Midterm 2

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INSTRUCTIONS:

- Please show ALL your work! Answers without supporting justification will not be given credit.
- Answer the questions in the green books provided.
- Write legibly and start each question on a new page. You can answer the problems out of order. In fact, I suggest working out the easier ones first.
- Please note that use of calculator, books, or notes is not allowed.
- If you write down the correct formula for an answer, you will get some partial credit regardless of whether you evaluated the exact values or not.
- Unless otherwise specified, you may use any valid method to solve a problem.

Full Name: ______________________________

Question	Points	Score
1	8	
2	10	
3	10	
4	15	
5	12	
Total:	55	

This exam has 5 questions, for a total of 55 points.
The maximum possible point for each problem is given on the right side of the problem.

1. Evaluate the following double integral after changing the order of integration $\boxed{8}$

$$\int_0^8 \int_{y^{1/3}}^2 \frac{y^2 e^{x^2}}{x^8}\, dx\, dy$$

Solution:

Observe that

$$\{(x,y) \mid 0 \le y \le 8, y^{1/3} \le x \le 2\} = \{(x,y) \mid 0 \le x \le 2, 0 \le y \le x^3\}$$

Hence,

$$\begin{aligned}
\int_0^8 \int_{y^{1/3}}^2 \frac{y^2 e^{x^2}}{x^8}\, dx\, dy &= \int_0^2 \int_0^{x^3} \frac{y^2 e^{x^2}}{x^8}\, dy\, dx \\
&= \int_0^2 \left[\frac{y^3}{3}\right]_0^{x^3} \frac{e^{x^2}}{x^8}\, dx \\
&= \frac{1}{3}\int_0^2 \frac{x^9 e^{x^2}}{x^8}\, dx \\
&= \frac{1}{3}\int_0^2 x e^{x^2}\, dx \\
&= \frac{1}{6}\int_0^4 e^u\, du \\
&= \frac{e^4 - 1}{6}
\end{aligned}$$

2. Find the volume of the wedge shaped solid that lies above the XY-plane and below the plane $z = x$ and within the cylinder $x^2 + y^2 = 4$. $\boxed{10}$

Solution:

The plane $z = x$ intersects the XY-plane along the Y-axis. Hence half of the plane is ‘above’ XY-palne. We consider the part that lies ’within’ the cylinder $x^2 + y^2 = 4$. Thus the base of the region in question is in fact the half-disc

$$R = \{(x,y) \mid x^2 + y^2 = 4, x \ge 0\}$$

Volume is obtained by integrating 'height' function over the base. Hence the answer is

$$\begin{aligned}\iint_R z\,dA &= \iint_R (x-0)\,dA \\ &= \int_{-\pi/2}^{\pi/2}\int_0^2 r\cos(\theta)\,r\,dr\,d\theta \\ &= \int_{-\pi/2}^{\pi/2} \frac{8}{3}\cos(\theta)\,d\theta \\ &= \frac{16}{3}\end{aligned}$$

3. Find the area inside one of the smaller loops of the polar curve $r = 1 + 2\cos(2\theta)$ pictured below. $\boxed{10}$

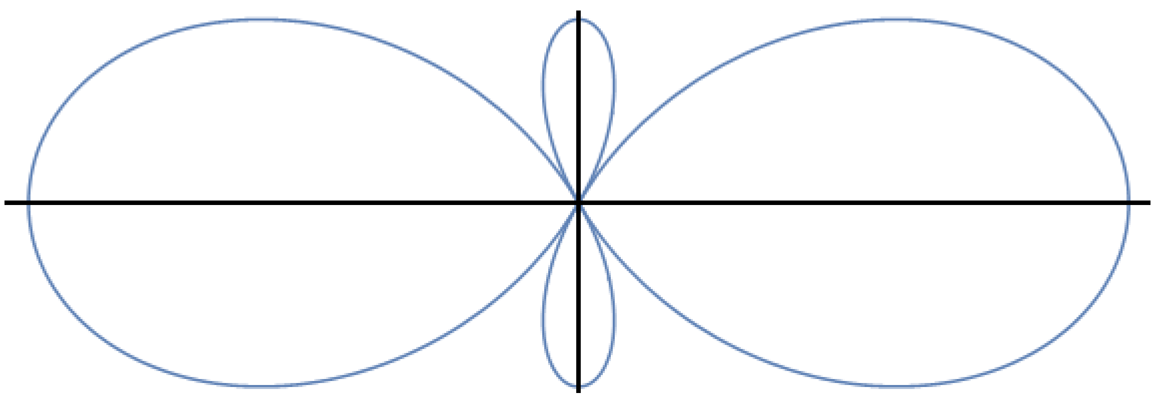

Figure 1: Question 3

Solution:

$$r = 0 \Longrightarrow \cos(2\theta) = -\frac{1}{2} \Longrightarrow 2\theta = 2\pi/3, 4\pi/3 \Longrightarrow \theta = \pi/3, 2\pi/3$$

