Graphing Rational Functions

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Definition

Quotient of two polynomials is called a *rational function*. The following are examples of rational functions

1. $\frac{1}{x}$
2. $\frac{x^3+5x-5}{x^8-3}$
3. $\frac{x^9-4x^4+2x}{x^7-3x^2+3}$

By definition, polynomials themselves are rational functions which have denominator 1. e.g. $x+1 = \frac{x+1}{1} = \frac{x+1}{x^0}$, so it's a rational function.

Roots and Asymptotes

In general, it is hard to graph rational functions. However, there is a very straightforward algorithmic way to draw them if both the numerator and denominator polynomial can be factored into *linear factors*. In this notes, we will explain how to find *asymptotes* and *roots* of such functions in order to sketch the graph.

We will begin by going through a particular example, then explain the general rules, and then do a more complicated example.

§ Roots.

Roots are points on the x-axis where a function is zero. These can be found out by solving the equation $f(x) = 0$. Thus for example, the roots of $f(x) = (x+2)(x-3)$ are -2 and 3.

Consider a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials. Then the roots of $f(x)$ are same as the roots of the numerator polynomial $P(x)$. For example, consider the rational function

$$f(x) = \frac{2x+4}{x-1}$$

Solving the equation $f(x) = 0$ for x, we get

$$\frac{2x+4}{x-1} = 0 \implies 2x+4 = 0 \implies x = \frac{-4}{2} = -2$$

If $P(x)$ can be easily factored into linear factors, we can readily find out what the roots are. *For the scope of this course, we will not discuss how to factor high degree polynomials. You can expect that I will provide the polynomial in factored form when talking about plotting rational functions.*

Before we talk about asymptotes, let's look at an example where we are plotting a polynomial with roots. Consider

$$f(x) = (x+3)^3(x-2)^2(x-5)$$

This polynomial intersects the x-axis at three points: $-3, 2,$ and 5. We have the following rules:

- If the degree of the linear factor is odd, the polynomial crosses the x-axis (from top-to-bottom or bottom-to-top) at the root.
- If the degree of the linear factor is even, the polynomial touches x-axis at the root and bounces back.

Thus, the graph should cross x-axis at -3 and 5; and should only touch the x-axis at 2. How do we know whether the polynomial crosses the axis from top-to-bottom or bottom-to-top at 5? We can figure that out by looking at the sign of $P(x)$.

Observe that when $x = 10^{100}$, some large quantity, all three factors of $f(x)$ become positive. Hence the graph starts from the positive side of the x-axis.

Furthermore, as x becomes larger and larger, $f(x)$ becomes larger and larger as well. We write this as $x \to \infty$, the function $f(x) \to \infty$. Similarly, by checking large negative values of x, we see that as $x \to -\infty$, the function $f(x) \to \infty$, since the product of two negative and one positive factors gives positive value.

So we now know how the ends of the graph looks like and where is intersects the x-axis.

1. We start above x-axis, from the direction of $+\infty$,
2. cross at 5 to go below x-axis (the negative side)
3. turn upwards and bounce off at 2,
4. stay below x-axis until we turn upwards to cross at -3,
5. cross x-axis and head off to $+\infty$ as $x \to -\infty$.

The rough picture then looks as follows.

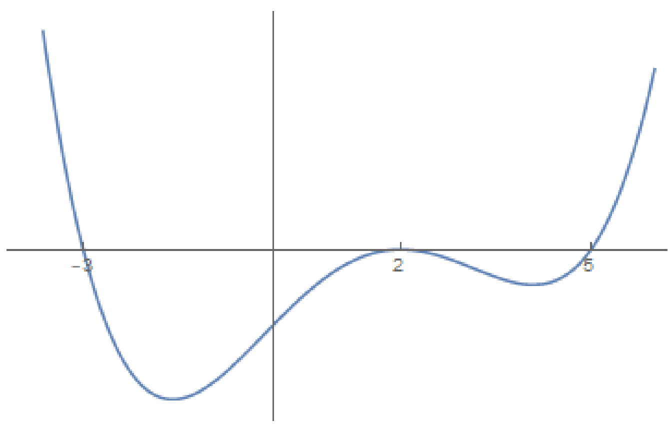


Figure 1: Rough Sketch

For comparison, here's the actual plot of the function by Mathematica. Admittedly it looks different from above picture. However up to scaling, the main features of the two pictures (e.g. roots, increasing/decreasing parts etc.) are same.

