MATH 1800 PROJECT 3: EPICYCLOIDS AND THE ROTARY ENGINE

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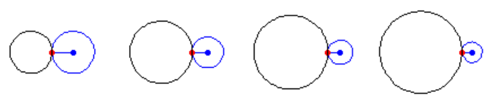
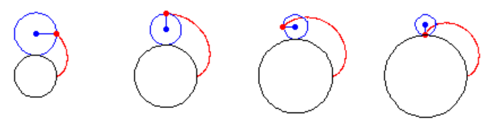
THE EPICYLOID

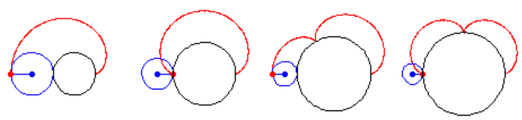
Consider a (black) circle of radius $\mathbf{R}$ with its center at the origin $\mathbf{O}$. A (blue) circle of radius r rolls around the *outside* of the circle of radius $\mathbf{R}$. See figure 1 for diagrams of different values of r. A (red) point $\mathbf{P}$ is located on the circumference of the rolling circle. The path traced out by $\mathbf{P}$ is called an epicycloid.



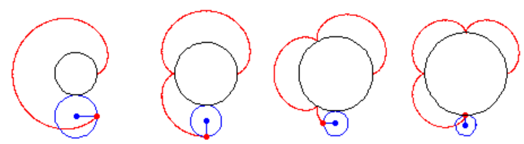
(a) $t = 0$



(b) $t = \pi/2$



(c) $t = \pi$



(d) $t = 3\pi/2$

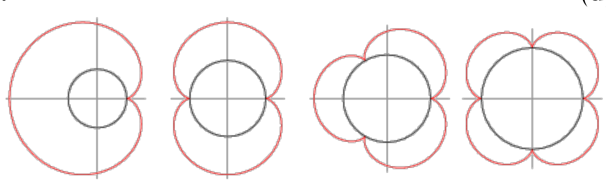


Figure 1

Assume that initially at time $t = 0$, the rolling circle sits to the right of the fixed circle and the point $\mathbf{P}$ is located at $(\mathbf{R}, 0)$. After the rolling circle has moved a bit, draw a line from the center $\mathbf{O}$ of the large circle to the point of contact with the rolling circle and let t be the angle the line makes with the positive $\mathbf{X}$–axis. If the location of $\mathbf{P}$ at this moment is given by $(x(t), y(t))$ (see figure 2), then the parametric equation of the *epicycloid* is given by

$$x(t) = (\mathrm{R}+r)\cos t - r\cos\frac{(\mathrm{R}+r)t}{r}$$

$$y(t) = (\mathrm{R}+r)\sin t - r\sin\frac{(\mathrm{R}+r)t}{r}$$

Note: If the initial location of $\mathbf{P}$ and the rolling circle is chosen differently, you will get the same shape with a different orientation and the form of the parametric equations will change slightly. For example, if the **sin** and **cos** functions are interchanged (**sin** $\leftrightarrow$ **cos**), we get a vertically oriented epicycloid. If we replace t with $t + \varphi$ we get an epicycloid that has been rotated by angle φ.

