Practice Problems for Midterm 1

Subhadip Chowdhury

Domain/Range, Inverse Function, Modification of graphs

- The domain of a function $f(x)$ is the set of x—values where the function is defined. The range of a function is the set of y—values the function can achieve.
- The domain and range of inverse are respectively the range and domain of the original.
- The graph of a function and inverse are reflections of each other at the $x = y$ line.
- e^x and $\ln x$ are inverse of each other. Similarly, $\sin x$ and $\arcsin x$. Similarly, $\tan x$ and $\arctan x$.
- A function has an inverse only if every horizontal line crosses its graph at most once.
- A function is *odd* if its graph is symmetric under a reflection across the origin. It's *even* if the graph is symmetric under reflection at the Y-axis.
- Given a graph of $f(x)$, we can get the following new ones by shrinking/shifting it.
  (i) $f(ax)$ - Shrink horizontally.
  (ii) $f(a+x)$ - Shift to the left by a.
  (iii) $af(x)$ - Stretch vertically.
  (iv) $a+f(x)$ - Shift upwards by a

Exercise 1

Draw the graph of $\sin x, \cos x, \tan x, \arcsin x, \arctan x, e^x, \ln x$ and determine their domain and ranges.

Exercise 2

Draw the graphs of $3\sin(2x+\pi)$, $2\arctan(x+3)$, $3+\ln\frac{x}{2}$. In each case, find the domain and range.

Exercise 3

Find possible equation of a polynomial whose graph looks as figure 1.

Exercise 4

Find possible equation of a trigonometric function of the form

$$A + B\sin(Cx+d)$$

whose graph looks as figure 2.

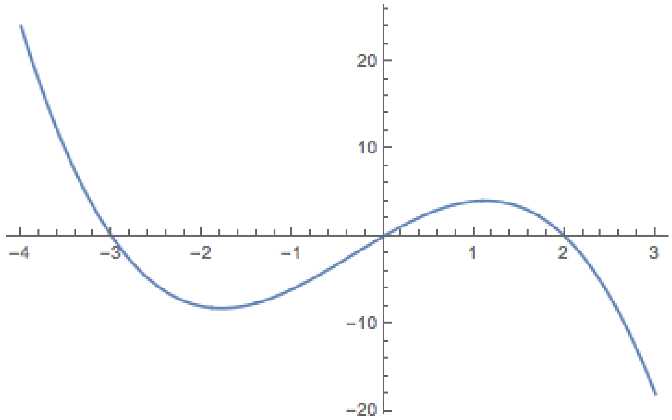


Figure 1

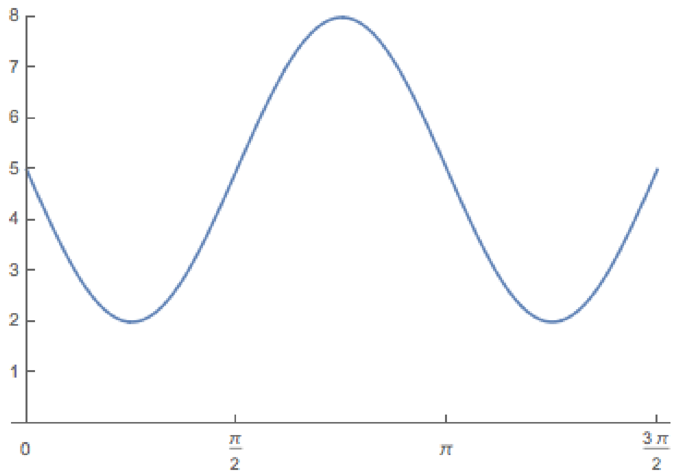


Figure 2

Exercise 5

Find possible equation of an exponential function of the form

$$Ae^{Bx}$$

whose graph looks as figure 3.