Let's do another example. Consider

$$g(x) = (2-x)^4(5-x)3$$

Note that we can rewrite $g(x)$ as

$$Q(x) = -(x-2)^2(x-5)$$

So it intersects x-axis at 2 and 5, only touches at 2 and crosses at 5. As $x \to \infty$, we find this time that $g(x) \to -\infty$. So it should start from below x-axis at the right end.

Following the steps above we can draw the following rough sketch,
which agrees well with the actual picture.

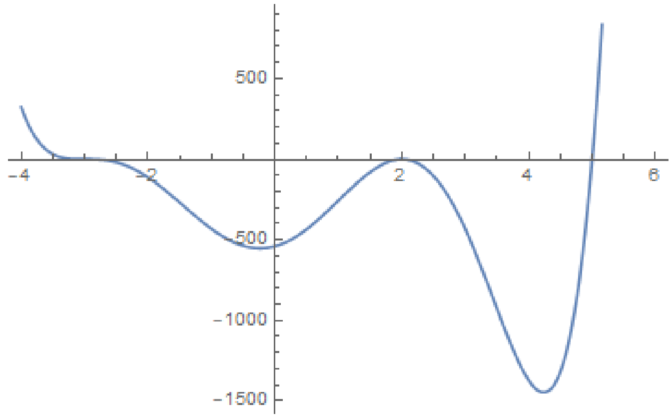


Figure 2: Actual Plot

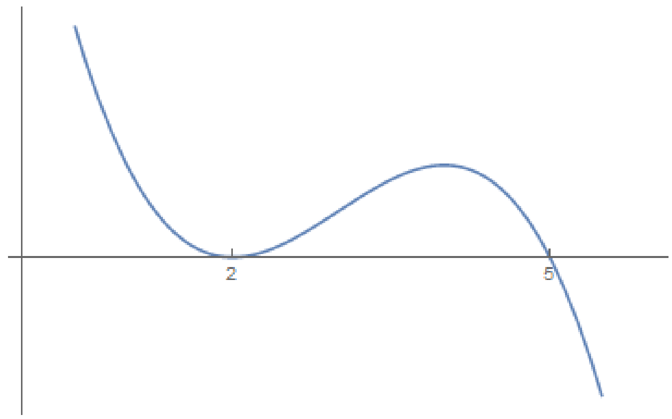


Figure 3: Rough Sketch

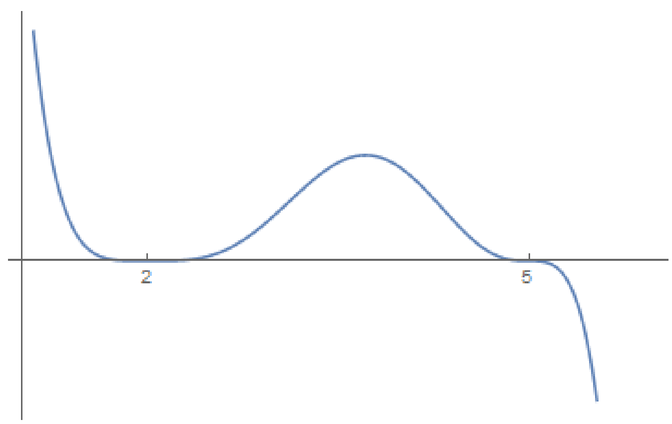


Figure 4: Actual Plot

Exercise 1: Can you give an approximate formula for the polynomial whose graph looks as in figure 5?

§ Asymptotes.

A rational function can have two kinds of Asymptotes, *vertical* and *horizontal*. Vertical asymptotes are easier to find. For a rational function, vertical asymptotes are vertical straight lines corresponding to the roots of the denominator.

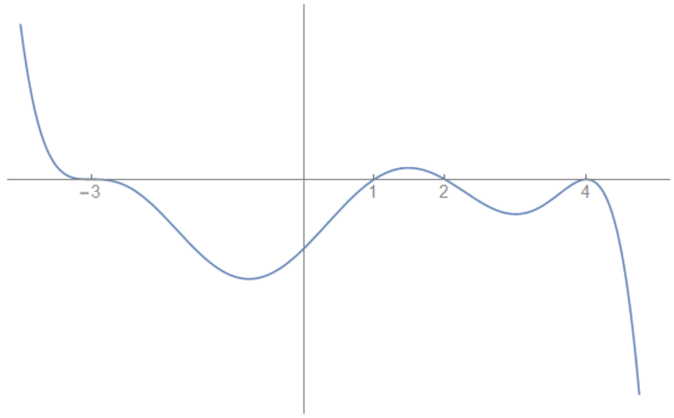


Figure 5: Exercise 1

Let's consider the following rational function.

