Teaching Portfolio

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Chapter 1

Teaching Experience and Reflections

§1. The College of Wooster

My time as a Visiting Assistant Professor at the College of Wooster has been one of the most important highlights of my Teaching Career so far, as I found the opportunities and had the prerequisite experience to experiment with new pedagogical techniques (such as discovery based learning, and alternate grading methods) and improve upon my own practices. Besides designing my own course curriculum, planning lectures and worksheets, designing and grading assessments, holding office hours etc., I also supervised Independent Studies (Senior Thesis) and Summer Research projects. In particular, I got to make significant contribution in redesigning the Calculus sequence; and personally create the syllabi for (and teach) two new courses being offered by the Mathematical and Computational Studies department over the last semester and the next. Brief description and reflection on each of the courses I have taught so far, are listed below.

1.1. Introduction to Topology, Math 330

- **Meeting Schedule** - Fall 2021 - 1.5 hours per week
- **Class Size** - 4

This was a tutorial course that met once a week for 1.5 hrs which was taught using a seminar style. The students were senior Math majors interested in Grad school. The content covered most of Point Set Topology without going into technicalities of separation axioms, and ended with brief introduction to Fundamental groups and Covering Spaces. Students used office hours to get further help on their reading but it was a heavily independent project. Their performance was assessed through homework problems sets which they were allowed to resubmit after my feedback.

1.2. Numerical Analysis, Math 327

- **Meeting Schedule** - Spring 2022 - 3 hours per week
- **Class Size** - 6

1.3. Putnam Seminar, Math 279

- **Meeting Schedule** - Fall 2022, Fall 2021 - 1 hour per week

- **Class Size** - 7

This was 0.25 credit problem seminar course that met once a week as we discussed various problem solving strategies to prepare for the Putnam Exam, and how to set the correct mindset for solving 'hard' math problems in general. This included discussing how to decide what part of Mathematics to use to ''start'' solving a problem, and how to persevere with a difficult question, even when no goal might immediately be within your sights. Students were graded on a S/NC scale based on participation.

1.4. Differential Equations, Math 221

- **Meeting Schedule** - Fall 2020 - 3 hours per week
- **Class Size** - 32

End of Semester Reflection: This course being an applied Math course, included a collection of group projects to work on during the semester where the students used the results learned in class to real-life problems. Most students liked these assignments, even more than the lectures or the exams. I spent a significant amount of time curating examples from Environmental Science, Epidemiology, Engineering, Physics, and Chemistry. I am happy to say that the effort did pay off. For groups, I decided to divide the students based on their time zones, which seemed to work best.

1.5. Transition to Advanced Mathematics, Math 215

- **Meeting Schedule** - Spring 2021, Fall 2021, Fall 2022 - 3 hours per week
- **Class Size** - 18 (FA21), 17 (SP21), 15 (FA22)

End of Semester Reflection: This course differs greatly from any previous Mathematical experiences a student may have had. One of the many differences is that students cannot learn how to construct logical arguments by listening to lectures, they must try it themselves. As such, the class was taught in a semi Inquiry-Based Learning format where students spent most classes working on problems themselves first. They were provided new definitions and theorems only when they needed them during the process of solving a new problem or for generalizing and emphasizing a certain technique. Most classes started with a wrap-up of the previous class for 15-20 minutes followed by hands-on active learning through provided worksheets for about 30 mins, the process repeated. I believe this method worked very well for this class and I hope to make the in-class discussions more student-led in future semesters.

When I taught the course again, the aspect that I improved upon was the Expository Paper writing requirement. By providing plentiful and transparent instructions on the various steps of writing a paper, and giving a detailed rubric from the beginning of the semester helped students learn how to read and understand new proofs instead of whether or not they were able to just write notes on it. The peer review was preceded by short in-class video presentations where the audience gives immediate feedback, first verbally and then in writing.

As a part of this course, students are expected to learn how to communicate their mathematical ideas in a way that is understandable by peers and logically infallible. At the same time, students must be allowed the freedom and time to come up with original ideas themselves as well. Unfortunately, this does present a time management issue of how long I can let the students explore ideas, against how much study material we need to cover. I believe a restructuring of the course material proved helpful in this regard, where the 'alphabet' (set theory) and the 'grammar' (logic) of Math were taught side-by-side and not one after another. This way, I could start talking about proof techniques a bit earlier, without having to go into too many technicalities. In later iterations, I further decided to cut down on some of the end-semester topics in favor of exploring more proofreading and writing practice.

The homework was mostly a source of practice problems where students collaborated together in writing the solutions. A specific collection of problems selected for a "Proof Portfolio" were specifically designated as an independent exercise. Students were given a chance to redo these assignments, and the EP based on feedback from me. The content-based questions were turned into Moodle multiple-choice quizzes while retaining the technique-based ones as homework. This way I was able to replace the traditional "two midterm and a final" system with a collection of weekly quizzes instead. A modified form of Specification Based Grading was used to ensure that all of these different aspects of the course are given equal attention by all students while allowing a variety of ways to assess their growth over the semester. A detailed syllabus of the course, along with the IBL style entire workbook is available on my website HERE.

1.6. Multivariate Calculus, Math 212

- **Meeting Schedule** - Spring 2022, Fall 2022 - 4.5 hours per week
- **Class Size** - 14 (SP22), 12 (FA22)

1.7. Foundations of Mathematical Computing, Math 130

- **Meeting Schedule** - Spring 2022 - 3 hours per week
- **Class Size** - 30

1.8. Theory of Integral Calculus, Math 125

- **Meeting Schedule** - Fall 2022, 2nd Half - 4.5 hours per week
- **Class Size** - 24

1.9. Theory of Differential Calculus, Math 115

- **Meeting Schedule** - Fall 2021, 2nd half - 4.5 hours per week
- **Class Size** - 31 (two sections)

End of Semester Reflection: For context, this course, along with another half semester course titled "Applied Differential Calculus" cover the content of a standard Calculus I sequence. The rationale behind the half-semester breaks was to ensure that the students who are taking the second half are only the ones interested in becoming a STEM major that goes beyond calculations, and cares more about justifications. This overall increased student quality and performance and at the same time allowed me to talk about the theoretical aspects of Calculus in more detail.

Most of the classes were taught using some form of active learning with one day of the week dedicated to problem-solving only. Lectures were never more than 5-10 minutes in a row, before we did practice problems or group exercises. Students were expected to interact during lectures to fill in the steps of a proof or a calculation. The course was graded using Mastery based grading and I felt both very comfortable and satisfied in this form of assessment having used it in the past. Student grades were based on portion of the learning targets completed fully instead of accumulated partial works. Students were given 5 or more chances per learning target to show evidence of continued proficiency. Since most feedback were formative, and I did not have to worry about partial credits while grading, the workload was fairly manageable with weekly take-home checkpoint quizzes. Homework deadlines were generous, these were a source of online and repeatable practice problems that were due by the end of the semester, but only 90% or above counted for any credits. Students were allowed to use tokens to extend deadlines on quizzes or other exceptions.

The shortcoming of a half-semester course however was that students who were not used to Mastery-Based Grading did not have too much time to get comfortable with this form of assessment and this led to a multitude of reassessment requests closer to the end of the semester. Although I took time to explain the methodology in the first class, in the future I will need to have another session clarifying the system around halfway through the course.

I also did not get to cover as many real-life extension problems on the course content as I would have liked compared to a full semester as the course overall felt a bit rushed. I will need to think about how to incorporate more 'fun' problems in a theory-based class to keep students more engaged.

1.10. Calculus & Analytical geometry II, Math 112

- **Meeting Schedule** - Spring 2021 - 3 hours per week
- **Class Size** - 21 + 27 (two sections)

End of Semester Reflection: Over multiple semesters, I have improved the implementation of Mastery-Based Grading for teaching this class, and most students seemed appreciative of the new method of learning. Besides rewarding productive failure, one of my main goals for implementing MBG was to grade the students based on the amount of learning objectives fully completed instead of accumulated partial works. In that aspect, I believe I was successful. However, I need to emphasize this point more next time I use MBG so that students are more open to the idea of not having partial credits.

1.11. Calculus & Analytical geometry I, Math 111

- **Meeting Schedule** - Fall 2021 - 3 hours per week
- **Class Size** - 27 + 25 (two sections)

End of Semester Reflection: Students liked that they were able to apply the theoretical knowledge during the applied lab work. However, I have since found that I need to make the instructions clearer, with step-by-step breakdowns regarding these projects. I should reduce my expectation regarding the ability of first-year students to do exploratory work outside the regular study material. I have since then changed these assignments to a completion-based grading, with a chance for reassessment, so that students feel encouraged to participate, while not having to fear the unknown.

§2. Bowdoin College

The 2018-2020 academic years were my first experience teaching at a liberal arts institution. As a Visiting Assistant Professor, I was responsible for designing my own course curriculum, planning lectures and worksheets, designing and grading exams, holding office hours, and assigning individual homeworks and group projects. I also coordinated and mentored several graders, teaching assistant and study group leaders. Brief description of each of the courses I have taught are listed below.

2.1. Ordinary Differential Equations, Math 2208

- **Meeting Schedule** - Fall 2019, Spring 2020 - 3 hours per week
- **Class Size** - 20

This course, intended for Junior and Senior Math majors, is designed as a gateway course for students interested in Applied Math. I taught this course using a hybrid discovery method. This was the first applied course I had taught at Bowdoin college. I spent a significant amount of time carefully designing lecture worksheets that consisted of leading questions which helped students figure out the content for themselves. Although a part of the lecture was spent by me presenting on board, a lot of it was spent doing group work on blackboards and experimenting with software. Besides classical methods for solving differential equations, the main emphasis was on modern, qualitative techniques for studying the behavior of solutions to differential equations. We also worked on applications of ODEs in catastrophe theory, flow-kick regimen, resonance, market economy and auto-catalytic biochemical oscillations through multiple lab sessions and projects. Some of the projects and worksheets can be found in my website.

2.2. Linear Algebra, Math 2000

- **Meeting Schedule** - Spring 2019 - 3 hours per week
- **Class Size** - 11

This course is intended for Sophomores and Juniors and is designed as a gateway course for Mathematics and interdisciplinary majors. The students taking this course were not expected to have experience with writing proofs. As such, in an effort to make the course less dry, we spent a significant amount of time looking at various applications drawn from flight networks, cryptography, error correcting codes, population dynamics, Markov chains and Google page-rank algorithm, computer graphics, and optimization techniques using least-squares approximations. The students were assessed with multi-part in-class and take-home final exams. The goal was to make sure they are proficient in conceptual and numerical techniques as well as are able to apply their knowledge to practical applications. Some of the projects and exams can be found in my website.

2.3. Multivariable Calculus, Math 1800

- **Meeting Schedule** - Fall 2018, Spring & Fall 2019, Spring 2020 - 4.5 hours per week
- **Class Size** - 12 on average

This course is one of my most favorite course to teach. It's aimed towards mathematically inclined students, mostly Freshmen and Sophomores, who have learned differential and integral calculus, and would like to broaden their horizon. I have taught it several times over the last few years and usually the class size is smaller compared to other sections because of less favorable meeting times (three times a week). However the smaller class size allows me incorporate a lot of group discussion style techniques fairly regularly. I can easily keep track of every students' performance and struggles, and could create individualized work for them to catch up with the rest of the class. I have created and refined lecture notes, worksheets, labs, and group projects for this class over the year all of which create an ecosystem where the students learn higher dimensional abstract concepts with relative ease as they get to approach it from numerous viewpoints. Students learned applications of regression techniques in data science, an introduction to the Gradient descent method of optimization techniques used in machine learning, practical modeling of climate change evidences, the Normal probability distribution, and mathematics behind rotary engines and rocket propulsion. Students heavily relied on demonstrations using *Mathematica* and *Desmos*, both to visualize three dimensional pictures of surfaces, vector fields etc. as well as to learn numerical approximation techniques.

2.4. Differential Calculus, Math 1600

- **Meeting Schedule** - Fall 2018 - 4.5 hours per week
- **Class Size** - 32

This course was aimed towards Freshmen and Sophomores from various backgrounds as one of the introductory Mathematics courses offered at Bowdoin College. For a lot of the students, this was the first college Math course and I wanted to make sure they learn the proper way to think about Math from the very beginning. Building on the traditional course structure, one of my main focus in teaching this course was to make sure students are able to interpret and describe symbolic equations using words and conversely be able to transform practical examples and word problems into mathematical models. Over the semester, I created a number of lab sessions which also helped solidify the abstract ideas by doing numerical estimations through Mathematica, and by describing how to implement various algorithms e.g. the Newton-Raphson method.

2.5. Independent Study

In 2019-2020 academic year, I mentored two undergraduate student projects and independent studies.

- The first one on *Asset revenue modeling using differential equations and machine learning* followed the structure from the Machine learning course by Andrew Ng on Coursera; but was modified to include linear algebra and ODE justifications and proofs befit for a senior math major undergraduate student.
- The second IS project was on *Non-euclidean geometry and tiling*, and followed the content from the chapter on *Tile Invariants for Tackling Tiling Questions* by Hitchman M.P. (2017), in "A Primer for Undergraduate Research" (doi). The student learned necessary Group theory background in the context of tiling in planes to handle combinatorial problems.

§3. University of Chicago

Besides my liberal arts teaching experience, I was also fortunate to have the opportunity of being an instructor during and after my PhD at the University of Chicago.

3.1. Proof-Based Methods

- **Meeting Schedule** - Summer 2018 - 6 per week for six weeks
- **Class Size** - 16

After finishing my PhD in summer 2018, I had the unique experience of teaching an *Introduction to Proof* style class to a group of academically talented incoming first-year students at UChicago through the *Chicago Academic Achievement Program* Summer academy, conducted by the *Center for College Student Success*. This class was designed to expose the students to the academic rigor expected of them as they enroll into introductory Math courses at the college, as well as provide a support framework to help them navigate through the new social and cultural norms.

As a class designed essentially to develop Math reasoning, we covered ideas and problem solving strategies from a broad area of topics such as Number Theory, Combinatorics, Graph Theory, Sequences, and limit Calculus. Besides the final exam, the students also were required to give a presentation in front of their peers which I believe helped them with their Mathematical writing and interaction skills. I tried to keep the atmosphere of the class as casual as possible so that they do not get overloaded with too much expectation. The syllabus for this class is available in my website.

3.2. Linear Algebra, Math 196

- **Meeting Schedule** - Summer 2017 - 6 hours per week for 5 weeks
- **Class Size** - 18 on average

This course was offered through the *Graham School of Continuing Liberal and Professional Studies* for computational linear algebra, intended primarily for students in the social sciences who have completed single and multivariable calculus sequence. However, the students weren't expected to have much experience with writing proofs and as such, we spent a lot of time working on examples from many disciplines, in particular ones that relate to their primary fields of interest.

3.3. Mathematical Methods for Social Science, Math 195

- **Meeting Schedule** - Fall 2017, Fall 2018 - 3 hours per week
- **Class Size** - 18 on average

The course consists of topics that are important for students who are planning to become majors in Economics, Political Science, Mathematical Linguistics etc. As such we covered vectors and multivariable calculus up to optimization, but instead of talking about Green's theorem, we covered linear programming next and finally sequences and series with the goal of learning Taylor approximations.

3.4. Independent Study

During Fall 2018, I mentored two interested talented students in my Math 195 course via an independent study of Game Theory and a project on *Least Unique Bid Auction*, where we discussed the usage of Lagrange Multipliers for finding complicated Probability estimates.

3.5. Standard Calculus Sequence, Math 150's

- **Meeting Schedule** - 3 hours per week
- **Class Size** - 15-30

As a graduate student college instructor, I taught independent section of Calculus courses in 2014-2017. The yearlong rigorous one-variable and multi-variable *standard Calculus* sequence (taught thrice) is designed for science, economics and Math majors. As the instructor of record, I was responsible for designing my own course curriculum, planning lectures, designing and grading exams, holding office hours, and assigning homework. I also mentored teaching assistants, and coordinated junior tutors.

3.6. Elementary Functions and Calculus, Math 133

- **Meeting Schedule** - 3 hours per week
- **Class Size** - 6

I taught a quarter long course on Vector calculus titled 'Elementary Functions and Calculus' to non-science (mostly History, English, and Theater) majors. Teaching students with very little technical background was an unparalleled learning experience.

§4. Other Academic Service

4.1. Undergraduate Mentoring

During the 2021-2022 academic year at the College of Wooster, I supervised two yearlong senior Independent Study (thesis) projects titled:

- *In Hot Water! Using Numerical Analysis to show the Effects of Climate Change on the Great Lakes*, and
- *The Infinity Conundrum: Understanding Topics in Set Theory and the Continuum Hypothesis*

Currently, I am supervising three more independent senior theses on

- *Tiling Invariants for families of two-dimensional regions*,
- *Curvature, Hyperbolic Plane, and Hilbert's Theorem*, and
- *Illustrated Story Book Generator using Transformers and VQGAN-CLIP*.

Previously, at Bowdoin College, I mentored an Intermediate (Junior) Independent Study project on *Machine Learning.* At the University of Chicago, I mentored eight undergraduate students (during 2014-2017) through the *Directed Reading Program* (DRP) and the summer *Research Experience for Undergraduates* (REU) on a wide array of topics from geometry, linear algebra, topology, dynamics of group action etc. We usually met students once or twice a week for 1.5 hrs, where the students would discuss a paper they have read and any original work they have done, followed by me outlining the next possible direction of approach and available useful literature. In both cases, I also helped them learn mathematical writing and coached them for an end-of-quarter presentation or written paper. A detailed list of students and their papers is *available in my CV* and on my webpage.

4.2. Math competitions and Problem Solving sessions

I co-organized a weekly Problem Solving Session at Bowdoin College where we worked on Math 'puzzles'. The goal was on one hand to teach important mathematical strategies in a fun setting and on the other, to train more enthusiastic students for the *Putnam Competition*.

4.3. As a Teaching Assistant and Grader

In the 2013-2014 academic year, I worked with professor Eugenia Cheng as a teaching assistant for a year-long *Honors Calculus* sequence and later worked as a grader for graduate courses on Algebraic Topology, Differential Topology, Differential Geometry, and Riemannian Geometry. Details on these are listed in my CV.

Chapter 2

Student Evaluations Numerical Summary

§1. The College of Wooster

Numerical averages of all student responses for all the courses I taught at College of Wooster for Spring 2022, Fall 2021 and Spring 2021 semesters are attached below. Fall 2020 evaluations are not included as there weren't enough responses received (due to COVID issues). All of the evaluations in full are available on request.

The College of Wooster
2122FA Course Evaluations

Course: MATH-21500-01-2122FA: Transition to Adv Mathematics FA21 (MATH-21500-01)
Instructor: Subhadip Chowdhury *
Response Rate: 15/19 (78.95 %)

Question		5	4	3	2	1	Mean	Std	Median
3. Please assess the following elements of Subhadip Chowdhury's instruction: The instructor stimulated my interest in the subject matter.	n	10	5	0	0	0	4.67	0.49	5.00
	%	66.67%	33.33%	0.00%	0.00%	0.00%			
3. Please assess the following elements of Subhadip Chowdhury's instruction: The instructor's explanations of the material helped me understand the course content.	n	10	5	0	0	0	4.67	0.49	5.00
	%	66.67%	33.33%	0.00%	0.00%	0.00%			
3. Please assess the following elements of Subhadip Chowdhury's instruction: The instructor provided helpful feedback.	n	10	5	0	0	0	4.67	0.49	5.00
	%	66.67%	33.33%	0.00%	0.00%	0.00%			
4. Please assess the following elements of the course and instruction: The course was organized in a way that helped me understand underlying concepts.	n	9	6	0	0	0	4.60	0.51	5.00
	%	60.00%	40.00%	0.00%	0.00%	0.00%			
4. Please assess the following elements of the course and instruction: The course was attentive to issues of diversity, equity and inclusion.	n	7	4	4	0	0	4.20	0.86	4.00
	%	46.67%	26.67%	26.67%	0.00%	0.00%			
4. Please assess the following elements of the course and instruction: The course materials (readings, instructional materials, assigned problems, laboratory experiments, videos, etc.) facilitated my learning.	n	8	7	0	0	0	4.53	0.52	5.00
	%	53.33%	46.67%	0.00%	0.00%	0.00%			

Scale: 5 = Strongly Agree, 4 = Agree, 3 = Neither Agree Nor Disagree, 2 = Disagree, 1 = Strongly Disagree

The College of Wooster
2122FA Course Evaluations

Course: MATH-11500-MC-2122FA: Theory Diff Calculus, 2nd Half FA21 Metacourse (MATH-11500-MC)
Instructor: Subhadip Chowdhury *
Response Rate: 27/31 (87.10 %)

Question		5	4	3	2	1	Mean	Std	Median
3. Please assess the following elements of Subhadip Chowdhury's instruction: The instructor stimulated my interest in the subject matter.	n	4	12	8	3	0	3.63	0.88	4.00
	%	14.81%	44.44%	29.63%	11.11%	0.00%			
3. Please assess the following elements of Subhadip Chowdhury's instruction: The instructor's explanations of the material helped me understand the course content.	n	7	8	8	4	0	3.67	1.04	4.00
	%	25.93%	29.63%	29.63%	14.81%	0.00%			
3. Please assess the following elements of Subhadip Chowdhury's instruction: The instructor provided helpful feedback.	n	8	9	8	2	0	3.85	0.95	4.00
	%	29.63%	33.33%	29.63%	7.41%	0.00%			
4. Please assess the following elements of the course and instruction: The course was organized in a way that helped me understand underlying concepts.	n	8	6	8	4	1	3.59	1.19	4.00
	%	29.63%	22.22%	29.63%	14.81%	3.70%			
4. Please assess the following elements of the course and instruction: The course was attentive to issues of diversity, equity and inclusion.	n	8	6	12	0	1	3.74	1.02	4.00
	%	29.63%	22.22%	44.44%	0.00%	3.70%			
4. Please assess the following elements of the course and instruction: The course materials (readings, instructional materials, assigned problems, laboratory experiments, videos, etc.) facilitated my learning.	n	7	8	8	4	0	3.67	1.04	4.00
	%	25.93%	29.63%	29.63%	14.81%	0.00%			

Scale: 5 = Strongly Agree, 4 = Agree, 3 = Neither Agree Nor Disagree, 2 = Disagree, 1 = Strongly Disagree

The College of Wooster
Spring 2022

Course: MATH-13000-01-2122SP: Mathematical Foundations of Computing SP22 (MATH-13000-01)

Instructor: Subhadip Chowdhury *

Response Rate: 25/29 (86.21 %)

[redacted]

Question		5	4	3	2	1	Mean	Std	Median
3. Please assess the following elements of Subhadip Chowdhury's instruction: The instructor stimulated my interest in the subject matter.	n	10	9	2	4	0	4.00	1.08	4.00
	%	40.00%	36.00%	8.00%	16.00%	0.00%			
3. Please assess the following elements of Subhadip Chowdhury's instruction: The instructor's explanations of the material helped me understand the course content.	n	15	6	1	2	1	4.28	1.14	5.00
	%	60.00%	24.00%	4.00%	8.00%	4.00%			
3. Please assess the following elements of Subhadip Chowdhury's instruction: The instructor provided helpful feedback.	n	15	6	1	2	1	4.28	1.14	5.00
	%	60.00%	24.00%	4.00%	8.00%	4.00%			
4. Please assess the following elements of the course and instruction: The course was organized in a way that helped me understand underlying concepts.	n	10	11	3	0	1	4.16	0.94	4.00
	%	40.00%	44.00%	12.00%	0.00%	4.00%			
4. Please assess the following elements of the course and instruction: The course was attentive to issues of diversity, equity and inclusion.	n	10	6	9	0	0	4.04	0.89	4.00
	%	40.00%	24.00%	36.00%	0.00%	0.00%			
4. Please assess the following elements of the course and instruction: The course materials (readings, instructional materials, assigned problems, laboratory experiments, videos, etc.) facilitated my learning.	n	11	8	3	2	1	4.04	1.14	4.00
	%	44.00%	32.00%	12.00%	8.00%	4.00%			

Scale: 5 = Strongly Agree, 4 = Agree, 3 = Neither Agree Nor Disagree, 2 = Disagree, 1 = Strongly Disagree

The College of Wooster
Spring 2022

Course: MATH-21200-01-2122SP: Multivariate Calculus SP22 (MATH-21200-01)
Instructor: Subhadip Chowdhury *
Response Rate: 14/18 (77.78 %)

Question		5	4	3	2	1	Mean	Std	Median
3. Please assess the following elements of Subhadip Chowdhury's instruction: The instructor stimulated my interest in the subject matter.	n	5	4	5	0	0	4.00	0.88	4.00
	%	35.71%	28.57%	35.71%	0.00%	0.00%			
3. Please assess the following elements of Subhadip Chowdhury's instruction: The instructor's explanations of the material helped me understand the course content.	n	5	4	3	2	0	3.86	1.10	4.00
	%	35.71%	28.57%	21.43%	14.29%	0.00%			
3. Please assess the following elements of Subhadip Chowdhury's instruction: The instructor provided helpful feedback.	n	5	8	0	1	0	4.21	0.80	4.00
	%	35.71%	57.14%	0.00%	7.14%	0.00%			
4. Please assess the following elements of the course and instruction: The course was organized in a way that helped me understand underlying concepts.	n	4	5	0	5	0	3.57	1.28	4.00
	%	28.57%	35.71%	0.00%	35.71%	0.00%			
4. Please assess the following elements of the course and instruction: The course was attentive to issues of diversity, equity and inclusion.	n	5	3	6	0	0	3.93	0.92	4.00
	%	35.71%	21.43%	42.86%	0.00%	0.00%			
4. Please assess the following elements of the course and instruction: The course materials (readings, instructional materials, assigned problems, laboratory experiments, videos, etc.) facilitated my learning.	n	5	7	1	1	0	4.14	0.86	4.00
	%	35.71%	50.00%	7.14%	7.14%	0.00%			

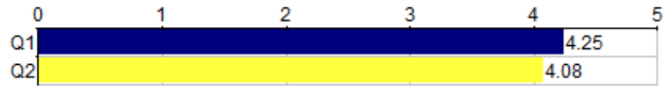
Scale: 5 = Strongly Agree, 4 = Agree, 3 = Neither Agree Nor Disagree, 2 = Disagree, 1 = Strongly Disagree

[redacted]

1.3. Spring 2021

2021SP-R: EXP Core (Revised Spring 2021) Survey 2021SP	The College of Wooster Wooster Ohio

Course:	MATH-21500 02 - Transition to Adv Mathematics	**Department:**	MATH
Responsible Faculty:	Subhadip Chowdhury	**Responses / Expected:**	12 / 17 (70.59%)

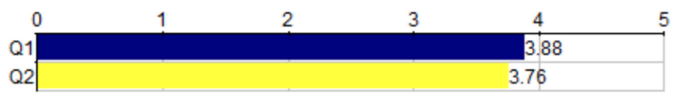


Core EXP Instructor		Subhadip Chowdhury						
		Responses					Individual	
		E	VG	G	F	P	N	Mean
Q1	I would rate the quality of instruction I received in this course as:	3	9	0	0	0	12	4.25
Q2	I would rate this course overall as:	3	7	2	0	0	12	4.08

Responses: [E] Excellent=5 [VG] Very Good=4 [G] Good=3 [F] Fair=2 [P] Poor=1

2021SP-R: EXP Core (Revised Spring 2021) Survey 2021SP	The College of Wooster Wooster Ohio

Course:	MATH-11200 01 - Calc & Analytic Geometry II	**Department:**	MATH
Responsible Faculty:	Subhadip Chowdhury	**Responses / Expected:**	17 / 27 (62.96%)

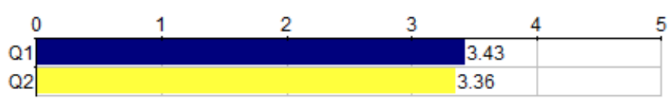


Core EXP Instructor		Subhadip Chowdhury						
		Responses					Individual	
		E	VG	G	F	P	N	Mean
Q1	I would rate the quality of instruction I received in this course as:	6	8	0	1	2	17	3.88
Q2	I would rate this course overall as:	6	6	2	1	2	17	3.76

Responses: [E] Excellent=5 [VG] Very Good=4 [G] Good=3 [F] Fair=2 [P] Poor=1

2021SP-R: EXP Core (Revised Spring 2021) Survey 2021SP	The College of Wooster Wooster Ohio

Course:	MATH-11200 02 - Calc & Analytic Geometry II	**Department:**	MATH
Responsible Faculty:	Subhadip Chowdhury	**Responses / Expected:**	14 / 21 (66.67%)



Core EXP Instructor		Subhadip Chowdhury						
		Responses					Individual	
		E	VG	G	F	P	N	Mean
Q1	I would rate the quality of instruction I received in this course as:	4	4	2	2	2	14	3.43
Q2	I would rate this course overall as:	2	6	3	1	2	14	3.36

Responses: [E] Excellent=5 [VG] Very Good=4 [G] Good=3 [F] Fair=2 [P] Poor=1

§2. Bowdoin College

Numerical averages of all student responses for all the courses I taught at Bowdoin College last three semesters (Fall 2019, Spring 2019 and Fall 2018) are attached below. All of the evaluations in full are available on request.

