Math 1800 Lab 5: Taylor Approximation and Second Derivative Test

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Why does the Second Derivative Test Work?

It combines 2 ideas:

(a) It's not too hard to classify critical points for *quadratic* polynomials in two variables, i.e. polynomials where no term is bigger than x^2, y^2, or xy.

(b) We can use Taylor series to approximate any (smooth) function by a quadratic polynomial.

Let's begin with the first point.

Understanding quadratics in two variables

1. Give a simple explanation why $f(x,y) = x^2 + y^2$ has a minimum at $(0,0)$.

   Hint: what is the minimum possible value this function could have?

2. For the simple function $f(x) = ax^2$, can you can tell whether it's concave up or down at $x = 0$ just by the sign of a? When a is positive, is it concave up or down? When a is negative?

Even though the example $x^2 + y^2$ is easy, let's think about it in a different way. A function $z = f(x,y)$ has a local minimum at a point if it's ''concave up in every direction'' at that point. To be more precise, when we take any vertical slice through this point, the resulting curve should be concave up.

Now let's convince ourselves that $f(x,y) = x^2 + y^2$ is concave up on every vertical slice going through $(0,0,0)$.

3. Show it's concave up for the two standard cross-sections $x = 0$ and $y = 0$.

4. Think about intersecting $z = x^2 + y^2$ with the vertical plane $y = x$ (you could call this the ''45-degree vertical plane'' I suppose if you looked down on it from above). Note that this plane does go through the origin. Justify that the slice of the surface lying on this plane is concave up. (Just plug $y = x$ into the equation!)

5. Now do the same thing with slice from the vertical plane $y = 2x$. And $y = -3x$. And $y = 0.02x$. It should be concave up on each of these slices.

6. Every vertical slice is found by intersecting the surface with the plane $y = mx$ for some m. (Or $x = 0$ if you want to think about the slice where m goes to infinity.) Plug $y = mx$ into $z = x^2 + y^2$ and get an equation of the form $z = ax^2$. What is a in terms of m? Argue that a is always positive.

This last part shows that the surface is concave up in every direction, thus we really do have a local minimum at the origin!

7. Repeat the same analysis as above for the quadratic function $z = g(x,y) = x^2 - y^2$, at the critical point $(0,0)$ and show that it is concave up in some directions and concave down in others. This explains why g has a saddle point at the origin.

8. Repeat the same analysis as above for the quadratic function $z = h(x,y) = x^2 + 3xy + y^2$ at the critical point $(0,0)$. Is it concave up in every direction, or are there some directions where it is concave down? Then say whether this critical point is a local max or min or saddle point.

9. One more specific example: try $z = f(x,y) = 2x^2 - 4xy + 3y^2$. When you plug in $y = mx$, you get $z = p(m) \cdot x^2$ where $p(m)$ is a quadratic in m. How can you tell if $p(m)$ takes on both positive and negative values?

   HINT: quadratic equation, discriminant. Justify that the critical point at $(0,0)$ is a local min.

The General Quadratic

10. Show that the quadratic $z = q(x,y) = \mathrm{A}x^2 + \mathrm{B}xy + \mathrm{C}y^2$ has a critical point at $(0,0)$.

11. Intersect the surface $z = q(x,y)$ with the vertical plane $y = mx$. As before, we get $z = p(m)x^2$ where $p(m)$ is a quadratic in m. Write down $p(m)$.

12. Convince yourself that if $p(m)$ takes on only positive values (for any choice of m), then the critical point is a local minimum. And that if $p(m)$ takes on only negative values, then the critical point is a local max. And that if $p(m)$ can be either positive or negative depending on the choice of m, then the critical point is a saddle point.

13. Show that the sign of the expression $\mathrm{B}^2 - 4\mathrm{AC}$ determines whether $p(m)$ takes on only positive (or only negative) values or both positive and negative values.

More general Quadratic?