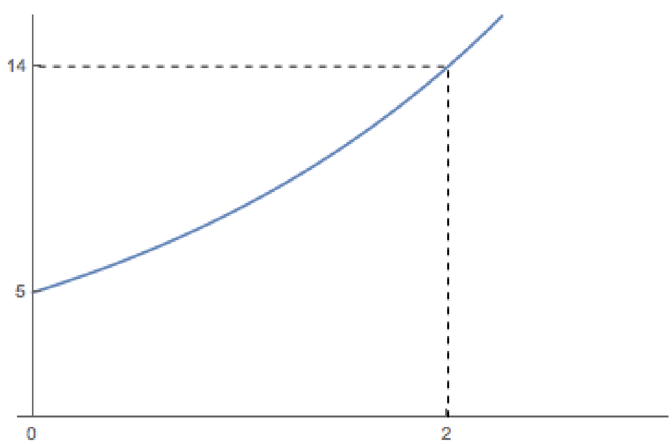


Figure 3

Exponential Growth/Decay, Logarithm

- An exponential growth/decay process is given by the equation

$$Q = Q_0 a^t \text{ or } Q = Q_0 e^{kt}$$

- a is called the growth/decay rate. k is called the *continuous* growth/decay rate. The constant a and k are related as $a = e^k$.
- If $a > 1$ or equivalently $k > 0$, the process is a growth. If $a < 1$ or equivalently $k < 0$, the process is a decay.
- If a problem gives percentage growth of $x\%$, then $a = 1 + \frac{x}{100}$. If it's decay, then $a = 1 - \frac{x}{100}$.
- In most cases, equations from exponential growth/decay word problems need to be solved using logarithm. Recall that,

$$a^x = b \implies x = \log_a b$$

- $\log_e x$ is also denoted as $\ln x$.
- The two most important properties of log that you will use for solving equations are
  - $\log_c(ab) = \log_c a + \log_c b$
  - $\log_c(a^b) = b \log_c a$

Exercise 1

A biologist is researching a newly-discovered species of bacteria. At time $t = 0$ hours, she puts one hundred bacteria into what she has determined to be a favorable growth medium. Six hours later, she measures 450 bacteria. Assuming exponential growth,

1. what is the growth rate?
2. how long does it take for the bacteria population to become 1600?

Exercise 2

Carbon-14 has a half-life of 5730 years. You are presented with a document which purports to contain the recollections of a Mycenaean soldier during the Trojan War. The city of Troy was finally destroyed about 3250 years ago. Carbon-dating evaluates the ratio of radioactive carbon-14 to stable carbon-12. Given the amount of carbon-12 contained a measured sample cut from the document, there would have been about 1.3×10^{-12} grams of carbon-14 in the sample when the parchment was new, assuming the proposed age is correct. According to your equipment, there remains 1.0×10^{-12} grams. Is there a possibility that this is a genuine document? Or is this instead a recent forgery? Justify your conclusions.

Rational Functions and Asymptotes

- Given a rational function of the form $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials, we can find the roots, vertical asymptotes and horizontal asymptotes as follows:

  (i) ROOTS - These are same as roots of $P(x)$.
  - If exponent of linear factor is odd, then graph crosses X-axis from one side to other.
  - If exponent of linear factor is even, then graph bounces off X-axis, and stays in same side.

  (ii) VERTICAL ASYMPTOTES - same as points where $Q(x)$ is zero, i.e. the roots of $Q(x)$.
  - If exponent of linear factor is odd, then graph goes to opposite vertical directions on both sides of asymptotes.
  - If exponent of linear factor is even, then graph goes to same vertical direction on both sides of asymptotes.

  (iii) HORIZONTAL ASYMPTOTES -
  - If degree of $P(x) =$ degree of $Q(x)$, then the horizontal asymptote is at a height given by the ratio of the coefficient of highest power of x in numerator and the coefficient of highest power of x denominator.
  - If degree of $P(x) <$ degree of $Q(x)$, then $x-$axis is a horizontal asymptote.
  - If degree of $P(x) >$ degree of $Q(x)$, then it has no horizontal asymptote.

Exercise 1

Match the following graphs with the rational functions.

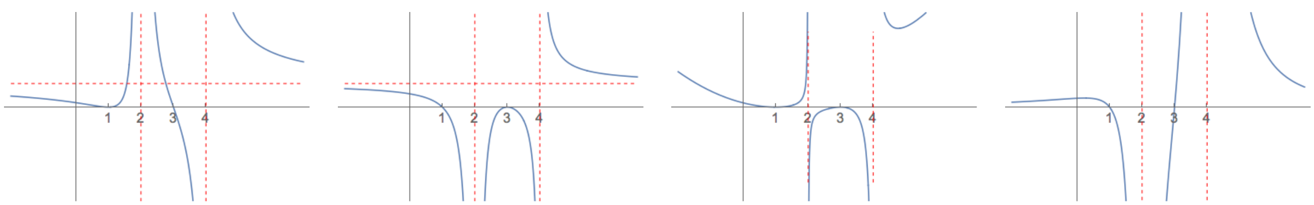


Figure 4: A,B,C,D

$$(I)\frac{(x-3)^2(x-1)^2}{(x-2)(x-4)} \qquad (II)\frac{(x-3)(x-1)}{(x-2)^2(x-4)^2} \qquad (III)\frac{(x-3)(x-1)^2}{(x-2)^2(x-4)} \qquad (IV)\frac{(x-3)^2(x-1)}{(x-2)^2(x-4)}$$

Exercise 2

Find the horizontal and vertical asymptotes of each of the functions above.

