Problem Set 3 Solutions

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Exercise 2: Set Notation

Problem 1.1. *Write the set of positive numbers that end in* 0 *i.e.* 10, 20, 30, 40, 50, ... *in descriptive notation.*

Solution. $\{10n | n \in \mathbb{N}\}$ ■

Exercise 3: Sets of Numbers

Problem 2.1. *Give example of two sets* $\mathscr{A}$ *and* $\mathscr{B}$ *such that neither* $\mathscr{A} \subset \mathscr{B}$ *nor* $\mathbb{B} \subset \mathbb{A}$ *.*

Solution. Some examples are:

- $\mathscr{A} = \mathbb{N}, \mathscr{B} = \{x | x \in \mathbb{R}, 0 \leq x \leq 1\} = [0, 1]$
- $\mathbb{Q}, \mathbb{R} \setminus \mathbb{Q}$
- $\{1, 2\}, \{2, 3\}$

■

You could have chosen any $\mathscr{B}$ and $\mathscr{A}$ such that $\mathscr{B}$ has at least one element not in $\mathscr{A}$, and $\mathscr{A}$ has at least one element not in $\mathscr{B}$. However, you could not have chosen the empty set $\emptyset = \{\}$ as one of your sets, since every set is a superset of the empty set.

Exercise 4: Closure of Number Sets Under Operations

Problem 3.1. *Consider two numbers m, $n \in \mathbb{N}$. Which of the four numbers $m+n$, $m-n$, $m \times n$, and m/n must be another element of $\mathbb{N}$? Answer the same question when $\mathbb{N}$ is replaced by $\mathbb{Z}$,$\mathbb{Q}$,$\mathbb{R}$ or the set of irrational numbers ($\mathbb{R} \setminus \mathbb{Q}$).*

In the table below, the entries are T if true, and a counterexample if false.

Set $\mathscr{A}$	$m+n \in \mathscr{A}$	$m-n \in \mathscr{A}$	$m \times n \in \mathscr{A}$	$m/n \in \mathscr{A}$
$\mathbb{N}$	T	$1-2$	T	$1/2$
$\mathbb{Z}$	T	T	T	$1/2$
$\mathbb{Q}$	T	T	T	T if $n \neq 0$
$\mathbb{R}$	T	T	T	T if $n \neq 0$
$\mathbb{R} \setminus \mathbb{Q}$	$\sqrt{2}+(-\sqrt{2})$	$\pi - \pi$	$\sqrt{2} \times \sqrt{2}$	$\frac{\pi}{\pi}$

Exercise 6: Irrational Sum

Problem 4.1. *Prove that the sum of a rational number and an irrational number is irrational.*

Claim. *The difference of two rational numbers is rational.*[1]

Proof of Claim. If $r, q \in \mathbb{Q}$, then $\exists a, b, c, d \in \mathbb{Z}$ such that $r = \frac{a}{b}$ and $q = \frac{c}{d}$. Consider $r - q = \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$. As the product and sum of integers are integers, we get that $ad - bc$ and bd are integers, and hence $r - q$ is an element of $\mathbb{Q}$. $\square$

Proof of the Exercise. Suppose, for the sake of contradiction, that $\exists r \in \mathbb{Q}$ and $x \in \mathbb{R} \backslash \mathbb{Q}$ such that $x + r \in \mathbb{Q}$. Now we showed above that the difference of two rational numbers is rational, i.e. if $m, n \in \mathbb{Q}$, then $m - n \in \mathbb{Q}$. So now let us take $m = r + x$ and $n = r$. Both $r + x$ and r are elements of $\mathbb{Q}$, by our assumption. Hence $(r + x) - r$, which is equal to simply x, is also an element of $\mathbb{Q}$. But this contradicts $x \in \mathbb{R} \setminus \mathbb{Q}$. $\square$

1. We could have started by proving that sum of integers are integers (by induction-can you do it?) and since multiplication is an integer number of additions, this would imply product of integers is integer (also can be proved by induction). Since rationals are defined using integers, we are going to directly assume these facts about integers for the purpose of this proof.

Exercise 7: Negation

Problem 5.1. *What is the negation of the following sentence? "There exists some natural number N such that all elements of set A are less than N."*

Solution. $\forall N \in \mathbb{N}$, $\exists x \in \mathbb{A}$ such that $x \geq N$.
The above statement reads: For all natural numbers N, there exists an element x in $\mathbb{A}$ that is greater than or equal to N. ■

Exercise 8: Sylvester Problem

Problem 6.1. *A finite set S of points in the plane has the property that any line through two of them passes through a third. Show that all the points lie on a line.*

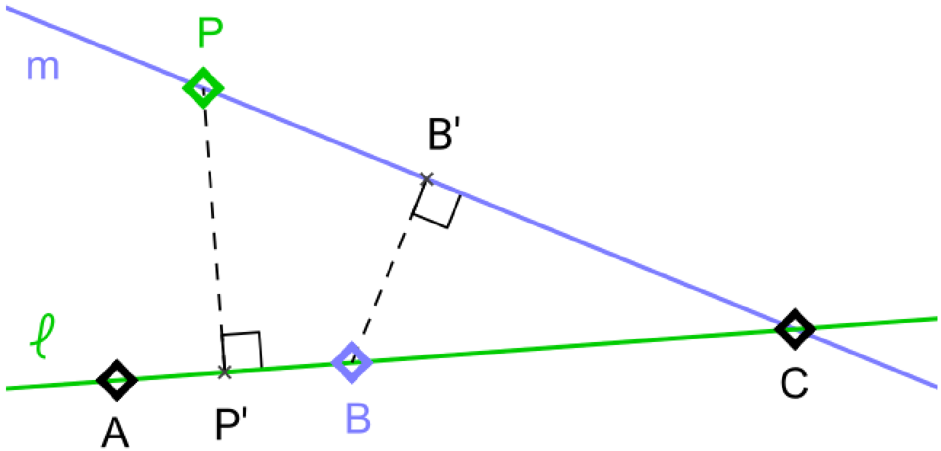

Figure 1: A drawing of an example geometry for the Sylvester problem.

