# Assignment 17 (2/26)

**Subhadip Chowdhury**

- *This homework is due at the beginning of class on **Friday** 3/2. You may cite results from class as appropriate. Unless otherwise stated, you must provide a complete explanation for your solutions, not simply an answer. You are encouraged to work together on these problems, but you must write up your solutions independently.*
- *Hand in the exercises only, not the reading material item. You are encouraged to think about the exercises marked with a ($\star$) or (†) if you have time, but you don't need to hand them in. If you correctly solve a (†)-marked problem, you will get a candy!*
- *Remember that you can always use the result of the previous assignment problems without proof to solve the new assignment problems.*
- *We finished talking about the Simplex Method and have started Sequences and Series (Chapter 11) from Stewart.*

## Important Points and Reading Materials

- SOME DEFINITIONS

  Before we proceed we define *Basic* and *Non-basic* variables. Each variable corresponds to a column in the table. If the column is cleared out and has only one non-zero element in it, then that variable is a *basic* variable. If a column is not cleared out and has more than one non-zero element in it, that variable is *non-basic*.

  After each iteration of the Simplex method, the current value of the objective function is obtained by setting all non-basic variables to zero. **Observe that by the process of the algorithm, the basic variables automatically get a zero coefficient in the last row at the end of each iteration (due to row-reduction).** Thus we continue pivoting until the coefficients for the non-basic variables in last row become non-negative; since then setting all non-basic variables to zero indeed gives the optimal solution.

- THE SIMPLEX METHOD - INITIALIZATION

  Consider a LPP in standard form

$$\begin{aligned}
&\text{Maximize } P = \sum_{i=1}^{n} c_i x_i \\
&\text{subject to } \sum_{i=1}^{n} a_{ij} x_i \leq b_j \text{ for } j = 1, 2, \ldots, m \\
&\qquad\qquad x_i \geq 0 \text{ for } i = 1, 2, \ldots, n
\end{aligned}$$

  where setting all non-basic variables to 0 does not give a feasible solution. Note that initially $x_1, x_2, \ldots, x_n$ are the non-basic variables. In such cases, we need to initialize the LPP by turning it into an equivalent LPP where setting all non-basic variables to 0 does infact give a feasible solution. The steps are as follows:

**Step 1.** Construct the Auxiliary LPP.

$$\begin{aligned}
&\text{Maximize } Q = -y \\
&\text{subject to } \sum_{i=1}^{n} a_{ij} x_i - y \leq b_j \text{ for } j = 1, 2, \ldots, m \\
&\qquad\qquad x_i \geq 0 \text{ for } i = 1, 2, \ldots, n \\
&\qquad\qquad y \geq 0
\end{aligned}$$

The main point to note here is that the original LPP has a feasible solution iff the aux. LPP has an *optimal* solution with $y = 0$ i.e. the maximum of $Q$ in the aux. LPP is equal to 0.

**Step 2.** Construct the table for the aux. lpp. Choose the column corresponding to $y$ as the Pivot Column.

Remark 1. Note that by construction, all entries in $y$-column except the last row is $-1$.

**Step 3.** Take the row with least $b_j$ (positive or negative, both allowed) to be the Pivot row.

Remark 2. *We are not calculating any ratio*, we are choosing the row corresponding to the "most infeasible variable" as Pivot row. The goal is to interchange $y$ with that variable to get a new set of basic variables.

**Step 4.** Do a Pivot step - row-reductions etc.

**Step 5.** For the next iteration onwards, we go back to our original method for choosing Pivot columns and rows. Thus the smallest negative number in obj.func. row decides the Pivot Column. The least positive ratio of $b_j/a_{ij}$ decides the Pivot row.

**Step 6.** Eventually you will end up with a table with the following properties.

- The last row (obj. func. row) reads $y + Q = 0$
- Setting the non-basic variables (those corresponding to columns with more than one non-zero entry) to zero gives a feasible solution (i.e. the inequalities are satisfied).

Remark 3. This must happen, because we know that $Q = 0$ is supposed to be an optimal solution for the aux. lpp.

**Step 7.** Now remove the $y$ column from the table and remove the obj. func. row. You have a table with one less column and one less row than before.

Remark 4. Dropping the variable $y$ from this new set of equation gives us a set of $m$ equations that are equivalent to the original set of $m$ equations. However, setting non-basics to 0 now gives a feasible solution to this new set of equation; so we can go back to the original objective of maximize $P$ with these constraints.

**Step 8.** Add a row corresponding to the original obj. func. at the end of the table. We can't start pivoting yet because the entries in the last row corresponding to the basic vector columns might not be zero. Make sure they are zero by doing row operations

Remark 5. This is to make sure the columns corresponding to basic variables has only one non-zero entry of 1.

