Class Project Ideas, Guidelines, and Rubric

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Contents

Introduction

This project is open-ended, and I want you to do something that interests you. I have below some project ideas, but they are not exhaustive, and the questions being asked are meant only to be prompts. You might only find certain parts interesting, and it might not be possible to answer all of the questions put forth. You might also consider covering a section of the book that we didn't discuss in class. If you have an idea, just let me know.

I will post a link on Canvas (not the course webpage) at 5PM July 20th for people to sign-up for projects. Sign-up slots will be available on a first come, first served basis. **If you would like to work on a topic other than those listed, email me your choice by midnight, July 20th.**

I think it is a very good idea to work with a partner, but you will not be required to do so. If you and one partner want to work on a project together, let me know when you email me your project topic. The maximum size of a group can be two.

Some of these projects are harder than others. Do a little research before deciding on a project! Wikipedia is a good starting place. You should look for historical and mathematical connections that you could expound upon. **I will ask you for a list of references and a rough outline of the project by July 22nd.**

Topic Ideas

Many, many thanks to Tori Akin, Asilata Bapat, and Jonathan Rubin for allowing me to use some of their project ideas.

Topic 2.1 ► Infinitely Many Infinities

We know that the set of integers is infinite, and so is the set of real numbers. But does it make sense to say that one infinity is bigger than the other? Well, mathematicians think yes! How do mathematicians decide which is bigger? What does this have to do with injective functions and surjective functions?

Order these sets from biggest to smallest: Integers, Even integers, Odd integers, Rational numbers, Real numbers. Is there anything bigger than the real numbers? What is the Hilbert Hotel? What is Cantor's Diagonal Argument?

Topic 2.2 ► Fibonacci Numbers

You could do many different projects with the Fibonacci numbers, as they satisfy many interesting identity relations. Below we list two possible direction, each of which can be a project on its own.

1. Look at the greatest common divisor of two Fibonacci numbers. You'll notice that the gcd is itself a Fibonacci number. In fact, if F_n denotes the n^{th} Fibonacci number, then $\gcd(F_n, F_m) = F_{\gcd(n,m)}$. Can you prove this by induction?

   For a start, what if we look at two adjacent Fibonacci numbers? Can you prove by induction that $(F_n, F_{n+1}) = 1$? What is *strong induction*? Can you show that $F_{m+n} = F_{m+1}F_n + F_m F_{n+1}$?

   If $m \mid n$ can we see that $F_m \mid F_n$?

2. Can you write any positive integer as a sum of Fibonacci numbers? What about non-consecutive Fibonacci numbers? Are there more than one way of doing this? **Zeckendorf's Theorem** says the following:

   Any positive integer N can be uniquely *expressed as the sum of one or more distinct Fibonacci numbers in such a way that the sum does not include any two consecutive Fibonacci numbers.*

   To prove this fact, you will need to use something called *strong induction*. You will also need to use the following identities (which you should prove)

   $$F_1 + F_3 + F_5 + \ldots + F_{2n-1} = F_{2n}, \quad 1 + F_2 + F_4 + F_6 + \ldots + F_{2n} = F_{2n+1}$$

   What is Fibonacci multiplication? Is this associative?

   What are nega-Fibonacci numbers?

There are many other patterns within the Fibonacci sequence. Try to spot some on your own, or read up on them from Wikipedia, and then use induction to prove the patterns continue indefinitely.

Topic 2.3 ► JOSEPHUS PROBLEM

Suppose 41 people are arranged to stand in a circle and numbered from 1 to 41. Then every 2^{nd} person is removed from the circle and the circle closes up after each removal. What is the number of the last remaining person? [1] Try it out by hand first for cases when you have $1, 2, 3, 4, 5, 6, \ldots, 17, 18$ total people. Do you see a pattern? When is the last survivor the person numbered 1? Can you prove your guess mathematically? What if we start with n people and remove every 2^{nd} person? What is the number $J(n)$ of the last survivor? Show that the function $J(n)$ satisfies the recurrence relations

$$J(2n) = J(n) - 1, \quad J(2n+1) = J(n) + 1, \quad J(1) = 1$$

Use these to calculate $J(41)$ and check that it matches with what you obtained before.

There is almost an explicit formula for $f(n)$. Can you give a proof using induction? What does it have to do with binary representations?