How much did this course contribute to your education?

Name	Resp	Mean
Overall	34	4.48
Multivariate Calculus	14	4.71
Ordinary Differ Equations	20	4.25

How effectively did the instructor make use of class sessions to advance your learning?

Name	Resp	Mean
Overall	34	4.34
Multivariate Calculus	14	4.79
Ordinary Differ Equations	20	3.90

How helpful to your learning was the feedback you received in the course?

Name	Resp	Mean
Overall	34	4.06
Multivariate Calculus	14	4.07
Ordinary Differ Equations	20	4.05

Did you communicate with the instructor outside of class?

Name	% Yes	% No
Overall	88.24%	11.76%
Multivariate Calculus	78.57%	21.43%
Ordinary Differ Equations	95.00%	5.00%

How helpful to your learning were these communications with the instructor outside of class?

Name	Resp	Mean
Overall	30	4.74
Multivariate Calculus	11	5.00
Ordinary Differ Equations	19	4.47

How accessible was the instructor outside of class?

Name	Resp	Mean
Overall	34	4.95
Multivariate Calculus	14	5.00
Ordinary Differ Equations	20	4.90

What is your overall rating of the instructor as a teacher?

Name	Resp	Mean
Overall	34	4.33
Multivariate Calculus	14	4.71
Ordinary Differ Equations	20	3.95

How much did this course contribute to your education?

Name	Resp	Mean
Overall	21	4.33
Linear Algebra	11	4.36
Multivariate Calculus	10	4.30

How effectively did the instructor make use of class sessions to advance your learning?

Name	Resp	Mean
Overall	21	4.20
Linear Algebra	11	4.00
Multivariate Calculus	10	4.40

How helpful to your learning was the feedback you received in the course?

Name	Resp	Mean
Overall	21	4.38
Linear Algebra	11	4.45
Multivariate Calculus	10	4.30

Did you communicate with the instructor outside of class?

Name	% Yes	% No
Overall	85.71%	14.29%
Linear Algebra	90.91%	9.09%
Multivariate Calculus	80.00%	20.00%

How helpful to your learning were these communications with the instructor outside of class?

Name	Resp	Mean
Overall	18	4.84
Linear Algebra	10	4.80
Multivariate Calculus	8	4.88

How accessible was the instructor outside of class?

Name	Resp	Mean
Overall	21	4.71
Linear Algebra	11	4.82
Multivariate Calculus	10	4.60

What is your overall rating of the instructor as a teacher?

Name	Resp	Mean
Overall	20	4.05
Linear Algebra	10	4.20
Multivariate Calculus	10	3.90

How much did this course contribute to your education?

Name	Resp	Mean
Overall	38	3.77
Differential Calculus	30	3.53
Multivariate Calculus	8	4.00

How effectively did the instructor make use of class sessions to advance your learning?

Name	Resp	Mean
Overall	38	3.67
Differential Calculus	30	3.33
Multivariate Calculus	8	4.00

How helpful to your learning was the feedback you received in the course?

Name	Resp	Mean
Overall	37	3.64
Differential Calculus	29	3.41
Multivariate Calculus	8	3.88

Did you communicate with the instructor outside of class?

Name	% Yes	% No
Overall	81.58 %	18.42 %
Differential Calculus	83.33 %	16.67 %
Multivariate Calculus	75.00 %	25.00 %

How helpful to your learning were these communications with the instructor outside of class?

Name	Resp	Mean
Overall	31	4.29
Differential Calculus	25	4.08
Multivariate Calculus	6	4.50

How accessible was the instructor outside of class?

Name	Resp	Mean
Overall	38	4.41
Differential Calculus	30	4.20
Multivariate Calculus	8	4.63

What is your overall rating of the instructor as a teacher?

Name	Resp	Mean
Overall	38	3.60
Differential Calculus	30	3.33
Multivariate Calculus	8	3.88

§3. University of Chicago

A numerical average of all student responses for all the courses I taught at UChicago is attached below. All of the evaluations in full are available on request.

Chowdhury, Subhadip

Course Evaluations

Key of Values

5 - strongly agree
4 - agree
3 - neutral
2 - disagree
1 - strongly disagree

#1: Instructor was organized
#2: Lectures were clear and understandable
#3: Lectures were interesting
#4: instructor exhibited a positive attitude towards the students
#5: Instructor was accessible outside of class
#6: I would recommend this instructor to others

Quarter	Course Number	Section	Instructor (Last)	Instructor (First)	#Eval	#Stud	#1	#2	#3	#4	#5	#6	Overall
Winter 2018	19520	59	Chowdhury	Subhadip	9	10	4.38	4.25	4.00	4.75	4.43	4.38	4.36
Autumn 2017	19520	41	Chowdhury	Subhadip	20	27	3.90	4.10	3.63	4.50	4.56	4.05	4.12
Winter 2017	15300	45	Chowdhury	Subhadip	18	20	3.61	3.56	3.44	3.67	4.28	3.28	3.64
Autumn 2016	15200	45	Chowdhury	Subhadip	28	28	3.79	3.75	3.57	4.00	4.56	3.57	3.87
Spring 2016	13300	22	Chowdhury	Subhadip	6	6	4.17	4.33	3.67	4.50	4.67	4.17	4.25
Winter 2016	15300	48	Chowdhury	Subhaddip	7	8	4.43	4.43	4.43	5.00	5.00	4.71	4.67
Autumn 2015	15200	45	Chowdhury	Subhadip	14	16	3.50	3.64	4.21	4.36	4.50	4.00	4.04
Spring 2015	15300	41	Chowdhury	Subhadip	7	7	4.43	4.29	4.29	5.00	4.86	4.57	4.57
Winter 2015	15200	41	Chowdhury	Subhadip	23	27	3.70	3.61	3.35	4.00	4.50	3.43	3.76
Autumn 2014	15100	41	Chowdhury	Subhadip	22	23	3.32	3.32	3.32	4.18	4.67	3.55	3.72

Chapter 3

Future Teaching Goals

Building upon my current experience, I have many specific ideas that I plan to incorporate into my role as a future instructor.

- My current SoTL interests concern how to utilize alternate assessment techniques for proof-based courses. I recently gave an invited talk in the Project NExT session o n ''Re-imagining Grading: The Why and the How" at JMM 2022 on using ''Techniques Grading in an IBL-stylr Intro to Proofs Course".
- I would like to learn and implement the discovery learning method and experiment with bolder pedagogical ideas.
- I intend to create my own theoretical Math summer internship programs or engage in existing programs that relate specifically to Applied methods and research experiences.
- I intend to develop and introduce new forms of assignments (e.g. videos or experimental) consistent with modern advancement in technology and welcome any opportunity of employing digital and computational tools to enrich my teaching.
- I plan to engage in outreach programs for incoming freshmen for a successful transition to college life, Math Circles for K-12 students, professional networking between different communities, etc.

Finally, by bringing in a wide variety of perspectives, I hope to impact and get support from my peers in designing approaches towards broader, more widely applicable, and more memorable learning.

Chapter 4

Professional Development Activities

§1. Pedagogy Conference

During the past few years, I have had the opportunity to attend several pedagogy conferences and workshops organized by *Chicago Center for Teaching*, *Center for Learning and Teaching* at Bowdoin and Bates College, the *Five Colleges of Ohio* consortium, and the *MAA mentoring network*.

In the summer of 2021 and 2022, I attended conferences on Mastery-Based Grading that helped me transition to alternate grading methods in my classroom that encourages productive failure and reward perseverance. Earlier in Spring 2022, I attended the *Project NExT* session on ''Reimagining Grading: The Whys and Hows'' and presented a talk on my use of a variant Spec-based grading model implemented in my Intro to Proof class.

These conferences have taught me about the ever-changing role of Math professors in the face of modern technological advances and how the teaching process has evolved to be relevant in the modern day and age. I also learned about new ways of encouraging active learning and ways to encourage proactive student involvement.

§2. Faculty Mentoring Cohorts

Over the academic years 2018-2021, I have been part of various faculty mentoring cohorts which involved professors from different departments visiting each others' classes and giving constructive feedback in an effort to gain new insights into teaching and students' learning. Although the process was not evaluative, it facilitated a reflective conversation with my colleagues regarding evidence of student learning and new techniques of student engagement.

The Math department at the College of Wooster and Bowdoin college organized weekly teaching seminars where we share our methods, coordinate reciprocal classroom visits, and get advice on handling unexpected issues in the classroom from more experienced faculty members. I regularly participated in these and once led a discussion from the *MAA Instructional Practices Guide*. These meetings have been invaluable for my professional development.

Appendices

The appendix section includes a small collection of sample assignments, worksheets, projects, labs, and exams I have written for students over the years. Most of the contents for all of my courses starting in 2018 are available through my webpage and my github page. Here are some direct links to full semester lecture notes for some of my courses:

- Numerical Analysis
- Transition to Advanced Mathematics
- Multivariable Calculus
- Theory of Integral Calculus

Appendix A

Sample Assignments and Projects

MATH 221 - Differential Equations

Project 2: Mathematical Epidemiology 101: The SIR Model and COVID-19

Fall 2020 **Subhadip Chowdhury** **Sep 25-29**

On December 31, 2019, the Chinese city of Wuhan reported an outbreak of a novel coronavirus (**COVID-19**) that has since killed over **979,000** people. As of Sep 23, 2020, over **32,000,000** infections* - spanning **215** countries [1] - have been confirmed by the World Health Organization (WHO). In this project, we try to understand the infectious disease models, exploring how the WHO and other groups are characterizing and forecasting the COVID-19 pandemic.

§A. What is the SIR model?

Epidemic spread can be modeled by a system of differential equations. Such models include the **SIR** model and its variations. The SIR model is a **compartmental disease model** describing the dynamics of infectious diseases. The letters in **SIR** represent the three compartments of the total population:

- **Susceptible** - Susceptible individuals have no immunity to the disease (immunity can come from prior exposure, vaccination, or a mutation that confers resistance). Susceptible individuals can move into the Infectious compartment through contact with an infectious person.
- **Infectious** - The group of infectious represents people who can pass the disease to susceptible people and can recover after a specific period. Note that it does not represent the *infected* people.
- **Removed** - People who recover from the disease get immunity so that they are not susceptible to the same illness anymore. Many SIR-based models assume that a recovered person remains immune (which is often appropriate if immunity is long-lasting, e.g., chicken pox or the disease is being modeled over a relatively short time period). As a matter of convenience, we include the group of people who do not recover but die in the 'Removed' group -- since they too can no longer contract the disease.

We assume that at any given moment, a person must be in exactly one compartment. However, because people can move between compartments, the number of people in each compartment changes over time. The SIR model captures population changes in each compartment with a system of ordinary differential equations (ODEs) to model the progression of a disease.

§B. Derivation of the Model

As the first step in the modeling process, we identify the independent and dependent variables. The independent variable is time t, measured in days. We are going to make the following simplifying assumptions:

Assumption I.

Assume that the total population size $\mathbf{N}(t)$ is a constant. This is reasonable if for example, a city is on lockdown. We also do not consider the effect of the natural death or birth rate because the model assumes the outstanding period of the disease to be much shorter than the average lifetime of a human.

*Interesting (macabre?) Note:** When I wrote this project last semester, the data was as follows: "...has since killed over **170,000** people. As of April 20, 2020, over **2,480,000** infections - spanning **210** countries..."

In our closed population of **N** individuals, say that **S** are susceptible, **I** are infectious, and **R** are recovered. Let

$$s = \frac{\mathbf{S}}{\mathbf{N}}, \quad i = \frac{\mathbf{I}}{\mathbf{N}}, \quad r = \frac{\mathbf{R}}{\mathbf{N}}$$

denote the fraction in each compartment.

Question 1. 1 point

Explain why, at each time t, we have

$$s(t) + i(t) + r(t) = 1. \tag{0}$$

End of Question 1

The complete SIR model is given by the following three dimensional system of ODEs:

$$\frac{ds}{dt} = -\beta s i \tag{1}$$

$$\frac{di}{dt} = \beta s i - \gamma i \tag{2}$$

$$\frac{dr}{dt} = \gamma i \tag{3}$$

$$\underset{s(t)}{\boxed{\textbf{Susceptible}}} \xrightarrow{\beta s i} \underset{i(t)}{\boxed{\textbf{Infectious}}} \xrightarrow{\gamma i} \underset{r(t)}{\boxed{\textbf{Recovered}}}$$

Below we will explain how to derive each of the three equations.

Assumption II.

We assume that the population is well-mixed. This means any infectious individual has a constant probability of contacting any susceptible individual. This is often the most problematic assumption, but is easily relaxed in more complex models by taking averages.

Assumption III.

We assume that the time-rate of change of $\mathbf{S}(t)$, the number of susceptibles, depends on four things:

- the number of individuals currently susceptible,
- the number of individuals currently infectious,
- the amount of contact between susceptibles and infectious, and
- the transmissibility of the disease.

In particular, suppose that each infectious individual has a fixed number of contacts per day and each contact has a fixed probability to transmit the disease. Not all these contacts are with susceptible individuals. Let's assume that on average, each infectious individual generates $\beta s(t)$ new infectious individuals per day. The constant β depends on the last two factors.

■ **Question 2.** **The Susceptible Equation, (1+1+1) points**

The rate of change of **S** over time is given by $\frac{d\mathbf{S}}{dt} = -\beta s(t)\mathbf{I}(t)$. Explain carefully how each term in the differential equation follows from assumptions II and III.

(a) Why is the factor of $\mathbf{I}(t)$ present?

(b) Where did the negative sign come from?

(c) Explain how this leads to the equation (1)

$$\frac{ds}{dt} = -\beta si$$

End of Question 2

Assumption IV.

Infectious individuals are assumed to recover with a constant probability at any time, which translates into a constant per capita recovery rate that we denote with γ. For example, if the average duration of infection is three days, then, on average, one-third of the currently infectious population recovers each day.

■ **Question 3.** **The Recovered Equation, 1 point**

Explain how the corresponding differential equation (3) for $r(t)$,

$$\frac{dr}{dt} = \gamma i$$

follows from the last assumption.

■ **Question 4.** **The Infectious Equation, 1 point**

Explain how we can use equations (1) and (3) together to conclude equation (2)

$$\frac{di}{dt} = \beta si - \gamma i$$

Which assumption about the model did you use to get this?

§C. Determining Outcomes: Graphical Representations

A common goal for modeling is to understand likely outcomes in the short term and the long term. These outcomes may be visualized via solutions to the differential equations, or via the differential equations directly. Before moving forward with further calculation, let's try to make some initial observations directly from the set of equations (1)-(3).

■ Question 5. **3 points**

Below are six graphs from the SIR model for a theoretical outbreak of COVID in a population of **3000** people. At the start of the outbreak, there is one Infectious person, and everyone else is Susceptible. There is one graph for each of the following: $s(t)$, $i(t)$, $r(t)$, ds/dt, di/dt, and dr/dt. Which is which? How do you know? Consider shapes of graphs, values on the vertical axis, and other information you believe to be relevant. In particular, two graphs look nearly identical. One is a derivative graph (ds/dt, di/dt, or dr/dt) and one is a solution graph ($s(t)$, $i(t)$, or $r(t)$). Reason through which graph is which. Explain your logic. **You are not allowed to use any technology for this part.**

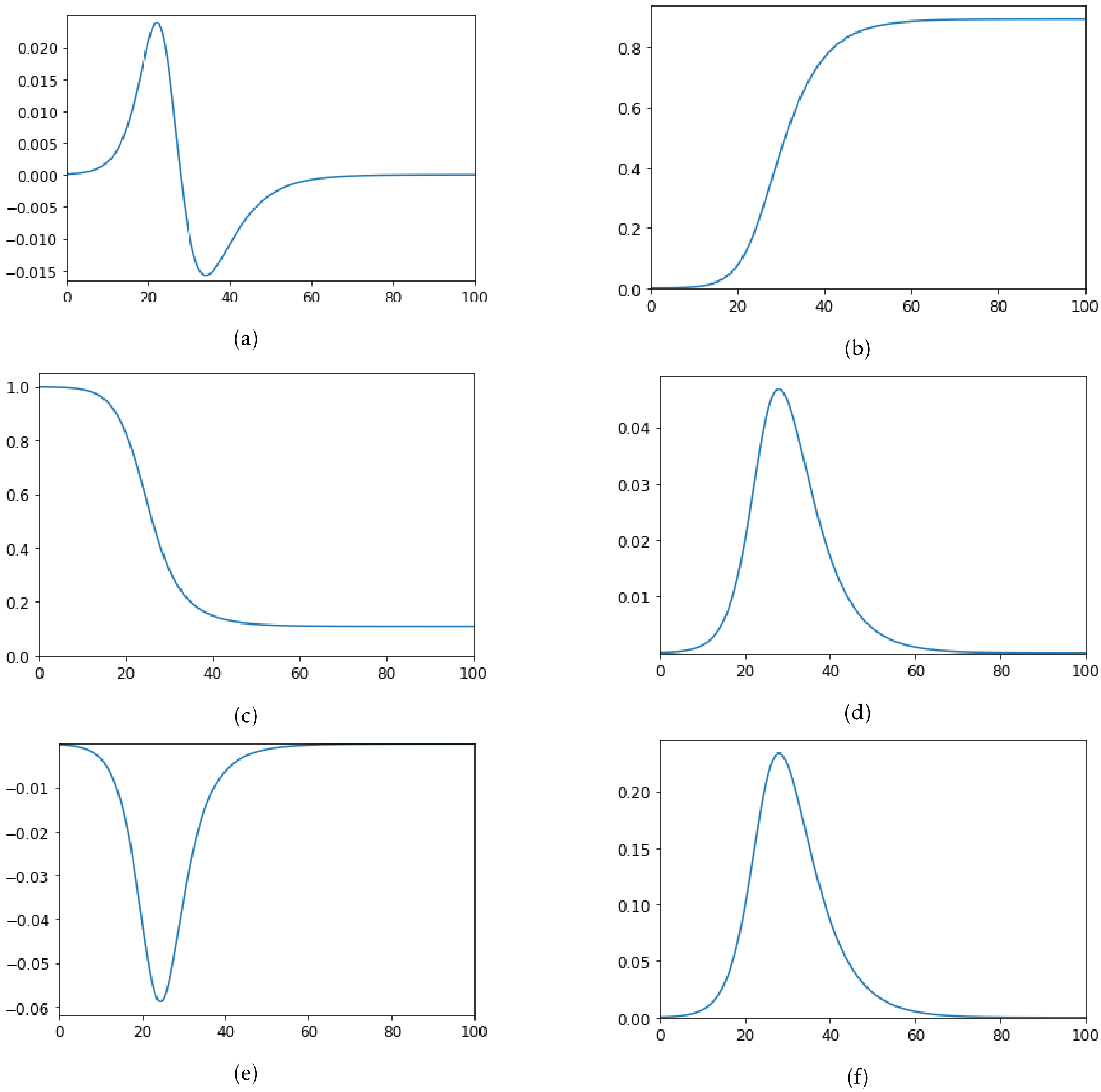


Figure 1

§D. Basic Reproduction Number $\mathbf{R_0}$

Definition D.1

The **basic reproduction number**, $\mathbf{R_0}$, also known as the **contact number**, is defined as the expected number of secondary cases produced by a single infection in a completely susceptible population.

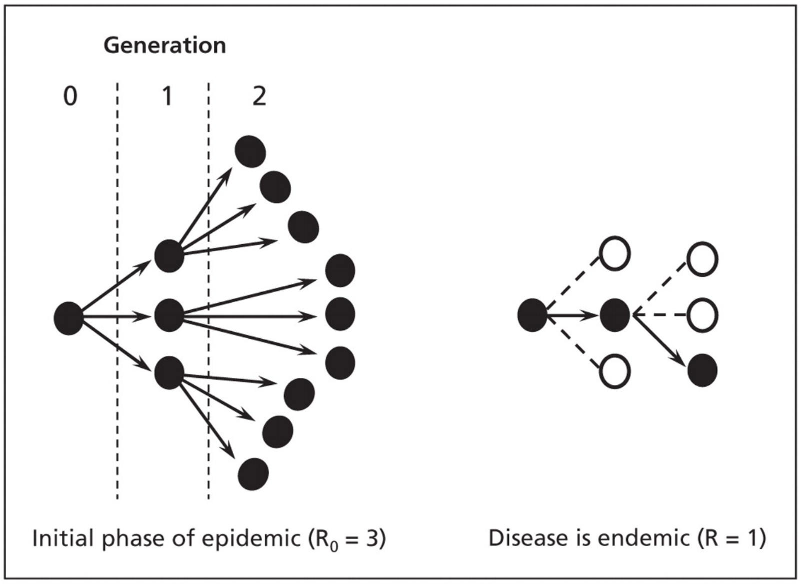


Figure 2: $\mathbf{R_0 > 1}$ is epidemic, $\mathbf{R_0 = 1}$ is endemic

$\mathbf{R_0}$ is a combined characteristic of the population and of the disease, it measures the relative contagiousness of a disease. It is important to understand that $\mathbf{R_0}$ is a dimensionless unitless number and not a rate. We can define

$$
\begin{aligned}
\mathbf{R_0} &= \underbrace{\underbrace{\left(\frac{\text{infection}}{\text{contact}}\right)}_{\text{transmissibility of disease}} \times \underbrace{\left(\frac{\text{contact}}{\text{time}}\right)}_{\text{average rate of contact}}}_{\beta} \times \underbrace{\underbrace{\left(\frac{\text{time}}{\text{infection}}\right)}_{\text{duration of infectiousness}}}_{\frac{1}{\gamma}} \\
&= \beta \times \frac{1}{\gamma}
\end{aligned}
$$

We have used assumption IV here: γ is roughly equal to the reciprocal of the number of days an individual is sick enough to infect others. Although γ is directly observable from patients, there is no direct way to observe β. Fortunately, there is an indirect way of calculating $\mathbf{R_0}$ that doesn't require us to know β beforehand.

§E. Qualitative/Analytical Approach

Note that the first two ODEs in the SIR system can be treated as a **2D** system by themselves, since they do not invlove a r variable. Then once we find $i(t)$ and $s(t)$, we can use that to find $r(t)$ (using equation 0). For a **2D** system, we can use PPLANE to draw the (s,i) phase portrait with s as the horizontal axis and i as the vertical axis. We only need to look at the range $[0,1]\times[0,1]$.[†]

Question 6. (2+2+2) points

For the two-dimensional system

$$\frac{ds}{dt} = -\beta si, \qquad \frac{di}{dt} = \beta si - \gamma i,$$

(a) find the equations of the nullclines and the equilibrium points.

(b) Using the nullclines or otherwise, explain why for an epidemic to occur (i.e. $i(t)$ increases from its initial value) we must have $s(0) > \frac{1}{R_0}$. This number $\frac{1}{R_0}$ is consequently called the **threshold value** of the model.

Question 7. 4 points

Find a differential equation for $\frac{di}{ds}$ from equations (2) and (1). [**Hint:** $\frac{di}{ds} = \left(\frac{di}{dt}\right)\Big/\left(\frac{ds}{dt}\right)$.]

