# Lab 4: Newton-Raphson Method of finding Roots

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- Sign on to an iMac with your username and password.
- Go to Blackboard, the Lab Handouts section and download the `Lab4_1600_Rootfinding.nb` notebook file.
- Make the window bigger if necessary by dragging the lower right corner. Choose at least **125%** from the lower left of the Untitled window for a comfortable viewing size.
- Do not forget to **Save** the notebook periodically.
- Follow the instructions in the paper copy of the handout. Fill in the blanks in the Mathematica notebook file as appropriate.
- At the end of Lab session, save and quit Mathematica. **Do not delete your file or any calculation you did.** Email the notebook file to `schowdhu@bowdoin.edu` with subject line `Lab4_Name1_Name2`.
- Then log off or restart the computer. Do NOT click **Shut Down**.

## Root-finding Algorithms

In mathematics and computing, a *root-finding algorithm* is an algorithm for finding roots of continuous functions. A *root* of a function $f(x)$ is a number $x$ such that $f(x) = 0$. Generally, the roots of a function cannot be computed exactly, nor expressed in closed form. So your calculator uses these root-finding algorithms to provide approximate answer when it solves an equation. Note that most root-finding algorithms do not guarantee that they will find all the roots. In particular, if such an algorithm does not find any root, that does not mean that no root exists.

There are mainly three different types of algorithms based on the technique they use. One of this is to use *Interpolation*, which we learned in lab 2. The second uses *Intermediate Value Theorem* to successively tighten a possible domain containing the root. In this lab we are going to look at the third kind of popular root-finding algorithm, that uses derivatives and Linear Approximation.

## Newton-Raphson Method

The Newton-Raphson method in one variable uses properties of derivative and tangents. The idea of the method is as follows: we start with an initial guess which is reasonably close to the true root, then the function is approximated by its tangent line, and we computes the $x$-intercept of this tangent line. This $x$-intercept will typically be a better approximation to the function's root than the original guess, and the method can be iterated.

Note that this method only applies to differentiable functions. Consider the function

$$f(x) = \sin(x) - \cos(x) + x$$

Fill in the blanks of the Mathematica notebook as you go along.

Step 1. Suppose our starting guess for the root is $x_0$.

Step 2. Find the derivative $f'(x)$ and the equation of the tangent line to the graph of $f(x)$ at $x = x_0$. Work it out on paper and copy it to the notebook.

Step 3. Find the point where above line intersects the $X$-axis. Call that point $x_1$. Can you express $x_1$ in terms of $x_0$? See the notebook for a hint.

Step 4. Find where the tangent line to graph of $f(x)$ at $x = x_1$ intersects the $X$-axis. We are going to call that $x_2$. Can you express $x_2$ in terms of $x_1$?

Step 5. We are going to continue to calculate $x_3, x_4, \ldots$ this way. Can you give a general formula that outputs $x_{n+1}$ as a function of $x_n$?

Suppose

$$x_{n+1} = P(x_n)$$

Find the function $P(x)$ and fill in Mathematica.

Step 6. Keep evaluating $x_n$ until $|f(x_n)| < 10^{-5}$.

When $|f(x_n)|$ is sufficiently small, by continuity of $f$, we get that $x_n$ must be sufficiently close to the actual root of $f(x)$. So we can take $x_n$ as an approximation for the actual root.

## Bisection Method (Optional)

Let $f$ be a continuous function defined on an interval $[a, b]$ where $f(a)$ and $f(b)$ have opposite signs. In this case $a$ and $b$ are said to bracket a root since, by the intermediate value theorem, the continuous function $f(x)$ must have at least one root in the interval $(a, b)$.

At each step the method divides the interval in two by computing the midpoint $c = \frac{a+b}{2}$ of the interval and the value of the function $f(c)$ at that point. Unless $c$ is itself a root (which is very unlikely, but possible) there are now only two possibilities: either $f(a)$ and $f(c)$ have opposite signs and bracket a root, or $f(c)$ and $f(b)$ have opposite signs and bracket a root. The method selects the subinterval that is guaranteed to be a bracket as the new interval to be used in the next step. In this way an interval that contains a zero of $f$ is reduced in width by $50\%$ at each step. The process is continued until the interval is sufficiently small.

We are going to use this method to approximate a root of the function $f(x) = \sin(x) - \cos(x) + x$.

Step 1. First take any two values $a$ and $b$ such that $f(a)$ is negative and $f(b)$ is positive.

Step 2. Evaluate $f(c_1)$ where $c_1 = \frac{a+b}{2}$.

Step 3. If $f(c_1)$ is positive, then $a$ and $c$ bracket a root. So we move on to calculating $c_2 = \frac{a+c_1}{2}$ and to the next step.

Otherwise, if $f(c_1)$ is negative, then $c_2 = \frac{c_1+b}{2}$.

If $f(c_1)$ is zero, then we have found our root and the process stops.

Step 4. Evaluate $f(c_2)$. Define $c_3$ as follows.

Suppose $f(c_1)$ was negative. Then in that case, $c_2 = (c_1 + b)/2$. Then if $f(c_2)$ is positive, then $c_3 = \frac{c_2+c_1}{2}$. Otherwise, if $f(c_2)$ is negative, then $c_3 = \frac{c_2+b}{2}$. If $f(c_2)$ is zero, then we have found our root and the process stops.

Step 5. Evaluate $f(c_3)$. Define $c_4$ similarly. Note that definition of $c_4$ will look like

$$c_4 = \frac{(c_3 + ??)}{2}$$

where $??$ is one of the two endpoints used for defining $c_3$.

Step 6. Continue finding $c_5, \ldots$ etc. until $|f(c_n)| < \frac{1}{1000}$.

When $f(c_n)$ is sufficiently close to $0$, we get that $c_n$ is sufficiently close to the actual root of $f(x)$. So $c_n$ serves as an approximation for the root.

[Acknowledgement: Most of these notes are courtesy of Wikipedia.]