Proof. For the sake of contradiction, suppose that not all the points in S are collinear. Consider the set of lines L which contains all the lines between any two points in S. For every line in L, there must exist some point in S that does *not* lie on the line, i.e. the distance d between them is positive.[2] Now, since the set L is finite (since S is finite), there must exist some line l and some point P not on it that achieve the minimum such distance d, which we shall call d_{min}. Refer to figure 1 for an example of such a situation. In the figure the square points are elements of S. P' is the foot of perpendicular from P to l and $d_{min} = |PP'|$.

Recall that any line through two points of S passes through a third. Hence, there are at least three points A, B, and C on l. Now, notice that the point P' divides l into two regions. By the Pigeonhole Principle, at least two of the three points A, B, and C lie in the same region; let's say B and C lie to the right of P' (in that order).[3]

We construct the line m through the points P and C. This still in L. We are going to consider the perpendicular distance from B to m, call it d'. This is the length BB' in above figure.

We are going to show that $d' < d_{min}$. There are many ways to show this. One way is to see that $\triangle BB'C$ and $\triangle PP'C$ are similar and hence the ratios of similar sides are equal. Since $|CP'| > |CB| > |CB'|$, we get that $\frac{|BB'|}{|PP'|} = \frac{|CB'|}{|CP'|} < 1$. Hence $d' < d_{min}$.

Another way is via a simple construction. Assume the stratight line through B that is parallel to PP', intersects m at D. Then $\triangle BB'D$ is a right angled triangle, implying that $|BB'| < |BD|$. But clearly $|BD| < |PP'|$, since B is between C and P'. Hence $d' = |BB'| < |PP'| = d_{min}$.

Thus we have shown $d' < d_{min}$. But this is a contradiction, as d_{min} was supposed to be the minimum such distance for all lines in the set L. □

2. Here, we define distance of a point to a line as the length of the perpendicular from the point to the line.
3. The rest of the proof still holds if one of the points is P'.

Exercise 9: Proof By Induction

Problem 7.1. *Prove that* $\forall n \in \mathbb{N}$,

$$1 \times 2 + 2 \times 3 + ... + n(n+1) = \frac{n(n+1)(n+2)}{3} \qquad (\star)$$

Proof. We will prove the identity by inducting on n. We will begin by proving that the above identity holds for $n = 1$. Observe that the right-hand side can be written $\frac{1(1+1)(1+2)}{3} = 2$ which is equal to the left-hand side, $1 \times 2 = 2$.

Now, let us assume that the identity holds for some $k \in \mathbb{N}$. Thus we have,

$$1 \times 2 + 2 \times 3 + ... + k(k+1) = \frac{(k)(k+1)(k+2)}{3}$$

Adding the next term, $(k+1)(k+1+1)$ to both sides, we have

$$\begin{aligned} 1 \times 2 + 2 \times 3 + ... + (k)(k+1) + (k+1)(k+2) &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\ &= (k+1)(k+2)\left(\frac{k}{3} + 1\right) \\ &= \frac{(k+1)(k+2)(k+3)}{3} \end{aligned}$$

Thus the identity holds for $n = k+1$ whenever the identity holds for $n = k$. Hence, by the principle of mathematical induction, the identity ($\star$) holds $\forall n \in \mathbb{N}$.

□

Exercise 10: Induction on Odds

Problem 8.1. *Guess a formula for the sum of the first n odd natural numbers and then prove it by induction.*

We claim that the formula for sum of the first n odd natural numbers is n^2.

Claim. $\forall n \in \mathbb{N}$,

$$1 + 3 + 5 + ... + 2n - 1 = n^2 \qquad (\dagger)$$

Proof. We will prove the identity by inducting on n. We observe that the above identity holds for $n = 1$: the right side is 1^2 which is equal to the sum up to the first odd natural number, i.e. 1.

Now, let us make the induction assumption that there is some $k \in \mathbb{N}$ for which this identity holds:

$$1 + 3 + 5 + ... + 2(k-1) = k^2$$

where we have summed up to the k^{th} odd natural number. Adding the $(k+1)^{th}$ odd number, $(2k+1)$, to both sides, we have as the left side

$$1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = 1 + 3 + 5 + \ldots + (2k + 1)$$

Moreover, the right side is

$$(k)^2 + (2k + 1) = k^2 + 2k.1 + 1^2 = (k + 1)^2$$

Thus the identity holds for $n = k + 1$ whenever the identity holds for $n = k$. Hence, by the principle of mathematical induction, the identity (†) holds $\forall n \in \mathbb{N}$. □

Exercise 11: Induction Step Fallacy (Extra Credit)

Problem 9.1. *Explain what is wrong with it the proof by induction of the following: All real numbers are equal. (see assignment for the incorrect proof).*

Solution. The induction step contains the fallacy. To be precise, consider the logic given in the induction step applied to the case $k = 1$. The base case is trivially true. So we are good so far, there is no issue in there. Next, as we attempt to make the inductive step, the proof reads: Applying the induction hypothesis to the first $k(= 1)$ numbers, we get

$$a_1$$

and

$$a_2.$$

Each of the chain of equalities in the fake proof in fact contains only one term. Since there is no common terms in the two 'equations', the conclusion $a_1 = a_2$ is false. So the argument doesn't work when $k = 1$, and our induction process never gets started. ■