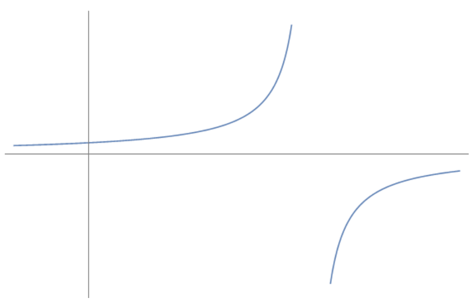
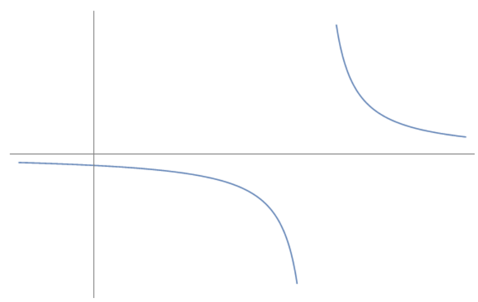
$$f(x) = \frac{(x+3)(x-1)}{(x-2)(x-3)}$$

Can we have a zero in the denominator of a fraction? No. So if I set the denominator of the above fraction equal to zero and solve, it tells me that x can not be 2 or 3. More precisely, as x approaches 2 or 3, the function $f(x) \to \pm\infty$. So this function has two vertical asymptotes at $x = 2$ and $x = 3$. Another way to think about them is that they are the points which do not belong to the domain of $f(x)$.

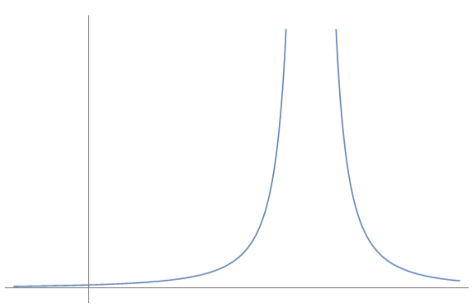
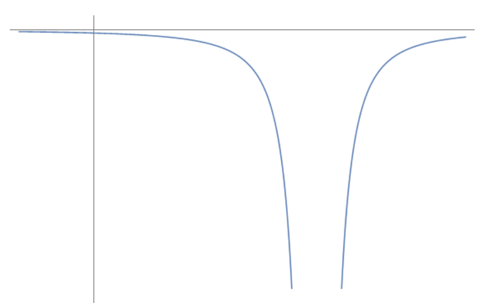
The proper definition is as follows. A line $x = K$ is a vertical asymptote of a function $f(x)$ if $f(x) \to \infty$ or $-\infty$ as $x \to K$.

Now the rule regarding odd/even exponent in the denominator of a rational function is:

- If the exponent of a linear factor in the denominator is odd, the graph will go to $+\infty$ on one side and $-\infty$ on the other side of the corresponding vertical asymptote.

- If the exponent of a linear factor in the denominator is even, the graph will either go to $+\infty$ on both sides or to $-\infty$ on both sides of the corresponding vertical asymptote.

Horizontal Asymptotes are different from vertical in some significant ways:

- A function can never cross a vertical asymptote, but it can cross a horizontal one.
- Vertical asymptotes indicate specific behavior of the graph as finite points. Horizontal asymptotes describe general behavior of the graph as $x \to \pm\infty$.

The proper definition is as follows. A line $y = L$ is a horizontal asymptote of a function $f(x)$ if $f(x) \to L$ as $x \to \infty$ or $x \to -\infty$. Note that a function may have different asymptotes as it goes towards $+\infty$ or $-\infty$.

Exercise 2: What's the maximum number of horizontal asymptotes a function can have? What about vertical asymptotes?

Exercise 3: What are the horizontal asymptotes of the function $f(x) = \arctan x$?

Exercise 4: What is the vertical asymptote of the function $f(x) = \ln x$?

For a rational function we can figure out the horizontal asymptotes by looking at the highest degree term in the numerator and denominator. Consider the example

$$f(x) = \frac{2x+3}{3x+6}$$

As $x \to \infty$, the quantity $2x + 3$ will be approximately equal to $2x$ and the quantity $3x + 2$ will be approximately equal to $3x$. So their ratio will be approximately equal to

$$\frac{2x}{3x} = \frac{2}{3}$$

Thus the function will have a horizontal asymptote at height $y = 2/3$, as $x \to \infty$. We can check that the same is true when $x \to -\infty$. In fact, this particular function has only one horizontal asymptote.

