Final Exam

August 7, 2018
Subhadip Chowdhury

Please show **all** your work! Answers without supporting work will not be given credit. No electronic equipment is allowed with you in the exam hall. Please answer in the green book provided to you. Write clearly and legibly. Start every problem on a different page. They can be out of order, in fact I suggest answering the easier problems first.

You can score a maximum of 80 points!

Time Limit: 1 Hour 20 minutes

Question	Points
1	15
2	10
3	15
4	6
5	15
6	9
7	10
8	0
Total:	80

This exam has 8 questions, for a total of 80 points. The maximum possible point for each problem is given on the right side of the problem.

1. Prove by mathematical induction that $n^3 + 2n$ is divisible by 3 for all natural numbers n. $\boxed{15}$

2. Prove that if a positive integer p has no factors that are less than or equal to $\sqrt{p}$, then p is a prime number. $\boxed{10}$

3. Suppose three distinct real numbers a, b, and c are in an Arithmetic Progression. If $\boxed{15}$

$$2(b-a)+x(c-b)^2+(c-a)^3=0$$

then find all possible values of x.

4. Find the least upper bound and the greatest lower bound of the following sets. Write DNE if it does not exist. No explanation is necessary.

   (a) $\{x \in \mathbb{R} \mid (x-2)(3-x) \geq 0\}$ $\boxed{2}$

   (b) $\{k \in \mathbb{Z} \mid k^2 < 4\}$ $\boxed{2}$

   (c) $\left\{ \frac{n-1}{n} \,\middle|\, n \in \mathbb{N} \right\}$ $\boxed{2}$

5. Determine which of the following statements are *True* or *False*. Very very briefly explain your reasoning.

   (a) Let $a, b, c \in \mathbb{Z}$. Then $(a+bc, b) = (a, b+ac)$. $\boxed{3}$

   (b) The highest power of 5 that divides 100! is 5^{25}. $\boxed{3}$

   (c) For any real number x, we have $\lfloor 2x+4 \rfloor = 2\lfloor x+2 \rfloor$. $\boxed{3}$

   (d) If $\sin\alpha = \sin\beta$, then $\alpha = \beta + 2\pi n$ for some integer n. $\boxed{3}$

   (e) The greatest lower bound of a set S is equal to the least upper bound of the set of all lower bounds of S. $\boxed{3}$

6. Using trigonometric identities, show that $\boxed{9}$

$$(1+\tan\alpha\tan\beta)^2 + (\tan\alpha - \tan\beta)^2 = \sec^2\alpha\sec^2\beta$$

for any angles α and β.

7. Give an $\epsilon - N$ proof of the following: $\boxed{10}$

$$\lim_{n\to\infty} \frac{1+2n}{3n-1} = \frac{2}{3}$$

8. Define a sequence $\{a_n\}_{n\in\mathbb{N}}$ as $\boxed{\text{5 (bonus)}}$

$$a_1 = 1, \quad a_n = a_1 + a_2 + \ldots + a_{n-1}$$

Prove that $a_n = 2^{n-2}$ for $n > 2$.