Rotation Number and Dynamics on the Circle

Subhadip Chowdhury

University of Chicago

https://subhadipchowdhury.github.io

May 2, 2018

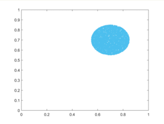
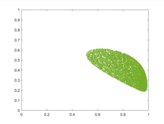
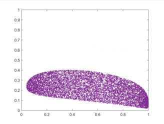
What is a Dynamical System?

- *"something"* that *"evolves"* over time.

- Needs two ingredients

  → **something:** A mathematical object, e.g. some geometric space

  → **rule** of evolution: A transformation of this space, repeated over and over. Time can be continuous or discrete.

 $\xrightarrow{f}$  $\xrightarrow{f}$  $\xrightarrow{f} \ldots$

Examples of Dynamical Systems

- Examples
  - Exponential growth or decay - Bacteria population, Radioactive emission

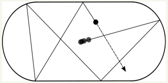
Examples of Dynamical Systems

- Examples
  - → Exponential growth or decay - Bacteria population, Radioactive emission
  - → Chaotic motion - Double pendulum

Examples of Dynamical Systems

- Examples

  - → Exponential growth or decay - Bacteria population, Radioactive emission
  - → Chaotic motion - Double pendulum
  - → Billiards on a table

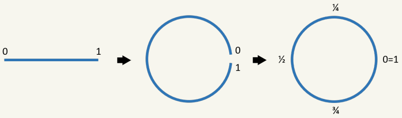


- Applications in Mathematics, Physics, Biology, Chemistry, Engineering, Economics ...

Dynamical Systems on the Circle

- **Something** = a Circle
- **Rule** = iterate a function (from the circle to itself) over time

Circle = interval $[0, 1]$ with 0 and 1 glued



- Think (scaled) polar coordinates.
- Example - The doubling map

$$f(x) = 2x \pmod 1$$

- $f(1/4) = 1/2$, $f(1/2) = 0$, $f(3/4) = 1/2$.
- **Orbit** - sequence of points generated by the iteration rule

Doubling Map

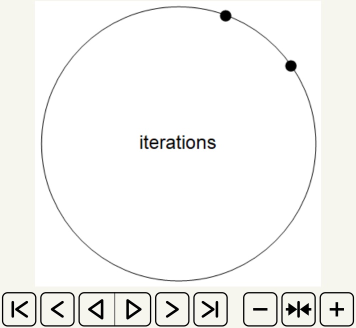


Figure: Starting point = $3/31$

$$3/31 \to 6/31 \to 12/31 \to 24/31 \to \underset{\substack{\| \\ 17/31}}{48/31} \to \underset{\substack{\| \\ 3/31}}{96/31}$$

- **Periodic orbit** - repeats itself in time

> When do you have a *Periodic orbit* of length n?

Rotation Map

- $R_\theta :=$ Rotation counterclockwise by θ

$$R_\theta(x) = x + \theta \pmod 1$$

→ $R_1(x) = x$.

Is $R_{0.5}(x)$ same as $R_\pi(x)$?

Orbit of a Rational rotation

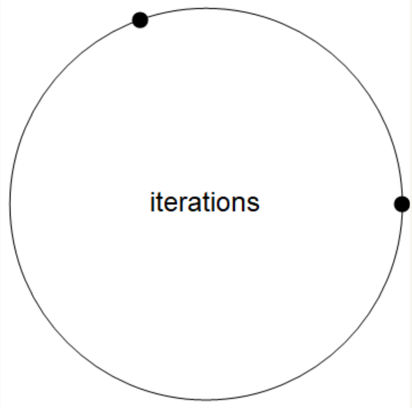


Figure: Rational rotation by $7/23$

Orbits of Rotation Map

- Orbit of a point is periodic if and only if θ is rational. In which case, every point has a periodic orbit.

→ In last example period length was 23.