**Step 9.** Proceed with choosing a pivot column and row normally and iterate until all entries in last row (except possibly the constant in the $\vec{b}$ column at the end) become positive.

**Step 10.** Set all non-basic variables to zero to get $\max P$.

- WORKED-OUT EXAMPLE, PROBLEM 2.3

  Since $(0, 0, 0)$ is not a feasible solution, we need to initialize. Introduce an artificial variable $y$ and two slack variables $s_1, s_2$. In the following table, the columns are correspond to $x_1, x_2, x_3, y, s_1, s_2, Q, b$ respectively. We get

| $x_1$ | $x_2$ | $x_3$ | $y$ | $s_1$ | $s_2$ | $Q$ | $b$ |
|---|---|---|---|---|---|---|---|
| Step 2, 3 | | | | | | | |
| $-1$ | $-1$ | $-1$ | $\boxed{-1}$ | $1$ | $0$ | $0$ | $-2$ |
| $2$ | $-1$ | $1$ | $-1$ | $0$ | $1$ | $0$ | $1$ |
| $0$ | $0$ | $0$ | $1$ | $0$ | $0$ | $1$ | $0$ |
| Step 4 | | | | | | | |
| $1$ | $1$ | $1$ | $1$ | $-1$ | $0$ | $0$ | $2$ |
| $2$ | $-1$ | $1$ | $-1$ | $0$ | $1$ | $0$ | $1$ |
| $0$ | $0$ | $0$ | $1$ | $0$ | $0$ | $1$ | $0$ |
| Step 4, 5 | | | | | | | |
| $1$ | $1$ | $1$ | $1$ | $-1$ | $0$ | $0$ | $2$ |
| $\boxed{3}$ | $0$ | $2$ | $0$ | $-1$ | $1$ | $0$ | $3$ |
| $-1$ | $-1$ | $-1$ | $0$ | $1$ | $0$ | $1$ | $-2$ |
| Step 5 | | | | | | | |
| $0$ | $1$ | $\frac{1}{3}$ | $1$ | $-\frac{2}{3}$ | $-\frac{1}{3}$ | $0$ | $1$ |
| $1$ | $0$ | $\frac{2}{3}$ | $0$ | $-\frac{1}{3}$ | $\frac{1}{3}$ | $0$ | $1$ |
| $0$ | $-1$ | $-\frac{1}{3}$ | $0$ | $\frac{2}{3}$ | $\frac{1}{3}$ | $1$ | $-1$ |
| Step 6 | | | | | | | |
| $0$ | $1$ | $\frac{1}{3}$ | $1$ | $-\frac{2}{3}$ | $-\frac{1}{3}$ | $0$ | $1$ |
| $1$ | $0$ | $\frac{2}{3}$ | $0$ | $-\frac{1}{3}$ | $\frac{1}{3}$ | $0$ | $1$ |
| $0$ | $0$ | $0$ | $1$ | $0$ | $0$ | $1$ | $0$ |

At this step, we drop the fourth column and the last row.

| | | | | | | |
|---|---|---|---|---|---|---|
| Step 7 | | | | | | |
| $0$ | $1$ | $\frac{1}{3}$ | $-\frac{2}{3}$ | $-\frac{1}{3}$ | $0$ | $1$ |
| $1$ | $0$ | $\frac{2}{3}$ | $-\frac{1}{3}$ | $\frac{1}{3}$ | $0$ | $1$ |

Now we are going to add a row corresponding to $P$. The columns now correspond to $x_1, x_2, x_3, s_1, s_2, P, b$.

| $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $P$ | $b$ |
|---|---|---|---|---|---|---|
| Step 8 | | | | | | |
| 0 | 1 | $\frac{1}{3}$ | $-\frac{2}{3}$ | $-\frac{1}{3}$ | 0 | 1 |
| 1 | 0 | $\frac{2}{3}$ | $-\frac{1}{3}$ | $\frac{1}{3}$ | 0 | 1 |
| $-2$ | 6 | 0 | 0 | 0 | 1 | 0 |

Need to make sure that the first and second entries in last row are 0 since those are in basic variable columns. We do $R_3 \mapsto R_3 + 2R_2 - 6R_1$.