Question 1

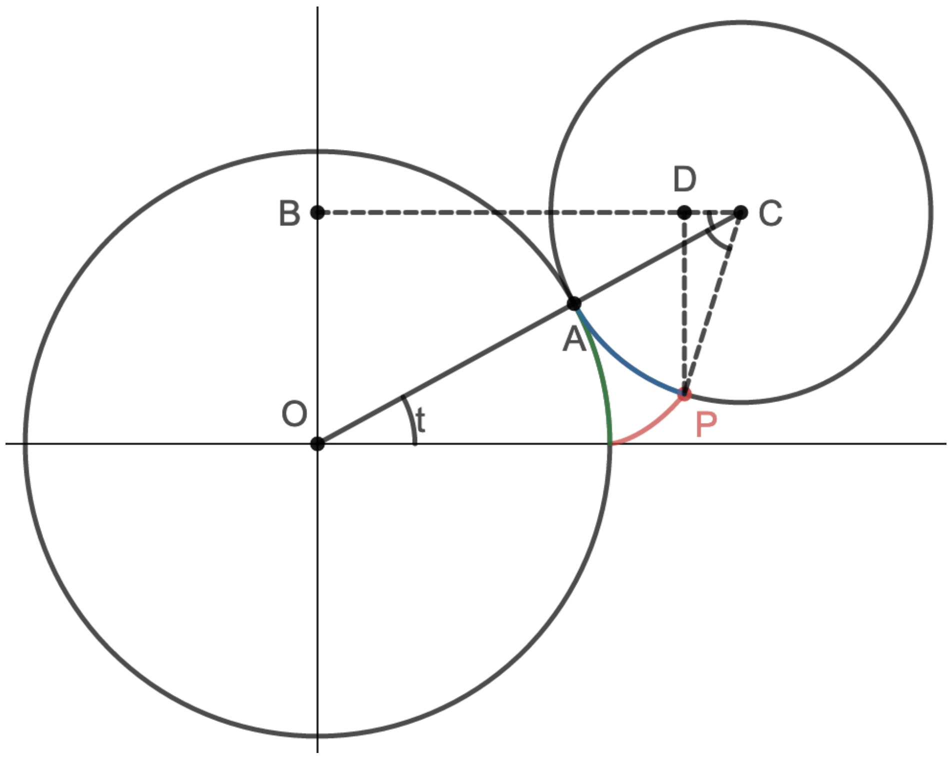


Figure 2

Use the diagram above to analyze the geometry of the epicycloid. Write the detailed steps for the derivation of the parametric form of the epicycloid.
[**Hints:** The length of the blue arc and the green arc are equal, why? Use this to derive the angle $\angle\mathbf{PCA}$ in terms of t. The x–coordinate of $\mathbf{P}$ is given by $\mathbf{BD}$ which is equal to $\mathbf{BC}-\mathbf{CD}$. The y–coordinate is equal to $\mathbf{OB}-\mathbf{PD}$.]

Question 2

In figure 1, you have picture of **4** epicycloids. You are told that each of the picture corresponds to a different value of $\frac{\mathbf{R}}{r}$. Can you identify the values?

The Wankel Rotary Engine

We are especially interested in a special case from among the epicycloids as it pertains to the revolutionary rotary engine. First let's slightly modify the starting location of the point we are tracing. Instead of being situated on the circumference of the rolling circle, suppose we trace the point **Q** located at a distance h from the center of the rolling circle. The curve traced out by **Q** is called an epitrochoid. Epicycloids are epitrochoids with $h = r$.

Question 3

Use figure 3 to show that the parametric equation of the *epitrochoid* is given by

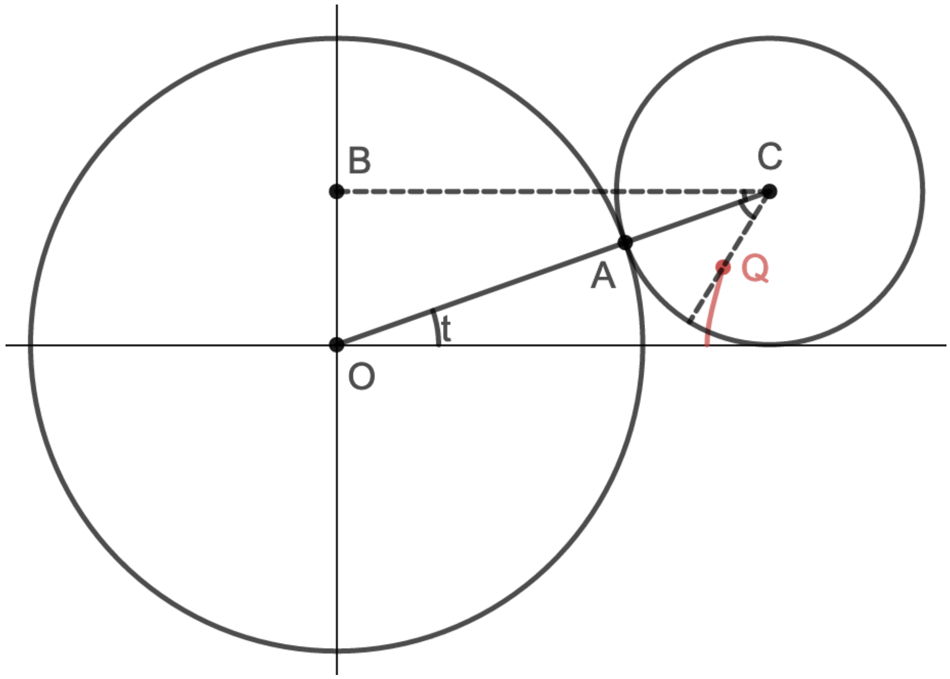


Figure 3

$$x(t) = (\mathbf{R}+r)\cos t - h\cos\frac{(\mathbf{R}+r)t}{r}$$

$$y(t) = (\mathbf{R}+r)\sin t - h\sin\frac{(\mathbf{R}+r)t}{r}$$

where $\mathbf{CQ} = h$.

Now that you have the general form of an epitrochoid, consider the case where the fixed circle is twice the size of the rolling one ($\mathbf{R} = 2r$) and $h = 4r/9$. This is the geometry of the Wankel rotary engine.

Question 4

Give the simplified parametric equations for this special case.

What makes this geometry especially useful is that an equilateral triangle fits perfectly within the epitrochoids no matter how the triangle it is rotated. We will show that an *equilateral triangle* can be inscribed in this epitrochoid independent of the value of t. See figure 4 below for a demonstration.

