Problem Set 1 Solutions

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Exercise 1: Odd-Degree Only at Endpoints

Claim. *Suppose we have a graph with an Eulerian Path on it. Prove that if a vertex has odd degree, then it must be either the starting or the ending point in the Eulerian path.*

Proof. We will prove that an odd-degree vertex is a starting or ending point by showing that it cannot be a point in the middle of the Eulerian path. For a Eulerian path, all edges are traversed exactly once. Therefore, an entry edge to a middle vertex cannot serve as the exit edge. Moreover, every entry to a middle vertex must be accompanied by an exit, for otherwise the point would be the start or end of the path. This requirement, that edges to a middle vertex come only in pairs, implies that its degree must be even. Hence a vertex with odd degree can only be the start or end of a Euler tour. □

Exercise 2: No Euler Tour

Theorem 2.1. *If a graph has more than two vertices of odd degree, then it has no Eulerian Path.*

Proof. Suppose, by contradiction, that there are more than two odd-degree vertices in a graph with a Eulerian path. Now, from the claim in exercise 1, we know that there are two possible locations for such vertices in the Euler path: the beginning or the end. But, if this is true, we can use the Pigeonhole Principle (PHP) to conclude that there are either multiple beginnings or multiple ends (or both). This is clearly a contradiction, since a path can only start once and end once. Therefore, there cannot be more than two odd-degree vertices in a graph that is traversed by a Eulerian path. □

Exercise 3: Sharing Friends

Problem 3.1. *There are* 20 *students present in a classroom. Assume that if student A is friends with student B, then student B is friends with student A. Prove that there are at least two students who have the same number of acquaintances in the room.*

Proof. We will use the Pigeonhole Principle to solve this problem. First, let's figure out the possibilities for the number of friends a student can have. Clearly, a student can be friends with at least 0 (i.e. no friends) and at most $n-1$ other students (i.e. everyone else). However, if say, student A has 0 friends, then none of the other students can have $(n-1)$ friends, because they are not friends with student A. Alternately if a student B has $(n-1)$ friends, then no one else can have 0 friends, because they are friends with student B. This reduces the possible friend-counts by one, since the possibilities $(n-1)$ and 0 cannot exist simultaneously for any given group of students. Thus there are a total of $(n-1)$ possibilities but a total of n students. Hence, by PHP, at least 2 students share the same number of friends. □

Exercise 4: Sharing Birthdays

Problem 4.1. *There are m students present in a conference. Find the least value of m to ensure that at least three students will have birthdays on the same day of the year (the birth-year doesn't have to same, only the month and day have to be same). Assume all years have* 365 *days.*

Proof. This is an application of the Generalized PHP with $n = 365$ and $k = 2$, leading to a minimum of $(nk+1) = 731$ for a three-person birthday coincidence to be certain. □

The above is a complete proof, but let us examine this case of PHP more closely. It is possible to choose a group of up to $730 = 2 \times 365$ people, such that no three students will have birthdays on the same day of the year. In particular, we can find 730 students in a way so that each of 365 birthdays contains exactly two people born on that day. However, in a group of 731 students, no matter how we choose them, three students will always have same birthday.