Assignment 14 (2/12)

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- *This homework is due at the beginning of class on **Friday** 2/16. You may cite results from class as appropriate. Unless otherwise stated, you must provide a complete explanation for your solutions, not simply an answer. You are encouraged to work together on these problems, but you must write up your solutions independently.*
- *Hand in the exercises only, not the reading material item. You are encouraged to think about the exercises marked with a ($\star$) or (†) if you have time, but you don't need to hand them in. If you correctly solve a (†)-marked problem, you will get a candy!*
- *Remember that you can always use the result of the previous assignment problems without proof to solve the new assignment problems.*
- *We are currently covering Chapter 15 from Stewart.*

Important Points and Reading Materials

INTEGRATION USING POLAR COORDINATES

- By polar coordinates, we mean expressing points on the planes in coordinates (r,θ) where $x = r\cos(\theta)$ and $y = r\sin(\theta)$. For example, the point $(2,2)$ in XY-coordinate is same as $(2\sqrt{2}, \pi/4)$ in (r,θ) coordinates.

  So integration using polar coordinates is essentially a change of variables to (r,θ) where the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ is equal to r.

- They are mainly used for the following two kinds of problems
  - When the integrand is radially symmetric.
    Examples of these kind of functions are $f(x,y) = g(x^2+y^2)$, where g is any continuous function of one variable. Essentially this means f depends only on $\sqrt{x^2+y^2} = r$, the distance of (x,y) from origin.
  - When region of integration is radially symmetric or defined using (r,θ)-coordinates.
    Here we may have questions of the following kind:
    * Calculate volume of a solid on a base that is radially symmetric. See Problem ?.
    * Calculate area inside a curve of the form $r = f(\theta)$.

- **How do we plot a curve of the form $r = f(\theta)$ on the XY-plane?**

  To plot a curve of the form $r = f(\theta)$, we calculate the value of $f(\theta)$ as θ goes from 0 to 2π and plot the points $(f(\theta)\cos(\theta), f(\theta)\sin(\theta))$ on XY-plane. This is equivalent to plotting the point at distance $f(\theta)$ from origin along the line that makes an angle θ with positive X-axis. However, here we regard r as a 'signed' radius. That means if for some value of θ, the value of r is negative, we go across origin and take the diametrically opposite point.

  As an example, consider the curve $r = \cos(2\theta)$, the four petal rose we drew in class. When θ is in $[-\pi/4, \pi/4]$, we get the right petal in the figure below as expected.

  On the other hand, when $\theta \in [\pi/4, 3\pi/4]$, we get the *bottom* petal! This is because $\cos(2\theta)$ is negative on $(\pi/4, 3\pi/4)$. So for example, when $\theta = \pi/3$, $\cos(2\theta) = \cos(2\pi/3) = -1/2$, and we plot the diametrically opposite point $(r,\theta) = (1/2, \pi/2 + 2\pi/3) = (1/2, 7\pi/6)$ in (r,θ) coordinates.

