D[z <= 6 y^2 && x^2 + 3 y^2 <= 12 && x >= 0 && y >= 0,
 {x, -0, 4}, {y, 0, 5}, {z, 0, 50}, AxesLabel → Automatic,
 PlotPoints → 60, PlotStyle → Opacity[0.5], Mesh → None,
 LabelStyle → Directive[Blue, Bold, Larger]]
```

Out[106]=



Fill in the limits of integration in each of the following to write an iterated integral that gives the volume of *E*.

In[107]:= int1 = $\int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} 1 \, dz \, dx \, dy$

Out[107]= 0

In[108]:= int2 = $\int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} 1 \, dx \, dz \, dy$

Out[108]= 0

In[109]:= int3 = $\int_{\square}^{\square}\int_{\square}^{\square}\int_{\square}^{\square} 1 \, dy \, dz \, dx$

Out[109]= 0

We can check whether all three of above integrals give the same numerical answer.

In[110]:=
```
N[int1]
N[int2]
N[int3]
```

Out[110]= 0.

Out[111]= 0.

Out[112]= 0.

Did you get the same answer for all three?

Response:

Appendix C

Sample Worksheets and Handouts

Math 2208: Ordinary Differential Equations

Lecture 9 Worksheet

Fall 2019 **Subhadip Chowdhury** **Oct 16**

Title: System of First Order ODEs

Summary: We will learn how about models that involve a system of first order differential equations. The most famous of these is the Predator-Prey model of Lotka and Volterra. Then we will use direction fields and phase portraits to do qualitative and quantitative analysis.

§A. General Form

The general form for a two-dimensional system of first order ODEs is

$$\frac{dx}{dt} = f(x,y,t)$$
$$\frac{dy}{dt} = g(x,y,t)$$

If f and g are linear in both x and y, the system is called *linear*, otherwise the system is called *nonlinear*.

Note that a second-order ODE can be converted to a two-dimensional system of first-order ODEs via some change in variables. For example, if

$$y'' = f(t, y(t)), y'(t))$$

is a second-order ODE, we can make the substitution $u_2(t) = y'(t)$ and $u_1(t) = y(t)$ to convert the ODE into a system of 2 first-order ODEs

$$\frac{du_1}{dt} = u_2(t)$$
$$\frac{du_2}{dt} = f(t, u_1(t), u_2(t))$$

Question 1.

Show that the third-order linear ODE

$$y''' + 3y'' + 2y' - 5y = \sin(t)$$

can be written as a linear system of three first-order ODEs.

§B. Lotka-Volterra Model

Probably the most famous system of ordinary differential equations of all time is the Lotka-Volterra predator-prey model. We will study a very special case of interaction between exactly two species, one of which -- the predators -- eats the other -- the prey. Such pairs exist throughout nature: lions and gazelles, birds and insects, pandas and eucalyptus trees, foxes and rabbits. To keep our model simple, we will make some assumptions:

- the predator species is totally dependent on a single prey species as its only food supply,
- the prey species has an unlimited food supply, and
- there is no threat to the prey other than the specific predator.

Let $x(t)$ denote the population of prey (rabbits) and let $y(t)$ denote the population of their predators (foxes). Let a, b, c and d be nonnegative parameters. One system of differential equations that might govern the changes in the population of these two species is

$$\frac{dx}{dt} = ax - bxy$$
$$\frac{dy}{dt} = cxy - dy$$

Question 2.

1. What do the constants a, b, c and d represent physically?
2. What is the significance of the xy terms in the model? Is the significance same for rabbits and foxes?
3. What happens to $x(t)$ if $b = c = d = 0$? How does this correspond to our assumptions?
4. What happens to $y(t)$ if $a = b = c = 0$? How does this correspond to our assumptions?
5. What does the model predict will happen if at any time one of the populations of the rabbits or the foxes becomes zero?

Question 3.

Are there any fixed points for the system of equations? In other words, what are the equilibrium solutions for the system?

§C. Phase Portrait

One of the ways to graph the solution of the system is to form the pair $(x(t), y(t))$ and think of it as a point in the xy-plane. In other words, the coordinates of the point are the values of the two populations at time t. As t varies, the pair $(x(t), y(t))$ sweeps out a parametric curve in the xy-plane. This curve is called the *solution curve*.

The xy-plane is called the *phase plane*, and it is analogous to the phase line for an autonomous first-order differential equation.

Observe that the solution curves that correspond to equilibrium solutions are really just points, and we refer to them as *equilibrium points*.

Definition 3.1: Nullclines

For a system

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

the $x-$*nullcline* is the set of points

$$\left\{(x,y)\middle|\frac{dx}{dt} = 0\right\}$$

i.e. the level curve in the phase plane where $f(x,y) = 0$. The $y-$nullcline is the level curve $g(x,y) = 0$.

At an equilibrium point, both $f(x,y)$ and $g(x,y)$ must be zero, hence

Theorem 3.1

The intersection of the nullclines are the equilibrium points.

A *Phase Portrait* of a system consists of the following information on the phase plane.

- the nullclines,
- the equilibrium points
- several solution curves corresponding to different initial conditions.

§D. Direction Field as a (normalized) Vector Field

Instead of thinking of $(x(t), y(t))$ as simply a combination of the two scalar-valued functions $x(t)$ and $y(t)$, we can consider the pair $(x(t), y(t))$ as a vector-valued function in the xy- plane. For each t, let $\vec{\mathbf{R}}(t)$ denote the column vector

$$\vec{\mathbf{R}}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

which corresponds to the position vector of the point $(x(t), y(t))$.

Then using this notation,

$$\frac{d\vec{\mathbf{R}}}{dt} = \begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \vec{\mathbf{F}}(\vec{\mathbf{R}})$$

where $\vec{\mathbf{F}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ is the vector field

$$\vec{\mathbf{F}}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$$

Then the solution curves on a phase plane are essentially the *flow lines* of this vector field. If we draw the field after normalizing each vector to some small magnitude and only care about the direction, it gives us a diagram called the *direction field*.

Along the x-nullcline, the x-component of the vector field is zero, and consequently the arrows in the direction field are vertical. They point either straight up or straight down. Similarly, on the y-nullcline, the y-component of the vector field is zero, so the vector field is horizontal. Arrows points either left or right.

Observe that the x–nullcline naturally divides the plane into regions where $f(x,y) > 0$ and $f(x,y) < 0$. Since $x' = f(x,y)$, $f(x;y) > 0$ means x is increasing which in turn means the arrows point rightward in the plane. Similarly, the y–nullcline shows us where y is increasing or decreasing. The following table will help you fill out the direction field once the nullclines are found.

	$f(x,y) < 0$	$f(x,y) = 0$	$f(x,y) > 0$
$g(x,y) < 0$	$\swarrow$	$\downarrow$	$\searrow$
$g(x,y) = 0$	$\leftarrow$	$\cdot$	$\rightarrow$
$g(x,y) > 0$	$\nwarrow$	$\uparrow$	$\nearrow$

§E. Examples

Consider the following Lotka-Volterra model:

$$\frac{dx}{dt} = 2x - 1.2xy$$
$$\frac{dy}{dt} = 0.9xy - y$$

We will use pplane to draw the direction field on the phase plane. In the following picture of direction field, x–axis is prey and y–axis is predator.

Question 4.

Draw the x– and y–nullclines of above system and show the direction of the direction field along the nullclines (up, down, left, right). Find the equilibrium point(s), if any.

Question 5.

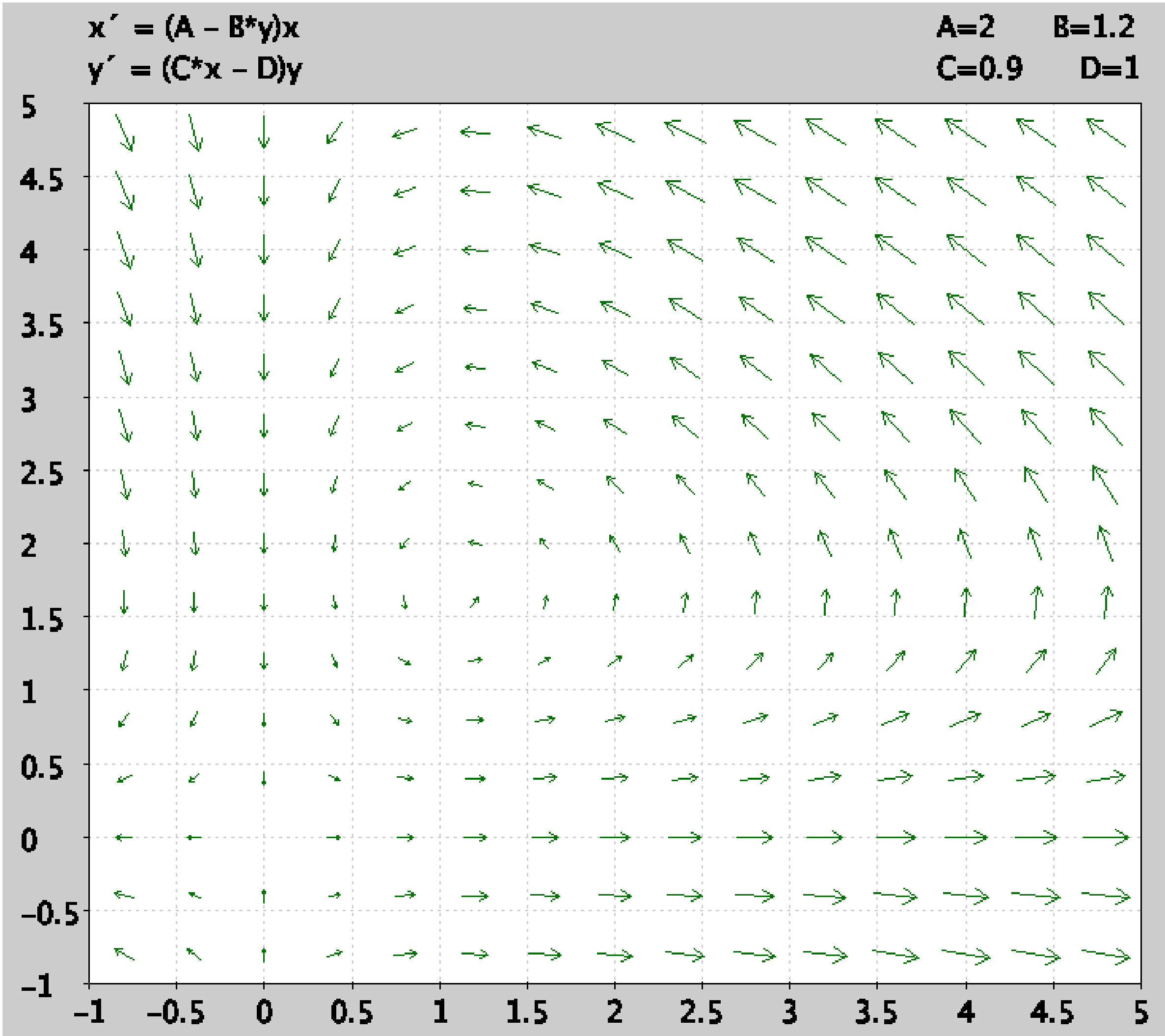
Draw the solution curve that starts at the initial condition $(x(0), y(0)) = (1, 0.5)$. Draw some more solution curves on your paper.