> If $\theta = p/q$, how many revolutions around the circle does it make before coming back to the starting point?

- What if θ is not rational? Does it ever *'come back'*?

Irrational Rotation

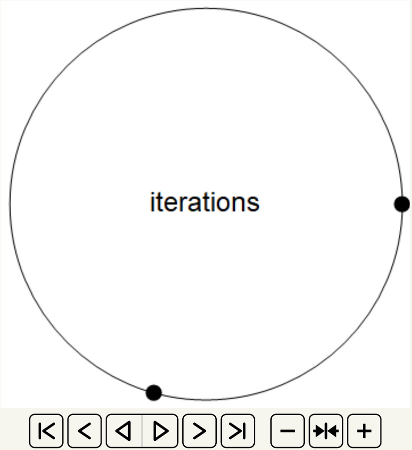


Figure: Irrational rotation by $\frac{1}{\sqrt{2}}$

Dense Orbit

Theorem

For an irrational rotation of the circle, the orbit of a point visits every subinterval. In other words, every orbit is dense.

Proof.

Exercise! Easy proof by contradiction.

□

Homeomorphisms of the Circle

- f: a map from the Circle to itself.
- Graph of f: can be drawn on the unit square.

- **Homeomorphism**
  - $\rightarrow$ Continuous
  - $\rightarrow$ one-to-one and onto
  - $\rightarrow$ inverse is continuous

- Example of homeomorphism: R_θ
- Non-Example - Doubling map

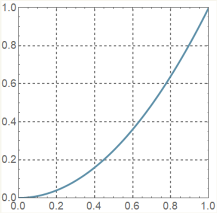


Figure: $f(x) = x^2$

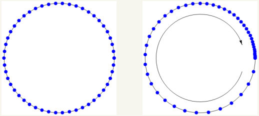
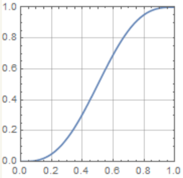


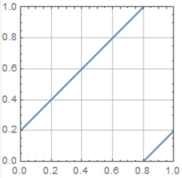
Figure: $f(x) = x^2$ on the Circle

More Examples

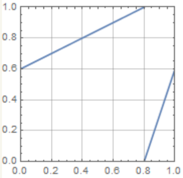
- $f\colon [0,1] \to [0,1]$ continuous bijective where $0 \equiv 1$ and $f(1) = f(0)$.



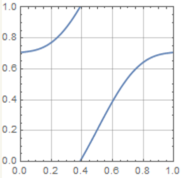
(a) $f(x) = x - \frac{\sin(2\pi x)}{2\pi}$



(b) $f(x) = x + 0.2 \pmod 1$



(c) A piecewise linear map



(d) A standard circle map

Poincaré's Rotation Number

- Want a *number* $\tau(f)$ to describe a map f.
- Average length of thread used!

$$\text{length used in first iteration} \to \frac{\text{length used in first two iteration}}{2}$$

$$\to \frac{\text{length used in first three iteration}}{3} \to \cdots \to \tau(f)$$

- $\tau(R_\theta) = \theta$.
- If f fixes a point then $\tau(f) = 0$.

Rational Rotation Number

Theorem

$\tau(f)$ is rational p/q if and only if f has a periodic point.

- Any periodic orbit has length q.
- Example

$$f(x) = x + \frac{1}{2} - \frac{1}{4\pi}\sin(2\pi x)$$

→ $\left\{0, \frac{1}{2}\right\}$ is a periodic orbit.

→ $\tau(f) = \frac{1}{2}$

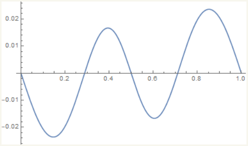


Figure: Graph of $f(f(x)) - x$

→ Another periodic orbit exists: $\{\sim 0.2886, \sim 0.7114\}$.

→ What if we start at another point?