The most general formula of a quadratic function is

$$\mathrm{Q}(x,y) = \mathrm{A}x^2 + \mathrm{B}xy + \mathrm{C}y^2 + \mathrm{D}x + \mathrm{E}y + \mathrm{G}$$

We can rewrite this by completing some squares as follows:

$$\mathrm{Q}(x,y) = \mathrm{A}\left(x + \frac{\mathrm{D}}{2\mathrm{A}}\right)^2 + \mathrm{C}\left(y + \frac{\mathrm{E}}{2\mathrm{C}}\right)^2 + \mathrm{B}xy + \underbrace{\left(\mathrm{G} - \frac{\mathrm{D}^2}{4\mathrm{A}} - \frac{\mathrm{E}^2}{4\mathrm{C}}\right)}_{\text{constant}}$$

For the following three questions, try experimenting in `Fall2019_1800_Lab5.nb`.

14. What is the effect on the graph of **Q** if we change **G**?

15. What is the effect on the graph of **Q** if we change **E**?

16. What is the effect on the graph of **Q** if we change **D**?

We should be able to conclude that the *shape* of **Q** is unaffected by different choices of **D**, **E** and **G**. So we can assume $\mathrm{D} = \mathrm{E} = \mathrm{G} = 0$. Hence the shape of a general quadratic can be fully understood by analyzing $q(x,y)$ from above.

THE TAYLOR QUADRATIC

Recall that the linear approximation of the function $f(x,y)$ at $(0,0)$corresponds to the tangent plane at $(0,0)$:

$$T(x,y) = f_x(0,0)x + f_y(0,0)y + f(0,0)$$

Note that a linear approximation essentially finds a linear function such that the slope of the function matches with the linear function in $x-$ and $y-$ direction. But clearly it can't distinguish between local max/min/saddle point.

So we want to find a better approximation of f by matching more derivatives. For that we'll need something more complicated than linear functions, a quadratic function.

Suppose $Q(x,y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + G$ such that

$$\begin{aligned}
Q(0,0) &= f(0,0)\\
Q_x(0,0) &= f_x(0,0)\\
Q_y(0,0) &= f_y(0,0)\\
Q_{xx}(0,0) &= f_{xx}(0,0)\\
Q_{yy}(0,0) &= f_{yy}(0,0)\\
Q_{xy}(0,0) &= f_{xy}(0,0)\\
Q_{yx}(0,0) &= f_{yx}(0,0)
\end{aligned}$$

17. Find A, B, C, D, E, G.

    These choices lead us to the *Taylor Quadratic* for $f(x,y)$ at $(0,0)$.

$$TQ(x,y) = \frac{f_{xx}(0,0)}{2}x^2 + f_{xy}(0,0)xy + \frac{f_{yy}(0,0)}{2}y^2 + f_x(0,0)x + f_y(0,0)y + f(0,0)$$

Or more generally the Taylor Quadratic approximation at a point (a,b) is given by,

$$TQ(x,y) = \frac{f_{xx}(a,b)}{2}(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{f_{yy}(a,b)}{2}(y-b)^2$$
$$+f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

Using the same argument as last section, we can then conclude that the shape of this quadratic is entirely determined by

$$4AC - B^2 = 4\frac{f_{xx}(a,b)}{2}\frac{f_{yy}(a,b)}{2} - (f_{xy}(a,b))^2 = \mathbf{det}[\text{Hessian}]$$

18. Apply the $4AC - B^2$ rule to determine the graph shapes of the Taylor quadratics:
    (a) $-2xy - 4y^2 - 3x$
    (b) $-3(x+1)^2 - 2(x+1)(y-1) - 4(y-1)^2 - 2(x+1) - 6(y-1)$

19. We can also rewrite $q(x,y)$ as

$$q(x,y) = Ax^2 + Bxy + Cy^2 = A\left(x + \frac{B}{2A}y\right)^2 + \left(C - \frac{B^2}{4A}\right)y^2$$

    So when $B^2 - 4AC = 0$, the y^2 term vanishes. What is the shape of the surface

$$z = A\left(x + \frac{B}{2A}y\right)^2?$$

20. What can you conclude when $B^2 - 4AC = 0$? Is it a local max/min/saddle point? Try experimenting in `Fall2019_1800_Lab5.nb`.