Question 6.

What happens when the initial condition approaches $(10/9, 5/3)$?

Question 7.

Pick one of the solution curves $(x(t), y(t))$. How do the graphs of $x(t)$ and $y(t)$ against t look like? Do you believe they would be periodic based on the behavior of the solution curves in the phase plane?



Question 8.

Can you explain why the two graphs should have the same period? Why does it makes sense that the increase in predator population lag behind (in time) the increase in prey population?

Question 9.

Find the equation of the nullclines of the following system, draw them in XY-plane, and find the equilibrium points.

$$\frac{dx}{dt} = x(2-x) - xy$$
$$\frac{dy}{dt} = xy - 1$$

Try drawing a rough diagram of the direction field on the phase plane $0 \leq x, y \leq 2$, using the table from section D. Here are some instructions: find an easily identifiable point in the plane which is not on one of the nullclines. Then evaluate (f, g) at this point and use this to draw a direction at that specific point. Then the rest of the arrows are easy to fill in using alternate choices.

Can you draw some sample solution curves?

Math 2000-B Handout 7: Geometry of Linear Transformations

Subhadip Chowdhury

As shown in Theorem 10 in Section 1.9, when a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ given, it can be identified with a matrix, and there is an easy way to get a formula for the function as follows. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ a linear transformation and let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ denote the columns of the $n \times n$ identity matrix.

Figure out what each $T(\mathbf{e}_i)$ should be and write each $T(\mathbf{e}_i)$ as a column vector. If you then define the matrix $A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \ldots \quad T(\mathbf{e}_n)]$, then it will be true that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$, and $A\mathbf{x}$ gives a formula for the function. In other words, given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ if you know its values at just the n independent vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ then its value at every point $\mathbf{x}$ is determined!

Example 1. The 2×2 linear transformation that maps $\vec{\mathbf{e}}_1$ to $\vec{\mathbf{e}}_1 + \vec{\mathbf{e}}_2$ and $\vec{\mathbf{e}}_2$ to $\vec{\mathbf{e}}_1 - \vec{\mathbf{e}}_2$ is $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Example 2. The 3×3 matrix transformation that maps $\vec{\mathbf{e}}_1$ to $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$, $\vec{\mathbf{e}}_2$ to $\begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix}$, and $\vec{\mathbf{e}}_3$ to $\begin{bmatrix} 5 \\ 4 \\ -1 \end{bmatrix}$ is $\begin{bmatrix} 3 & 6 & 5 \\ -2 & 0 & 4 \\ 1 & 7 & -1 \end{bmatrix}$.

Exercise 1. *(a) The function that reflects $\mathbb{R}^2$ across the line $y = -x$ is a linear transformation. What are the images of $\vec{\mathbf{e}}_1$ and $\vec{\mathbf{e}}_2$ under this map? What is the matrix of this linear transformation?*

(b) Write a 2×2 matrix that reflects $\mathbb{R}^2$ across the line $y = x$.

A matrix transformation always maps a line onto a line or a point, and maps parallel lines onto parallel lines or onto points. (See exercises 25-28 in Section 1.8.) In the following exercises, you will verify these things for a particular matrix.

Exercise 2. *Let $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.*

(a) Explain why the function $T(\mathbf{x}) = M\mathbf{x}$ maps the x-axis onto the line $y = x$, and why it maps the line $y = 2$ onto the line $y = x + 2$.

[Hint: Use the parametric equation form of a straight line.]

(b) Sketch and label the lines and their images on a graph.

Because a matrix transformation maps parallel lines to parallel lines, it will map any parallelogram to another parallelogram. (The parallelogram could be degenerate-one line segment or a single point.) When a linear transformation and parallelogram are given, the easy way to draw the image of the parallelogram is to plot the images of its four vertices and connect those points to make a parallelogram.

Define the *standard unit square* to be the square in $\mathbb{R}^2$ whose vertices are $(0,0), (1,0), (1,1)$, and $(0,1)$. When you want to visualize what a 2×2 matrix transformation does geometrically, it is particularly useful to sketch the image of this standard square. Seeing how this square gets moved or distorted shows what the transformation does to the x-axis and y-axis and thus gives a good idea what the transformation does geometrically to the whole plane. Recall that any linear transformation maps the origin to itself (why?), so you only need to figure out where the transformation maps the other three vertices.

Example 3. The image of the standard unit square under the matrix from exercise 2 looks like figure 1.

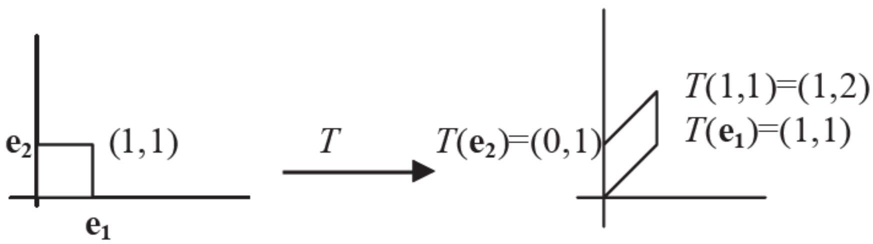


Figure 1

So you can see that the y-axis stays fixed and the x-axis is mapped onto the line $y = x$, causing a vertical *shear* of the plane. See more examples like this in Table 3 in Section 1.9.

Example 4. Find a matrix M which maps the standard unit square to the parallelogram with vertices $(0,0), (3,1), (2,2), (-1,1)$. To do this, sketch the parallelogram and recognize that $M\vec{e}_1$ and $M\vec{e}_2$ must be $(3,1)$ and $(-1,1)$, or vice versa. (Why?) So either of the matrices $\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ or $\begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$ will work.

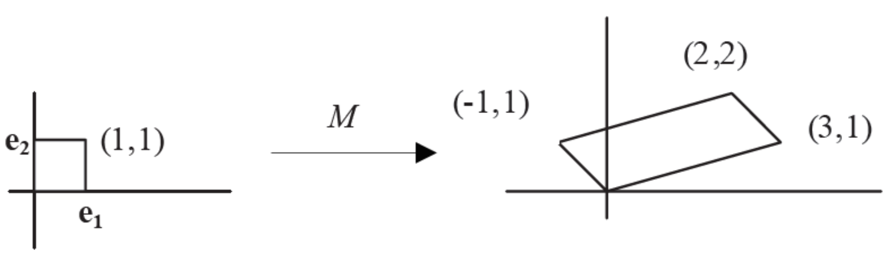


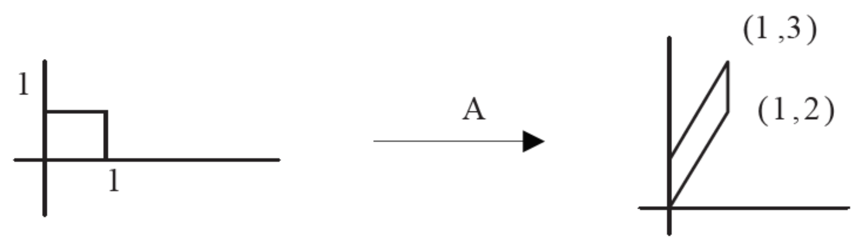
Figure 2

Calculate the image of $(1,1)$ under these matrices and verify that it's $(2,2)$.