Solve it using separation of variables method and show that $i(s)$ has the general formula

$$i(s) = -s + \frac{\ln s}{R_0} + c \tag{4}$$

where c is some arbitrary constant. This is the equation of the general solution curve in the phase plane!

End of Question 7

Since the initial conditions are $(s(0), i(0)) \approx (1,0)$, we can use equation (4) to write

$$0 \approx -1 + \frac{\ln 1}{R_0} + c \implies c \approx 1$$

Now take limit as $t \to \infty$ on both sides of equation (4) and use the fact that $\lim_{t\to\infty} i = 0$ (why?) to get

$$0 = \lim_{t\to\infty}\left(-s + \frac{\ln s}{R_0} + 1\right) \implies R_0 = \lim_{t\to\infty}\left(\frac{\ln s}{s-1}\right) \tag{5}$$

Thus we can find the numerical value of R_0 by collecting data about $s_\infty = \lim_{t\to\infty} s(t)$ in real-life. For countries who have stabilised their COVID transmissions, this number can be found easily and we can consequently find approximate value of R_0. Unfortunately, in the USA we are not yet in the $t\to\infty$ part of the curve[2]; and so the R_0 value gets frequently updated.

Question 8. 2+1 points

Use $\beta = 0.5, \gamma = 0.2$ in your PPLANE phase portrait. Draw a solution curve that starts approximately near $(1,0)$.

Attach a picture of the phase portrait with the solution curve.

[†] Strictly speaking, we should only look at the region $s + i \le 1$ since $s+i+r=1$.

Use the picture to estimate the value of s_∞. Use this value of s_∞ in the limit in equation (5) to find $\mathbf{R_0}$. Your answer should be approximately equal to $\frac{\beta}{\gamma}$.

End of Question 8

One of the most fascinating observation we can make about s_∞ from the phase portrait is that it's not equal to $\mathbf{0}$. Indeed, there is always a fraction of the population who never get infected! This is one of the fundamental insights of mathematical theory of epidemics.

Question 9. 2 points

Assume $i(0) > 0$ and $\mathbf{R_0} > 1$.[‡] Show that $s_\infty = \lim_{t\to\infty} s(t)$ is strictly larger than $\mathbf{0}$.

[**Hint:** First note that s_∞ has to be between $\mathbf{0}$ and $\mathbf{1}$ (why?), so it's positive. Why can't s_∞ be equal to zero?]

§F. Numerical/Qualitative Approach

You will need to save and attach your Python Output pictures for different questions of this section. So make sure to provide meaningful labels and titles in the pictures.

You can download and use the `ODE_2D_System.ipynb` file from Moodle as a reference. You will need to convert the code to be used for a 3D system. Alternately, you can use `PPLANE`.

Here is a neat trick: Copy all the pictures onto a single page of Word document and convert it into pdf!

Question 10. 6 points

Write a program that draws the graphs of $s(t), i(t)$, and $r(t)$ vs. t for $0 \le t \le 100$, either together or separately. You should use different color for each curve. Use the following values for the parameters and the initial conditions:

- $\beta = 0.5$ and $\gamma = 0.2$
- $s(0) = 2999/3000$ and $i(0) = 1/3000$, i.e. we are assuming that is one in a $\mathbf{3000}$ person is infectious at time $t = 0$.
- $r(0) = 0$

Attach the picture(s). They should look like some of the curves from question 5.

Question 11. (2+2+1 points)

Use your plots to answer the following questions.

(a) What are the long term ($t = 100$) approximate values of $i(t)$ and $s(t)$? How does the long term value of $s(t)$ compare to your answer from question (8)?

(b) What is the maximum value of $i(t)$? Find the approximate value of t when it happens.

End of Question 11

For the rest of this section, we are going to focus our experimentation on the infectious-fraction, $i(t)$, since that function tells us about the progress of the epidemic. We are going to vary the parameters β and γ and find its effect on the solution curve $i(t)$ vs. t.

‡If $\mathbf{R_0} < 1$, on average, an infectious person infects less than one person. I.e. the disease is expected to stop spreading.

■ Question 12. (2+1) points

First let's experiment with changes in β when γ is fixed at 0.2.

(a) Plot the graphs[§] of $i(t)$ with β values $0.5, 0.7, 0.9, \ldots, 1.5$. Describe how changing β affects the graph of $i(t)$.

(b) Explain briefly why the changes you see are reasonable from your intuitive understanding of how β affects the epidemic model.

(c) **(BONUS 2 points)** Modify the program to plot all the graphs for consecutive values of β in a single picture and attach it. All the required code can be found in the Jupyter notebooks for 2D Systems and for one-parameter family of ODEs.

■ Question 13. (2+1+2) points

Now let's experiment with changes in γ when β is fixed to 0.5.

(a) Plot the graphs[¶] of $i(t)$ with γ values $0.2, 0.3, 0.4, 0.5, 0.6$. Describe how these changes affect the graph of $i(t)$.

(b) Explain the changes you see in terms of your intuitive understanding of how γ affects the model.

(c) There is a change in the behavior of the $i(t)$ graph for a certain value of γ in the given range. What is the change? What happens to $\frac{di}{dt}$ and $i(t)$ when γ is bigger than this particular value? Use equation (2) to justify your answer.

(d) **(BONUS 2 points)** Modify the program to plot all the graphs for consecutive values of γ in a single picture and attach it.

§G. **Flattening the curve**

■ Question 14. 4 points

In our planetary response to COVID-19, we have come up with many different ways to reduce the $\mathbf{R_0}$: via **social distancing**, via **quarantining the infectious**, or via **providing better treatment and healthcare**. Explain which of these correspond to changing β and which ones correspond to γ? Describe what the phrase "flatten the curve" means! Which curve are we talking about here? Why is it important to flatten the curve?

End of Question 14

So what happens to our model when we change β? While public health responses can reduce the value of β, a single social event on a college campus weekend may cause an increase in the value of β. Changes in β lead to changes in how many people become sick, and changes in the model provide a fascinating moment for building insight into the relationship between a differential equation and its solution.

As a first adjustment, consider $\beta = \beta(t)$ to be piecewise constant

$$\beta(t) = \begin{cases} 0.5 & \text{for } 0 \le t \le 20, \\ 0.2 & \text{for } t \ge 20 \end{cases}$$

This may represent suddenly changed health policies once authorities realize an outbreak has begun.

[§] You don't need to attach the pictures unless you want the bonus point from (c).

[¶] You don't need to attach the pictures unless you want the bonus point from (d).

Question 15. **3 points**

Determine which graph is which below. The choices are $s(t)$, $i(t)$, $r(t)$, ds/dt, di/dt, and dr/dt.

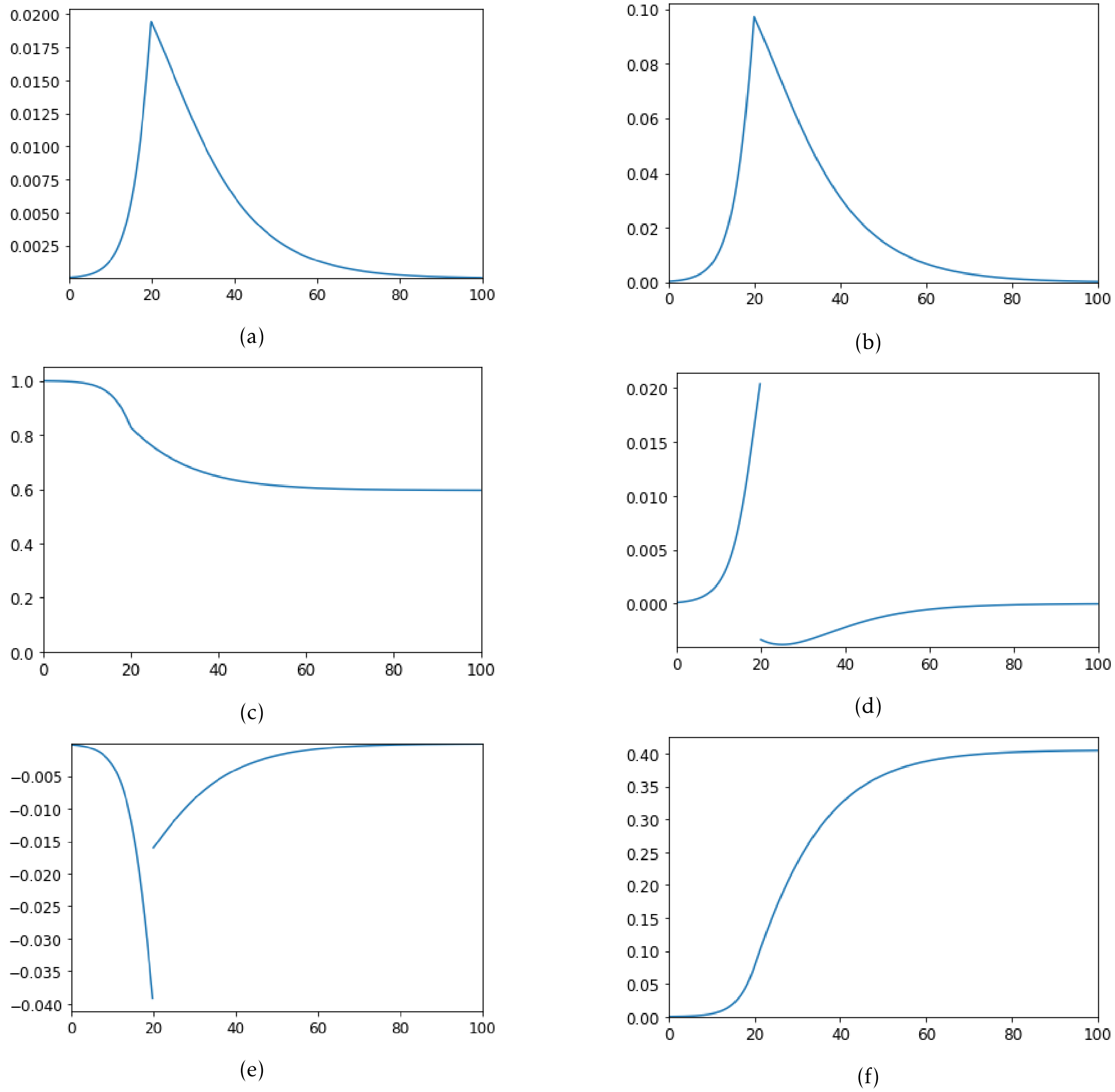


Figure 3

Question 16. **Optional, Bonus 6 points**

Finally, try a more free-form approach.

- What would happen if $\beta(t)$ were constantly decreasing, that is, if $\beta(t)$ were a straight line with negative slope? (Be sure $\beta(t)$ remains nonnegative throughout the time of your outbreak.)
- What would happen if $\beta(t)$ were periodic? Can you construct a periodic function for $\beta(t)$, with a

period of seven days, to represent weekly variation in infectivity? Again, keep $\beta(t) > 0$.

- What other $\beta(t)$ functions could you consider?

In all variations, think through what the graphs of $s(t)$, $i(t)$, $r(t)$, ds/dt, di/dt, and dr/dt should look like. Reason through the answers first, and explain your reasoning carefully. Then, use Python to test your claims by creating the graphs. Ask me if you are unsure how to code variable β in Python.

§G. References

[1] https://www.worldometers.info/coronavirus/

[2] https://covid19.healthdata.org/united-states-of-america

[3] Meredith Greer (2018), "6-007-S-FunctionsAndDerivativesInSIRModels," https://www.simiode.org/resources/4884.

[4] David Smith and Lang Moore, "The SIR Model for Spread of Disease - The Differential Equation Model", https://www.maa.org/press/periodicals/loci/joma/the-sir-model-for-spread-of-disease-the-differential-equation-model

§H. COVID-19 (Optional)

Figure (4) gives an idea of how COVID-19 compares to other infectious diseases. Note that it is extremely hard to estimate $\mathbf{R_0}$ accurately, the picture below only provides a possible upper and lower bound for the value.

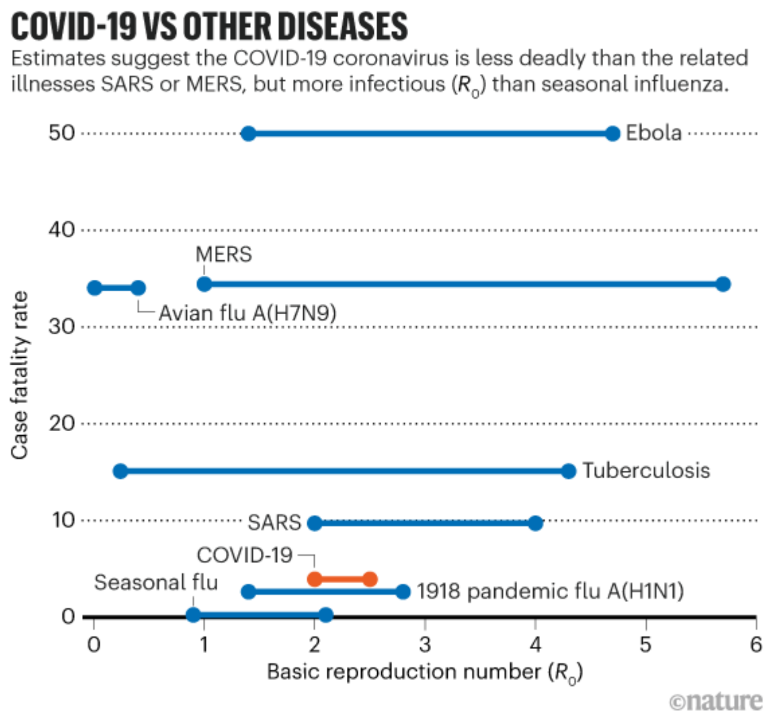


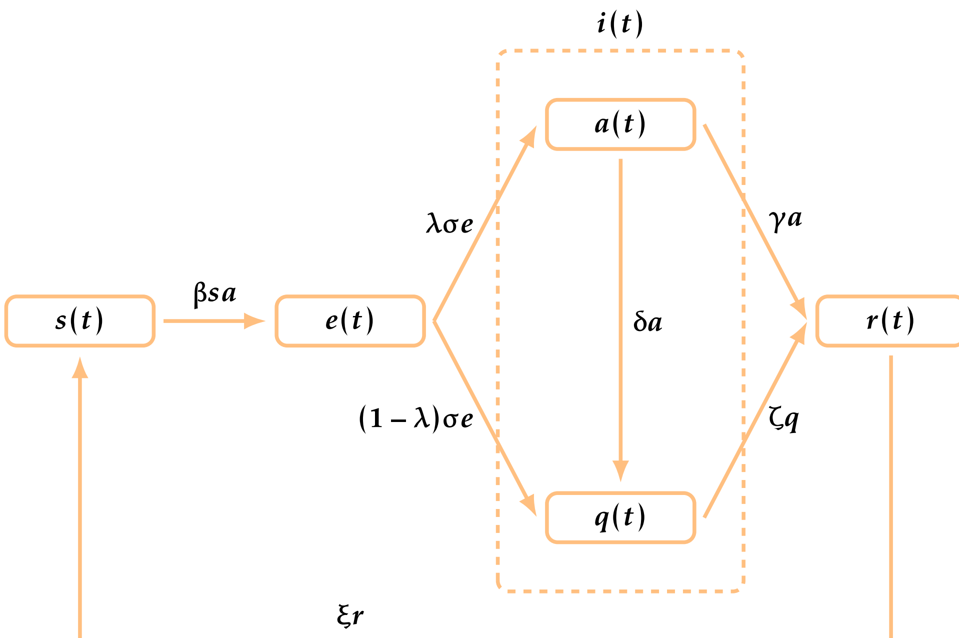
Figure 4: Source: Nature 579, 482-483 (2020)

The SIR model is fairly simplistic in nature and needs modification for complex diseases such as COVID-19. One of the more commonly used is the SEIR model, the E stands for **(Exposed)**. It takes into account the fact that some diseases (e.g. COVID-19) have a latent (or incubation) period, during which individuals have been infected but are not yet infectious themselves (i.e. cannot infect others). Another approach is to use a SIQR model, where the Q stands for **(Quarantined)**. There are also models where β varies over time.

We are going to consider a SEAQRS model. The 'A' stands for **(Asymptomatic)**.

- When someone first contracts the disease they go from S to E.
- Once the incubation period is over, people move from E to I at a rate σ approximately equal to the reciprocal of the duration of incubation.
- We are breaking I in to two parts: people who show their symptoms go to Q, people who don't move to A. People can move from A to Q but not conversely.
- Both I and Q lead to R.
- Some part of R individuals return to S status due to loss of immunity.

I have provided a graphical representation below. There are more underlying assumptions as before that I am not going to list here. You are encouraged to think about what some of those could be to make this model more accurate. I have purposefully made the model a bit over-complicated to give you an idea of what a general model looks like. Some questions that you could consider are:



- what does λ represent?
- why does the arrow between s and e says $\beta s a$?
- why is γ and ζ different? Note that ζ is usually bigger than γ.

Here are some stats for COVID-19. The average incubation period has been approximated to have a median of 5.1 days. So $\sigma \approx \frac{1}{5}$. According to a NYT article, $\lambda \approx \frac{1}{4}$. The majority of individuals that contract COVID-19 resolve symptoms within two weeks, so we can take $\zeta \approx \frac{1}{14}$.

Question 17. Optional, Bonus 4 points

Write down the system of ODEs corresponding to this model.

Question 18. Optional, Bonus 6 points

Use the Python file to draw the $i(t)$ vs. t graph, where $i = a + q$. Discuss how the peak of i changes with respect to β and ζ. Use them to demonstrate the effectiveness of quarantine.

§A Project 1: Fourier Coefficients

A.1 What this project is about

This project introduces Fourier coefficients and Fourier polynomials of simple functions to illustrate an important application of integrals that uses the technique of integration by parts and properties of definite integrals.

A.2 Prerequisites and tech requirements

Before starting this project, you should

- have a solid understanding of Integration by Parts,
- be familiar with the summation (sigma) notation,
- have some knowledge of what the graphs of basic trigonometric functions look like, and
- know how to use `DESMOS` to plot functions.

A.3 Grading criteria

This project will be graded based on the EMPX rubric (see the 'Assessment' document for details). You can check your Moodle gradebook to see your grade and view feedback left by the professor. These appear as text annotations on your PDF submission or as general comments next to the grade. Grades of **E** or **M** may not have much feedback. Grades of **P** or **X** always have feedback, so please look carefully for this.

In order to earn an **E** or **M**, your submission must:

- show all of your work neatly and in a ordered manner.
- back up any claim you make with sufficient proof.
- explain your reasoning in a way that could be understood by a classmate who understands the mathematical concepts but has no familiarity with the particular problem being solved.

In short, readers of your work should not have to fill in any details or guess your thought process.

> **Note:** Completing questions 1001-1004 correctly will earn you an M. You must additionally complete question 1005 correctly to earn an E.

A.4 Project Task

Suppose we have a continuous function $f(x)$ on the interval $[-\pi, \pi]$. Then for each integer $i = 0, 1, 2, \ldots$, we can define the i^{th} **Fourier coefficients** of the function f by the formulae:

$$a_i = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(ix)\,\mathrm{d}x \qquad \text{and} \qquad b_i = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(ix)\,\mathrm{d}x$$

Note that the values a_i and b_i are all constants, not functions. But their values depend on the choice of f.

Definition 0.1.64

We will define the N^{th} **"Fourier polynomial"** of $f(x)$ to be

$$P_N(x) = \frac{a_0}{2} + \sum_{i=1}^{N} [a_i \cos(ix) + b_i \sin(ix)] \qquad \text{for } N = 1, 2, \ldots.$$

Note: The Fourier Polynomial is not actually a polynomial function! It's a misnomer.

Fourier coefficients and polynomials are named for the French mathematician Joseph Fourier. In a speech for the French Academy of Sciences in 1807, **Fourier proposed that these polynomials could be used to approximate arbitrary functions on the interval** $[-\pi, \pi]$. He was led to study these polynomials and make his bold statement by studying heat flow.

Unfortunately, the great French mathematicians of the time did not agree and did not take his ideas seriously. They were wrong. Today, the ideas introduced by Fourier are used to study phenomena that exhibit wavelike (periodic) behavior, such as sound and light. These ideas have been used in physics, engineering, and even economics. This problem is an introduction to some of Fourier's work.

Warning: For the following exercises, any time a problem asks you to plot something in DESMOS, you must include a screenshot of that plot in your final report.

Question 1001.

Let $f(x) = x$. Either using integration by parts, or by using DESMOS *find the first six sets of Fourier coefficients $a_0, b_0, a_1, b_1, \ldots, a_5, b_5$.*

(a) *Do you see any pattern in your answers? Can you guess a general formula for a_i and b_i, in terms of i if necessary.*

(b) *Write down the first three Fourier polynomials P_1, P_2, and P_3, P_4, and P_5 associated to f in expanded form (i.e. not using the sigma notation).*

(c) *Use* DESMOS *to plot the function $f(x)$ and these five Fourier polynomials.*

(d) *What can you conclude from the pictures? Does it look like $P_n(x) \to f(x)$ as n gets larger?*

Question 1002.

Now suppose f is an odd function that is continuous on the interval $[-c, c]$.

(a) *Use the area interpretation of definite integral and the definition of an odd function to explain why*

$$\int_{-c}^{c} f(x)\, dx = 0$$

(b) *If $f(x)$ is an odd function, is $g(x) = f(x)\cos(ix)$ an odd function? Why?*

(c) *If f is any odd function on $[-\pi, \pi]$, what can you say about the Fourier coefficients a_i of f?*

Question 1003.

Next, let's consider the case of even functions that are continuous on an interval $[-c, c]$. After thinking about what happened for the odd functions, what can you conclude about the Fourier coefficients b_i of an even function? Explain.

So far, the domain of our function was chosen to be an interval of the form $[-c, c]$, not the entire real number line. But that's enough if we are working with periodic functions!

Question 1004.

*Why did Fourier think that these "polynomials" would be a good technique for approximating **periodic** functions?*

This process of taking a periodic function and approximating it as a sum of sinusoidal functions is called **Fourier Transform**. File formats such as MP3 and JPEG depend on splitting up a function into its component sine waves! Click here to visit an interactive website where you can get a visual idea of what Fourier transform does.

Consider a periodic function $g(x)$ of period 2π, i.e. it repeats after every 2π. Between $[-\pi, \pi)$, the function is defined as

$$g(x) = \begin{cases} -1, & -\pi \le x < 0 \\ 1, & 0 \le x < \pi \end{cases}$$

This is clearly a discontinuous function. The graph of the function for all Real numbers looks like the black curve in fig. 2 and it repeats on both ends (note that the vertical parts are just for drawing purposes, they are not part of the graph, since a graph cannot have a vertical line it). We will call it the **Square Wave Function**.

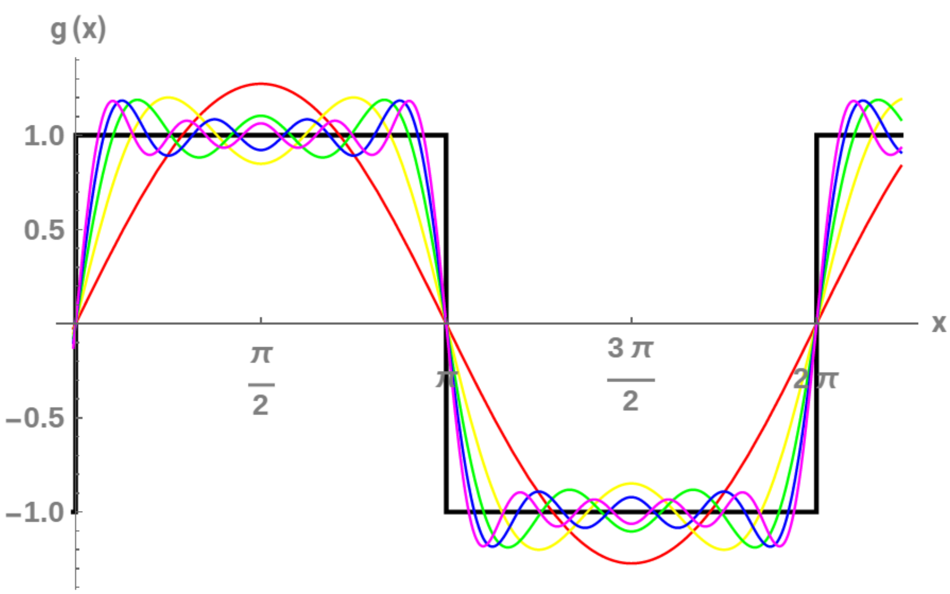


Figure 2: Graph of $g(x)$ and its approximations by Fourier polynomials

If you have checked out the website linked above, you know that this function can be approximated as a sum of sinusoidal functions. The approximations are the Fourier Polynomials! Let's find out mathematically what the coefficients are.

■ Question 1005.

□

Let $g(x)$ be the square wave function.

(a) *Find a formula for the i^{th} Fourier coefficients a_i and b_i for $g(x)$, in terms of i. Note that $g(x)$ is an odd function, so your work is essentially halved.*

(b) *Plot the fifth Fourier polynomial P_5 associated to g using* DESMOS*. It should match one of the colored curves in* *fig. 2*.

HINT: *P_5 for g should be sum of three sine functions.*

(c) *What happens when we plot higher order Fourier polynomials? Do you believe they will approximate the Square Wave function better?*

Theory of Differential Calculus

Application/Extension Problem 1

Fall 2021 **Subhadip Chowdhury** **Math 115**

§A. What this AEP is about

In mathematics and computing, a **root-finding algorithm** is an algorithm for finding roots of continuous functions. A **root** of a function $f(x)$ is a number x such that $f(x) = 0$. In general, if the function $f(x)$ is not overly simple, the roots of $f(x)$ cannot be computed exactly, nor expressed in closed form. So we might ask, how does your calculator (or a computer) find roots of functions or solutions of equations in general?

