Assignment 4 (7/2)

Subhadip Chowdhury

- *This homework is due at the beginning of class on* ***Friday*** 7/6. *You are encouraged to work together on these problems, but you must write up your solutions independently.*

Proof by Contradiction

There are usually two kinds of proofs you may encounter in Math:

- Prove that a statement Q is true.
- Prove that if P is true then Q is true. In other words, $P \implies Q$.

Note that the first kind of proof can be rewritten as "Math is Consistent $\implies Q$ is true". Thus all proofs are essentially of the form $P \implies Q$.

Now, as we mentioned in class, a proof by contradiction is essentially proving the contrapositive. This works because an implication and its contrapositive are equivalent to each other. So we will start by assuming Q is false and show that this implies P is false. We can also think of it as starting by assuming the opposite of "$P \implies Q$" is true, and arrive at some absurdity.

The opposite of "$P \implies Q$" is "P but not Q". let's explain this using an example. 6 divides n implies 3 divides n. The opposite of this sentence is 6 divides n but 3 does not divide n. More precisely, if the first sentence is

For all integers n, if 6 divides n, then 3 divides n.

Then the opposite (also called negation) of that is

There exists a natural number n such that 6 divides n but 3 does not divide n.

As you can see the opposite of "for all" is "there exists"; we only need to rewrite using proper grammar.

Exercise 7. *What is the negation (opposite) of the following sentence: "There exists some natural number N such that all elements of set $\mathscr{A}$ are less than N."*

Note that, in the exercise above, I am not asking you to prove anything, just write the opposite. Let us now discuss the format of a proof by contradiction using two examples:

Example 1.1. Prove that there is no least positive rational number.

Example 1.2. Prove that for all positive integers n, if n^2 is odd then n is odd.

We will write the proofs side by side to emphasize the steps. Recall from last assignment, that the steps are

Step 1. *Negate the conclusion.* If you are trying to show Q is true, then start by assuming Q is false. If you want to show $P \implies Q$, start by assuming not Q.

Step 2. *Analyze the consequences of this premise.* Assuming Q is false i.e. not Q, explore the logical implications that would follow.

Step 3. *Look for a contradiction.* A contradiction is something that doesn't make sense given the negated conclusion premise.

Step 1.	Assume, for the sake of contradiction, that there is a least positive rational number, call it r.	Assume, for the sake of contradiction, there exists a natural number n such that n^2 is odd but n is even. Since n is even, we can find an integer k such that $n = 2k$.
Step 2.	Consider the number $s = \frac{r}{2}$. Observe that s is a positive rational number. By construction, we also have $s < r$. Thus we have a positive rational number smaller than r.	Then $n^2 = 4k^2 = 2(2k^2)$. Hence n^2 is an even number.
Step 3.	This clearly contradicts the minimality of r. $\square$	But n^2 was assumed to be odd. Hence we have a contradiction. $\square$

While proving the first example, we employed a strategy known as the "*Extremal Principle*". We can describe it as follows.

We are trying to prove that a mathematical object e.g. a set of numbers with certain properties does not exist. We assume, for the sake of contradiction, that it does. The extremal principle tells us to pick an element which maximizes or minimizes some function, e.g. we might just want to maximize or minimize its absolute value. Then we perform some slight perturbation (variation) of this element to produce another element of the set that further increase or decrease the given function. This gives us a contradiction.

Recall the example from last class. That was one of the examples of using the Extremal principle.

Recap of the Proof from last time: For the sake of clarity, let us recap the proof we discussed last time. You should try to identify the relevant steps in the following proof.

Problem 1.3 (Exercise 5 from last assignment)**.** *Show that $\sqrt{2}$ is not a Rational number.*

Proof. Suppose, for the sake of contradiction, that $\sqrt{2}$ is a Rational number. Then we can find natural numbers n such that $n\sqrt{2}$ is an integer. Let S be the set of such natural numbers n. Then S has a least element. Let k be the least element of S.

Consider the number $m = (\sqrt{2} - 1)k$. Observe that

$$m\sqrt{2} = 2k - k\sqrt{2}$$

Now $k\sqrt{2}$ is an integer since $k \in S$ and $2k$ is an integer since $k \in \mathbb{N}$. So $2k - k\sqrt{2} \in \mathbb{Z}$. Hence the left hand side of the above equality, $m\sqrt{2}$ must also be an integer. Also since $k\sqrt{2}$ is an integer, so is $k\sqrt{2} - k = m$. In fact, since $\sqrt{2} > 1$, we get that m is a natural number. Thus we conclude that m is a natural number such that $m\sqrt{2}$ is an integer and by definition of the set S, we must have $m \in S$.

On the other hand, $m = (\sqrt{2} - 1)k < k$. But this contradicts our assumption that k is the least element of S.

Hence our assumption is false, and $\sqrt{2}$ is in fact not a Rational number. $\square$

Although the next problem is an exercise, you can look up the solution on internet if you get stuck. But please give it an honest try first, it has a really short nice geometric proof.