Exercise 3. *Each of the following seven matrices is one of the special, simple types described in Sections 1.8 and 1.9. Each determines a linear transformation of* $\mathbb{R}^2$. *For each, sketch the image of the standard unit square, label the vertices of the image, and describe how the matrix is transforming the plane. To get you started, answers are given for the first matrix.*

(A) $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

Description: *A vertical shear. It leaves the y-axis fixed and increases the slope of all other lines through the origin.*



(B) $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

(C) $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(D) $D = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

(E) $E = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(F) $F = \begin{bmatrix} \cos(45°) & -\sin(45°) \\ \sin(45°) & \cos(45°) \end{bmatrix}$

(G) $G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Exercise 4. *(a) Draw the image of the standard unit square under the transformations* EFA *and* FCA. *Here* A, C, E, F *are from the last exercise.*

Exercise 5. *Sketch the parallelogram with vertices* $(0,0), (4,2), (0,-4), (4,-2)$ *and write two different* 2×2 *matrices* X *and* Y *which would transform the standard unit square into this parallelogram.*

Exercise 6. *Sketch the parallelogram with vertices* $(1,1), (1,2), (3,1), (3,2)$. *Explain why no* 2×2 *matrix transformation could map the standard unit square onto this figure.*

Exercise 7. *Let* $T : \mathbb{R}^n \to \mathbb{R}^n$ *a linear transformation. Prove that* T *is completely determined by its values on any* n *linearly independent vectors.*

Hint: *Here is an outline of the proof to get you started. Let* $\vec{v}_1, \vec{v}_2 \ldots, \vec{v}_n$ *be linearly independent vectors in* $\mathbb{R}^n$, *and suppose* $T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_n)$ *are known. Let* $\vec{x}$ *be any element of* $\mathbb{R}^n$. *Explain why* $span\{\vec{v}_1, \vec{v}_2 \ldots, \vec{v}_n\} = \mathbb{R}^n$, *so* $\vec{x}$ *equals some linear combination of* $\vec{v}_1, \vec{v}_2 \ldots, \vec{v}_n$. *Then show* $T(x)$ *can be calculated using* $T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_n)$.

MATH 1800 HANDOUT 7: APPLICATIONS AND EXTENSIONS OF GREEN'S THEOREM

Subhadip Chowdhury

Careful Statement of Green's Theorem

Green's Theorem is a statement about only the 2D plane, not 3D space. Here's the setup: Let C be a piecewise smooth simple closed curve that is the boundary of a simply-connected region D in the plane. Let $\vec{\mathrm{F}} = P\,\vec{\mathrm{i}} + Q\,\vec{\mathrm{j}}$ be a smooth vector field defined on all of D and C.

Notes on terminology: To say a curve is *simple* means that it doesn't intersect itself, and to say a curve is *closed* means that it is a closed loop, that it starts and ends at the same point. A region is called *simply-connected* if it is just one piece (connected) and doesn't have any holes in it. A vector field is said to be *smooth* if it has continuous first partials.

Since C is the boundary of D, we write $C = \partial D$ and stop referring to C explicitly. When we consider the boundary curve of a simply-connected region, we always orient the curve so that the region is on the left as we follow the curve. (Note: this is harder to specify when one considers 3D surfaces with boundaries.)

Green's Theorem says

$$\oint_{\partial D} \vec{\mathrm{F}} \cdot d\vec{\mathrm{r}} = \iint_D \mathrm{curl}(F)\, dA = \iint_D (Q_x - P_y)\, dA.$$

You may also see this written the Leibniz notation for partial derivatives.

$$\oint_{\partial D} P\, dx + Q\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Using Green's Theorem to Evaluate Difficult Line Integrals

■ If you're asked to find the line integral of an ugly vector field over a closed curve, you should look to see if Q_x and P_y are drastically more simple. If so, use Green's Theorem!

Exercise 1

Evaulate

$$\int_C (2y + \sqrt{9 + x^3})\, dx + (5x + e^{\tan^{-1} y})\, dy,$$

where C is the circle $x^2 + y^2 = 4$ in the plane oriented counterclockwise.

■ If you're asked to find the line integral over a closed curve that is clearly the boundary of a nice region, it's often a good idea to use Green's Theorem to switch to the double integral over the interior.

Exercise 2

Integrate $\vec{\mathrm{F}} = xy\,\vec{\mathrm{i}} + e^x\,\vec{\mathrm{j}}$ *over the boundary of the rectangle determined by* $0 \le x \le 2$, $0 \le y \le 3$, *oriented clockwise around the boundary.*

Exercise 3

Find the line integral of $\vec{\mathrm{F}} = 3xy\,\vec{\mathrm{i}} + 2x^2\,\vec{\mathrm{j}}$ *over the curve* C *defined as follows: follow the curve* $y = x^2 - 2x$ *from* $(0,0)$ *to* $(3,3)$*, then follow the line* $y = x$ *from* $(3,3)$ *back to* $(0,0)$.

Exercise 4

Evaluate

$$\oint_C \left(3y - e^{\sin x}\right) dx + \left(7x + \sqrt{y^4 + 1}\right) dy$$

where C *is the circle* $x^2 + y^2 = 9$, *oriented clockwise.*

■ Consider the following problem.

Find the line integral of

$$\vec{\mathrm{G}}(x, y) = \left(x + y\right)\vec{\mathrm{i}} + \left(2x + y\ln\left(\csc\sqrt{1 - y^5}\right)\right)\vec{\mathrm{j}}$$

over C_1, *the upper half of the unit circle from* $(1,0)$ *to* $(-1,0)$.

The problem is that the vector field is ugly, so parametrizing C_1 is just going to lead to an impossible integral. So we would like to use Green's Theoerem, but this isn't a closed curve!

Here's how to fix that issue. Let C_2 be the straight line segment from $(-1,0)$ to $(1,0)$. Now $C_1 + C_2$ is a closed loop.

(a) Let R be the region enclosed by $C_1 + C_2$. Use Green's Theorem to compute $\displaystyle\oint_{C_1+C_2} \vec{\mathrm{G}} \cdot d\vec{\mathrm{r}}$.

(b) Parametrize C_2 and directly calculate $\displaystyle\int_{C_2} \vec{\mathrm{G}} \cdot d\vec{\mathrm{r}}$. (Note that $y = 0$ everywhere on C_2, which is helpful.)

(c) Write $\displaystyle\oint_{C_1+C_2} \vec{\mathrm{G}} \cdot d\vec{\mathrm{r}} = \int_{C_1} \vec{\mathrm{G}} \cdot d\vec{\mathrm{r}} + \int_{C_2} \vec{\mathrm{G}} \cdot d\vec{\mathrm{r}}$ and use your answers to part (a) and (b) to finish off the problem and find the line integral of $\vec{\mathrm{G}}$ along C_1.

Exercise 5

Evaluate the integral.

Calculating Area with Green's Theorem

Consider the following vector fields:

$$\vec{\mathbf{F}}_1 = x\,\vec{\mathbf{j}}, \qquad \vec{\mathbf{F}}_2 = -y\,\vec{\mathbf{i}}, \qquad \vec{\mathbf{F}}_3 = -\tfrac{1}{2}y\,\vec{\mathbf{i}} + \tfrac{1}{2}x\,\vec{\mathbf{j}}.$$

What is $Q_x - P_y$ for each of these fields? Applying Green's Theorem to a region D, we get that

$$\oint_{\partial D} \vec{\mathbf{F}}_1 \cdot d\vec{\mathbf{r}} \;=\; \oint_{\partial D} \vec{\mathbf{F}}_2 \cdot d\vec{\mathbf{r}} \;=\; \oint_{\partial D} \vec{\mathbf{F}}_3 \cdot d\vec{\mathbf{r}} \;=\; \iint_D 1\,dA \;=\; \text{Area of } D.\ (!)$$

Exercise 6

An ellipse with semi-major axis a and semi-minor axis b is parametrized by $x = a\cos t,\ y = b\sin t$ for $0 \le t \le 2\pi$. Use $\vec{\mathbf{F}}_3$ to find the area inside this ellipse.

Exercise 7

Let C be the curve parametrized by $\vec{\mathbf{r}}(t) = (t^2-3)\,\vec{\mathbf{i}} + (t^3-4t+1)\,\vec{\mathbf{j}}$, $-2 \le t \le 2$. This is a closed loop. Use Green's Theorem and $\vec{\mathbf{F}}_1$ to find the area inside this loop.

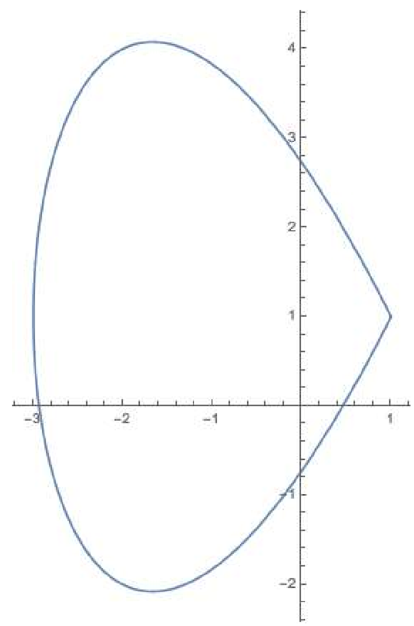


Figure 1: Exercise 7

Extended Versions of Green's Theorem

Although we have proved Green's Theorem only for the case where D is simple, we can now extend it to the case where D is a *finite union of simple regions*. For example, if D is the region shown in Figure 2, then we can write $D = D_1 \cup D_2$, where D_1 and D_2 are both simple. The boundary of D_1 is $C_1 + C_3$ and the boundary of D_2 is $C_2 + (-C_3)$. so, applying Green's Theorem to D_1 and D_2 separately, we get

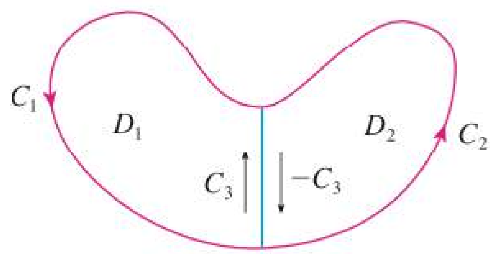


Figure 2

$$\oint_{C_1+C_3} Pdx + Qdy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\oint_{C_2+(-C_3)} Pdx + Qdy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Adding the two integrals above, we get,

$$\oint_{C_1+C_2} Pdx + Qdy = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Note that we can similarly extend the theorem to regions that are finite *intersections* of simple regions.

