Problem Set 15,16 Solutions

Questions? Corrections? email jjudge@uchicago.edu (John)

edited by Subhadip Chowdhury

The University of Chicago, CAAP 2018: Proof-Based Methods in Calculus (Chowdhury)

August 2, 2018

Exercise 1: Find LUB and GLB

Problem 1.1. *(11.1.n). What are the LUB and GLB of the following sets? Write DNE if it does not exist. No proof is necessary.*

(a) $(-\infty, 1)$
(b) $\{x|x^3 \leq 8\}$
(c) $\{2\frac{1}{2}, 2\frac{1}{3}, 2\frac{1}{4}, \ldots\}$
(d) $\{0.9, 0.99, 0.999, 0.9999, \ldots\}$
(e) $\{x|x^2 + x + 2 \leq 0\}$
(f) $\{x||x-1| > 2\}$

Solution. In the form GLB, LUB:

(a) DNE, 1
(b) DNE, 2
(c) 2, $2\frac{1}{2}$
(d) 0.9, 1
(e) DNE, DNE (This set is $\mathbb{R}$ since $x^2 + x + 2$ is non-negative for all real numbers x.)
(f) DNE, DNE (This set is $(-\infty, -1) \cup (3, \infty)$.)

■

Exercise 2: Irrational LUB's for Sets of Rationals

Problem 2.1. *(11.1.30). Find an example to show that the lub of a set of rational numbers may not be rational.*

Solution. Consider the set containing the elements of this particular monotonically increasing sequence:

$$\{3, 3.1, 3.14, 3.141, 3.1415, \ldots\}$$

where each subsequent rational element includes one more digit of π than the previous. This converges to the irrational number π.

For another example, consider the set

$$\{x \mid x^2 < 2, x \in \mathbb{Q}\}$$

The lub is $\sqrt{2}$, which isn't rational.

Yet another example, consider the set containing the elements of this sequence:

$$(1+\frac{1}{1})^1, (1+\frac{1}{2})^2, (1+\frac{1}{3})^3, ..., (1+\frac{1}{i})^i ...$$

where $i \in \mathbb{N}$. In fact, the limit of this sequence is defined as e, which can be shown to be irrational. ■

Exercise 3: Recursive Sequence Properties

Problem 3.1. *Consider a sequence defined as* $a_1 = 2$*,* $a_n = 1 - \frac{1}{a_{n-1}}$.

(a) Prove by induction that $a_n \leq 3$ *for all* n.

(b) Show that $\{a_n\}_{n\in\mathbb{N}}$ *is a periodic sequence.*

(c) Find the least upper bound of the sequence.

Solution. (a)

We will prove by inducting on $n \in \mathbb{N}$. For $n = 1$, observe that $a_1 = 2$, and $2 < 3$. So the statement that $a_n \leq 3$ holds for $n = 1$. Now let us make the induction assumption that $\exists k \in \mathbb{N}$ such that $a_k \leq 3$. Then consider

$$a_{k+1} = 1 - \frac{1}{a_k}$$

Using our induction assumption, we know

$$\frac{1}{a_k} \geq \frac{1}{3}$$

which implies

$$-\frac{1}{a_k} \leq -\frac{1}{3}$$

So we can write

$$1 - \frac{1}{a_k} \leq 1 - \frac{1}{3}$$

which is less than 3.

Thus, the inequality holds for $n = k + 1$ whenever it is true for $n = k$. By the principle of mathematical induction, it is true $\forall n \in \mathbb{N}$

(b)

Because the recurrence relation depends only on the single previous term, we need to show that a single value occurs twice in the sequence, for then all subsequent terms are predetermined and must revisit a periodic finite subsequence.

In fact, we only need to calculate the first four terms of the sequence to find a revisited term:

$$2, \frac{1}{2}, -1, 2$$

(c) 2

As found in part (b), the sequence infinitely repeats only three unique terms. So the LUB of the sequence is the LUB of $\{-1, \frac{1}{2}, 2\}$.

■

Exercise 5: Negation

Problem 4.1. *Write the negation of the following statements.*

(a) For all $x \in P$, there exists $y \in Q$ such that, R is true for all $z \in S$.

(b) For all $\epsilon > 0$, there exists a natural number N such that $\frac{1}{n} < \epsilon$ for all $n > N$.

(c) For all $\epsilon > 0$, there exists a natural number N such that, $n > N$ then $\frac{1}{n} < \epsilon$.

(d) For all $\epsilon > 0$, there exists a natural number N such that, $|a_n - l| < \epsilon$ for all $n > N$.

(e) For all $\epsilon > 0$, there exists a $\delta > 0$ such that, if $|x - c| < \delta$ then $|f(x) - l| \leq \epsilon$.

Solution. Negations of statements:

(a) There exists some $x \in P$ such that for all $y \in Q$, $\exists z \in S$ such that R is false.

(b) There exists some $\epsilon > 0$ such that for all natural numbers N, we have some $n > N$ for which $\frac{1}{n} \geq \epsilon$.

(c) There exists some $\epsilon > 0$ such that for all natural numbers N, $\exists n > N$ such that $\frac{1}{n} \geq \epsilon$. (same as part (b)).

(d) There exists some $\epsilon > 0$ such that for all natural numbers N, we can find some $n > N$ such that $|a_n - l| \geq \epsilon$.

(e) There exists some $\epsilon > 0$ such that, for all $\delta > 0$, there exists x satisfying $|x - c| < \delta$ such that, $|f(x) - l| > \epsilon$.

■

Exercise 6: LUB, GLB Proofs

Problem 5.1. *Do exercise 11.1.(32a): Show that the lub of a set of negative numbers cannot be positive.*

Proof. Suppose, for the sake of contradiction, that the LUB of a set S of negative numbers is M, where $M > 0$. Consider the number $\frac{M}{2}$. Clearly $\frac{M}{2} > 0$. Since S is a set of negative numbers, $\frac{M}{2}$ is thus an upper bound for S. But then we have an upper bound of S that is smaller than the LUB of S. That is a contradiction. □

We can use the theorem regarding LUB that we did in class to give a second proof.

Proof. Suppose, for the sake of contradiction, that the LUB of a set S of negative numbers is M, where $M > 0$. Then $\forall \epsilon > 0$, there exists $x \in S$ such that $M - \epsilon < x \leq M$. So, we should be able to find such an $x \in S$ when we fix $\epsilon = \frac{M}{2}$, which is indeed a positive ϵ as required. This implies that there should be some $x \in S$ such that

$$\frac{M}{2} = M - \frac{M}{2} < x$$

But x must be negative. The fact that it is greater than the positive quantity $\frac{1}{2}M$ is a contradiction. □