Lab 4: Stationary Points

Subhadip Chowdhury

A stationary point for a function of two variables is a point where both first partial derivatives equal zero. In this lab you will investigate the stationary points for the following functions:

$$f(x,y) = x^3 - 2\,x\,y - 4\,y^2 - 3\,x$$
$$g(x,y) = x^2 + 6\,x\,y + y^2 + 14\,x + 10\,y$$
$$h(x,y) = 16\,x^2 + 8\,x\,y + y^2$$

Computing Stationary Points

Not only will *Mathematica* calculate the partial derivatives for you, but it also has a built-in `NSolve` command that you can use to find the points where the derivatives are both zero.

- Define the first function by `enter`ing the command

```
f[x_,y_] := x^3 - 2*x*y - 4*y^2 - 3*x
```

- Solve for the stationary points by entering the command

```
NSolve[{D[f[x, y], x] == 0, D[f[x, y], y] == 0}, {x, y}]
```

  Alternately, you can also type

```
NSolve[Grad[f[x,y],{x,y}]=={0,0}, {x, y}]
```

- Record these stationary points in the table below for future reference.
- We can define a function in Mathematica that will take f, g or h as input and produce the list of stationary points directly as follows. Type

```
statlist[func_] := (temp = func; NSolve[Grad[temp, {x, y}] == {0, 0}, {x, y}])
```

  Check that `statlist[f[x,y]]` produces the same list of points as above. The advantage of doing this step is that, now we can define `g` and `h` similar to `f` and reuse the code for finding stationary points without rewriting previous outputs.

- Plot the x and y cross-sections through each of the stationary points, and use these to record the signs of the two second partial derivatives in the table. Note that you can plot an x cross-section through a point (a, b) by entering the command

```
Plot[f[a,y],{y,b-0.1,b+0.1}]
```

  Similarly you can plot a y cross-section by fixing y and varying x. Change a and b appropriately to the points from the list of stationary points you obtained above.

- Using only the cross-sections you just plotted, try to classify the stationary points as a local maximum, a local minimum, a saddle, or undetermined.
- Now repeat this procedure for the other functions and fill in the table below. (The third function $h(x,y)$ has an entire line of stationary points, and you should choose one of these for your investigation.)

Function:	$f(x,y)$		$g(x,y)$	$h(x,y)$
Stationary points: $x =$ $y =$				
Sign of $\frac{\partial^2}{\partial x^2}$:				
Sign of $\frac{\partial^2}{\partial y^2}$:				
Classification:				

Using Second Derivative test

- We can define the determinant of the Hessian of the function $f(x,y)$ as

```
DH[x_, y_] := (D[D[f[x, y], x], x] D[D[f[x, y], y], y])
          - (D[D[f[x, y], x], y])^2
```

- Find $DH[x,y]$. Next find the determinant for other two functions by changing f to g and h.
- We can evaluate the determinant at each of the stationary point above as follows. Type

```
DH[x,y]/.statlist[f[x,y]]
```

  to evaluate $DH(x,y)$ at the stationary points of f. Note the '`/.`' operator. It's called `ReplaceAll`.
- Record the values at each stationary point as above. Check that the signs agree with your classification in the table above using the second derivative test.

Exercise 1

Did you classify any stationary points incorrectly using just the cross-sections? What do you think went wrong?

Exercise 2

Is the following "second derivative test" always valid?

> Consider a stationary point (a,b) for any function $f(x,y)$.
> If $f_{xx}(a,b) > 0$ and $f_{yy}(a,b) > 0$, then (a,b) is a local minimizer.
> If $f_{xx}(a,b) < 0$ and $f_{yy}(a,b) < 0$, then (a,b) is a local maximizer.

Exercise 3

Draw a contour plot of the function to check your classification graphically. Note that you can make a contour plot in the neighborhood of a point (a,b) by entering the command

```
ContourPlot[f[x,y],{x,a-0.1,a+0.1}, {y,b-0.1,b+0.1}]
```