Dissection Of a Square Into Triangles Of Equal Area

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Abstract

In what follows,We discuss an interesting problem with rather easy looking statement published in American Mathematical Monthly in 1967,proposed by Fred Richman and John Thomas and solved by Paul Monsky in 1970.It Proves that if a square is dissected into triangles having equal area then number of triangles obtained must be even.It uses p-adic numbers and Sperner's Lemma in two Dimension.

1 Introduction and A Bit of History

In 1965 Fred Richman at the University of New Mexico at Las Cruces wanted to include a geometric question in an examination he was preparing.But not being able to solve it himself,he chose not to put it on the exam. However,he mentioned it to colleagues and his Bridge Partner,John Thomas.Thomas recalls that

> Everyone to whom the problem was put(myself included) said something like 'that is not my area but the question surely must have been considered and the answer is probably well known'...When I sent the paper to *Mthematics Magazine*,the referee's reaction was predictable.He thought the problem might be fairly easy(although he could not solve it) and was possibly well known(although he could find no reference to it).

The problem was submitted to the *Monthly* :

Can a square S be divided into an odd number of non-overlapping triangles T_i ,all of the same area?

In 1968,The answer was shown to be **no**, by Thomas, provided $S = [0,1] \times [0,1]$ and coordinates of vertices of T_i are rational no. with odd denominators.
Paul Monsky settled the question in 1970, three years after it was published in the *Monthly* by removing restrictions assumed in the the proof given by Thomas in 1968.[3]

2 Tools for the Proof

There are two parts to the proof : one conbinatorial, other valuation theoretic.The combinatorial argument, which is a generalisation of the argument in [3],itself may be made to prove the special case considered in [3]. But to handle the case of arbitary vertices,it becomes necessary to argue with ”congruences mod 2 in reals”.That’s where we require valuation theory and 2-adic absolute value of rationals.In the course of proof extension of valuation plays a remarkable and unexpected role.

2.1 Combinatorial Part

first we define some terminology we are going to use:
Suppose $\Re$ be a region in the plane bounded by a simple closed polygon. Suppose $\Re$ is divided into **m** nonoverlapping triangles T_i.

1. By a **‘vertex’** we shall mean vertex of some T_i.
2. By a **‘face’** we shall mean a face of some T_i or $\Re$.
3. Two vertices are called **‘adjacent’** if they are in the same face and the line segment joining them contains no other vertices.
4. A **‘basic segment’** is a segment joining two adjacent vertices.

Note that the boundary of each T_i is a union of nonoverlapping basic segments; the same is true for the boundary of $\Re$.

Suppose now that the vertices have been colored in three different colours RED,BLUE and GREEN. We shall say that a basic segment is of **‘type RB’** if it has one end point colored RED and the other BLUE.

Lemma 1. *If a triangle T_i has vertices of at most 2 colors (only counting the 3 extremal vertices) and any if any face has points with atmost two colors then T_i has an even number of* RB *basic segments.*

Proof. Consider all possible colorings of the extremal vertices of T_i. In case neither Red color is not there or in case Blue color is not there, by the assumption of having a most two colors in any face, there are exactly zero RB basic segments. If the the three extremal vertices have R-R-B or B-B-R (there cant be any other case) coloring then observe the following:

Suppose we have n vertices on a face having end points colored Red and Blue respectively. Clearly vertices are colored either Red or Blue. Then number of basic segments of type RB *is odd.*

By the observation and the fact that In R-R-B or B-B-R type coloring there are exactly two faces having end points colored Red and Blue respectively in each one of them, we have an $odd + odd = even$ number of basic segments of type RB in T_i. ■

Corollary 1. *Suppose that no face contains vertices of all three types and that $\Re$ has an odd number of faces of type* RB*.Then some T_i has vertices of all three colors.*

Proof. We count the number of RB basic segments on all the faces of all triangles and equate that to number of RB segments on boundary + $2\times$ RB segments in interior. [Clearly for any basic segment not on boundary of $\Re$ will be counted twice.] So the number of RB segments on boundary of $\Re$ is congruent to sum of the boundaries of the T_i modulo 2. Thus there must exist a triangle with odd number of RB segments in its face. So by the Lemma, that triangle has vertices of all three colors. ■

2.2 Valuation Theory

Now we discuss our other tool for the proof, *Valuation Theory* which is algebraic. A *Valuation* on a field **F** is a function ϕ from **F** to $\mathbb{R} \cup \{\infty\}$ satisfying the following properties:

1. $\phi(xy) = \phi(x) + \phi(y)$
2. $\phi(x + y) \geq min\{\phi(x), \phi(y)\}$
3. $\phi(x) = \infty$ if and only if $x = 0$.