Exercise 8

Evaluate

$$\oint_C y^2 dx + 3xy dy$$

where C *is the boundary of the semiannular region* D *in the upper half plane between* $x^2 + y^2 = 1$ *and* $x^2 + y^2 = 4$.

Green's Theorem can be also extended to apply to regions with holes, that is, regions that are not simply-connected. Observe that the boundary C of the region D in Figure 3 consists of two simple closed curves C_1 and C_2. We assume that these boundary curves are oriented so that the region D is always on the left as the curve C is traversed. *Thus the positive direction is counterclockwise for the outer curve* C_1 *but clockwise for the inner curve* C_2. If we divide D into two regions D' and D'' by means of the lines shown in Figure 3 and then apply Green's Theorem to each of D' and D'', we get

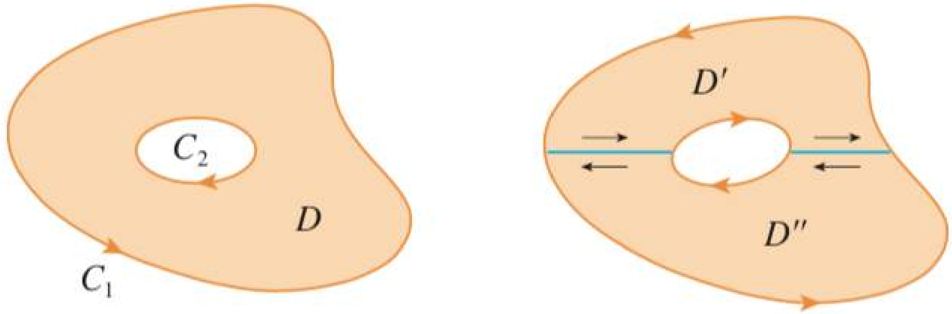


Figure 3

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$
$$= \oint_{\partial D'} (P dx + Q dy) + \oint_{\partial D''} (P dx + Q dy)$$

Since the line integrals along the common boundary lines are in opposite directions, they cancel each other out and we get

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_{C_1} (P dx + Q dy) + \oint_{C_2} (P dx + Q dy) = \oint_{C_1 + C_2} P dx + Q dy$$

Exercise 9

If $\vec{\mathbf{F}}(x, y) = \dfrac{-y\vec{\mathbf{i}} + x\vec{\mathbf{j}}}{x^2 + y^2}$*, show that* $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 2\pi$ *for every positively oriented closed path* C *that encloses the origin.*

Exercise 10

Evaluate $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ *where*

$$\vec{\mathbf{F}}(x, y) = \frac{2xy\vec{\mathbf{i}} + \left(y^2 - x^2\right)\vec{\mathbf{j}}}{\left(x^2 + y^2\right)^2}$$

and C *is a positively oriented closed path that encloses the origin.*

Appendix D

Sample Exams

Final Exam

Ordinary Differential Equations -- Math 2208

Subhadip Chowdhury

Due: Noon, Dec 16, 2019

Instructions:

- This is a take-home exam. As such, your written arguments will be held to a higher standard than on a sit-down in-class exam. Please submit clear and carefully composed solutions, and explain the concepts you are using and the connections among them. As always, points may be deducted for any unjustified steps, and generous partial credit will be given if you explain your thought process to me.
- You may consult and use our course materials while taking this exam, including the textbook, class notes, your problem sets, and any of the worksheets on Blackboard. You are not allowed to use the web links (for eignevalues or TD plane etc.) on worksheets.
  You may **NOT** use a calculator or any graphing tool.
  You may **NOT** use `dfield` or `pplane` or `Octave`.
  You may **NOT** consult the internet or discuss problem specifics with other people.
  You may email me to ask questions.
  If you are not sure if some resource is allowed, please ask!
- When submitting your exam, print and staple this first page on top, and sign the "Honor Signature" to indicate that you followed Bowdoin's Honor Code with respect to this exam.

Full Name: ______________________________

Honor Signature: ______________________________

Section Number	A	B	C	D	E	Total
Available Points	**30**	**25**	**35**	**35**	**20**	**145**
Your Score						

§A. Air Resistance and Terminal Velocity

A paratrooper jumps out of an airplane at a sufficiently high altitude (with initial downward velocity $v(0) = 0$), falls freely for 20 seconds and then opens her parachute. Assume her mass is m.

Her vertical motion is subject to two forces:

- a downward gravitational force $\mathbf{F_G} = mg$ and
- a force $\mathbf{F_R}$ of air resistance that is proportional to velocity (so that $\mathbf{F_R} = kv$) and of course directed opposite to the direction of motion of the body (i.e. upward).

Newton's law of motion says that the net force acting on the paratrooper is equal to her mass times her acceleration.

■ Question 1 (5 points).

Show that at time t, the velocity $v(t)$ of the paratrooper can be found by solving the ODE

$$\frac{dv}{dt} = -\rho v + g$$

where $\rho = \frac{k}{m}$.

■ Question 2 (10 points).

For the first 20 seconds, without the parachute opened, ρ is given by 0.5. Find $v(20)$. Assume $g = 10\, m/s^2$.

■ Question 3 (10 points).

With the parachute open, ρ increases to 1.5. Find a formula for $v(t)$ for $t > 20$.

■ Question 4 (5 points).

Show that as $t \to \infty$, the paratrooper's velocity does not increase indefinitely. Instead, it approaches a finite limiting velocity, called the *terminal velocity*. Find the terminal velocity of the paratrooper.

§B. Boundary Value Problems

We know that the solution of a second-order linear differential equation is uniquely determined by two initial conditions. In particular, the solution of the initial value problem

$$y'' + py' + qy = 0, \qquad y(a) = 0, y'(a) = 0 \tag{1}$$

has a unique solution $y(t) \equiv 0$. The situation is quite radically different for a problem such as

$$y'' + py' + qy = 0, \qquad y(a) = 0, y(b) = 0 \tag{2}$$

The difference between the problems in equations (1) and (2) is that in (2) the two conditions are imposed at two different points a and b with (say) $a < b$. In (2) we are to find a solution of the differential equation on the interval (a, b) that satisfies the conditions $y(a) = 0$ and $y(b) = 0$ at the endpoints of the interval. Such a problem is called an endpoint or *boundary value problem*.

Consider the Boundary Value Problem (BVP)

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y(\pi) = 0$$

Question 1 (5 points).

Let $\lambda = 3$. Show that the only solution to above BVP in this case is the trivial solution $y(t) = 0$ for all t.

Question 2 (4 points).

Show that if $\lambda = 0$, then the only solution to the BVP is the trivial solution.

Question 3 (6 points).

Show that if $\lambda < 0$, then the only solution to the BVP is the trivial solution.

Question 3 (10 points).

Show that if $\lambda > 0$, then the BVP has a non-trivial solution if and only if λ is of the form

$$\lambda = n^2, \qquad n = 1, 2, 3, \ldots$$

§C. Competing Species

The blue crab is native to the US Atlantic coast, but there is concern that the population is on the decline. The European green crab is an invasive species (recently introduced to the US in the ballast waters of ships), that competes with the blue crab. Assume that the interaction between the blue crab, x, and the green crab, y, is modeled by (up to some scaling):

$$\frac{dx}{dt} = x(100 - x) - 2xy$$

$$\frac{dy}{dt} = y(400 - 6y) - xy$$

Question 1 (8 points).

Find equation of the nullclines and the four equilibrium solutions (three of which are non-negative, representing biologically relevant values).

Question 2 (7 points).

Assume x and y are non-negative. Draw the direction field in the first quadrant of the phase plane. what does the model predict about the long-term fate of the two species?

Question 3 (8 points).

Suppose an intervention effort is launched to preserve the blue crab by harvesting a proportion h of the green crabs, so the equations modeling the system become (this is up to scaling, so h can be any positive number):

$$\frac{dx}{dt} = x(100 - x) - 2xy$$
$$\frac{dy}{dt} = y(400 - 6y) - xy - hy$$

It would be nice if we could find harvesting values h so that the species can co-exist. Calculate the new equilibrium solutions and observe that for some values of h, the fourth equilibrium solution can be found at a point in the first quadrant (it will represent co-existence of the species, since both populations will have positive values). For what range of h values does the fourth equilibrium solution have positive coordinates? Find the exact h values.

Question 4 (12 points).

Suppose that you found in the previous problem that for $h_1 < h < h_2$ there is a fourth equilibrium solution in first quadrant of the phase plane. Show that, if $h_1 < h < h_2$ then the equilibrium solution where both blue and green crab co-exist is the **only** stable equilibrium solution among the four possible options. Thus, no mater what the initial value (as long as it is positive), the green crab and blue crab will co-exist. (yay!)

For this problem, you can assume that the Jacobian (associated to linearization) at each equilibrium point always has two real eigenvalues.

[Hint: Try to answer using the (T, D)-plane. Don't calculate the eigenvalues.]

[Note: The algebra will be a little messy in this question. So be patient during your calculation.]

§D. Damped Harmonic Oscillator with Sinusoidal Forcing

Consider a damped harmonic oscillator with sinusoidal forcing whose equation is given by

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + y = \cos(\omega t) \tag{$\star$}$$

where

- $y(t)$ denotes the displacement at time t,
- $p > 0$ is the damping constant,
- the spring constant is 1, and
- $\omega > 0$ is the forcing frequency.

Question 1 (9 points).

Find a particular solution $y_0(t)$ to above differential equation ($\star$) using the *method of undetermined coefficient*. Your solution should be in terms of ω and p.

