# Math 2208: Ordinary Differential Equations

## Project 1: The Spruce Budworm

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A well known journal article [1] concerns the outbreak of the North American spruce budworm. These worms are considered a pest and can kill off an entire forest region if the population reaches outbreak status. For the rest of this project we will try to answer the question why an outbreak can happen using a mathematical model. Note, an outbreak means there is a sudden jump in the population of the insect.

In the paper [1], Ludwig proposes an elegant model of the spruce budworm as follows. First, since the budworm population evolves much faster (they can increase their density **5** fold in one year) than the surrounding forest (balsam fir tree has a life span of 100-150 years); it is reasonable to assume that the parameters of our model associated to the forest change very slowly. Indeed, for our analysis, we begin with a logistic growth model with fixed $k$ and $\mathbf{N}$ where we consider these forest parameters as constants.

Next we modify the model to account for predation. The primary predators of the spruce budworm are birds. We introduce a harvesting *function* $h(p)$ to get the model

$$\frac{dp}{dt} = kp\left(1 - \frac{p}{\mathrm{N}}\right) - h(p) \tag{1}$$

Let us first examine the predation term $h(p)$ that was introduced in the paper [1]. The following assumptions were made based on practical observation:

(A) If the budworm density $p$ reaches a critical value (which is called the saturation point) then the predation starts to level off (birds are eating as fast as they can, so it doesn't matter if there is more food available to them).

(B) We assume there is very little predation if the budworm population $p$ is small (birds dont bother looking for them and find other sources of food instead).

(C) If there are enough budworms (but still smaller than the saturation point) the birds learn budworms are an available food source, and then actively search for them.

$\boxed{1}$ On your piece of paper, sketch what a graph of $h(p)$ should look like so that assumptions (A)-(C) are satisfied.

$\boxed{2}$ Consider the predation term

$$h(p) = \beta\frac{p^2}{\alpha^2 + p^2}$$

Graph this function in Desmos (www.desmos.com). You will have the option to add a slider for each parameter, which you should do. Slide the various sliders to get a feeling for how the parameters $\alpha$ and $\beta$ change the function. Is this function consistent with your sketch? What is different?

$\boxed{3}$ The parameters $\alpha$ and $\beta$ are related to the saturation of the predation. One controls the predation level itself (the '$h$-value') and the other controls when the leveling off starts (the '$p$-value'). Which parameter ($\alpha$ or $\beta$) controls which aspect of the saturation?

$\boxed{4}$ **Numerical Exploration of Solutions:**

*For the remainder of this project, we will assume that* $\beta = \alpha = 1$ *and* $\mathrm{N} = 10$. *We'll keep* $k$ *as a parameter (remember that* $k$ *describes the intrinsic growth rate of the spruce budworm which depends on the amount of foliage).*

(a) With the foliage parameter $k = 0.5,$ suppose the system is in a 'happy' or refuge state. Meaning the population is in a balance with its environment. There is no concern the budworms will destroy the entire forest. Use the `ODE45` function to approximate the solution of equation 1 with the initial value $p(0) = 0.1$ for $0 \leq t \leq 200$ and use time step size of $\Delta t = 0.1$ Refer to the `ODE45_example.m` on Blackboard for a reminder how to use `ODE45`.

What density (population) value is approached in $t = 200$?

(b) Let's suppose the forest canopy grows slightly, and now the foliage parameter $k$ is slightly larger. Repeat part (a) with $k = 0.53$, but before you hit run read the following:

Use the '`hold on`' command so that your old graph is not erased (that way you can compare the graphs). Recall that the command '`clf`' will clear the figure, so make sure that command is NOT in your code.

What density (population) value is approached? Was there a significant change?

(c) Let's suppose the forest canopy grows again slightly, and now the foliage parameter $k$ is slightly larger. Repeat part (a) with $k = 0.55$, but before you hit run read the following:

Get this new graph to be superimposed with the other two, but use the color green for this graph.

What density (population) value is approached? Was there a significant change?

(d) Let's suppose the forest canopy grows again slightly, and now the foliage parameter $k$ is slightly larger. Repeat part (a) with $k = 0.58$.

