Assignment 11 (2/2)

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- *This homework is due at the beginning of class on **Friday** 2/9. You may cite results from class as appropriate. Unless otherwise stated, you must provide a complete explanation for your solutions, not simply an answer. You are encouraged to work together on these problems, but you must write up your solutions independently.*
- *Hand in the exercises only, not the reading material item. You are encouraged to think about the exercises marked with a ($\star$) or (†) if you have time, but you don't need to hand them in. If you correctly solve a (†)-marked problem, you will get a candy!*
- *Remember that you can always use the result of the previous assignment problems without proof to solve the new assignment problems.*
- *We are currently covering Chapter 15 from Stewart.*

Important Points and Reading Materials

- Definition of Double Integral and Estimation using Riemann Sum
  - Note that the definition of definite integral in one dimension can be easily generalized to two dimension as follows:
    * Given a function $f(x,y)$ defined over a **rectangle** R, we can define the Riemann sum

      $$R_f^*(P) = \sum_{j=1}^{m}\sum_{i=1}^{n} f(x_i^*, y_j^*)(x_{i+1-x_i})(y_{j+1}-y_j)$$

      where P is a partition of R into subrectangles by vertical lines at $x_0, x_1, \ldots, x_n$ and horizontal lines at $y_0, y_1, \ldots, y_m$; and (x_i^*, y_j^*) is an arbitrary point in the $(i,j)^{th}$ subrectangle. Note that there are a total of mn subrectangle.
    * We also use the notation ΔA to denote $(x_{i+1-x_i})(y_{j+1}-y_j)$. When the order of integration is not clear, we use dA instead of $dx\,dy$ or $dy\,dx$.
    * Then the double integral of f over R is

      $$\iint_R f = \iint_R f(x,y)dA = \lim_{\|P\|\to 0} R_f^*(P)$$

      where $\|P\|$ is equal to the area of the largest subrectangle in P.
  - (The Midpoint Rule) Consider the case when (x_i^*, y_j^*) is midpoint of the subrectangles; i.e. $x_i^* = \bar{x}_i = \frac{x_i + x_{i+1}}{2}$ and similarly $\bar{y}_j$. Then for big enough n and m, we can approximate

    $$\iint_R f(x,y)dA \approx \sum_{j=1}^{m}\sum_{i=1}^{n} f(\bar{x}_i, \bar{y}_j)(x_{i+1-x_i})(y_{j+1}-y_j)$$

  - (Fubini's Theorem) If f is continuous on the rectangle $R = \{(x,y) \mid a \le x \le b, c \le y \le d\}$, also denoted as $[a,b]\times[c,d]$, then

    $$\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

  - The continuity assumption in above statement is sufficient, but not necessary. However, we will be dealing with continuous functions from now on.

- When $R = [a,b]\times[c,d]$, we can use the unifrom partition P where each subrectangle has equal area $= \frac{(b-a)(d-c)}{nm}$ and we can write

$$\lim_{m,n\to\infty} \frac{(b-a)(d-c)}{nm} \sum_{j=1}^{m}\sum_{i=1}^{n} f\left(a + \frac{(b-a)i}{n}, c + \frac{(d-c)j}{m}\right) = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

- In fact, it is not necessary that we use the point $\left(a + \frac{(b-a)i}{n}, c + \frac{(d-c)j}{m}\right)$ as (x_i^*, y_j^*). We can also, for example, use the midpoints of the subrectangles. We work out some examples below.

- Application of Fubini's Theorem
  - We use Fubini's theorem to transform a double integral over a rectangle to an iterated integral. However, it is not always clear what should be the order of integration. We should choose the order that makes the integration easier to solve.
  - For example, consider the integral

$$\iint\limits_{[0,\pi]\times[1,2]} x\sin(xy)dA$$

    Here if we do integral with respect to x first, we need to do by-parts integration; which is arguably harder. However if we integrate with respect to y first, we can treat x as constant, making the integration much easier. Look in the book for more of such examples.
  - In the case when $f(x,y)$ is of the form $g(x)h(y)$ i.e. if we can separate out the x and the y parts, then

$$\iint\limits_{[a,b]\times[c,d]} f(x,y)dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

- Integral as volume
  - Similar to one dimension, the double integral of f over a region R gives the volume of the solid over R, bounded by the graph of $f(x,y)$ and XY-plane, i.e. 'under' the graph of f.

Problems

Exercise 0⋆

Show that the Dirichlet function

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \notin \mathbb{Q} \end{cases}$$

is NOT integrable.

Exercise 1

Suppose we want to calculate the limit

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \frac{i}{n}$$

Note that this is equal to $\int_0^1 x\,dx = 1/2$. Because, we can write the limit as

$$\lim_{n\to\infty} \frac{1-0}{n}\sum_{i=1}^{n} f\left(0 + \frac{1-0}{n}\right)$$

where $f(x) = x$, which by Riemann sum limit formula, equals the integral.

Similarly,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^2}{n^3} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{i^2}{n^2} = \int_0^1 x^2 dx = 1/3$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{i^2+n^2}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{\frac{i}{n}}{\frac{i^2}{n^2}+1}=\int_0^1\frac{x}{x^2+1}dx=\frac{\ln 2}{2}$$

Can you calculate the following limits?

(a)

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n-i}{ni}$$

(b)

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{(2i+1)}{2n^2}$$

Exercise 2

Similar to problem 1, calculate the following limit using a double integral.

$$\lim_{m,n\to\infty}\frac{1}{nm}\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{i(m-j)}{nm}$$

Exercise 3

Use the midpoint rule (see above) with $m=n=2$ to estimate

$$\iint_R \frac{x-1}{y-2}dA$$

where $R=[0,4]\times[4,8]$. Try not to use a calculator.

Exercise 4⋆

Find

$$\iint_{[0,\pi]\times[1,2]} x\sin(xy)dA$$

Look in the write-up above regarding the best way to do this.

Exercise 5

Calculate the following double integrals.

(a)

$$\iint_R ye^{-xy}dA$$

where $R=[0,2]\times[0,3]$.

(b)

$$\iint_R \frac{xy^2}{x^2+1}dA$$

where $R=[0,1]\times[-3,3]$.

Exercise 6⋆

Find volume of the solid that lies under the hyperbolic paraboloid (aka, pringle) $z=3y^2-x^2+2$ and above the rectangle $R=[-1,1]\times[1,2]$

Exercise 7

Find volume of the solid enclosed by the surface $z = 1 + e^x \sin y$ and the planes $x = \pm 1, y = 0, y = \pi$, and $z = 0$.

Exercise 8

The average value of a function $f(x, y)$ over a rectangle R is defined as

$$\frac{\iint_R f}{area(R)}$$

Use this to work out exercise 15.1.7. Note that there is no exact correct answer here. Your answer should be close to 248 and 15.5 respectively.