Math 1600 Handout : Practice Problems for Chain Rule, Squeeze Theorem and Derivative of Trigonometric Functions

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Chain Rule

The *Chain Rule* states that

$$(f(g(x)))' = f'(g(x))g'(x)$$

Using the rule twice we similarly get

$$f(g(h(x))) = f'(g(h(x)))g'(h(x))h'(x)$$

As an example, consider the function

$$P(x) = \sqrt{e^{\frac{x^2}{5}} - 1}$$

The order of operation here is as follows

$$x \longrightarrow \frac{x^2}{5} \longrightarrow e^{\frac{x^2}{5}} - 1 \longrightarrow \sqrt{e^{\frac{x^2}{5}} - 1}$$

We broke our function into exact steps as above because we want to able to take derivative at each step using the simpler rules we have learned so far. We would like to write $P(x)$ as $f(g(h(x)))$. Note that h is applied first to x, and then g and then f. So in the above sequence of steps, we can identify f, g and h as follows:

$$x \xrightarrow{h} \underbrace{\frac{x^2}{5}}_{h(x)} \xrightarrow{g} \underbrace{e^{\frac{x^2}{5}} - 1}_{g(h(x))} \xrightarrow{f} \underbrace{\sqrt{e^{\frac{x^2}{5}} - 1}}_{f(g(h(x)))}$$

where

$$\begin{aligned}
&h(x) = \frac{x^2}{5} \implies h'(x) = \frac{2x}{5} \\
&g(x) = e^x - 1 \implies g'(x) = e^x \implies g'(h(x)) = e^{\frac{x^2}{5}} \\
&f(x) = \sqrt{x} = x^{\frac{1}{2}} \implies f'(x) = \frac{1}{2}x^{-1/2} \implies f'(g(h(x))) = \frac{1}{2}(e^{\frac{x^2}{5}} - 1)^{-1/2}
\end{aligned}$$

So,

$$\begin{aligned}
P'(x) &= f'(g(h(x)))g'(h(x))h'(x) \\
&= \frac{1}{2}(e^{\frac{x^2}{5}} - 1)^{-1/2} e^{\frac{x^2}{5}} \frac{2x}{5}
\end{aligned}$$

Exercise 1

Let $f(x)$ be a function with

$$f(1) = 1, \quad f(2) = 2, \quad f'(1) = 3, \quad f'(2) = 5.$$

If $g(x) = 2f(2x) + f(x)$, what is $g'(1)$?

ANS: 23

Exercise 2

Let f be a function which is differentiable on the entire real line. Find the derivative of $f(x^3) - (f(x))^3$.

ANS: $3x^2 f'(x^3) - 3(f(x))^2 f'(x)$

Exercise 3

Suppose $f(x)$ and $g(x)$ and their derivatives have the values given in the table.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	1	5	2	-5
1	3	-2	0	1
2	0	2	3	1
3	2	4	1	-6

Let $h(x) = f(g(x))$. Find the following.

a) $h'(0)$ b) $h'(1)$ c) $h'(2)$ d) $h'(3)$

ANS: $-10, 5, 4, 12$

Squeeze Theorem

Let $f(x), g(x)$, and $h(x)$ be three functions such that

$$f(x) \leq g(x) \leq h(x) \text{ for all } x$$

and assume that

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$$

Then $\lim_{x \to c} g(x) = L$.

Exercise 4

(Problem 1.9.36) Find $\lim_{x \to 0} f(x)$ if, for all x, we have

$$4\cos(2x) \leq f(x) \leq 3x^2 + 4$$

Exercise 5

(Problem 1.9.37) Find $\lim_{x \to \infty} f(x)$ if, for all x, we have

$$\frac{4x^2 - 5}{x^2} \leq f(x) \leq \frac{4x^6 + 3}{x^6}$$

Exercise 6

Find $\lim\limits_{x\to\infty} \dfrac{x+\sin x}{x-\cos x}$.

Exercise 7

Problems **1.9.(76 − 83)**.

Derivative of Trigonometric Functions

As an application of Squeeze theorem, we proved in class

- (1.9.56) $\lim\limits_{x\to\infty} \dfrac{\sin x}{x} = 0$.
- We used a Euclidean Geometry and Trigonometry argument to show that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

- We claimed that you need above result to find the derivative of $\sin x$. We show the steps below: Let $f(x) = \sin x$. Note that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

Now we need to use a Trigonometric formula for sum of two angles. The formula is as follows:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

With $\alpha = x$ and $\beta = h$, we can then simplify above limit as follows:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x\cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0} \frac{\cos x \sin h}{h} \\
&= \sin x \left(\lim_{h\to 0} \frac{(\cos h - 1)}{h} \right) + \cos x \left(\lim_{h\to 0} \frac{\sin h}{h} \right)
\end{aligned}$$

Note the last step carefully. The $\sin x$ and $\cos x$ terms come out of the limit because the limit is about h, not x. So we are only keeping the part that's a function of h inside the limit.

Using squeeze theorem we got that

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

Next we are going to calculate $\lim_{h \to 0} \frac{(\cos h - 1)}{h}$. We are going to use a sort of 'multiply by the conjugate' method.

$$\begin{aligned}
\lim_{h \to 0} \frac{(\cos h - 1)}{h} &= \lim_{h \to 0} \frac{(\cos h - 1)(\cos h + 1)}{h(\cos h + 1)} \\
&= \lim_{h \to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} \\
&= \lim_{h \to 0} \frac{-\sin^2 h}{h(\cos h + 1)} \\
&= -\left(\lim_{h \to 0} \frac{\sin h}{h}\right)\left(\lim_{h \to 0} \frac{\sin h}{\cos h + 1}\right) \\
&= -(1)\left(\frac{\sin 0}{\cos 0 + 1}\right) \\
&= -(1)\left(\frac{0}{1+1}\right) \\
&= 0
\end{aligned}$$

Now we substitute these results back into our calculation for the derivative to get,

$$f'(x) = \sin x \left(\lim_{h \to 0} \frac{(\cos h - 1)}{h}\right) + \cos x \left(\lim_{h \to 0} \frac{\sin h}{h}\right) = \sin x \times 0 + \cos(x) \times 1 = \cos x$$

We conclude that

- Derivative of $\sin x$ is $\cos x$.
- Using Chain rule and the fact that $\sin(\pi/2 - x) = \cos x$ we get that derivative of $\cos x$ is $-\sin x$.
- Using quotient rule we get that derivative of $\tan x$ is $\sec^2 x$.

Exercise 8

Let a and b be real numbers such that $f(x) = ax \sin x + b \cos x$ and $f'(x) = x \cos x$ for every real number x. What are the values of a and b?
Solution: Note that

$$f'(x) = ax \cos x + a \sin x - b \sin x = ax \cos x + (a - b) \sin x$$

So if this is equal to $x \cos x$, then $a - b = 0$ and $a = 1$. Then $a = b \implies b = 1$.

ANS: $a = 1, b = 1$

Exercise 9

Let $f(x)$ be a continuous and differentiable function defined as

$$f(x) = a \sin^2 x + b \cos x$$

where a and b are real numbers. If $f(\pi/2) = 2$ and $f'(\pi/2) = 3$, what are the values of a and b?

ANS: $a = 2, b = -3$

Exercise 10

Let $f(x)$ and $g(x)$ be continuous and differentiable functions such that

$$f(x) = \sin(g(x)) + \cos x$$

If $g(0) = \pi$ and $g'(0) = \frac{\pi}{4}$, find the value of $f'(0)$.

ANS: $-\pi/4$