Math 2208: Ordinary Differential Equations

Project 2: An Application from Economics: Modelling Profit

Fall 2019 **Subhadip Chowdhury** **Due: Nov 6**

Two cafes, "Paul's High Test Coffee" and "Bob's Gourmet Tea" plan to open on the same block. It is common for stores of the same type to open close to each other, which is a partially explained by *Hotelling's Model of Spatial Competition*. According to Hotelling, when competing on location, each business wants the central point as it is the most strategic spot that allows it to be as close to as many customers as possible. Since every business has the same mindset, they will be competing with one another which eventually causes similar businesses to end up in a cluster focused on one specific point. The question is, how will these businesses fare?

We will take a differential equation approach to gain insight into the profitability of these two cafes in a cluster. Let

$$x(t) = \text{ daily profit of Paul's cafe at time } \mathbf{t}$$
$$y(t) = \text{ daily profit of Bob's cafe at time } \mathbf{t}$$

That is, if $x(t) > 0$, then Paul's cafe is making money, but if $x(t) < 0$, then Paul's cafe is losing money. Likewise for the profit of Bob. Like always, we start with a very simple model.

$$\frac{dx}{dt} = ax + by \tag{1}$$

$$\frac{dy}{dt} = cx + dy \tag{2}$$

where $a, b, c,$ and d are parameters. The rate of change of Paul's profits depends linearly on both Paul's profits and Bob's profits (and nothing else). The same assumptions apply to Bob's profits. Linear models are usually only valid for values near the break-even point. For the sake of this problem if $x(t) \to \infty$ we will say Paul's business will flourish; if $x(t) \to -\infty$ Paul will go bankrupt and if $x(t) \to 0$ Paul will break even. Likewise for Bob.

$\boxed{1}$ If you havent done so, please READ THE TEXT above.

Note: You will be asked to generate a number of graphs that you will be asked to print out and include in the report you turn in. **You could collect all the images and put several into a single page of, e.g. a Word document. Be sure to give your graph labels and titles that you can refer to.**

$\boxed{2}$ For this problem, consider $a = 2, b = -1, c = -1, d = 1$.

(a) Discuss the interaction between the two stores with these parameter values.

(b) What would you guess the fate of the two stores to be (without doing any mathematical analysis)?

(c) Use the website https://matrixcalc.org/en/vectors.html to compute the eigenvalues and eigenvectors of the corresponding matrix for the given parameter values.

(d) **Qualitative approach:** Draw the phase portrait (including nullclines, equilibrium points, straight line solutions etc.) *by hand* and classify it as a sink, source, or saddle.

(e) **Analytical approach:** Write down the general solution formula and find the solution that has the initial value $x(0) = 1/2$ and $y(0) = 1$. You can use the same website as above to solve a system of linear equation (click on `Solving systems of linear equations' on the left or go to `https://matrixcalc.org/en/slu.html`).

(f) **Numerical approach:** Go to Blackboard and open `ODE45_system_Example.m`. This is a modified version of the `ODE45` code that we used earlier in the semester that will solve systems of ODEs. Inspect the code, and modify so that it solves the system with the given parameter values and has the initial value $x(0) = 1/2$ and $y(0) = 1$. Run the code and save the graphs (somewhere where you can access them later, like on Onedrive, or to your e-mail). Include a printout of your graphs when you turn this project in.

(g) Using either the qualitative, analytical, or numerical approach, answer the following questions. (you don't have to use all approaches!) In your answer, include any work or reference to graphs that you used to justify your answer.

   i. Is there an initial profit for Bob and Paul such that both will approach the break even point as $t \to \infty$?

   ii. How much more initial profit must Bob have over Paul if Bob's store is to flourish and Paul is to go Bankrupt? Try to answer in terms of the ratio $\frac{y(0)}{x(0)}$.

$\boxed{3}$ **Using** `pplane`**:**

(a) If you are on an iMac in the lab, go to Applications and find `pplane`. Otherwise, download and open the `pplane` app from Blackboard.

(b) Enter the system that corresponds to Eq. (1) and (2) with the parameter values $a = 2, b = -1, c = -1, d = 1$. Note you can type e.g. $\mathtt{a * x}$ as long as you define $\mathtt{a}$ in the parameter expression part.

(c) Click `graph phase plane`. On the window, a direction field will be produced. If you click anywhere on the graph, it will create a forward and backward trajectory for the solution with the initial value where you clicked. Take a screen shot of the generated phase plane with some sample solution curves. Include a printout of your graphs when you turn this project in.

(d) How does the generated phase plane compare to the one you drew for problem 2d.

$\boxed{4}$ Now consider $a = 2, b = -1, c = -3, d = 1$. You can use any of the above tools to analyze this case.

(a) How does this interaction differ from problem 2?

(b) What would you guess the fate of the two stores to be (without doing any mathematical analysis)?

(c) Is there an initial profit for Bob and Paul such that both will approach the break even point as $t \to \infty$?

(d) How much more initial profit must Bob have over Paul if Bob's store is to flourish and Paul is to go Bankrupt?

$\boxed{5}$ **Bifurcation:** Consider the one parameter family of system of linear ODEs

$$\frac{d\vec{r}}{dt} = \begin{bmatrix} 2 & -1 \\ c & 1 \end{bmatrix} \vec{r}(t)$$

where c is a real number.

(a) We observed that the qualitative behavior of the system has changed as the parameter c was changed from $c = -1$ to $c = -3$ since the phase portrait changed from a ________________ to a ________________. (Fill in blanks with either saddle, sink or source).

Thus, for some value of c between $-3 < c < -1$ a *bifurcation* has occurred.

(b) Suppose the eigenvalues of the matrix $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ c & 1 \end{bmatrix}$ are λ_1 and λ_2. Explain why

$$\lambda_1 \lambda_2 = \det(\mathbf{A})$$

(c) Observe that the bifurcation happened when the product of the eigenvalues changed from positive to negative. Find the value of c for which the product is zero. This must be the bifurcation point!

(d) Use pplane to graph and describe the phase portrait when c is equal to the bifurcation value. Include the graph with your report.

$\boxed{6}$ **Complex Eigenvalues:**

(a) Using the pplane software, draw the phase plane with the parameters $a = 2, b = 3, c = -3, d = -3.$

i. Describe the dynamics (that is, describe what the solutions look like). Does this system have a straight line solution?

ii. Find the eigenvalues (use the website).

iii. Use ODE45 to solve the system with these parameters with the initial values $x(0) = 1/2$ and $y(0) = 1$. **Include a printout of all graphs generated.**

(b) Repeat part (a) for $a = 2, b = 3, c = -3, d = -2$.

(c) Repeat part (a) for $a = 2, b = 3, c = -3, d = -1$.