Math 2208: Ordinary Differential Equations

Lecture 12 Worksheet

Fall 2019 **Subhadip Chowdhury** **Oct 30**

Title: Linear Systems with Complex Eigenvalues

Summary: We'll continue to explore the various scenarios that occur with linear systems of ODEs. This time dealing with those that possess complex eigenvalues.

§A. Complex Numbers

How do we solve the equation $x^2+1=0$? Clearly, there is no real number that satisfies the equation. However, if we allow ourselves to expand the criteria for being a 'number', we can assume that $\pm\sqrt{-1}$ would be acceptable solutions to this equation. But since $\sqrt{-1}$ does not belong to the set of $\mathbb{R}$eal numbers, we would need pursue a new way of thinking about numbers to understand $\sqrt{-1}$.

Let's give this new number a name. We will denote $\sqrt{-1}$ by i. Can we use i to solve quadratic equations? Consider the equation $at^2+bt+c=0$. Quadratic formula says:

$$t=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

So if $b^2-4ac<0$, let's say $b^2-4ac=-m^2$, we can rewrite above formula as

$$t=\frac{-b\pm\sqrt{-m^2}}{2a}=\frac{-b\pm mi}{2a}$$

Thus if we expand the collection of numbers to include i and in particular numbers of the form $x+iy$ (where $x,y\in\mathbb{R}$), every quadratic equation becomes solvable regardless of the sign of the discriminant. This new collection (set) of numbers is called the set of $\mathbb{C}$*omplex Numbers*. If $z=x+iy$ is a complex number, we say that x is the "Real" part of z, and it is denoted as $x=\mathfrak{Re}(z)$. Similarly we say y is the "Imaginary" part of z, and we denote it as $y=\mathfrak{Im}(z)$ (although there is nothing imaginary about it!).

How do we 'draw' complex numbers? Since the $\mathbb{R}$eal number line is evidently not big enough to contain the $\mathbb{C}$omplex numbers, we will need to go outside the line to 'draw' such a number. We will also need to redefine our rules of arithmetic to allow addition or multiplication of Complex numbers.

We observe that if $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$, then

$$z_1+z_2=(x_1+x_2)+i(y_1+y_2)$$

This gives us an idea to represent the complex number $x+iy$ as the point (x,y) (or the vector $\langle x,y\rangle$) in the 2D plane where the X−axis (i.e. the $y=0$ line) corresponds to the Real number line. We check that the sum of two complex numbers indeed follows the parallelogram law of adding vectors (as we are just adding the two components separately).

What about multiplying two complex numbers? Check that

$$z_1z_2 = x_1x_2 + i^2y_1y_2 + ix_1y_2 + iy_1x_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$$

This does not look familiar. However, if we change to polar coordinates, this becomes much simpler looking! Recall that the polar coordinate of a point (x, y) is given by (r, θ) where

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Using the new coordinate system, if $z_1 \equiv (r_1, \theta_1)$ and $z_2 \equiv (r_2, \theta_2)$ then the product simplifies due to trigonometric identities and becomes

$$z_1z_2 \equiv (r_1r_2, \theta_1 + \theta_2)$$

As a consequence, if $z \equiv (r, \theta)$, then $z^2 \equiv (r^2, 2\theta)$. And similarly, if $z \equiv (r, \theta)$, then $\sqrt{z} \equiv \left(\sqrt{r}, \frac{\theta}{2}\right)$.

Using this new arithmetic, we observe that since the real number -1 corresponds to the the point $(1, \pi)$ in polar coordinates, we must have

$$i = \sqrt{-1} \equiv (1, \pi/2)$$

So we can place the new mystery number i at a distance 1 from the origin in the positive Y−axis direction which is consistent with the idea of associating $z = x + iy$ to the point (x, y) (in Cartesian coordinates) in the plane.

Definition 1.1

The modulus of a complex number $z = x + iy$ is $|z| = \sqrt{x^2 + y^2}$.

The conjugate of a complex number $z = x + iy$ is $\bar{z} = x - iy$.

Theorem 1.1

If the roots of a quadratic polynomial with real coefficients are not real, then they are conjugate complex numbers.

Question 1.

Express the following numbers in the form $a + ib$.

1. $(1 + 2i)(1 - 2i)$
2. $\dfrac{3}{i}$
3. $(1 + 5i)(i - 2)$

Theorem 1.2: Euler's Formula

Euler's formula, named after Leonhard Euler, is a mathematical formula in complex analysis that establishes the fundamental relationship between the trigonometric functions and the complex exponential function. Euler's formula states that for any real number θ:

$$e^{i\theta} = \cos\theta + i\sin\theta =: cis\ \theta$$

When $x = \pi$, Euler's formula evaluates to Euler's identity, the "greatest" Math identity:

$$e^{i\pi} + 1 = 0$$

Question 2.

Use Euler's formula to show that $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$.

