Math 1800 Project 9: Normal Density Integral

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In statistical applications it is important to know the exact value of the area under the bellshaped curve $y = e^{-t^2/2}$, i.e., we need to evaluate the improper integral

$$k = \int_{-\infty}^{\infty} e^{-t^2/2} dt$$

Using integral k, the standard normal density function becomes

$$f(t) = \frac{1}{k} e^{-t^2/2}.$$

In this project, as a byproduct of our ability to evaluate double integrals using polar coordinates, we will obtain an exact value for k.

Notice that, by symmetry, we have

$$k = \int_{-\infty}^{\infty} e^{-t^2/2} dt = 2\int_{0}^{\infty} e^{-t^2/2} dt$$

In addition, by definition of the improper integral from 0 to ∞, we also have

$$\int_{0}^{\infty} e^{-t^2/2} dt = \lim_{a\to\infty} \int_{0}^{a} e^{-t^2/2} dt$$

(a) Let a be positive, and let D_a be the square domain $[0, a] \times [0, a]$. Show that

$$\iint_{D_a} e^{-(x^2+y^2)/2} dA = \left(\int_{0}^{a} e^{-t^2/2} dt\right)^2$$

(b) Use part (a) to show that

$$\lim_{a\to\infty} \iint_{D_a} e^{-(x^2+y^2)/2} dA = \frac{1}{4}k^2$$

(c) Now designate by S_a the quarter-disk of radius a consisting of points with polar coordinates satisfying $0 \leq \theta \leq \pi/2$ and $0 \leq r \leq a$, and designate by T_a the quarter-disk of radius $\sqrt{2}a$ consisting of points with polar coordinates satisfying $0 \leq \theta \leq \pi/2$ and $0 \leq r \leq \sqrt{2}a$. Explain geometrically why

$$\iint_{S_a} e^{-(x^2+y^2)/2} dA \leq \iint_{D_a} e^{-(x^2+y^2)/2} dA \leq \iint_{T_a} e^{-(x^2+y^2)/2} dA$$

(d) Transform

$$\iint_{S_a} e^{-\left(x^2+y^2\right)/2} dA$$

into an iterated integral in polar coordinates, and evaluate this integral exactly.

(e) Transform

$$\iint_{T_a} e^{-\left(x^2+y^2\right)/2} dA$$

into an iterated integral in polar coordinates, and evaluate this integral exactly.

(f) Show that the integrals in parts (d) and (e) both approach the same limiting value as a approaches infinity.

(g) Use the results of parts (a) through (f) to find the value of k.