Math 1800-C Handout 5: Double Integral Using Polar Coordinates and Probability Density Function

Subhadip Chowdhury

Change to Polar Coordinates in a Double Integral: If f is continuous on a polar region of the form

$$D = \{(r,\theta) | \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

then

$$\iint\limits_D f(x,y)\, \mathrm{d}A = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, \mathrm{d}r\, \mathrm{d}\theta$$

Exercise 1

Consider a function $f(x,y)$ defined as follows.

$$f(x,y) = \begin{cases} 8 & \text{if } x^2+y^2 \le 6^2 \\ \frac{48}{\sqrt{x^2+y^2}} & \text{if } 6^2 \le x^2+y^2 \le 16^2 \end{cases}$$

The region under $f(x,y)$ and above the XY-plane looks like a circus tent as in figure **1**. Find the volume of the tent.

Exercise 2

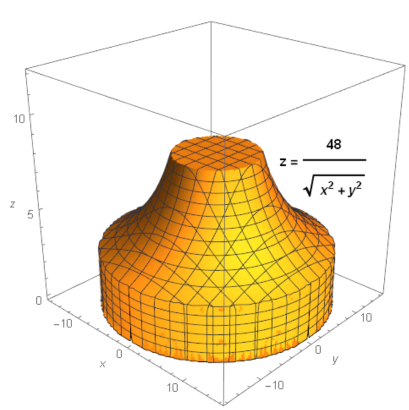
Evaluate the integral $\iint\limits_R (3x+4y^2)\, \mathrm{d}A$, where R is the annulus $1 \le x^2+y^2 \le 4$.

Exercise 3 (Optional)

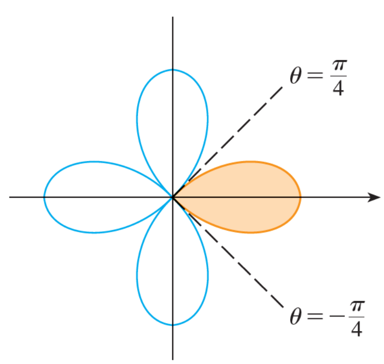
Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos(2\theta)$. See figure 2.

Exercise 4

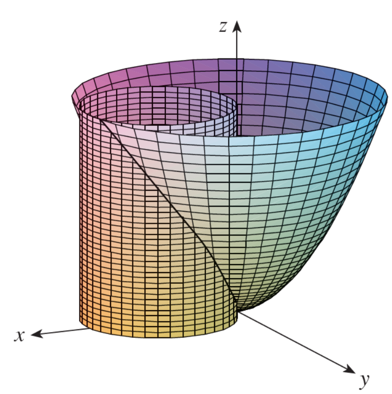
Find the volume of the solid that lies under the paraboloid $z = x^2+y^2$, above the XY-plane, and inside the cylinder $x^2+y^2 = 2x$. See figure 3.



(a) Figure 1



(b) Figure 2



(c) Figure 3

Probability Density Function: A function $p(x, y)$ is called a (joint) probability density function for x and y if

- The double integral of p over the entire XY-plane (in either Cartesian or Polar coordinates) is equal to $\mathbf{1}$.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y)\, dx\, dy = \int_{0}^{\infty} \int_{0}^{2\pi} p(r\cos(\theta), r\sin(\theta))\, r\, d\theta\, dr = 1$$

- The values of $p(x, y)$ are always non-negative.

Exercise 5

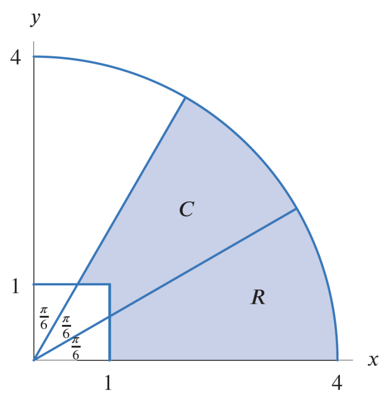
A point is chosen at random from the region R in the XY-plane containing all points (x, y) such that $-1 \leq x \leq 1$ and $-2 \leq y \leq 2$ with uniform probability. Given that ''uniform'' means the probability density function $p(x, y)$ is constant on R, find that constant.

Exercise 6

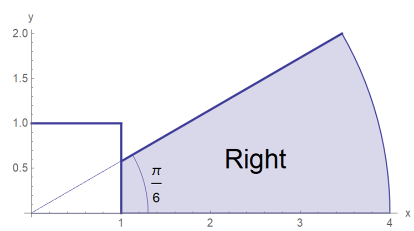
Figure 4 represents a baseball field, with the bases at $(1, 0)$, $(1, 1)$, $(0, 1)$, and home plate at $(0, 0)$. The outer bound of the outfield is a piece of a circle about the origin with radius $\mathbf{4}$. When a ball is hit by a batter we record the spot on the field where the ball is caught. Let $p(r, \theta)$ be a function in the plane that gives the density of the distribution of such spots. Write an expression that represents the probability that a hit is caught in

(a) The right field (region R)

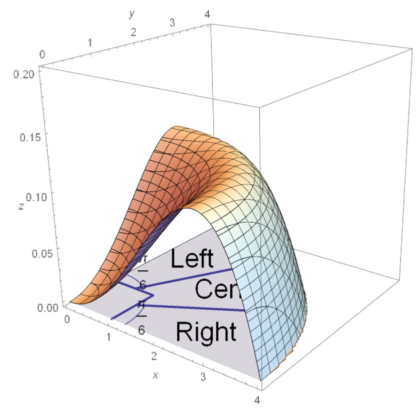
(b) The center field (region C)



(a) Figure 4



(b) Figure 5



(c) Figure 6

Assume that the probability density function is given in Cartesian coordinate as follows. Figure 6 gives a graph of this function.

$$p(x, y) = \begin{cases} \frac{3}{512\pi} \left(16\left(x^2 + y^2\right) - \left(x^2 + y^2\right)^2\right) & \text{if } (x, y) \text{ is in the field} \\ 0 & \text{otherwise} \end{cases}$$

(c) Rewrite the probability density function in polar coordinates.

(d) Check that the given function is indeed a probability density function.

(e) Calculate the probability that the ball is caught in region R.

[Acknowledgement: The pictures are from Multivariable Calculus books by Stewart, by Hughes-Hallett, Gleason, McCallum and by Professor Adam Levy]