Assignment 21 (11/20)

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*This homework is due at the beginning of class on **Wednesday** 11/22. You may cite results from class as appropriate. Unless otherwise stated, you must provide a complete explanation for your solutions, not simply an answer. You are encouraged to work together on these problems, but you must write up your solutions independently.*

You are encouraged to think about the problems marked with a ($\star$) if you have time, but you don't need to hand them in.

Remember that you can always use the result of the previous assignment problems without proof to solve the new assignment problems.

Problem 0

1.1 The Fundamental Theorem of Linear Programming

Consider the optimization problem

$$\text{Maximize } \sum_{i=1}^{n} c_i x_i \text{ subject to the set of } m \text{ constraints } \sum_{i=1}^{n} a_{ji} x_i \leq b_j \text{ for } j = 1, 2, \ldots, m$$

Denote the set of points $(x_1, x_2, \ldots, x_n)$ that satisfy the set of m equations by P. If P is **bounded**, and $x^\star = (x_1^\star, x_2^\star, \ldots, x_n^\star)$ is an optimal solution, then the fundamental theorem of LP says that

1. either $x^\star$ lies on an extreme point (vertes) of P,
2. or $x^\star$ lies of a face of P.

Here P is called a polytope (an n-dimensional analog of a polygon). Thus P has m faces, each corresponding to one of the m equations. Observe that n of the faces meet at an extreme point (vertex) of P. Thus the theorem provides us a with a straightforward (albeit lengthy) algortihm for finding $x^\star$ as follows:

Step 1. Take n of the m equations at a time and solve the system of n equations for n variable.

Step 2. For each of the solutions from step 1, if it satisfies the other $m - n$ inequalities, then it is a vertex of P. This step is to make sure that the intersection of the n faces actually lies in P.

Step 3. Calculate $\sum_{i=1}^{n} c_i x_i$ at each of the vertices from previous step.

Step 4. Assuming that a solution exists, the maximum among all values from step 2 is the required answer.

Remark.

If you know permutation and combinations, and are familiar with the factorial notation, you may recognize that the number of ways of choosing n equations from a set of m equations is

$$\binom{m}{n} = \frac{m!}{n!(m-n)!} = \frac{m(m-1)\cdots 2\cdot 1}{[n(n-1)\cdots 2\cdot 1][(m-n)(m-n-1)\cdots 2\cdot 1]}$$

assuming $m \geq n$ and 0 otherwise. So we have at most these many points in step 1.

1.2 Example

Let's optimize $3x_1 + 2x_2$ subject to the constraints

$$-x_1 + 3x_2 \leq 12 \tag{1}$$
$$x_1 + x_2 \leq 8 \tag{2}$$
$$2x_1 - x_2 \leq 10 \tag{3}$$
$$x_1 \geq 0 \tag{4}$$
$$x_2 \geq 0 \tag{5}$$

Here $m = 5, n = 2$. Following our algorithm above, we need to solve 2 equations at a time.

$$\begin{cases} -x_1 + 3x_2 & = 12 \\ x_1 + x_2 & = 8 \end{cases} \implies (x_1, x_2) = (3, 5)$$

$$\begin{cases} x_1 + x_2 & = 8 \\ 2x_1 - x_2 & = 10 \end{cases} \implies (x_1, x_2) = (6, 2)$$

$$\begin{cases} -x_1 + 3x_2 & = 12 \\ 2x_1 - x_2 & = 10 \end{cases} \implies (x_1, x_2) = (42/5, 34/5)$$

$$\begin{cases} -x_1 + 3x_2 & = 12 \\ x_1 & = 0 \end{cases} \implies (x_1, x_2) = (0, 4)$$

$$\begin{cases} x_1 + x_2 & = 8 \\ x_1 & = 0 \end{cases} \implies (x_1, x_2) = (0, 8)$$

$$\begin{cases} 2x_1 - x_2 & = 10 \\ x_1 & = 0 \end{cases} \implies (x_1, x_2) = (0, -10)$$

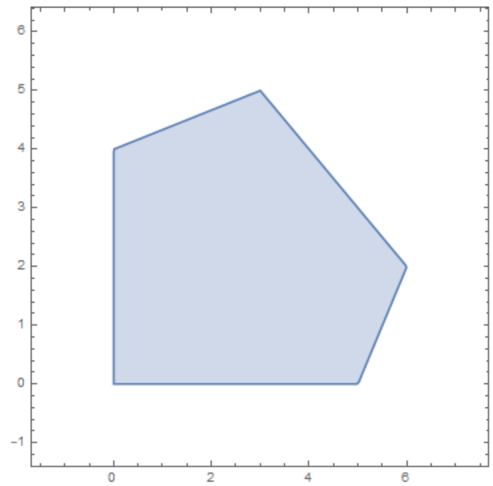
$$\begin{cases} -x_1 + 3x_2 & = 12 \\ x_2 & = 0 \end{cases} \implies (x_1, x_2) = (-12, 0)$$

$$\begin{cases} x_1 + x_2 & = 8 \\ x_2 & = 0 \end{cases} \implies (x_1, x_2) = (8, 0)$$

$$\begin{cases} 2x_1 - x_2 & = 10 \\ x_2 & = 0 \end{cases} \implies (x_1, x_2) = (5, 0)$$

$$\begin{cases} x_1 &= 0 \\ x_2 &= 0 \end{cases} \implies (x_1, x_2) = (0, 0)$$

Note that if you plot the region, there are only 5 vertices. That is because each of the other 5 solutions correspond to intersection point of two straight lines that lie outside the polygon.



When we can't draw the plot, how do we check which one of our solutions are actual vertices? This is step 2 of our algorithm.

Check that, $(0, -10), (-12, 0)$ can't be our solution because of equation 4, 5. Next, $(8, 0)$ is not a solution since it does not satisfy equation 3. Similarly, $(42/5, 34/5)$ does not satisfy equation 2, and $(0, 8)$ does not satisfy equation 1. Evrery other point satisfies the rest of the three equations.

So our possible candidates for an optimal solution are $(3, 5), (6, 2), (0, 4), (5, 0)$, and $(0, 0)$. The values obtained by $3x_1 + 2x_2$ at each of these points are $19, 22, 8, 15$, and 0. Hence the maximum value is 22, achieved at $(6, 2)$ and the minimum value is 0, obtained at the origin.

Problem 2

Do problems 2.(2, 9) from Vanderbei.