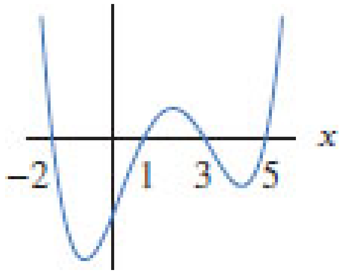
Exercise 3

Find the horizontal and vertical asymptotes of

$$\frac{(3x-2)^3(4-3x)}{(x-5)^2(x^2+1)}$$

Exercise 4

Find possible polynomial formulas for the following graphs.



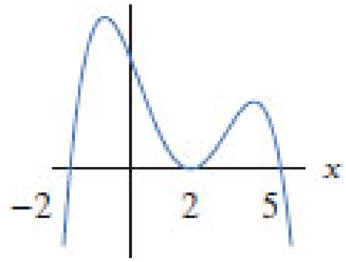


Figure 5

Word Problems on Trigonometric Functions

- Given a trigonometric function of the form

$$f(x) = A + B\sin(C(x - D))$$

(i) A is the average of the lowest and highest value of $f(x)$.

(ii) B is the amplitude.

(iii) The period is $\frac{2\pi}{C}$.

(iv) When $x = D$, the function $f(x)$ is equal to A, the average.

Exercise 1

On July 1, 2007, high tide in Boston was at midnight. The water level at high tide was 9.9 feet; later, at low tide, it was 0.1 feet. Assume the next high tide is at exactly 12 noon and that the height of the water is given by a sine or cosine curve.

(a) find a formula for the water level in Boston as a function of time.

(b) write a formula for the water level in Boston on a day when the high tide is at 2 pm.

Exercise 2

The Bay of Fundy in Canada has the largest tides in the world. The difference between low and high water levels is 15 meters (nearly 50 feet). At a particular point the depth of the water, y meters, is given as a function of time, t, in hours since midnight by

$$y = D + A\cos(B(t - C)).$$

(a) What is the physical meaning of D?

(b) What is the value of A?

(c) What is the value of B? Assume the time between successive high tides is 12.4 hours.

(d) What is the physical meaning of C?

Exercise 3

Use a calculator to find all solutions to $\sin x = -0.4$ with $-\pi \leq x \leq \pi$.

Limit and Continuity

- a function is *continuous* on an interval if its graph has no breaks, jumps, or holes in that interval; i.e. you can draw the graph without lifting your pen from the paper.
- We write
$$\lim_{x\to c} f(x) = L$$
if the values of $f(x)$ approach L as x approaches c.
- For a continuous function $f(x)$, we have
$$\lim_{x\to c} f(x) = f(c)$$
- Polynomials, trigonometric, and exponential functions are continuous on $(-\infty, \infty)$. Rational functions are continuous on any interval in which their denominators are not zero. Functions created by adding, multiplying, or composing continuous functions are also continuous.
- Continuity of a function depends on the domain. Thus for example, $f(x) = \frac{1}{x}$ is continuous on $(1, 2)$ but discontinuous on $(-1, 1)$.
- When calculating limits, if plugging in $x = c$ gives a $\frac{0}{0}$ form, then we need to do some algebraic manipulation (e.g. factorization, rationalizing the numerator/denomiator by multiplying with the conjugate etc.) to simplify our function until it reaches a form that's not $\frac{0}{0}$.
- The left-hand limit $\lim_{x\to c-} f(x)$ is the value that $f(x)$ approaches as x approaches c *from the left*. Similarly we can define the right hand limit $\lim_{x\to c+} f(x)$.
- If the left-hand limit and right-hand limits are equal to each other then they are also equal to the actual limit. If the two one-sided limits are not equal then limit does not exist.
- Another way we might get limit does not exists is if the function is oscillating, e.g. $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$.