Similarly if the function was

$$g(x) = \frac{(x-2)(3x+4)}{(5x-1)(7-x)}$$

then as $x \to \infty$, the quantity $(x-2)(3x+4)$ will be approximately equal to $3x^2$ and the quantity $(5x-1)(7-x)$ will be approximately equal to $-5x^2$ (this is because the quadratic term is much more larger than the linear terms as x becomes large). So their ratio will be approximately equal to $-\frac{3}{5}$ and the function will have a horizontal asymptote at height $y = -3/5$.

What if the function was of the form

$$h(x) = \frac{(x-1)^2}{(x+2)^3}$$

Then as $x \to \infty$, the function is approximately $\dfrac{x^2}{x^3} = \dfrac{1}{x}$, which goes to 0. Thus $y = 0$ is the horizontal asymptote.

Finally, if the function was of the form

$$h(x) = \frac{(x-1)^5}{(x-7)^3}$$

Then as $x \to \infty$, the function is approximately $\dfrac{x^5}{x^3} = x^2$, which goes to ∞. So in this case, we have no horizontal asymptote.

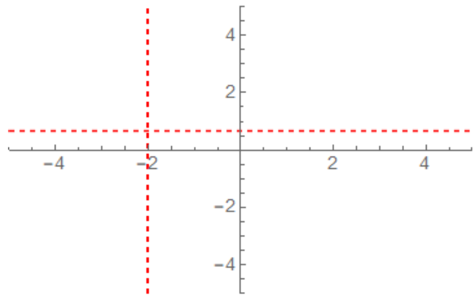
To summarize,

- If degree of numerator = degree of denominator, then the horizontal asymptote is at a height given by the ratio of the coefficient of highest power of x in numerator and the coefficient of highest power of x denominator.
- If degree of numerator < degree of denominator, then $x-$axis is a horizontal asymptote.
- If degree of numerator > degree of denominator, then it has no horizontal asymptote.

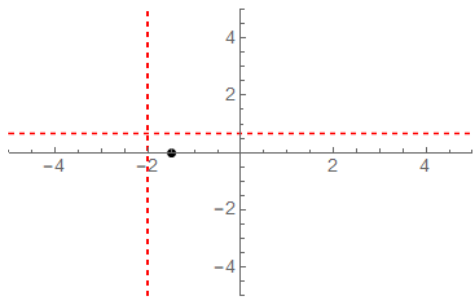
Drawing Rough Sketch

So now that we have learned about horizontal and vertical asymptotes, how do we use them to plot a rational function? Let's do it step by step for the example $f(x) = \frac{2x+3}{3x+6}$.

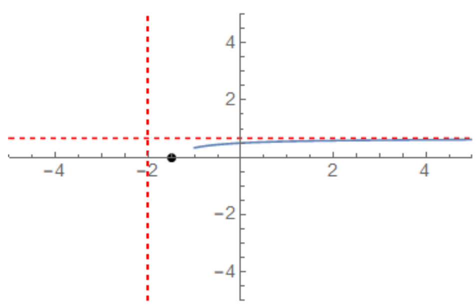
Step 1. **Find and draw the vertical asymptote and the horizontal asymptotes.** In this case, the vertical asymptote is at $x = -2$, and the horizontal one at $y = \frac{2}{3}$.



Step 2. **Find where it crosses the x—axis.** In this case, the point is $x = -3/2$.



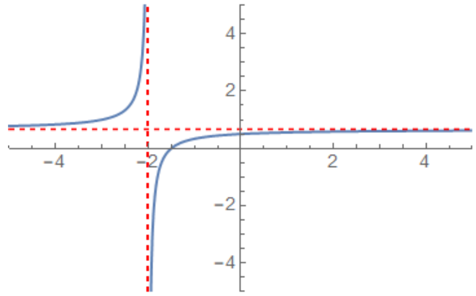
Step 3. **Pick either $+\infty$ or $-\infty$ to start plotting the graph. Find whether the function approaches the horizontal asymptote from above or below by checking some values.** In this case, we see that if $x = 100$, $f(x) = 203/306 < 2/3$. This tells us that the function approaches the horizontal asymptote from below (since it's less than $2/3$) as $x \to \infty$. We picked the value 100 because it's a large enough number (compared to the roots, vertical asymptotes etc) to give us an idea of the general end behavior of $f(x)$. Any large number will work.



Step 4. So we start below $y = 2/3$ on the right, continue left until we cross the x—axis at $x = -3/2$. Now we can't cross x-axis any more, and we still need to approach either $+\infty$ or $-\infty$ when $x = -2$, since that's a vertical asymptote. So the graph goes to $-\infty$ to the right of $x = -2$.

