Assignment 2 (9/28)

Subhadip Chowdhury

Problem 1

Review of Comparisons:

Implication	$P \implies Q$	First statement implies the second
Inverse	$\neg P \implies \neg Q$	Negation of both Statements
Converse	$Q \implies P$	Reversal of both statements
Contrapositive	$\neg Q \implies \neg P$	Reversal and negation of both statements
Negation	$\neg(P \implies Q)$	Contradicts the implication

The "Negation" can also be written as "P is true but not Q". Note that it is **not** a "if-then" comparison.

Examples:

Take the statement "All red objects have color." This can be equivalently expressed as "If an object is red, then it has color."

The **contrapositive** is "If an object does not have color, then it is not red." This follows logically from our initial statement and, like it, it is evidently true.

In other words, the contrapositive is logically equivalent to a given conditional statement

The **inverse** is "If an object is not red, then it does not have color." An object which is blue is not red, and still has color. Therefore in this case the inverse is false.

The **converse** is "If an object has color, then it is red." Objects can have other colors, of course, so, the converse of our statement is false.

The **negation** is "There exists a red object that does not have color." This statement is false because the initial statement which it negates is true.

[Source: Wikipedia!]

Assignment:

Write the converse, inverse, negation and contrapositive of the following statement:

If n is divisible by 4, then n is divisible by 2.

We will be discussing functions and bijections in more details in chapter 7.

- **Definition(Domain and Range):**

  Given a function $f : X \to Y$, the set X is the domain of f; the set Y is the codomain of f. The image (sometimes called the range) of f is the set of all values assumed by f for all possible x; this is the set of all values of the form $f(x)$ for all $x \in X$, denoted set theoretically as $\{f(x) \mid x \in X\}$.

- **Definition(Injective function):**(Definition 7.1.1)

  A function $f : A \to B$ is said to be injective if $f(a) = f(b) \implies a = b$ for all $a, b \in A$.

- **Definition(Surjective function):** A function $f : A \to B$ is called surjective if for every $b \in B$, there exists some $a \in A$ such that $f(a) = b$. This is the case when range is same as codomain.

Problem 2

Find the largest subset of $\mathbb{R}$ that can be the domain of the following functions. Then decide whether or not they are injective. Determine the range of each function as well.

(a) $x + \frac{1}{x}$

(b) $\frac{x}{|x|}$

(c) $\frac{1}{(x+1)^{2/3}}$

Problem 3

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \frac{(1-x)(x-2)(x-6)}{x^2(x-5)}$$

Is f surjective onto $\mathbb{R}$?

Problem 4

In each of the following case, give examples of a function f and two sets A, B, both subsets of $\mathbb{R}$; such that $f : A \to B$ satisfies:

(a) $f'(x) > 0$ for all $x \in A$ and f is surjective onto B, but f is not injective.

(b) $f : A \to B$ is not injective but $f : A' \to B$ is injective for some subset $A' \subseteq A$.

(c) $B = \mathbb{R}$, $f : A \to \mathbb{R}$ is bijective and f has infinitely many discontinuities.

(d) f is not a constant function and $f(A') = B$ for infinitely many subsets $A' \subseteq A$.

$\boxed{\textbf{Problem 5}}$

(a) Prove that the function $f : [0, \infty) \to [0, \infty)$ defined as $f(x) = x^2$ is surjective.

(b) Prove that the function $g : \mathbb{N} \to \mathbb{N}$ defined as $g(n) = n^2$ is NOT surjective.

$\boxed{\textbf{Problem 6}}$

Problems 3.1.$(42, 52, 59)$.

$\boxed{\textbf{Problem 7}}$

Let a function $f : \mathbb{R} \to \mathbb{R}$ satisfy the equation

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$. Prove that

1. if f is continuous at $x = 0$, then it is continuous for all $x \in \mathbb{R}$.
2. If f is differentiable everywhere then $f(x) = xf(1)$ for all $x \in \mathbb{R}$.

$\boxed{\textbf{Problem 8}}$

Problem 4.3.$(40, 42)$.