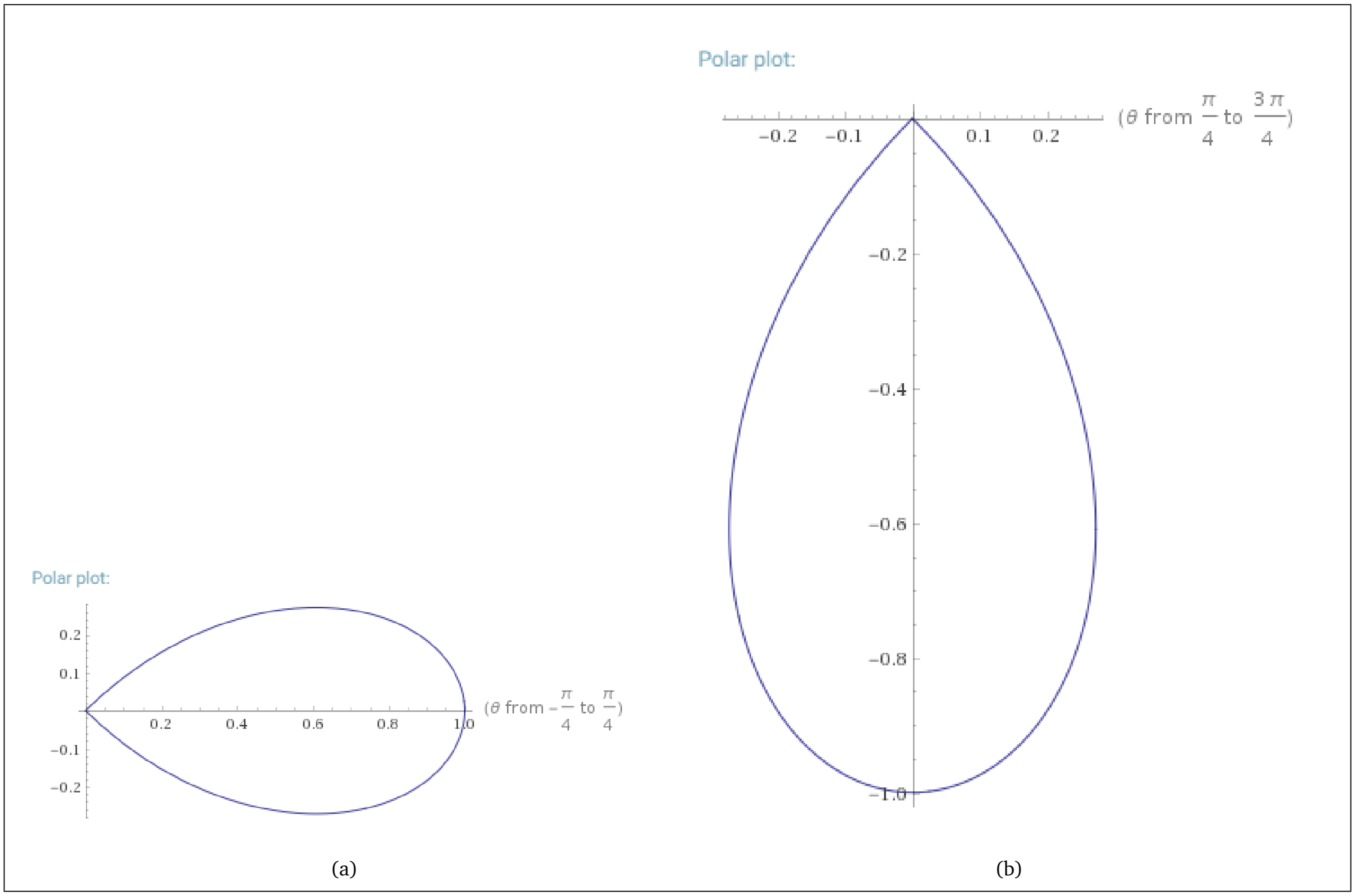


Figure 1

Also compare the plot when $\theta \in [0, \pi/2]$.

