Math 2208: Ordinary Differential Equations

Lecture 16 Worksheet

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Title: Equilibrium Point Analysis

Summary: We will analyze equlibrium points of non-linear systems using a technique called linearization which transforms the behavior of nonlinear systems of ODEs back into our familiar linear systems of ODEs.

§A. The Van der Pol Equation

An important nonlinear system of ODEs which occurs in Physics is the Van der Pol Oscillator Equation

$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = 0$$

which can be written as a non-linear system of first order ODEs as

$$\begin{aligned}\frac{dx}{dt} &= y\\ \frac{dy}{dt} &= -x + \mu(1-x^2)y\end{aligned}$$

For this lecture, we will assume $\mu = 1$.

Question 1.

We will draw the phase portrait for the Van der Pol system using `pplane`.

1. What happens to solutions that start near the origin at $(0,0)$?
2. What about solutions that start (relatively) far away at $(3,3)$?
3. Take a close-up look of the phase portrait near the point $(0,0)$. Can we call $(0,0)$ an equilibrium point? What kind of equilibrium does it resemble most closely?

Linearization. Let's use the idea of linear approximations to explain the behavior near the origin of the Van der Pol system. Suppose x and y are very small, for example say less that 0.01, then the nonlinear term x^2y will be less than 10^{-6} in magnitude, much less than either x or y. We can therefore write a linearized approximate version of the Van der Pol system as follows:

$$\begin{aligned}\frac{dx}{dt} &= y\\ \frac{dy}{dt} &= -x + y\end{aligned}$$

Question 2.

Find the eigenvalues of the associated matrix and use that to classify the type of equilibrium at $(0,0)$.

§B. Linearization near an Isolated Equilibrium

Definition 2.1

An equilibrium point (x_e, y_e) is called *isolated* if some neighborhood of it contains no other equilibrium point.

Tangent plane Approximation. Let's assume (x_e, y_e) is an isolated equilibrium of

$$x' = f(x,y) \qquad y' = g(x,y) \tag{1}$$

and f and g are continuously differentiable in a neighborhood of (x_e, y_e).

Question 3.

What is the equation of the tangent plane $L_f(x,y)$ to the graph of $f(x,y)$ at (x_e, y_e)?

Question 4.

If $(x,y) \to (x_e, y_e)$, the function $f(x,y)$ can be well approximated by the local linearization $L_f(x,y)$. Similarly, we can approximate $g(x,y)$ by $L_g(x,y)$ near the the point (x_e, y_e). Replace $f(x,y)$ and $g(x,y)$ in above system by $L_f(x,y)$ and $L_g(x,y)$ respectively.

Question 5.

Consider the change of variable $u = x - x_e$ and $v = y - y_e$. Check that $u' = x'$ and $v' = y'$.

Rewrite the system (1) in terms of the variables u and v. This process is called *linearization*.

Question 6.

Show that the linearization of system (1) at (x_e, y_e) is a linear system of the form

$$\vec{U}'(t) = J\vec{U}(t)$$

where $\vec{U}(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$ and the coefficient matrix J is the *Jacobian*

$$J(x_e, y_e) = \begin{bmatrix} f_x(x_e, y_e) & f_y(x_e, y_e) \\ g_x(x_e, y_e) & g_y(x_e, y_e) \end{bmatrix} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}\Bigg|_{(x_e, y_e)}$$

Question 7.

Consider the non-linear system

$$\frac{dx}{dt} = (x-1)(y+1)$$
$$\frac{dy}{dt} = (y-2)(x+2)$$

1. What are the equilibrium points?
2. Find the Jacobian at each equilibrium point.
3. Calculate the eigenvalue of the Jacobian matrices to find out the type of equilibrium at each point.
4. Draw the phase portrait using `pplane` to confirm your observation.

§C. The Significance of Isolated Equilibrium

Question 8.

Show that the linearization has an isolated equilibrium point at the origin if and only if both of its eigenvalues are non-zero.

Question 9.

Consider the nonlinear system

$$\frac{dx}{dt} = (x-2y)x$$
$$\frac{dy}{dt} = (x-2)y$$

1. Find the equilibrium points.
2. Find the corresponding linear system near each critical point.
3. Find the eigenvalues of each linear system. What conclusions can you then draw about the nonlinear system?
4. Draw a phase portrait of the nonlinear system to confirm your conclusions, or to extend them in those cases where the linear system does not provide definite information about the nonlinear system.