Practice Problems
Math 195

Subhadip Chowdhury

Following are a set of practice problems for the Final exam. They are not in any particular order of difficulty. Please make sure you know how to solve all of them before them final.

Problem 1

Find whether the following series converge:

1. $$\sum_{n=1}^{\infty} \frac{\sqrt{n^3+\sin(n)}}{\sqrt[3]{n^7+n^5}}$$

   [HINT: Squeeze theorem (comparison) + limit comparison]

2. $$\sum_{n=1}^{\infty} \frac{(1/n)^n}{(e^{2/n}-1)^n}$$

   [HINT: root test + L'Hospital]

3. $$\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)!}$$

   [HINT: ratio test]

4. Show that the following series converges and find its sum.

   $$\sum_{n=1}^{\infty} \frac{5^{2n-1}}{3^{3n+2}}$$

   [HINT: Geometric series]

Problem 2

1. Show that the curve $\vec{r}(t) = \frac{3}{2}(t^2+1)\hat{\imath} + (t^4+1)\hat{\jmath} + t^3\hat{k}$ is perpendicular to the ellipsoid $x^2 + 2y^2 + 3z^2 = 20$ at the point $(3,2,1)$.

   [HINT: At what value of t does the curve intersect the ellipsoid?]

2. Find the equation of the normal line to above ellipsoid at $(3,2,1)$.

[HINT: $r'(t)$ is the tangent to the curve. gradient (i.e. normal) is perpendicular to the surface. So if the curve and the surface are perpendicular, the tangent (to curve) and the normal (to surface) are parallel.]

Problem 3

Find all the points on the surface

$$x + y + z + xy - x^2 - y^2 = 0$$

where the tangent plane is parallel to the plane $8x - 4y - 2z = 7$.

[HINT: i.e. the gradient to the surface is normal to the plane.]

Problem 4

Consider the function

$$f(x,y,z) = \ln(\sin(xy)) + \cos(xz) + e^{yz}.$$

1. In which direction does f increase most rapidly at $(\pi/2, 1, 1)$?
2. Find the directional derivative of f at $(\pi/2,1,1)$ towards $(\pi/2,5,4)$.

Problem 5

Let

$$f(x,y) = \sin(x-y) + \cos(x+y), \quad x = \frac{s}{t}, \quad y = (\ln(s))^2$$

Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ when $s = 1, t = 1$.

Problem 6

1. Evaluate

$$\iint\limits_{\Omega} e^{x^2}\, dA$$

   where Ω is the triangular region bounded by the X—axis, $2y = x$, and $x = 2$.

2. Evaluate

$$\int_z^{\pi/2} \int_0^{\sin z} 3x^2 \sin y \, dx \, dy$$

Problem 7

Determine whether the following series converge or diverge. In each case, clearly mention what test(s) or theorem you are using. Check your answers with wolfram alpha.

1. $$\sum \frac{\sqrt{(ln(n))^3 + \ln(n^3)}}{n^3+1}$$

   [Comparison and Limit-comparison]

2. $$\sum \left(2 + \frac{1}{n}\right)^n$$

   [root test]

3. $$\frac{1}{2} + \frac{2}{3^2} + \frac{4}{4^3} + \frac{8}{5^4} + \frac{16}{6^5} + \cdots$$

   [ratio test]

4. $$\sum \sin\left(\frac{\pi}{4n^2}\right)$$

   [comparison test]

5. $$\sum \frac{(2k+1)^{2k}}{(5k^2+1)^k}$$

   [root test]

6. $$\sum \frac{1}{k}\left(\frac{1}{\ln k}\right)^{3/2}$$

   [integral test]

Problem 8

Let

$$f(x,y) = \frac{x^3 + xy}{y\sqrt{4y - x^2}}$$

1. Find and describe the domain of the function $f(x,y)$. Also draw picture.

2. Show that

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

does not exist by considering one path to the origin along the Y axis and another path along the parabola $y = x^2$.

[HINT: One of the limit is 0, another $2/\sqrt{3}$]

Problem 9

Let $n \in \mathbb{N}$. Expand $(x-1)^n$ in powers of x. [HINT: i.e. find its Taylor series at 0.]