Below we will describe two such algorithms your calculator might use to provide **approximate answer** when it solves for a root. Note that most root-finding algorithms apriori assumes that a root exists. The algorithm itself does not guarantee that it will find any or all the roots. In particular, if such an algorithm does not find any root, that does not necessarily mean that a root does not exist!

§B. Prerequisites and tech requirements

Before starting this AEP, you should

- know what a continuous function looks like,
- know what it means to have a root of a function,
- know the conclusion of Intermediate Value Theorem and how it is applied,
- understand the interpretation of derivative as slope of tangent,
- know how to find equation of the tangent line to a given curve $y = f(x)$ at a point $(a, f(a))$.
- be comfortable using `DESMOS` to evaluate a function repeatedly at different values.

Please come talk to me if you are not comfortable with any of these topics.

§C. Submission Instructions and Grading criteria

To submit this AEP:

- Create a handwritten or typed document with your solution. Convert the document/picture of the document to JPG or PDF format using an app or software. Please do not submit MS Word documents.
- Upload the file to the appropriate AEP assignment link on Moodle.

This AEP will be graded based on the EMPX rubric. You can check your Moodle gradebook to see your grade and and view feedback left by the professor. These appear as text annotations on your PDF submission or as general comments next to the grade. Grades of **E** or **M** may not have much feedback. Grades of **P** or **X** always have feedback, so please look carefully for this.

In order to earn an **E** or **M**, your submission must:

- show all of your work neatly and in a ordered manner.
- back up any claim you make with sufficient proof.

- explain your reasoning in a way that could be understood by a classmate who understands the mathematical concepts but has no familiarity with the particular problem being solved.

In short, readers of your work should not have to fill in any details or guess your thought process.

> **Important Note.** There will be one chance to revise and resubmit this AEP within a week from when it is first graded and returned to you.

§D. Learning Targets covered

Besides regular PE credits, getting an **E** or **M** in this AEP will also earn you an 'S' in learning target L3 (IVT).

§E. AEP Task

Consider the polynomial $P(x) = x^3 + 3x - 1$. Your goal is to find a zero of this function: i.e., a number a so that $P(a) = 0$. Although there is an algebraic technique for finding a zero of a cubic polynomial, we are going to approximate a zero.

GOAL: We want to manually find an approximate value for a within 10^{-2} of the actual value of a.

Feel free to use DESMOS or a calculator throughout this AEP to calculate the value of $P(x)$ as needed. Note however that this AEP is describing how a computer or calculator finds a root of polynomial. So if you just copy the root from DESMOS, that defeats the purpose of the exercise.

FIRST ALGORITHM: BISECTION METHOD

Question 1. ☐

To begin, show that the equation $P(x) = 0$ has at least one solution in the interval $[-1, 1]$. You must give a good justification that such a solution exists.

Our goal is find the location of the root. One way to approximate the root is to **bisect** the interval $[-1, 1]$ (i.e. break it into two equal halves). Then we can aks which one of those halves is the root in.

Question 2. ☐

(a) *Determine whether $P(x) = 0$ has a solution in $[-1, 0]$ or $[0, 1]$ (as you did in the previous step), and*

(b) *then repeat the process with the new interval containing a solution (i.e. bisect it again).*

Note that each time you bisect the interval, you get an interval half the length of the previous one.

Question 3. ☐

How many times do you need to repeat the bisection process to have a sufficiently accurate (see the goal) answer? Don't just give a number here; write down and show the steps.

What's your final approximation? Your answer should look like a fraction of the form $\frac{a}{b}$, and not like a decimal expansion.

SECOND ALGORITHM: USING DERIVATIVE

Calculus gives us another way to perform the search. We are going to use the interpretation of derivative as the slope of tangent at a point.

Question 4. ☐

(a) *Let $y = f(x)$ be a function of x. What is the slope of of the line L tangent to the graph of f at $(x_1, f(x_1))$?*

(b) *What is the equation of the line L?*

(c) *Suppose the tangent line L intersects the x–axis at x_2. Find x_2 in terms of x_1 and f. See fig. 1 for a picture.*

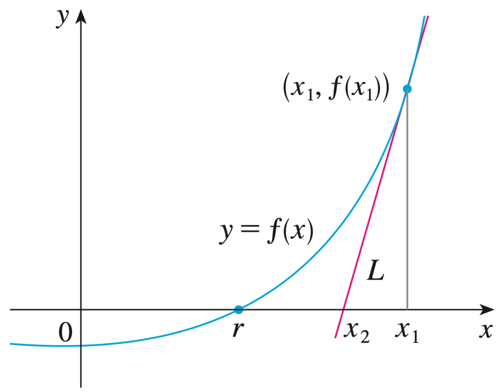


Figure 1

The main idea behind the second algorithm is that this new x-intercept will typically be a better approximation to the function's root than the original guess, and the method can be iterated.

Let's return to the original problem of $P(x) = x^3 + 3x - 1$. We will begin with one of the end points of the original interval; this is your original guess a.

Let $x_1 = 1$.

Question 5.

(a) *Apply the operation in the previous step, obtaining the x-intercept of the tangent line at* $(a, P(a))$ *as a new guess, which hopefully is a better approximation to a solution of the equation* $x^3 + 3x - 1 = 0$ *than the end point you started with.*

What is x_2*?*

(b) *Is this answer within the desired margin of error (i.e. within* 10^{-2}*) from the answer you obtained using the bisection method?*

(c) *When you get an answer within the margin of error you may stop. Otherwise, repeat the operation, this time beginning with your latest guess* x_2 *as the new a and find* $x_3, x_4, \ldots$ *so on.*

Comparison

Question 6.

Compare the two techniques for finding a solution.

(a) *Which is easier to understand in your opinion? Why?*

(b) *Which is faster; that is, which leads to an answer within the desired degree of accuracy in the fewest number of iterations?*

MATH 2000 PROJECT 5: MARKOV CHAINS, THE PERRON-FROBENIUS THEOREM AND GOOGLE'S PAGERANK ALGORITHM*

Subhadip Chowdhury

- **Purpose:** To analyze Markov chains and investigate steady state vectors.
- **Prerequisite:** Eigenvalues and eigenvectors.
- **Resources:** Use Mathematica as needed. You might also want to take a look at http://setosa.io/ev/markov-chains/ and http://setosa.io/markov/index.html

Web Surfing

Definition 1. A **Stochastic matrix** (aka Markov Matrix) is a square matrix, all of whose entries are between 0 and 1 (inclusive), and such that the entries in each column add up to 1.

We can think of the matrix entries as probabilities of different events happening. For example, consider the matrix

$$A = \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.4 & 0.2 \\ 0.1 & 0.5 & 0.6 \end{bmatrix}.$$

(Note that the entries in each column add up to 1.) We can use this matrix to make a *very very* simple model of the internet, as follows.

Pretend there are only 3 domains on the internet: reddit.com, google.com, and instagram.com (Hereafter referred to as Domains 1, 2, and 3, respectively.) In every five-minute period, say, the people surfing this web have some probability of switching to another domain. For example, let's say that, in a given five-minute period, out of all the people clicking around on Domain 1, 70% will remain on Domain 1, 20% will end up clicking on a link to Domain 2, and 10% will end up on Domain 3. Notice that these are precisely the numbers in the first column of A. I.e., the first column encodes what happens to the people surfing Domain 1. Similarly, the second and third columns tells the probabilities of what happens to the people on Domain 2 and Domain 3, respectively.

Another way to say the same thing: if we identify matrix entries in the standard way, where a_{ij} represents the entry in row i and column j, then a_{ij} here is the probability that someone surfing Domain j will end up on Domain i five minutes from now.

*Most of this project is made using or copied from Lay's Linear Algebra book and Interactive Linear Algebra by Dan Margalit, Joseph Rabinoff.

Definition 2. The matrix A is called the *Transition Matrix* of this system and the columns of A are called the *Transition Probability Vectors*.

By our definition, the transition matrix is a stochastic matrix. We will see more examples of stochastic matrices later.

Exercise 1

Draw a graph illustrating this situation: make a vertex(node) for each of Domain 1, 2, and 3, and draw arrows between the nodes, each labelled with the probability of moving from one bubble to the next. You might want to check out the links above for some pretty neat animations.

Suppose initially, at time $t = 0$, **50%** of the surfers are on Domain **1**, **30%** are on Domain **2**, and **20%** are on Domain **3**. Encode this by the vector

$$\vec{\mathbf{x}}_0 = \begin{bmatrix} 0.5 \\ 0.3 \\ 0.2 \end{bmatrix}.$$

Then $\vec{\mathbf{x}}_1 = A\vec{\mathbf{x}}_0$ tells us what proportion of the surfers are on each domain after one time increment. And $\vec{\mathbf{x}}_2 = A\vec{\mathbf{x}}_1$ tells us where the surfers are after two time increments. And so on.

Exercise 2

Compute $\vec{\mathbf{x}}_1$ and $\vec{\mathbf{x}}_2$.

Note that for each of $\vec{\mathbf{x}}_0$, $\vec{\mathbf{x}}_1$, and $\vec{\mathbf{x}}_2$, the entries add up to 1. This makes sense for $\vec{\mathbf{x}}_0$, since its entries are probabilities covering all the cases. But it's not so obvious for $\vec{\mathbf{x}}_1$ and $\vec{\mathbf{x}}_2$.

Exercise 3

Show in general that, given a Markov matrix M and a vector $\vec{\mathbf{v}}$ whose entries add up to 1, the entries of $M\vec{\mathbf{v}}$ also add up to 1.

Definition 3. The sequence of vectors $\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \vec{\mathbf{x}}_3, \ldots, \vec{\mathbf{x}}_n, \ldots$ is called a **Markov chain**.

It lets us track the evolution of this system, seeing where the populations end up over time. However, in order to find something like $\vec{\mathbf{x}}_{100}$, we would need to compute A^{100}. In other words, we need eigenstuff and diagonalization!

The first claim is that A (or any Markov matrix) always has an eigenvalue of 1. It's not trivial to find a $\vec{\mathbf{v}}$ such that $A\vec{\mathbf{v}} = \vec{\mathbf{v}}$ in general, but...

Exercise 4

Find an easy nonzero $\vec{\mathbf{v}}$ such that $A^T\vec{\mathbf{v}} = \vec{\mathbf{v}}$. This shows that 1 is an eigenvalue of A^T.
Hint: use the fact that, since we've transposed, the entries in each *row* of A^T add up to 1.

Recall that A and A^T have the same eigenvalues. Thus $\mathbf{1}$ is an eigenvalue of A as well.

Exercise 5

(a) Find the characteristic polynomial of A, and use the fact that we already know $(\lambda - 1)$ will appear in the factorization to find the other eigenvalue(s) of A.

(b) Find the eigenspace for each eigenvalue of A, and write down a nice eigenbasis for $\mathbb{R}^3$.

(c) Write down the diagonalization $A = BDB^{-1}$.

(d) Compute A^{100} and $\vec{\mathbf{x}}_{100}$.

(e) Find $\lim\limits_{n\to\infty} \vec{\mathbf{x}}_n$.

(f) In the long-term, what percentage of surfers end up on reddit.com, what percentage end up on google.com, and what percentage end up on instagram.com?

(g) Did it matter here what our particular initial distribution $\vec{\mathbf{x}}_0$ was? If $\mathbf{100\%}$ started at reddit.com, would we still end up with the same percentages on each domain over the long term?

Steady State and the Perron-Frobenius Theorem

The eigenvalues of stochastic matrices have very special properties.

Proposition 1. *Let A be a stochastic matrix. Then:*

(a) $\mathbf{1}$ *is an eigenvalue of* A.

(b) If λ is a (real or complex) eigenvalue of A, then $|\lambda| \leq 1$.

As we observed in the last section in exercise $\mathbf{4}$, we can prove that $\mathbf{1}$ is always an eigenvalue of a stochastic matrix. Let's prove the second part. We will restrict to the case of real eigenvalues for the sake of this project.

Exercise 6

1. Let λ be any real eigenvalue of A. Explain why we can always find a vector $\vec{\mathbf{x}}$ such that $A^T\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$.

2. Let $\vec{\mathbf{x}} = [x_1 \quad x_2 \quad \dots \quad x_n]^T$. Choose x_j with the largest absolute value, so that $|x_i| \le |x_j|$ for all i. Explain the steps in the following chain of inequality

$$|\lambda| \cdot |x_j| = \left|\sum_{i=1}^{n} a_{ij}x_i\right| \le \sum_{i=1}^{n} (a_{ij} \cdot |x_i|) \le \left(\sum_{i=1}^{n} a_{ij}\right) \cdot |x_j| = 1 \cdot |x_j|$$

Hence we can conclude that $|\lambda| \le 1$.

Definition 4. We say that a matrix A is *positive* if all of its entries are positive numbers.

For a positive stochastic matrix A, one can show that if $\lambda \ne 1$ is a (real or complex) eigenvalue of A, then $|\lambda| < 1$. The $1-$eigenspace E_1 of a stochastic matrix is very important.

Definition 5. If A is a stochastic matrix, then a *steady-state vector* (or equilibrium vector) for A is a probability vector $\vec{\mathbf{q}}$ such that

$$A\vec{\mathbf{q}} = \vec{\mathbf{q}}$$

In other words, it is an eigenvector $\vec{\mathbf{q}}$ of A with eigenvalue 1, such that the entries are positive and sum to 1.

The Perron-Frobenius theorem describes the long-term behavior of such a process represented by a stochastic matrix. Its proof is complicated and is beyond the scope of this project.

Theorem 2 (Perron-Frobenius Theorem)**.** *Let A be a positive stochastic matrix. Then A admits a unique steady state vector $\vec{\mathbf{q}}$, which spans the $1-$eigenspace E_1. Further, if $\vec{\mathbf{x}}_0$ is any initial state and $\vec{\mathbf{x}}_{k+1} = A\vec{\mathbf{x}}_k$ then the Markov chain $\{\vec{\mathbf{x}}_k\}$ converges to $\vec{\mathbf{q}}$ as $k \to \infty$.*

Why is this nontrivial? For two reasons:

- Apriori, we did not know whether all the entries of the eigenvector corresponding to the eigenvalue 1 are positive. We also did not know about the geometric multiplicity of the eigenvalue 1. P-F theorem tells us that in fact, $\dim(E_1) = 1$ and we can find a vector $\vec{\mathbf{q}} \in E_1$ such that all entries of $\vec{\mathbf{q}}$ are positive and sum to 1!

- If a Markov process has a positive transition matrix, the process will converge to *the* steady state $\vec{\mathbf{q}}$ *regardless of the initial state.*

We say that a sequence of vectors $\{\vec{\mathbf{x}}_k\}$ converges to a vector $\vec{\mathbf{q}}$ as $k \to \infty$ if the entries in $\vec{\mathbf{x}}_k$ can be made as close as desired to the corresponding entries in $\vec{\mathbf{q}}$ by taking k sufficiently large.

Exercise 7

Let $A = \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$ be a stochastic matrix.

(a) Find the eigenvalues and corresponding eigenvectors of A.

(b) Using the eigenvector corresponding to the eigenvalue 1, find the steady-state vector $\vec{\mathbf{q}}$ of A.

Let's try to give a visual interpretation of the linear transformation defined by the matrix above. This matrix A is diagonalizable; we have $A = CDC^{-1}$ for

$$C = \begin{pmatrix} 7 & -1 & 1 \\ 6 & 0 & -3 \\ 5 & 1 & 2 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -.2 & 0 \\ 0 & 0 & .1 \end{pmatrix}$$

The matrix D leaves the $x-$coordinate unchanged, scales the $y-$ coordinate by $-1/5$, and scales the $z-$coordinate by $1/10$. Repeated multiplication by D makes the $y-$ and $z-$coordinates very small, so it *sucks all vectors into the $x-$axis.*

The matrix A does the same thing as D, but with respect to the coordinate system defined by the columns $\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \vec{\mathbf{u}}_3$ of C. This means that A *sucks all vectors into the $1-$eigenspace*, without changing the sum of the entries of the vectors.

Google's PageRank Algorithm

In 1996, Larry Page and Sergey Brin invented a way to rank pages by importance. They founded Google based on their algorithm. Here is how it works (Roughly). Each web page has an associated importance, or *rank*. This is a positive number. If a page P links to n other pages $Q_1, Q_2, \ldots, Q_n$, then each page Q_i inherits $\frac{1}{n}$ of P's importance.

Definition 6. Consider an Internet with n pages. The Rank matrix is the $n \times n$ matrix A whose i, j-entry is the importance that page j passes to page i.

Observe that the rank matrix is a stochastic matrix, assuming every page contains a link: if page i has m links, $m \leq n$, then the ith column contains the number $\frac{1}{m}$, a total of m times, and the number zero in the other entries.

The goal is to find the steady-state rank vector of this Rank matrix. We would like to use the Perron-Frobenius theorem to find the rank vector. Unfortunately, the Rank Matrix is not always a positive stochastic matrix.

Here is Page and Brin's solution. First we fix the rank matrix by replacing each zero column with a column of $\frac{1}{n}$s, where n is the number of pages.

So for example,

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{becomes} \quad A' = \begin{pmatrix} 0 & 0 & 1/3 \\ 0 & 0 & 1/3 \\ 1 & 1 & 1/3 \end{pmatrix}$$

The modified Rank Matrix A' is always stochastic.

Now we choose a number p in $(0,1)$, called the damping factor. (A typical value is $p = 0.15$.)

Definition 7 (The Google Matrix)**.** Let A be the Rank Matrix for an Internet with n pages, and let A' be the modified Rank Matrix. The *Google Matrix* is the matrix

$$G = (1-p)\cdot A' + p \cdot B \quad \text{where} \quad B = \frac{1}{n}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

Exercise 8

Show that G is a positive stochastic matrix.

If we declare that the ranks of all of the pages must sum to one, then we find:

Definition 8 (The 25 Billion Dollar Eigenvector)**.** The PageRank vector is the steady state of the Google Matrix.

This exists and has positive entries by the Perron-Frobenius theorem. The hard part is calculating it: in real life, the Google Matrix has zillions of rows.

Transition to Advanced Mathematics

Expository Paper

Fall 2022 | **Subhadip Chowdhury** | **Math 215**

Writing an expository paper is the biggest writing assignment of Math 215 course, and I want you to choose something that interests you. You will be presenting a self contained discussion of a mathematical topic, with your peers as the intended audience, including at least one proof involving appropriate sophistication, complete with an annotated bibliography.

You will be given the opportunity to conduct a thorough drafting process for this paper by participating in peer reviews. That is, each of you will help your classmates by reviewing your final drafts leading up to final paper submission. I will also help you throughout the process by having regular check-ins.

I have included some project ideas at the end of this document, the topics were chosen to include areas that we do not normally get to discuss in traditional mathematics courses. This is by no means an exhaustive list. If you have an idea for a topic that is not on the list, please discuss it with me. Note that you may not choose a topic on which you have previously written a paper for another course or experience.

§A. Learning Objectives

- Find online or offline resources to research on a particular topic.
- Distinguish academic, peer-reviewed, and authentic references from incomplete or misleading ones.
- Read a proof beyond the level of introductory Mathematics and understand the notations and the logic behind it.
- Show that you understand the proof by explaining it in your own words, on paper and verbally through class presentations.
- Explain how your chosen topic relates to other ideas in 'advanced' mathematics and put it in context.
- Give constructive feedback to your peers' presentations and incorporate others' suggestion into your own work.

§B. The Timeline

Step 1. Topic Choice

You have an option of either **working on one topic by yourself** or **working together in a group of two**. Note that if you are working as a group of two, your group will need to complete two papers – however, you can work together freely during the research, writing, and presentations. **In either case, no topic may be repeated, and as such, topics will be assigned on a first-come, first-served basis.**

On **Monday, Oct 17** I will post a sign-up link in Moodle. Please indicate your choice by choosing the topic you wish to write about. If you wish to add/change a topic, discuss it with me beforehand, and then choose 'other' from the list. *If you do not choose a topic by the end of that week, you will fail the course.*

Step 2. Summary and Discussion

You will need to have a short meeting (5-8 minutes) with me between **Monday, Oct 17** and **Friday, Oct 28** to discuss your topic. *You should come prepared to this meeting with at least one source, and a few ideas* about what you would like to include in your paper. Ideally, You should prepare about half a page summary of what your plans are before the meeting, but you do not need to submit anything in Moodle.

> **Note:** Wikipedia is *not* a valid source, but there is often a list of valid sources at the bottom of an article. You should find a book or peer-reviewed journal as a reference before the meeting.

Step 3. Outline and Annotated Bibliography

The next step is to submit an LaTeX outline of your paper along with an annotated bibliography. **In your outline, you need to highlight the main theorem you wish to prove in your paper.** You can change your choice later, but talk to me first if you plan to do so. The deadline for this step is **Friday, Nov 4**.

The entries in an annotated bibliography give all of the bibliographic information about the book, article, or webpage, as well as a brief description of the source.

- Your bibliography must include at least two sources, *at least one of which should be an academic journal or (text)book.*
- In order to write your annotated bibliography, you will have to do more than just find your sources. You will also have to describe how and why you used your source.

Below is an example of an entry in an annotated bibliography:

> Laura Taalman. *Taking Sudoku Seriously.* Math Horizons, pages 5--7, September 2007.
>
> This is an introductory article that gives an overview of the Sudoku puzzle, complete with concise terminology and the rules of the puzzle. It discusses the number of possible Sudoku boards and puzzles, as well as remaining open questions and generalizations of Sudoku. This article provides most of the introductory material needed for the paper.

Step 4. Full First Draft and Presentation

A full first draft (doesn't need to be final version, can have grammatical mistakes etc.) is due by the end of the Thanksgiving week, **Friday, Nov 25**. In the week following that, you will be required to give a short (5 min) presentation of your topic in front of your peers.

What will be the format of the presentation? You should create a slideshow or screen recording (e.g. on a tablet) of **at most 5 min length** that will be played during class. You must upload this video recording (or a link to the video) to the "EP Presentations" channel in our class Team on MS Teams. If you record on Teams, your video is saved on MS Stream, get that link.

Make sure that your video is playable after you upload it. If you believe your video needs subtitles, you should add them. If you record on Teams, subtitles are automatically generated.

What to Include in your presentation? You may include in your presentation any compelling/relevant material. You will not have time to give all of the details of a proof. So only provide an outline or important key ideas.

The presentations should be understandable to the class who have never seen the material before, so majority of your time should spent on describing the terminology. This may require pictures or tables. See the rubric to check what to focus on.

If creating a slideshow, keep the number of slides low, and make sure your slides are not too dense with information. Your audience should learn about your topic mostly by listening, not by reading the slides. However, there should be still enough keywords in the slides to follow the train of thought. As a general rule of thumb, avoid writing full English sentences in your slides and only write phrases.

One of your primary goals is to engage your audience! It may be the case that you do not get to talk about the proof of the main question at all. Instead, you should give enough context to the problem so that the audience finds it something to care or be excited about.

Step 5. Peer Review

During the presentation week, you will be required to review the draft and presentation of two of your peers. Obviously, for you to give and receive meaningful feedback, you will need to have a full first draft of your own paper submitted before the start of the presentation week.

Since your partners will rely on your feedback in order to improve their papers, you will need to complete your review of their papers in a timely manner. The deadline for this step is **Wednesday, Dec 7, 5PM EST**.

You will be provided an assessment form for each of your partners' papers you are reviewing. You do not need to assign a numerical score. Note that you are not grading or being graded by your classmates. Instead, I will be grading how meaningful your feedbacks to your partners are. You will receive full credit for this phase of the expository paper if you provide a meaningful review of your peers' paper during the workshop.

Step 6. Response to Peer Review and Final Submission

The last phase of the EP writing process is to submit the final version of your EP after you have made any necessary adjustments according to any feedback you received.

Your submission packet will consist of the following files:

- Your responses to the peer reviews you received.
- The final version of your EP.

The deadline for this step is **Wednesday, December 14, 5PM EST**.

§C. Grading

The grading for the final presentation and the expository paper will be according to the rubric on the following page.

