Assignment 3 (7/2)

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- *This homework is due at the beginning of class on* ***Friday*** 7/6. *You are encouraged to work together on these problems, but you must write up your solutions independently.*

Example on Invariance Principle

We discussed the following problem in class. Write a complete proof of it.

Exercise 1 (Do not submit this)

The numbers 1, 0, 1, 0, 0, 0 are written around a circle (clockwise, say). You may increase (or decrease) two neighboring numbers by 1 at the same time. Is it possible to equalize all numbers by a sequence of such steps?

Set Theory

- A *Set* is a collection of similar objects. Let $\mathscr{S}$ be a set.
- The objects in a set are called *elements*. E.g. if $\mathscr{S}$ consists of all positive numbers, then 3 is an element of $\mathscr{S}$. This is denoted by $3 \in \mathscr{S}$.
- The number of elements in a set is called its *cardinality*. So if $\mathscr{S}$ consists of the numbers $1,2,3$, then it has cardinality 3, denoted as: $|\mathscr{S}| = 3$.
- There are two ways of writing a set. The first is to list all the elements. E.g.
  - $\mathscr{A} = \{1,2,3\}$
  - $\mathscr{B} = \{1,2,3,4,5,\ldots\}$
  - $\mathscr{C} = \{1,4,9,16,25,\ldots\}$

  One clear disadvantage of this notation is, for example, it's not clear what the next elements of $\mathscr{C}$ are. We might guess that $\mathscr{C}$ contains all the positive square numbers, but we cannot be sure. This is where the second way of denoting a set comes in handy. We write

  $$\mathscr{C} = \{n^2 \mid n \text{ is a positive number}\}$$

  This reads: "The set $\mathscr{C}$ consists of all numbers of the form n^2 where n is a positive number."

Exercise 2

Write the set of positive numbers that end in 0 i.e. $\{10,20,30,40,50,\ldots\}$ in the second way.

Here are some of the most famous number sets.

- The set of Natural numbers, $\mathbb{N} = \{1,2,3,4,\ldots\}$
- The set of Integers, $\mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$
- The set of Rational numbers, $\mathbb{Q} = \{p/q \mid p,q \in \mathbb{Z} \text{ and } q \neq 0\}$
- The set of Real numbers, $\mathbb{R}$, usually depicted as the number line.

There is a concept of bigger set in the following sense. We say a set $\mathscr{A}$ is a subset of $\mathscr{B}$, denoted $\mathscr{A} \subset \mathscr{B}$ if all elements of $\mathscr{A}$ are also elements of $\mathscr{B}$. In other words the set $\mathscr{A}$ is contained inside $\mathscr{B}$.

Exercise 3

> Give example of two sets $\mathscr{A}$ and $\mathscr{B}$ such that neither $\mathscr{A} \subset \mathscr{B}$ nor $\mathscr{B} \subset \mathscr{A}$.

With this notation, we get the following chain of inclusions:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

The set of Real numbers that are not rational are called *Irrational* numbers. Examples include $\sqrt{2}, \pi, e$ etc. It is in general very hard to prove that a given number is not rational.

Exercise 4

> Consider two numbers $m, n \in \mathbb{N}$. Which of the four numbers $m+n, m-n, m \times n$, and m/n must be another element of $\mathbb{N}$?
> Answer the same question when $\mathbb{N}$ is replaced by $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or the set of irrational numbers.

Logical Comparisons

We will explain implication, inverse, converse, and contrapositive using examples.

- Start with a statement P e.g. it is raining outside. This may be true or false.
- If another statement Q follows from this statement P, then we write $P \implies Q$, which reads "P implies Q".
  For example, the statement above implies the statement: the roads are wet.
- Sometime multiple statements together imply one statement. E.g. It is raining Outside AND I am walking on the road together imply my shoes are wet.
- Recall the example from the first quiz. Theresa loves dogs. If Spot is a dog then Theresa loves Spot. Here the two statements "Theresa loves dogs" and "Spot is a dog" together imply "Theresa loves Spot". Observe that only one of the first two statement does not imply the third by itself.
- Given an implication $P \implies Q$,
  - the *inverse* is not $P \implies$ not Q.
  - the *converse* is $Q \implies P$
  - the *contrapositive* is not $Q \implies$ not P
- We can show that an implication and its Contrapositive are equivalent i.e. either both are true or both are false. Similarly the Inverse and Converse are equivalent.
- An example where the implication is true but the converse is false: "If a quadrilateral is a rectangle, then it has two pairs of parallel sides."
- An example where both the implication and the statement are true: "If two angles are congruent, then they have the same measure."
- We say P if and only if Q if both $P \implies Q$ and $Q \implies P$.Or equivalently, $P \implies Q$ and not $P \implies$ not Q. Or equivalently, not $Q \implies$ not P and $Q \implies P$..
- The sentence "P is necessary for Q" is same as saying $Q \implies P$.
- The sentence "P is sufficient for Q" is same as saying $P \implies Q$.
- For example, consider the following:
  - Given $n \neq 2$, it is necessary for n to be odd for n to be a prime. This is equivalent to saying " Given $n \neq 2$, n is a prime $\implies n$ is odd".
  - Also try to rewrite the following statement: It is necessary for an American Citizen to be of age > 35 to be the President.
  - On the other hand, consider the following example: "n being divisible by 4 is sufficient for n to be even." Here the implication is "4 divides $n \implies n$ is even."

Finally, we describe an example to explain 'Proof by Contradiction'. We will do a lot more examples next time. Observe that a 'Proof by Contradiction' is same as proving the contrapositive.

Exercise 5 (Do not submit this)

Explain the reasoning for each of the steps in the following proof.

Problem: Show that $\sqrt{2}$ is not a Rational number.

Solution: Suppose, for the sake of contradiction, that $\sqrt{2}$ is a Rational number.
Then we can find natural numbers n such that $n\sqrt{2}$ is an integer.
Let S be the set of such natural numbers n. Then S has a least element. Let k be the least element of S.
Consider the number $m = (\sqrt{2} - 1)k$. Observe that

$$m\sqrt{2} = 2k - k\sqrt{2}$$

Now $k\sqrt{2}$ is an integer since $k \in S$ and $2k$ is an integer since $k \in \mathbb{N}$. So $2k - k\sqrt{2} \in \mathbb{Z}$.
Hence the left hand side of the above equality, $m\sqrt{2}$ must also be an integer.
More precisely, $m\sqrt{2} \in \mathbb{N}$ since it is a positive integer.
Thus m is in fact a natural number such that $m\sqrt{2}$ is an integer.
Therefore, by definition of the set S, we must have $m \in S$.
On the other hand, $m = (\sqrt{2} - 1)k < k$. But this contradicts our assumption that k is the least element of S.
Hence our assumption is false, and $\sqrt{2}$ is in fact not a Rational number. □

To summarize, a *Proof by Contradiction* Process is as follows:

Step 1. *Negate the conclusion.* If you are trying to show Q is true, then start by assuming Q is false. If you want to show $P \implies Q$, start by assuming not Q.

Step 2. *Analyze the consequences of this premise.* Assuming Q is false i.e. not Q, explore the logical implications that would follow.

Step 3. *Look for a contradiction.* A contradiction is something that doesn't make sense given the negated conclusion premise.

Step 4. Conclude using the famous Sherlock Holmes quote:

"...when you have eliminated the impossible, whatever remains, however improbable, must be the truth"

Exercise 6

Prove that the sum of a rational number and an irrational number is irrational.
[HINT: Prove by contradiction.]