MATH 1800 PROJECT 1: DISTANCES

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In this project, we will use Dot product and Cross product of vectors to derive formula for calculating distances between points, lines and planes. We will use the notation $d(\cdot,\cdot)$ to denote distance.

DISTANCE BETWEEN TWO POINTS

To begin with, the distance between two points $\mathbf{P}$ and $\mathbf{Q}$ with position vectors $\vec{\mathbf{P}}$ and $\vec{\mathbf{Q}}$ is simply given by

$$d(\mathbf{P},\mathbf{Q}) = ||\vec{\mathbf{Q}} - \vec{\mathbf{P}}|| = ||\overrightarrow{\mathbf{PQ}}||$$

where $||\cdot||$ denotes the magnitude of a vector.

Question 0

Find the distance between $(-5,2,4)$ and $(-2,2,0)$.

DISTANCE FROM A POINT TO A PLANE

The distance of a point $\mathbf{P}$ from a plane Σ is defined as the length of the perpendicular from $\mathbf{P}$ to Σ. Suppose the plane Σ passes through a point $\mathbf{Q}$ and has normal vector $\vec{n}$.

Question 1

Explain using a picture why the distance from $\mathbf{P}$ to Σ is the length of the projection of $\overrightarrow{\mathbf{PQ}}$ onto $\vec{n}$. Then

$$d(\mathbf{P},\Sigma) = \frac{|\overrightarrow{\mathbf{PQ}}\cdot\vec{n}|}{||\vec{n}||}$$

Question 2

Find the distance of the point $(7,1,4)$ from the plane $2x+4y+5z=9$.

Question 3

Without the absolute sign in the numerator of the distance formula, your answer in question (2) would have been negative. What does the negative sign signify here?

Distance from a point to a line

The distance of a point $\mathbf{P}$ from a line $\mathcal{L}$ is defined as the length of the perpendicular from $\mathbf{P}$ to $\mathcal{L}$. Suppose the line $\mathcal{L}$ passes through a point $\mathbf{Q}$ and is parallel to a vector $\vec{u}$ (i.e. its parametric equation looks like $\vec{r}(t) = \overrightarrow{\mathrm{Q}} + t\vec{u}$).

Question 4

Use the definition of cross product to derive the following formula:

$$d(\mathrm{P}, \mathcal{L}) = \frac{||\overrightarrow{\mathbf{PQ}} \times \vec{u}||}{||\vec{u}||}$$

Question 5

Find the distance of the point $(2,3,1)$ from the straight line $\vec{r}(t) = (1,1,2) + t\langle 5,0,1\rangle$.

Question 6

What is the equation of the plane which contains the point $\mathbf{P}$ and the line $\mathcal{L}$?

Distance between two straight lines

Suppose the two straight lines $\mathcal{L}_1$ and $\mathcal{L}_2$ are given by

$$\vec{r}_1(t) = \overrightarrow{\mathrm{P}} + t\vec{u} \quad \text{and} \quad \vec{r}_2(t) = \overrightarrow{\mathrm{Q}} + t\vec{v}$$

i.e. the straight lines pass through $\mathbf{P}$ (and $\mathbf{Q}$ respectively) and is parallel to $\vec{u}$ (and $\vec{v}$ respectively).

Question 7

Draw a picture and explain using geometry why the distance between the two straight lines is given by

$$d(\mathcal{L}_1, \mathcal{L}_2) = \frac{|\overrightarrow{\mathbf{PQ}} \cdot (\vec{u} \times \vec{v})|}{||\vec{u} \times \vec{v}||}$$

Question 8

Find the distance between the lines $\vec{r}_1(t) = (2,1,4) + t\langle -1,1,0\rangle$ and $\vec{r}_2(t) = (-1,0,2) + t\langle 5,1,2\rangle$.

Distance between two planes

Before deriving the formula, observe that the distance between two planes is non-zero iff the two planes are parallel to each other, in which case they have the same normal vector $\vec{n} = \langle a,b,c\rangle$. Suppose the two planes Σ_1 and Σ_2 are given by

$$ax + by + cz = d \quad \text{and} \quad ax + by + cz = e$$

Question 9

Show that the distance formula is given by

$$d(\Sigma_1, \Sigma_2) = \frac{|d - e|}{||\vec{n}||}$$

Question 10

Find the distance between the planes $5x + 4y + 3z = 8$ and $5x + 4y + 3z = 1$.

Question 11

Find the distance between the planes $x + 3y - 2z = 2$ and $5x + 15y - 10z = 30$.