MATH 215: Presentation Grading Rubric

Mathematical Content	**/(2+2)**
• Did the speaker demonstrate adequate understanding of the content? (2=adequate, 1=marginal, 0=unsatisfactory) • Was the amount and sophistication of content presented appropriate for the task? (2=adequate, 1=marginal, 0=unsatisfactory)	
Presentation Style	**/(1+1+1+1+1+1)**
• Was the voice of the speaker of appropriate volume and clear? (1=yes, 0=no) • Was the pace of the presentation appropriate? (1=yes, 0=no) • Did the speaker make good use of the board and/or prudent use of media/slides? (1=yes, 0=no) • Was the time allotted for the presentation used judiciously? (1=yes, 0=no) • Did the speaker demonstrate sufficient preparation and practice for the presentation? (1=yes, 0=no) • Did the speaker engage appropriately with the audience? (1=yes, 0=no)	
Clarity and Organization	**/(1+1+1+1)**
• Was there a clear overall organization to the presentation? (1=yes, 0=no) • Were sufficient and clear examples given when appropriate? (1=yes, 0=no) • Was sufficient motivation for the mathematics given when appropriate? (1=yes, 0=no) • Were the explanations of terminology and the statement of theorems clearly presented as appropriate? (1=yes, 0=no)	
Total	**/14**

MATH 215: Final Paper Grading Rubric

<table>
<tr><th>Introduction</th><th>/(2+2)</th></tr>
<tr><td colspan="2">• Is the topic introduced in a clear and compelling manner? (2=adequate, 1=marginal, 0=unsatisfactory)
• Does the introduction provide a logical framework for the paper? (2=adequate, 1=marginal, 0=unsatisfactory)</td></tr>
<tr><th>Background</th><th>/(3+3)</th></tr>
<tr><td colspan="2">• Is an appropriate amount of background content (definitions, terminology, lemmas etc.) included for the reader to understand the paper? (3=strong, 2=adequate, 1=marginal, 0=unsatisfactory)
• Are all the mathematical notation and terminology defined correctly and explained clearly? (3=strong, 2=adequate, 1=marginal, 0=unsatisfactory)</td></tr>
<tr><th>Proof Structure</th><th>/(1+2+1+3)</th></tr>
<tr><td colspan="2">• Does the paper contain at least one proof of a result pertinent to the topic? (1=yes, 0=no)
• Is the result being proven clearly stated? (2=adequate, 1=marginal, 0=unsatisfactory)
• Is the proof prefaced with a brief description of the proof strategy? (1=yes, 0=no)
• Is the proof easy-to-read and written using the correct LaTeX environment? (3=strong, 2=adequate, 1=marginal, 0=unsatisfactory)</td></tr>
<tr><th>Tables/Figures/Diagrams</th><th>/(2+1+1)</th></tr>
<tr><td colspan="2">• Does the paper include sufficient figures, tables, diagrams, equations etc. to make the content clear? (2=adequate, 1=marginal, 0=unsatisfactory)
• Are all the above properly labeled and captioned? (1=yes, 0=no)
• Are all the above properly cited (if not the author's own)? (1=yes, 0=no)</td></tr>
<tr><th>Examples</th><th>/(1+2+1)</th></tr>
<tr><td colspan="2">• Does the paper include at least one example different from those in the author's sources? (1=yes, 0=no)
• Is the choice of examples simple and illuminating enough to make the content clearer? (2=adequate, 1=marginal, 0=unsatisfactory)
• Is the example written using the proper LaTeX environment? (1=yes, 0=no)</td></tr>
<tr><th>Mathematical Content</th><th>/(4+4+4)</th></tr>
<tr><td colspan="2">• Is the mathematical content correct? (4=E, 3=M, 2=P, 1=X, 0=N)
• Does the author demonstrate a clear understanding of the mathematics? (4=exceptional, 3=strong, 2=adequate, 1=marginal, 0=unsatisfactory)
• Is the sophistication of the mathematics discussed appropriately challenging for the course level? (4=exceptional, 3=strong, 2=adequate, 1=marginal, 0=unsatisfactory)</td></tr>
<tr><th>Annotated Bibliography</th><th>/(2+2)</th></tr>
</table>

• Does the paper acknowledge external sources with appropriate citations? (2=adequate, 1=marginal, 0=unsatisfactory) • Does the bibliography have sufficient and appropriate annotations? (2=adequate, 1=marginal, 0=unsatisfactory)	
Writing and organization	**/(3+3+2+2+2+2)**
• Is the purpose of the paper and the topic clear and consistent throughout? (3=strong, 2=adequate, 1=marginal, 0=unsatisfactory) • Is the paper organized in a logical manner that makes it easy to read? (3=strong, 2=adequate, 1=marginal, 0=unsatisfactory) • Are there smooth transitions between paragraphs and sections? (2=adequate, 1=marginal, 0=unsatisfactory) • Does the paper provide some conclusion or applications of the main result? (2=adequate, 1=marginal, 0=unsatisfactory) • Does the paper use correct grammar, punctuation, and spelling? (2=adequate, 1=marginal, 0=unsatisfactory) • Is the paper of the appropriate length (5-7 pages in the provided template, without including tables, figures, bibliography etc.)? (2=adequate, 1=marginal, 0=unsatisfactory)	
Total	**/56**

§D. General Requirements

(a) The expository paper is to be roughly 5-7 pages in length (not including the annotated bibliography) and written for your fellow students in this class. If your paper contains many diagrams, adjust the length accordingly (i.e. the length increases).

(b) The paper has to be typeset in LaTeX. Use 12 pt. font, double-spacing, with margins of 1'' on the top, bottom, right, and left. (The spacing and margins are already set up for you in the LaTeX template that has been posted on Moodle.)

(c) Your paper should be expository, combining lots of mathematical content (not necessarily new mathematics) with some interesting (but minimal) historical notes.

(d) **Follow the guidelines in "Mathematical Writing Practices"** (Appendix C in the lecture notes). For example, any time you define a concept, introduce a variable, state a result, or present a proof, it should be done clearly and typeset so it is easy to find. Be sure to proof-read your paper many times before turning it in!

(e) The paper must contain the following components:

- **A title and an introduction, in which you explain what the paper is about and why the reader should continue reading.**

  For example, this can include some background information on the problem you are studying. Who has studied the problem? Which mathematicians have contributed to the solution? Why is this problem interesting/important? Are there any connections to history, politics, culture? How are the mathematical conventions of other cultures/times relevant? **Be sure to focus on the Math, not the Mathematician.**

  The prompts vary in the type of background information that is most appropriate. I don't expect you to give an exhaustive account of all mathematical connections. I want you to tell me a compelling story. Give me a reason to care about the problem, or demonstrate that some prominent mathematical figures were interested in the math.

- **Appropriate background material, including notation, terminology, and definitions.**

  Remember that this is not talking about historical background. Rather, in this section, you need to introduce and explain the tools you plan to use in your proof.

- **One or more proofs of some results pertinent to the topic, prefaced by a short descriptions of the basic proof strategy.**

  You must give a complete, clear, and understandable proof. **You don't need to come up with the proof on your own;** you should be able to find references that explain the math. But make sure you understand the logic behind the proof and can present it clearly to the audience.

  For some projects it's more obvious what to prove than others. However in some other topics, there are some options as far as which proofs to include. You should look through several references and decide which math is most related to the class/would be best to discuss.

  > **An Important Note:** You will be graded based on whether or not your choice of proof is of an appropriate difficulty level for this class. Talk to me before you begin.

- **Appropriately labeled and captioned mathematical tables/diagrams (at least one), properly cited if not your own.**

- **Appropriate examples (at least one), different from those in your sources.**

  Motivate the math by working out simple and illuminating examples in detail. Try to find examples that demonstrate key aspects of the proof that you will write down abstractly. The abstraction will make more sense if you ground it in concrete numbers.

- **A conclusion.**

  This might include some applications or further reading instructions, generalizations of the results etc.

- **An annotated bibliography**

  Your work should be well cited. See above for details.

(f) Your final submission will include your written response to peer reviews and the final paper. This way it will be clear how you incorporated feedback into the final paper.

§E. EP Topic Ideas

This document is being provided ahead of time so that you can do a little research before deciding on a project! Wikipedia is a good starting place. From there, you should look for any mathematical connection that you could expound upon.

Be careful when choosing a topic, some of these projects are harder than others, and may require additional preliminary ideas from combinatorics, linear algebra, group theory, analysis etc. I will help you with resources and can help you in office hours, but it will be mostly up to you to understand them (think of it as a mini IS). If you are interested, you may choose to work on a variation or only part of the prompt. Please discuss your ideas with me during the 'Summary and Discussion' week if you wish to do so.

Cantor Set and Fractals

Main Objective: How do we construct the Cantor Set? What is a self-similar set? What is the Box-Counting Dimension? How can we use box-counting to define the fractal dimension of the Cantor set? Give some other examples of Fractals and their fractal dimensions.

Further Exploration: What are some interesting properties of the box-counting dimension? What is an Iterated Function System and what does it have to do with fractals?

Lebesgue Measure and Vitali sets

Main Objective: What is Lebesgue measure? What are Vitali sets? How can we use Vitali sets to prove the existence of a nonmeasurable set?

Further Exploration: What is the Banach-Tarski Paradox? Why is it called a paradox? Is it really a paradox? How do we resolve the paradox? Optionally, read this article to get a rough overview of the topic.

Appendix B

Sample Worksheets and Handouts

Calculus & Analytical Geometry II

Lecture 10-11 Worksheet

Spring 2021 **Subhadip Chowdhury** **Math 112**

Just as we can use definite integrals to find area of specific regions, we can also use it to find the volume of three dimensional solids.

It is particularly straightforward when the cross sections of the solids have a consistent shape (e.g. a circle or square). So we will start with a type of solids known as **Solids of Revolution.**

§A. What is a Solid of Revolution

Start with a two dimensional region, e.g. the area under the graph of a function $y = f(x)$. Now revolve it around a straight line, e.g. the x–axis. The three dimensional solid you obtain this way is called a **solid of revolution**. The line around which you rotate is called the **axis of revolution.**

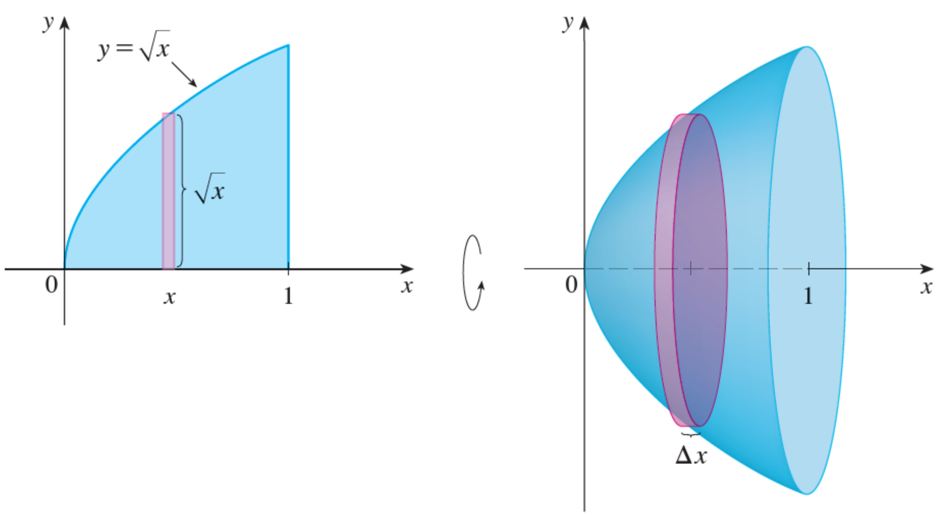
Example A.1

Suppose we wish to calculate the volume of the solid obtained by rotating the region under $y = \sqrt{x}$ about the x–axis for $0 \le x \le 1$.

- **Step 1. Visualize the Solid** Graph the function and the line it is to be rotated about. Label the curves. Make a 3-dimensional sketch of the solid.

  Go to this website to get a 3-D visual.

- **Step 2. Picture a slice** perpendicular to XY-plane. Each slice will produce a disk. Make a rough sketch of what a slice will look like.



(a) (b)

- **Step 3.Express the volume of each slice** as volume of a really thin cylinder. So the volume of the slice is the area of the face ($A(x) = \pi f(x)^2$, because it's a circle!) times its thickness (Δx). But the radius of the circle depends on the function.

In this example, the volume of each slice is

$$A(x)\Delta x = \pi(\sqrt{x})^2 \Delta x$$

- **Step 4: Express the Volume of the Solid as an Integral and Solve.** The volume of the solid is the sum of the volumes of the individual disks. This is a **Riemann sum**, which becomes our integral! The limits of the integral are the boundary values of the variable of integration. Set up and evaluate the integral.

$$V = \int_a^b A(x)\,dx = \underline{\hspace{10cm}}$$

§B. The Disk and the Washer method

The process of calculating volumes of solids of revolutions as in the last example is called the **Disk Method**. Let's try some more examples to solidify (pun intended) the concept!

Question 1.

Calculate the volume of the solid obtained by rotating around the x-axis the region bounded by $y = x^3$, the x-axis, $x = 0$ and $x = 3$.

□

Question 2.

Calculate the volume of the solid obtained by rotating around the x-axis the region bounded by $y = \dfrac{1}{x+1}$, $y = 0$, $x = 0$, and $x = 4$. Give an exact answer.

□

Question 3.

Calculate the volume of the solid obtained by rotating around the x-axis the region bounded by $y = e^{-x} + 1$, $y = 0$, $x = 0$, and $x = 4$. Give an exact answer.

□

Rotating About y-axis

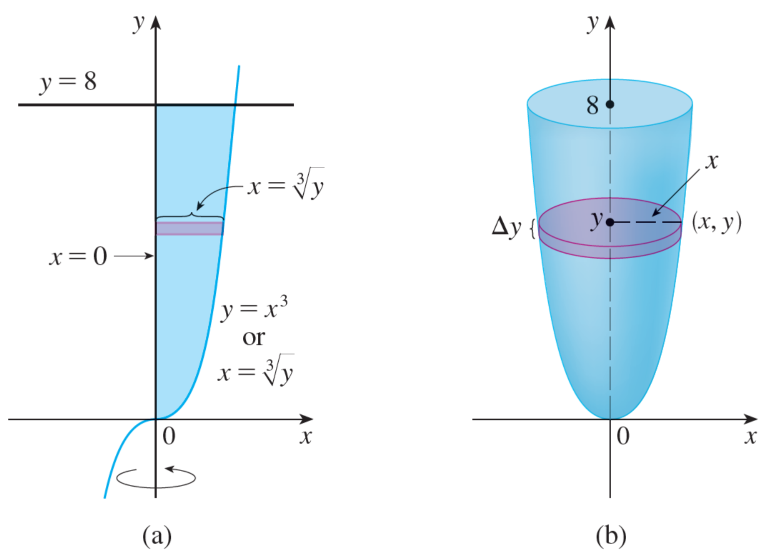
Each of the above problems had you computing the volume of a solid formed by rotating about the x-axis. But we can form a solid by rotating about the y-axis too! Let's work out an example first.

Example B.2

Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.

- **Step 1.** Visualize the solid using the geogebra applet.
- **Step 2.** Slice the solid into disks **perpendicular to the axis of revolution.**

  Note that the radius of the disk is given by the value of x, as a function of y, So we will need to solve for x from the equation $y = x^3$. Thus the radius of the slice pictured below is $x = g(y) = \sqrt[3]{y}$.



- **Step 3.** The volume of the slice is given by $A(y)\Delta y = \pi g(y)^2 \Delta y = \pi (y^{1/3})^2 \Delta y$.
- **Step 4.** The total volume is the limit of the Riemann sum with bounds on y. Note that the upper and lower bound on the integral must be obtained from the picture using algebra. In this case, the lower bound of y is obtained when $x = 0$, i.e. when $y = 0^3 = 0$. The upper bound is given in the problem as $y = 8$. So the integral is

$$V = \int_0^8 \pi y^{2/3}\, dy = \underline{\qquad\qquad\qquad\qquad}$$

Question 4.

☐

Calculate the volume of the solid obtained by rotating about the y-axis the region bounded by $y = \frac{1}{x+1}$, $y = \frac{1}{5}$, and $y = 1$.

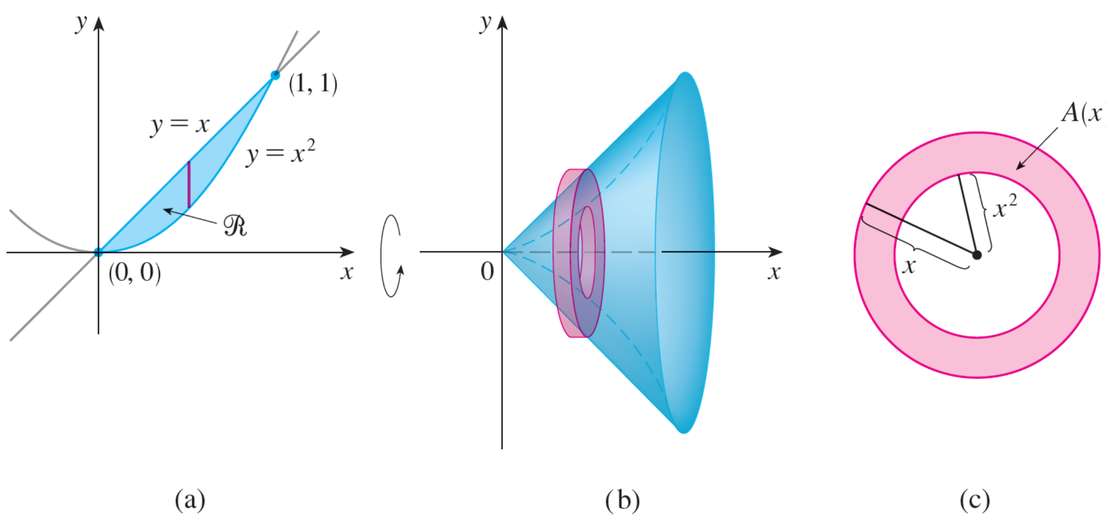
The Washer Method

The washer method is a modified version of the disk method for cases when we have a region bounded by two curves. Let's take a look at an example.

Example B.3

The region $\mathscr{R}$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

- **Step 1.** First Visualize the region $\mathscr{R}$ and the then solid. This time, the solid of revolution looks like a 'container'. It's outer surface is formed by revolving the curve $y = x$, and the inner surface is formed by revolving the curve $y = x^2$.



- **Step 2.** Picture the slices perpendicular to the axis of revolution, in this case the x–axis. The cross-sections no longer look like full disks. Instead they have the shape of a *washer* (an annular ring) with an inner radius x^2 and an outer radius x.

- **Step 3.** The volume of a thin slice is the area of the washer times the thickness. Evidently, the area of washer cross-section by subtracting the area of the inner circle from the area of the outer circle:

$$A(x) = \pi x^2 - \pi(x^2)^2 = \pi(x^2 - x^4)$$

- **Step 4.** The bounds on the integral are obtained from the point of intersection of the two curves $y = x$ and $y = x^2$. Therefore we have

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^2 - x^4)\,dx = \underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

Rotating about other straight lines parallel to the axes

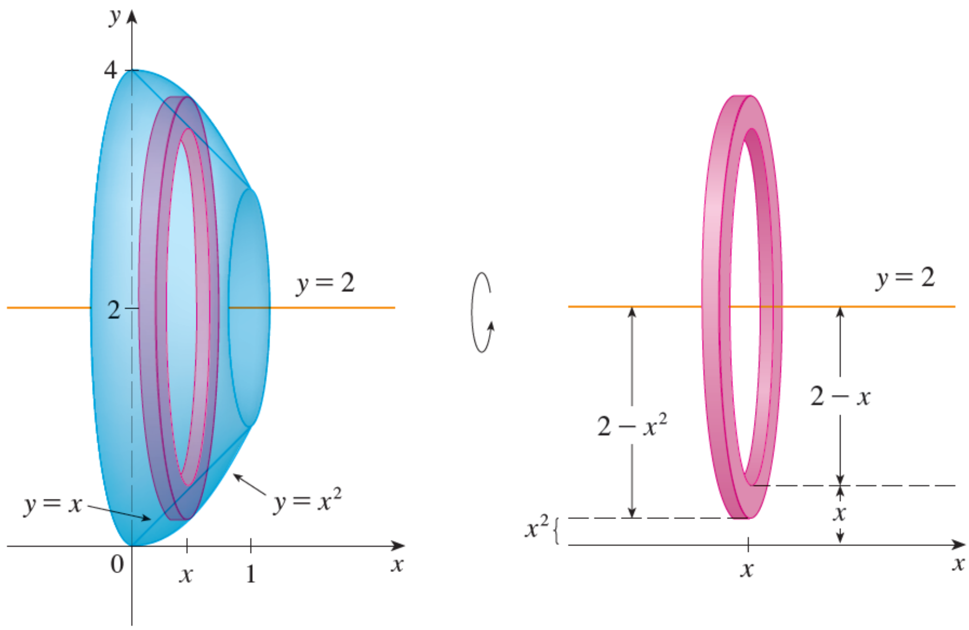
We don't have to necessarily rotate a region about the x-axis or y-axis to get a solid of revolution! Typically, this will result in a solid with a hole in it, thus utilizing the washer method. Give that a try in the next example, where we use the same region as the previous example, but rotate about a different line.

Question 5.

☐

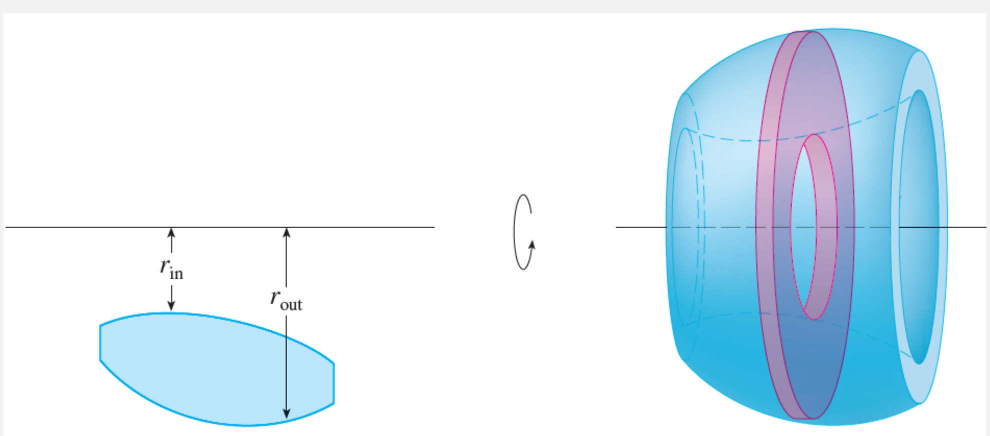
Find the volume of the solid obtained by rotating the region in example 3 about the line $y = 2$.

Here's a picture to help you get started.



Note: In general if the cross-section is a washer, we find the inner radius r_{in} and outer radius r_{out} as the distance from the axis of revolution using a sketch and compute the area of the washer by subtracting the area of the inner disk from the area of the outer disk:

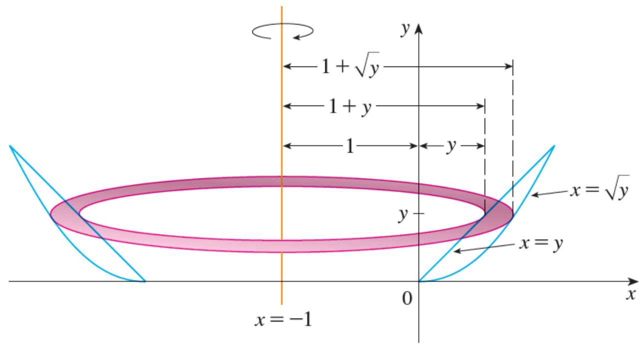
$$A = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2 = \pi\left(r_{out}^2 - r_{in}^2\right)$$



■ Question 6.

☐

Find the volume of the solid obtained by rotating the region in example 3 about the line $x = -1$.



■ Question 7.

☐

Set up the integrals needed to find the volume of the solid generated by revolving the region bounded by $y = 2x^2$, $y = 0$, and $x = 2$ about the given line. After you have set up all four, then you can evaluate (or leave the evaluating until after class, as setting these up is generally the trickiest part!).

(i) the y–axis

(ii) the x–axis

(iii) the line $y = 8$

(iv) the line $x = 2$

■ Question 8.

☐

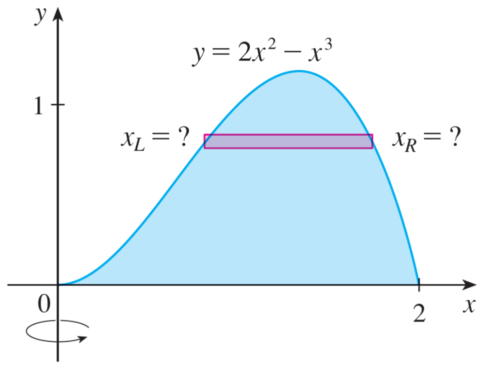
Set up the integrals needed to find the volume of the solid generated by revolving the region bounded by $y = e^x$, $y = 0$, $x = -1$, and $x = 1$ about the given line. Leave evaluating until after class.

(i) the line $x = 1$

(ii) the line $y = -5$

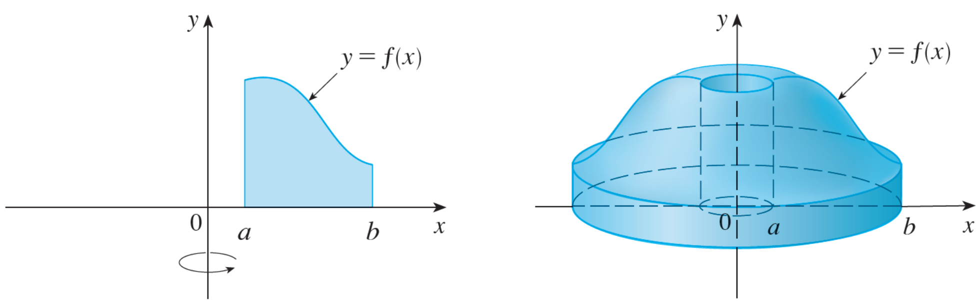
§C. The Shell Method

Some volume problems are very difficult to handle by the disk or washer method. For instance, let's consider the problem of finding the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

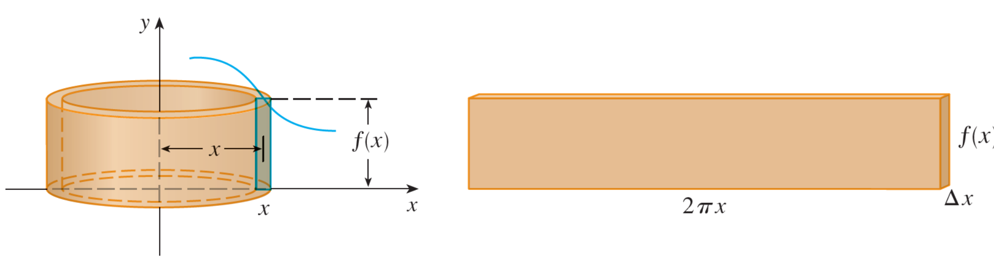


If we slice perpendicular to the y-axis, we get a washer. But to compute the inner radius and the outer radius of the washer, we'd have to solve the cubic equation $y = 2x^2 - x^3$ for x in terms of y; that's not easy. Fortunately, there is a method, called the method of cylindrical shells, that is easier to use in such a case. We will mention it briefly for the sake of completion.

Suppose we wish to find the volume of a solid S obtained by rotating about the y-axis the region bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$, where $b \geq a > 0$.

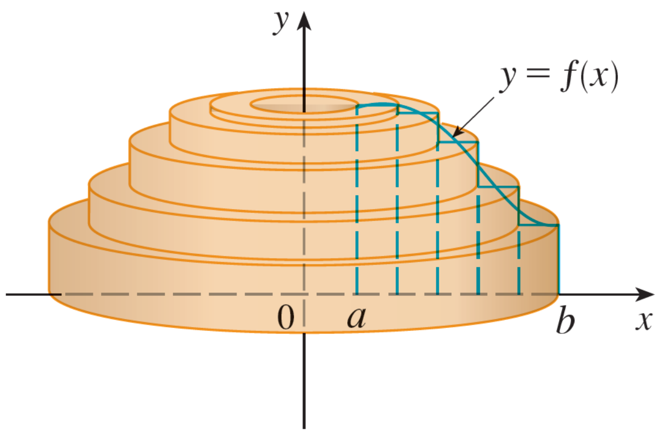


We are going to calculate this by dividing the region into thin rectangles (thickness=Δx) with height $f(x)$ and finding the volume of the cylinder formed by revolving it.