Again, try to get this new graph to be superimposed with the other three, but use the color red for this graph).

What density (population) value is approached? Was there a significant change?

(e) Oh no! It looks like there was an outbreak! However, you remember that $k = .55$ led to happy state, so you call the local forester to reduce the canopy so that the foliage is reduced. Now, the foliage parameter is back to $k = 0.55.$ Do you think the population will go back down to the safe level you had in part (c)?

(f) Set your parameter $k$ back to $k = 0.55$ in your code, and set the initial value to $p(0) = 7.8346$ (since the population has exploded to this amount). Use the color green again for your plot. Then run your code.

What density (population) value is approached? Did it go back to the 'happy' state you had in part (c)?

(g) Print and attach your Matlab/Octave graph. Make sure to provide meaningful labels in the graph.

## 5 Equilibrium solutions:

(a) To try to make sense of all this craziness, we will use a qualitative analysis, starting with phase lines. One equilibrium solution of this model corresponds to extinction i.e. $p = 0.$ Show that the other *non-extinction equilibrium* solution(s) satisfies,

$$k\left(1 - \frac{p}{10}\right) - \frac{p}{1 + p^2} = 0$$

(remember, we set $\mathbf{N} = 10$).

(b) Let's use a graph to estimate the roots (i.e. equilibrium solutions). Graph

$$f(x) = k\left(1 - \frac{x}{10}\right) - \frac{x}{1 + x^2}$$

in Desmos. Add the slider for $k$ and set the max and min value of $k$ to be $0 \leq k \leq 1$ . If you add another graph $g(x) = 0$ Desmos will show points where the intersections are. You can then hover over those points to see the approximate coordinate values.

**Note: for the next three problems you will be asked to draw phase lines. Please put them next to each other on your piece of paper.**

(c) Set the value $k = 0.5$. Use the Desmos graph to estimate the equilibrium solutions and their stability. Then draw the phase line on your piece of paper. Label the equilibria with approximate numerical values. According to your phase line, if $p(0) = 0.1$ what is the long term behavior of $p(t)$**?** How does this fit with your observation in question 3.4.(a)?

(d) Set the value $k = 0.55$. Use the Desmos graph to estimate the equilibrium solutions and their stability. Then draw the phase line on your piece of paper. Label the equilibria with approximate numerical values. If $p(0) = 0.1$ what is the long term behavior of $p(t)$**?** How does this fit with your observation in question 3.4.(c)?

(e) Set the value $k = 0.58$. Use the Desmos graph to estimate the equilibrium solutions and their stability. Then draw the phase line on your piece of paper. Label the equilibria with approximate numerical values. If $p(0) = 0.1$ what is the long term behavior of $p(t)$**?** How does this fit with your observation in question 3.4.(d)?

(f) Refer again your phase line in 3.5.(d). If $p(0) = 7.8346$ what is the long term behavior of $p(t)$**?** How does this fit with your observation in 3.4.(f)?

## 6 Bifurcation Diagram:

(a) On your piece of paper, draw axes for the equilibrium values $p_e$ (the vertical axis) and the parameter $k$ (on the horizontal axis). Using your graph in Desmos with $k$ being very small (close to $0$) sketch the equilibrium values and how they change as you increase $k$ with your slider gradually. Watch closely for bifurcations!

(b) Now redraw your graph with the following convention: Use dashed lines if the corresponding equilibrium solution is unstable (a source) and use a solid line if it is stable (a sink). Then label the bifurcation values in your graph (there should be two) and write down the corresponding numerical values (you can use the Desmos graph to estimate them).

(c) Use you bifurcation diagram to explain why, as the foliage amount gradually increases, there is the risk of an outbreak. Use your bifurcation diagram to explain how much you would have to reduce the parameter $k$ to in order to get out of the outbreak state.

# References

[1] Ludwig, D., D. D. Jones, and C. S. Holling. ''Qualitative Analysis of Insect Outbreak Systems: The Spruce Budworm and Forest.'' Journal of Animal Ecology 47, no. 1 (1978): 315-32. doi:10.2307/3939.