Assignment 15 (2/19)

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- *This homework is due at the beginning of class on* ***Friday*** *2/23. You may cite results from class as appropriate. Unless otherwise stated, you must provide a complete explanation for your solutions, not simply an answer. You are encouraged to work together on these problems, but you must write up your solutions independently.*
- *Hand in the exercises only, not the reading material item. You are encouraged to think about the exercises marked with a (⋆) or (†) if you have time, but you don't need to hand them in. If you correctly solve a (†)-marked problem, you will get a candy!*
- *Remember that you can always use the result of the previous assignment problems without proof to solve the new assignment problems.*
- *We are currently covering the chapter 1 from Vanderbei.*

Important Points and Reading Materials

- Linear Programming - Generic Example
  - Understand what a linear programming problem is, and how it can often show up in real life situations
  - Know how to solve a two variable linear programming problem by graphing the inequalities. Can you reasonably do this with more than two variables?
  - Understand why the maximum/minimum values of a linear programming problem have to occurs at the corners of the solution region (we discuss more on this below).
- Matrices and Vectors - notations in LPP
  - A $m \times n$ matrix is a collection of mn numbers written in m rows and n columns. E.g. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -5 & 0 \end{bmatrix}$ is a 3×2 matrix.
  - A vector with n-coordinates (i.e. a vector in n-dimension) can be treated as a $n \times 1$ matrix with each entry corresponding to different coordinates.
  - Matrix Multiplication
    - Product of matrices A and B is defined as the matrix C such that the $(i,j)^{th}$ entry of C is sum of the product of i-th row entries from A and j-th column entries from B.
    - In particular, this is only defined when number of columns in A = number of rows in B.
    - If A is $m \times n$ and B is $n \times p$, then C is $m \times p$.
    - Abstractly, $C_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$, for $1 \leq i \leq m, 1 \leq j \leq p$
    - $AB \neq BA$ in general.
  - Consider the optimization problem

$$\text{Optimize } \sum_{i=1}^{n} c_i x_i \text{ subject to the set of } m \text{ constraints } \sum_{i=1}^{n} a_{ji} x_i \leq b_j \text{ for } j = 1, 2, \ldots, m$$

    With the above notations, this can be rephrased as

$$\text{Optimize } \mathbf{c}^t \mathbf{x} \text{ subject to } A\mathbf{x} \leq \mathbf{b}$$

    where

    - $\mathbf{c}$ is the $n \times 1$ vector $\begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_n \end{bmatrix}$,

* $\mathbf{x}$ is the $n \times 1$ vector $\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$,
* A is the $m \times n$ matrix whose (j,i)-th entry is a_{ji}, the coefficient of x_i from the jth equation in the LPP, (note that $1 \le j \le m, 1 \le i \le n$)
* and $\mathbf{b}$ is the $m \times 1$ vector $\begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$.

- You should check that the matrix product $A\mathbf{x}$ indeed gives the left hand side of the set of m equations above.
- The matrix A is called the **coefficient matrix** and the $m \times (n+1)$ matrix $\begin{bmatrix} A \mid \mathbf{b} \end{bmatrix}$ which is made by attaching the column vector $\mathbf{b}$ as the last column of A, is called the **Augmented matrix**. We will be working with these next time.

NOTE: Any linear constraint can be always expressed in $A\mathbf{x} \le \mathbf{b}$ form. For example, say we have the constraint

$$2x + 3y \ge 40$$

We can rewrite it in the form

$$-2x - 3y \le -40$$

Alternately, if we have a constraint of the form

$$4x + 5y = 13$$

we can rewrite this as a set of two inequalities

$$4x + 5y \le 13$$

$$-4x - 5y \le -13$$

It is customary to write all of our constraints in the LPP using '$\le$' sign, so that we can express it in short-hand using matrices.

- Fundamental Theorem of Linear Programming

  THEOREM: Consider the optimization problem

$$\text{Minimize } \mathbf{c}^t\mathbf{x} \text{ subject to } \mathbf{x} \in P$$

Where $P = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \le \mathbf{b}\}$. If P is a bounded polyhedron (a $n-$dimensional polygon) and $\mathbf{x}^*$ is an optimal solution to the problem, then $\mathbf{x}^*$ is either an extreme point (vertex) of P, or lies on a face $F \subset P$ where all points in F are optimal solutions.

