Math 1800-C Handout 3: Parametrized Curves and Chain Rule

Subhadip Chowdhury

Exercise 1

1. Find parametric equations for the line through the points $(6, 1, 1)$ and $(9, 1, 4)$. Call this line L_1.
2. Find parametric equations for the line through the points $(-4, 4, 0)$ and $(-6, 5, 1)$. Call this line L_2.
3. Find parametric equations for the line through the points $(6, -1, -5)$ and $(2, 1, -3)$. Call this line L_3.
4. Verify that L_2 and L_3 are parallel. (Their direction vectors should be parallel.) Are they the same line? How could you tell?
5. Do lines L_1 and L_2 intersect? If so, where?
6. Find the intersection of L_1 with the plane given by the equation $2x + y + 3z = 7$.
7. **Challenge:** Find the point on the plane $2x + y + 3z = 7$ which is closest to the origin.
8. **Challenge:** Find the point on L_2 closest to the origin.

Solution:

1.7. Normal vector to the plane is $\langle 2, 1, 3\rangle$. Equation of line through origin parallel to normal is $x = 2t, y = t, z = 3t$. This intersects the plane when $2(2t) + t + 3(3t) = 7 \implies t = 1/2$.

There are also possible ways of doing this using Projection Vectors or using Lagrange Multipliers.

1.8. Points on L_2 are $(-4 - 2t, 4 + t, t)$. Closest to origin when $\sqrt{(-4 - 2t)^2 + (4 + t)^2 + t^2}$ is minimized or equivalently when $(-4 - 2t)^2 + (4 + t)^2 + t^2$ is minimized. Taking derivative wrt t and setting that equal to zero gives $t = -2$.

Exercise 2

Parametrize the line segment connecting $(3, 7)$ to $(5, -2)$, so that $t = 0$ corresponds to $(3, 7)$ and $t = 1$ corresponds to $(5, -2)$.
Hint: Find the vector from the first point to the second, and parametrize by scaling the vector!
Solution: $x = 3 + 2t, y = 7 - 9t$.

Exercise 3

Suppose the curve given by $\vec{r}(t) = \langle \cos(\pi t), \sin(\pi t), t\rangle$ intersects the paraboloid $z = x^2 + y^2$ at a point $P = (x_0, y_0, z_0)$.

1. Find the coordinates of P.
2. Find equation of the tangent plane to the paraboloid at P.
3. What is the equation of the tangent line to the curve $\vec{r}(t)$ at P?
4. What is the angle of intersection between the curve and the paraboloid? This is the angle between the tangent line in part (3) and the plane in part (2).

Solution: At the intersection, $t = 1 \implies P = (-1, 0, 1)$. Gradient is $\langle 2x, 2y, -1 \rangle = \langle -2, 0, -1 \rangle$. Tangent plane is $-2x - z = 1$. Tangent line to $\vec{r}(t)$ is parallel to $\vec{r}'(t) = \langle -\pi \sin(\pi), \pi \cos(\pi), 1 \rangle = \langle 0, -\pi, 1 \rangle$. Equation is $x = -1, y = -\pi t, z = 1 + t$. The angle between the tangent line and the plane is

$$arccos \frac{\langle 0, -\pi, 1 \rangle \cdot \langle -2, 0, -1 \rangle}{|| \langle 0, -\pi, 1 \rangle || \, || \langle -2, 0, -1 \rangle ||} - \pi/2$$

The $-\pi/2$ comes because the **arccos** part gives angle between the tangent line and the normal to the plane, at it turns out ot be an obtuse angle.

Exercise 4

Let $z = f(x, y) = x^2 + y^3$, and $x = x(s, t)$ and $y = y(s, t)$; i.e., x and y are functions of s and t. Suppose that when $(s, t) = (0, 1)$, we have:

$$x(0,1) = -1, \quad x_s(0,1) = -4, \quad x_t(0,1) = -7, \quad y(0,1) = 2, \quad y_s(0,1) = 10, \quad y_t(0,1) = 5.$$

Compute $\dfrac{\partial z}{\partial t}$ at $(s, t) = (0, 1)$.

Exercise 5

If $u = f(x, y)$, where $x = r \cos \theta$ and $y = r \sin \theta$, show that

$$\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 = \left(\frac{\partial u}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial u}{\partial \theta} \right)^2$$

Solution: Start from the right hand side. Evaluate $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial \theta}$ using chain rule and use $\sin^2 \theta + \cos^2 \theta = 1$ to simplify.