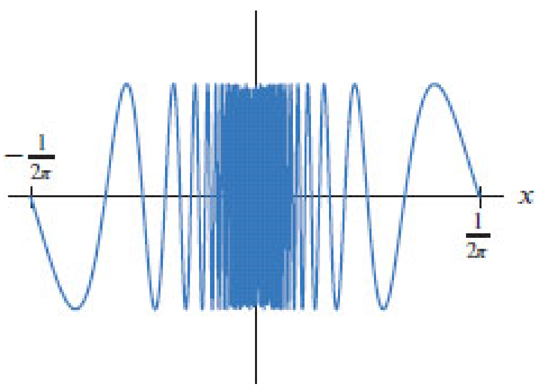


Figure 6

Exercise 1

For the following picture, find

- $\lim_{x\to c-} f(x)$
- $\lim_{x\to c+} f(x)$

- $\lim_{x \to c} f(x)$
- $f(c)$

at $c = -2, -1, 0, 3$. Write DNE if it doesn't exist.

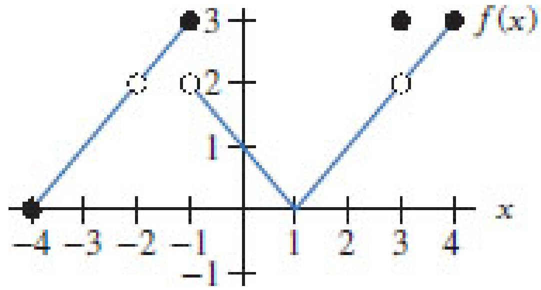


Figure 7

Exercise 2

Let $f(x) = \cos x - x$. Use the intermediate value theorem to prove that there is a number c with $0 \le c \le 1$, such that $f(c) = 0$.

Exercise 3

Let $g(x)$ be continuous with $g(0) = 3, g(1) = 8, g(2) = 4$. Use the Intermediate Value Theorem to explain why $g(x)$ is not invertible.

Exercise 4

Find the constants a and b, so that the following piece-wise defined function is continuous everywhere.

$$f(x) = \begin{cases} a - bx & x \le 1 \\ x^2 & 1 < x < 2 \\ b + ax & x \ge 2 \end{cases}$$

Exercise 5 (Multiple Choice Questions)

(1) Let f, g, and h be functions satisfying

$$\lim_{x \to 3} f(x) = 2, \quad \lim_{x \to 3} g(x) = 2, \quad \lim_{x \to 3} h(x) = -3$$

What is $\lim_{x \to 3} \left[\left(f(x)^{g(x)} \right)^{h(x)} \right]$?

(A) $\frac{1}{64}$

(B) $\frac{1}{2}$

(C) $\sqrt[3]{4}$

(D) $\sqrt[8]{2}$

(E) $2^{\sqrt[3]{2}}$

(2) If $\lim_{x\to 1} f(x) = 0$ and $\lim_{x\to 1} g(x) = 0$, what is $\lim_{x\to 1} \dfrac{f(x)}{g(x)}$?

(A) 0

(B) 1

(C) ∞

(D) This limit does not exist

(E) The value of the limit cannot be determined from the given information.

(3) Find $\lim_{x\to 2} \dfrac{x^2-4}{x-2}$.

(A) -2

(B) 0

(C) 2

(D) 4

(E) This limit does not exist.

(4) Find $\lim_{x\to 2} \dfrac{x^2-x-2}{x^2+x-6}$.

(A) -3

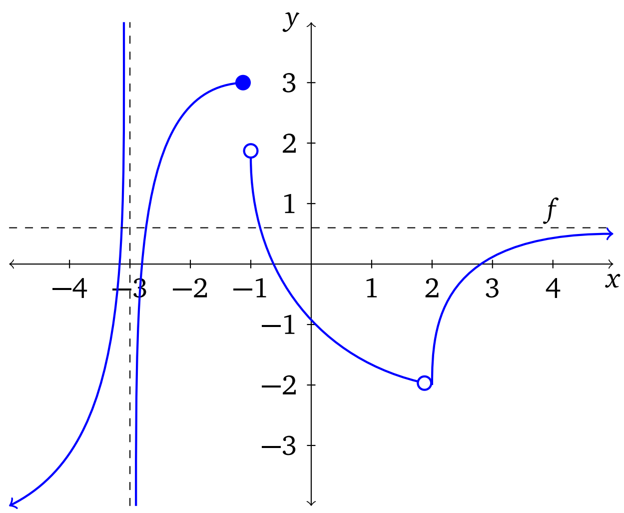
(B) -1

(C) $\dfrac{1}{5}$

(D) $\dfrac{3}{5}$

(E) This limit does not exist.

(5) Consider a function $f : \mathbb{R} \to \mathbb{R}$ whose graph is as follows:



Which of the following statements is true?

