Math 2208: Ordinary Differential Equations

Lecture 17 Worksheet

Fall 2019 **Subhadip Chowdhury** **Nov 20**

Title: Almost Linear Systems

Summary: We are going to show that if an almost linear system is used to model a physical situation, then -- apart from two sensitive cases -- the qualitative behavior of the system near a critical point can be determined by examining its linearization.

§A. Stability of Almost Linear Systems

Definition 1.1

A nonlinear system is called *almost linear* at an isolated equilibrium point $P = (x_e, y_e)$ if its linearization at P has an isolated equilibrium point at the origin $(0,0)$.

Theorem 1.1: Stability of Almost Linear System

Consider a *almost linear system* whose linearization at a point $P = (x_e, y_e)$ is a linear system with associated matrix A. Let λ_1 and λ_2 be the eigenvalues of A and assume $\det(A) \neq 0$ or equivalently, $(0,0)$ is an isolated equilibrium. Then the equilibrium point P is identical in type to that of $(0,0)$ in its linearization, except in two cases:

1. If $\lambda_1 = \lambda_2 \in \mathbb{R}$, then the linearization has a defective node at the origin, but the original almost linear system has either a node or a spiral point at P.
2. If λ_1 and λ_2 are purely imaginary, then the linearization has a center at the origin, but the original almost linear system has either a center or a spiral.

Question 1.

Determine the type of the critical point $(4,3)$ of the almost linear system

$$\frac{dx}{dt} = 33 - 10x - 3y + x^2$$

$$\frac{dy}{dt} = -18 + 6x + 2y - xy$$

■ **Question 2.**

Next consider the following two systems

$$x' = y - (x^2 + y^2)x, \qquad y' = -x - (x^2 + y^2)y \tag{1}$$

$$x' = y + (x^2 + y^2)x, \qquad y' = -x + (x^2 + y^2)y \tag{2}$$

(a) Check that both of them have isolated equilibrium point at $(0,0)$ and the same linearization at $(0,0)$.

(b) Check that the corresponding linear system in both cases have a center at the origin. On the other hand, we are going to show that system (1) has a spiral sink whereas system (2) has a spiral source at the origin.

(c) Consider the first system. The corresponding vector field can be written as sum of two vector fields as follows:

$$\vec{\mathbf{V}}(x,y) = \Big(y - (x^2 + y^2)x, -x - (x^2 + y^2)y\Big) = \vec{\mathbf{V}}_1(x,y) - (x^2 + y^2)\vec{\mathbf{V}}_2(x,y)$$

where $\vec{\mathbf{V}}_1(x,y) = (y, -x)$ and $\vec{\mathbf{V}}_2(x,y) = (x,y)$.

(d) Check that $\vec{\mathbf{V}}_1(x,y)$ is tangent to the position vector $\langle x,y\rangle$ at any point. Similarly, $\vec{\mathbf{V}}_2(x,y)$ is parallel to the position vector $\langle x,y\rangle$. So the vector field $\vec{\mathbf{V}}(x,y)$ always has a inward pointing radial component.

Consequently, all flow lines spiral slowly toward the origin!

(e) For similar reasons, check that all flow lines of the vector field corresponding to system (2) spiral outward from the origin.

§B. Possible Trajectories

Under some general hypotheses it can be shown that there are four possible trajectories for a nondegenerate solution curve of the autonomous system

$$x' = f(x,y) \qquad y' = g(x,y)$$

The four possibilities are as follows:

(a) $(x(t), y(t))$ approaches a critical point as $t \to \infty$.

(b) $(x(t), y(t))$ is unbounded with increasing t.

(c) $(x(t), y(t))$ is a periodic solution with a closed trajectory.

(d) $(x(t), y(t))$ spirals toward a closed trajectory, called a *limit cycle*, as $t \to \infty$.

§C. Van der Pol Equation revisited

Recall the Van der Pol equation from last lecture

$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = 0$$

which can be written as a non-linear system of first order ODEs as

$$\begin{aligned}\frac{dx}{dt} &= y\\ \frac{dy}{dt} &= -x + \mu(1-x^2)y\end{aligned}$$

Note that $(0,0)$ is always an equilibrium to above system. Let's draw the phase portrait for a couple of different values of μ and observe if the stability changes at μ. Use `pplane` and set the window to $-6 \le x, y \le 6$ for a comfortable close-up of the origin.

Question 3.

(a) Find the Jacobian and evaluate it at $(0,0)$.

(b) Suppose $\mu = 3$. What kind of stability do you observe at $(0,0)$?

Visually classify the trajectories of other solution curves into the four cases above according to possible initial conditions. Which of the cases can you find examples of?

(c) Next set $\mu = 1$. What kind of stability do you observe at $(0,0)$? Can you describe the trajectories of other solution curves? Which of above four cases can you find examples of?

(d) Repeat above analysis for $\mu = -1$ and $\mu = -3$.

(e) Clearly all four stability types are distinct. So at least three bifurcations happened as μ increased from -3 to 3. Draw the corresponding curve in (T, D)-plane and find the bifurcation values.