Assignment 9 (7/12)

Subhadip Chowdhury

- *This homework is due at the beginning of class on* ***Thursday*** 7/19. *You are encouraged to work together on these problems, but you must write up your solutions independently.*

Geometric Progression

A sequence where the ratio of consecutive terms is constant, is called an Geometric Progression (GP). We need two data to define an GP. We need the starting value, call it a; and we need the common ratio, call it r. Then we have the following formula:

- The nth term of the GP is given by
$$t_n = ar^{n-1}.$$
- Sum of the first n terms in the GP is given by
$$S_n = t_1 + t_2 + \ldots + t_n = a\frac{r^n - 1}{r-1}$$
If $r = 1$, the sum is $S_n = an$.

The formula for the sum was obtained using the identity:

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \ldots + x^2 + x + 1)$$

Exercise 1 (do not turn this in)**.** *If three numbers a, b, c are in a GP, then $b^2 = ac$.*

Remark 1.1 (Important Observation). If each term of a GP is multiplied by the same non-zero quantity, then the resulting sequence is also a GP. This is because if

$$a, ar, ar^2, ar^3, \ldots$$

is a GP then so is

$$ka, kar, kar^2, kar^3, \ldots$$

Remark 1.2 (Important Observation). If

$$a_1, a_2, a_3, \ldots$$

and

$$b_1, b_2, b_3, \ldots$$

are two GPs, then so are the sequences

$$a_1 b_1, a_2 b_2, a_3 b_3, \ldots$$

and

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$$

Remark 1.3. Similar result holds for adding or subtracting two APs termwise.

Exercise 2. *Suppose the ratio of the sum of first three terms of a GP to the sum of the sum of first six terms is $\frac{125}{152}$. Find the common ratio of the GP.*

Exercise 3. *The first, second and seventh term of a certain AP are the first three terms of a GP. If their sum is* 93*, then find the numbers.*

Exercise 4. *One side of an equilateral triangle is* 24 *cm. The mid-points of its sides are joined to form a triangle. Again another triangle is formed by joining the mid-points of the sides of this triangle and this process is continued. Determine the sum of the area of the triangles after you have repeated the process* 100 *times. (To clarify, we have drawn a total of* 100 *triangles, including the starting one with side length* 24 *cm.) Your answer should be in closed-form, but you do not have to simplify the powers.*

Exercise 5 (Extra Credit, Hard Problem)**.** *Five distinct* 2*-digit positive integers are in a geometric progression. Find the middle term.*

Finding An Explicit Formula for the Fibonacci sequence (To be continued)

Recall that the Fibonacci sequence is defined as

$$\begin{aligned} &F_1 = F_2 = 1 \\ &F_n = F_{n-1} + F_{n-2} \text{ for } n > 2 \end{aligned} \tag{$\star$}$$

We are going to find an explicit formula for the nth term F_n. Before that let's analyze the equation ($\star$). Suppose there is some GP that has property ($\star$) i.e. each term (starting with the third) is sum of the previous two terms. We can write the terms of the GP as

$$a, ar, ar^2, ar^3, \ldots, ar^{n-1}, \ldots$$

Equation ($\star$) says

$$\begin{aligned} &ar^{n-1} = ar^{n-2} + ar^{n-3} \\ \implies &r^{n-1} = r^{n-2} + r^{n-3} \\ \implies &r^2 = r + 1 \\ \implies &r^2 - r - 1 = 0 \\ \implies &r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

We will use α, β to denote the two roots. The constant α is also known as the *Golden Ratio*.

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}$$

Thus we have proved that the two sequences

$$1, \alpha, \alpha^2, \alpha^3, \alpha^4, \ldots$$

and

$$1, \beta, \beta^2, \beta^3, \beta^4, \ldots$$

satisfy the property ($\star$) i.e. each term (starting with the third) is sum of the previous two terms. In fact observe that if a sequence satisfies ($\star$), and each term is multiplied by the same non-zero quantity, then the resulting sequence also satisfies ($\star$). This fact, along with observation 1.1 prove the following theorem.

Theorem 2.1. *Suppose we have a Geometric Progression $\{a_i\}_{i \in \mathbb{N}}$ that satisfies*

$$a_n = a_{n-1} + a_{n-2} \quad \textit{for } n > 2$$

Then the sequence must be in one of the following two formats:

$$c, c\alpha, c\alpha^2, c\alpha^3, c\alpha^4, \ldots \quad \textit{for some constant } c$$

or

$$d, d\beta, d\beta^2, d\beta^3, d\beta^4, \ldots \quad \textit{for some constant } d.$$

Finally, we make the following claim that we will prove next time.

Claim. *A sequence* $\{z_i\}_{i\in\mathbb{N}}$ *defined as*

$$z_i = c\alpha^i + d\beta^i$$

for some constants c *and* d *has the property that each term (starting with the third) is sum of the previous two terms.*

Exercise 6. *Suppose a Geometric Progressions* $\{x_i\}_{i\in\mathbb{N}}$ *satisfies*

$$x_{n+1} = 7x_n - 12x_{n-1} \quad \textit{for } n > 2$$

Find all such sequences.