Step 5. **Check the degree of the factor corresponding to the asymptote to figure out which end (top or bottom) the graph will start from once we go past the vertical asymptote.** The degree of $(3x + 6)$ is odd, so the function approaches $+\infty$ to the right of $x = -2$.

Step 6. **Continue this way at every root and vertical asymptote until you get to the other end.** Since there are no more roots for $f(x)$, the function starts to approach the horizontal asymptote past $x = -2$. In fact, since started from $+\infty$ in the left side of -2, we will approach the horizontal asymptote from the top as $x \to -\infty$.



Here is one more example. Consider the function

$$h(x) = \frac{(x-3)^2(x+5)}{(x-2)(2x+4)^2}$$

The important things to consider are:

- The vertical asymptotes are at $x = 2$ and $x = -2$.
- The roots are at $x = 3$ and $x = -5$.
- The function crosses x—axis at -5, but bounces off at 3.
- The function approaches ∞ on one side of $x = 2$ and $-\infty$ on the other side.
- The function either approaches $+\infty$ on both sides of -2 or it approaches $-\infty$ on both sides of -2.
- The horizontal asymptote is at a height $y = \frac{x^2 x}{x(2x)^2} = \frac{1}{4}$.
- If $x = 10$, $h(x) = \frac{7^2 \times 15}{8 \times 24^2} < 1/4$. So we start below the horizontal asymptote.
- If we continue left, we bounce off at $x = 3$, go up to ∞ at $x = 2$ from the right, start from the other end $-\infty$ at the left of $x = 2$.
- Then we must go to $-\infty$ again at $x = -2$ since we cannot cross the x-axis between 2 and -2. But on the other side of $x = -2$, we start from $-\infty$ this time since $(2x+4)$ has an even power in the denominator.
- Then we cross again at $x = -5$ and go above x-axis.
- But we still have to approach $y = 1/4$, so we turn downwards and approach the horizontal asymptote from above as $x \to -\infty$.

The final picture looks like figure 6.
Here are some more problems for you to practice.

Exercise 5: For the following rational functions, find the roots, vertical asymptotes, and horizontal asymptotes, if any. Then draw a rough sketch.

(a)

$$\frac{(x-1)(x+1)^2}{x^2(x+3)}$$

(b)

$$\frac{(x-2)(x+5)^3(x-6)^2}{(x-4)^4(x+1)^3}$$

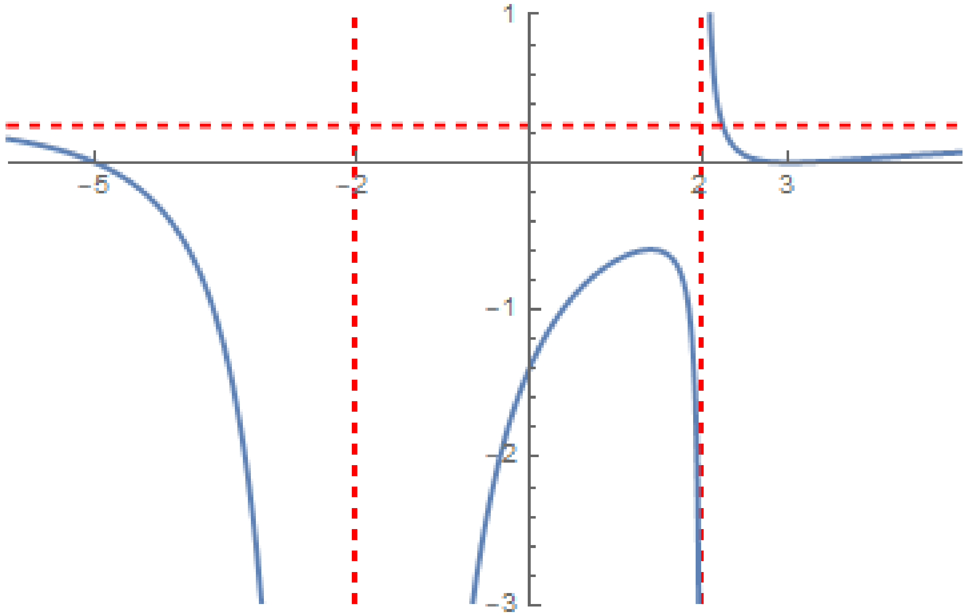


Figure 6: $h(x) = \frac{(x-3)^2(x+5)}{(x-2)(2x+4)^2}$