The volume of the cylinder is given by

$$\underbrace{2\pi x}_{\text{circumference}} \cdot \underbrace{f(x)}_{\text{height}} \cdot \underbrace{dx}_{\text{thickness}}$$



Then we sum up all this volumes, take a limit of the Riemann sum and voila! We have an integral formula

$$V = \int_a^b 2\pi x f(x)\,dx$$

Question 9.

Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

Question 10.

Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the y-axis.

Chapter 4 | Relations and Functions

We'll now a make a change in the focus of the course. Up to this point, our focus has been on learning techniques to write proofs. Along the way we've had to learn a few new concepts to have something new to write proofs about. The remainder of our time will be devoted to learning new math concepts, but these are still fundamental to all of mathematics. In this chapter, we focus on **Relations** and **functions**. You may be already familiar with some or both of these notions, but we will explore them in a more abstract and more rigorous way than you have (likely) ever done before. All the while we will, of course, write proofs and continue to focus on improving our writing.

§4.1 Relation on a Set

We first seek to motivate our upcoming definition for a **relation** by means of an example.

Question 188. □

(a) *Let's pretend that you're teaching an elementary school class and you need to teach the students how to properly use the less than symbol "<." To keep things simple, we'll focus entirely on the integers in the set* $A = \{1,5,7,10\}$. *The students need to see some examples, so write down every possible correct use of the "<" symbol for elements in the set* A *(e.g.,* $5 < 7$ *is one example).*

(b) *Repeat the same process for the "=" symbol.*

In the question above, there's nothing particularly special about the "<" and "=" symbols. That is, the symbols themselves don't really have any meaning. It's the elements that we put the symbols between that really matter. To sum up: the only way to distinguish between two relations on a given set is to know an **ordered pair** that belongs to one of the relations but not to the other. Hence, when we talk of **relations**, we are really talking about Cartesian product of **sets**.

Definition 4.1.117

Let A be a set. Then a relation R on the set A is a collection of ordered pairs of elements of A; that is, a subset $R \subseteq A \times A$. If $(a,b) \in R$ we write aRb and say aloud "a is related to b". If $(a,b) \notin R$, then we write $a \not R b$.

Question 189. □

Let $A = \{0,1,2,3\}$ *and consider a relation* $R \subseteq A \times A$ *defined by congruence modulo* 3*:*

$$xRy \iff x \equiv y \pmod 3.$$

Write out the relation R *as a set of ordered pairs.*

Here's a less numerical example.

Example 4.1.118

- Let P denote the set of all people with accounts on Facebook. Define a relation via x $y \iff x$ is friends with y. Then is a relation on P .
- Compare this to the set Q of all people with accounts on Instagram. Define via x $y \iff x$ follows y. Then is a relation on Q.

There is an interesting distinction between the two relations above. Observe that x y automatically implies y x. But x y does not necessarily imply y x.

We can often represent relations using graphs. Given a finite set A and a relation R on A, a **digraph** (short for **directed graph**) is a discrete graph having the members of A as vertices and a directed edge from x to y if and only if $x\mathrm{R}y$.

Example 4.1.119

Figure 4.1 depicts a digraph that represents a relation R given by

$$\mathrm{R} = \{(a,b),(a,c),(b,b),(b,c),(c,d),(c,e),(d,d),(d,a),(e,a)\}.$$

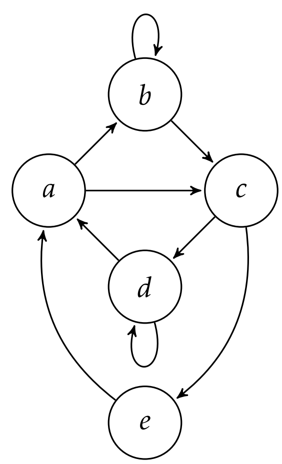


Figure 4.1: An example of a digraph for a relation R

■ Question 190.

Consider the digraph in fig. 4.2 below. Write the sets A *and* R.

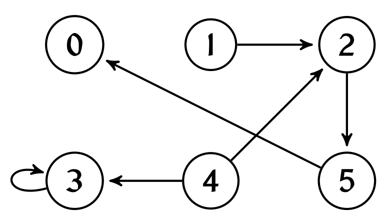


Figure 4.2: Digraph for Question 3

Example 4.1.120

When we write $x^2 + y^2 = 1$, we are implicitly defining a relation. In particular, the relation is the set of ordered pairs (x, y) satisfying $x^2 + y^2 = 1$. In set notation:

$$\{(x, y) : x^2 + y^2 = 1\}.$$

A picture depicting this relation (a set) in $\mathbb{R}^2$ is the standard unit circle.

Example 4.1.121

The "less than" relation $<$ on $A = \mathbb{Z}$ can be written in set-builder notation as:

$$\{(x, y) \in \mathbb{Z} \times \mathbb{Z} : y - x \in \mathbb{N}\}.$$

Note how we write the relation without referencing the symbol '$<$' or '$>$'. We know from our algebra of numbers that, if $x < y$, then $0 < y - x$. Hence, we define the relation using only sets and arithmetic here.

Question 191.

Consider the relation $R = (\mathbb{R} \times \mathbb{R}) \setminus \{(x, x) : x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}$. *What familiar relation on* $\mathbb{R}$ *is this? Explain.*

Chapter 24 | Triple Integrals

Just as we defined single integrals for functions of one variable and double integrals for functions of two variables, so we can define triple integrals for functions of three variables.

§24.1 Volume and Mass as an Iterated Integral

Consider first a 3D solid W that looks like a box and is described by the bounds

$$a \le x \le b, \qquad c \le y \le d, \qquad p \le z \le q$$

Then the triple integral $\iiint\limits_W f(x,y,z)dV$ can be written as an iterated integral

$$\iiint\limits_W f(x,y,z)\,\mathrm{d}V = \int_a^b \int_c^d \int_p^q f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

■ **Question 155.** □

What does the integral $\int_0^1 \int_0^1 \int_0^1 2\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$ *represent?*

(a) *Twice the volume of a cube of side* 1.

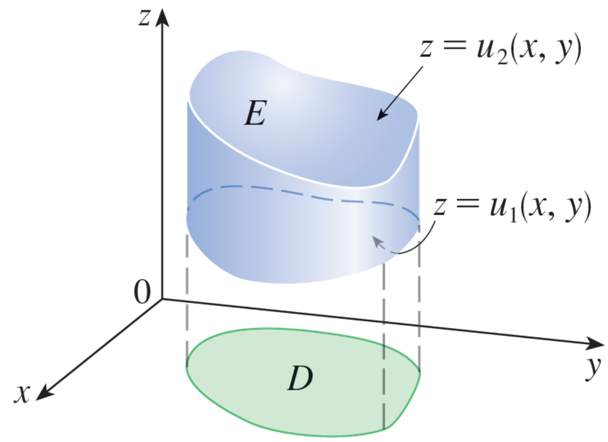
(b) *The volume of a cube of side length* 1.

(c) *Twice the volume of a sphere of radius* 1.

(d) *The volume under the plane* $z = 2$ *and over a square of side length 1 in the xy-plane.*

> **Note:** In general, the volume of a three-dimensional solid W is given by $\iiint\limits_W dV$. If the density at point (x,y,z) is given by $f(x,y,z)$, then it's mass is given by $\iiint\limits_W f(x,y,z)dV$.

Now consider a space region E bounded below by the surface $z = u_1(x,y)$ and above by the surface $z = u_2(x,y)$, and whose 'shadow' (i.e. projection) in the XY-plane is a region D.

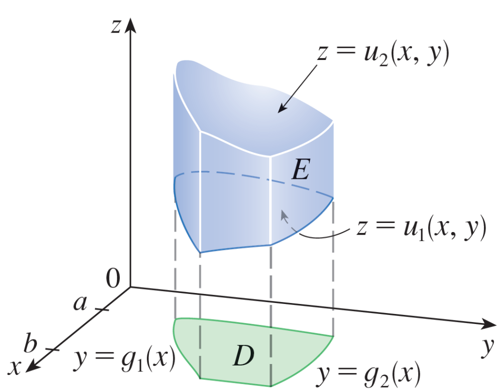


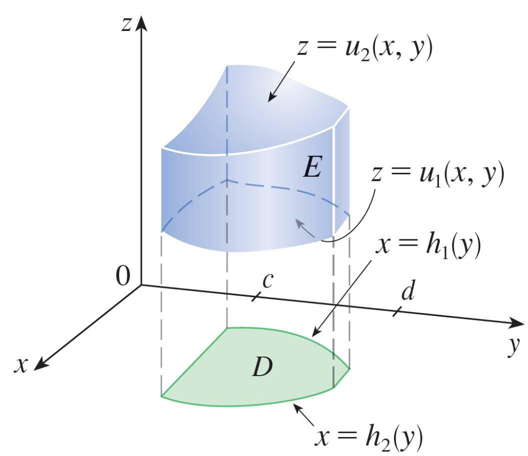
Then similar to the double integrals, the mass of the region E can be evaluated as

$$\iiint\limits_{E} f(x,y,z)\,\mathrm{d}V = \iint\limits_{D}\left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,\mathrm{d}z\right]\mathrm{d}A$$

Question 156.

Now suppose further that the region D *looks like either type I or type II region as defined earlier.*





In each case, rewrite the triple integral as an iterated integral involving dz, dx, *and* dy *in some order.*

> **Note:** Other orders of integration are possible and sometimes necessary to make the integration **feasible**. In general, the rules on the limits on a triple integral are as follows:
>
> - The limits for the outer integral are constants.
> - The limits for the middle integral can involve only one variable (that in the outer integral).
> - The limits for the inner integral can involve two variables (those on the two outer integrals).

Question 157.

For each integral (i) - (iv) that makes sense, match it with its region of integration, I or II.

(i) $\displaystyle\int_1^3\int_{y-1}^2\int_0^y f(x,y,z)\,dz\,dx\,dy$ *(ii)* $\displaystyle\int_1^3\int_0^y\int_2^{y-1} f(x,y,z)\,dx\,dy\,dz$

(iii) $\displaystyle\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$ *(iv)* $\displaystyle\int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dx\,dy$

I. *The region below the plane $z=y$ and above the triangle with vertices $(0,1),(2,1),(2,3)$ in the xy-plane.*

II. *The region between the upper hemisphere of $x^2+y^2+z^2=1$ and the xy-plane.*

§24.2 Changing the Order of Integration - distinct viewpoints

Example 24.2.44

Consider the triple integral $\iiint\limits_E f(x,y,z)\,dV$ where E is the solid in the first octant bounded by the surface $z=2xy$ and the planes $y=x$ and $x=1$.

- Use CalcPlot3D to add a "region" to your picture.
- Change the 2D Top to the function $y=x$ and the Bottom to $y=0$ (because we are in the first octant)
- Change the bound for x to $0\le x\le 1$
- Change the 3D Top to $f(x,y)=2xy$.
- Increase the number of gridline to make your picture smoother.
- Click on the "**Format Axes**" panel and change it to "**Quick View**" to *First Octant*.
- Change the coordinate cutoffs until you get a nice picture.

We will start with a $z-$viewpoint. Rotate the solid until the $z-$axis is coming out of the screen directly towards you. In this viewpoint:

The ceiling has the equation $z=u_2(x,y)=$ ______________________________

The floor has the equation $z=u_1(x,y)=$ ______________________________

Check that the shadows (projection) of the ceiling and the floor onto the $xy-$plane match up perfectly. In fact, the projection of the vertical walls down to the $xy-$plane can be described as a triangle which is in fact both a type I and a type II region.

Written as a type I region, the bounds on the projection are: _____ $\le y\le$ _____ , _____ $\le x\le$ _____ .

Written as a type II region, the bounds on the projection are: _____ $\le x\le$ _____ , _____ $\le y\le$ _____ .

So the triple integral can be written as an iterated integral of the form

______________________ $f(x,y,z)\,dz\,dy\,dx$, and ______________________ $f(x,y,z)\,dz\,dx\,dy$

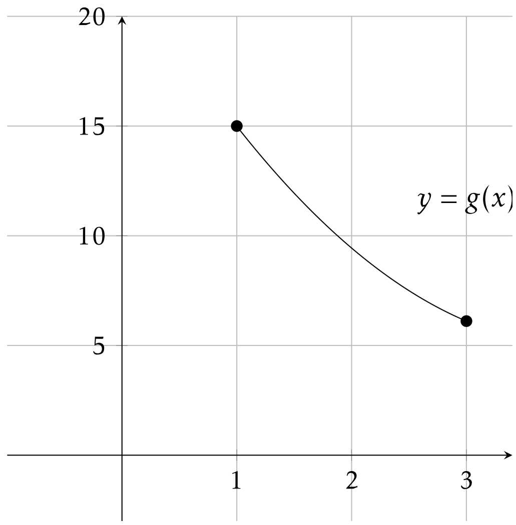
Chapter 5 | Application of Derivatives Part I - MVT and L'Hôpital's Rule

§5.1 Mean Value Theorem

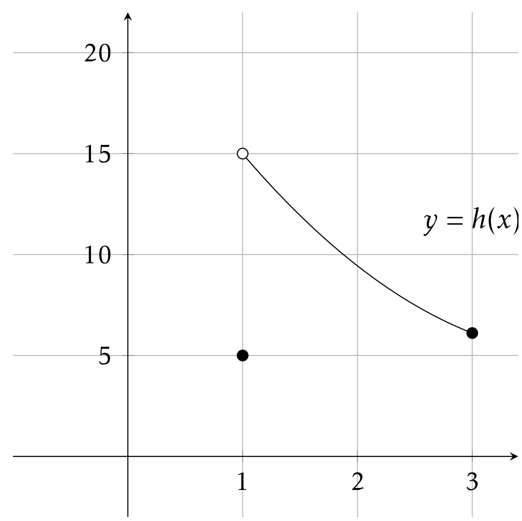
Question 70. ☐

Draw the secant line between the endpoints for the given interval $[a,b]$. Can you identify a point c, with $a < c < b$, such that the slope of the tangent line to the graph at $x = c$ is equal to the slope of the secant line between a and b?

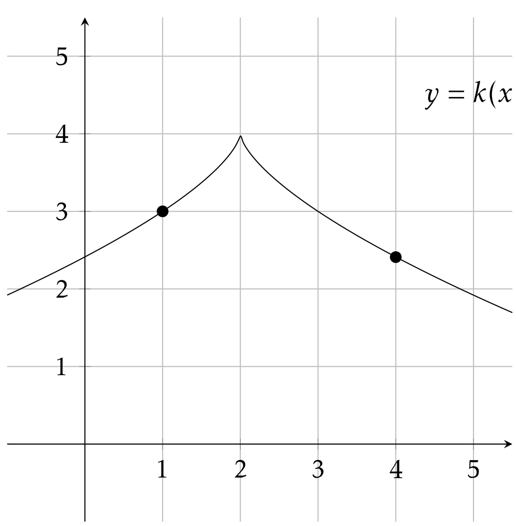
$g(x)$ **on the interval** $[1,3]$



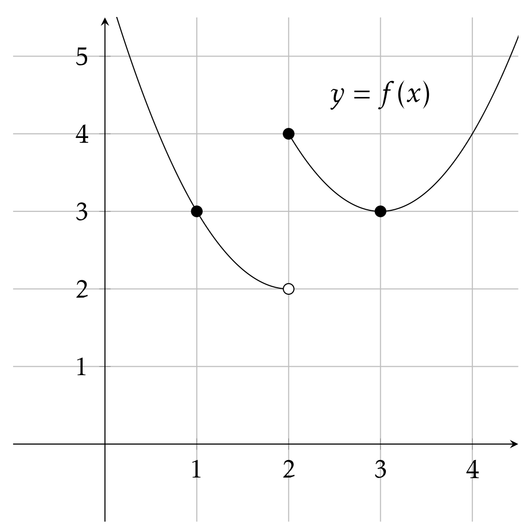
$h(x)$ **on the interval** $[1,3]$



$k(x)$ **on the interval** $[1,4]$



$f(x)$ **on the interval** $[1,3]$



Question 71. ☐

Using your observations from these four cases, make a conjecture regarding when it is possible to find such a point c. In other words, what properties does the function need to have?

Theorem 5.1.42

If f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then there exists at least one number c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

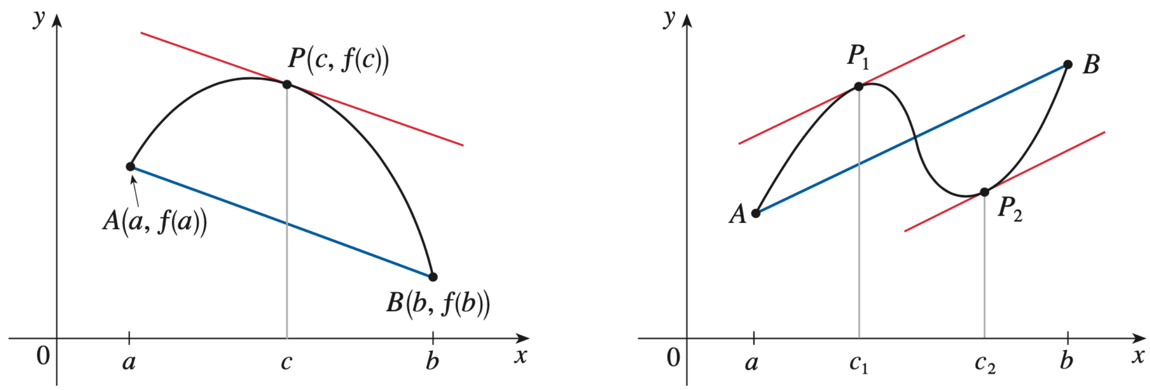


Figure 5.1: The function f attains the slope of the secant between a and b as the derivative at the point(s) $c \in (a,b)$.

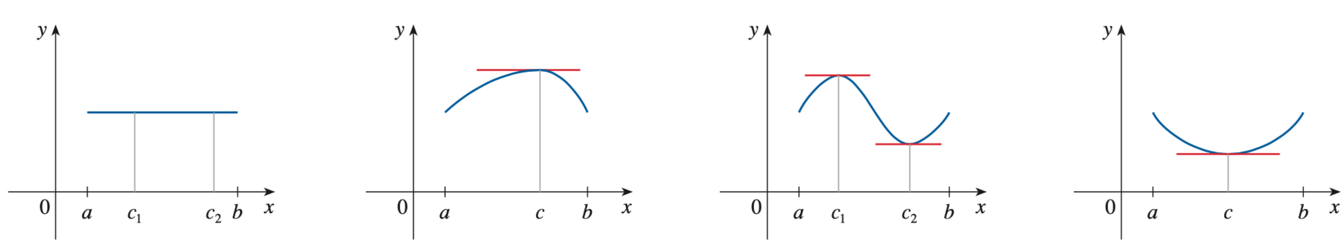
■ Question 72.

An elevator starts at ground level at time $t = 0$ seconds. At $t = 20$ seconds, the elevator has risen 100 feet. What does the Mean Value Theorem tell you about this situation? (Be specific to this case.)

A special case of the Mean Value Theorem is called Rolle's Theorem.

Theorem 5.1.43: Rolle's Theorem

Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). If $f(a) = f(b)$, then there is at least one number c in (a,b) such that $f'(c) = 0$.



Sketch of Proof: The proof of Rolle's theorem follows from the Extreme Value Theorem which says that a continuous function on a closed interval must attain its extremum at some c. This fact along with the fact that local extrema are critical points gives us $f'(c) = 0$ at such points. ■

■ **Question 73.** ☐

Like other Math theorems, the MVT is a "If-Then" statement. There are some hypotheses and there is a conclusion. Can you identify which part is which?

5.1.1 Applications

■ **Question 74.** ☐

Let $g(x) = |x^2 - 1|$. Graph this function using Desmos and answer the questions below.

(a) *Do the hypotheses of the MVT hold on $[0,3]$? Does the conclusion hold? Explain.*

(b) *Do the hypotheses of the MVT hold on $[1,3]$? Does the conclusion hold? Explain.*

(c) *Do the hypotheses of the MVT hold on $[-1,3]$? Does the conclusion hold? Explain.*

Exploration Activity

It is important to think about the chain of logic for a theorem. Let me use the Cats analogy. Consider the statement: If we have a cat, then we have a mammal. Note that the converse isn't true. Just because an animal is a mammal, it doesn't necessarily mean it's a cat. Indeed a dog is also a mammal. So the conclusion can be valid even when the hypothesis isn't. Relating to the problem above, in part (a), the conclusion is valid, even when the hypothesis isn't.

Similarly, when the hypothesis doesn't hold, we can't really say whether the conclusion holds or not. For example, if your animal is not a cat, we do not know if it is a mammal or not, it could be an octopus, or it could be a dog. Relating to the problem above, the hypothesis doesn't hold in both (a) and (c); but for one of them the conclusion holds, for the other it doesn't.

■ **Question 75.** ☐

Does the MVT apply to $g(x) = x^{1/3}$ on $[0,8]$? Why or why not? If so, find all values of c that satisfy the theorem.

■ **Question 76.** ☐

Explain why $h(x) = x^3 + 6x + 2$ satisfies the hypotheses of the MVT on the interval $[-1,3]$. Then find all values of c in $[-1,3]$ guaranteed by the theorem.

■ **Question 77.** **Show a Write-up**

Considering the following situation. You are driving a car on a highway, traveling at the speed limit of 55 mph. At 10 : 17am, you pass a police car on the side of the road, presumably checking for speeders. At 10 : 53am, 39 miles from the first police car, you pass another police car. You are of course obeying the speed limit and traveling exactly 55 mph. However, you are shocked when the police turn on their lights and pull you over. The officer claims you were speeding at some point in the last 39 miles. Is the officer telling the truth, or needlessly pulling you over?

■ **Question 78.** **Show a Write-up**

Let $f(x) = \dfrac{1}{x^2}$. Show analytically why there cannot exist a number c in $(-1,1)$ such that

$$f(1) - f(-1) = 2f'(c).$$

Does this contradict the MVT? Explain.

MATH 221 - Differential Equations

Lecture 41 Worksheet

Fall 2020 **Subhadip Chowdhury** **Nov 20**

Title: Lorenz Equations

Summary: We will see an example of **chaos** in three dimensional ODE systems.

---

§A. Higher-Order Linear Systems

Consider a $n-$dimensional linear system of ODEs of the form $\vec{\mathbf{R}}'(t) = \mathbf{A}\vec{\mathbf{R}}(t)$ where $\mathbf{A}$ is a $n \times n$ matrix whose (i, j)th element is a_{ij} and $\vec{\mathbf{R}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ is a $n \times 1$ vector.

The strategy for finding solutions to the system $\vec{\mathbf{R}}'(t) = \mathbf{A}\vec{\mathbf{R}}(t)$ is the same as for systems of two equations. In particular, if λ is an eigenvalue of $\mathbf{A}$ with eigenvector $\vec{v}$ and $\vec{r}(t) = e^{\lambda t}\vec{v}$, we can still show that $\vec{r}(t)$ is a solution to our system.

It is easy to check that the **Linearity Principle** also holds in higher dimension. So a general solution can be found as a linear combination of linearly independent solutions.

Question 1.

Find the general solution to the system of ODEs

$$\begin{aligned} x' &= -5x - 8y - 2z \\ y' &= 5x + 12y + 4z \\ z' &= -11x - 19y - 5z \end{aligned}$$

You can use `WolframAlpha` to find the eigenvalues and eigenvectors of matrices. Type

```
Eigenvalues[{-5,-8,-2},{5,12,4},{-11,-19,-5}]
```

into the query field.

§B. The Geometry of Solutions

Over the semester, we classified all possible geometry of an equilibrium point for planar systems using the trace-determinant plane. Another way to classify equilibrium points is by their stability.

Definition B.1

If every solution that starts close to an equilibrium stays close to that equilibrium for all time, then the equilibrium solution is called **stable**. Otherwise, it is called **unstable**.

This definition is a bit vague but we can formalize this if needed. Here 'close' means within some neighborhood of bounded radius.

Question 2. □

(a) Identify the types of equilibrium solutions in $\mathbb{R}^2$ that are stable according to the above definition from the list below:

Nodal source, Spiral source, Saddle, Nodal Sink, Spiral Sink, Center

(b) Fill in the blanks with either the word "**stable**" or "**unstable**":

If an equilibrium solution has eigenvalues with real parts that are non-positive, then the equilibrium solution is ________________.

If an equilibrium solution has at least one eigenvalue with a positive real part, then the equilibrium solution is ________________.

(c) It turns out, the conclusion you made in part (b) holds even for systems with more than two dependent variables. For example, if we have a $n \times n$ matrix $\mathbf{A}$ and a n−dimensional system $\vec{\mathbf{R}}' = \mathbf{A}\vec{\mathbf{R}}$, we can compute the eigenvalues of $\mathbf{A}$. By inspecting the real parts of the eigenvalues only, we can determine the stability. Does this make sense, yes or no? ________________.

Although the stability of equilibrium solutions for linear systems in more than two variables is easy to determine, the geometry is a bit more complicated. For a system in three dimension, the solution curves live in $\mathbb{R}^3$, and there is simply a lot more room to move around in three dimensions than in two dimensions! Note that the origin is still the **only** equilibrium solution for a non-degenerate system of linear differential equations in three variables.

