MATH 1800-B HANDOUT 5: PRACTICE PROBLEMS

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■ Exercise 1.

Show that every plane that is tangent to the cone $x^2 + y^2 = z^2$ passes through the origin.

■ Exercise 2.

Where does the normal line to the paraboloid $z = x^2 + y^2$ at the point $(1, 1, 2)$ intersect the paraboloid a second time?

■ Exercise 3.

Find the angle of intersection between the curve given by its parametric equation $\vec{r}(t) = \langle t, 2t^2 \rangle$, and the parabola $y = x^2 + 4$.

■ Exercise 4.

The length of a side of a triangle is increasing at a rate of 3 in/s, the length of another side is decreasing at a rate of 2 in/s, and the contained angle θ is increasing at a rate of 0.05 radian/s. How fast is the area of the triangle changing when $x = 40$ in, $y = 50$ in, and $\theta = \pi/6$?

■ Exercise 6.

Let $p = g(u, v)$ be a differentiable function of two variables. Let $u = \frac{x}{y}$ and $v = \frac{y}{z}$. Show that

$$x\frac{\partial p}{\partial x} + y\frac{\partial p}{\partial y} + z\frac{\partial p}{\partial z} = 0$$

■ Exercise 7.

Consider the function

$$F(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 1$$

defined on the disk D of radius $\sqrt{2}$ centered at the origin i.e.

$$D = \{(x, y) \mid x^2 + y^2 \leq 2\}.$$

Follow the steps below to find the absolute maximum and minimum of f on D.

(a) Find all the stationary critical points of F. [HINT: There are 4 such points.]

(b) Find all of the second order partial derivatives of F and write down the determinant of the Hessian matrix as a function of x and y. Don't calculate its value at any specific point yet.

(c) In the list of critical points from part (a), identify the ones lying *inside* D (excluding the boundary).

(d) Classify each of the point(s) in part (c) as a local maximum, local minimum, or a saddle point using the Hessian.

(e) Evaluate F at the critical point(s) from part (c).

(f) Use Lagrange multiplier to find the maximum and minimum of $F(x, y)$ subject to the constraint $x^2 + y^2 = 2$. Note that this gives the extreme values of F on the boundary circle of D.

(g) Compare the extreme values of F from part (e), and the extreme values of F from part (f), to find the absolute maximum and minimum of $F(x, y)$ on D.

■ Exercise 8.

Do the same steps to find all local and global extrema of the function

$$f(x, y) = 2x^3 + 2y^3 - 3x^2 - 3y^2 + 6$$

on the disc $D = \{(x, y) \mid x^2 + y^2 \leq 4\}$.

■ Exercise 9.

Consider a function $f(x, y)$ defined as follows.

$$f(x, y) = \begin{cases} 8 & \text{if } x^2 + y^2 \leq 6^2 \\ \frac{48}{\sqrt{x^2+y^2}} & \text{if } 6^2 \leq x^2 + y^2 \leq 16^2 \end{cases}$$

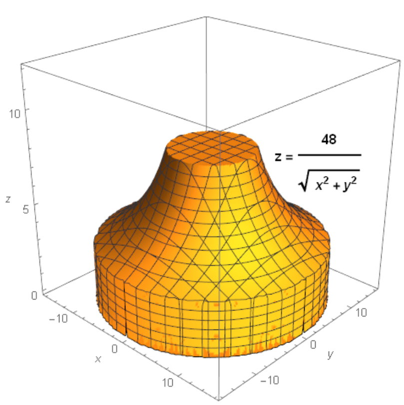
The region under $f(x, y)$ and above the XY-plane looks like a circus tent as in figure 1. Find the volume of the tent.

■ Exercise 10.

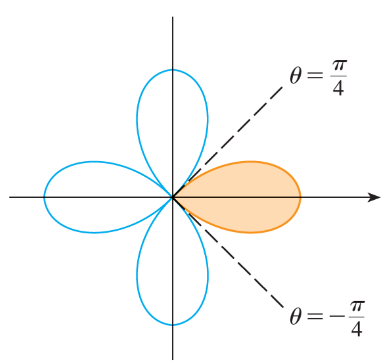
Evaluate the integral $\iint\limits_R (3x + 4y^2)\, dA$, where R is the annulus $1 \leq x^2 + y^2 \leq 4$.

■ Exercise 11.

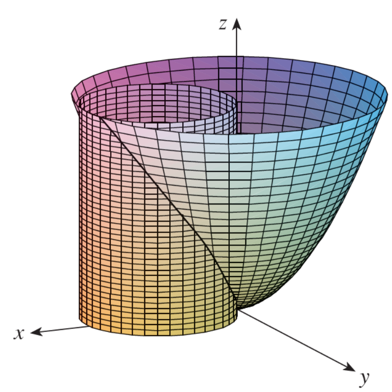
Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the XY-plane, and inside the cylinder $x^2 + y^2 = 2x$. See figure 3.



(a) Figure 1



(b) Figure 2



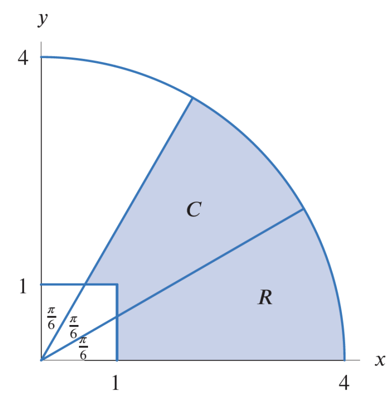
(c) Figure 3

■ Exercise 12.

