

MATH 2208: ORDINARY DIFFERENTIAL EQUATIONS

LECTURE 10 WORKSHEET

Fall 2019

Subhadip Chowdhury

Oct 21

TITLE: Autonomous Linear System of ODEs

SUMMARY: We will begin to analyze systems for ODEs which have the form $\frac{d\vec{r}}{dt} = A\vec{r}$.

§A. Linear Systems and Matrix Notation

The *dimension* of a system of ODEs is equal to the number of dependent variables in the system. A two-dimensional linear system of ODE has the form

$$\begin{aligned}\frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy\end{aligned}$$

where a, b, c , and d are some constants. Recall that we can rewrite this system in the form $\frac{d\vec{r}}{dt} = \vec{F}(\vec{r})$, where $\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ is a column vector and \vec{F} is the vector field

$$\vec{F}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

If we think of \vec{F} as a linear transformation from $\mathbb{R}^2 \rightarrow \mathbb{R}^2$, we can then write

$$\frac{d\vec{r}}{dt} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = A\vec{r}$$

where A is the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A\vec{r}$ is regular matrix multiplication.

Theorem 1.1

If A is a matrix with nonzero determinant, then the only equilibrium point for the linear system of ODEs $\frac{d\vec{r}}{dt} = A\vec{r}$ is the origin.

§B. The Linearity Principle

The following two properties are easy to check using the matrix notation.

- Given a solution $\vec{r}(t)$ of the system, $k\vec{r}(t)$ is also a solution for any real number k .
- If $\vec{r}_1(t)$ and $\vec{r}_2(t)$ are solutions of the system, then $\vec{r}_1(t) + \vec{r}_2(t)$ is also a solution.

■ Question 1.

Consider the linear system $\frac{d\vec{r}}{dt} = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} \vec{r}$.

1. Show that it has $\vec{r}_1(t) = \begin{bmatrix} 2e^{-3t} \\ e^{-3t} \end{bmatrix}$ and $\vec{r}_2(t) = \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix}$ as solutions.
2. Is $\vec{r}_3(t) = 5\vec{r}_1(t) - 3\vec{r}_2(t)$ also a solution?
3. Is there a solution to the system that solves the following initial value problem?

$$\frac{d\vec{r}}{dt} = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} \vec{r}, \quad \vec{r}(0) = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

Theorem 2.1: General Solution to 2D Linear System of ODEs

Suppose $\vec{r}_1(t)$ and $\vec{r}_2(t)$ are solutions of the linear equations $\frac{d\vec{r}}{dt} = A\vec{r}$, where A is a 2×2 matrix. If $\vec{r}_1(0)$ and $\vec{r}_2(0)$ are linearly independent, then for any initial condition $\vec{r}(0) = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$, we can find a specific solution to the initial value problem

$$\frac{d\vec{r}}{dt} = A\vec{r}, \quad \vec{r}(0) = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$$

which looks like $k_1\vec{r}_1(t) + k_2\vec{r}_2(t)$, some linear combination of the two solutions $\vec{r}_1(t)$ and $\vec{r}_2(t)$.

§C. Straight Line Solutions

Consider the system from question 2. Observe that

$$\vec{r}_1(t) = e^{-3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } \vec{r}_2(t) = e^{2t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

■ Question 2.

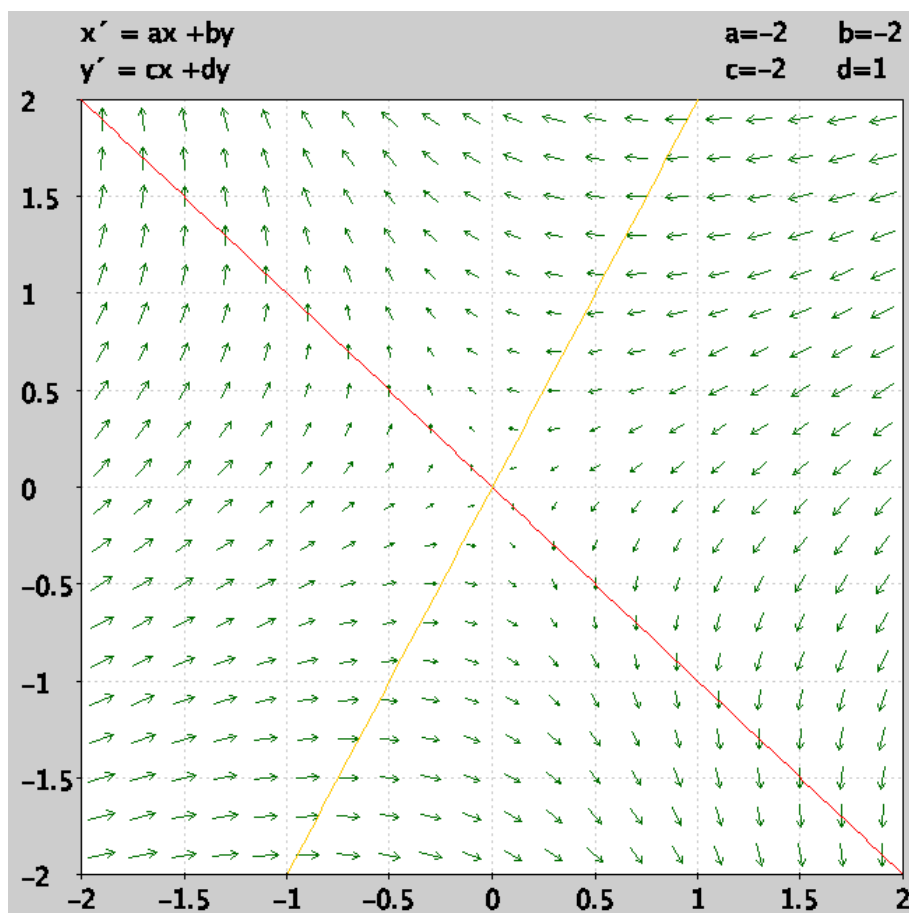
Do you notice anything special about the column vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ in relationship to the matrix $A = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix}$? What happens if you multiply each vector by A ?

■ Question 3.

Consider the direction field for the system below.

1. What do you think the yellow and the red lines are?
2. Can you identify the solution curves $\vec{r}_1(t)$ and $\vec{r}_2(t)$ in it?

- For those solution curves, does it matter what your initial condition is?
- Does one of the solutions seem more “attractive” than the other?
- Describe how other solutions are behaving.



- Use the general formula of the solution (in terms of some arbitrary constants k_1 and k_2) to find $\frac{dy}{dt}$ and $\frac{dx}{dt}$ and use that to calculate $\frac{dy}{dx}$. Now let $t \rightarrow \infty$ and find the limit. Does it depend on whether or not $k_2 = 0$? Is this consistent with your observation about the behavior of the solutions?
- What about $t \rightarrow -\infty$, (i.e. in the reverse direction of the arrows)? Is that consistent with your observation about the behavior of the solutions?

■ Question 4.

Consider the system $\frac{d\vec{r}}{dt} = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \vec{r}$. Find the eigenvalues λ_1 and λ_2 and the corresponding eigenvectors \vec{v}_1, \vec{v}_2 of the matrix $\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}$.

Show that the general solution can be written as $\vec{r}(t) = k_1 e^{\lambda_1 t} \vec{v}_1 + k_2 e^{\lambda_2 t} \vec{v}_2$ and confirm that it is actually a solution of $\frac{d\vec{r}}{dt} = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \vec{r}$.