§C. Three Dimensional Systems

We want to analyze the so-called Lorenz equations, which is a famous system of differential equations derived by Edward Lorenz when studying convection rolls in the atmosphere,

$$\frac{dx}{dt} = \alpha(y-x) \tag{1}$$
$$\frac{dy}{dt} = x(\rho-z)-y \tag{2}$$
$$\frac{dz}{dt} = -\beta z+xy \tag{3}$$

where α, β and ρ are real parameter values. Note that $(x,y,z)=(0,0,0)\equiv \mathbf{O}$ is an equilibrium solution to this ODE for all parameter values.

$$(x,y,z) = \left(\sqrt{\beta(\rho-1)} \equiv \mathbf{P}, \sqrt{\beta(\rho-1)}, \rho-1\right)$$

and

$$(x,y,z) = \left(-\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}, \rho-1\right) \equiv \mathbf{Q}$$

are equilibrium solutions if $\rho > 1$. There are no other equilibria.

Linearization in three dimensions

Question 3. □

(a) Write down the Jacobian matrix in terms of x, y, z. You don't have to evaluate the Jacobian at any equilibrium solution.

(b) Let $\alpha = 10$, $\rho = 5$ and $\beta = 8/3$. Download and open the file `solvelorenz.ipynb`. This file computes the eigenvalues for the Jacobian at each of the three equilibrium points $\mathbf{O}, \mathbf{P}$ and $\mathbf{Q}$.

Determine the type and stability of each of the three equilibrium points.

Numerical Analysis

(c) The Jupyter notebook also computes a solution to the Lorenz equations from $t = 0$ to 100. The green dot is the initial value, and the red dot is the solution at $t = 100$.

   (i) Consider a solution curve with the initial value $x(0) = 1$, $y(0) = 0, z(0) = 0$. What is (x, y, z) at $t = 100$? Is this consistent with your observation about the stability of the equilibrium points?

   (ii) Consider a solution curve with the initial value $x(0) = 1.000001$, $y(0) = 0, z(0) = 0$. What is (x, y, z) at $t = 100$? Was there a significant change?

(d) Repeat all of the above parts with $\alpha = 10$, $\rho = 28$ and $\beta = 8/3$. Be sure to change the relevant values in the Jupyter notebook. Did you get anything surprising?

What you just observed is exactly what Edward Lorenz did in the 60s. It was something never seen, or even thought of before. The equations are completely deterministic (no statistical variation), yet the solutions can change drastically, even if the initial conditions are changed only slightly. All solutions live on that funny looking butterfly surface, which is now called a strange attractor. This is an example of **Chaos**. Weather systems have this same chaotic property (infinite sensitivity to initial conditions). What does this tell you about long term weather forecasts?

Appendix C

Sample Exams

THEORY OF INTEGRAL CALCULUS

QUIZ 3

Fall 2022 **Subhadip Chowdhury** **Math 125**

Directions

- Answer the learning targets that you need to take and feel ready to take.
- If you have already earned two **'M'** on a Learning Target, do not attempt a problem for that Target!
- You can skip a Target if you need more time to practice with it, and take it on the next checkpoint.
- **Write your full name, legibly,** at the bottom of this page below the Academic Integrity statement.
- Write your answer on this questionnaire in the space provided.
- You are not allowed to use a computer or a calculator or a textbook. You are allowed a one-sided handwritten letter-sized paper note.
- You do not need to simplify your answers any further once you have a numerical expression. For example, $2\pi + 3$ or $\frac{9^3-7}{2}$ is an acceptable solution.
- Unless explicitly stated otherwise, you must **show your work or explain your reasoning** clearly on each item of each problem you do. **Your answer script should not look like scratchwork.**

Responses that consist of only the final answer without any supporting work, or where the work is insufficient or difficult to read, or which have significant gaps or omissions (including parts left blank) will be given a grade of ''N''.

Circle the list of Learning Targets you are attempting in this quiz:

A1, A2, A3, B1, B2, C1, C2, D1

Please write your full name **(this is not a signature, so don't scribble)** legibly to indicate your commitment to the Wooster Ethic and Academic Integrity.

I have neither given nor received any unauthorized aid on this assignment:

. . . .

Do not open the questionnaire until you are instructed to do so.

§ Learning Target A1 (u-sub and by parts Integration)

Requirements. You must correctly complete *both parts of question 1* to get an 'M' on **A1**.

Question 1.

☐

Evaluate the following integrals. If you use u-substitution, you must clearly specify the function you are substituting as u. If you use integration by parts, you must clearly specify which function you are considering as the first function u and which one is the second function dv. Write out explicitly what du and v are.

(a) $\displaystyle\int_0^1 x^2 2^x \,\mathrm{d}x$

(b) $\displaystyle\int \frac{1}{x}\cos(\ln x)\,\mathrm{d}x$

§ Learning Target A2 (Trigonometric Integrals)

Question 2.

☐

Evaluate the following integral: $\boxed{\int \sec^5\theta \tan^3\theta \, d\theta}$. *Show all the steps.*

§ Learning Target A3 (Trigonometric Substitutions)

Question 3.

☐

Evaluate the following integral: $\boxed{\int \frac{1}{x^2\sqrt{9-x^2}} \, dx}$ *Show all the steps.*

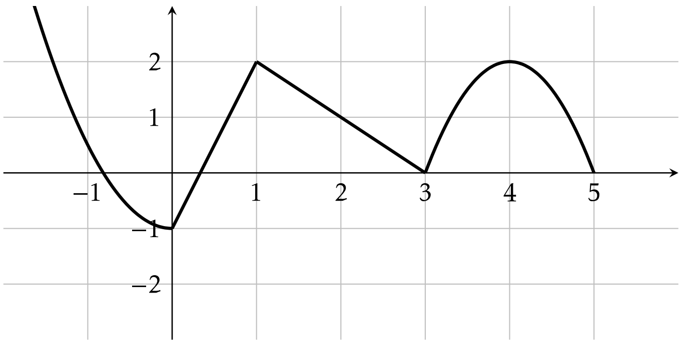
§ Learning Target B1 (Riemann Sum)

Requirements. You must correctly complete *both questions in this section* to get an 'M' on **B1**.

Question 4.

☐

Below is the graph of a continuous function f on the interval $[-2, 4]$.



x	-1	0	1	2	3	4	5
$f(x)$	0.5	-1	2	1	0	2	0

Suppose that we want to approximate the value of $\int_{-1}^{5} f(x)\,dx$ *using the midpoint Riemann sum* M_3.

(a) *Use the graph on the left to draw the rectangles that are used in this approximation.*

(b) *Next, compute the exact numerical value of* M_3.

Question 5.

☐

Given the expression below for R_n, *express the limit of the expression as* $n \to \infty$ *as a definite integral* $\int_a^b f(x)\,dx$.

$$R_n = \frac{1}{n}\sum_{i=1}^{n} \cos\left(3 + \frac{4i}{n}\right)$$

You must correctly identify $a, b,$ and f. You do not need to evaluate the integral.

§ Learning Target B2 (Improper Integrals)

Requirements. You must correctly complete *both parts of question 6* to get an 'M' on **A1**.

Question 6.

☐

Evaluate each of the following integrals. If the integral diverges, say so. Do not use ∞ as a number.

(a) $\displaystyle\int_0^\infty xe^{-x^2}\,\mathrm{d}x$

(b) $\displaystyle\int_1^3 \frac{1}{(x-2)^{1/3}}\,\mathrm{d}x$

§ Learning Target C1 (Sequence Basics)

Requirements. You must correctly complete *both questions in this section* to get an 'M' on **C1**.

■ Question 7. □

Find the first five terms of each of the following sequences. You do not need to show your work, but your answers must be correct.

(a) $a_n = 2(3^n) + 3n$ *where* $n = 1, 2, 3, \ldots$

(b) $b_1 = 2$, $b_n = 2b_{n-1} + 3$ *for* $n \geq 2$.

■ Question 8. □

The four angles of a quadrilateral are in an Arithmetic Progression. If the common difference of the AP is $10°$*, find the four angles.*

§ Learning Target C2 (Limit and Convergence of Sequences)

■ Question 9. □

Determine whether each sequence converges or diverges. If it converges, find the limit.

(a) $a_n = \arctan(e^n)$

(b) $b_n = \ln\left(\dfrac{n+5}{\sqrt{100+n^2}}\right)$

§ Learning Target D1 (Geometric Series)

Question 10.

Identify which one pf the following series are geometric series, circle your choices. For those that are geometric, find the sum, if it exists. If the series diverges, say so.

(i) $1 - \dfrac{5}{3} + \dfrac{25}{9} - \dfrac{125}{27} + \dots$ *(ii)* $\displaystyle\sum_{k=1}^{\infty} \frac{3}{k^3}$ *(iii)* $\displaystyle\sum_{n=1}^{\infty} 2^{2n} \cdot 5^{2-n}$

Mathematical Foundations of Computing

Checkpoint 4

Spring 2022 **Subhadip Chowdhury** **Math 130**

Directions

- Do only the Checkpoint problems that you need to take and feel ready to take. If you have already earned **'M'** on a Learning Target, do not attempt a problem for that Target! You can skip a Target if you need more time to practice with it, and take it on the next checkpoint.
- Note that you cannot get a **'E'** on a checkpoint quiz. Those are only for learning targets covered in homework assignments.
- Do not write on this question paper; do all your work on the provided blank white paper instead. **Ask me for extra paper if needed.** I will bring a stapler to the class to staple your loose answer papers.
- **Please turn in solutions for the questions in order** (for example, do not turn in work for question 2 after work for question 1). The easiest way to do this is to put each learning target on its own page.
- I have enough supply of white paper, so you should not feel an urge to save space. Write spaciously, use the full width to write your sentences and **do not write in more than one column per page.**
- Using a pen with black or blue ink is preferred over a pencil. If possible, clearly mark your final answers with a box around it.
- You will take this question paper back home with you. Do not write anything on it that you wish to be graded.
- Unless explicitly stated otherwise, you must **show your work or explain your reasoning** clearly on each item of each problem you do. **Your answer script should not look like scratch-work.** Responses that consist of only answers with no work shown, or where the work is insufficient or difficult to read, or which have significant gaps or omissions (including parts left blank) will be given a grade of "X".

Please write your full name (not a signature, so don't scribble) **legibly at the top of your answer script (not here)** to acknowledge your commitment to the Wooster Ethic and Academic Integrity.

Write down the list of Learning Targets you are attempting in this checkpoint quiz **at the top of your answer script**. The available learning targets are:

LP4Q, MA1Q, RF1Q, RF2Q, RI1Q, RI2Q

Do not open the questionnaire until you are instructed to do so.

§ Learning Target LP4Q - Proof Techniques

Requirements. You must correctly answer **all questions in this section** to get an 'M' on **LP4Q**.

Question 1.

☐

Consider the following proposition:

$$\text{If } X \text{ and } Y \text{ are subsets of } Z\text{, then either } X \cup Y = \varnothing \text{ or } X \cap Y \neq \varnothing.$$

Construct just the outline of a proof by the methods given below. Note that you are not required to provide any details of the proof. Instead, you only need to clearly state the initial assumption and the final goal of your proof.

(a) *Outline a proof of the proposition that uses proof by contradiction technique.*

(b) *Write the converse of the proposition; then outline a proof of the converse that uses the proof by contrapositive technique.*

§ Learning Target MA1Q - Matrix Arithmetic

Requirements. You must correctly answer **all questions in this section** to get an 'M' on **MA1Q**.

Question 2.

☐

Let $A = \begin{bmatrix} a & a \\ b & b \end{bmatrix}$ *where a and b are integers. If* $A^2 = \begin{bmatrix} 4 & 4 \\ 12 & 12 \end{bmatrix}$, *then find all possible choices for* A.

§ Learning Target RF1Q - Relations

Requirements. You must correctly answer **all questions in this section** to get an 'M' on **RF1Q**.

Question 3.

☐

Define a relation R *on the set* $\{1,2,3,4,5,6,7\}$ *as* aRb *iff*

$$(a+1) \mid (b-1).$$

Draw the digraph for this relation R.

§ Learning Target RF2Q - Functions

Requirements. You must correctly answer **all questions in this section** to get an 'M' on **RF2Q**.

Question 4.

☐

Below are three functions. ***For each, (i) state whether the function is injective, then (ii) state whether it is surjective, then (iii) state whether it is bijective.***

If a function fails to have one or more of these properties, give a ***specific example*** *that shows why. Otherwise, you do not need to explain your reasoning unless it helps you; but your answers must be correct.*

(a) $f : \{1,2,3,4\} \to \{1,2,3,4\}$ *given by* $f(a) = (a\%4) + 1$

(b) $g : \mathbb{Z} \to \mathbb{Z}$ *given by* $g(n) = n + 1$

(c) $h : \mathbb{N} \to \mathbb{N}$ *given by* $h(n) = n + 1$

§ Learning Target RI1Q - Sequence Definitions

Requirements. You must correctly answer **at least 4 out of 5 questions in this section** to get an 'M' on **RI1Q**.

Question 5.

☐

For each of the following sequences, (i) identify it as either an Arithmetic or a Geometric sequence; and (ii) find a closed-form formula for the n^{th} term. You do not need to show your work, but your results must be correct.

(a) $4, 6, 9, 13.5, 20.25, \ldots$

(b) $2, 5, 8, 11, 14, \ldots$

Question 6.

☐

List the first four (4) terms of each of the following sequences. You do not need to show your work, but your answers must be correct.

(a) $a_n = 2(3^n) + 3$ *where* $n = 1, 2, 3, \ldots$.

(b) $c_0 = 2$, *and* $c_n = 2c_{n-1} + 3$ *if* $n > 0$.

§ Learning Target RI2Q - Solving Recurrence Relations

Requirements. You must correctly answer **all questions in this section** to get an 'M' on **RI2Q**. You must show all significant work and complete all numerical computations. Up to two (2) simple errors (addition/subtraction mistakes etc.) are allowed but the process must be completed to give a final answer.

Question 7.

☐

Solve the recurrence relation given by

$$a_0 = 1, \quad a_1 = 4, \quad a_n = a_{n-1} + 12a_{n-2} \ \forall n \geq 2.$$

Multivariable Calculus

Exam 3

Fall 2022 **Subhadip Chowdhury** **Math 212**

Directions

- If you have already earned two **'M'** on a Learning Target, do not attempt a problem for that Target!
- Write your answer on the white pages provided. *Ask me for extra paper if needed.*
- please try to write the Learning Targets in order (for example, do not turn in work for D2 after work for D3).
- **Write your full name, legibly,** at the top of your answer script.
- You are not allowed to use a calculator. You may use one page of handwritten notes (one-sided).
- You do not need to simplify your answers any further once you have a numerical expression. For example, $2\pi + 3$ or $\frac{9^3-7}{2}$ is an acceptable solution.
- Unless explicitly stated otherwise, you must **show your work or explain your reasoning** clearly on each item of each problem you do. **Your answer script should not look like scratchwork.**

Warning: Responses that consist of only the final answer without any supporting work, or where the work is insufficient or difficult to read, or which have significant gaps or omissions (including parts left blank) will be given a grade of "N".

The following Learning Targets are available in this exam. Please keep the answers to all problems in a particular learning target together. The best way to do this would be to start each learning target on a separate page.

D1, D2, D3, E1, E2

Please add your full name **(not a signature, so don't scribble)** legibly to the top of your answer script to indicate your commitment to the Wooster Ethic and Academic Integrity.

"I have neither given nor received any unauthorized aid on this assignment."

Do not open the questionnaire until you are instructed to do so.

§ Learning Target D1 (Double Integrals)

Question 1.

In evaluating a double integral over a region D*, a sum of iterated integrals was obtained as follows:*

$$\iint_{\mathrm{D}} f(x,y)\,\mathrm{d}A = \int_0^1 \int_0^{2y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_1^3 \int_0^{3-y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

(a) *Sketch the region* D.

(b) *Rewrite the double integral as a single iterated integral with reversed order of integration.*

(c) *Evaluate the double integral when* $f(x,y) = x$.

§ Learning Target D2 (Change of Variables)

Question 2.

Consider the region $\mathcal{R}$ *in the upper half-plane bounded by the four circles of radii* 1 *and* 2*, centered at* $(1,0)$ *and* $(-1,0)$*, as pictured below.*

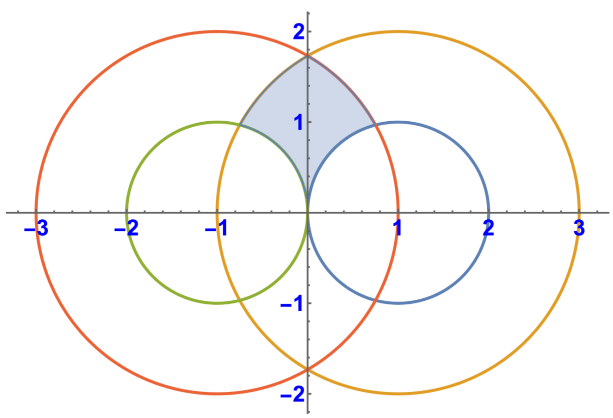


Figure 1

Use an appropriate change of variable $u = f(x,y)$ *and* $v = g(x,y)$ *and calculate the associated Jacobian* $\dfrac{\partial(x,y)}{\partial(u,v)}$ *to evaluate*

$$\iint_{\mathcal{R}} y\,\mathrm{d}x\,\mathrm{d}y.$$

Hint: The equation of a circle of radius r centered as (h,k) is given by $(x-h)^2 + (y-k)^2 = r^2$.

§ Learning Target D3 (Triple Integrals)

Question 3.

*Consider the region T bounded by the elliptical cylinder $4x^2 + y^2 = 4$, the plane $2x + z = 2$, the plane $y = 2$, and the $xy-$plane. Write **two** different iterated integral equivalent to the triple integral $\iiint\limits_{T} 1\,dV$ that*

(a) *integrates in the order* $\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z$.

(b) *integrates in the order* $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

You do not need to actually evaluate the integral.

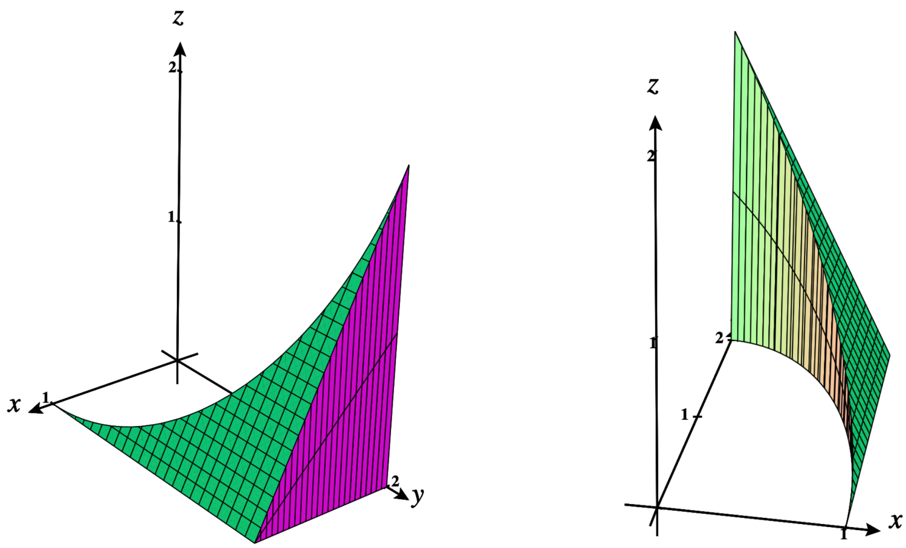


Figure 2: The region T from two different viewpoints

§ Learning Target E1 (Vector Fields and Differential Operators)

Note: You must complete most of the questions in this section correctly to get an 'M' in E1.

Question 4.

Consider the vector field

$$\vec{\mathbf{F}}(x,y,z) = \langle e^x \sin y, e^y \sin z, e^z \sin x\rangle$$

For each of the following, state whether the required expression is meaningful. If it is, evaluate it. If it is not, explain why.

(i) $\operatorname{curl}(\operatorname{grad}(\vec{\mathbf{F}}))$.   *(ii)* $\operatorname{div}(\operatorname{div}(\vec{\mathbf{F}}))$ *(iii)* $\operatorname{curl}(\operatorname{curl}(\vec{\mathbf{F}}))$

Question 5.

Match the Vector Fields in figure 3 with their formula. No justification is necessary.

(a) $\langle\cos(x+y), x\rangle$ *(b)* $\langle\sin(y), \cos(x)\rangle$ *(c)* $\langle\sin(x+y), y\rangle$ *(d)* $\langle\cos(x+y), \sin(x-y)\rangle$

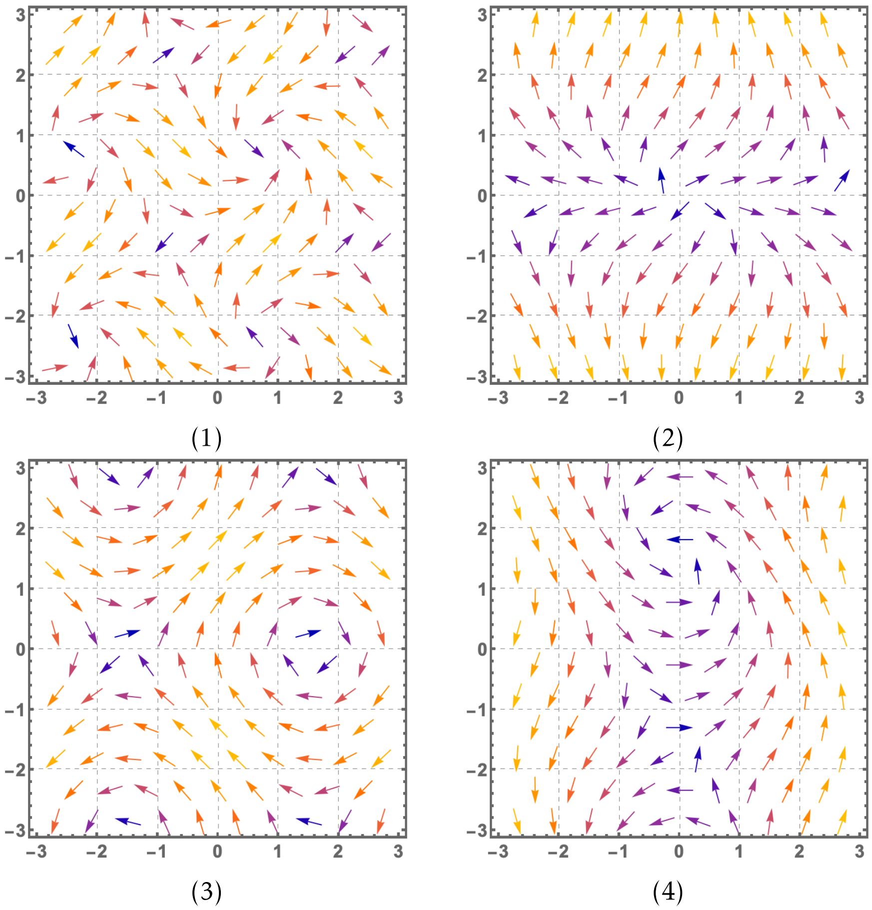


Figure 3

§ Learning Target E2 (Line Integrals)

Note: You need to complete most of the questions in this section correctly to get an 'M' in E2.

Question 6.

☐

We will consider two different parametrizations of the straight line joining $(0,0)$ *to* $(1,1)$*. Let* C_1 *be the parameterization given by* $\vec{r}_1(t) = \langle t, t\rangle$ *for* $0 \le t \le 1$*. Let* C_2 *be the parameterization by* $\vec{r}_2(t) = \langle t^3, t^3\rangle$ *for* $0 \le t \le 1$*. If* $\vec{\mathbf{F}} = x\hat{i} + y\hat{j}$*, then which of the following is true? Give a brief justification for your choice.*

(a) $\displaystyle\int_{C_1} \vec{\mathbf{F}}\cdot d\vec{r} < \int_{C_2} \vec{\mathbf{F}}\cdot d\vec{r}.$

(b) $\displaystyle\int_{C_1} \vec{\mathbf{F}}\cdot d\vec{r} = \int_{C_2} \vec{\mathbf{F}}\cdot d\vec{r}.$

(c) $\displaystyle\int_{C_1} \vec{\mathbf{F}}\cdot d\vec{r} > \int_{C_2} \vec{\mathbf{F}}\cdot d\vec{r}.$

Question 7.

☐

Consider the vector field $\vec{\mathbf{F}} = \langle 4x^3y^2 - 2xy^3, 2x^4y - 3x^2y^2 + 4y^3\rangle$*.*

(a) *Is* $\vec{\mathbf{F}}$ *a gradient vector field?*

(b) *Evaluate the line integral of* $\vec{\mathbf{F}}$ *along the parameterized curve* $\mathcal{C}$ *given by* $\vec{r}(t) = \langle t + \sin(\pi t), 2t + \cos(\pi t)\rangle$*,* $0 \le t \le 1$*.*

Appendix D

Sample Syllabi

Multivariate Calculus

Syllabus

Fall 2022 | **Subhadip Chowdhury** | **Math 212**

§A. What is this class?

Calculus III or Multivariate Calculus is kind of like a greatest hits compilation from Calculus I and II, but a remixed version for higher dimensions! In real life, most processes depend on more than one input - if you have ever filled a tax form, you know how many inputs it requires. The same is true all the way from quadratic equations to rocket science. So, whether we do optimization using differentiation or calculate volume using integration, it's all for functions of more than one variable. To ensure the new analogues make sense, we define some new ideas such as vectors and parametric curves along the way. Finally, everything comes together to like an IKEA furniture and culminates in the idea of vector calculus, which ties all of Multivariable Calculus together into several neat little theorems.

Please make use of my office hours and plan to work hard. My classes have a high workload (as all math classes usually do!), so make sure that you stay on top of your assignments and get help early. Remember that part of doing real math is productive failure: you'll try things that don't work; learn something from that failure; and try something new that works a bit better. And... after a while, you will figure it out, and come out with a much stronger understanding of the structure of mathematics.

Additional details on some parts of the syllabus are available on Moodle.

§B. Key Information

Course Info

- **Class Meetings:** MWF 9:00 AM - 9:50 AM (EST), Taylor 206
- **Lab Meetings:** Th 9:30 AM - 10:50 AM (EST), Taylor 206

How to contact me

- **Email:** schowdhury@wooster.edu
- **Phone:** 330-263-2473
- **Office:** Taylor 307

Office Hours

See Moodle for Up-to-date hours. You can also stop by any time my door is open, or email me to set up an individual meeting.

