Assignment 6 (1/17)

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- *This homework is due at the beginning of class on **Friday** 1/26. You may cite results from class as appropriate. Unless otherwise stated, you must provide a complete explanation for your solutions, not simply an answer. You are encouraged to work together on these problems, but you must write up your solutions independently.*
- *Hand in the exercises only, not the reading material part. You are encouraged to think about the exercises marked with a ($\star$) or (†) if you have time, but you don't need to hand them in. If you correctly solve a (†)-marked problem, you will get a candy!*
- *Remember that you can always use the result of the previous assignment problems without proof to solve the new assignment problems.*
- *We are currently covering Chapter 14 from Stewart.*

Important Points and Reading Materials

- Directional Derivatives - Partial Derivatives
  - Understand the definition of directional derivative $D_{\vec{u}}f(\vec{r})$, pay particular attention to the fact that by convention, we choose $\vec{u}$ to be a unit vector.
  - How does this relate to partial derivatives i.e. what $D_{\hat{i}}f$ and $D_{\hat{j}}f$ are.
  - Understand that partial derivatives do not give enough information about how a function changes - they only tell us what happens when you move parallel to the axes. In general you need infinitely many pieces of information - corresponding to infinitely many directions and directional derivatives - to determine a satisfactory picture of the 'derivative' of f.
  - Note that for a function $z = f(x, y)$, we will be interchangeably using the notations $\frac{\partial z}{\partial x}, \frac{\partial f}{\partial x}(x, y), f_x(x, y)$ etc.
  - Know how to compute partial derivatives. For example, taking $\frac{\partial}{\partial x}f(x, y)$ means treating y as a constant and taking a one-variable-derivative of f with respect to x.
  - Understand why the formula $D_{\langle a,b\rangle}f = af_x + bf_y$ can be expressed as $D_{\vec{u}}f = \nabla f \cdot \vec{u}$, where ∇f is the gradient vector.
  - The direction f changes most rapidly is when $D_{\vec{u}}f$ is maximized, and this happens along ∇f. What is the maximum rate of change? The maximum rate is $\|\nabla f\|$.
  - Recall that we can define higher order partial derivatives by taking repeated partials with respect to different variables. Thus an expression of the form $\frac{\partial^2}{\partial x \partial y}f(x, y)$ means that we first take partial w.r.t. y, and then wrt x. E.g.
  $$\frac{\partial^2}{\partial x \partial y}\sin(xy) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}(\sin(xy))\right) = \frac{\partial}{\partial x}(x\cos(xy)) = -yx\sin(xy) + \cos(xy)$$
  The last step above is product rule. Do not mix up this order.
  - All of the above notions can be suitably generalized to a function of more than two variables.
  - Equation of tangent plane at a point (x_0, y_0, z_0) to the graph $z = f(x, y)$ is given by
  $$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$
  We will prove this formula next time and also elaborate more on the topic of tangent planes to surfaces.

Problems

Exercise 1

Find the directional derivative of $f(x, y) = x^2 \ln y$ at $(3, 1)$ in the direction of the vector $\langle -5, 12\rangle$.

Exercise 2⋆

(14.6.19) Find the directional derivative of $f(x,y)=\sqrt{xy}$ at $(2,8)$ towards $(5,4)$.

Exercise 3

(14.6.26) Find the maximum rate of change of $f(x,y,z)=\arctan(xyz)$ at $(1,2,1)$ and the direction in which it occurs.

Exercise 4

(14.6.29) Find all the points at which the direction of fastest change of the function

$$f(x,y)=x^2+y^2-2x-4y$$

is $\langle 1,1\rangle$.

Exercise 5

(14.6.35) Suppose f is a function of 2 variables that has continuous partial derivatives. The directional derivative of f at $(1,3)$ towards $(3,3)$ is 3 and towards $(1,7)$ is 26. Find the directional derivative of f at $(1,3)$ towards $(6,15)$.

Exercise 6

(14.3.56) Find all the second order partial derivatives of the function $f(r,\theta)=e^{-2r}\cos\theta$. Note that there are 4 of them.

Exercise 7

(14.3.75) Verify that the function $u(t,x)=e^{\alpha^2k^2t}\sin(kx)$ satisfies the *heat conduction equation* $u_t=\alpha^2u_{xx}$.

Exercise 8

(14.3.83) The total resistance R produced by three conductors with resistance R_1, R_2, and R_3 connected in a parallel electrical circuit is given by the formula

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}$$

Find $\frac{\partial R}{\partial R_1}$.

Exercise 9†

The gas law for an ideal gas of mass n, pressure P, volume V, and temperature T is

$$PV=nRT$$

where R is a constant. Show that

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P}=-1$$

Exercise 10

If $f(x,y)=\sqrt[3]{x^3+y^3}$, find $f_x(0,0)$.