Lab 5: Volume Integration

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In this lab, we'll use *Mathematica*'s `Integrate` command to compute double and triple integrals. Note that the "outer" integral bounds in double integration are listed first as follows:

$$\int_c^d \int_a^b f(x,y)\, \mathrm{d}x\, \mathrm{d}y \quad \leftrightarrow \quad \texttt{Integrate[f[x,y],\{y,c,d\},\{x,a,b\}]}$$

Exercise 1

Let Ω be the region inside the circle $x^2+y^2=1$. We calculated the volume integral $\iint\limits_{\Omega} (x^2+y^2)^{3/2}\, \mathrm{d}A$ using polar coordinates. Calculate it in Cartesian coordinates using Mathematica's `Integrate` command. Compare the answers.

Exercise 2 (Cautionary Example 1)

Let Ω be the interior of the triangle with vertices $(0,0)$, $(1,1)$ and $(0,1)$. Express the volume integral $\iint\limits_{\Omega} e^{y^2}\, \mathrm{d}A$ as a Cartesian double integral in two different ways. Calculate whichever of the two you can do by hand, and (try to) calculate the other with *Mathematica*'s `Integrate` command. Compare the answers.

Exercise 3 (Cautionary Example 2)

Let Ω be the interior of the square with vertices $(0,0)$, $(1,0)$, $(1,-1)$, and $(0,-1)$. Use *Mathematica*'s `Plot3D` command to draw the graph of

$$f(x,y) = \frac{x+y}{(x-y)^3}$$

and estimate (without calculation) $\iint\limits_{\Omega} f(x,y)\, \mathrm{d}A$, the (signed) volume between the graph of $\frac{x+y}{(x-y)^3}$ and the xy-plane over Ω. Hint: use symmetry.

Now express the volume integral $\iint\limits_{\Omega} f(x,y)\, \mathrm{d}A$ as a Cartesian double integral in two different ways, and use *Mathematica* to compute each. Did you find anything surprising? Can you explain what's going on?

Exercise 4

We can also calculate volume using triple integrals as follows. Consider the space region T bounded below by the surface $z = f_{bottom}(x,y)$ and above by the surface $z = f_{top}(x,y)$, and whose 'shadow' (i.e. projection) in the XY-plane is a region R. Then the volume of T is

$$\iiint\limits_{T} \mathrm{d}V = \iint\limits_{R} \left[f_{top}(x,y) - f_{bottom}(x,y)\right] \mathrm{d}A = \iint\limits_{R} \left(\int_{f_{bottom}(x,y)}^{f_{top}(x,y)} \mathrm{d}z\right) \mathrm{d}A$$

Depending on the region, it might be easier to use one form over other.

For the following problems consider the region T bounded by the surfaces $z = x^2$, $y + z = 10$, and $y = 0$. Type the following in *Mathematica* to see the three surfaces. You might need to adjust the x-,y-, and z-ranges to see everything.

```
ContourPlot3D[{z == x^2, y + z == 10, y == 0}, {x, -5, 5}, {y, -1, 11}, {z, -1, 11},
                ContourStyle -> Opacity[0.5], AxesLabel -> Automatic]
```

(a) What are the 'vertical' walls of this region?

(b) What is the equation of f_{top}? What is the equation of f_{bottom}?

(c) Type the `RegionPlot3D` command and plug in the equations with correct inequalities to see T.

```
RegionPlot3D[y >= 0 && y + z <= 10 && z >= x^2,
            {x, -4, 4}, {y, -1, 11}, {z, -1, 11},
            AxesLabel -> Automatic, PlotPoints -> 50, PlotStyle -> Opacity[0.5]]
```

(d) Set f_{top} equal to f_{bottom} to find the intersection curve of those two surfaces.

(e) Type the following command to see the intersection curves that bound the "shadow in the xy-plane", the region R.

```
ContourPlot[{y == 0, x^2 + y == 10}, {x, -4, 4}, {y, -1, 11}]
```

(f) Type the `RegionPlot` command and and plug in above equations with the correct inequalities to see R.

```
RegionPlot[y >= 0 && x^2 + y <= 10, {x, -4, 4}, {y, -1, 11}]
```

(g) Set up the triple integral giving the volume of T.

Exercise 5

Follow the same steps as in Exercise 4 to set up the triple integral giving the volume of the solid bounded by the following surfaces. Use *Mathematica* to visualize if you need to. Note that you can change the order of integration while setting it up, so that the 'height' is along X- or Y-axis, and the base is in YZ or XZ-plane if required.

a) $x^2 + z^2 = 4, \quad y = -1, \quad y = 1$

b) $x = y^2, \quad z = 0, \quad x + z = 1$

c) $x = 0, \quad x + z = 1, \quad 3 + z = y^2$

d) $x = 0, \quad y = 0, \quad z = 4, \quad x + y - z = 0$

Exercise 6

One physical interpretation of triple integrals is as follows. Consider a solid T with variable density $\rho(x, y, z)$ at the point $(x, y, z) \in T$. Then the mass of T is given by $\iiint_T \rho \, \mathrm{d}V$. Note that with this interpretation, the volume of T is numerically equal to its mass if it has a constant density 1 everywhere. This is why the statements in Exercise 4 make sense.

Consider a cylindrical solid T bounded by $x^2 + y^2 = 4$ and the planes $z = \pm 2$, whose density at a point (x, y, z) is given by the function $\rho(x, y, z) = 4 + 5x^2yz^2$.

(a) Set up the triple integral that gives the mass of T.

(b) Evaluate it using only geometric interpretation and symmetry.