Exercise 8 (The Sylvester Problem)**.** *A finite set S of points in the plane has the property that any line through two of them passes through a third. Show that all the points lie on a line.*

*[*HINT: *Suppose all points are not on the same line. Out of all the possible lines you could draw through any two of the points, consider the line that has the least distance from a point not on it.]*

Proof by Induction

Proof by induction applies to problems where we are required to show that certain statement $P(n)$ holds true for all natural numbers n. To give an example, suppose we need to show $f(n) = g(n)$ for all natural numbers n. Here f and g are some functions of n.

We first check that $P(1)$ is true i.e. $f(1) = g(1)$. This is called the *base case*, or the base step. Next we make the assumption (called the *Induction Assumption or the Induction Hypothesis*) that $P(k)$ is true i.e. $f(k) = g(k)$

for some k. Using the assumption, we prove that then $P(k+1)$ must be true i.e. $f(k+1) = g(k+1)$ (this is the *Induction Step*). Since starting at 1 and by traversing the 'next' number, we can eventually cover all natural numbers, we can then conclude that $P(n)$ is true for all n i.e. $f(n) = g(n)$ for all n.

Let's work out some examples.

Example 2.1. Show that $1+2+3+\ldots+n = \frac{n(n+1)}{2}$ for all natural numbers n.

Proof. We will prove the identity by inducting on n.

Base Case: When $n = 1$, the LHS$= 1$ and the RHS$= \frac{1(1+1)}{2} = 1$. Hence the identity is true for $n = 1$.

Induction Assumption: Assume that the identity is true for some natural number k.

Induction Step: By our induction assumption we have,

$$1+2+3+\ldots+k = \frac{k(k+1)}{2}$$

Adding $(k+1)$ to both sides we get,

$$\begin{aligned}1+2+3+\ldots+k+(k+1) &= \frac{k(k+1)}{2} + (k+1)\\ &= (k+1)(k/2+1)\\ &= \frac{(k+1)(k+2)}{2}\\ &= \frac{(k+1)((k+1)+1)}{2}\end{aligned}$$

Thus the identity holds when $n = k+1$.

Hence by the Induction Principle, the identity is true for all natural number n. □

The most important step here the manipulations in the induction step. Starting with the induction assumption, we first introduce additional data to bring it to a format that would resemble the LHS of the identity for $n = (k+1)$. Then we simplify the RHS and show that it resembles the RHS of the identity for $n = (k+1)$. Let's do another example.

Example 2.2. Show that $2+2^2+2^3+2^4+\ldots+2^n = 2^{n+1}2$ for all natural numbers n.

Proof. We will prove the identity by inducting on n.

Base Case: When $n = 1$, the LHS$= 2$ and the RHS$= 2^2 - 2 = 1$. Hence the identity is true for $n = 1$.

Induction Hypothesis: Assume that the identity is true for some natural number k.

Induction Step: By our induction assumption we have,

$$2+2^2+\ldots+2^k = 2^{k+1} - 1$$

Adding 2^{k+1} to both sides we get,

$$\begin{aligned}2+2^2+\ldots+2^k+2^{k+1} &= 2^{k+1} - 1 + 2^{k+1}\\ &= 2.2^{k+1} - 1\\ &= 2^{k+2} - 1\\ &= 2^{(k+1)+1} - 1\end{aligned}$$

Thus the identity holds when $n = k+1$.

Hence by the Induction Principle, the identity is true for all natural number n. □

Clearly, proof by induction works whenever we have a concept of 'next' number. Thus it does not work for Rationals, but it works for the set of all odd natural numbers for example, or for the set of positive multiples of a number.

Exercise 9. *Show that*

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

for all natural numbers n.

Exercise 10. *Guess a formula for the sum of the first n odd natural numbers and then prove it by induction.*

Exercise 11. *The following is a famous fallacy that uses Mathematical induction. Explain what is wrong with it!*

Claim. *All real numbers are equal.*

Proof. To prove the claim, we will prove by induction that, for all $n \in \mathbb{N}$, the following statement holds:

$$\text{For any set of } n \text{ real numbers } a_1, a_2, a_3, \ldots, a_n, \text{ we have } a_1 = a_2 = a_3 = \ldots = a_n$$

Base step: When $n = 1$, the statement is trivially true.

Induction Hypothesis: Suppose above statement is true for some $k \in \mathbb{N}$.

Induction Step: By induction assumption, any set of k real numbers are equal to each other. Now start with a set of $(k+1)$ real numbers $a_1, a_2, \ldots, a_{k+1}$. Applying the Induction hypothesis to the first k numbers we get

$$a_1 = a_2 = \ldots = a_k$$

Next, applying the Induction hypothesis to the last k numbers we get

$$a_2 = a_3 = \ldots = a_{k+1}$$

Combining above two, we get that

$$a_1 = a_2 = a_3 = \ldots = a_k = a_{k+1}$$

Thus we have show that the statement holds for $n = k+1$ and the induction step is complete.
Hence by the Induction Principle, all real numbers are equal. □

Further Reading Materials

Here are some online puzzles that you may find interesting.

- More fake induction proofs:
  HERE
- An online True/False logic game:
  HERE
- The famous Knights and Knaves puzzle:
  HERE
- The NYT Confirmation Bias puzzle:
  HERE