PROOF: Suppose, for the sake of contradiction, that $\mathbf{x}^* \in \text{int}(P)$, i.e. not on the boundary. Then there exists some $\epsilon > 0$ such that the ball of radius ϵ centered at $\mathbf{x}^*$ is contained in P, that is all points that are within distance ϵ from $\mathbf{x}^*$ are within the boundary of P. Therefore the point

$$\mathbf{x}^* - \frac{\epsilon}{2}\frac{\mathbf{c}}{||\mathbf{c}||} \in P$$

since $\frac{\epsilon}{2}\frac{\mathbf{c}}{||\mathbf{c}||}$ has length $< \epsilon$. However, we observe that

$$\mathbf{c}^t\left(\mathbf{x}^* - \frac{\epsilon}{2}\frac{\mathbf{c}}{||\mathbf{c}||}\right) = \mathbf{c}^t\mathbf{x}^* - \frac{\epsilon}{2}\frac{\mathbf{c}^t\mathbf{c}}{||\mathbf{c}||} = \mathbf{c}^t\mathbf{x}^* - \frac{\epsilon}{2}||\mathbf{c}|| < \mathbf{c}^t\mathbf{x}^*.$$

Hence $\mathbf{x}^*$ is not an optimal solution, a contradiction. Therefore, $\mathbf{x}^*$ must live on the boundary of P.

One important point in above proof is as follows. Note that the proof so far says that x^* is in the boundary. But that does not necessarily imply it's a vertex or a face. It could have been, for example, the midpoint of line joining two vertex or just one point in a face! But that doesn't happen due to the linear nature of P. We can use the fact that P is a polytope to show that x^* is infact a vertex or a 'whole' face. See Wikipedia or other sources for a more complete proof.

- Solving LPP using Geometry
  - When the LPP has only two variables, it is reasonable to solve it using Geometry. In particular, we first draw the set of points that satisfy all the inequalities (called the set of *feasible* solutions), and then evaluate the *objective* function (the function we are trying to optimize) at the vertices of the feasible region. By the fundamental theorem, one of these is bound to be our solution.
  - We did one such example in class. Another can be found in the book on page 23.

Problems

Read the first chapter from Vanderbei for another real life application of LPP. For the following questions, I ask that you first transform the word problem into a LPP of the form

$$\text{Optimize } \sum_{i=1}^{n} c_i x_i \text{ subject to the set of } m \text{ constraints } \sum_{i=1}^{n} a_{ji} x_i \leq b_j \text{ for } j = 1, 2, \ldots, m$$

In particular, you should correctly identify the variables and the constraints, and write all the constraints using $\leq$ sign.

Exercise 1

Problem 1.$(1, 2)$ from Vanderbei. Do not solve these LPPs.

Exercise 2

A calculator company produces a scientific calculator and a graphing calculator. Long-term projections indicate an expected demand of at least 100 scientific and 80 graphing calculators each day. Because of limitations on production capacity, no more than 200 scientific and 170 graphing calculators can be made daily. To satisfy a shipping contract, a total of at least 200 calculators much be shipped each day.

If each scientific calculator sold results in a \$ 2 loss, but each graphing calculator produces a \$ 5 profit, how many of each type should be made daily to maximize net profits?

[Ans: 100 scientific calculators and 170 graphing calculators.]

Exercise 3

A gold processor has two sources of gold ore, source A and source B. In order to keep his plant running, at least three tons of ore must be processed each day. Ore from source A costs \$20 per ton to process, and ore from source B costs \$10 per ton to process. Costs must be kept to less than \$80 per day. Moreover, Federal Regulations require that the amount of ore from source B cannot exceed twice the amount of ore from source A. If ore from source A yields 2 oz. of gold per ton, and ore from source B yields 3 oz. of gold per ton, how many tons of ore from both sources must be processed each day to maximize the amount of gold extracted subject to the above constraints?

[Ans: maximum yield of gold is 16oz. by processing 2 tons of ore from source A and 4 tons from source B.]

Exercise 4

A publisher has orders for 600 copies of a certain text from San Francisco and 400 copies from Sacramento. The company has 700 copies in a warehouse in Novato and 800 copies in a warehouse in Lodi. It costs $5 to ship a text from Novato to San Francisco, but it costs $10 to ship it to Sacramento. It costs $15 to ship a text from Lodi to San Francisco, but it costs $4 to ship it from Lodi to Sacramento. How many copies should the company ship from each warehouse to San Francisco and Sacramento to fill the order at the least cost?

[HINT: least cost is $4600.]

Exercise 5

Solve problems $2.(2, 5)$ from Vanderbei using Geometric method.