| | | | | | | |
|---|---|---|---|---|---|---|
| Step 8 | | | | | | |
| 0 | 1 | $\frac{1}{3}$ | $-\frac{2}{3}$ | $-\frac{1}{3}$ | 0 | 1 |
| 1 | 0 | $\boxed{\frac{2}{3}}$ | $-\frac{1}{3}$ | $\frac{1}{3}$ | 0 | 1 |
| 0 | 0 | $-\frac{2}{3}$ | $\frac{10}{3}$ | $\frac{8}{3}$ | 1 | $-4$ |

Keep pivoting. $R_2 \mapsto \frac{3}{2}R_2, R_1 \mapsto R_1 - \frac{1}{3}R_2, R_3 \mapsto R_3 + \frac{2}{3}R_2$.

| | | | | | | |
|---|---|---|---|---|---|---|
| Step 9 | | | | | | |
| $-\frac{1}{2}$ | 1 | 0 | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ |
| $\frac{3}{2}$ | 0 | 1 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{3}{2}$ |
| 1 | 0 | 0 | 3 | 3 | 1 | $-3$ |

The algorithm has ended since there are no more negative coefficient in last row. The optimal value of $P$ is $-3$. This is obtained when $x_1 = s_1 = s_2 = 0$. Using the equations

$$-1/2x_1 + x_2 - 1/2s_1 - 1/2s_2 = 1/2$$

and

$$3/2x_1 + x_3 - 1/2s_1 + 1/2s_2 = 3/2$$

we get $x_2 = 1/2, x_3 = 3/2$.

- SEQUENCES AND SERIES

  Review your notes from previous courses on Sequence/Series. Next lecture, I will quickly recap section 11.1 and 11.2 main results before moving on to 11.3,4,6. Section 11.7 is a recap of the first 6 sections. Halfway through Friday I plan to start 11.8, discuss 11.9 on Monday and parts of 11.10 on Wednesday. There will probably be a last quiz on next week Wednesday.

  Review the following concepts before this Friday:

  - Definition of Convergence of Sequence - Squeeze Theorem (p738), Monotonic Sequence Theorem (p742)
  - Definition of Convergence of Series - Examples (Geometric and Harmonic series)
  - Tests for determining convergence:
    * $n^{th}$ term divergence test - p753, thm 6,7
    * integral test - p761; specific case - $p-$series
    * comparison test - p767
    * limit comparison test - p769
    * alternating series test - p772
    * ratio test - p779
    * rrot test - p781

    For each of these tests, you should know three things:
    * When the test is applicable
    * How to perform the test, i.e. what to check
    * What are the conclusions

    In particular you should be able to at least recall the statement and one example for each test.

  Below are some warm-up problems from the first two sections.

# Problems

## Exercise 1

Problem 2.4 from vanderbei.

## Exercise 2

(11.1.17,14) Find a formula for the general term $a_n$ of the sequences, assuming the pattern continues.

$$\left\{\frac{1}{2}, -\frac{4}{3}, \frac{9}{4}, -\frac{16}{5}, \dots\right\}$$

$$\{4, -1, 1/4, -1/16, 1/64, \dots\}$$

## Exercise 3

Determine whether the following sequences converge or diverge. Briefly explain your reasoning. If it converges, find the limit.

(a)

$$a_n = \frac{\sqrt{n^3+4n-3}}{\sqrt[3]{n^7-6n^3+5}}$$

(b) (11.1.37)

$$a_n = \frac{(2n-1)!}{(2n+1)!}$$

(c) (11.1.43)

$$a_n = \frac{\cos^2 n}{2^n}$$

(d) (11.1.81)

$$a_1 = 1,\ a_{n+1} = 3 - \frac{1}{a_n} \text{ for } n \geq 1$$

## Exercise 4⋆

Below are some sequences defined recursively. Assume that the sequences are convergent. Find the limit.

(a) $a_1 = 1, a_n = \frac{1}{2}\left(a_{n-1} + \frac{R}{a_{n-1}}\right) \quad \forall n \geq 2$

(b) $a_1 = 1, a_n = \sqrt{6 + a_{n-1}} \quad \forall n \geq 2$

(c) $a_1 = 1, a_{n+1} = a_n + \cos(a_n) \quad \forall n \geq 1$

## Exercise 5⋆

The Fibonacci sequence is defined as

$$F_1 = 1,\ F_2 = 1,\ F_n = F_{n-1} + F_{n-2}$$

(a) Evaluate $F_3, \ldots, F_6$.

(b) Define $r_n = \frac{F_{n+1}}{F_n}$. Assuming that the sequence $\{r_n\}$ converges, find its limit.

(c) Find $\lim\limits_{n\to\infty} \dfrac{F_{n+3}}{F_n}$. [Hint: For any convergent sequence $\{a_n\}$,we have $a_n \to l \Rightarrow a_{n+1} \to l$]