Practice Problems and review notes

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- The following are a set of practice/review problems for Midterm 2. Over this week, I will be going over these problems in office hour, class etc. Some of these problems might be on the final exam itself.
- Make sure you can solve all of the problems listed below and ask me via email or in person if you have questions.
- Apart from these problems, you should also go through the extra problems (i.e. those outside the book) that I had assigned in homeworks over the quarter. Ask me if you need clarification with any of those.

Problem 1

$$\int_{1/e}^{e^2} \left| \frac{\ln(x)}{x} \right| dx$$

[Hint: Be careful about any sign change which affects the absolute($|.|$) function.]

Problem 2

$$\int \frac{1-x^2}{x(2-x^2)} dx$$

[Hint: Multiply Num and Den by x. Then take $u = 1 - x^2$.]

Problem 3

Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 1 - e^{-x}$. What is the domain of f^{-1}?

Problem 4

1. Prove the following property of definite integrals:
$$\int_0^a f(x)dx = \int_0^a f(a-x)dx.$$
2. Use it to evaluate
$$\int_0^{\pi/4} \ln(1 + \tan(\theta))d\theta.$$

$\boxed{\textbf{Problem 5}}$

Find

$$\int_0^{\pi/2} \frac{1}{1+\tan x} dx$$

[SOLUTION]

Note that

$$\begin{aligned}
\int_0^{\pi/2} \frac{1}{1+\tan x} dx &= \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx \\
&= \int_0^{\pi/2} \frac{\cos(\pi/2 - x)}{\cos(\pi/2 - x) + \sin(\pi/2 - x)} dx \\
&= \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx
\end{aligned}$$

Hence

$$\begin{aligned}
2\int_0^{\pi/2} \frac{1}{1+\tan x} dx &= \int_0^{\pi/2} \frac{\cos x + \sin x}{\cos x + \sin x} dx \\
&= \int_0^{\pi/2} 1 dx \\
&= \pi/2 \\
\implies \int_0^{\pi/2} \frac{1}{1+\tan x} dx &= \pi/4
\end{aligned}$$

Use the same kind of logic to evaluate

(a)

$$\int_0^{\pi/2} \frac{a + b\tan x}{1+\tan x} dx$$

(b)

$$\int_0^{\pi/2} \frac{\sin^6 x - \cos^6 x}{\sin^7 x + \cos^7 x} dx$$

(c)

$$\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}} dx$$

[Ans: $\pi/12$]

(d)

$$\int_0^{\pi} \frac{x \sin x}{1+\sin x} dx$$

[Ans: $\frac{\pi}{2}(\pi - 2)$]

Problem 6

$$\int \frac{\tan x}{\sec x + \tan x} dx$$

Problem 7

(a)

$$\int \frac{1}{1 + \cos(2x) + \sin(2x)} dx$$

[Hint: Use double angle formulae. Then divide both Num and Den by $\cos^2 x$.]

(b) Find

$$\int \frac{1 + \sin(2x)}{1 + \cos(2x)} dx$$

[Hint: Separate into two integrals and then use double angle formulae.]

(c)

$$\int \frac{1 + \cos x}{\sin(2x)} dx$$

Problem 8

$$\int \frac{1 + x^2}{x^4 + 1 - 2x^2} dx$$

[Hint: Divide Num and Den by x^2.]

Problem 9

Suppose f has an inverse function. Recall that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Suppose $f(3) = -6$, and $f'(3) = 2/3$. If $g = 1/(f^{-1})$, what is $g'(-6)$?
[Hint; Use Chain rule and quotient rule to find out g'.]

Problem 10

Consider the region Ω bounded by the two lines

$$y - x = 4, \qquad 3x + y = 12,$$

and the $X-$axis and the $Y-$axis [i.e. Ω is a quadrilateral.] Suppose Ω is revolved around the $X-$axis to obtained a solid S.

1. Express the volume of S as an integral in terms of x. Do **not** evaluate the integral.
2. Express the volume of S as an integral in terms of y. Do **not** evaluate the integral.

Problem 11

Consider the function $f : [1, 2] \to \mathbb{R}$ defined as

$$f(x) = e^x - \ln(x).$$

Is f an injective function? Justify your answer.

Problem 12

Consider the function $g : \mathbb{R} \setminus \{1, 2\} \to \mathbb{R}$ defined by

$$g(x) = \frac{(1-2x)^2}{(x-1)^2(x-2)}.$$

Is g a surjective function? Justify your answer.

Problem 13

Find the following integral

$$\int \frac{\sin x - \cos x}{1 + \sin(2x)} dx.$$

Problem 14

Find

$$\int \frac{4 - 5\sin^3 x}{\cos^2 x} dx$$

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Problem 15

Evaluate

$$\int_0^{\pi/2} \log(\tan\theta + \cot\theta) d\theta.$$