Problem Set 13 Solutions

Questions? Corrections? email jjudge@uchicago.edu (John)

edited by Subhadip Chowdhury

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Exercise 1: Induction on n

Problem 1.1. *Show that the following statements are true for all natural numbers n by inducting on n.*

(a) $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(4n^2-1)}{3}$
(b) $\frac{1}{1\times2\times3} + \frac{1}{2\times3\times4} + \ldots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$
(c) $2^n > n$
(d) $9^n + 7$ is divisible by 8

(a)

Proof. We will prove by inducting on $n \in \mathbb{N}$. For $n = 1$, the left side is 1^2, and the right side is $\frac{1(4(1)-1)}{3} = 1$. So the identity holds for $n = 1$.

Now let us make the induction assumption that $\exists k \in \mathbb{N}$ such that $1^2+3^2+5^2+\ldots+(2k-1)^2 = \frac{k(4k^2-1)}{3}$. Consider the left side expression, and add the $(k+1)^{th}$ term (the square of the next odd number) to it:

$$1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2 + (2k+1)^2$$

Meanwhile, the right side becomes of the form below, where we can factor the first term, add to the second, and factor out a common factor of $(2k+1)$:

$$\frac{k(4k^2-1)}{3} + (2k+1)^2 = \frac{(2k+1)[k(2k-1)+3(2k+1)]}{3}$$

$$= \frac{(2k+1)(2k+3)(k+1)}{3}$$

$$= \frac{(2(k+1)-1)(2(k+1)+1)(k+1)}{3}$$

which we can write in the form desired to complete the induction step

$$= \frac{(k+1)[4(k+1)^2-1\}}{3}$$

Thus, the identity is true for $n = k+1$ whenever it is true for $n = k$. By the principle of mathematical induction, it is true $\forall n \in \mathbb{N}$. $\square$

(b)

Proof. We will prove by inducting on $n \in \mathbb{N}$. For $n = 1$, the left side of the base case is $\frac{1}{1\times2\times3} = \frac{1}{6}$, and the right side is $\frac{1(1+3)}{4(1+1)(1+2)} = \frac{1}{6}$. So the identity holds for $n = 1$.

Now let us make the induction assumption that $\exists k \in \mathbb{N}$ such that $\frac{1}{1\times2\times3} + \frac{1}{2\times3\times4} + ... + \frac{1}{k(k+1)(k+2)} = \frac{k(k+3)}{4(k+1)(k+2)}$. Consider the left side expression, adding the $(k+1)^{th}$ term.

$$\frac{1}{2\times3\times4} + ... + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$

This is equal to the right side, with the same term added:

$$\frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} = \frac{k(k+3)^2+4}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(k^3+6k^2+9k+4)}{4(k+1)(k+2)(k+3)}$$

The numerator can be factored to $(k+1)^2(k+4)$, so this is

$$= \frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

which we can write in the form desired to complete the induction step

$$= \frac{(k+1)[(k+1)+3]}{4[(k+1)+1][(k+1)+2]}$$

Thus, the inequality is true for $n = k+1$ whenever it is true for $n = k$. By the principle of mathematical induction, it is true $\forall n \in \mathbb{N}$. □

(c)

Proof. We will prove by inducting on $n \in \mathbb{N}$. For $n = 1$, observe that $2^1 = 2$, and $2 > 1$. So the identity holds for $n = 1$.

Now let us make the induction assumption that $\exists k \in \mathbb{N}$ such that $2^k > k$. Now consider

$$2^{k+1} = 2(2^k)$$

Applying the inequality we assumed,

$$2(2^k) > 2k$$

which is greater than or equal to $k+1$, since $k \geq 1$.

Thus, the identity is true for $n = k+1$ whenever it is true for $n = k$. By the principle of mathematical induction, it is true $\forall n \in \mathbb{N}$ □

(d)

Proof. We will prove by inducting on $n \in \mathbb{N}$. For $n = 1$, observe that $9^1 + 7 = 16$, and $8|16$. So the identity holds for $n = 1$. Now let us make the induction assumption that $\exists k \in \mathbb{N}$ such that $8|(9^k + 7)$. Then consider the expression

$$9^{k+1} + 7 = 9(9^k) + 7$$

$$= 8(9^k) + (9^k + 7)$$

Now 8 divides into the first term since 9^k is an integer. And $8|(9^k + 7)$ by our induction assumption. So 8 divides into the sum of these two terms.

