MATH 2208: ORDINARY DIFFERENTIAL EQUATIONS

LECTURE 8 WORKSHEET

Fall 2019 **Subhadip Chowdhury** **Oct 2**

TITLE: Some analytical techniques for solving first order ODEs

SUMMARY: We will learn about integrating factors, the change of variable method, and exact ODEs.

§A. Integrating Factor

Consider a linear ODE of the form $y' = \varphi(t)y + \psi(t)$. To use the technique of *Integrating Factors*, we will first rewrite it into the following form:

$$\frac{dy}{dt} + P(t)y = Q(t)$$

What's the idea?

Think about the product rule for differentiating the function $\mu(t)y(t)$.

$$\frac{d(\mu(t)y(t))}{dt} = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y(t)$$

So if we stare at both of the last two equations hard enough and long enough, we might think about rewriting the ODE as

$$\mu(t)\frac{dy}{dt} + \mu(t)P(t)y = \mu(t)Q(t)$$

whose left hand side 'sort of' looks like the product rule. So if we could find a function $\mu(t)$ such that

$$\mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y(t) = \mu(t)\frac{dy}{dt} + \mu(t)P(t)y,$$

we would be able to rewrite the ODE as

$$\frac{d(\mu(t)y(t))}{dt} = \mu(t)Q(t)$$

which can be easily solved as

$$\mu(t)y(t) = \int \mu(t)Q(t)dt \implies y(t) = \frac{1}{\mu(t)}\int \mu(t)Q(t)dt$$

So what's $\mu(t)$?

Note that we need to find $\mu(t)$ such that

$$\frac{d\mu}{dt} = \mu(t)P(t)$$

■ **Question 1.**

Find $\mu(t)$.

Theorem 3.1

We call $\mu(t)$ the *integrating factor*. With the formula for $\mu(t)$ you obtained above, the complete formula for $y(t)$ is given by

$$y(t) = \frac{1}{e^{\int P(t)dt}} \int \left(Q(t) e^{\int P(t)dt} \right) dt$$

■ **Question 2.**

Solve $y' = -2ty + 4e^{-t^2}$.

■ **Question 3.**

Recall the salt-mixing problem from your first assignment. Below we have the same ODE but for the sake of simplifying the calculation, I have changed all the constants to 1. Solve

$$\frac{dy}{dt} = 1 - \frac{y}{1+t}, \quad y(0) = 1$$

■ **Question 4.**

For what value(s) of the parameter r is it possible to find explicit formulas (without integrals) for the solution to the ODE

$$y' = t^r y + 4$$

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§B. Change of Variable

Often, a first-order ODE that is neither separable nor linear can be simplified to one of these types by making a change of variables. Here are some important examples:

Homogeneous Equation: If $\frac{dy}{dt} = f(t,y)$ where $f(kt,ky) = f(t,y)$, use the change of variables $z = \frac{y}{t}$ or equivalently, $y = zt$.

■ **Question 5.**

Consider the ODE

$$\frac{dy}{dt} = \frac{y-t}{y+t}$$

Change the ODE such that the dependent variable becomes $z = y/t$ instead of y. What do you get? Why is this a better form than what you started with?

Bernoulli Equation: This is an ODE of the form $\frac{dy}{dt}+P(t)y = Q(t)y^b$, $(b \neq 1)$. This looks almost like a linear ODE but not quite. However, consider the change of variable $z = y^{1-b}$.

■ Question 6.

Consider the ODE

$$\frac{dy}{dt}+y = e^t y^2$$

Change the ODE such that the dependent variable becomes $z = \frac{1}{y}$ instead of y. What do you get? Why is this a better form than what you started with?

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§B. Exact ODEs

An ODE

$$Q(x,y)y' + P(x,y) = 0$$

is called an *exact equation* if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ in a region of the xy -plane. Consider the vector field

$$\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$$

Then the exactness of above ODE is equivalent to saying $\mathbf{curl}(\vec{F}) = 0$. However, we know that smooth irrotational vector fields are gradient vector fields. So we can find a function $H(x,y)$ such that $\vec{F} = \nabla H$. In other words,

$$\frac{\partial H}{\partial x} = P \text{ and } \frac{\partial H}{\partial y} = Q$$

Then we can rewrite the ODE as

$$\frac{\partial H}{\partial y}\frac{dy}{dx} + \frac{\partial H}{\partial x}\frac{dx}{dx} = 0$$

or equivalently using chain rule,

$$\frac{dH}{dx} = 0$$

which has the solution $H(x,y) = c$ for some constant c.

■ Question 7.

Solve the ODE

$$(2yx^2+4)\frac{dy}{dx} + (2y^2x-3) = 0.$$