We note the following properties of ϕ:

- $\phi(1) = \phi(1.1) = \phi(1) + \phi(1) \Rightarrow \phi(1) = 0$
- $\phi(1) = \phi(-1. - 1) = \phi(-1) + \phi(-1) \Rightarrow \phi(-1) = 0$
- $\phi(-x) = \phi(-1.x) = \phi(-1) + \phi(x) \Rightarrow \phi(x) = \phi(-x)$
- $\phi(\frac{1}{x}) = -\phi(x)$
- If $\phi(x) < \phi(y)$ then

$$\begin{aligned}\phi(x) &= \phi(x + y - y) \\ &\geq min\{\phi(x + y), \phi(-y)\} \\ &= min\{\phi(x + y), \phi(y)\} \\ &\geq min\{\phi(x), \phi(y)\}\text{[By Property (ii)]} \\ &= \phi(x)\end{aligned}$$

  Thus $\phi(x) = min\{\phi(x+y), \phi(y)\}$. Since $\phi(x) < \phi(y)$, we get $\phi(x) = \phi(x+y)$
- $\phi(x/y) = \phi(x) - \phi(y)$

2.2.1 *p-adic* Valuation

One of the various examples of valuation ϕ is defined on the set of rationals $\mathbb{Q}$ as follows:

- Select any prime p.
  Define ϕ_p as follows:
- First define $\phi_p(0) := \infty$
- If $r \in \mathbb{Q}, r \neq 0$, Write r in the form $p^n \frac{a}{b}$,where $n, a, b \in \mathbb{Z}$ and $p \not| a, p \not| b$
- Define $\phi_p(r) = n$, which can be thought of as 'the number of times p divides r'.

ϕ_p is called *p-adic valuation* on $\mathbb{Q}$.

2.2.2 Extension of Valuation

It is possible to extend any p-adic valuation on $\mathbb{Q}$ to $\mathbb{R}$.
Consider the special case $p = 2$.
By the Theorem of extension of valuation[1],There exists an extension ϕ on the field of real number of the valuation ϕ_2 on $\mathbb{Q}$ such that $\phi(2) > 0$.

3 Combining Two Parts Of The Proof

For $\phi := \phi_2$ on $\mathbb{R}$ We decompose the whole coordinate Plane into three sets as follows:

- $S_0 = \{(x, y)|\phi(x) > 0 \text{ and } \phi(y) > 0\}$.Color these points RED.
- $S_1 = \{(x, y)|\phi(x) \leq 0 \text{ and } \phi(y) \geq \phi(x)\}$.Color these points BLUE.
- $S_2 = \{(x, y)|\phi(x) > \phi(y) \text{ and } \phi(y) \leq 0\}$.Color these points GREEN.

We observe the following properties:

Lemma 2. *Suppose $a \in S_0$.Then P & $P + a$ have same color for all $P \in \mathbb{R}^2$.*

Proof. If $P \in S_0$,the claim is obvious.Otherwise let $P \equiv (x, y)$ and $P' \equiv (x', y')$ be such that $P \in S_1$ and $P' - P \in S_0$ i.e. $\phi(x' - x) > 0$ and $\phi(y' - y) > 0$.Then $\phi(x' - x) > \phi(x) \Rightarrow 0 \geq \phi(x) = \phi(x' - x + x) = \phi(x')$ and $\phi(y') \geq min\{\phi(y' - y), \phi(y)\} \geq min\{0, \phi(x)\} = \phi(x) = \phi(x')$. So $P' \in S_1$.For $P \in S_2$ the argument is similar. ■