Thus, what we are trying to prove is true for $n = k + 1$ whenever it is true for $n = k$. By the principle of mathematical induction, it is true $\forall n \in \mathbb{N}$ □

Exercise 2: Trigonometric Induction

Problem 2.1. *If $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$ for all $n \geq 1$, show that $a_n = 2cos\frac{\pi}{2^{n+1}}$ HINT: Use induction on n and the trigonometric identity $cos(2\theta) = 2cos^2\theta - 1$*

Proof. We will prove by inducting on $n \in \mathbb{N}$. For $n = 1$, observe that $a_1 = \sqrt{2}$, and according to the closed form, $a_1 = 2cos\frac{\pi}{2^{1+1}} = 2(\frac{1}{\sqrt{2}}) = \sqrt{2}$. So the formula holds for $n = 1$. Now let us make the induction assumption that $\exists k \in \mathbb{N}$ such that $a_k = 2cos\frac{\pi}{2^{k+1}}$. Then consider

$$a_{k+1} = \sqrt{2 + a_k}$$

$$= \sqrt{2 + 2cos\frac{\pi}{2^{k+1}}}$$

Squaring both sides, we get

$$a_{k+1}^2 = 2 + 2cos\frac{\pi}{2^{k+1}}$$

$$= 2(1 + cos(2\frac{\pi}{2^{k+2}}))$$

Using the trigonometric identity for $cos(2\theta)$, we have

$$= 4(cos^2(\frac{\pi}{2^{k+2}}))$$

which means that, taking the positive square root of both sides,

$$a_{k+1} = 2(cos(\frac{\pi}{2^{k+2}}))$$

Thus, the formula is true for $n = k+1$ whenever it is true for $n = k$. By the principle of mathematical induction, it is true $\forall n \in \mathbb{N}$ □

Exercise 3: Divisibility Induction on Odds

Problem 3.1. *If n is an odd natural number, prove that $n(n^2-1)$ is divisible by 8.*

I will give two proofs. The first is shorter. The second uses induction and can also prove Exercise 4 immediately.

Proof. The odd integer n can be written as $2k+1$ where $k \in \mathbb{Z}$. Therefore, we can write only the second factor of the expression of interest as

$$(n^2-1) = (2k+1)^2 - 1$$

$$= 4k(k-1)$$

Now, k and $k-1$ are consecutive integers, so one must be even. Hence, $2|k(k-1)$, and so we have $8|4k(k-1)$, and thus $8|n(n^2-1)$.

□

Proof. Every positive odd natural number can be written in the form $n = 2m-1$ with $m \in \mathbb{N}$, so we will induct on $m \in \mathbb{N}$. Note that once we find that this holds for all positive odd numbers, the negative odd numbers follow the same proof, with signs changed appropriately. When $m = 1$, observe that $8|1(1^2-1)$ holds true, since $8 \times 0 = 0$, and 0 is an integer.

Let us make the induction assumption the $\exists k \in \mathbb{N}$ such that $8|(2k-1)[(2k-1)^2-1]$. Now $(2k-1)$ is odd, so it does not have divisibility by 2, which is the only prime factor of 8. Thus, we have $8|[(2k-1)^2-1]$. Consider the quantity

$$(2k+1)[(2k+1)^2-1] = (2k-1)[(2k+1)^2-1] + 2[(2k+1)^2-1]$$

Rewrite $2k+1$ as $(2k-1)+2$, then distribute

$$= (2k-1)[(2k-1)^2+8k-1] + 2[(2k-1)^2+8k-1]$$

Distribute again, once in each term:

$$= (2k-1)[(2k-1)^2-1] + 8k(2k-1) + 2[(2k-1)^2-1] + 16k$$

We can further change the expression, combining the last three terms,, combining the last three terms, to be

$$= (2k-1)[(2k-1)^2-1] + 2(4k^2-4k+1) + 16k - 2$$

$$= (2k-1)[(2k-1)^2-1] + 8k(2k-1) + 8k(k+1)$$

$$= (2k-1)[(2k-1)^2-1] + 8k(3k)$$

The first term $(2k-1)[(2k-1)^2-1]$ is divisible by 8 due to our induction assumption.The second term, $24k^2$, is divisible by 8, as $3k^2$ is an integer. Thus, 8 divides evenly into the sum of these terms, by the additive property of divisibility. This completes our induction step.

Thus, the identity is true for $m = k+1$ whenever it is true for $m = k$. By the principle of mathematical induction, it is true $\forall m \in \mathbb{N}$ □

Exercise 4: Divisibility Induction on Odds II (Extra Credit)

Problem 4.1. *If n is an odd natural number, prove that $n(n^2-1)$ is divisible by 24.*

Besides the following proof, this can also be proven with the first proof of Exercise 3, combined with the fact that every 3 consecutive integers has one divisible by 3.