Question 2 (6 points).

Suppose your solution $y_0(t)$ is of the form

$$y_0(t) = a\cos(\omega t) + b\sin(\omega t)$$

(a) Suppose the polar form of the complex number $z = a + ib$ is given by $re^{i\theta}$. What's the polar form of $a - ib$ in terms of r and θ?

(b) Calculate the real part of $(a - ib)e^{i\omega t}$ in two different ways to show that the solution $y_0(t)$ can be also written as

$$y_0(t) = r\cos(\omega t - \theta)$$

Question 3 (10 points).

Show that regardless of initial conditions, all general solutions of the differential equation ($\star$) converge to $y_0(t)$ for large values of t.

[Hint: Show that the two-dimensional system of first order differential equations corresponding to the *associated homogeneous equation* always has a sink of some type at the origin. What do the general solutions of the nonhomogeneous system ($\star$) look like?]

Question 4 (3 points).

We conclude that if damping is present, in the long-term, every solution of ($\star$) oscillates with frequency ω and amplitude r. Write r in terms of a and b and consequently as a function of ω and p.

Question 5 (7 points).

Fix p and let $r(\omega)$ be the amplitude of the particular solution when the system is forced at frequency ω. We say that *Practical resonance* occurs when $r(\omega)$ achieves its maximum as a function of ω.

(a) Show that if $p < \sqrt{2}$, then practical resonance occurs at $\omega = \sqrt{1 - \frac{p^2}{2}}$.

(b) Show that if $p > \sqrt{2}$, then no practical resonance occurs.

§E. Existence and Uniqueness

Consider the differential equation $y' = f(t,y)$ where $f(t,y)$ is continuously differentiable in t and y i.e. the partial derivatives are continuous functions. Suppose $f(t,y)$ is a periodic function of t with period T i.e.

$$f(t+\mathrm{T},y) = f(t,y) \qquad \text{for all } t \text{ and } y$$

and suppose there are constants p,q with $p < q$ such that

$$f(t,p) > 0, \quad f(t,q) < 0 \qquad \text{for all } t$$

Question 1 (10 points).

(a) If $y_p(t)$ is a solution curve such that $y_p(0) = p$, then explain why $y_p(\mathrm{T}) > p$.

(b) If $y_q(t)$ is a solution curve such that $y_q(0) = q$, then explain why $y_q(\mathrm{T}) < q$.

[Hint: What does the direction field look like?]

Question 2 (10 points).

Show that there is a periodic solution $\tilde{y}(t)$ with period T such that $p < \tilde{y}(0) < q$.

[Hint: Why is it enough to find $\tilde{y}$ such that $\tilde{y}(\mathrm{T}) = \tilde{y}(0)$?]

Note: An example of such a function is $f(t,y) = \sin(t) - y$. However, you are not allowed choose a fixed example to prove above question statements. You have to prove that the statements are true for all functions satisfying the given assumptions.

Final Exam Part 1

Linear Algebra -- Math 2000

Subhadip Chowdhury

Due: May 17, 2019, 8:30 AM

Instructions:

- This is a take-home exam. As such, your written arguments will be held to a higher standard than on a sit-down in-class exam. Please submit clear and carefully composed solutions, and explain the concepts you are using and the connections among them. As always, points may be deducted for any unjustified steps, and generous partial credit will be given if you explain your thought process to me.
- You may consult and use our course materials while taking this exam, including the textbook, class notes, your problem sets, and any of the handouts or Mathematica code on Blackboard. If you use Mathematica, please print and attach your commands and output to your exam. If you are not sure if some resource is allowed, please ask! You may NOT consult the internet or discuss problem specifics with other people. You may email me to ask questions.
- when submitting your exam, staple this packet on top, and sign the ''Honor Signature'' to indicate that you followed Bowdoin's Honor Code with respect to this exam.

Full Name: ______________________________

Honor Signature: ______________________________

Total Points Available: 125

Section Number	1	2	3	4	5	6	7	Total
Available Points	**18**	**20**	**12**	**20**	**20**	**8**	**27**	**125**
Your Score								

§1. The Pell Sequence

The Fibonacci sequence is arguably the most famous integer sequence in Mathematics, making surprise appearances in everything from seashell patterns to the Parthenon. It is defined recursively as follows: each term of the sequence is equal to the sum of the previous two. A generalization of Fibonacci Sequence called $c-$Fibonacci sequence is defined as follows:

$$\begin{aligned} p_0 &= 1 \\ p_1 &= 1 \\ p_n &= cp_{n-1} + p_{n-2},\ n \geq 2 \end{aligned}$$

where c is any natural number. The $2-$Fibonacci sequence is also known as the *Pell Sequence*. In this section, we will try to figure out the long term behaviour of the Pell sequence by calculating an explicit formula for p_n. In the following questions, *we will assume* $c = 2$.

Question 1 (3 points). *For $n \geq 1$, define the vector $\vec{\mathbf{u}}_n$ as*

$$\vec{\mathbf{u}}_n = \begin{bmatrix} p_{n-1} \\ p_n \end{bmatrix}$$

Find a 2×2 matrix A such that

$$\vec{\mathbf{u}}_{n+1} = A\vec{\mathbf{u}}_n \text{ for all } n \geq 1.$$

We can conclude that

$$\begin{bmatrix} p_{n-1} \\ p_n \end{bmatrix} = \vec{\mathbf{u}}_n = A^{n-1}\vec{\mathbf{u}}_1 = A^{n-1}\begin{bmatrix} p_0 \\ p_1 \end{bmatrix} = A^{n-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So if we can calculate A^{n-1}, we can easily give a formula for p_n.

Question 2 ((3+4+3+3+2) points). *(a) Find the eigenvalue(s) of A.*

(b) Find an eigenbasis.

(c) Write down the diagonalization $A = SDS^{-1}$ i.e. find $S, D,$ and S^{-1}.

(d) Find A^{n-1}.

(e) Find $\vec{\mathbf{u}}_n$ and a formula for p_n.

Question 3 (Bonus 2 points). *Show that*

$$\lim_{n\to\infty} \frac{p_{n+1}}{p_n} = 1 + \sqrt{5}$$

The number $1 + \sqrt{5}$ is called the *Silver Ratio*.

§2. Nul space, Col space and Rank

Question 4 ((3+3+4+3) points). *For some real matrix A, the following vectors form a* basis *for its column space and null space:*

$$\mathbf{Col}(A) = \mathbf{span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}\right\}$$

$$\mathbf{Nul}(A) = \mathbf{span}\left\{\begin{bmatrix}1\\-1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}2\\1\\-1\\0\\0\end{bmatrix}\right\}$$

(a) *What is the size $m \times n$ of A? What is* $\mathbf{rank}(A)$*?*

(b) *What are the dimensions of* $\mathbf{Col}(A^T)$, $\mathbf{Nul}(A^T)$, *and* $\mathbf{Row}(A)$*?*

(c) *Give a matrix A with this* $\mathbf{Col}(A)$ *and* $\mathbf{Nul}(A)$. *There are many possible answers.*

(d) *Give a vector $\vec{\mathbf{b}}$ for which $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has at least one solution, and give all the solutions $\vec{\mathbf{x}}$ for your A from the previous part.*

Hint: you should not have to do Row-reduction.

Question 5 ((4+3) points). *Let A be a $m \times n$ matrix and let B be a $n \times p$ matrix.*

(a) *Explain why* $\mathbf{rank}(AB) \leq \mathbf{rank}(A)$.

Hint: What is the relation between the column space of AB and that of A?

(b) *Explain why* $\mathbf{rank}(AB) \leq \mathbf{rank}(B)$.

Hint: What can you say about $\mathbf{rank}\left((AB)^T\right)$*?*

§3. Characteristic Polynomial

Question 6 (7 points). *Let A be an 5×5 diagonalizable matrix and let*

$$p_A(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + t^5$$

be the characteristic polynomial of A. Show that

$$\mathbf{det}(A) = c_0 \quad \text{and} \quad \mathbf{tr}(A) = -c_4.$$

Question 7 (5 points). *If A is an invertible and diagonalizable $n \times n$ matrix, show that A^{-1} can be written as a linear combination of the matrices $A, A^2, A^3, \ldots, A^{n-1}$.*

Hint: Use the Cayley-Hamilton theorem from assignment 22, chapter 5 supplementary exercise 7.

§4. Matrix of a Linear Transformations with respect to a Basis

Question 8 ((1+5+6) points). *Consider the vector space $\mathbb{V}$ consisting of 3×3 matrices A such that $A^T = -A$. These type of matrices are called* skew-symmetric.

(a) *Show that the diagonal entries of any skew-symmetric matrix are zero.*

(b) *Find the dimension of $\mathbb{V}$ and a basis $\mathfrak{B}$ for $\mathbb{V}$.*

(c) *Let $T : \mathbb{V} \to \mathbb{V}$ be a linear transformation defined by*

$$T(M) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} M - M \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

for all $M \in \mathbb{V}$. Find the $\mathfrak{B}$-matrix of T for your choice of $\mathfrak{B}$ from part (b).

Question 9 ((6+2) points). *Consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined as*

$$T(f(x)) = f(1) + (x-1)f'(1)$$

(a) *Consider the basis $\mathfrak{B}$ of $\mathbb{P}_2$ given by $\mathfrak{B} = \{1, (x-1), (x-1)^2\}$. Find the $\mathfrak{B}$-matrix of T.*

(b) *Is T an isomorphism?*

§5. Orthogonality

Question 10 ((3+2+5+1) points). *Let $\vec{\mathbf{v}}$ be a* unit vector *in $\mathbb{R}^3$ and let $H = I - 2\vec{\mathbf{v}}\vec{\mathbf{v}}^T$.*

(a) *Show that $H^2 = I$.*

(b) *Show that H is an orthogonal matrix.*

(c) *Find the eigenvalues of H and describe the eigenspaces geometrically.*

(d) *Can you give a geometric description of the linear transformation corresponding to H?*

Hint: What is $H\vec{\mathbf{u}}$ for $\vec{\mathbf{u}} \perp \vec{\mathbf{v}}$?

