Quiz 4

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Full Name:

1. **CIRCLE** the correct option (only one) in each of the following. In problems (a)-(c), 'lub' stands for least upper bound and 'glb' stands for greatest lower bound.

   (a) Let $M = lub$ of a nonempty bounded subset S of $\mathbb{R}$. Then M is also equal to $\boxed{2}$

   A. the glb of the set of all lower bounds (in $\mathbb{R}$) of S.
   B. the lub of the set of all lower bounds (in $\mathbb{R}$) of S.
   C. the glb of the set of all upper bounds (in $\mathbb{R}$) of S.
   D. the lub of the set of all upper bounds (in $\mathbb{R}$) of S.

   (b) Consider the following four intervals where $a < b$ are both fixed real numbers: $\boxed{2}$

   $$(a, b), [a, b], [a, b), \text{ and } (a, b]$$

   Which of the following is true about the upper and lower bounds of these intervals?

   A. The glb for the intervals $[a, b)$ and $[a, b]$, while same to each other, differs from the glb of the intervals $(a, b]$ and (a, b), which in turn are the same. Similarly, The lub for the intervals $(a, b]$ and $[a, b]$, while same to each other, differs from the lub of the intervals $[a, b)$ and (a, b), which in turn are same.
   B. The intervals $[a, b]$ and $[a, b)$ have a glb and the intervals $(a, b]$ and (a, b) do not. Further, the intervals $[a, b]$ and $(a, b]$ have a lub, and the intervals $[a, b)$ and (a, b) do not.
   C. $[a, b]$ is the only interval among the four intervals that has a glb and a lub.
   D. All of them have the same lub and glb.

   (c) Suppose S is a nonempty bounded subset of $\mathbb{R}$. Denote by $-S$ the set $\{-s \mid s \in S\}$. Which of the following is true about S? $\boxed{2}$

   A. $lub(-S) = lub(S)$ and $glb(-S) = glb(S)$
   B. $lub(-S) = glb(S)$ and $glb(-S) = lub(S)$
   C. $lub(-S) = -lub(S)$ and $glb(-S) = -glb(S)$
   D. $lub(-S) = -glb(S)$ **and** $glb(-S) = -lub(S)$

   (d) If $\tan\theta = -\frac{4}{3}$, then $\sin\theta$ can be $\boxed{2}$

   A. $-4/5$ but not $4/5$
   B. $4/5$ but not $-4/5$
   C. either $-4/5$ **or** $4/5$

   (e) $\tan\left(\frac{30\pi}{4} + \theta\right) =$ $\boxed{2}$

   A. $\tan\theta$
   B. $-\tan\theta$
   C. $\cot\theta$
   D. $-\cot\theta$

2. Suppose x is a real number such that $\cos x + \sin x = \sqrt{2}\cos x$. Prove that $\cos x - \sin x = \sqrt{2}\sin x$. $\boxed{10}$

Solution: From the first equation, we get

$$\cos x(\sqrt{2}-1) = \sin x \implies \cos x = \frac{\sin x}{\sqrt{2}-1}$$

Substituting this value of $\cos x$ in the LHS we get

$$\begin{aligned}
\cos x - \sin x &= \frac{\sin x}{\sqrt{2}-1} - \sin x \\
&= \left(\frac{1}{\sqrt{2}-1} - 1\right)\sin x \\
&= \left(\frac{1-\sqrt{2}+1}{\sqrt{2}-1}\right)\sin x \\
&= \left(\frac{2-\sqrt{2}}{\sqrt{2}-1}\right)\sin x \\
&= \left(\frac{\sqrt{2}(\sqrt{2}-1)}{\sqrt{2}-1}\right)\sin x \\
&= \sqrt{2}\sin x
\end{aligned}$$

Here is another solution.

Solution: Dividing both sides of the first equation by $\cos x$, we get that

$$\cos x(\sqrt{2}-1) = \sin x \implies 1 + \tan x + \sqrt{2} \implies \tan x = \sqrt{2}-1$$

Dividing both sides of the second equation by $\sin x$, we get the equivalent equation

$$\cot x - 1 = \sqrt{2}$$

Now

$$\begin{aligned}
\tan x &= \sqrt{2}-1 \\
\implies \cot x &= \frac{1}{\sqrt{2}-1} \\
&= \frac{\sqrt{2}+1}{(\sqrt{2}-1)(\sqrt{2}+1)} \\
&= \frac{\sqrt{2}+1}{2-1} \\
&= \sqrt{2}+1 \\
\implies \cot x - 1 &= \sqrt{2}
\end{aligned}$$

Here is another solution.

Solution: Squaring both sides of the first equation we get,

$$\begin{aligned}
(\cos x+\sin x)^2 &= 2\cos^2 x\\
\implies \cos^2 x+\sin^2 x+2\cos x\sin x &= 2\cos^2 x\\
\implies \sin^2 x+2\cos x\sin x &= \cos^2 x\\
\implies 2\sin^2 x+2\cos x\sin x &= \cos^2 x+\sin^2 x\\
\implies 2\sin^2 x &= \cos^2 x+\sin^2 x-2\cos x\sin x\\
\implies 2\sin^2 x &= (\cos x-\sin x)^2\\
\implies \sqrt{2}\sin x &= \cos x-\sin x
\end{aligned}$$

3. Suppose the equation $(5x^2-4x+2)+m(4x^2-2x-1)=0$ has no solution. Find all possible values of m. $\boxed{10}$

Solution: We can rewrute the equation as

$$(5+4m)x^2+(-4-2m)x+(2-m)=0$$

Hence the discriminant is

$$\begin{aligned}
D &= (-4-2m)^2-4(5+4m)(2-m)\\
&= 16+4m^2+16m-40-12m+16m^2\\
&= 20m^2+4m-24
\end{aligned}$$

If the polynomial has no roots, then $D<0$. Then

$$20m^2+4m-24<0 \implies 5m^2+m-6<0 \implies (5m+6)(m-1)<0$$

Hence the possible values of m are $-\frac{6}{5}<m<1$.

4. In a Geometric Progression, the $(m+n)^{th}$ term is p and the $(m-n)^{th}$ term is q. $\boxed{5 \text{ (bonus)}}$

Show that its m^{th} term is $\sqrt{pq}$.

Solution: The formula for m^{th} term of a GP is ar^{m-1}. So we are given

$$ar^{m+n-1}=p$$

$$ar^{m-n-1}=q$$

Then

$$pq = a^2 r^{m+n-1+m-n-1} = a^2 r^{2m-2} = \left(ar^{m-1}\right)^2$$

Hence $\sqrt{pq} = ar^{m-1}$, the m^{th} term.