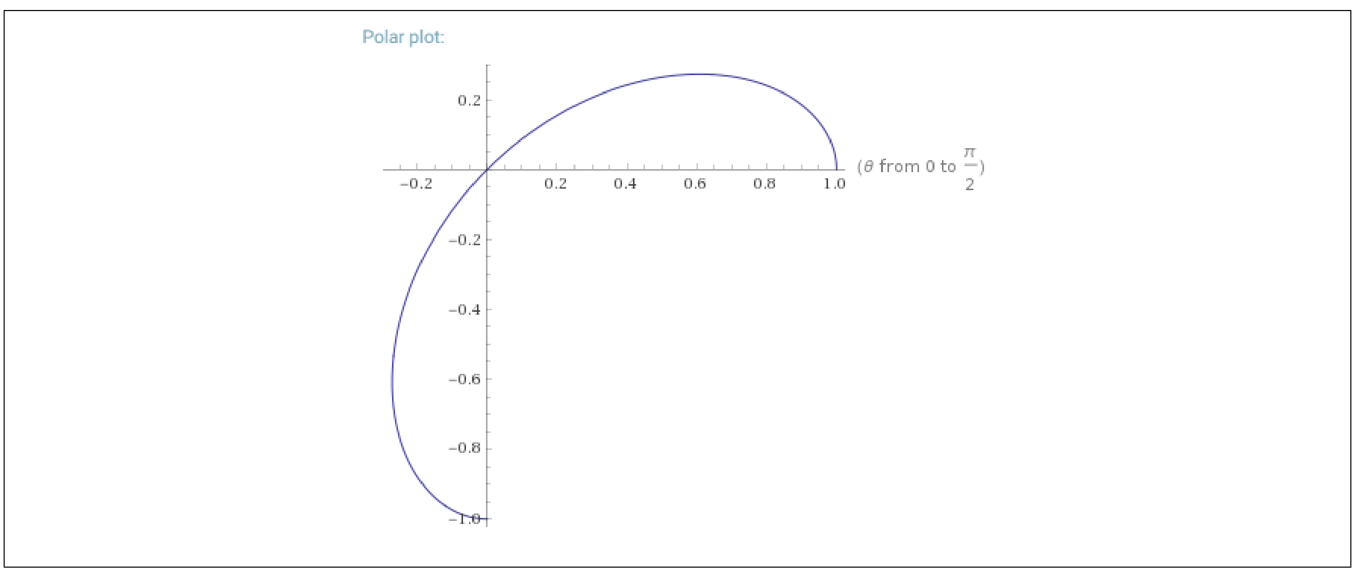


Figure 2

However note that this interpretation is for plotting purpose only. While doing an integral, the negative sign is automatically taken care of.

- Suppose we want to calculate area inside a curve of the form $r = f(\theta)$. This can be done using following steps:
  - First calculate the values of θ when $r = 0$ i.e. the points when the curve passes through the origin. This is very important because the curve $r = f(\theta)$ might repeat the same path for different values of θ; and so to calculate the correct area inside, we need to avoid counting the same region twice.

To clarify this point, consider the case of the curve $r = 2\cos(\theta)$. Here $r = 0$ implies $\theta = \pi/2$ or $-\pi/2$. So as θ goes from $-\pi/2$ to $\pi/2$, the position vector r traces the circle $x^2 + y^2 = 2x$. Observe that $\cos\theta$ is negative when $\theta \in (\pi/2, 3\pi/2)$. So, when $\theta \in (\pi/2, 3\pi/2)$, the point $(2\cos\theta, \theta)$ should be in fact plotted as the point $(-2\cos\theta, \theta + \pi)$, which is a point on the circle $x^2 + y^2 = 2x$ again. In other words, we trace the same path as before if we increase θ from $\pi/2$ to $3\pi/2$. So while setting up the double integral, if we take 0 and 2π as our limits, we will have *twice* the required area.

– Next note that dr and $d\theta$ are not in perpendicular direction to each other as dx and dy. So area "under" a curve $r = f(\theta)$ is NOT of the form $\int_{\theta_1}^{\theta_2} f(\theta)d\theta$. In fact it can be shown that the infinitesimal area in polar coordinates is $\frac{r^2}{2}d\theta$. I will give a proof of this fact using Riemann sums in class Wednesday. For now, we solve this problem using a different approach.

  Observe that area of a region Ω in XY-plane is numerically equal to volume of a solid of constant height 1 with Ω as the base. In other words,

$$\text{Area of } \Omega = \iint_{\Omega} 1\, dA$$

  After a change of variable, dA becomes $r dr d\theta$. So in particular, area inside a curve of the form $r = f(\theta)$ is equal to

$$\int_{\alpha}^{\beta}\int_{0}^{f(\theta)} r\, dr\, d\theta = \int_{\alpha}^{\beta}\left[\frac{r^2}{2}\right]_0^{f(\theta)} d\theta = \int_{\alpha}^{\beta}\frac{(f(\theta))^2}{2}\, d\theta$$

  where α and β are appropriate angles where $r = 0$.

– As an example, to calculate area inside the 4—petal rose from class, $r = \cos(2\theta)$; we take $\alpha = \pi/4$ and $\beta = \pi/4$ and then multiply the answer by 4. Here we could have also calculated the area using an integral from 0 to 2π and it gives the same answer.

  On the other hand to calculate area inside $r = 2\cos\theta$ we must integrate from $-\pi/2$ to $\pi/2$. Integrating from 0 to 2π gives two times the correct answer.

Problems

Exercise 1

Sketch the region whose area is given by the following integrals, and then evaluate the integral.

