Assignment 17 (7/26)

Subhadip Chowdhury

- *This homework is due at the beginning of class on* ***Tuesday*** 7/31. *You are encouraged to work together on these problems, but you must write up your solutions independently.*

Limit of a Sequence

Definition 1.1. We say that a sequence $\{a_n\}_{n\in\mathbb{N}}$ has limit l, if

$$\text{for all } \epsilon > 0, \text{ there exists } N \in \mathbb{N} \text{ such that, if } n > N \text{ then } |a_n - l| < \epsilon.$$

We denote this by writing

$$\lim_{n\to\infty} a_n = l$$

Note that $|a_n - l| < \epsilon \iff -\epsilon < a_n - l < \epsilon \iff l - \epsilon < a_n < l + \epsilon$. Thus the limit definition says the following. If I provide you with any positive real number ϵ, you can give me a threshold N for the indices, beyond which a_n lies in the interval $(l-\epsilon, l+\epsilon)$. If you change ϵ, the threshold might move, in fact reducing ϵ usually moves it to the right; but a threshold N will always exist.

Example 1.2. Consider the sequence $a_n = 1/n$. It seems to be the case that as n becomes large enough a_n approaches 0. Let us prove that this is indeed the case.

Proof. We want to show that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that, if $n > N$ then $\left|\frac{1}{n}\right| < \epsilon$. In other words, given ϵ, we want to find a formula for N, in terms of ϵ, such that $|1/n| < \epsilon$ for $n > N$.

Choose $N = \left\lceil \frac{1}{\epsilon} \right\rceil$. Then $N > 1/\epsilon \implies 1/N < \epsilon$. Now if $n > N$, then $1/N > 1/n \implies \epsilon > 1/N > 1/n$. Hence we have found a formula for N in terms of ϵ that satisfies the requirements. $\square$

Observe that a problem that asks you to prove a limit statement using $\epsilon - N$ essentially asks you to find a formula for N in terms of ϵ. Usually we work backward to get this formula. We will discuss more examples tomorrow.

Look up the first two pages of section 11.3. We will explain the two examples tomorrow.

Homework

Read the notes. Read the relevant sections from the book as mentioned in the note.