Problem 10

Expand $(1+2x)^{-4}$ in powers of x.

Problem 11

Find the interval of convergence for

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 3^k}(x+2)^k$$

[HINT: i.e. do Root/ratio test to find out for what values of x, this series is convergent.]

Problem 12

Find the area of the triangle $\triangle PQR$ where $P = (0,-2,0), Q = (4,1,-2)$, and $R = (5,3,1)$.

[HINT: Use cross products.]

Problem 13

Use scalar triple product to determine volume of the parallelepiped with adjacent edges AB, AC, AD where $A = (1,0,1), B = (2,3,0), C = (-1,1,4)$, and $D = (0,3,2)$.

In general if the triple product is zero, then the three vectors are coplanar.

Problem 14

(Chapter 14 review problem 35) If $u = x^2y^3 + z^4$, where $x = p + 3p^2, y = pe^p$, and $z = p\sin p$, find $\frac{\partial u}{\partial p}$.

Problem 15

(Chapter 14 review problem 50) Find parametric equations of the tangent line at the point $(-2,2,4)$ to the curve of intersection of the surface $z=2x^2-y^2$ and the plane $z=4$.

[SOLUTION: The tangent line that we are trying to find is perpendicular to the gradient of both the surface and the plane at (-2,2,4). So it's their cross product.]

Problem 16

(Chapter 14 review problem 52) Find local max/min/saddle points of

$$f(x,y)=x^3-6xy+8y^3$$

Problem 17

(Chapter 14 review problem 60) Use Lagrange multipliers to find maximum and minimum of $f(x,y)=1/x+1/y$ subject to the constraint $1/x^2+1/y^2=1$.

Problem 18

Chapter 14 review problem 65.

[HINT: Take a and $b\,(b>a)$ as sides of the rectangle. Then sum of the top two sides is $b\sec\theta$. Use Lagrangian multipliers.]

Problem 19

Chapter 15 review problem $27,45,55,57$.

[HINT: For 57, the 4 sides of the square tell you what the change of variable should be.]

Problem 20

A store sells two types of toys, A and B. The store owner pays \$8 and \$14 for each one unit of toy A and B respectively. One unit of toys A yields a profit of \$2 while a unit of toys B yields a profit of \$3. The store owner estimates that no more than 2000 toys will be sold every month and he does not plan to invest more than \$20,000 in inventory of these toys. How many units of each type of toys should be stocked in order to maximize his monthly total profit profit? Solve using graphical method.

[Ans: 1333 toys of type A and 667 toys of type B]

Problem 21

A farmer plans to mix two types of food to make a mix of low cost feed for the animals in his farm. A bag of food A costs \$10 and contains 40 units of proteins, 20 units of minerals and 10 units of vitamins. A bag of food B costs \$12 and contains 30 units of proteins, 20 units of minerals and 30 units of vitamins. How many bags of food A and B should the consumed by the animals each day in order to meet the minimum daily requirements of 150 units of proteins, 90 units of minerals and 60 units of vitamins at a minimum cost? Solve using graphical method.

[Ans: 3.75 bags of food A and 0.75 bags of food B]

Problem 22

Solve using Simplex Method. Use this link to check your answer:

`https://www.zweigmedia.com/RealWorld/simplex.html`

maximize $P = x_1 + 2x_2 - x_3$ subject to $x_1, x_2, x_3 \geq 0$ and

$$\begin{aligned} 2x_1 + x_2 + x_3 &\leq 14 \\ 4x_1 + 2x_2 + 3x_3 &\leq 28 \\ 2x_1 + 5x_2 + 5x_3 &\leq 30 \end{aligned}$$

[Ans: 2 pivots, $P = 13$ at $x_1 = 5, x_2 = 4, x_3 = 0$]

Problem 23

Solve using Simplex method. Maximize $P = x + 2y + 3z$ subject to

$$x, y, z \geq 0, \quad 7x + z \leq 6, \quad x + 2y \leq 20, \quad 3y + 4z \leq 30$$

[Ans: $P = 22; x = 0, y = 2, z = 6$]