Topic 2.4 ► HOLLYWOOD MATH

Over the years there have been many instances of "Math" being used in Hollywood movies to tackle real-life situations. Below are two famous examples, each of which are instances of deeper mathematical theorems. You can do a project out of either of them. Knowledge of the movie plot is not required!

1. **Die Hard: With a Vengeance**

   McClane (Bruce Willis) and Zeus (Samuel L. Jackson) find a puzzle with a bomb, beside a fountain in a city park. They are given a five gallon jug and a three gallon jug, and have to measure out exactly four gallons of water on a scale to keep a bomb from exploding. How would you do it?

   Are there any other way to do it? Given and 11 gallon and a 4 gallon jug, can you make exactly one gallon?

   If you start with a p gallon jug and q gallon jug ($p < q$), for what n can you measure out n gallon? What does this have to do with gcd and the Euclidean Algorithm?

2. **Speed**

   A city bus in L.A. has been rigged with a bomb that arms when the speed exceeds 50 miles/hour and will explode if it then drops below 50 miles/hour. While driving on the freeway, the bus is forced to jump a 50 foot gap where the road is unfinished. Being Hollywood, of course the jump is successful, but just how realistic is this?

   First of all, the bus has no chance of success unless it launched off a ramp at some angle. From the movie, the initial speed of the bus is 68 miles/hour. If we assume that there is no air resistance, then the mass of the bass doesn't factor into calculation. Estimate at what angle the bus must have left the road.

   What does this have to do with Projectile Motions? Why does the path look like a parabola? Find the general formula for the flight time, maximum height obtained, and the range covered by a projectile launched with velocity v at an angle θ (assuming no air resistance). Can you apply this theory to soccer or baseball?

If you find other fun non-trivial math uses in movies, let me know!

[1] The problem is named after Flavius Josephus. According to Josephus' account, he and his 40 soldiers were trapped in a cave by Roman soldiers. They chose suicide over capture, and settled on the above method. Everyone kills the person who is standing to their right and still alive. Josephus himself wasn't too keen on suicide,and wanted to know who would be last survivor, so that he can surrender!

Topic 2.5 ► Chicken McNugget Problem

The Chicken McNugget Problem (aka Postage Stamp Problem, aka Frobenius Coin Problem) was thought by Henri Picciotto while dining with his son at McDonald's. We will consider a modified version here. Suppose McDonald's sells chicken nuggets in boxes containing either 3 or 5 nuggets. This means you can't get a order of exactly 7 nuggets in any way. What's the largest number of McNuggets that you cannot get as a collection of these boxes? Try out some examples.

The Frobenius version (after the mathematician Ferdinand Frobenius) asks for the largest monetary amount that cannot be obtained using only coins of specified denominations. Are there any formula?

What is Bézout's Identity? What is the Frobenius number of a set? What if the set contained all entries in an AP or a GP?

Topic 2.6 ► Fermat and Beyond!

In 1637 Fermat wrote a note in the margins[2] of a copy of Arithmetica about a fact of mathematics that wasn't proved until 1994! What was the fact?

No three positive integers a, b, and c can satisfy the equation

$$a^n + b^n = c^n$$

for any integer n greater than 2.

The proof of this fact is more than 150 pages long, and not something to take on for this class. Do not try to prove Fermat's Last Theorem in your presentation. However, you can look into some related math. In particular, let's consider the case when $n = 2$.

When $n = 2$, the set {a, b, c} satisfying $a^2 + b^2 = c^2$ is a Pythagorean triple. We know that many of those exist. How many? Infinitely many? What does it mean to be a "primitive" triple? How can we generate "primitive" triples? Are there any formula?

What are *Diophantine Equations*? Can you give some other examples?

Topic 2.7 ► Cutting a Round Cake

Suppose you have a circular cake. You make 5 cuts along straight lines with a knife in such a way that the lines intersect each other in a total of 8 points (intersection on the boundary is not counted) and no three cuts pass through the same point. How many pieces of cake do you have?

Make the cuts along 5 different lines that intersect again at some 8 points (with no three lines concurrent). Now how many pieces do you have now? Can you prove that you will always have the same number of pieces no matter how you choose the lines?

