Final Exam

AUGUST 7, 2018
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Please show **all** your work! Answers without supporting work will not be given credit. No electronic equipment is allowed with you in the exam hall. Please answer in the green book provided to you. Write clearly and legibly. Start every problem on a different page. They can be out of order, in fact I suggest answering the easier problems first.

You can score a maximum of 80 points!

Time Limit: 1 Hour 20 minutes

Question	Points
1	15
2	10
3	15
4	6
5	15
6	9
7	10
8	0
Total:	80

This exam has 8 questions, for a total of 80 points. The maximum possible point for each problem is given on the right side of the problem.

1. Prove by mathematical induction that $n^3 + 2n$ is divisible by 3 for all natural numbers n. [15]

Solution:

We will prove the claim by inducting on n. For the base case, observe that $1^3 + 2 \times 1 = 3$ is divisible by 3.

Now assume that $k^3 + 2k$ is divisible by 3 for some natural number k. We write $k^3 + 2k = 3l$ for some integer l.

Then

$$(k+1)^3+2(k+1) = k^3+3k^2+3k+1+2k+2 = (k^3+2k)+(3k^2+3k+3) = 3(l+k^2+k+1)$$

i.e. $(k+1)^3 + 2(k+1)$ is divisible by 3.

Hence by by the principle of mathematical induction, the claim is true for all $n \in \mathbb{N}$.

2. Prove that if a positive integer p has no factors that are less than or equal to $\sqrt{p}$, then p is a prime number. [10]

Solution: Assume, for the sake of contradiction that p is not a prime number. Then p is either equal to 1 or a composite number. Clearly $p \neq 1$ since $\sqrt{1} = 1$ is a factor of 1. Hence we get that p is a composite number.

Then p must have a factor other than 1 and p, call it k. We are given that $k > \sqrt{p}$. Observe that

$$k > \sqrt{p} \implies \frac{p}{k} < \sqrt{p}$$

But if $k \mid p$, then $\frac{p}{k}$ is another factor of p. This is a contradiction, since we have found a factor of p less than $\sqrt{p}$.

3. Suppose three distinct real numbers a, b, and c are in an Arithmetic Progression. If [15]

$$2(b-a) + x(c-b)^2 + (c-a)^3 = 0$$

then find all possible values of x.

Solution: Let us denote the common difference of the AP by d. Then

$$b - a = d, \quad c - b = d, \quad c - a = 2d$$

So we can rewrite the equation as

$$2d + xd^2 + (2d)^3 = 0 \implies 2 + xd + 8d^2 = 0$$

This is a quadratic equation in d, which must have real root(s) since the AP exists. Hence the discriminant is non-negative. The discriminant $= x^2 - 4 \times 2 \times 8 = x^2 - 64$. So

$$x^2 - 64 \geq 0 \implies x \leq -8 \text{ or } x \geq 8.$$

4. Find the least upper bound and the greatest lower bound of the following sets. Write DNE if it does not exist. No explanation is necessary.

(a) $\{x \in \mathbb{R} \mid (x-2)(3-x) \geq 0\}$ 2

Solution: lub $= 3$, glb $= 2$.

(b) $\{k \in \mathbb{Z} \mid k^2 < 4\}$ 2

Solution: lub $= 1$, glb $= -1$.

(c) $\left\{ \frac{n-1}{n} \,\middle|\, n \in \mathbb{N} \right\}$ 2

Solution: lub $= 1$, glb $= 0$.

5. Determine which of the following statements are *True* or *False*. Very very briefly explain your reasoning.

(a) Let $a, b, c \in \mathbb{Z}$. Then $(a + bc, b) = (a, b + ac)$. 3

Solution: TRUE. $(a + bc, b) = (a, b) = (b, a) = (b + ac, a)$.