(A) $\lim_{x\to -3} f(x)$ does not exist, $\lim_{x\to 2} f(x)$ does not exist, $\lim_{x\to +\infty} f(x)$ does not exist

(B) $\lim_{x\to -3} f(x)$ does not exist, $\lim_{x\to 2} f(x)$ exists, $\lim_{x\to +\infty} f(x)$ does not exist

(C) $\lim_{x\to -3} f(x)$ does not exist, $\lim_{x\to 2} f(x)$ exists, $\lim_{x\to +\infty} f(x)$ exists

(D) $\lim_{x\to -3} f(x)$ does not exist, $\lim_{x\to 2} f(x)$ does not exist, $\lim_{x\to +\infty} f(x)$ exists

(E) $\lim_{x\to -3} f(x)$ exists, $\lim_{x\to 2} f(x)$ exists, $\lim_{x\to +\infty} f(x)$ exists

(6) Let

$$f(x) = \begin{cases} \dfrac{2}{x} & \text{if } x < 1 \\ \sqrt{x}+1 & \text{if } 1 \le x \le 4 \\ x^2 - 3x & \text{if } x > 4 \end{cases}$$

At what values of x is f discontinuous?

(A) $x = 0$

(B) $x = 0$ and $x = 4$

(C) $x = 1$ and $x = 4$

(D) $x = 0$, $x = 1$, and $x = 4$

(E) The function f is continuous at all values of x.

(7) Let $f(x) = \dfrac{x^2-1}{|x^2-1|}$. For what values of x is f NOT continuous?

(A) $x = -1$ only

(B) $x = 1$ only

(C) $x = -1$ and $x = 1$

(D) This function is not continuous for any value of x.

(E) This function is continuous for all values of x.

(8) Which of the following statements is true?

(A) A curve can never cross an asymptote.

(B) An asymptote is a line which a curve approaches as x tends to infinity.

(C) A continuous function can not have any vertical asymptotes.

(D) If $f(x)$ has a vertical asymptote at $x = c$, then $\lim_{x\to c} f(x) = \pm\infty$.

(E) Asymptotes are always parallel to either the x-axis or the y-axis.

Derivative and Slopes

- Given a function $f(x)$, the slope of the tangent to the graph at $(x, f(x))$ is the derivative $f'(x)$.
- $f'(x) = 0$ when the tangent is horizontal. This happens at the maximum or minimum point.
- $f'(x) \geq 0$ when the function is increasing.
- $f'(x) \leq 0$ when the function is decreasing.
- $f'(x) \to \infty$ or $-\infty$ if the tangent becomes almost vertical. This happens near a vertical asymptote.
- $f'(x) \to 0$ if the tangent becomes almost horizontal. This happens near a horizontal asymptote.
- $f'(x)$ doesn't exist if the tangent abruptly changes direction. This happens at a cusp. If you are drawing a graph of $f'(x)$ this is denoted as a jump discontinuity.

Exercise 1

Match the expressions $(i)-(vii)$ to the slopes of the lines on the graph of $y = f(x)$ below. Note that you can use a line more than once, or not at all. No explanation is necessary.

(i) Instantaneous velocity at $x = a$

(ii) Average velocity between $x = a$ and $x = b$.

(iii) $f'(b)$

(iv) $\frac{f(b)-f(a)}{b-a}$

(v) $\frac{f(b)}{b}$

(vi) $\lim\limits_{u \to a}($ Average velocity between $x = u$ and $x = a)$

(vii) $\lim\limits_{h \to 0} \frac{f(b+h)-f(b)}{h}$

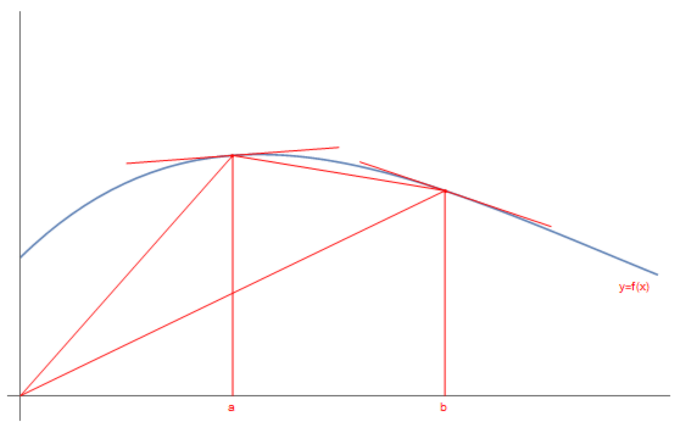


Figure 8

Exercise 2

Problems 2.3.(9, 41, 42, 52, 53).