We will derive this in the following steps. Let

$$\mathbf{Q}_0 = (x(t), y(t)), \quad \mathbf{Q}_1 = (x(t+2\pi/3), y(t+2\pi/3)), \quad \mathbf{Q}_2 = (x(t-2\pi/3), y(t-2\pi/3))$$

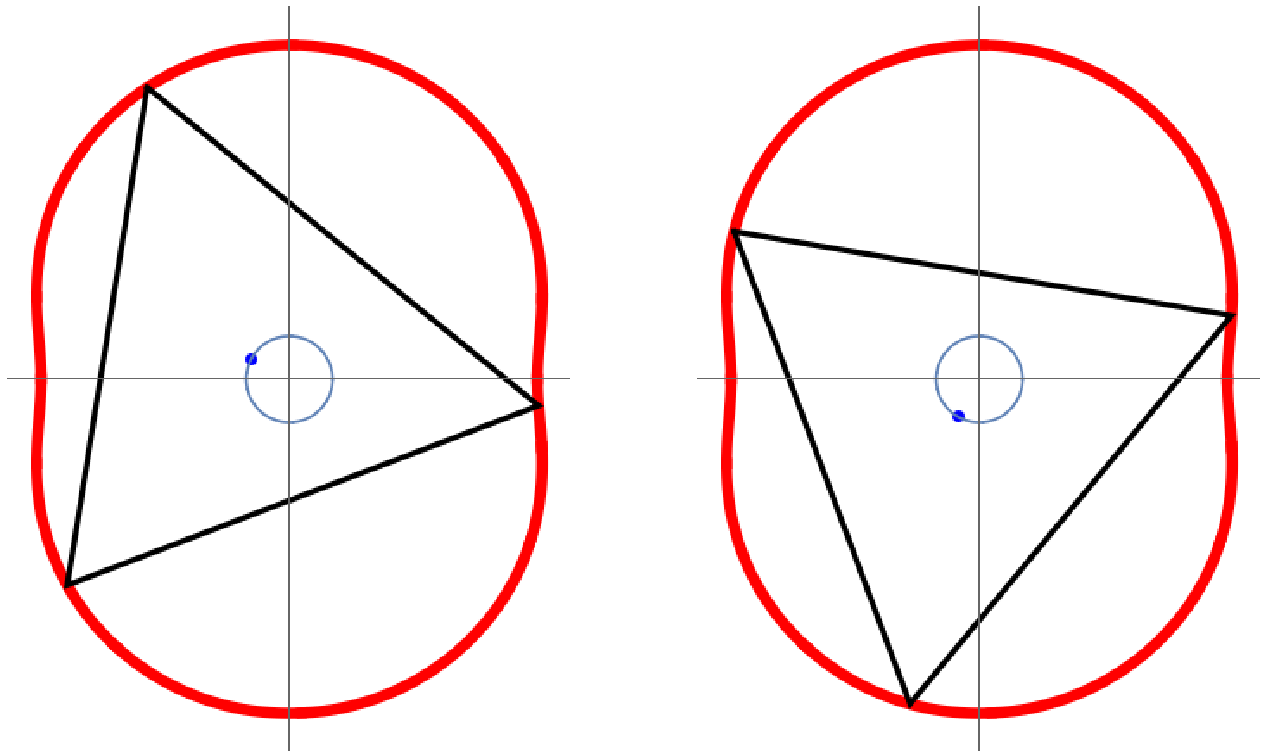


Figure 4

Question 5

(a) Show that the epitrochoid makes one revolution for $t \in [0, 2\pi]$.

(b) Since the three points Q_0, Q_1, Q_2 are $2\pi/3$ apart they are evenly spaced in $[0, 2\pi]$. Thus, if their pairwise distances are equal then they must form an equilateral triangle. Show that

$$\mathrm{dist}(Q_0, Q_1) = \mathrm{dist}(Q_1, Q_2) = \mathrm{dist}(Q_2, Q_0)$$

(c) Furthermore, the centroid of the equilateral triangle always lies on a circle of radius h. Show that the average of Q_0, Q_1 and Q_2 is always at a distance h from the origin.

This is the principle of the Wankel rotary engine. When the equilateral triangle rotates with its vertices on the epitrochoid, its centroid sweeps out a circle whose center is at the center of the epitrochoid. The space between the triangle and the epitrochoid is the firing chamber. A gif of the engine in action can be found here.

Question 6(Bonus)

How would the shape of the epicycloids (and epitrochoids) change if R/r is a rational number but not an integer? Try plotting the curve with $R = 3, r = 2$. Does it complete a full revolution for $t \in [0, 2\pi]$?
What happens if R/r is an irrational number. Try plotting the curve with $R = e, r = 1$. How long does it take to complete a full revolution?