Lab 4: Stationary Points

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A stationary point for a function of two variables is a point where both first partial derivatives equal zero. In this lab you will investigate the stationary points for the following functions:

$$f(x,y) = x^3 + 3xy + y^3$$
$$g(x,y) = x^2 + 6xy + y^2 + 14x + 10y$$
$$h(x,y) = 16x^2 + 8xy + y^2$$

§1. Computing Stationary Points

Not only will *Mathematica* calculate the partial derivatives for you, but it also has a built-in `NSolve` command that you can use to find the points where the derivatives are both zero.

(1) Define the first function by entering the command

```
f[x_,y_] := x^3 + 3*x*y + y^3
```

(2) Solve for the stationary points by entering the command

```
NSolve[Grad[f[x,y],{x,y}]=={0,0}, {x, y}, Reals]
```

(3) We can define a function `StatPtList` in Mathematica that will take f, g or h as input and produce the list of stationary points directly as follows. Type

```
StatPtList[func_] := NSolve[Grad[func[x,y], {x, y}] == {0, 0}, {x, y}, Reals]
```

Check that `StatPtList[f]` produces the same list of points as above. The advantages of doing this step are as follows.

- First, to find the stationary points of `g` and `h`, we can skip writing a long command as in part (2). Instead, we can directly get a list by using `StatPtList[g]` and `StatPtList[h]`.

- Secondly, we have given the list of stationary points a name, that we can refer to later.

(4) Record these stationary points in the table below for future reference. *Note that the third function $h(x,y)$ has an entire line of stationary points, and you should choose any one of these for your investigation.*

(5) We are going to plot the cross-sections parallel to YZ and XZ planes through each of the stationary points, and use these record the signs of the two second order partial derivatives in the table. Recall that you can plot a cross-section parallel to YZ plane through a point (a,b) by entering the command

```
Plot[f[a,y],{y,b-0.1,b+0.1}]
```

Similarly you can plot a cross-section parallel to XZ plane by fixing y and varying x. To speed up our process, we are going to define a function `CrossSectionPlot` as follows:

```
CrossSectionPlot[func_, pt_] := (
    Plot[func[pt[[1]], y], {y, pt[[2]] - 0.01, pt[[2]] + 0.01},
    ImageSize -> Medium, AxesLabel -> {y, func},
    PlotLabel -> Row[{"Cross Section at " , pt , " parallel to XZ plane"}]]

    Plot[func[x, pt[[2]]], {x, pt[[1]] - 0.01, pt[[1]] + 0.01},
    ImageSize -> Medium, AxesLabel -> {x, func},
    PlotLabel -> Row[{"Cross Section at " , pt , " parallel to YZ plane"}]]
  )
```

Spend a couple of moments to read the function definition above carefully. Then enter the following command to get the cross section plots for f.

```
CrossSectionPlot[f,{a,b}]
```

where you need to change a and b appropriately to the points from the list of stationary points you obtained above. Next change f to g and h and the points a and b appropriately to the corresponding stationary points to repeat this procedure. *Use the concavity of the curves in these plots to fill out the table below.*

(6) Using only the cross-sections you just plotted, try to classify the stationary points as a local maximum, a local minimum, a saddle, or undetermined.

Function:	$f(x,y)$		$g(x,y)$	$h(x,y)$
Stationary points: $x =$				
$y =$				
Sign of $\frac{\partial^2}{\partial x^2}$:				
Sign of $\frac{\partial^2}{\partial y^2}$:				
Classification:				

§2. Using Second Derivative test

(8) We can define the determinant of the Hessian of the function $f(x, y)$ as

```
DH[func_,x_, y_] := D[func[x, y], x, x] * D[func[x, y], y, y]
                    - (D[func[x, y], x, y])^2
```

(9) Find `DH[f,x,y]`. Next find the determinant for other two functions by changing f to g and h.

(10) We can evaluate the determinant at each of the stationary point above as follows. Type

```
DH[f,x,y]/.StatPtList[f]
```

to evaluate $DH(x, y)$ at the stationary points of f. Note the '`/.`' operator. It's called `ReplaceAll`.

(11) Record the values at each stationary point as above. Check whether or not the signs agree with your classification in the table above using the second derivative test.

■ Exercise 1.

Did you classify any stationary points incorrectly using just the cross-sections? What do you think went wrong?

■ Exercise 2.

Draw contour plots of the functions f, g and h to check your classifications graphically. Choose a big enough domain so that it contains all the stationary points you are looking at. Recall that the command for Contour Plot looks like

```
ContourPlot[f[x,y],{x,a,b}, {y,c,d}]
```

Do you see what went wrong in exercise 1?

■ Exercise 3.

Is the following "second derivative test" always valid?

Consider a stationary point (a, b) for any function $f(x, y)$.

If $f_{xx}(a, b) > 0$ and $f_{yy}(a, b) > 0$, then (a, b) is a local minima.

If $f_{xx}(a, b) < 0$ and $f_{yy}(a, b) < 0$, then (a, b) is a local maxima.

■ Exercise 4 (Spring Break Homework).

Repeat all of the above steps for $k(x, y) = x^3 - 6\,x\,y + 3\,x\,y^2$ and classify its stationary points.

■ Exercise 5 (Optional).

Study the commands in step (5) and (10) more carefully and read the documentations of the commands used. Try the following command:

```
Map[CrossSectionPlot[k, #] &, {x, y} /.  StatPtList[f]]
```

Can you figure out what it is doing?