Math 1600 Handout : Practice Problems for Chain Rule, Derivative of Inverse Functions, and Implicit Differentiation

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Derivative of Inverse function

Given an invertible function $f(x)$, the derivative of the inverse function of $f(x)$ is given by

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Using above rule we showed in class that

- Derivative of $\ln x$ is $\frac{1}{x}$.
- Derivative of $\arcsin x$ is $\frac{1}{\sqrt{1-x^2}}$.
- Derivative of $\arctan x$ is $\frac{1}{1+x^2}$.

To figure out the derivative of $\arccos x$, we could do the same calculation as above, or we could use a trigonometric identity as follows.

Recall that $\sin(\pi/2 - \theta) = \cos\theta$. Let $x = \cos\theta$. Then $\theta = \arccos(x)$ and $\pi/2 - \theta = \arcsin(x)$. Hence

$$\arcsin x + \arccos x = \pi/2$$

Now taking derivative of both sides of above equation with respect to x, we get

$$(\arcsin x)' + (\arccos x)' = 0 \implies (\arccos x)' = -(\arcsin x)' = -\frac{1}{\sqrt{1-x^2}}$$

Exercise 1

Let g denote the inverse function of f. Suppose

$$f(3) = -6, \quad f'(3) = 2/3, \quad f(-6) = 2, \quad f'(2) = 1,$$

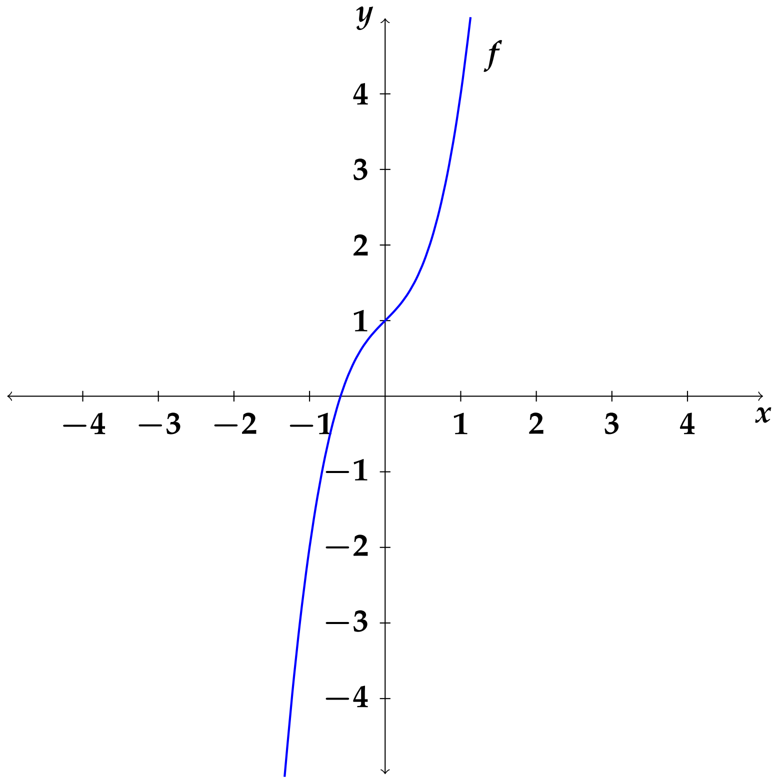
$$f'(-6) = 3, \quad f'(-1) = -6, \quad f'(-6) = 5$$

What is $g'(-6)$?

Ans: $3/2$.