Question 11 ((6+3) points). *Let P be a 3×3 matrix that corresponds to the linear transformation that projects every vector in $\mathbb{R}^3$ orthogonally onto the plane $x + 2y + 2z = 0$.*

(a) *Find an orthogonal basis of $\mathbb{R}^3$ consisting of three eigenvectors of P.*

Hint: Investigate the geometry of the eigenspaces.

(b) *Find the matrix P.*

§6. Vector Space and Dimension

Let $B = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ be a diagonal 3×3 matrix, and let V be the vector space defined as

$$V = \{A \in \mathbb{R}^{3\times 3} \mid AB = BA\}$$

i.e. V consists of all 3×3 matrices A that commute with B.

Question 12 (8 points). *Find the possible dimensions of V. In each case, give a basis of V.*

Hint: Consider the following cases separately: (1) when $x = y = z$, (2) any two of x, y, z are equal but the third is not, (3) none of them are equal to each other.

§7. The Least-Square Approximation problem

One of the fundamental problems in Machine Learning is to find the best approximate solution to equations of the form $A\vec{x} = \vec{b}$ where A be a $m \times n$ matrix and $\vec{b}$ is a $m \times 1$ vector and the system is possibly inconsistent. Recall that the space $\mathbf{Col}(A)$ is a subspace of $\mathbb{R}^m$, defined as

$$\mathbf{Col}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$$

So if $\vec{b} < \mathbf{Col}(A)$, then $A\vec{x} = \vec{b}$ is not consistent, but we can still try to find an $\vec{x}$ that will minimize the "error". We define a *least-square solution* $\vec{x}^*$ to be an $\vec{x} \in \mathbb{R}^n$ such that

$$||A\vec{x}^* - \vec{b}|| \leq ||A\vec{x} - \vec{b}||$$

for all $\vec{x} \in \mathbb{R}^n$. In other words, $\vec{x}^*$ is such that $\vec{b}^* = A\vec{x}^*$ is the point in $\mathbf{Col}(A)$ that is closest to $\vec{b}$.

Geometrically speaking, $\vec{b}^* = A\vec{x}^*$ is the orthogonal projection of $\vec{b}$ onto $\mathbf{Col}(A)$ i.e

$$\vec{b}^* = \mathbf{proj}_{\mathbf{Col}(A)}\vec{b}.$$

Question 13 ((2+2) points). *(a) Explain why $\vec{b} - \vec{b}^*$ is in $(\mathbf{Col}(A))^\perp$.*

(b) Explain why part (a) is equivalent to the equation $A^T\vec{b} = A^TA\vec{x}^*$.

Hint: What is $(\mathbf{Col}(A))^\perp$ equal to?

(c) If $A\vec{x} = \vec{b}$, explain why $\vec{b}^* = \vec{b}$.

The equation obtained in part (b) above is called the *Normal Equation* for $A\vec{x} = \vec{b}$. By definition, the Normal Equation is always consistent, however it may have more than one solution i.e. there may be more than one best approximation. One particular case of interest is when $\mathbf{Nul}(A) = \{\mathbf{0}\}$.

Question 14 ((8+3+2) points). *Let A be an $m \times n$ matrix.*

(a) *Show that* $\mathbf{Nul}(A) = \mathbf{Nul}(A^T A)$.

Hint: It is easy to show that $\mathbf{Nul}(A) \subseteq \mathbf{Nul}(A^T A)$*. To show the opposite containment* $\mathbf{Nul}(A^T A) \subseteq \mathbf{Nul}(A)$*, use orthogonal complements.*

(b) *Show that* $\mathbf{rank}(A) = \mathbf{rank}(A^T A)$.

(c) *Show that if* $\mathbf{Nul}(A) = \{0\}$*, then* $A^T A$ *is an* $n \times n$ invertible *matrix.*

So if $\mathbf{Nul}(A) = \{0\}$, we get $A^T A$ is invertible and in that case the normal equation has a unique solution

$$\vec{\mathbf{x}}^* = \left(A^T A\right)^{-1} A^T \vec{\mathbf{b}}. \qquad (\star)$$

Scientists are often interested in fitting a function of a certain type, e.g. a polynomial to experimental data they have gathered. Since physical experiments often have unavoidable errors, there may not be an exact polynomial that satisfies given data-set.

Consider for example an experiment where we know that the quantity V is supposed to be a quadratic polynomial in variable T because of theoretical reasons. Let's write

$$V = a + bT + cT^2 = p(T)$$

During the experiment we note down a list of input versus output (T, V) data-points as follows:

Observation No.	T	V
1	-1	8
2	0	8
3	1	4
4	2	16

Our goal is to find a, b and c that will give the best approximate polynomial with least amount of error. In other words,

Question 15 (10 points). *Use equation* ($\star$) *to find the least-square solution* $\vec{\mathbf{x}}^* = \begin{bmatrix} a^* \\ b^* \\ c^* \end{bmatrix}$ *to the system of equations*

$$V_i = p(T_i) \text{ for } i = 1, 2, 3, 4$$

where (T_i, V_i) *is the data-point from observation* i*.*

A picture of $p^*(T) = a^* + b^*T + c^*T^2$ is given below.

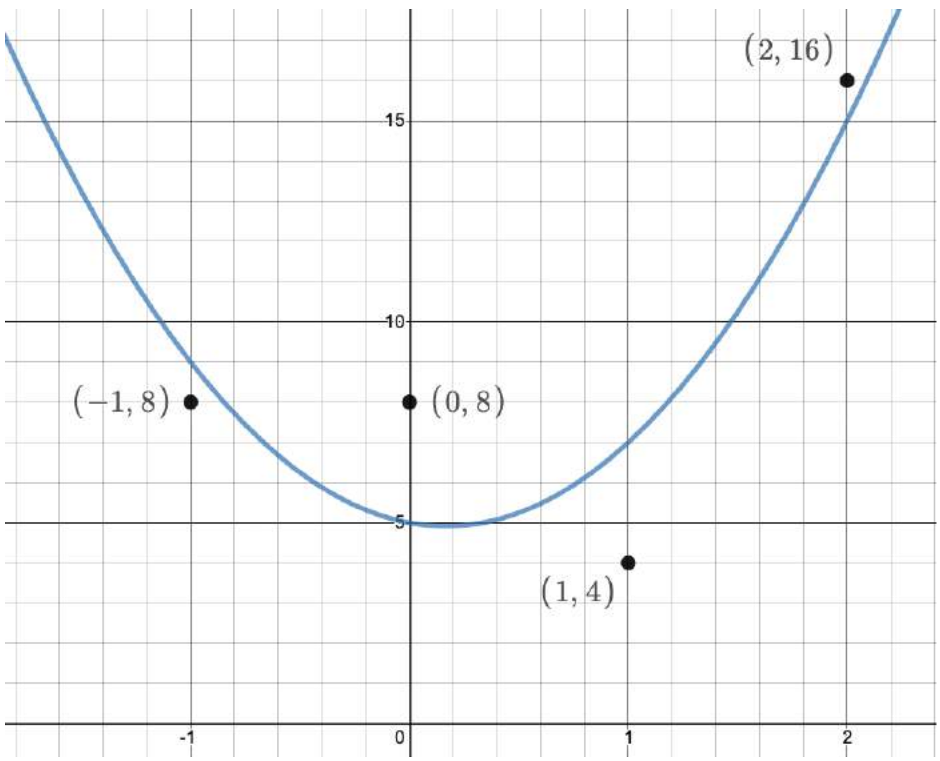


Figure 1

Final Exam Part 2

May 17, 2019
Subhadip Chowdhury

Instructions:

- Please show ALL your work! Answers without supporting justification will not be given credit.
- Answer the questions in the blue books provided.
- Write legibly and clearly mark the answer.
- Please note that use of any books or notes is not allowed. You are allowed to use the two pages of handwritten letter-sized note that you brought. Use of calculators are not allowed.
- If you write down the correct formula/procedure to find an answer, you will get some partial credit regardless of whether you evaluated the exact values or not.
- Unless otherwise specified, you may use any valid method to solve a problem.

Full Name: ______________________________

Question	Points	Score
1	12	
2	12	
3	16	
4	4	
5	4	
6	4	
Total:	52	

This exam has 6 questions, for a total of 52 points.
The maximum possible point for each problem is given on the right side of the problem.

1. For each of the following statements, find out whether it is '**always true**', '**sometimes true**', or '**always false**'. Give a brief explanation for your answer. If you think an answer is 'sometimes true', give an example or criterion when it's false.

   (a) If we have **2020** vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{2020} \in \mathbb{R}^{2019}$, then each $\vec{v}_i$ can be written as a linear combination of the other **2019** vectors. $\boxed{3}$

   (b) If V_k denotes the set consisting of all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ such that $xyz = k$ for some real constant k, then V_k is a subspace of $\mathbb{R}^3$. $\boxed{3}$

   (c) If all the diagonal entries of an $n \times n$ matrix A are odd integers and all other entries are even integers, then A is an invertible matrix. $\boxed{3}$

   (d) If A and B are invertible $n \times n$ matrices then AB and BA are similar matrices. $\boxed{3}$

2. Is it possible or impossible to find examples of each of the following? If possible, please provide an example. If impossible, please explain why.