§B. Two Complex Eigenvalues

Recall that the general solution to the ODE

$$\frac{d\vec{\mathbf{R}}}{dt} = \mathrm{A}\vec{\mathbf{R}}$$

can be written as

$$\vec{\mathbf{R}}(t) = k_1 e^{\lambda_1 t}\vec{\boldsymbol{v}}_1 + k_2 e^{\lambda_2 t}\vec{\boldsymbol{v}}_2$$

where λ_is are the eigenvalues of A and $\vec{\boldsymbol{v}}_i$ are the eigenvectors corresponding to λ_i. What happens if $\lambda_i \in \mathbb{C}$?

Question 3.

Consider the ODE $\dfrac{d\vec{\mathbf{R}}}{dt} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\vec{\mathbf{R}} = \mathrm{A}\vec{\mathbf{R}}$.

1. Find the eigenvalues and corresponding eigenvectors of A. By an earlier exercise, we know they will be of the form $\alpha \pm i\beta$ for some real numbers α and β.
2. Let $\lambda = \alpha + i\beta$ and name the corresponding eigenvector $\vec{\boldsymbol{v}}$. Then $\vec{\mathbf{R}}_1(t) = e^{(\alpha+i\beta)t}\vec{\boldsymbol{v}}$ is a solution to our ODE. Use Euler's formula to rewrite your solution in the form

$$\vec{\mathbf{R}}_1(t) = \vec{\mathbf{R}}_{\mathfrak{Re}}(t) + i\vec{\mathbf{R}}_{\mathrm{Im}}(t)$$

   where $\vec{\mathbf{R}}_{\mathfrak{Re}}(t)$ and $\vec{\mathbf{R}}_{\mathrm{Im}}(t)$ are real-valued functions of t.
3. Check that $\vec{\mathbf{R}}_{\mathfrak{Re}}(t)$ and $\vec{\mathbf{R}}_{\mathrm{Im}}(t)$ are also solutions to the ODE.
4. Conclude that a real-valued general solution to the ODE is of the form

$$\vec{\mathbf{R}}(t) = k_1\vec{\mathbf{R}}_{\mathfrak{Re}}(t) + k_2\vec{\mathbf{R}}_{\mathrm{Im}}(t)$$

   for some real numbers k_1 and k_2.

§C. Classification of Solutions in case of Complex Eigenvalues

Your simplification using the Euler's formula should convince you that the effect of the exponential term on solutions depends on the sign of α whereas β determines the periodic nature of the solutions.

Question 4.

Discuss the nature of the solution curves in the following three cases:

$$(a)\alpha > 0 \qquad (b)\alpha < 0 \qquad (c)\alpha = 0$$

In each case, what is the "natural" period of the solution curves?

Group Work

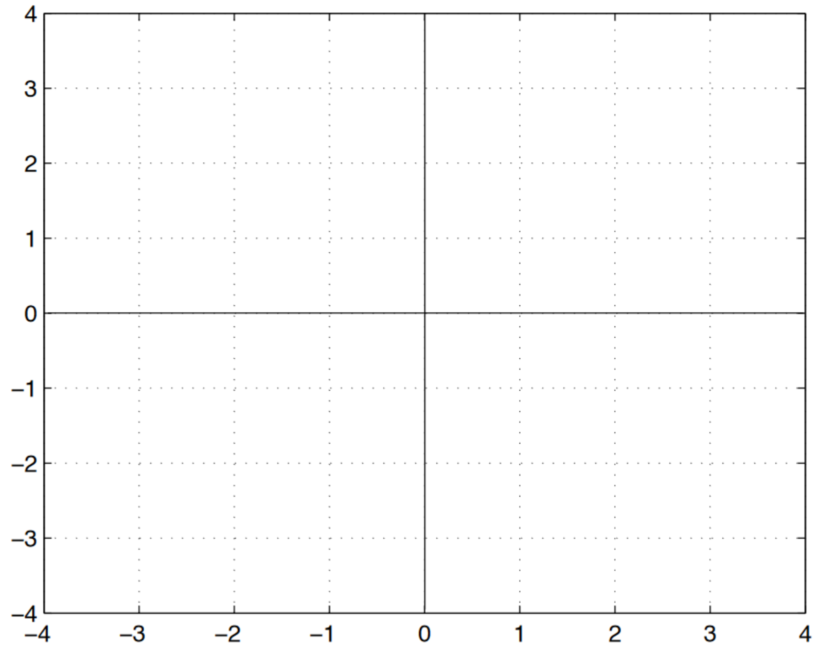
Come up with your own examples of 2×2 matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ that have complex eigenvalues and the corresponding ODE for each of the three cases named above. Use pplane to help you sketch the phase portrait for each case on the given axes.

It may be helpful to recall that the eigenvalues are given by the formula

$$\lambda = \alpha \pm i\beta = \frac{\mathrm{tr}(A) \pm \sqrt{\mathrm{tr}(A)^2 - 4\det(A)}}{2}$$

Case 1: $\alpha < 0$ (Spiral Sink). $A = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$

Case 2: $\alpha > 0$ **(Spiral Source).** $A = \begin{bmatrix} & \\ & \end{bmatrix}$



Case 3: $\alpha = 0$ **(Center).** $A = \begin{bmatrix} & \\ & \end{bmatrix}$