Assignment 7 (7/9)

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- *This homework is due at the beginning of class on **Tuesday** 7/17. You are encouraged to work together on these problems, but you must write up your solutions independently.*

Fundamental Theorem of Arithmetic

Theorem 1.1 (Fundamental Theorem of Arithmetic)**.** *Every integer greater than* 1 *can be represented as the product of prime number(s) and this representation is unique up to the order of the prime factors.*

Example 1.2. $720 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 5 = 2^4 \cdot 3^2 \cdot 5$. The statement 'up to the order of the factors' means we can also write it as $2^4 \cdot 5 \cdot 3^2$. Thus we can change the order in which we write the primes, but the theorem says that the primes themselves (and their powers) are unique for each natural numbers.

Recall that n factorial, denoted $n!$, is defined as

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-2)(n-1)n$$

Thus $3! = 6, 5! = 120$ etc. Factorial is not defined for negative numbers and fractions.

Problem 1.3. *How many zeroes are at the end of* 2018!*?*

Solution. Note that the number of zeroes at the end of 2018! is equal to the highest power of 5 that divides 2018!. This is because $10 = 2 \times 5$ and 5 appears more time as a factor than 2 in 2018!. Now we will have one factor of 5 for every multiple of 5 up to 2018, one more for each multiple of 25, one more for each multiple of 125, and so on. Make sure you understand why.

Now there are 403 multiples of 5 up to 2018, 80 for $25 = 5^2$, 16 for $125 = 5^3$, and 3 for $625 = 5^4$. Note that, for example, a multiple of a prime power is also a multiple of a lower power of the prime (e.g. multiple of 125 is also multiple of 25). Thus we only need to count the total number of multiples **once** for each prime power.

So the number of zeroes at the end of 2018! is

$$403 + 80 + 16 + 3 = 502.$$

■

Definition 1.4. The *greatest integer function* or *box* function or *floor* function, denoted $[x]$ is defined to be the greatest integer less than or equal to x.

To avoid ambiguity, we are going to use the notation $\lfloor \cdot \rfloor$ to mean the box function and reserve the brackets for their usual meaning. Note that both are acceptable mathematical notation, but you should stick to one of them within one problem.

Example 1.5. $\lfloor 2.5 \rfloor = 2, \lfloor 3.3 \rfloor = 3, \lfloor 2 \rfloor = 2, \lfloor 0 \rfloor = 0, \lfloor -1.5 \rfloor = -2, \lfloor -2 \rfloor = -2$.

The graph of the box function looks as follows:

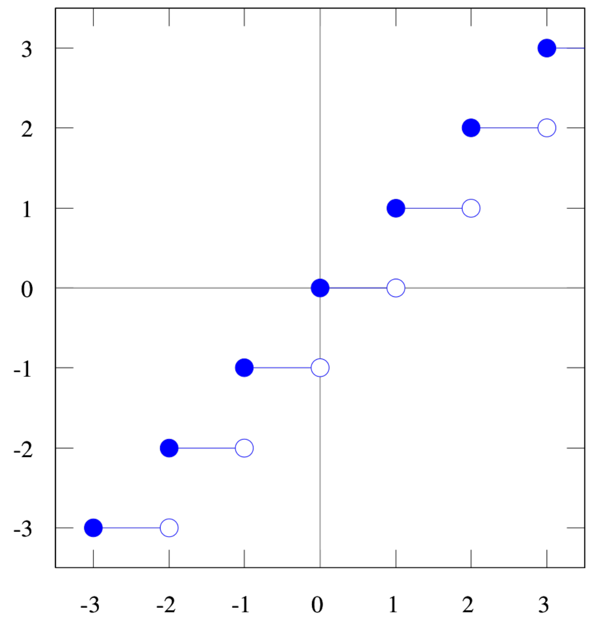

Figure 1: The Floor Function, courtesy: Wikipedia

Legendre's formula: Using the floor function, we can give a nice formula for the general case of problem 1.3. Following the exact same logic as in the solution, we get that the exponent of the highest power of p that divides $n!$, denoted ν_p, is given by

$$\nu_p = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \left\lfloor \frac{n}{p^4} \right\rfloor + \dots \qquad (\star)$$

Note that the sum above is finite, since the summands eventually will be zero for large enough power on p.

Exercise 1. *We can define the* fractional part *of x, denoted $\{x\}$, to be $\{x\} = x - \lfloor x \rfloor$. Draw the graph of the fractional part function.*

Exercise 2. *The* ceiling function*, denoted $\lceil x \rceil$, is defined to be the least integer greater than or equal to x. Draw the graph of the ceiling function.*

Exercise 3. *Sketch rough graphs of the functions*

$$f(x) = \left\lfloor \frac{1}{x} \right\rfloor$$

and

$$g(x) = \frac{1}{\lfloor x \rfloor}.$$

Exercise 4. *Solve the following equation*

$$\lfloor 3.2 + \lfloor 2.5x - 7.9 \rfloor \rfloor = -5.$$

Exercise 5. *Prove that $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$ for all real number x.*

Thus the FTA (also known as, the Unique-Prime-Factorization theorem) tells us that every natural number can be factored all the way to prime numbers. Using this theorem, we can prove the following observation.

Proposition 1.6. *Given a prime number p and a natural number n, if $p \mid n^2$, then $p \mid n$.*

Proof. Let

$$n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}$$

be the unique prime factorization of n, where p_i are prime numbers and α_i are natural numbers denoting the exponent of p_i respectively.

Then

$$n^2 = p_1^{2\alpha_1} p_2^{2\alpha_2} p_3^{2\alpha_3} \cdots p_k^{2\alpha_k}$$

Hence n and n^2 share the same prime factors, only with different exponents. Thus if $p \mid n^2$, then p must be equal to one of the p_i's, and consequently $p \mid n$. □

We will use proposition 1.6 to give another short proof of the following claim.

Claim. *$\sqrt{2}$ is irrational.*

Proof. Assume, for the sake of contradiction, that $\sqrt{2}$ is rational. Then we can find integers a and b such that $(a,b) = 1$, $b \neq 0$ and $\sqrt{2} = \frac{a}{b}$. Then

$$\begin{aligned} 2 &= \frac{a^2}{b^2} \\ \Longrightarrow 2b^2 &= a^2 \\ \Longrightarrow 2 &\mid a^2 \\ \Longrightarrow 2 &\mid a \\ \Longrightarrow 4 &\mid a^2 \\ \Longrightarrow 4 &\mid 2b^2 \\ \Longrightarrow 2 &\mid b^2 \\ \Longrightarrow 2 &\mid b \end{aligned}$$

But that means 2 is a common factor of a and b which contradicts $(a,b) = 1$. □

Exercise 6. *Prove that $\sqrt{7}$ is irrational.*

Exercise 7. *Show that $x^2 - 4y = 2$ does not have any integer solution. In other words, show that if x and y are integers then $x^2 - 4y$ can not be equal to 2.* [HINT: *Use proposition 1.6*]

Exercise 8 (Extra Credit)**.** *Given a natural number n, prove that*

$$4 \textit{ divides } \frac{(2n)!}{(n!)^2} \Longrightarrow n \textit{ is not a power of } 2.$$

[HINT: *Prove the contrapositive. Use Legendre's Formula ($\star$).*]

Remark 1.7. Note that $(2n)! \neq 2 \times n!$. In fact, $(2n)! = 2n(2n-1)(2n-2)\cdots 3 \cdot 2 \cdot 1$.

Remark 1.8. The converse of above exercise is true as well. That is much harder to prove.