Problem Set 3 Solutions

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Exercise 2: Set Notation

Problem 1.1. *Write the set of positive numbers that end in* 0 *i.e.* $10, 20, 30, 40, 50, \ldots$ *in descriptive notation.*

Solution. $\{10n | n \in \mathbb{N}\}$ ■

Exercise 3: Sets of Numbers

Problem 2.1. *Give example of two sets* $\mathscr{A}$ *and* $\mathscr{B}$ *such that neither* $\mathscr{A} \subset \mathscr{B}$ *nor* $\mathbb{B} \subset \mathbb{A}$ *.*

Solution. Some examples are:

- $\mathscr{A} = \mathbb{N}, \mathscr{B} = \{x | x \in \mathbb{R}, 0 \leq x \leq 1\} = [0, 1]$
- $\mathbb{Q}, \mathbb{R} \setminus \mathbb{Q}$
- $\{1, 2\}, \{2, 3\}$

■

You could have chosen any $\mathscr{B}$ and $\mathscr{A}$ such that $\mathscr{B}$ has at least one element not in $\mathscr{A}$, and $\mathscr{A}$ has at least one element not in $\mathscr{B}$. However, you could not have chosen the empty set $\emptyset = \{\}$ as one of your sets, since every set is a superset of the empty set.

Exercise 4: Closure of Number Sets Under Operations

Problem 3.1. *Consider two numbers $m, n \in \mathbb{N}$. Which of the four numbers $m+n$, $m-n$, $m \times n$, and m/n must be another element of $\mathbb{N}$? Answer the same question when $\mathbb{N}$ is replaced by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or the set of irrational numbers ($\mathbb{R} \setminus \mathbb{Q}$).*

In the table below, the entries are T if true, and a counterexample if false.

Set $\mathscr{A}$	$m+n \in \mathscr{A}$	$m-n \in \mathscr{A}$	$m \times n \in \mathscr{A}$	$m/n \in \mathscr{A}$
$\mathbb{N}$	T	$1-2$	T	$1/2$
$\mathbb{Z}$	T	T	T	$1/2$
$\mathbb{Q}$	T	T	T	T if $n \neq 0$
$\mathbb{R}$	T	T	T	T if $n \neq 0$
$\mathbb{R} \setminus \mathbb{Q}$	$\sqrt{2} + (-\sqrt{2})$	$\pi - \pi$	$\sqrt{2} \times \sqrt{2}$	$\frac{\pi}{\pi}$

Exercise 6: Irrational Sum

Problem 4.1. *Prove that the sum of a rational number and an irrational number is irrational.*

Claim. *The difference of two rational numbers is rational.*[1]

Proof of Claim. If $r, q \in \mathbb{Q}$, then $\exists a, b, c, d \in \mathbb{Z}$ such that $r = \frac{a}{b}$ and $q = \frac{c}{d}$. Consider $r - q = \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$. As the product and sum of integers are integers, we get that $ad - bc$ and bd are integers, and hence $r - q$ is an element of $\mathbb{Q}$. $\square$

Proof of the Exercise. Suppose, for the sake of contradiction, that $\exists r \in \mathbb{Q}$ and $x \in \mathbb{R} \backslash \mathbb{Q}$ such that $x + r \in \mathbb{Q}$. Now we showed above that the difference of two rational numbers is rational, i.e. if $m, n \in \mathbb{Q}$, then $m - n \in \mathbb{Q}$. So now let us take $m = r + x$ and $n = r$. Both $r + x$ and r are elements of $\mathbb{Q}$, by our assumption. Hence $(r + x) - r$, which is equal to simply x, is also an element of $\mathbb{Q}$. But this contradicts $x \in \mathbb{R} \setminus \mathbb{Q}$. $\square$

1. We could have started by proving that sum of integers are integers (by induction-can you do it?) and since multiplication is an integer number of additions, this would imply product of integers is integer (also can be proved by induction). Since rationals are defined using integers, we are going to directly assume these facts about integers for the purpose of this proof.