What is a *convex* set of points in the plane? Consider any *convex* region in the plane crossed by l lines with p interior points of intersection. Assume the no three lines are concurrent. Experiment with some different values of l and p and check the number of disjoint regions created, call it r, in each case. Can you guess a relationship between l, p, and r. It is very simple (linear). Can you prove it using induction on l?

Suppose we replace the convex region by the whole plane $\mathbb{R}^2$ (i.e. we are allowing regions to be unbounded). Then what is the *maximum* number of regions created by n intersecting lines on the plane?

What about n intersecting planes in space?

[2] Fermat mentioned the theorem and then wrote "Cuius rei demonstrationem mirabilem sane detexi hanc marginis exiguitas non caperet." That roughly translates as "I have discovered a truly remarkable proof of this theorem which this margin is too small to contain." To this day, we don't know if he *actually* had a proof!

Topic 2.8 ► Friends and Strangers

Six people meet at a party. Consider any two of them. They might be meeting for the first time-in which case we will call them mutual strangers; or they might have met before-in which case we will call them mutual acquaintances. Can you show that either at least three of them are (pairwise) mutual strangers or at least three of them are (pairwise) mutual acquaintances? What does this problem have to do with Graph theory and PHP?

What is a complete graph? What is an edge coloring? What are Ramsey numbers? What are some known bounds on Ramsey numbers?

Topic 2.9 ► The Art Gallery problem

Suppose you have an art gallery containing priceless paintings and sculptures. You would like it to be supervised by security guards, and you want to employ enough of them so that at any one time the guards can between them oversee the whole gallery. How many guards will you need? We will make some assumptions to make the problem easier.

Let's assume that the floor plan of the gallery is a simple polygon, a shape that is bounded by straight line segments that do not cross each other, and which doesn't contain any holes. Let's also assume, as is often the case in real galleries, that the guards don't move around and are placed in the corners, or vertices, of this polygon. Try out some examples. Draw polygons of $5, 9, 13$ sides and try to place the guards. Can you find the minimum? Try more examples and see if you can identify a pattern.

What does this problem have to do with graph theory? What is a triangulation? What is a *dominating set*?

What is the *fortress problem*?

Topic 2.10 ► Knight's Tour

A knight's tour is a sequence of moves of a knight on a chessboard such that the knight visits every square only once. If the knight ends on a square that is one knight's move from the beginning square (so that it could tour the board again immediately, following the same path), the tour is closed, otherwise it is open.

Can you find a closed knight's tour on the 8×8 chessboard? (This is hard, look up the solution online or write a computer program to find one for you!)

What about an open knight's tour on 4×3 board? Find all of them.

Prove that a $4 \times n$ board ($n \in \mathbb{Z}$) has no closed Knight's tour.

Given a $m \times n$ board, when does a closed knight tour exist?

What are Hamiltonian Paths?

Topic 2.11 ► Rigging a Bachet's Game

Anna and Ben play a game. Initially there are n stones on the table. Anna and Ben will move alternately. Anna always starts. A legal move consists of removing at least one but not more than k stones from the table. The winner is the one to take the last stone. Assume that both players are genius and always make the best decision each round. Can you choose n and k in a way such that Anna always has a foolproof strategy to win?

Consider a variation of this game. The number of stones that can be removed in a single move must be a member of a certain set of numbers, call it L. Thus in the game above $L = \{1, 2, \ldots, k\}$. Assume L always contains 1 and thus the game never stalls.

If $L = \{1, 2, 4, 8, \ldots\}$, any power of 2, can you choose n in a way so that the first player always has a winning strategy?

What if $L = \{1, 2, 3, 5, 7, 11, 13, \ldots\}$, the set of prime numbers and 1?

Topic 2.12 ► Kakeya Needle Problem

A *Kakeya needle set* (sometimes also known as a Kakeya set) is a region in the plane with the property, that a unit length line segment can be rotated continuously through 180 degrees within it, returning to its original position with reversed orientation. Clearly, a disk of diameter 1 is an example of a Kakeya needle set (Why?). Can you find a set that has smaller area?

What is a convex set? What's the lowest possible area of a convex Kakeya Needle set?

What is the construction by Besikovitch which shows that there is no lower bound for the area of such a region if we allow non-convex sets? What is the 'sprouting' method?