Required Study Materials

- **Textbook:** We will mainly use lecture notes and activities written especially for this class. You can use Calculus Volume 3 - OpenStax as a reference. The text is open-source and freely available online.
- **Computing Software:** We will use CalcPlot3D.

Class announcements

- **Available on:** `https://moodle-2223.wooster.edu/`
  Check Moodle and your Wooster email at least once before and after each class.

Additional college policies are listed in a separate document called Academic Policies, Procedures & Support Services.

Contents

§C. Catalog description

This course covers analytic geometry of functions of several variables, limits and partial derivatives, multiple and iterated integrals, non-rectangular coordinates, change of variables, line and surface integrals and the theorems of Green and Stokes. [MNS, Q, QL]

Prerequisites: MATH-10500, MATH-11000, MATH-11500, MATH-12000, and MATH-12500, minimum grade C-

§D. How do I earn a grade

Our course is graded by a methodology called Learning-Based Grading system, also called standards-based or mastery-based grading, in which the different assignments do not have fixed weight percentages or numerical scores. Instead, you earn your grade by showing appropriate engagement with the course (by attending classes and completing homework) and demonstrating evidence of skill on the learning objectives that describe the major ideas covered by each assignment. These objectives, called **Learning Targets (LT)**, are listed in section L and may be updated throughout the semester.

To succeed in this class, you are expected to develop two kinds of skills simultaneously: *computational facility* and *conceptual understanding*. Both are essential to demonstrate fluency on the material. You will also work to improve your communication skills—with each other, with me, and with the rest of the world.

D.I. Types of Assignments

More details on each type of assignment are given in the later parts of the syllabus. Check MOODLE for details.

- **Daily Work** Multivariable calculus has a plethora of interrelated skills that mesh together to form a network of ideas. The best way to master this network is through daily practice and engagement with the class.

  Most days, you will be expected to read a section of the textbook before coming to class. This will be your primary first exposure to new material. During most classes, collaboration and problem-solving will occupy around half of our meeting time. You will be given one or more questions to address, and we will discuss different approaches to answering these questions developed within the class.

- **CalcPlot3D Labs**: Computer literacy is a crucial skill in today's world. The computational and graphing capabilities of modern computers also make it easier to explore complex mathematical ideas without always getting bogged down in calculation. We will be using the visualization software **CalcPlot3D** (`https://c3d.libretexts.org/CalcPlot3D/index.html`) to complete around 5 or 6 projects this semester that will be scored based on the EMRN scale explained in fig. 1.

  Most projects will be group based. For each project, you will have the possibility to revise your write-up once to improve your score based on my comments.

- **Checkpoint Quizzes (almost every week)**: Rather than midterm or final exams, we will have in-class checkpoint quizzes every week (covering new materials). These will be your first attempt at assessing each Learning Target. Check MOODLE for details.

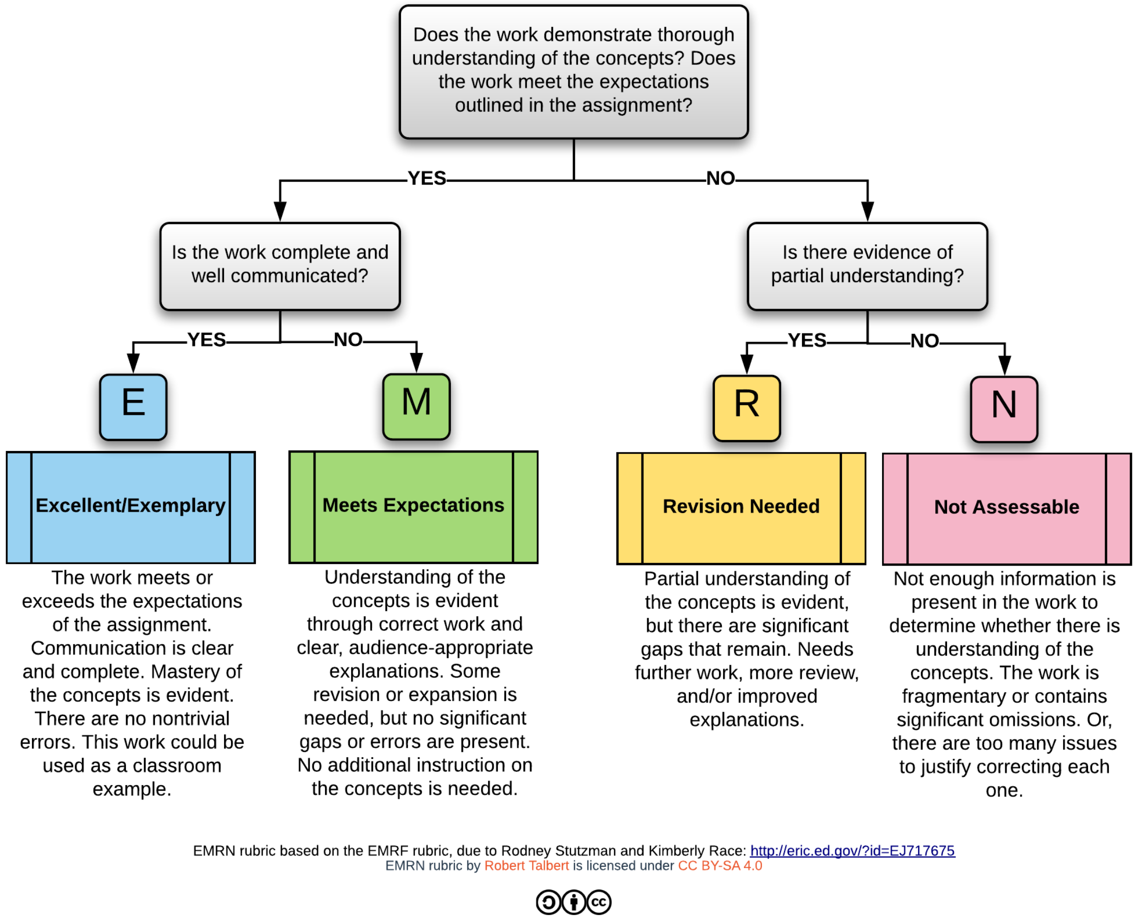


Figure 1: EMRN Rubric

- **Exams (every ~ 3 weeks)**: Additionally, there will be four in-class exams during the semester. On these exams, you will not receive a grade but rather you will have your second possible assessment on the LTs covered since the previous exam. The quizzes and exams will be given during the lab period.
- **Reassessments**: After a LT has been assessed on an exam, if you wish to have a reassessment in order to improve your score, contact me to arrange a time. You will need to let me know ahead of time which LT you wish to have reassessed. I will reassess up to two LTs per student per week. Check MOODLE for details.
- **Edfinity Homework (One set for every LT)**: We will be using Edfinity for most homework assignments this semester. These are online homework assignments, to help build your computational skills. There will be at least one Homework set per learning target. Check MOODLE for details.

D.2. How are Learning Targets scored?

When you submit most work, I will evaluate it relative to quality standards made clear on each assignment. If your work meets the standard, then you will receive credit for it. Otherwise, you will get helpful feedback and, on most items, the chance to reflect on the feedback, revise your work, and then reassess your understanding.

This feedback loop represents and supports the way that people learn. *Learning happens over time, as we revisit ideas and reflect on them.* In this class, your final grade will reflect how well you eventually understand each topic. You can make mistakes without penalty, as long as you eventually demonstrate fluency of the topic.

My hope is that this method of grading will keep you clearly informed as to the expectations of the class and how well you are meeting them, while also removing the (often distracting) elements of linear grading that uses letters or total points. If you have questions or concerns at any time, please feel free to discuss them with me.

Description of Scores

Each LT represents about 1–2 days of classwork. They are listed in section L approximately in the chronological order we will cover them. Each LT will be graded either M, R, or N based on fig. 1.

> **Note:** You can not earn a **E** on a learning target.

- To show **PROFICIENCY** in a learning target, you must earn a **M** grade on that LT during a quiz, exam, or reassessment.
- To show **EXPERTISE** in a learning target, a second grade of **M** must be earned on that same LT during a quiz, exam, or reassessment.

> **Note:** You may sometimes earn a **R*** in a Quiz. This mark indicates work that contains an error which I think is minor, but I need to talk with you about it. Come to my office to discuss a **R*** within one week after it is returned. If you can convince me that the error was minor and explain how to fix it, then I will update the **R*** to an **M** for free - it does not use up a reassessment attempt. If I don't hear from you within one week, a **R*** automatically becomes a **R**.

You will always have opportunities to show improvement, until the end of the course. Do not put off learning the material, however; later skills depend on earlier ones, and it will be hard to catch up if you fall too far behind.

There are also two general Learning Targets in section L that are evaluated a bit differently:

- G1: **Participation.** Your active participation in class is crucial both to your own learning and to the success of all. Attendance is therefore required, as is a willingness to share ideas and to make brief presentations.
  - **EXPERTISE** - Continued, thoughtful participation in asking and answering questions, group-work, completing labs and Edfinity homework on time.
  - **PROFICIENCY** - No issues with attendance, and participation is adequate.
  - **I**nadequate - Issues with attendance,participation, or maintaining respectful classroom environment.
- G2: **Basic Calculus.** Perform algebraic calculations and interpret the results for the following list of concepts: limits, derivatives, integrals. Convert between the graphs and equations of conic sections.

  We will **NOT** devote any class time to this material and this will be assessed through Edfinity homework. A score of 100% will be **EXPERTISE** and 95% – 99% will be **PROFICIENCY**.

How is the final letter grade determined?

At the end of the semester, I am required to submit to the college a letter grade reflecting your achievement in this class. Here is how that grade will be determined.

To determine your course base grade (the letter **A/B/C/D/E/F** without plus/minus modifications), use the following table. *To earn a grade, you must complete all the requirements in the column for that grade*; your base grade is the highest grade level for which all the requirements have been met or exceeded.

Category	**A**	**B**	**C**	**D**
Homework Credits	90%	80%	70%	60%
Learning Targets (20, including 6 core and 2 general)	**Proficient** on at least 18; **Expert** on at least 8	**Proficient** on at least 16; **Expert** on at least 5	**Proficient** on at least 14; **Expert** on at least 3	**Proficient** on at least 12
Core Learning Targets (6)	**Expert** on all 6	**Proficient** on all 6; **Expert** on at least 4	**Proficient** on all 6; **Expert** on at least 2	**Proficient** on all 6

Note: If you do not meet all of the criteria for a **D**, your grade will be an **F**. There is no **A+** or **D-**.

I will set +/- grades based on how close you are to the next higher (or lower) letter grade. For example, if you meet all criteria for an **A** except that you are expert on 5 core LTs and 5 other LTs, that may be an **A-**. If you are instead missing something bigger, e.g. proficient in 17 LTs, expert on 5 core LTs and 3 other LTs, that may be a **B+**. Similar considerations apply to completing two out of three requirements in a column. Please contact me towards the end of the semester if you want to review your current grades.

§E. Tokens

Each student starts the semester with **2 tokens** (and can have a max of 2 tokens at any time), which can be used to purchase exceptions to the course rules. The token menu is below. To spend a token, send me an email. Everything listed here costs 1 token:

- Reassess three different Learning Targets in the same week.
- Extend the deadline of a lab/project report by 24 hrs.
- Take a checkpoint quiz as a timed take-home exam on Moodle instead of taking it in class. *It will still need to be finished by midnight on the same day.*

Please note that tokens may not be ‘‘stacked"; for example, you aren’t allowed to spend 2 tokens and assess four Learning Targets in the same week or extend the deadline of a project report by 48 hours.

I will update the number of remaining tokens per student as they are used. Note that any leftover token at the end of the course will be counted towards class engagement, but has no value towards your scores.

E.I. Resetting the total number of Tokens

- Completing Edfinity assignments A1-B3 by Sep 30 midnight will reset your remaining tokens to 2. Please email me if you have done so, since I will not be checking Edfinity scores regularly.
- Similarly, completing Edfinity assignments A1-E3 by Nov 15 midnight will reset your remaining tokens to 2.
- There may be occasional bonus challenge problems that you can answer to earn extra tokens over the semester.

§F. Policies

F.I. Attendance and Absence

Attendance is crucial to success in this class. Your best chance to discuss new material, ask questions, and avoid confusion is during class. So, don't miss class! You are responsible for all material and announcements from class, even in case of absence. Much of this information will be available on MOODLE. Please check in with me and with your classmates when you are back.

That said, life happens. We get the flu (or COVID!). Relatives need your help. When this happens, do what you need to do. I trust that you are an adult and will make the best choices that you can. I appreciate it if you can notify me in advance of an absence, if possible.

If you think you will miss more than one class in a row, you should contact me beforehand to let me know, and meet me afterwards to discuss how you can catch up and move forward in the course. If you miss four or more classes, I will send out an academic alert. If you miss more than two weeks of classes, you should contact the Dean Jen Bowen and/or Amber Larson, Director of the Academic Resource Center. They can help you consider options for completing or dropping the course.

F.2. Early and Late Work

Early Work

Checkpoint Quizzes: You can arrange to take a quiz up to two days early if you have an conflicting extracurricular college event on that day, and you got permission from the Dean's office or your coach. *In such cases, you must give me a heads-up at least a week in advance.* Contact me directly regarding other cases.

Make-up Work

- **Edfinity Homework cannot be turned in late.**
- Lab Reports can be turned in late up to one day using a token.
- Checkpoint Quizzes may not be taken late, but since they are based on getting competency on learning targets, you may have an opportunity to assess the same target on a later exam or office hour with no penalty.

If you have significant extenuating circumstances that cause you to miss multiple assignments (even with tokens), see me to discuss arrangements.The Academic Resource Center, which is in APEX (Gault library) offers a variety of academic support services such as time management and class preparation, ELL peer tutoring, coordinating accommodations for students with diagnosed disabilities, etc. Please see the Academic Policies, Procedures & Support Services document for further details or go to the ARC website.

F.3. Special Accommodations

The Academic Resource Center, which is in APEX (Gault library) offers a variety of academic support services such as time management and class preparation, ELL peer tutoring, coordinating accommodations for students with diagnosed disabilities, etc. Please see the **Academic Policies, Procedures & Support Services** document for further details or go to the ARC website.

F.4. Email Responses

I do my best to reply to emails promptly and helpfully. However, I receive a lot of email. To help both you and me, here are some specific expectations about emails:

- If you email me between 8:00 am and 6:00 pm on a weekday, I'll reply to you on the same day.
- If you email me in the evening or overnight (after 6:00 pm), I will reply to you the next weekday.
- If your email asks a question that is answered in the Syllabus or on Moodle (such as in an announcement or an assignment sheet), I may reply by directing you to read the appropriate document.

Se MOODLE for further instructions and examples of good prefessional emails.

§G. How to get help?

G.1. My Office Hours

Please come see me during my office hours if you have questions or just want to discuss something from class. These will be most effective if you have spent some time formulating your questions beforehand - often you will answer your own questions during that process! You can also contact me via Email or MS Teams with your questions. See the email response section above for my 'business hours'!

G.2. Teaching Assistant Office Hours

Brayden Beathe-Gateley (class of '24) is your TA for this course. Brayden will not be present during classes but will hold weekly office hours outside the classroom. You can ask them for help with Edfinity homework and for going over past checkpoint quizzes.

See Moodle for office hour times and further instructions.

§H. Academic Integrity and Collaboration

In this class, your primary goal in this course is to develop a deep personal understanding and expertise in Multivariable Calculus. Collaboration and cooperation are extremely helpful in the learning process, and we will have many opportunities for collaborative work. However, there are some portions of our class that must be done independently.

The College's understanding and expectations regarding issues of academic honesty are fully articulated in the Code of Academic Integrity as published in The Scot's Key and form an essential part of the implicit contract between the student and the College. The Code provides framework at Wooster to help students develop and exhibit honesty in their academic work. You are expected to know and abide by these rules.

In this class, we will use the following definition of plagiarism:

Definition 8.1

Plagiarism is the act of submitting the work of someone else as if it were your own. Specifically, this action misleads the instructor to think that the work is the result of learning and understanding by the student named on the paper, when in fact the understanding truly belongs to someone else. This may apply to an entire solution, or individual parts of a solution.

In Math 212, collaboration is permitted and even encouraged in some circumstances! However, *you may only collaborate with students currently enrolled in math 212*. In all cases where collaboration has occurred, you must acknowledge this clearly:

Acknowledging collaboration: In all work, specifically lab projects, you must clearly state the name(s) of the person(s) you collaborated with on each problem.

H.I. Specific academic honesty expectations

It is often unclear what exactly "collaboration" means when working on assignments. The following section should clarify what my expectations are regarding this and give guidelines for avoiding plagiarism in assignments. The list is intended to be helpful but not exhaustive. If you are unsure about the appropriateness of some form of assistance on an assignment, you should always ask me.

- **Edfinity Homework:** On homework problems, every step of every solution should be one that you understand yourself and that you have generated on your own. You are permitted to discuss big ideas and hints with your classmates and your TA.

  Any collaboration should occur only when your collaborator is at essentially the same stage of the problem solution as yourself. In particular, if you have not yet started problem #4 and you ask a friend (who has already completed it), "How did you do problem 4 ?", this counts as **plagiarism.**

- **Lab/Project Reports:** You will be divided into groups during lab sessions to work on a lab project. You are permitted to work within that group only. Note that each student will need to complete and submit their own lab report. As such, I expect that everyone in a group understands all aspects of the report. If a student was not present or did not have significant contribution to the project, this should be noted in the report.

- **Outside resources in general:** On all work, unless directly stated otherwise, the only resources you may use are our class notes (including any worksheet on Moodle) and the approved textbook (see the first page). You are not permitted to go looking for completed solutions to problems in other texts or resources. **In particular, use of internet resources is completely off limits for completing homework problems.** Often, full solutions for our homework problems can be found online. If you see such a solution prior to submitting homework, there is essentially no way that you can claim to have an original solution. Evidence of using internet sources in your work will result in a minimum penalty of earning a **N** on the relevant objectives.

- **Copying**: Copying a solution, or any part of a solution, from any source (friend, internet, book, etc.) in any setting, constitutes **plagiarism**.
- **Past students:** On any assignment, basing your work on the efforts of another student who previously completed this course, or one like it, is considered **plagiarism**.
- **Other instructors, the Math Center (ZIs), and TA:** Before the relevant deadlines, you are not allowed to discuss any Checkpoint Quiz problem with the ZIs in the Math Center, our TA, or seek the help of an instructor or tutor (other than me). You are encouraged to seek their help **after** you have submitted an assignment and need help checking or understanding a concept. If you seek their help before submission, this will be considered **plagiarism**.

H.2. Consequences of academic dishonesty

Evidence of dishonest behavior on any assignment will be grounds for a minimum penalty of earning an **X** on all relevant objectives for that assignment. Other penalties may include permanently failing the relevant objectives (regardless of other work) or, in severe cases, failure of the course. **Peers who willingly assist others in acts of plagiarism are equally guilty and will suffer similar penalties.** In all cases, the guidelines established in The Scot's Key will be followed. I reserve the right to discuss the nature and origins of any assignment with any student prior to assigning a grade.

H.3. A positive note

Remember that I want you to be successful. That is, I want you to develop a deep, personal understanding of the material we study so that you become a better student of mathematics who can go on to do well in all of your future endeavors. Every part of this course structure - including both collaborative work and restrictions on collaboration - are intended to help you with this. You will often struggle, and that's intentional - struggle (and eventual success!) is essential to learning. Indeed, productively failing (and learning from it) is part of your final grade.

In all aspects of the course, please understand that I am generous with hints and am always willing to discuss problems with you. I will never simply give you an answer, but I will offer direction and guidance that will assist you in coming up with a solution on your own. This is by far the most satisfying way to solve a problem, and the difficulty is well worth it. You are always welcome to discuss your questions or concerns with me at any time.

§I. Disclaimer

I reserve the right to make changes to this syllabus, if needed. Any changes will be announced to the class in a timely manner.

§J. Tentative course schedule

Week	Monday	Wednesday	Thursday	Friday
1 (Aug 24)		Syllabus Overview + 3D Coordinate Geometry	**Lab 0 (Intro to CalcPlot3D) +** Vectors in 3D	Dot Product, Angle, Projection
2 (Aug 29)	Cross Product, Area, Volume	Lines and Planes	**Lab 1 (Lines and Planes) + Project 1 (Distances)**	**Quiz 1**
3 (Sep 5)	Coordinate Systems	Parametrized Curves - Equation of Motion	Calculus of Vector valued Functions	**Quiz 2**
4 (Sep 12)	Geometry of Vector valued Functions	Functions of several variables	**Lab 2 (3D Graphing and Contour Plots)**	**Quiz 3**
5 (Sep 19)	Partial Derivatives and Local Linearity	Gradients and Directional Derivatives	**Exam 1**	**Lab 3 (Gradient Vector, Tangent Plane, and Directional Derivative)**
6 (Sep 26)	Three-dimensional Gradient and Tangent Plane	**Practice**	Chain Rule	**Quiz 4**
7 (Oct 3)	Local Optimization	Constrained Optimization	**Lab 4 (Stationary Points) + Practice**	**Quiz 5**
Fall Break				
8 (Oct 17)	Global Optimization	Definite Integral of Functions of Two Variables	**Exam 2**	Type I/II regions, Fubini's theorem
9 (Oct 24)	Double Integral in Polar Coordinates	Triple Integrals	**Lab 5 (Volume Integration)**	**Quiz 6**
10 (Oct 31)	Change of Variables - Jacobian	**Practice**	**Lab 6 (Vector Fields) + Project 3**	**Quiz 7**
11 (Nov 7)	Vector Fields contd.	Differential Operators	Differential Operators contd.	**Quiz 8**
12 (Nov 14)	Line Integrals on Parameterized Curves	Gradient Fields - Path-Independent	**Exam 3**	**Practice**
13 (Nov 21)	**Practice + Setting up for Green's theorem**	**Thanksgiving Break**		
14 (Nov 28)	Green's Theorem	Applications of Green's Theorem	**Practice**	**Quiz 9**
15 (Dec 5)	Parameterized Surface	Surface Integrals, Flux	Stoke's Theorem	**Quiz 10**

Exam 4 will be on the day of the final exam and will cover the material after Exam 3. You can also schedule meetings to reassess up to two (2) learning Targets.

§L. Math 212 Learning Targets

The $\boxed{\star}$ - marked learning targets are considered the most important (''core'') learning targets. They are not scored differently, but are used to determine final grades. See section D.2 for details.

- G1: **Participation.** Your active participation in class is crucial both to your own learning and to the success of all. Attendance is therefore required, as is a willingness to share ideas, engage in group work, and to make brief presentations as needed.
- G2: **Basic Calculus.** Perform algebraic calculations and interpret the results for the following list of concepts: limits, derivatives, integrals. Convert between the graphs and equations of conic sections.

- A1: **Vector Algebra.** Compute, graph, and interpret vector addition/subtraction, scalar multiplication, magnitude, dot products, cross products, angle between vectors, area of a parallelogram, and volume of a parallelopiped.
- A2 $\boxed{\star}$: **Lines and Planes.** Find the equation of a line or plane satisfying given conditions.
- A3: **Coordinate Systems.** Describe points, graphs of surfaces, and regions of space using rectangular, cylindrical, or spherical coordinates; convert representations (coordinates or equations) between coordinate systems.
- A4: **Vector-Valued Functions and Space Curves.** Convert between parametric and geometric descriptions of curves; find parametrizations of graphs and other implicit curves; apply concepts of single-variable calculus (limits, derivatives, integrals) to parametric curves in $\mathbb{R}^2$ or $\mathbb{R}^3$.
- A5 $\boxed{\star}$: **Calculus with Curves.** Describe motion via vector-valued functions; find the tangent, arc length, curvature, and acceleration of a curve.

- B1: **Surfaces and Level Sets.** Sketch or predict the appearance of the graph of a function or curve, planes, and other common quadric surfaces based on a formula or other description; interpret as level sets of a function; use computer software to examine shapes of graphs and level sets.
- B2: **Partial Derivative and Local Linearization.** Evaluate and interpret the partial derivatives of a two or three variable function, normal vector to a surface generated by $f(x,y)$, linear approximation.
- B3 $\boxed{\star}$: **Gradient and Directional Derivatives.** Geometric interpretation of the gradient and directional derivatives; find equation of tangent planes to implicit surfaces.
- B4: **Chain Rule.** Apply and explain equality of mixed partial derivatives, including sufficient conditions for such equality to hold; compute partial derivative of composite functions using chain rule.

- C1: **Unconstrained Optimization.** Locate the critical points and classify local extrema of a multivariate function.

- C2 $\boxed{\star}$: **Constrained Optimization.** Use Lagrange multipliers to find max/min given a constraint, possibly on the boundary of a region.

- D1: **Double integrals.** Set up and evaluate double integrals as iterated integrals for type I/II regions; be able to switch the order of integration as necessary.
- D2: **Change of variables.** Compute and interpret the Jacobian of a coordinate change function; use the Jacobian to transform and evaluate double integrals.
- D3 $\boxed{\star}$: **Triple integrals.** Set up and evaluate integrals over three dimensional regions as iterated integrals, change order of integration when useful, apply triple integrals to compute geometric quantities such as volume and mass.

- E1: **Vector fields and Differential Operators.** Sketch and visualize vector fields and their flow lines using a computer; compute and interpret divergence and curl; explain how differential operators (div, grad, curl) are related.
- E2: **Line integrals.** Compute line integrals of a vector valued function along a parameterized curve; determine if a Vector Field is conservative, find a potential function, and apply Fundamental Theorem of Line Integral.
- E3: **Parametrized surfaces and Surface Integral.** Find and use parametrizations of surfaces to set up and evaluate surface integrals of scalar fields and vector fields (flux); explain the role of orientation in flux integrals.
- E4 $\boxed{\star}$: **Green's and Stokes's Theorem**. Explain what the boundary of a region in $\mathbb{R}^2$ or $\mathbb{R}^3$ means; use Green's theorem and Stokes's theorem to compute line integrals over closed curves in in $\mathbb{R}^2$ or $\mathbb{R}^3$.