Math 2208: Ordinary Differential Equations

Lecture 18 Worksheet

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Title: Gradient System

Summary: We shall continue our analysis of non-linear systems by learning about gradient systems.

Gradient System

Definition 1.1

A system of differential equations is called a *gradient system* if there exists a continuously differentiable, scalar-valued function $V(x,y)$ such that for every (x,y), we have

$$\frac{dx}{dt} = \frac{\partial V}{\partial x}, \qquad \frac{dy}{dt} = \frac{\partial V}{\partial y}$$

i.e. $\dfrac{d\vec{R}}{dt} = \nabla V$, where $\vec{R} = \begin{bmatrix} x \\ y \end{bmatrix}$. In that case, $V(x,y)$ is called the *potential function* of the system.

Question 1.

(a) If $(x(t), y(t))$ is a solution curve of the gradient system, then show that $\dfrac{dV}{dt} \geq 0$ along the solution curve. [Hint: Use chain rule!]

(b) If $\dfrac{dV}{dt}$ is identically 0 along the curve, show that the trajectory is actually a fixed point.

(c) Explain why the equilibrium points of a gradient system are the stationary critical points of potential function.

(d) Explain why the solution curves of a gradient system are perpendicular to the level curves of $V(x,y)$.

Properties of Gradient Systems

Theorem 2.1

The phase portrait of a gradient system cannot have a closed orbit as a solution curve.

We will give a proof by contradiction. Suppose, for the sake of contradiction, there is a closed orbit in the phase portrait. Denote the closed path by $\mathcal{C}$ and let $\vec{r}(t)$, $0 \le t \le \tau$ be a parametrization of $\mathcal{C}$. We can also assume that $\mathcal{C}$ is not just a fixed point.

Question 2.

Use fundamental theorem of line integrals to explain why

$$\oint_{\mathcal{C}} \nabla V \cdot d\vec{r} = 0$$

Question 3.

On the other hand, show that we can rewrite the line integral as

$$\oint_{\mathcal{C}} \nabla V \cdot d\vec{r} = \int_0^{\tau} \frac{dV}{dt} dt$$

Observe that the integral in question 3 is always positive which contradicts our observation from question 2! So $\mathcal{C}$ does not exist!

Theorem 2.2

The phase portrait of a gradient system cannot have a spiral sink, spiral source or centers.

Question 3.

Calculate $T^2 - 4D$ of the Jacobian.

Question 4.

show that the system

$$\frac{dx}{dt} = x^2 + 3xy, \qquad \frac{dy}{dt} = 2x + y^3$$

is not a gradient system.

Question 5.

Consider the system

$$x' = \sin y, \qquad y' = x\cos y$$

Show that it is a gradient system and hence does not have closed orbits or spirals in its phase portrait.