Problem Set 20 Solutions

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The University of Chicago, CAAP 2018: Proof-Based Methods in Calculus (Chowdhury)

August 5, 2018

You did not need to submit these exercises, but here are solutions to help you review for the final exam. Note that there are many choices of N or δ that work.

Exercise 1

Give an ϵ-N proof of the following.

(a)

$$\lim_{n\to\infty} \frac{3}{n^2} = 0$$

Proof. Fix an $\epsilon > 0$. Choose $N = \lceil \sqrt{\frac{3}{\epsilon}} \rceil$. Then, when $n > N$, we have

$$\begin{aligned} n &> \lceil \sqrt{\frac{3}{\epsilon}} \rceil \\ &> \sqrt{\frac{3}{\epsilon}} \\ \implies n^2 &> \frac{3}{\epsilon} \\ \implies \frac{3}{n^2} &< \frac{3}{\frac{3}{\epsilon}} = \epsilon \end{aligned}$$

Thus we have found an N such that if $n > N$, then $|\frac{3}{n^2} - 0| < \epsilon$. Hence the claim is proved. $\square$

(b)

$$\lim_{n\to\infty} \frac{n-1}{2+n} = 1$$

We have to prove that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that, if $n > N$ then $\left|\frac{n-1}{2+n} - 1\right| < \epsilon$. Let's work out what n we need in roughwork. Observe that

$$
\begin{aligned}
&\left|\frac{n-1}{2+n}-1\right|<\epsilon \\
\iff &\left|\frac{n-1-2-n}{2+n}\right|<\epsilon \\
\iff &\left|\frac{-3}{2+n}\right|<\epsilon \\
\iff &\frac{3}{2+n}<\epsilon \\
\iff &n+2>\frac{3}{\epsilon} \\
\iff &n>\frac{3}{\epsilon}-2
\end{aligned}
$$

So we want $N=\lceil 3/\epsilon-2\rceil$.

Proof. Fix $\epsilon>0$. Choose $N=\lceil 3/\epsilon-2\rceil$. Then if $n>N$, we have $n>3/\epsilon-2$ and hence

$$
\begin{aligned}
&n+2>\frac{3}{\epsilon} \\
\implies &\frac{3}{2+n}<\epsilon \\
\implies &\left|\frac{-3}{2+n}\right|<\epsilon \\
\implies &\left|\frac{n-1-2-n}{2+n}\right|<\epsilon \\
\implies &\left|\frac{n-1}{2+n}-1\right|<\epsilon
\end{aligned}
$$

Hence, by definition, the limit is 1. □

Exercise 2

Prove

(a)

$$\lim_{x\to 2}(3-x)=1$$

Proof. For a fixed $\epsilon>0$, choose $\delta=\epsilon$. Then if $|x-2|<\delta$, we have

$$|(3-x)-1|=|x-2|<\delta=\epsilon$$

□

(b)

$$\lim_{x\to -2}(3x+5)=-1$$

Proof. For a fixed $\epsilon > 0$, choose $\delta = \frac{\epsilon}{3}$. Then if $|x+2| < \delta$, we have

$$|3x + 5 - (-1)| = 3|(x+2)| < 3\frac{\epsilon}{3} = \epsilon$$

$\square$

Exercise 3

Consider the sequence

$$1, -1, 1, -1, 1... \tag{1}$$

Prove that the limit does not exist.

Proof. Suppose, for the sake of contradiction, that a limit L of the alternating sequence exists. Then, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n > N$, then $|a_n - L| < \epsilon$. So, such an N should exist when $\epsilon = \frac{1}{2}$. All odd terms are 1, so we know that, for some odd $k > N$, we have $|a_k - L| < \frac{1}{2}$. So L must lie within the interval $(\frac{1}{2}, \frac{3}{2})$. But then, consider $|a_{k+1} - L|$. The number $k+1$ is even, so $a_{k+1} = -1$. But then $|a_{k+1} - L| > \frac{3}{2}$, which is a contradiction. $\square$

Exercise 4

If

$$\lim_{x \to c} f(x) = l$$

then prove that

$$\lim_{x \to c} (2f(x) - 1) = 2l - 1$$

We are given that $\forall \epsilon > 0$, $\exists \delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - l| < \epsilon$.

We want to prove that $\forall \epsilon' > 0$, $\exists \delta' > 0$ such that if $|x - c| < \delta'$, then $|2f(x) - 1 - (2l - 1)| < \epsilon'$.

We are going to use the following tactics. So we start with fixing an ϵ'. Use ϵ' to choose ϵ. For that ϵ, we have a δ. Use that δ to choose δ'.

Proof. Fix $\epsilon' > 0$. Let $\epsilon = \frac{\epsilon'}{2}$.

We know $\forall \epsilon > 0$, we can choose $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - l| < \epsilon$.

Choose $\delta' = \delta$. Then if $|x - c| < \delta'$, then $|x - c| < \delta$ and hence

$$|f(x) - l| < \epsilon \implies |2f(x) - 2l| < 2\epsilon \implies |2f(x) - 1 - (2l - 1)| < 2\epsilon \implies |2f(x) - 1 - (2l - 1)| < \epsilon'$$

Hence we have shown that for arbitrary choice of $\epsilon' > 0$, there exists a $\delta' > 0$ such that if $|x - c| < \delta'$, then $|(2f(x) - 1) - (2l - 1)| < \epsilon'$. This completes the proof. $\square$

Remark 4.1. For practice, use the same method to prove

$$\lim_{2x \to 2c} (1 - 3f(x)) = 1 - 3l.$$