Assignment 8 (7/10)

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- *This homework is due at the beginning of class on **Tuesday** 7/17. You are encouraged to work together on these problems, but you must write up your solutions independently.*

Sequence

A *sequence* can be defined as an ordered set of real numbers. There is a more precise definition using functions, but we will skip it for now until we define functions precisely.

We write

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

to denote a sequence of real numbers. Here a_1 is the first term in the sequence, a_2 is the second etc. and more generally, a_n, the term with *index* n, is called the nth term. We sometimes also use the notation $\{a_i\}_{i\in\mathbb{N}}$ to denote above sequence.

Sequences can be defined in two ways.

(1) We can provide an *explicit* "closed-form" formula for the nth term. Some examples are

- (a) $a_n = 1/n$ for all n. This is the sequence $1, 1/2, 1/3, 1/4, \ldots$
- (b) $a_n = \frac{n}{n+1}$ for all n. This is the sequence $1/2, 2/3, 3/4, 4/5, \ldots$
- (c) $a_n = n^2$ for all n. This is the sequence $1, 4, 9, 16, \ldots$

(2) Alternately, we can define a sequence *recursively*. This means, we define the nth term of the sequence using the term(s) before it. Some examples are

- (a) $a_n = \frac{a_{n-1}}{2}$ for all $n > 1$ and $a_1 = 1$. This is the sequence $1, 1/2, 1/4, 1/8, \ldots$
  Observe that we need two data to define a sequence this way. We need the defining relation and we need the starting points.
- (b) $a_1 = 0$ and $a_n = 3a_{n-1} + 4$ for all $n > 1$. This is the sequence $0, 4, 16, 52, \ldots$
- (c) The *Fibonacci Sequence* is one of the famous examples of integer sequences. It is defined as:

$$F_1 = F_2 = 1, \quad F_n = F_{n-1} + F_{n-2} \text{ for all } n \geq 3$$

  This is the sequence $1, 1, 2, 3, 5, 8, 13, \ldots$.

We can use Mathematical Induction to prove properties of sequences defined recursively. We will work out an example below. The format of these kind of proof is essentially the same as proving identities, but note the differences carefully.

Example 1.1. Suppose $a_1 = 1$ and $a_{n+1} = \frac{n+1}{2n} a_n$. Then show that $a_n = \frac{n}{2^{n-1}}$ for all n.

Proof. We will prove that $a_n = \frac{n}{2^{n-1}}$ for all n by inducting on n.

Base Case: When $n = 1$, we have $a_1 = \frac{1}{2^{1-1}} = \frac{1}{2^0} = \frac{1}{1} = 1$, which matches with our given data. Hence the formula works for $n = 1$.

Induction Hypothesis: Assume that $a_k = \frac{k}{2^{k-1}}$ for some $k \in \mathbb{N}$.

Induction Step: We are going to show that $a_{k+1} = \frac{k+1}{2^{k+1-1}} = \frac{k+1}{2^k}$. We observe that,

$$\begin{aligned}
a_{k+1} &= \frac{k+1}{2k} a_k && \text{[By definition]} \\
&= \left(\frac{k+1}{2k}\right)\left(\frac{k}{2^{k-1}}\right) && \text{[By induction hypothesis]} \\
&= \frac{k+1}{2(2^{k-1})} \\
&= \frac{k+1}{2^k}
\end{aligned}$$

Thus the formula is valid for $n = k+1$ whenever it is valid for $n = k$. Hence, by the principle of mathematical induction, the formula is proved. □

The main difference from before was in the induction step. Here we are not 'adding' the $(k+1)$th term, since we don't have an identity or a sum. Instead we are using the recursion definition to start and then plugging in the value of a_k from the induction hypothesis. Then, after simplification we should get a term that matches given formula for $n = k+1$.

Exercise 9. *Use mathematical induction to prove that if*

$$a_1 = 1 \text{ and } a_{n+1} = 2a_n + 1 \text{ for } n \geq 1$$

then $a_n = 2^n - 1$ *for all* $n \in \mathbb{N}$.

Exercise 10. *Give the first* 6 *terms of the following sequences and then guess a formula for the nth term. You don't have to prove the formula.*

(a) $a_1 = 1, \quad a_2 = 3, \quad a_{n+1} = 2a_n - a_{n-1} \quad$ *for* $n \geq 2$.

(b) $a_1 = 1, \quad a_2 = 3, \quad a_{n+1} = 3a_n - 2a_{n-1} \quad$ *for* $n \geq 2$.

Definition 1.2. A sequence is called

$$\begin{aligned}
\textit{increasing} &\text{ if } a_{n+1} > a_n \text{ for all } n \in \mathbb{N} \\
\textit{nonincreasing} &\text{ if } a_{n+1} \leq a_n \text{ for all } n \in \mathbb{N} \\
\textit{decreasing} &\text{ if } a_{n+1} < a_n \text{ for all } n \in \mathbb{N} \\
\textit{nondecreasing} &\text{ if } a_{n+1} \geq a_n \text{ for all } n \in \mathbb{N}
\end{aligned}$$

Exercise 11. *Define a sequence* $\{a_i\}_{i\in\mathbb{N}}$ *as*

$$a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2 + a_n} \quad \text{for } n \geq 1$$

Show by induction that $a_n < 2$ *for all* $n \in \mathbb{N}$.

Arithmetic Progression

A sequence where the difference of consecutive terms is constant, is called an Arithmetic Progression (AP). We need two data to define an AP. We need the starting value, call it a; and we need the common difference, call it d. Then we have the following formula:

- The nth term of the AP is given by
$$t_n = a + (n-1)d.$$
- Sum of the first n terms in the AP is given by
$$S_n = t_1 + t_2 + \ldots + t_n = \frac{n}{2}\left(2a + (n-1)d\right).$$

Remark 2.1. Note that if a problem asks for three numbers in an AP, we usually take them as $a-d, a, a+d$. If it asks for 5 numbers, we should take $a-2d, a-d, a, a+d, a+2d$. This makes calculations easier.

Exercise 12. *The fifth term of an AP is* 1 *and the* 31*st term is* -77. *Find the* 20*th term.*

Exercise 13. *(a) The mth term of an AP is n and the nth term is m. Show that the* $(m+n)$*th term is zero.*

(b) The sum of first m terms of an AP is n and the sum of first n terms is m. Show that the sum of the first $(m+n)$ *terms is* $-(m+n)$.

Exercise 14. *A student decides to pay off her student loan of* \$36000 *in* 40 *annual instalments which form an arithmetic progression. When* 30 *of the instalments are paid, she gives up and flees the country, leaving one-third of the debt unpaid. Find the value of the first instalment.*

Exercise 15 (Extra Credit)**.** 150 *workers were engaged to build a wall in a certain number of days. Due to some reason, four workers dropped on the second day, four more on the third day and so on. It took* 8 *more days than initially planned to finish the wall. Assuming all workers work at the same rate, find the number of days in which the work was completed.* [HINT: *Use AP and unitary method!*]

Remark 2.2. I am not writing notes for quadratic factorization. This is precalculus knowledge that you can learn yourself. Here is a Khan Academy video explaining strategies of factoring quadratics. This website is in fact a great math resource for a variety of topics. We will talk about factoring higher degree polynomials using the vanishing factor method next time. Please refresh your memory about polynomial long division before that.