(a)

$$\int_{\pi/4}^{3\pi/4}\int_{1}^{2} r\, dr\, d\theta$$

(b)

$$\int_{\pi/2}^{\pi}\int_{0}^{2\sin\theta} r\, dr\, d\theta$$

Exercise 2

Evaluate the given double integral using polar coordinates.

(a) ($\star$)

$$\iint_R (3x + 8y^2)\, dA$$

where R is the region bounded by the circles $r = 1$ and $r = 2$ and the lines $\theta = 0$ and $\theta = \pi/2$ (i.e. R is a quarter-ring).

(b)

$$\iint_D x\, dA$$

where D is the region in the first quadrant that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 2x$.

Exercise 3

We are going to evaluate the *Gaussian integral*

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx$$

First of all note that e^{-x^2} is an integrable function, since it is continuous. Infact, it can be shown that the definite integral is finite. However we do not have any closed form formula using elementary functions for the antiderivative of e^{-x^2}, so we can't do the indefinite integration!

(a) We use the following trick by Poisson,

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

Can you give some justification for the two equalities above?

(b) Use polar coordinates to evaluate the last integral and show that it is equal to π.

(c) Hence

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

Consequently show that

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = 1$$

If you have some statistics background, you may recognize the integrand above as the probability density of the normal distribution with mean μ and standard deviation σ.

Exercise 4

Find volume of the solid that is bounded above by the cone $z = \sqrt{x^2+y^2}$ and below by the sphere $x^2+y^2+z^2 = 1$.

Exercise 5

A cylindrical drill of radius 1 is used to bore a hole through the center of a sphere of radius 4. Find the volume of the ring shaped solid that remains.

Exercise 6

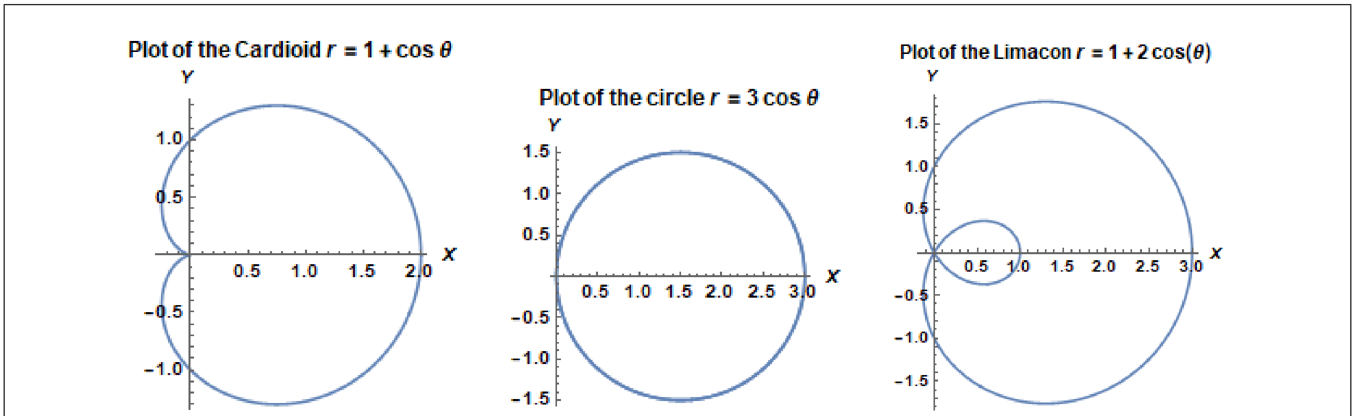
Use a double integral to find area of the following regions.

(a) ($\star$) One petal of the rose $r = \cos(3\theta)$ and the total area of the rose. [HINT: how many petals does the rose have?]

(b) The region inside the cardioid $r = 1+\cos(\theta)$ and outside $r = 3\cos\theta$.

(c) The region inside the outer loop but outside the inner loop of the limaçon $r = 1+2\cos\theta$.

The curves are pictured in the next page.


Plot of the Cardioid $r = 1 + \cos\theta$
Plot of the circle $r = 3\cos\theta$
Plot of the Limacon $r = 1 + 2\cos(\theta)$
Y
X
0.5
1.0
1.5
2.0
2.5
3.0
−0.5
−1.0
−1.5

(a) (b) (c)

Figure 3