Problem Set 20 Solutions

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You did not need to submit these exercises, but here are solutions to help you review for the final exam. Note that there are many choices of N or δ that work.

Exercise 1

Give an ϵ-N proof of the following.

(a)

$$\lim_{n\to\infty} \frac{3}{n^2} = 0$$

Proof. Fix an $\epsilon > 0$. Choose $N = \lceil \sqrt{\frac{3}{\epsilon}} \rceil$. Then, when $n > N$, we have

$$\begin{aligned} n &> \lceil \sqrt{\frac{3}{\epsilon}} \rceil \\ &> \sqrt{\frac{3}{\epsilon}} \\ \implies n^2 &> \frac{3}{\epsilon} \\ \implies \frac{3}{n^2} &< \frac{3}{\frac{3}{\epsilon}} = \epsilon \end{aligned}$$

Thus we have found an N such that if $n > N$, then $|\frac{3}{n^2} - 0| < \epsilon$. Hence the claim is proved. $\square$

(b)

$$\lim_{n\to\infty} \frac{n-1}{2+n} = 1$$

We have to prove that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that, if $n > N$ then $\left|\frac{n-1}{2+n} - 1\right| < \epsilon$. Let's work out what n we need in roughwork. Observe that

$$
\begin{aligned}
& \left|\frac{n-1}{2+n}-1\right|<\epsilon \\
\iff & \left|\frac{n-1-2-n}{2+n}\right|<\epsilon \\
\iff & \left|\frac{-3}{2+n}\right|<\epsilon \\
\iff & \frac{3}{2+n}<\epsilon \\
\iff & n+2>\frac{3}{\epsilon} \\
\iff & n>\frac{3}{\epsilon}-2
\end{aligned}
$$

So we want $N=\lceil 3/\epsilon-2\rceil$.

Proof. Fix $\epsilon>0$. Choose $N=\lceil 3/\epsilon-2\rceil$. Then if $n>N$, we have $n>3/\epsilon-2$ and hence

$$
\begin{aligned}
& n+2>\frac{3}{\epsilon} \\
\implies & \frac{3}{2+n}<\epsilon \\
\implies & \left|\frac{-3}{2+n}\right|<\epsilon \\
\implies & \left|\frac{n-1-2-n}{2+n}\right|<\epsilon \\
\implies & \left|\frac{n-1}{2+n}-1\right|<\epsilon
\end{aligned}
$$

Hence, by definition, the limit is 1. □

Exercise 2

Prove

(a)

$$\lim_{x\to 2}(3-x)=1$$

Proof. For a fixed $\epsilon>0$, choose $\delta=\epsilon$. Then if $|x-2|<\delta$, we have

$$|(3-x)-1|=|x-2|<\delta=\epsilon$$

□

(b)

$$\lim_{x\to -2}(3x+5)=-1$$

Proof. For a fixed $\epsilon > 0$, choose $\delta = \frac{\epsilon}{3}$. Then if $|x+2| < \delta$, we have

$$|3x+5-(-1)| = 3|(x+2)| < 3\frac{\epsilon}{3} = \epsilon$$

□

Exercise 3

Consider the sequence

$$1, -1, 1, -1, 1... \tag{1}$$

Prove that the limit does not exist.

Proof. Suppose, for the sake of contradiction, that a limit L of the alternating sequence exists. Then, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n > N$, then $|a_n - L| < \epsilon$. So, such an N should exist when $\epsilon = \frac{1}{2}$. All odd terms are 1, so we know that, for some odd $k > N$, we have $|a_k - L| < \frac{1}{2}$. So L must lie within the interval $(\frac{1}{2}, \frac{3}{2})$. But then, consider $|a_{k+1} - L|$. The number $k+1$ is even, so $a_{k+1} = -1$. But then $|a_{k+1} - L| > \frac{3}{2}$, which is a contradiction. □

Exercise 4

If

$$\lim_{x \to c} f(x) = l$$

then prove that

$$\lim_{x \to c} (2f(x) - 1) = 2l - 1$$

We are given that $\forall \epsilon > 0$, $\exists \delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - l| < \epsilon$.

We want to prove that $\forall \epsilon' > 0$, $\exists \delta' > 0$ such that if $|x - c| < \delta'$, then $|2f(x) - 1 - (2l - 1)| < \epsilon'$.

We are going to use the following tactics. We start with fixing an ϵ'. Use ϵ' to choose ϵ. For that ϵ, we have a δ. Use that δ to choose δ'.

Proof. Fix $\epsilon' > 0$. Let $\epsilon = \frac{\epsilon'}{2}$.

We know that for this ϵ, we can find a $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - l| < \epsilon$.

take that δ and set $\delta' = \delta$. Then if $|x - c| < \delta'$, then $|x - c| < \delta$ and hence

$$|f(x) - l| < \epsilon \implies |2f(x) - 2l| < 2\epsilon \implies |2f(x) - 1 - (2l - 1)| < 2\epsilon \implies |2f(x) - 1 - (2l - 1)| < \epsilon'$$

Hence we have shown that for arbitrary choice of $\epsilon' > 0$, there exists a $\delta' > 0$ such that if $|x - c| < \delta'$, then $|(2f(x) - 1) - (2l - 1)| < \epsilon'$. This completes the proof. □

Remark 4.1. For practice, use the same method to prove

$$\lim_{2x \to 2c} (1 - 3f(x)) = 1 - 3l.$$