   (a) A real number t such that $A = \begin{pmatrix} 2 & t & 0 \\ t & 2 & t \\ 0 & t & 2 \end{pmatrix}$ has eigenvalue 0. $\boxed{3}$

   (b) A square matrix whose nullspace is $\textbf{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\}$. $\boxed{3}$

   (c) Two real invertible 3×3 matrices A and S such that $S^T A S = -A$. $\boxed{3}$

   (d) Two 3×3 nonzero diagonalizable matrices that have same eigenvalues but are not similar. $\boxed{3}$

3. Choose the correct answer(s) from the given choices. No explanation is necessary.

   There may be more than one correct choice. You will get partial credit for circling correct but incomplete set of choices, full credit for correct and complete set of choices, and zero credit if you circle any incorrect choice.

   (a) Consider the following operations on $\mathbb{P}_2$. Which of them is/are linear transformation(s)? $\boxed{4}$

      α. $T(p(x)) = p(x) - p(1)$

      β. $T(p(x)) = p(x) - 1$

      γ. $T(p(x)) = x - p(1)$

      δ. $T(p(x)) = x^2 p\left(\frac{1}{x}\right)$

(b) Which of the following statements is/are true for all $n \times n$ matrices A and B? 4

α. $\mathbf{Col}(A) = \mathbf{Col}(AB)$

β. $\mathbf{Col}(A) = \mathbf{Col}(BA)$

γ. $\mathbf{Nul}(A) = \mathbf{Nul}(AB)$

δ. $\mathbf{Nul}(A) = \mathbf{Nul}(BA)$

(c) Suppose A and B are two 4×4 matrices and let $\mathbf{rank}(A) = 2$, $\mathbf{rank}(B) = 3$. Then which of the following are possible values of $\mathbf{rank}(A + B)$? 4

α. 0

β. 2

γ. 3

δ. 5

(d) Consider the system of linear equations 4

$$\begin{cases} x + y + z = 1 \\ 2x \quad\;\; + z = 2 \\ -x + y + az = b \end{cases}$$

Then,

α. If $a \neq 0$ then the system has exactly one solution regardless of the value of b.

β. If $a = 0$ then the system has no solution regardless of the value of b.

γ. If $a = 0$ then the system has no solution for exactly one value of b.

δ. If $a = 0$ then the system has infinitely many solutions for exactly one value of b.

4. If $A = Q^T D Q$ for a diagonal matrix D and an orthogonal matrix Q, then explain why the eigenvectors of A are the rows of Q. 4

5. Suppose A is a 3×3 matrix with eigenvalues $\lambda = 0, 1, -1$ and corresponding eigenvectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$. Find a basis of $\mathbf{Col}(A)$ in terms of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$, and $\vec{\mathbf{v}}_3$. 4

6. The matrix 4

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

is diagonalizable and $H^2 = 4I$. Find the eigenvalues of H and their algebraic multiplicities.

Hint: You should not need to calculate any determinant.

Final Exam

SUBHADIP CHOWDHURY

INSTRUCTIONS:

- Please show ALL your work! Answers without supporting justification will not be given credit.
- Answer the questions in the blue books provided.
- Start every question on a new page.
- Write legibly and clearly mark the answer.
- Please note that use of any textbook or note is not allowed. You are allowed to use the one two-sided page of handwritten letter-sized note that you brought with you. Use of calculator is not allowed.
- If you write down the correct formula for an answer, you will get some partial credit regardless of whether you evaluated the exact values or not.
- Unless otherwise specified, you may use any valid method to solve a problem.

Full Name: ______________________________

Question	Points	Score
1	10	
2	4	
3	4	
4	4	
5	5	
6	10	
7	14	
8	12	
9	8	
10	9	
11	10	
Total:	90	

This exam has 11 questions, for a total of 90 points.
The maximum possible point for each problem is given on the right side of the problem.

1. For the following problems, determine whether each statement is '**T**rue' or '**F**alse'. If it is true, explain your reasoning **briefly**. If it is false, explain why or give a counterexample that disproves the statement.

   (a) The scalar triple product $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$ between three vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ is zero if and only if two or more of the **3** vectors are parallel. $\boxed{2}$

   (b) If $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \mathbf{2019}\vec{\mathbf{w}}$ then $(\mathbf{Proj}_{\vec{\mathbf{v}}}\,\vec{\mathbf{u}}) \times (\mathbf{Proj}_{\vec{\mathbf{u}}}\,\vec{\mathbf{v}}) = \lambda\vec{\mathbf{w}}$ for some real number λ. $\boxed{2}$

   (c) Every flow line of the vector field $\vec{\mathbf{F}}(x, y) = x\hat{\imath} - y\hat{\jmath}$ lies on a level curve of the function $f(x, y) = xy$. $\boxed{2}$

   (d) Different parameterizations of the same curve result in identical tangent vectors at a given point on the curve. $\boxed{2}$

   (e) For a particle moving along the flow line $\vec{\mathbf{r}}(t)$ of a gradient vector field $\vec{\mathbf{F}} = \nabla f$, the velocity $\vec{\mathbf{r}}'(t)$ is zero at a stationary critical point of f. $\boxed{2}$

---

2. Compute the line integral of $\vec{\mathbf{F}}(x, y) = \left\langle 5y + 3y^2, 6xy + y^5 \right\rangle$ along the boundary of the green *Cyclops region* given in figure 1. There are four boundary curves, *oriented as shown in the picture*: a large ellipse of area **16**, two circles of area **2** (the eye) and **1** (the eyeball) as well as a small ellipse (the mouth) of area **3**. $\boxed{4}$

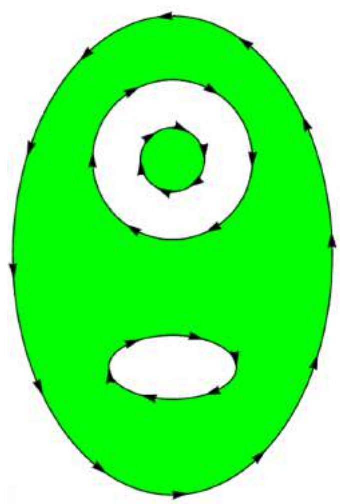


Figure 1

3. Match the Vector Fields in figure 2 with their formula. No explanation is necessary. 4

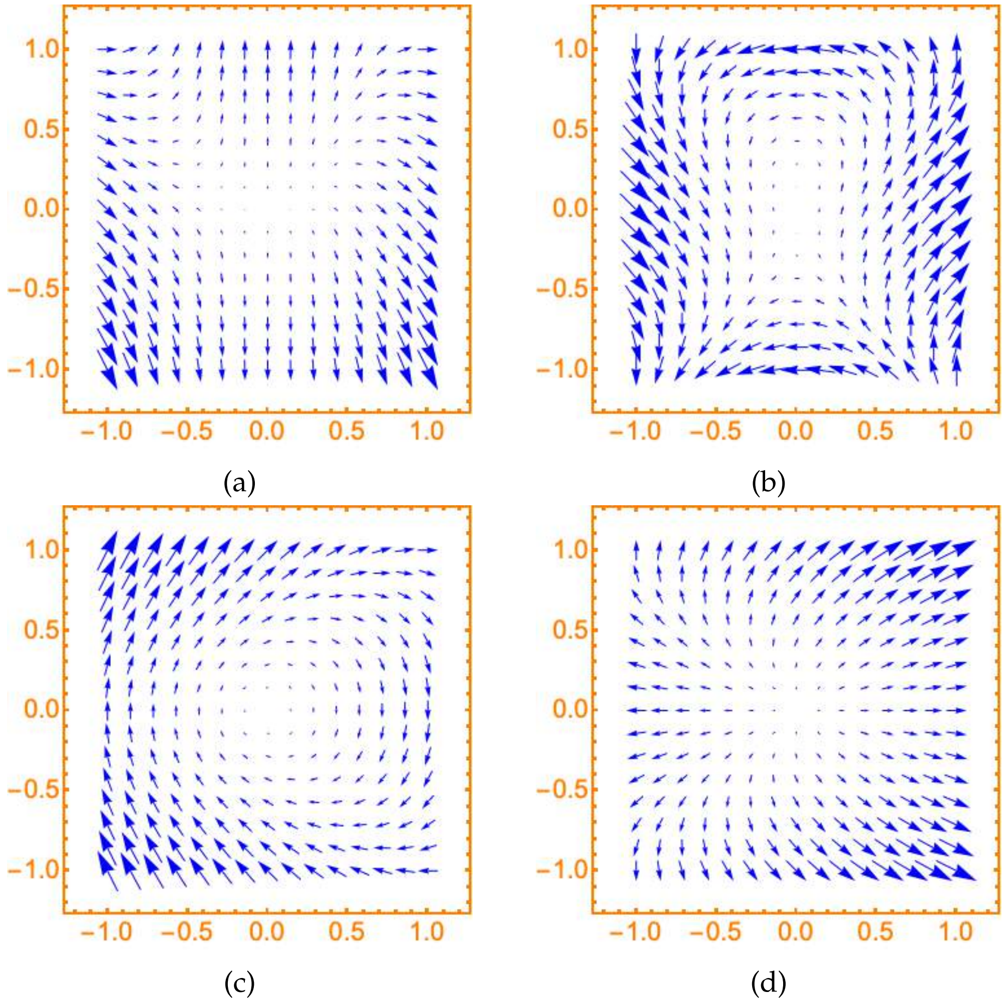


Figure 2

(1) $\langle y, y^2 - x\rangle$, **(2)** $\langle x^2, y - x^2\rangle$, **(3)** $\langle x^2 - y^2, x\rangle$, **(4)** $\langle x + y^2, y\rangle$,

---

4. Let $L(x, y)$ be the linearization of a function $f(x, y)$ and let $\vec{\mathbf{r}}(t)$ be a paramterized curve. Fill in the blanks with following three options. Options may be used more than once. No explanation is necessary.