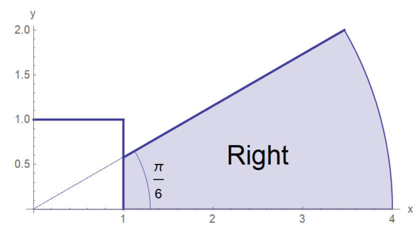
Figure 4 represents a baseball field, with the bases at $(1,0)$, $(1,1)$, $(0,1)$, and home plate at $(0,0)$. The outer bound of the outfield is a piece of a circle about the origin with radius 4. When a ball is hit by a batter we record the spot on the field where the ball is caught. Let $p(x,y)$ be a function in the plane that denotes the fraction of times a ball is caught at (x,y). Write an integral in polar coordinates that represents the total fraction of times a hit is caught in

(a) The right field (region R)

(b) The center field (region C)



(a) Figure 4



(b) Figure 5

■ Exercise 13.

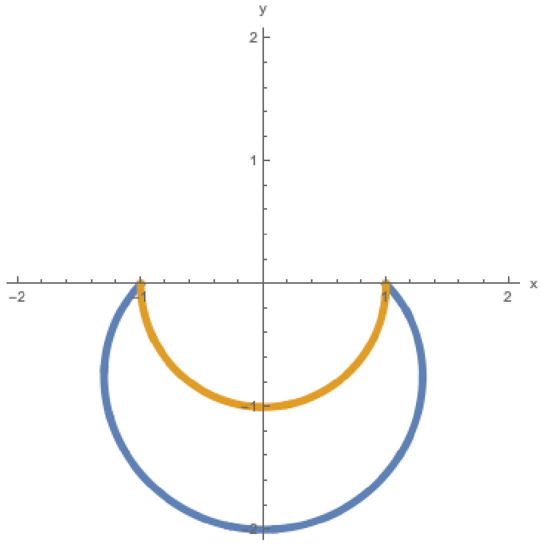
Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos(2\theta)$. See figure 2.

Solution:

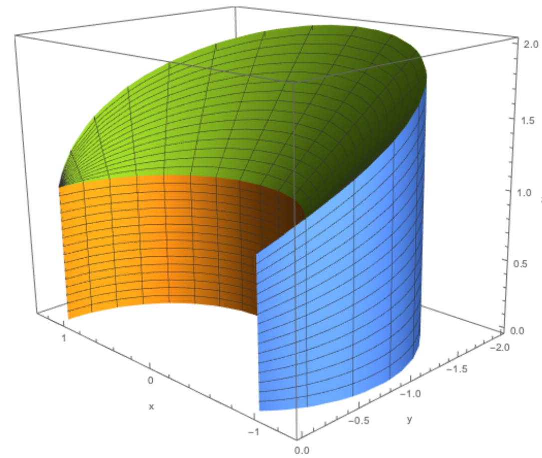
$$\begin{aligned}
\int_{-\pi/4}^{\pi/4}\int_0^{\cos(2\theta)} r\,dr\,d\theta &= \int_{-\pi/4}^{\pi/4} \frac{\cos^2(2\theta)}{2}\,d\theta \\
&= \int_{-\pi/4}^{\pi/4} \frac{\cos(4\theta)+1}{4}\,d\theta \\
&= \left(\frac{\sin(4\theta)}{16} + \frac{\theta}{4}\right)\Bigg|_{-\pi/4}^{\pi/4} \\
&= \pi/16 + \pi/16 \\
&= \pi/8
\end{aligned}$$

■ Exercise 14.

Find volume of the solid under the paraboloid $z = x^2 + y^2$, and over a region Ω that is inside the cardioid $r = 1 - \sin\theta$, but outside the circle $r = 1$.



(a) Domain



(b) Solid

Solution:

$$
\begin{aligned}
\int_{\pi}^{2\pi}\int_{1}^{1-\sin\theta} r^2 r\,dr\,d\theta =& \frac{1}{4}\int_{\pi}^{2\pi}(1-\sin\theta)^4 d\theta \\
=& \frac{1}{4}\int_{\pi}^{2\pi}(\sin^4(\theta)-4\sin^3(\theta)+6\sin^2(\theta)-4\sin(\theta)+1)d\theta \\
=& \frac{1}{4}\int_{\pi}^{2\pi}\left(\left(\frac{1-\cos(2\theta)}{2}\right)^2\right)d\theta+ \\
& \frac{1}{4}\int_{\pi}^{2\pi}\left(-4(1-\cos^2\theta)\sin(\theta)+6\frac{1-\cos(2\theta)}{2}-4\sin(\theta)+1\right)d\theta \\
=& \frac{1}{16}\int_{\pi}^{2\pi}\left(1-2\cos(2\theta)+\frac{\cos(4\theta)+1}{2}\right)d\theta+ \\
& \frac{1}{4}\int_{\pi}^{2\pi}4(1-\cos^2\theta)d(\cos(\theta))+ \\
& \frac{1}{4}\int_{\pi}^{2\pi}\left(6\frac{1-\cos(2\theta)}{2}-4\sin(\theta)+1\right)d\theta \\
=& \cdots \\
=& \frac{10}{3}+\frac{27\pi}{32}
\end{aligned}
$$