Assignment 6 (7/6)

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- *This homework is due at the beginning of class on **Tuesday** 7/10. You are encouraged to work together on these problems, but you must write up your solutions independently.*

Prime Numbers

Definition 1.1. A *prime number p* is a positive integer greater than 1 whose only positive divisors are 1 and p. Positive integers greater than 1 that are not primes are called *composite numbers*.

Theorem 1.2. *Every composite number is divisible by some prime number.*

Proof. We will sketch the main idea behind the proof. Let n be a composite number, so it has some positive divisor(s) apart from 1 and n. Consider the set of all such divisors and the least element in it. If that is not prime then it has a smaller divisor larger than 1 that also divides n. Contradiction. □

Theorem 1.3. *There are infinitely many prime numbers.*

Proof. We will sketch the main idea behind the proof. If the set of prime is finite, say $\{p_1, p_2, \ldots, p_n\}$, consider the number $N = p_1 p_2 p_3 \ldots p_n + 1$. Then N must be a composite number that's not divisible by any prime. Contradiction. □

Definition 1.4. Odd prime numbers that come next to one another in counting (i.e. their difference is 2) are called *twin primes*. If we have three odd primes in a row, they are called *triple primes*.

Theorem 1.5. *The set of primes $\{3, 5, 7\}$ is the only set of triple primes.*

Exercise 5. *Prove theorem 1.5.*
    [HINT: *The main idea behind the proof is that every third odd number is divisible by 3. Now use PHP or some other reasoning to prove that any set of three consecutive odd numbers must contain one multiple of 3. You can also see the book for a different proof.*]

Exercise 6. *Show that if p is the smallest prime factor of n, and $p > \sqrt[3]{n}$, then $\frac{n}{p}$ is either a prime or equal to 1.*

Exercise 7. *Can you find 3 consecutive composite integers? How about four? Five? Can you find a formula that will produce n consecutive composite integers?*   [HINT: *Think about using $n! = 1 \cdot 2 \cdot 3 \cdots n$.*]

Exercise 8. *Suppose c and d are positive integers such that $c = dq + r$ for some integers q and r. Then show that $(c, d) = (d, r)$.*   [HINT: *Use the result from exercise 3.*]

Exercise 9 (Extra Credit). *(a) Factorize $a^4 + 4b^4$.*

(b) Show that $4^{545} + 545^4$ is not a prime number.

(c) Show that $n^4 + 4^n$ is not a prime number for any natural number n.