(b) The highest power of 5 that divides 100! is 5^{25}. 3

Solution: FALSE. The exponent of the highest power is equal to

$$\lfloor 100/5 \rfloor + \lfloor 100/25 \rfloor + \lfloor 100/125 \rfloor + \ldots = 20 + 4 + 0 = 24$$

(c) For any real number x, we have $\lfloor 2x + 4 \rfloor = 2\lfloor x + 2 \rfloor$. 3

Solution: FALSE. Consider $x = 0.5$. Then $\lfloor 2x + 4 \rfloor = 5$. But $2\lfloor x + 2 \rfloor = 2 \times 2 = 4$.

(d) If $\sin\alpha = \sin\beta$, then $\alpha = \beta + 2\pi n$ for some integer n. 3

Solution: FALSE. Consider $\alpha = \pi/3$ and $\beta = 2\pi/3$. Then $\sin\alpha = \sin\beta$ but $\alpha - \beta = -\pi/3 \neq 2\pi n$ for any integer n.

(e) The greatest lower bound of a set S is equal to the least upper bound of the set of all lower bounds of S. 3

Solution: TRUE. The glb of a set S is the largest lower bound. So among all the lower bound of S, the glb is the least upper bound.

6. Using trigonometric identities, show that 9

$$(1+\tan\alpha\tan\beta)^2 + (\tan\alpha - \tan\beta)^2 = \sec^2\alpha\sec^2\beta$$

for any angles α and β.

Solution:

$$\begin{aligned}(1+\tan\alpha\tan\beta)^2 + (\tan\alpha - \tan\beta)^2 =& 1+\tan^2\alpha\tan^2\beta + 2\tan\alpha\tan\beta + \\ & \tan^2\alpha + \tan^2\beta - 2\tan\alpha\tan\beta \\ =& 1+\tan^2\alpha + \tan^2\beta + \tan^2\alpha\tan^2\beta \\ =& (1+\tan^2\alpha)(1+\tan^2\beta) \\ =& \sec^2\alpha\sec^2\beta\end{aligned}$$

7. Give an $\epsilon - N$ proof of the following: 10

$$\lim_{n\to\infty}\frac{1+2n}{3n-1} = \frac{2}{3}$$

Solution: We want to prove that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n > N$, then $\left|\frac{1+2n}{3n-1} - \frac{2}{3}\right| < \epsilon$.

Fix $\epsilon > 0$. Choose $N = \left\lceil \frac{1}{3}\left(\frac{5}{3\epsilon}+1\right)\right\rceil$. Then

$$\begin{aligned} n > N &\implies n > \frac{1}{3}\left(\frac{5}{3\epsilon}+1\right) \\ &\implies 3n-1 > \frac{5}{3\epsilon} \\ &\implies \frac{3(3n-1)}{5} > \frac{1}{\epsilon} \\ &\implies \frac{5}{3(3n-1)} < \epsilon \\ &\implies \frac{3+6n-6n+2}{3(3n-1)} > \epsilon \\ &\implies \left|\frac{1+2n}{3n-1} - \frac{2}{3}\right| < \epsilon \end{aligned}$$

Hence, by definition,

$$\lim_{n\to\infty} \frac{1+2n}{3n-1} = \frac{2}{3}.$$

8. Define a sequence $\{a_n\}_{n\in\mathbb{N}}$ as **5 (bonus)**

$$a_1 = 1, \quad a_n = a_1 + a_2 + \ldots + a_{n-1}$$

Prove that $a_n = 2^{n-2}$ for $n > 2$.

Solution: We will prove the claim by inducting on n. Observe that $a_2 = a_1 = 1$, hence for $n = 3$, we get

$$a_3 = a_1 + a_2 = a_1 + a_1 = 2 = 2^{3-2}$$

Now assume that the claim is true for some natural number k, i.e. $a_k = 2^{k-2}$.

Then

$$\begin{aligned} a_{k+1} =& a_1 + a_2 + \ldots a_{k-1} + a_k \\ =&(a_1 + a_2 + \ldots a_{k-1}) + a_k \\ =& a_k + a_k \\ =& 2a_k \\ =& 2 \times 2^{k-2} \\ =& 2^{k-1} \\ =& 2^{(k+1)-2} \end{aligned}$$

Hence by by the principle of mathematical induction, the claim is true for all $n > 2$.