Proof. Every positive odd natural number can be written in the form $n = 2m-1$ with $m \in \mathbb{N}$, so we will induct on $m \in \mathbb{N}$. Note that once we find that this holds for all positive odd numbers, the negative odd numbers follow the same proof, with signs changed appropriately. When $m = 1$, observe that $24|1(1^2-1)$ holds true, since $24 \times 0 = 0$, and 0 is an integer.

Let us make the induction assumption the $\exists k \in \mathbb{N}$ such that $24|(2k-1)[(2k-1)^2-1]$.

In Exercise 3, we simplified and obtained

$$(2k+1)[(2k+1)^2-1] = (2k-1)[(2k-1)^2-1] + 8k(3k)$$

The first term is divisible by 24 by the induction assumption. The second term is divisible by 24 since k^2 is an integer. Hence, the sum of the two terms is divided evenly by 24. This completes our induction step.

Thus, the identity is true for $m = k+1$ whenever it is true for $m = k$. By the principle of mathematical induction, it is true $\forall m \in \mathbb{N}$ □

Exercise 5: Trigonometric Identity

Problem 5.1. *If $sin\theta + cos\theta = \lambda$ for some angle θ, prove that $sin\theta - cos\theta = \pm\sqrt{2-\lambda^2}$.*

Proof. If we square both sides for the equation of our given value of λ, we get (using the identity $cos^2\theta + sin^2\theta = 1$ for all θ)

$$1 + 2sin\theta cos\theta = \lambda^2$$

Now consider the expressions $\pm\sqrt{2-\lambda^2}$, and use the above equation as a substitution. We will just examine the positive expression for now:

$$\sqrt{2-\lambda^2} = \sqrt{2-(1+2sin\theta cos\theta)}$$

$$= \sqrt{1-2sin\theta cos\theta}$$

Again using $cos^2\theta + sin^2\theta = 1$:

$$= \sqrt{cos^2\theta - 2sin\theta cos\theta + sin^2\theta}$$

$$= \sqrt{(cos\theta - sin\theta)^2}$$

The value of $cos\theta - sin\theta$ may be positive or negative, but its square will always be positive. Therefore $sin\theta - cos\theta$ will always match one of the expressions $\pm|cos\theta - sin\theta|$, which we have shown to be equivalent to $\pm\sqrt{2-\lambda^2}$. □

Exercise 6: Trigonometric Inequality Identity

Problem 6.1. *Prove that* $2sin^2\theta + 3cos^2\theta \geq 2$ *for all* θ.

Proof. We can examine the left side of the inequality, $2sin^2\theta + 3cos^2\theta$, and substitute in twice the identity $cos^2\theta + sin^2\theta = 1$ for all θ:

$$2sin^2\theta + 3cos^2\theta = 2 + cos^2\theta$$

Now, notice that $-1 \leq cos\theta \leq 1$, as cosine can be considered a measure of a leg of a right triangle with hypotenuse of length 1. The square of $cos\theta$ must be non-negative, so $0 \leq cos^2\theta \leq 1$. Applying this to our simplified expression, we have

$$2 + 0 \leq 2 + cos^2\theta \leq 2 + 1$$

which means that the original expression is also no less than 2:

$$2sin^2\theta + 3cos^2\theta \geq 2$$

□

Exercise 7: Trigonometric Inequality Identity II (Extra Credit)

Problem 7.1. *Prove that* $sin^4\theta + cos^4\theta \geq \frac{1}{2}$ *for all* θ.

Proof. If we add $2cos^2\theta sin^2\theta$ to the left side of the inequality, we have

$$sin^4\theta + 2cos^2\theta sin^2\theta + cos^4\theta = (sin^2\theta + cos^2\theta)^2$$

which is simply $1^2 = 1$. Now all that remains is to show that the modified right side, $\frac{1}{2} + 2cos^2\theta sin^2\theta$, is less than or equal to 1.

Now $2cos^2\theta sin^2\theta$ can be written, by the addition of angles identity, as $\frac{1}{2}sin^2(2\theta)$. We want to show that this is less than or equal to $\frac{1}{2}$ for all θ. But we know that $-1 \leq sin2\theta \leq 1$ for all θ, and so $0 \leq sin^2(2\theta) \leq 1$. So $\frac{1}{2}sin^2(2\theta) \leq \frac{1}{2}$. □