MATH 1800 PROJECT 2: CONIC SECTIONS AND QUADRIC SURFACES

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CONIC SECTIONS

A *conic section* (or simply conic) is a curve obtained as the intersection of the surface of a cone with a plane. The three types of conic sections are the *hyperbola*, the *parabola*, and the *ellipse*. The circle is type of ellipse, and is sometimes considered to be a fourth type of conic section.

A cone has two identically shaped parts called *nappes*. One nappe is what most people mean by ‘‘cone", and has the shape of a dunce hat. It can be thought of as the surface of revolution of a straight line around an axis.

- If the intersecting plane is parallel to the axis of revolution of the cone, then the conic section is a hyperbola.
- If the plane is parallel to the generating line, the conic section is a parabola.
- If the plane is perpendicular to the axis of revolution, the conic section is a circle.
- If the plane intersects one nappe at an angle to the axis (other than $90°$), then the conic section is an ellipse.

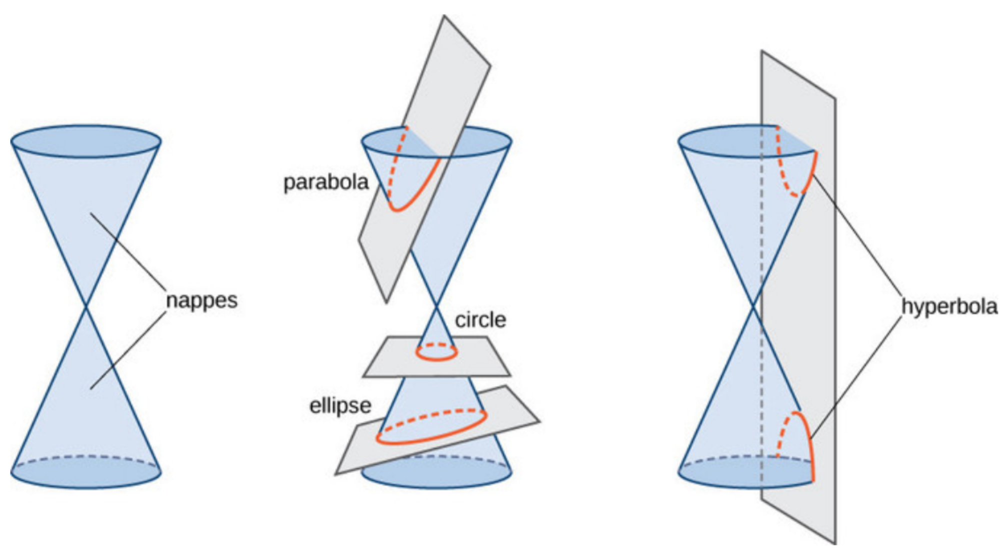

Figure 1: Conic Sections

Observe that when the intersecting plane is parallel to the axis of revolution and passes through the vertex of the cone, the conic section becomes a pair of straight lines (also known as a degenerate hyperbola).

Standard Equations in Cartesian Coordinates

The **Major Axis** is the chord between the two vertices: the longest chord of an ellipse, the shortest chord between the branches of a hyperbola. The **Minor Axis** is the shortest chord of an ellipse.

Conic Type	Standard Equation	Major Axis	Minor Axis
Circle	$x^2+y^2=r^2, \quad r\geq 0$	$2r$	$2r$
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad a,b>0$	$2a$	$2b$
Parabola	$y^2=4ax$	N/A	N/A
	$x^2=4ay$	N/A	N/A
Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \quad a,b>0$	$2a$	N/A
	$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1, \quad a,b>0$	$2b$	N/A
Pair of Straight Lines	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0, \quad a,b>0$	N/A	N/A
	$\iff y=\pm\frac{b}{a}x$	N/A	N/A

- In each of the above cases the center of the conic is at the origin. If the curve is translated h units horizontally and k units vertically, its new equation is obtained by replacing x with $(x-h)$ and y with $(y-k)$.

Question 1

Use your precalculus memory or your favorite computer graphics software (e.g. Desmos) to draw a picture of each of the above conic sections. Clearly denote the center, radius, major axis, minor axis etc. and specify their lengths in terms of a, b etc. as applicable.

Question 2

For each of the following curves, find out what kind of conic section it is.

(a) $(x-3)^2+(y-4)^2=y^2$

(b) $(x-3)^2+(y-4)^2=2y^2$

(c) $(x-3)^2+(y-4)^2=\frac{y^2}{2}$

(d) $(x-3)^2+(y-4)^2=1$

Question 3

Find the value(s) of p for which does the curve $\frac{x^2}{9-p}+\frac{y^2}{p-3}=1$ looks like a

(a) Circle.