Hence as θ goes from $\pi/3$ to $2\pi/3$, we traverse one of the (the bottom one) smaller loops. Consequently, the required area is equal to

$$\begin{aligned}
\int_{\pi/3}^{2\pi/3} \frac{r^2}{2} d\theta &= \frac{1}{2}\int_{\pi/3}^{2\pi/3} (1+2\cos(2\theta))^2 d\theta \\
&= \frac{1}{2}\int_{\pi/3}^{2\pi/3} (1+4\cos(2\theta)+4\cos^2(2\theta)) d\theta \\
&= \frac{1}{2}\int_{\pi/3}^{2\pi/3} [1+4\cos(2\theta)+2(1+\cos(4\theta))]\, d\theta \\
&= \frac{1}{2}\left[\theta + \frac{4\sin(2\theta)}{2} + 2\left(\theta + \frac{\sin(4\theta)}{4}\right)\right]_{\pi/3}^{2\pi/3} \\
&= \frac{1}{2}\left[\pi/3 + 2\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) + 2\left(\pi/3 + \frac{1}{4}\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right)\right)\right] \\
&= \frac{\pi}{2} - \frac{3\sqrt{3}}{4}
\end{aligned}$$

4. We want to find all local and global extrema of the function

$$f(x,y) = 2x^3 + 2y^3 - 3x^2 - 3y^2 + 6$$

on the disc $D = \{(x,y) \mid x^2 + y^2 \leq 4\}$. This problem has 3 steps.

(a) **Classify** every critical point (as local max/min/saddle pt.) inside the disc using the Hessian. You get part marks for correctly finding the critical points and the Hessian. $\boxed{6}$

Solution:

$$\nabla f = 0 \Longrightarrow \begin{cases} 6x^2 - 6x = 0 \\ 6y^2 - 6y = 0 \end{cases} \Longrightarrow \begin{cases} x = 0, 1 \\ y = 0, 1 \end{cases} \Longrightarrow (x,y) = (0,0), (0,1), (1,0), (1,1)$$

The Hessian is equal to

$$H = \begin{bmatrix} 12x - 6 & 0 \\ 0 & 12y - 6 \end{bmatrix}$$

Hence $D = \det(H) = (12x-6)(12y-6)$.

Crit. pt.	D	f_{xx}	Class
$(0,0)$	36	< 0	Local Max
$(1,0)$	-36		Saddle
$(0,1)$	-36		Saddle
$(1,1)$	36	> 0	Local Min

(b) Find the extrema of f on the **boundary** circle $x^2 + y^2 = 4$ using Lagrange multiplier. $\boxed{6}$

Solution:

Let $g(x,y) = x^2 + y^2$. Then

$$\nabla f = \lambda \nabla g \implies \begin{cases} 6x^2 - 6x = 2\lambda x \\ 6y^2 - 6y = 2\lambda y \end{cases} \implies \begin{cases} x = 0 \implies y = \pm 2 \\ y = 0 \implies x = \pm 2 \\ x = y \implies x = y = \pm\sqrt{2} \end{cases}$$

$$f(0,2) = f(2,0) = 16 - 12 + 6 = 10$$
$$f(0,-2) = f(-2,0) = -16 - 12 + 6 = -22$$
$$f(\sqrt{2},\sqrt{2}) = 4\sqrt{2} + 4\sqrt{2} - 6 - 6 + 6 = 8\sqrt{2} - 6 \approx 5.2$$
$$f(-\sqrt{2},-\sqrt{2}) = -4\sqrt{2} + 4\sqrt{2} - 6 - 6 + 6 = -8\sqrt{2} - 6 \approx -17.2$$

Hence the extreme values of f on the boundary are 10 and -22.

(c) Determine the global maxima and minima of f on D by comparing answers from the last two parts. 3

Solution:

Crit. pt.	$f(x,y)$
$(0,0)$	6
$(1,0)$	5
$(0,1)$	5
$(1,1)$	4

Hence the global maxima for f on D is 10, and the global minima is -22.

5. Evaluate 12

$$\iint_D \frac{y^2}{x}\,dx\,dy$$

where D is the region between the parabolas $x = 1 - y^2$ and $x = 3(1 - y^2)$. Use the change of variables $(x,y) = T(u,v) = (v(1-u^2), u)$.

[HINT: Drawing a picture might help to figure out the bounds for the integral.]

Solution:

We first need to figure out the preimage of D under the transformation $T(u,v) = (x,y) = (v(1-u^2),u)$. So we find out the preimages of the boundary of D.

Note that

$$x = 1 - y^2 \Leftrightarrow v(1-u^2) = 1 - u^2 \Leftrightarrow v = 1$$

Similarly the other boundary parabola's preimage is $v = 3$.

Next we need to find the bounds on u. But by definition, $u = y$, and the bound on y are obtained by observing that the two parabolas intersect when $x = 0$. Hence $1 - y^2 = 0 \implies y = \pm 1$. Thus u goes from -1 to 1.

The Jacobian of T is given by

$$J = \begin{bmatrix} v(-2u) & 1-u^2 \\ 1 & 0 \end{bmatrix}$$

So determinant of the Jacobian is $= (u^2-1)$ and $|\det(J)| = 1-u^2$ since $-1 \leq u \leq 1$.

Thus the given integral

$$\begin{aligned}
\iint_D \frac{y^2}{x}\,dx\,dy &= \int_{-1}^{1}\int_{1}^{3} \frac{u^2}{v(1-u^2)}(1-u^2)\,dv\,du \\
&= \int_{-1}^{1}\int_{1}^{3} \frac{u^2}{v}\,dv\,du \\
&= \int_{-1}^{1} u^2(\ln 3 - \ln 1)\,du \\
&= (\ln 3 - 0)(1/3 + 1/3) \\
&= \frac{2\ln 3}{3}
\end{aligned}$$