Solutions(and Hints) to some Practice Problems and some useful notes

Subhadip Chowdhury

- Compound interest is not in the syllabus.
- Neither 7.2.5 nor 7.2.6 is definition of logarithm. The paragraph in between these two equation which says "the function that we have labelled..." is the relevant part.
- **Definition(logarithm function):** The logarithm function $\ln(x)$ is defined as

  $$\ln(x) = \int_1^x \frac{1}{t} dt$$

  for all $x \in [0, \infty)$.
- Note that e is defined to be the number such that $\ln(e) = 1$. Thus you can NOT define ln using e in turn!
- Definition 7.4.2 and 7.4.3 together define the Exponetial function.
- **Definition(Exponential function):** For every real number x, the Exponential function $Exp(x)$ is defined as $Exp(x) = e^x$ where e^x is defined to be the *unique real* number such that

  $$\ln(e^x) = x.$$
- **Definition(Injective function):** (Definition 7.1.1)

  A function $f : A \to B$ is said to be injective if $f(a) = f(b) \implies a = b$ for all $a, b \in A$.
- **Definition(Surjective function):** A function $f : A \to B$ is called surjective if for every $b \in B$, there exists some $a \in A$ such that $f(a) = b$.
- **Definition(Inverse function):** Read 7.1.3.
- **Solution to Problem** 4, **Practice set** 2**:** g is neither injective nor surjective onto $\mathbb{R}$ in general. For example consider $f(x) = x^2$. Then $g(x) = e^{x^2}$. Then $g(-1) = g(1)$, so g is not injective. Also, $e^{x^2} > 0\, \forall x$. Hence $\nexists x$ such that $g(x) = -1$. So g is not surjective onto $\mathbb{R}$.
- Note that in problem 4, g is surjective onto $[0, \infty)$.
- **Solution to Problem** 5, **Practice set** 2**:** If $f(x)$ is continuous and injective then f is monotone. Now

  $$g'(x) = e^{f(x)} f'(x)$$

  Since $e^{f(x)}$ is *always positive*, $g'(x)$ has the same sign as $f'(x)$ for all x. Hence $g(x)$ is monotone. Clearly $g(x)$ is contionuous. Hence $g(x)$ is injective.

- **Solution to Problem** 6**, Practice set** 2**:** The function $g : \mathbb{N} \to \mathbb{N}$ is NOT surjective if there exists $b \in \mathbb{N}$ such that for all $a \in \mathbb{N}$, we have $g(a) \neq b$.

  $\boxed{\text{Note that this is the negation of the definition of Surjectivity.}}$

  Now, take $b = 3$. Clearly $b \neq g(a) = a^2$ for any $a \in \mathbb{N}$. Hence g is NOT surjective onto $\mathbb{N}$.

- **Hints to Problem** $7(c)$**, Practice set** 1**:**Use double angle formula and separate into two integrals by separating the numerator. The integrand becomes $\csc(2x) + \frac{1}{2}\csc(x)$.

- **Hints to Problem** 8**, Practice set** 1**:** Write the denominator as a square and divide numerator and denominator by x^2. Take $u = x - \frac{1}{x}$.

- **Hints to Problem** 9**, Practice set** 1**:** Note that

$$\sin(x) + \sqrt{3}\cos(x) = 2\left(\frac{1}{2}\sin(x) + \frac{\sqrt{3}}{2}\cos(x)\right) = 2\sin\left(x + \frac{\pi}{3}\right)$$

  Hence the integral in part (a) becomes $\frac{1}{2}\int \csc\left(x + \frac{\pi}{3}\right)dx$. Similarly the integral in part (b) becomes $\frac{1}{\sqrt{2}}\int \csc\left(x + \frac{\pi}{4}\right)$

- **Hints to Problem** 11**, Practice set** 1**:**

  Option 1: Change the integral into an integral in terms of cos and sin.

  Option 2: Use the fact that $\csc^2(x) - \cot^2(x) = 1$. Then we can write

$$\begin{aligned}\int \cot^3(x)\csc^4(x)dx &= \int \cot^3(x).(1 + \cot^2(x)).\csc^2(x)dx \\ &= -\int u^3(1 + u^2)du\end{aligned}$$

  where $u = \cot(x) \implies du = -\csc^2(x)dx$.

- **Hints to Problem** 12**, Practice set** 1**:** Note that if $f(x) = [\sqrt{x}]$, then

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1) \text{ because } 0 \leq \sqrt{x} < 1 \\ 1 & \text{if } x \in [1,4) \text{ because } 1 \leq \sqrt{x} < 2 \\ 2 & \text{if } x \in [4,9) \text{ because } 2 \leq \sqrt{x} < 3 \\ 3 & \text{if } x \in [9,16) \text{ because } 3 \leq \sqrt{x} < 4 \\ 4 & \text{if } x \in [16,25) \text{ because } 4 \leq \sqrt{x} < 5 \end{cases}$$

  Thus the required integral is equal to

$$\int_0^1 0.dx + \int_1^4 1.dx + \int_4^9 2.dx + \int_9^{16} 3.dx + \int_{16}^{25} 4.dx = \ldots$$