Math 1800 Handout 8: 3D Curl and Divergence

Subhadip Chowdhury

Curl

Let $\vec{\nabla} = \frac{\partial}{\partial x}\vec{\mathbf{i}} + \frac{\partial}{\partial y}\vec{\mathbf{j}} + \frac{\partial}{\partial z}\vec{\mathbf{k}}$. This is called a "differential operator". Given a vector field $\vec{\mathbf{F}} = P\,\vec{\mathbf{i}} + Q\,\vec{\mathbf{j}} + R\,\vec{\mathbf{k}}$, we define the curl of $\vec{\mathbf{F}}$ to be the *vector field*

$$\mathbf{curl}\,\vec{\mathbf{F}} = \vec{\nabla} \times \vec{\mathbf{F}}.$$

Exercise 0

Write out the formula for $\mathbf{curl}\,\vec{\mathbf{F}}$.

Exercise 1

Now suppose $\vec{\mathbf{F}}$ is a gradient field. I.e., $\vec{\mathbf{F}} = P\,\vec{\mathbf{i}} + Q\,\vec{\mathbf{j}} + R\,\vec{\mathbf{k}} = g_x\,\vec{\mathbf{i}} + g_y\,\vec{\mathbf{j}} + g_z\,\vec{\mathbf{k}}$ for some function $g(x, y, z)$. Use Clairaut's theorem to prove that $\mathbf{curl}\,\vec{\mathbf{F}} = \mathbf{0}$.

This statement is sometimes written as

$$\vec{\nabla} \times (\nabla g) = \mathbf{0}.$$

Exercise 2

Show that

$$\vec{\mathbf{F}}(x, y, z) = y^2z^3\vec{\mathbf{i}} + 2xyz^3\vec{\mathbf{j}} + 3xy^2z^2\vec{\mathbf{k}}$$

is a conservative vector field.

Find a function f such that $F = \nabla f$.

Exercise 3

Show that Green's Theorem can be rewritten as

$$\oint_{\partial R} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} \;=\; \iint_R \left((\mathbf{curl}\,\vec{\mathbf{F}}) \cdot \vec{\mathbf{k}}\right) dA.$$

This is called the **vector form** of Green's Theorem. It generalizes to 3D situations in the form of Stokes' Theorem!

Divergence

Since we have defined the differential operator $\vec{\nabla}$ which looks like a vector, we might naturally ask what is the dot product of the operator and the vector field. if the **curl** can be interpreted of as a *vector derivative* of the vector field, we define the *scalar derivative* of the vector field as follows.

If $\frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}$, and $\frac{\partial R}{\partial z}$ exist, then the divergence of $\vec{\mathbf{F}}$ is defined to be the *scalar* quantity

$$\mathbf{div}\,\vec{\mathbf{F}} = \vec{\nabla} \cdot \vec{\mathbf{F}} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Exercise 4

Show that If $\vec{\mathbf{F}} = P\vec{\mathbf{i}} + Q\vec{\mathbf{j}} + R\vec{\mathbf{k}}$ is a vector field defined on $\mathbb{R}^3$ and $P, Q,$ and R have continuous second-order partial derivatives, then

$$\mathbf{div}\,\mathbf{curl}\,\vec{\mathbf{F}} = 0.$$

This is also sometimes written as

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{\mathbf{F}}) = 0.$$

Exercise 5

Show that the vector field

$$\vec{\mathbf{F}}(x, y, z) = xz\vec{\mathbf{i}} + xyz\vec{\mathbf{j}} - y^2\vec{\mathbf{k}}$$

cannot be written as the curl of another vector field, that is, $F \neq \mathbf{curl}\, G$.

Exercise 6

Show that Green's Theorem can also be written in (yet another vector form) as

$$\oint_{\partial R} \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds = \iint_R \mathbf{div}\,\vec{\mathbf{F}}\, dA$$

where $\vec{\mathbf{n}}$ is the outward unit normal vector to ∂R. It generalizes to 3D situations in the form of Divergence Theorem!

the reason for the name divergence can be understood in the context of fluid flow. If $\vec{\mathbf{F}} = P\vec{\mathbf{i}} + Q\vec{\mathbf{j}} + R\vec{\mathbf{k}}$ the velocity of a fluid (or gas), then $\mathbf{div}\,\vec{\mathbf{F}}$ represents the net rate of change (with respect to time) of the mass of fluid (or gas) flowing from the point (x, y, z) per unit volume. In other words, $\mathbf{div}\,\vec{\mathbf{F}}$ measures the tendency of the fluid to diverge from the point (x, y, z). if $\mathbf{div}\,\vec{\mathbf{F}} = 0$ then $\vec{\mathbf{F}}$ is said to be *incompressible.*

Laplace Operator

For the sake of completion we also mention another differential operator that occurs when we compute the divergence of a gradient vector field.

$$\mathbf{div}(\nabla f) = \vec{\nabla} \times (\nabla f)$$

is abbreviated as $\nabla^2 f$, and the operator ∇^2 is called the Laplace operator .