Assignment 6 (7/6)

Subhadip Chowdhury

- *This homework is due at the beginning of class on **Thursday** 7/12. You are encouraged to work together on these problems, but you must write up your solutions independently.*

Prime Numbers

Definition 1.1. A *prime number* p is a positive integer greater than 1 whose only positive divisors are 1 and p. Positive integers greater than 1 that are not primes are called *composite numbers*.

Theorem 1.2. *Every composite number is divisible by some prime number.*

Proof. We will sketch the main idea behind the proof. Let n be a composite number, so it has some positive divisor(s) apart from 1 and n. Consider the set of all such divisors and the least element in it. If that is not prime then it has a smaller divisor larger than 1 that also divides n. Contradiction. □

Theorem 1.3. *There are infinitely many prime numbers.*

Proof. We will sketch the main idea behind the proof. If the set of prime is finite, say $\{p_1, p_2, \ldots, p_n\}$, consider the number $N = p_1 p_2 p_3 \ldots p_n + 1$. Then N must be a composite number that's not divisible by any prime. Contradiction. □

Definition 1.4. Odd prime numbers that come next to one another in counting (i.e. their difference is 2) are called *twin primes*. If we have three odd primes in a row, they are called *triple primes*.

Theorem 1.5. *The set of primes $\{3, 5, 7\}$ is the only set of triple primes.*

Exercise 5. *Prove theorem 1.5.*
[HINT: *The main idea behind the proof is that every third odd number is divisible by* 3. *Now use PHP or some other reasoning to prove that any set of three consecutive odd numbers must contain one multiple of* 3. *You can also see the book for a different proof.*]

Exercise 6. *Show that if p is the smallest prime factor of n, and $p > \sqrt[3]{n}$, then $\frac{n}{p}$ is either a prime or equal to* 1.

Exercise 7. *Can you find 3 consecutive composite integers? How about four? Five? Can you find a formula that will produce n consecutive composite integers?* [HINT: *Think about using $n! = 1 \cdot 2 \cdot 3 \cdots n$.*]

Exercise 8. *Suppose c and d are positive integers such that $c = dq + r$ for some integers q and r. Then show that $(c, d) = (d, r)$.* [HINT: *Use the result from exercise* 3.]

Exercise 9 (Extra Credit)**.** *(a) Factorize $a^4 + 4b^4$.*

(b) Show that $4^{545} + 545^4$ is not a prime number.

(c) Show that $n^4 + 4^n$ is not a prime number for any natural number n.