Orbit of $f(x)$

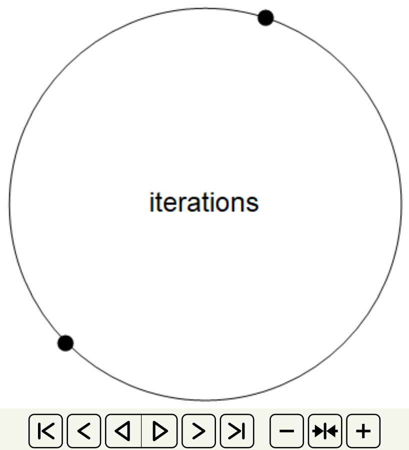


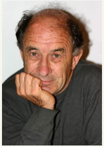
Figure: Starting point $= 0.2$

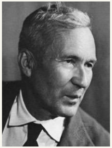
▷ Not every point has a periodic orbit.

Standard circle map and Arnold Tongue

- Start with rigid rotation: $f(x) = x + \alpha$
- Next introduce small perturbation.
- **Standard circle map, a.k.a. Arnold map**

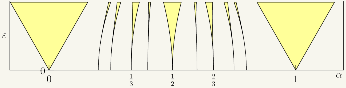
$$f(x) = x + \alpha - \frac{\epsilon}{2\pi}\sin(2\pi x) \pmod 1$$





Check that f is a homeomorphism when $0 \leqslant \epsilon \leqslant 1$.

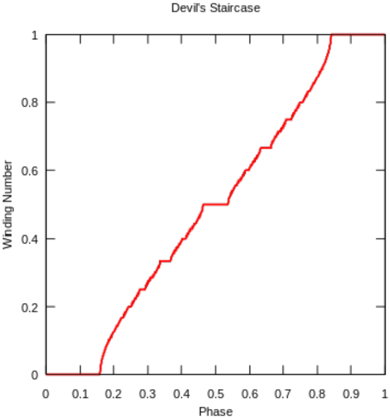
(a) Heatplot of $\tau(f)$ against α as X axis and ϵ as Y axis



(b) Phase-locked regions for rational τ, $0 \leqslant \alpha, \epsilon \leqslant 1$

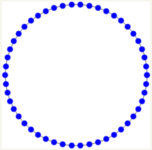
Figure: courtesy of Wikipedia

Devil's Staircase

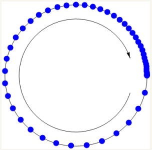


Two homeomorphisms

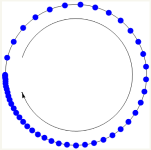
- What if we have two circle maps? Call them f and g.
- Suppose $\tau(f) = 0 = \tau(g)$. $\implies$ They each fix a point.
- Suppose f fixes p, and g fixes q.
- **Question:** Must $f \circ g$ fix a point? What can we say about $\tau(f \circ g)$?



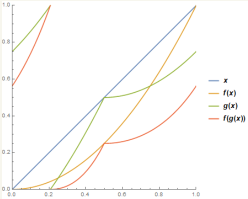
(a) Starting configuration



(b) After applying f



(c) After applying g



Ziggurat

More generally, if we know $\tau(f)$ and $\tau(g)$, what can we say about $\tau(f \circ g)$?

- $\tau(f \circ g)$ can take a range of values. What's important is the **maximum**.

- **The Jankins–Neumann ziggurat**

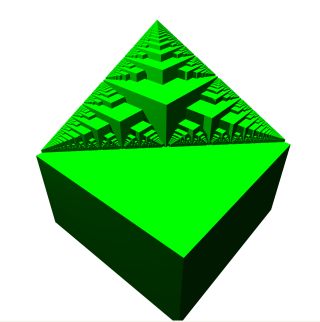


Figure: Plot of $\max\{\tau(f \circ g)\}$ against $\tau(f)$ and $\tau(g)$

Thank you.

Questions? Email me at subhadip@math.uchicago.edu.