Exercise 2

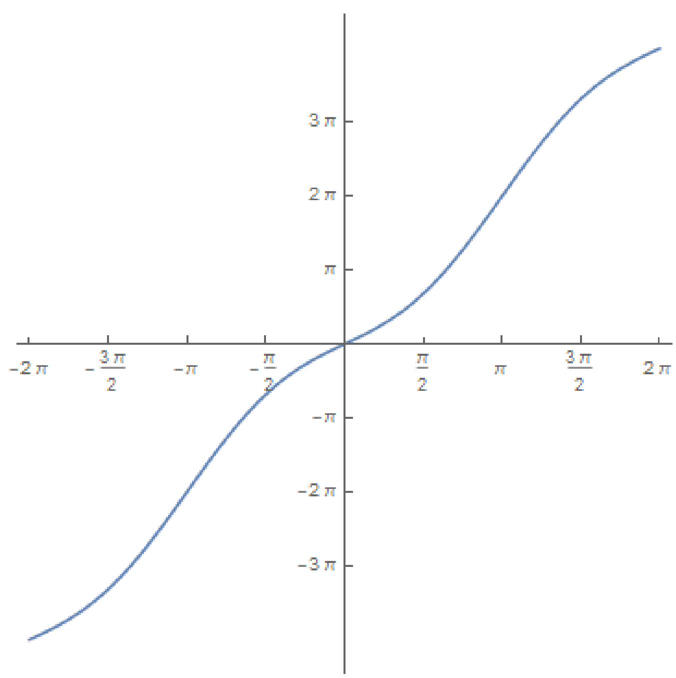
Let $g(x)$ be the inverse function of $f(x) = 2x^3 + x + 1$. What is $g'(4)$?



ANS: **1/7.**

Exercise 3

Let $f(x) = 2x - \sin(x)$ (graphed below) and let $g(x)$ be the inverse function of $f(x)$. Then find $g'(2\pi)$.



ANS: **1/3.**

Chain Rule

Exercise 4

Find the derivative of the following functions using Chain rule. You might also need to use product rule or quotient rule.

(a) $(e^{-x} - 6\pi)(5x^3 + \tan x)$ Ans: $\left(e^{-x} - 6\pi\right)\left(15x^2 + \sec^2(x)\right) - e^{-x}\left(5x^3 + \tan(x)\right)$

(b) $\cos(\ln\theta)$ Ans: $-\frac{\sin(\log(\theta))}{\theta}$

(c) $\sin^3(e^{7t} - t)$ Ans: $3\left(7e^{7t} - 1\right)\sin^2\left(e^{7t} - t\right)\cos\left(e^{7t} - t\right)$

(d) $7e^{2x^5 - \sin(x^3)}$ Ans: $7e^{2x^5 - \sin\left(x^3\right)}\left(10x^4 - 3x^2\cos\left(x^3\right)\right)$

(e) $\ln(e^x - \tan(x^3))$ Ans: $\frac{e^x - 3x^2\sec^2\left(x^3\right)}{e^x - \tan\left(x^3\right)}$

Linear Approximation and Tangent Lines

The equation of the tangent line to the graph of $f(x)$ at $x = a$ is given by

$$y = f(a) + f'(a)(x - a)$$

We define the 'linear approximation' of $f(x)$ near $x = a$ to be

$$L(x) = f(a) + f'(a)(x - a)$$

Exercise 5

Problems $3.7.(11, 12, 16)$.

Exercise 6

Find the point on the curve $y = 2x^2 - x + 1$ where the tangent is parallel to the line $y = 3x + 9$.

Ans: $(1, 2)$

Exercise 7

Find the equation of the line tangent to $f(x) = x^3 + 3x^2$ at $x = -1$.

Ans: $y = -3x - 1$

Implicit Differentiation

Derivative of y with respect to x is $\frac{dy}{dx}$. So derivative of $f(y)$ with respect to x is

$$\frac{d}{dx}f(y) = \frac{d}{dy}f(y)\frac{dy}{dx} = f'(y)\frac{dy}{dx}$$

Exercise 8

Find the point on the curve $x^2 + y^2 - 2x = 3$ where the tangent is parallel to the x-axis.

ANS: $(1, \pm 2)$

Exercise 9

Find all points on the curve given by $(y-1)^3 = x^2 - 1$ where the tangent line is vertical.

ANS: $(-1, 1), (1, 1)$.

Exercise 10

Let $x^2 + y^2 = 25$. Then find $\dfrac{d^2y}{dx^2}$.

ANS: $-\dfrac{x^2+y^2}{y^3}$

Exercise 11

If $x\sin(xy) + 2x^2 = 0$, then find $\frac{dy}{dx}$.