Lemma 3. *Any line in $\mathbb{R}^2$ has points of at most 2 colors.*

Proof. Suppose $L \in \mathbb{R}^2$ be a line which contains RED,BLUE,GREEN, points a, b, c. By Lemma 2 we can assume a = 0. So L has the form $y = \lambda x$, and if $b = (b_x, b_y), c = (c_x, c_y)$, then $\phi(\frac{b_y}{b_x}) = \phi(\frac{c_y}{c_x}) = \phi(\lambda)$ (modify if $c_x b_x = 0$). But b and c are different colors, which gives a contradiction (BLUE means the ratio is ≥ 0, GREEN$means \leq 0$).■

Now The key lemma in discussion of dissection of a square into triangles:

Lemma 4. *Let T be a triangle in coordinate plane whose vertices are $(x_0, y_0), (x_1, y_1)$, (x_2, y_2) where $(x_i, y_i) \in S_i$. Then*

$$\phi(\textbf{\textit{area of T}}) \leq -\phi(2)$$

.

Proof. Since translation by $a = (-x_0, -y_0) \in S_0$ doesnot change the color of (x_i, y_i), we may assume WLOG that $x_0 = y_0 = 0$.[Note that $(0, 0) \in S_0$] Then area of T is absolute value of $\frac{1}{2}(x_1 y_2 - x_2 y_1)$.Now $\phi(x_1) \leq 0$ and $\phi(x_1) \leq \phi(y_1)$. Also $\phi(y_2) \leq 0$ and $\phi(y_2) < \phi(x_2)$. Hence $\phi(\text{area of T}) = \phi(1/2) + \phi(x_1 y_2 - y_1 x_2) = -\phi(2) + \phi(x_1 y_2) \leq -\phi(2)$ ■

Lemma 5. *Let D be a m-equidissection of a polygon of area A let ϕ be a valuation* *on the real numbers as defined in the beginning of this section. If D contains a triangle with each vertex in different S_i relative to ϕ then*

$$\phi(m) \geq \phi(2A).$$

Proof. Let T be a complete triangle in the equidissection.Its area is A/m. Then by Lemma 4,

$$\phi(A/m) \leq -\phi(2)$$

or

$$\phi(m) \geq \phi(A) + \phi(2) = \phi(2A).\blacksquare$$

We now finish the proof by combining all above results:

Theorem 1. (Richman-Thomas-Monsky)
`In any m-equidissection of a square m is even.`

Proof. Consider the square whose vertices are $(0,0),(0,1),(1,0),(1,1)$. Relative to the valuation $\phi = \phi_2$ the vertices are colored RED,GREEN,BLUE and BLUE,respectively as shown in the figure. Consider a typical dissection of the square. By Lemma 3, vertices on the bottom edge can be colored either RED or BLUE,on the top edge either GREEN or BLUE, on the left edge either GREEN or RED,on the right edge either GREEN or BLUE[since $\phi(y) \leq 0 for 0 \leq y \leq 1$.] Also the square has oonly one RB face (the bottom face), so it has any odd no. of RB faces.Then By corollary 1 of Lemma 1 there is a triangle with each vertex in seperate S_i. Then by Lemma 5,

$$\phi(m) \geq \phi(2 \times \text{arae of square}) = \phi(2) = 1.$$

So,

$$2|m \implies \text{m is even.}\blacksquare\blacksquare$$

4 Some Generalisation

The square can be generalised into many directions: to n-dimensional cubes,to regular polygons,to trapezoids,to quadrilaterals,to centrally symmetric polygon and so on.
We present here only one more result along this line which takes on n-dimensional cubes:

Theorem 2.(Mead,1979[2]) *If the n-dimensional cube is divided into simplices of equal volumes then their number must be a multiple of n!.*

Our case is clearly that for $n = 2$.■

References

[1] Serge Lang. *Algebra*. Addison-Weasley, 1993.

[2] D. G. Mead. Dissection of a hypercube into simplices. *Proc. Amer. Math. Soc.*, 76:302–304, 1979.

[3] John Thomas. A dissection problem. *Math. Mag.*, 41:187–190, 1968.