Problem Set 2 Solutions

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The University of Chicago, CAAP 2018: Proof-Based Methods in Calculus (Chowdhury)

July 3, 2018

Exercise 5: Periodic Sequence

Problem 1.1. *In the sequence* $1, 1, 2, 3, 5, 8, 3, 1, 4, \ldots$ *each term starting with the third is the sum of the two preceding terms, but we only write the unit digit. Thus* 5 *and* 8 *is followed by the unit digit of* 13*, which is* 3*. Prove that the sequence is periodic, i.e. it will start repeating after some time.*

Proof. Let us consider the elements in pairs (a, b), where a is immediately followed by b in the sequence. Note that only this pair, and nothing else, determines the element after b, which is the unit's digit of $(a + b)$; call this c. The same can be said for the pair (b, c), and so on for the rest of the sequence, infinitely.

Whenever the sequence repeats a pair (a, b), it must have followed a finite sequence $S = a, b, c, d \ldots, g, h, a, b$. The sequence will now proceed to repeat S starting from c, and will do so again and again infinitely, showing periodic behavior. Hence, all that remains to show is that the sequence must repeat a pair of elements.

Let us count the number of possible pairs (a, b). The elements are single-digit integers, so there are ten possibilities for a, and ten for b. Thus, there are 100 possible pairs (a, b) that the sequence can cover without repeating.

However, after iterating only 102 elements into the sequence, we can create 101 pairs of elements. With only 100 possible pairs, by the Pigeonhole Principle (PHP), we can conclude that a pair has been repeated in the sequence. $\square$

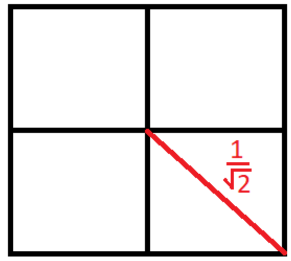

Figure 1: A square of side-length 1, divided into four equal squares whose diagonals are length $\frac{1}{2} \times \sqrt{2}$

Exercise 6: Five Points

Problem 2.1. *Consider any five points $P1$, $P2$, $P3$, $P4$, and $P5$ in the interior of a square S of side length 1. Show that we can always find two of them at distance at most $\frac{1}{\sqrt{2}}$ apart.*

Proof. Consider the division of the square S in Fig. 1. The side-length of each small square is $\frac{1}{2}$. Using the fact that the diagonal of a sqaure is $\sqrt{2}$ times its side length (can you prove this?), we conclude that the diagonal of each of the small squares is

$$\frac{1}{2} \times \sqrt{2} = \frac{1}{\sqrt{2}} \tag{1}$$

Now, we have 5 points and 4 small squares. By PHP, at least one of the small squares has at least two points. Inside this small square, any two points can be no further apart than the diagonal of length $\frac{1}{\sqrt{2}}$, since opposite corners are the farthest such two points can be from each other.

□

Exercise 7: Extra Credit

Please note that this problem uses Mathematical ideas that are more advanced that are being covered in class right now. In particular, we will explain some of the tools used in this proof next week.

Problem 3.1. *Any positive integer n has a multiple that looks like* $11\ldots100\ldots0$*,*

Proof. Consider $1, 11, 111, 1111, \ldots, 111\cdots1$, where the last number consists of $(n+1)$ ones, for some $n \in \mathbb{N}$. We will call these numbers $k_1, k_2, k_3, \ldots, k_{n+1}$.

As the hint suggests, let us consider the remainders of all the k_i's when it is divided by n. The possibilities for a remainder after division by n are $0, 1, 2, \ldots, (n-1)$, but we have computed $(n+1)$ remainders. By PHP, at least two of these remainders must be equal; suppose that these two are k_i and k_j. Then, $|k_i - k_j|$ is divisible by n. If you are sure how this step follows, we will explain it next week.

But note that $|k_i - k_j|$ is a number that begins with one or more ones and ends in zero or more zeros. Note that our use of PHP tell us that k_j and k_i are not equal, so if one of them is zero, the other is not. Thus, their difference is nonzero.

□