MATH 1800 PROJECT 8: ROCKET SCIENCE*

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Many rockets, such as the *Pegasus XL* currently used to launch satellites and the *Saturn V* that first put men on the moon, are designed to use three stages (engines) in their ascent into space. A large first stage initially propels the rocket until its fuel is consumed, at which point the stage is jettisoned to reduce the mass of the rocket. The smaller second and third stages function similarly in order to place the rockets payload into orbit about the earth. (With this design, at least two stages are required in order to reach the necessary velocities, and using three stages has proven to be a good compromise between cost and performance.) Our goal here is to determine the individual masses of the three stages, which are to be designed to minimize the total mass of the rocket while enabling it to reach a desired velocity.

For a single-stage rocket consuming fuel at a constant rate, the change in velocity resulting from the acceleration of the rocket vehicle has been modeled by

$$\Delta v = -c \ln\left(1 - \frac{(1-S)M_r}{P+M_r}\right)$$

where

- M_r is the mass of the rocket engine including initial fuel,
- P is the mass of the payload,
- S is a structural factor determined by the design of the rocket (specifically, it is the ratio of the mass of the rocket vehicle without fuel to the total mass of the rocket with payload),
- and c is the *(constant)* speed of exhaust relative to the rocket.

Now consider a rocket with *three* stages and a payload of mass A. Assume that outside forces are negligible and that c and S remain constant for each stage. If M_i is the mass of the ith stage, we can initially consider the rocket engine (i.e. the first stage) to have mass M_1 and its payload to have mass $M_2 + M_3 + A$; the second and third stages can be handled similarly.

1. Show that the velocity attained after all three stages have been jettisoned is given by

$$v = c\left[\ln\left(\frac{M_1+M_2+M_3+A}{SM_1+M_2+M_3+A}\right) + \ln\left(\frac{M_2+M_3+A}{SM_2+M_3+A}\right) + \ln\left(\frac{M_3+A}{SM_3+A}\right)\right]$$

2. We wish to minimize the total mass $M = M_1 + M_2 + M_3$ of the rocket engine subject to the constraint that the final velocity v from Problem **1** is equal to some desired velocity v_f. The method of Lagrange multipliers is appropriate here, but difficult to implement using the current expressions.

   To simplify, we define variables N_i so that the constraint equation may be expressed as

$$v_f = c(\ln N_1 + \ln N_2 + \ln N_3).$$

   Thus for example $N_1 = \frac{M_1+M_2+M_3+A}{SM_1+M_2+M_3+A}$. Show that

*Source: Stewart's Multivariable Calculus

$$\begin{aligned} \frac{M_1+M_2+M_3+A}{M_2+M_3+A} &= \frac{(1-S)N_1}{1-SN_1} \\ \frac{M_2+M_3+A}{M_3+A} &= \frac{(1-S)N_2}{1-SN_2} \\ \frac{M_3+A}{A} &= \frac{(1-S)N_3}{1-SN_3} \end{aligned}$$

and conclude that

$$\frac{M+A}{A} = \frac{(1-S)^3 N_1 N_2 N_3}{(1-SN_1)(1-SN_2)(1-SN_3)}$$

3. But now M is difficult to express in terms of the N_i's. So instead, we are going to find N_i's that minimize $\ln \frac{M+A}{A}$. To see why this works, show that $\ln \frac{M+A}{A}$ is an increasing function of M. So $\ln \frac{M+A}{A}$ is minimized at the same place where M is minimized.

4. Now write $\ln \frac{M+A}{A}$ as a function of $N_1, N_2,$ and N_3 by using the result of part 2. Then use Lagrange multipliers to minimize $\ln \frac{M+A}{A}$ subject to the constraint $c(\ln N_1 + \ln N_2 + \ln N_3) = v_f$. Find expressions for the values of N_i in terms of v_f where the minimum occurs.

   [Hint: Use properties of logarithms to help simplify the expressions. You should be getting $N_1 = N_2 = N_3$.]

5. Show that the minimum value of M as a function of v_f is

$$M = \frac{A(1-S)^3 e^{v_f/c}}{\left[1 - Se^{v_f/(3c)}\right]^3} - A$$

6. If we want to put a three-stage rocket into orbit 100 miles above the earths surface, a final velocity of approximately $17,500$ mph is required. Suppose that each stage is built with a structural factor $S = 0.2$ and an exhaust speed of $c = 6000$ mph.

   (a) Find the minimum total mass M of the rocket engines as a function of A.

   (b) Find the mass of each individual stage as a function of A. (They are not equally sized!)