Topic 2.13 ► The Kiss Precise

Can you draw 4 circles such that each of them is tangent to all of the others? *Descartes' theorem* states that for every four kissing, or mutually tangent, circles, the radii of the circles satisfy a certain quadratic equation. What is it?

What is *curvature* of a circle? What happens when the radius of one (or more) of the circle becomes really large and diverges towards infinity? What are some special cases of Descartes' theorem?

Who is Frederick Soddy? What is *'The Kiss Precise'*? Who is Thorold Gosset?

Topic 2.14 ► Continued Fraction

Continued fractions

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

offer an intriguing, alternative method for representing real numbers. They are also neat calculation tools. For example, we can use continued fractions to compute the value of $\sqrt{2}$ by hand! Find the continued fraction representations of some irrational numbers x (such as $\sqrt{2}$), and use them to estimate x, correct out to three decimal places. Look up the continued fractions of e and π.

The "form" of a continued fraction also is interesting. Why are rational numbers precisely those real numbers with finite continued fraction expansions? What can you say about periodic continued fractions? What do the continued fraction expansions of roots of quadratic equations with integer coefficients look like?

How can you use continued fractions to give rational approximations of real numbers? How good of an approximation can you get?

Topic 2.15 ► Base Representations of Real numbers

Just as natural numbers can be represented in base 10, or some other base $b > 1$, so too can real numbers. In base b, the expression

$$a_n a_{n-1} a_{n-2} \cdots a_1 a_0 . a_{-1} a_{-2} a_{-3} \cdots$$

is notation for the infinite series

$$a_n b^n + a_{n-1} b^{n-1} + \cdots + a_1 b^1 + a_0 b^0 + a_{-1} b^{-1} + a_{-2} b^{-2} + \cdots$$

What is an infinite series? What is the value of an infinite geometric series? What is the comparison test and why does it imply that all b-ary expansions converge?

The "form" of a b-ary expansion can tell us many things. Why are rational numbers precisely those reals whose decimal expansions either terminate or are eventually periodic? How do you compute the exact value of a periodic decimal? What about other bases?

What is phinary?

Topic 2.16 ► Cauchy Functional Equation

Equations for unknown functions are called functional equations. One of the famous example is as follows.

We want to find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R} \tag{$\star$}$$

Start with the limit definition of derivatives and simplify $f'(x)$ using the given relation. What does that tell you about $f(x)$?

Now suppose we know that the function f satisfies the relation ($\star$) from above and is continuous (you are not told whether or not f is differentiable). Now what can you say about f? Try the following steps.

First find $f(0)$. suppose $f(1) = c$. Now can you define f on natural numbers? Is f an even function or odd function? Can you extend your definition to larger number sets? How do you use the continuity property?

What are the other Cauchy Functional Equations? What are their solutions if we assume differentiability?

Topic 2.17 ► Pancake Theorem

You and your brother are having breakfast. Your dad serves one chocolate and one buttermilk pancake on the same plate side by side. Both of you love either type of pancake equally and you decide to cut each pancake in half. Can you bisect both pancakes simultaneously with a single straight cut?

We are making some assumptions here. Assume that the pancakes have even thickness everywhere. Assume they are lying on a plane, side-by-side, not overlapping each other. Assume you have sufficiently long and sharp knife (or a sword, if you want!).

What are continuous functions? What is the *Intermediate Value Theorem*? Can you some more interesting examples of IVT?

What is the *Ham Sandwich Theorem*?

Topic 2.18 ► Euler's Number

You invest \$1 at 100% yearly interest; so that, at the year end you have \$2. What if, instead the bank gave you 50% interest every six months? How much has your investment grown if you compound monthly? daily? hourly? continuously? What does it mean to compound continuously? How is this related to e, Euler's number?

What are some other ways to define e (Using a continued fraction? Using an infinite sum?) What are some of nice properties of the function e^x?

Can you show that e is irrational?

Topic 2.19 ► Roots of Polynomials

Tickets for a basketball game are on sale for \$30. At this price, $10,000$ people will attend. For every 10 cents that the price drops, 50 additional people will attend the game. At what price should the tickets be sold to maximize profits?

This problem (any many other optimization problems) can be solved by *completing the square*. Use the method of completing squares to derive the *quadratic formula*. What is the geometric significance of this?

