Assignment 20 (3/5)

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- *This homework is due at the beginning of class on* ***Wednesday*** 3/7*. You may cite results from class as appropriate. Unless otherwise stated, you must provide a complete explanation for your solutions, not simply an answer. You are encouraged to work together on these problems, but you must write up your solutions independently.*
- *Hand in the exercises only, not the reading material item. You are encouraged to think about the exercises marked with a* ($\star$) *or* (†) *if you have time, but you don't need to hand them in. If you correctly solve a* (†)*-marked problem, you will get a candy!*
- *Remember that you can always use the result of the previous assignment problems without proof to solve the new assignment problems.*
- *We are currently covering Sequences and Series (Chapter 11) from Stewart.*

Important Points and Reading Materials

- POWER SERIES
  - Know how to use Root and Ratio test to find radius of convergence of Power series.
  - What is the interval of convergence of a power series? Note that just knowing the radius of convergence is not enough to get the interval. You need to check the boundary points separately.
  - if the radius of convergence is r, the interval of convergence could be any of $(r,r),[r,r),(r,r]$ or $[r,r]$.
- REPRESENTING FUNCTIONS BY POWER SERIES
  - What is the power series representation of $\frac{1}{1-x}$? What is the radius of convergence?
  - If $f(x)=\sum_{n=1}^{\infty} a_n x^n$, then what are the power series representations of $f'(x), f(2x), f(x^3), xf(x), \int f(x)dx$ etc.
- TAYLOR SERIES
  - Understand that saying the taylor series expansion of $f(x)$ is $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ does NOT imply that $f(x)=\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$. It is possible for the series not to converge at some point where $f(x)$ is defined. Alternately, $f(x)$ might be defined at a point where the series does not converge. These two things are equal only when we know a priori that $f(x)$ has a power series representation at a point. See for example problem 11.10.84.
  - A list of some of the most common Taylor series expansions are on page 808 in the book.

Problems

Exercise 1

Find the radius of convergence and Interval of convergence for the following series.

1. ($\star$)(11.8.8)
$$\sum n^n x^n$$
2. ($\star$)(11.8.12)
$$\sum \frac{(-1)^{n-1}}{n5^n} x^n$$
3. (11.8.17)
$$\sum \frac{(x+2)^n}{2^n \ln n}$$

4. (11.8.23)

$$\sum n!(2x-1)^n$$

5.

$$\sum \frac{e^n}{n^3}(x-1)^n$$

Exercise 2

Suppose $\sum a_k(x+2)^k$ converges at $x=4$. At what other values of x must the series converge? Does it necessarily converge at $x=-8$?

Exercise 3

Starting from

$$\frac{1}{1-x} = 1+x+x^2+x^3+\dots$$

we can calculate the power series representation for a lot of other functions. For example,

$$\frac{2}{3-x} = \frac{2}{3}\left(\frac{1}{1-\frac{x}{3}}\right) = \frac{2}{3}\left[1+\left(\frac{x}{3}\right)+\left(\frac{x}{3}\right)^2+\left(\frac{x}{3}\right)^3+\dots\right] = \frac{2}{3}+\frac{2x}{3^2}+\frac{2x^2}{3^3}+\frac{2x^3}{3^4}+\dots$$

Note that above expansion is valid only when $\left|\frac{x}{3}\right| < 1$ i.e. the radius of convergence is 3 centered at 0. The interval of convergence is $(-3,3)$, the series doesn't converge when $x=-3,3$. Another example is as follows:

$$\frac{x}{1-2x^2} = x\frac{1}{1-2x^2} = x(1+2x^2+4x^4+8x^6+16x^8+\dots) = x+2x^3+4x^5+8x^7+\dots$$

Here the interval of convergence is $(-1/\sqrt{2}, 1/\sqrt{2})$ since $|2x^2| < 1 \implies x \in (-1/\sqrt{2}, 1/\sqrt{2})$ and the series doesn't converge at the boundary points.

Finally, we can also differentiate or integrate to get

$$\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\dots$$

and

$$\ln(1-x) = -x-x^2/2-x^3/3-x^4/4-\dots$$

For both of these series, radius of convergence is 1, same as the original series.

Now similarly, find a power series representation of the following functions and find their radius of convergence.

1. (11.9.6)

$$\frac{4}{2x+3}$$

2. ($\star$) (11.9.8)

$$\frac{x}{2x^2+1}$$

3. (11.9.13c)

$$\frac{x^2}{(1+x)^3}$$

4. (11.9.23)

$$\ln\frac{1+x}{1-x}$$

Exercise 4

Again by the same method as last problem, but now going backwards, find the sum of the following series:

1. ($\star$)(11.9.40)

$$\sum_{n=2}^{\infty} n(n-1)x^n$$

2. (11.9.40)

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

Exercise 5($\star$)

Find the power series expansion of $\arctan(x)$.

[HINT: Start with $1/(1-x)$ to get a series expansion for $1/(1+x)$ and then $1/(1+x^2)$, then integrate. Ans: $x - x^3/3 + x^5/5 - x^7/7 + \ldots$. Observe that you can also obtain this series by using Taylor series formula for the coefficients, but that would be significantly harder.]

Exercise 6

1. Expand $f(x) = \ln(1+x)$ in powers of $(x-1)$ i.e. find the Taylor series at $x = 1$. You don't have to write a closed form formula for the n−th term, just write the first couple of terms.

2. What is the radius of convergence of this series?

Exercise 7

Expand $\sin(\pi x/2)$ in powers of $(x-1)$.