- $\nabla f(\vec{\mathbf{r}}(0))$
- $\vec{\mathbf{r}}'(0)$
- $\vec{\mathbf{r}}(t) - \vec{\mathbf{r}}(0)$

(a) $L(\vec{\mathbf{r}}(t)) = f(\vec{\mathbf{r}}(0)) + \boxed{} \cdot \boxed{}$ 2

(b) $\left.\dfrac{d}{dt} f(\vec{\mathbf{r}}(t))\right|_{t=0} = \boxed{} \cdot \boxed{}$ 2

Please continue to next page...

5. Match each of the following iterated integrals with its domain of integration. No explanation is necessary. $\boxed{5}$

1. $\displaystyle\int_0^1 \int_y^1 \int_0^y f(x,y,z)\,dz\,dx\,dy$
2. $\displaystyle\int_0^1 \int_0^y \int_y^1 f(x,y,z)\,dz\,dx\,dy$
3. $\displaystyle\int_0^1 \int_y^1 \int_0^x f(x,y,z)\,dz\,dx\,dy$
4. $\displaystyle\int_0^1 \int_0^y \int_x^1 f(x,y,z)\,dz\,dx\,dy$
5. $\displaystyle\int_0^1 \int_0^y \int_x^y f(x,y,z)\,dz\,dx\,dy$

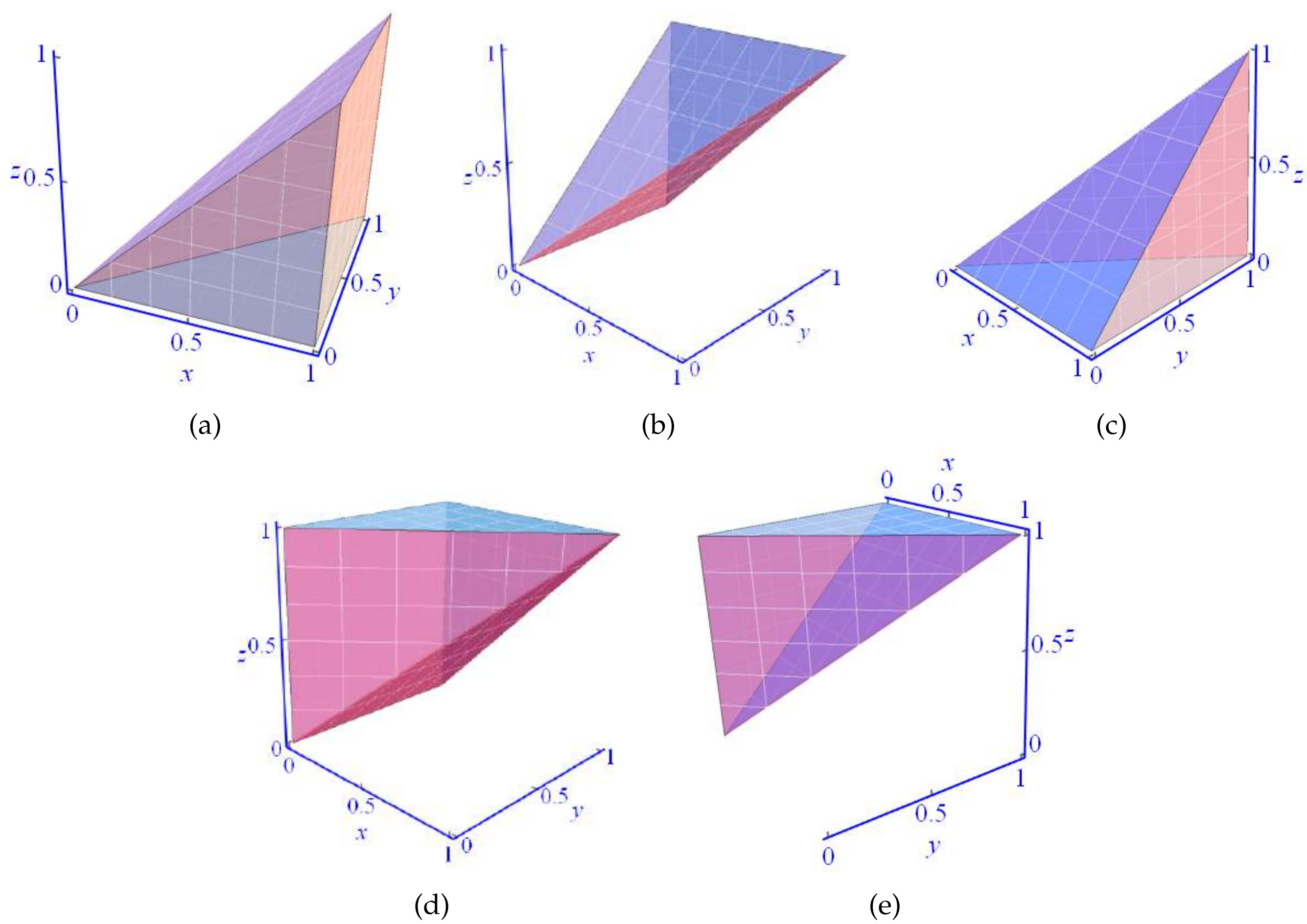


Figure 3: The five regions

6. Find the line integral of the vector field 10

$$\vec{\mathbf{F}} = \langle 45x^4y^2 - 2y^6 + 3,\ 18x^5y - 12xy^5 + 7\rangle$$

along the curve

$$r = 2\sin(\theta) + \sqrt{\left|5\cos^2\theta\ - 1\right|}$$

from $\theta = 0$ to $\theta = 3\pi/2$. A polar plot of the curve is given below in figure 4.

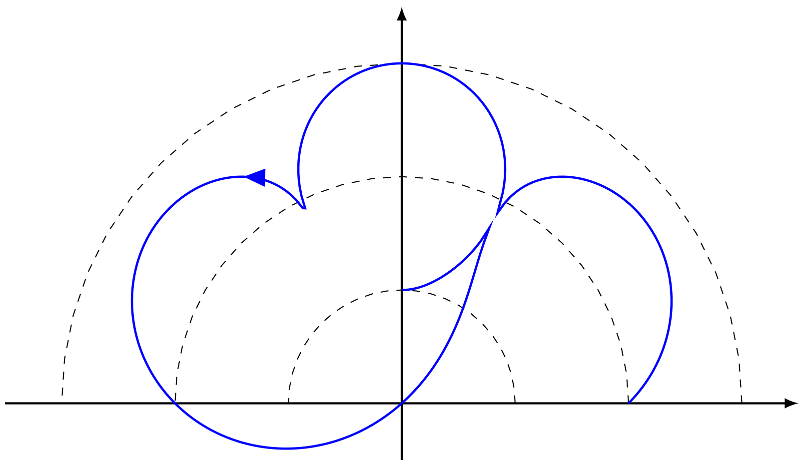


Figure 4

7. Let $\vec{\mathbf{F}}$ be the vector field 14

$$\vec{\mathbf{F}} = \left\langle xy^2,\ -x^2y\right\rangle$$

Let C be the blue curve given in figure 5. It consists of two circular arcs, one horizontal line segment, and one vertical line segment. Calculate the line integral $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ using Green's theorem.

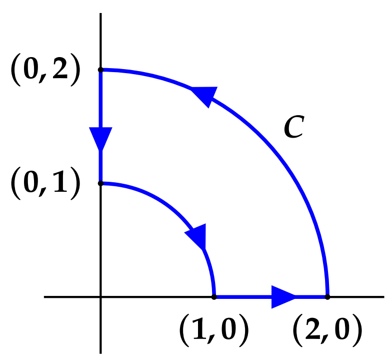


Figure 5

8. (a) Find the equations of the tangent planes at the points $A = (1, 1, 2)$ and $B = (-2, 1, 5)$ to the surface $\boxed{6}$

$$x^2 + y^2 - z = 0.$$

(b) Check that $(-1/2, 1, -1)$ lies on both of the tangent planes above. Find the parametric equation of the line of intersection of these two tangent planes. $\boxed{6}$

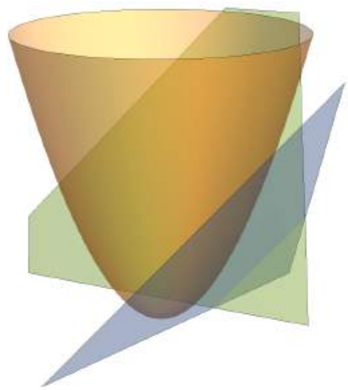


Figure 6

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9. Use the Lagrange multipliers to find the points (x, y) where $\boxed{8}$

$$f(x, y) = 4x^2y + y^3$$

achieves its maximum and minimum on the ellipse

$$g(x, y) = 2x^2 + y^2 = 6.$$

---

10. The function $F(x, y) = x^2y - 4xy + 3x^2 + 12y^2$ has three critical points, at $x = 0, x = 1$, and $x = 5$. $\boxed{9}$

(a) Find the values of y at these three critical points.

(b) Classify each critical point as a maximum, minimum, or saddle point.

---

11. Let

$$\iint_R f(x, y)\,dA = \int_1^2 \int_{2-x}^{\sqrt{2-x}} \frac{1}{2y^3 - 3y^2 + 5}\,dy\,dx$$

(a) Sketch the region R. $\boxed{3}$

(b) Rewrite the double integral as an iterated integral with the order interchanged. $\boxed{3}$

(c) Evaluate the integral. $\boxed{4}$

End of exam.