Math 1800 Project 3: Three Dimensional Pythagorean Theorem

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A tetrahedron is a solid with four vertices, $P, Q, R,$ and $S,$ and four triangular faces, as shown in the figure 1.

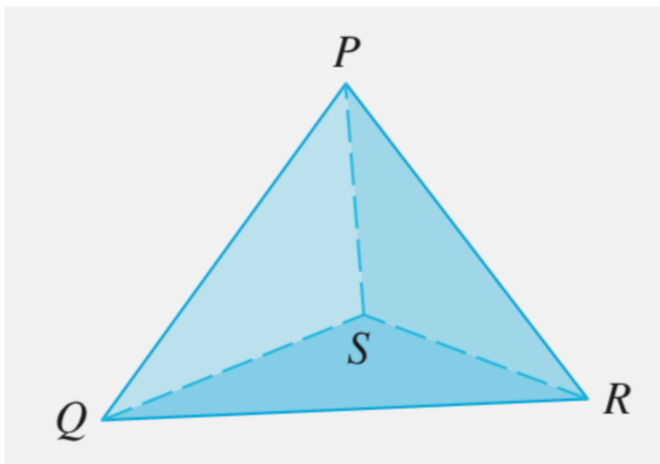

Figure 1

(a) Let $\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \vec{\mathbf{u}}_3$, and $\vec{\mathbf{u}}_4$ denote the four vectors $\overrightarrow{PQ}, \overrightarrow{QR}, \overrightarrow{RS}$, and $\overrightarrow{SP}$ respectively. What is $\vec{\mathbf{u}}_1 + \vec{\mathbf{u}}_2 + \vec{\mathbf{u}}_3 + \vec{\mathbf{u}}_4$ equal to?

(b) Let $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3,$ and $\vec{\mathbf{v}}_4$ be vectors with lengths equal to the areas of the faces opposite the vertices $P, Q, R,$ and S, respectively, and directions perpendicular to the respective faces and pointing outward. Show that

$$\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2 + \vec{\mathbf{v}}_3 + \vec{\mathbf{v}}_4 = \vec{\mathbf{0}}$$

[Hint: Write $\vec{\mathbf{v}}_i$'s in terms of $\vec{\mathbf{u}}_i$'s using cross products. Be careful about signs. Factorize and use the result of part (a).]

(c) The volume V of a tetrahedron is one-third the distance from a vertex to the opposite face, times the area of that face.

  (i) Find a formula for the volume of a tetrahedron in terms of $\vec{\mathbf{u}}_i$'s.

  (ii) Find the volume of the tetrahedron whose vertices are $P(1,1,1), Q(1,2,3), R(1,1,2)$, and $S(3,-1,2)$.

(d) Suppose the tetrahedron in the figure has a trirectangular vertex S. (This means that the three angles at S are all right angles.) Let $A, B,$ and C be the areas of the three faces that meet at S, and let D be the area of the opposite face $\triangle PQR$. Thus $D = \|\vec{\mathbf{v}}_4\|$ etc. Using the result of part (b), or otherwise, show that

$$D^2 = A^2 + B^2 + C^2$$

(This is a three-dimensional version of the Pythagorean Theorem.)

[Hint: For a vector $\vec{\mathbf{f}}$, recall that $\vec{\mathbf{f}} \cdot \vec{\mathbf{f}} = \|\vec{\mathbf{f}}\|^2$.]