(b) Ellipse.

(c) Hyperbola.

Quadric Surface

Equations of surfaces in three dimension are of the form $f(x,y,z) = c$. One special case of interest is when $f(x,y,z)$ is a polynomial which is not linear and quadratic at most, containing terms involving $x, y, z, x^2, y^2,$ and z^2 only. This kind of surface is called a *Quadric Surface*. Quadric surfaces are often used as example surfaces since they are relatively simple. There are nine different basic quadric surfaces listed below. A catalog of the equations and pictures of the quadric surfaces is available on page 691 of your textbook.

- **Cylinders:** A cylinder basically has no control over one of the variables. Take some sort of a curve in the plane, and draw a family of parallel lines so that each of the lines intersects the curve in a point. For example, a cylinder over the line $(0,t,t)$ would be all points of the form (s,t,t) for any values of s and t. This is a plane (a plane is a cylinder over a line!), and has the equation $y = z$. Some basic variations are
  - **elliptical cylinder:** A cylinder over an ellipse
  - **parabolic cylinder:** A cylinder over a parabola
  - **hyperbolic cylinder:** A cylinder over a hyperbola
- **ellipsoid**, the three-dimensional analogue of the ellipse. A sphere is an uniform ellipsoid.
- **elliptic paraboloid**, a sort of cup or a bowl
- **hyperbolic paraboloid**, looks like a horse-saddle or a pringle
- **cone**, take a straight line intersecting the $z-$axis and consider its surface of revolution around the $z-$axis
- **hyperboloids:** In three dimensions there are two different analogs of hyperbolas. The word "sheet" is used in an antique, specialized sense with surfaces: it means one connected "piece" of a surface. So a hyperboloid with one sheet is a surface with one (connected) piece, and a hyperboloid with two sheets is a surface with two (connected) pieces.
  - **hyperboloid of one sheet**, obtained by revolving a hyperbola around its minor axis. The surface is connected but there is a hole in it.
  - **hyperboloid of two sheet**, obtained by revolving a hyperbola around its major axis. This surface has two pieces.

Remark. In each case, note the direction of the axes relative to the surfaces and how the corresponding variables show up in the equation. For example, $z = 2y^2 - x^2$ is a hyperbolic paraboloid that goes downward in the $x-$axis direction and upwards in $y-$axis direction. The equation $y = 2x^2 - z^2$ is also a hyperbolic paraboloid that goes downward in the $z-$axis direction and upwards in $x-$axis direction. Similarly, $y = x^2 + z^2$ is an elliptical paraboloid that opens in the $y-$axis direction.

Question 4

For each of the following quadric surfaces, describe what kind of conic sections are obtained by taking its cross-sections parallel to the **YZ**−plane, **XZ**−plane and **XY**−plane. Then use the catalog to pick the term from the list above which seems to most accurately describe the surface and draw a rough picture of the surface (mark the origin and axis ticks) without using any graphing tool.

(a) $\frac{x^2}{9} - \frac{y^2}{16} = z$.

- Intersection with $x = k$:
- Intersection with $y = k$:
- Intersection with $z = k$:

(b) $\frac{x^2}{4} + \frac{y^2}{25} + \frac{z^2}{9} = 1$.

- Intersection with $x = k$:
- Intersection with $y = k$:
- Intersection with $z = k$:

(c) $\frac{x^2}{4} + \frac{y^2}{9} = \frac{z}{2}$.

- Intersection with $x = k$:
- Intersection with $y = k$:
- Intersection with $z = k$:

(d) $\frac{z^2}{4} - x^2 - \frac{y^2}{4} = 1$.

- Intersection with $x = k$:
- Intersection with $y = k$:
- Intersection with $z = k$:

(e) $x^2 + \frac{y^2}{9} = \frac{z^2}{16}$.

- Intersection with $x = k$:
- Intersection with $y = k$:
- Intersection with $z = k$:

(f) $\frac{x^2}{9} + y^2 - \frac{z^2}{16} = 1$.

- Intersection with $x = k$:
- Intersection with $y = k$:
- Intersection with $z = k$:

Question 5

Identify and sketch the following surfaces.

(a) $9y^2 + 4z^2 = 36$

(b) $y^2 + 2y + z^2 = x^2$

(c) $4x^2 - y^2 + z^2 + 9 = 0$

Question 6 (A non-basic Quadric Surface)

Google Geogebra 3D calculator. Use the website to plot the surface

$$x^2 - 17z^2 - 2y^2 - 2xz - 12yz - 1 = 0$$

Which of the nine basic quadric surface does this resemble most closely? Can you explain how the equation of the surface might tell us what kind of surface it is, without using a graphing software?

[HINT: Complete the squares.]