Are there similar formula for roots of a cubic polynomial? How about a quartic or a quintic?

What is the *Abel-Ruffini Theorem*? What is an *Algebraic solution*?

What is the *Newton-Raphson method*?

Topic 2.20 ► Other Possible Topics

Magic Squares, Euler's Totient function, Catalan Numbers, Hall's Marriage Problem, Pascal Triangle, something that interests you!!

Components

Paper

Format

Papers should be $5-10$ pages total (double-spaced, 12pt professional font, 1 inch margin). You may use a word processor with an equation editor, e.g. Microsoft Word (Using LaTeX is not necessary). If you do want to learn and use LaTeX, here is a 30-minute introduction.

Historical/Background Information

Give the background information on the problem you are studying. Who has studied the problem? Which mathematicians have contributed to the solution? Why is this problem interesting/important? Are there any connections to history, politics, culture? How are the mathematical conventions of other cultures/times relevant?

The prompts vary in the type of background information that is most appropriate. I don't expect you to give an exhaustive account of all historical connections. I want you to tell me a compelling story. Give me a reason to care about the problem, or demonstrate that some prominent historical figures were interested in the math.

Examples

Motivate the math by working out simple and illuminating examples in detail. For example, in the Fibonacci number project, you should compute out a few instances of the pattern you have in mind. Or for the Infinitely many Infinities project, you should write down an explicit bijection to show that there are as many integers as even integers. Try to find examples that demonstrate key aspects of the proof that you will write down abstractly. The abstraction will make more sense if you ground it in concrete numbers.

Proof

In each of the projects you should prove (at least some of) the relevant mathematical facts. You should give a complete, clear, and understandable proof. You don't need to come up with the proof on your own; you should be able to find references that explain the math. Make sure you understand the math and can present it clearly to the audience.

For some projects it's more obvious what to prove than others. For example, in the Cake cutting project, you should prove the relationship among l, r, and p. However, for example, in the Knight's Tour project, I think there are some options as far as which proofs to include. You should look through several references and decide which math is most related to the class/would be best to discuss.

Citations

Your work should be well cited. With your paper, you should turn in a complete reference list in either MLA or APA format. Wikipedia is a good starting place for understanding the math/finding other references, but you should have reputable sources in your final reference list.

Presentation

Time

The presentation should be $10-20$ minutes long (timing will depend on the number of groups).

Topics

You may include in your presentation any compelling/relevant material. You might not have time to give all of the details of a proof. You'll need to decide what to include and what to leave out. The presentations should be understandable. Don't blow through high-level math and leave your audience in the dust. Ask questions if you can. Engage your audience! Choose exciting material to present!

Media

Sometimes the best way to convey your point is by keeping things simple. Depending on your presentation, you might want to give a talk using the whiteboard exclusively. You may also combine the use of the board with other media, if that works for your presentation. The classroom has HDMI and VGA ports, if you would like to use the projector. You are welcome to make a slide show, but please keep the number of equations per slide low. In the past, groups have made and shown their own videos. Depending on your project (e.g., Josephus Problem, Bachet's game, Knight's tour etc.), you might want to write and run some code. Or you could mix any of these styles. Do whatever works best for your project.

Timeline

Email me your project choice. Let me know if you will work with a partner.	Friday, July 20
Presentation day and time assigned	Saturday, July 21
Email me a list of references and tell me what proof(s) you want to include.	Midnight, Sunday, July 22
Progress Meetings	Thursday & Friday, July 26-27
In class Presentations	Thursday & Friday, August 2-3
Project Report Due	Midnight, Sunday, August 5
Extra time for presentations, if needed.	Monday, August 6

Rubric

	Paper	
Background	Address important/relevant facts	5
	Piques interest, motivates study	5
Example(s)	Choice of Example: Simple and illuminating	5
	Accuracy of Mathematics	5
	General Understanding	5
Proof	Sophistication of Mathematics: Includes appropriately challenging math	10
	Accuracy of Mathematics	5
	General Understanding	5
General	Citations	5
Total		50

	Presentation	
Time	Judicious use of time	5
Content	Background: Piques interest	10
	Examples: Clear presentation, accurate math, solid understanding	10
	Proof: Clear presentation, accurate math, solid understanding	15
Style	Well Prepared, good use of board, prudent use of other media	10
Total		50