Math 2208: Ordinary Differential Equations

Lecture 5 Worksheet

Fall 2019 **Subhadip Chowdhury** **Sep 18**

Title: Existence and Uniqueness of Solutions
Summary: We will investigate the conditions which guarantee existence and/or uniqueness of solutions to the initial value problem $y' = f(t,y), y(t_0) = y_0$.

§1. Do Problems Always Have Solutions?

Think about the equation $2x^5 - 10x + 5 = 0$. Does it have a solution? How do we know? Discuss!

§2. Existence and Uniqueness Of Particular Solutions

The main questions we would like to be able to answer when analyzing IVPs are:

1. **Existence:** Does the differential equation possess solutions which pass through the given initial condition? and
2. **Uniqueness:** If such a solution does exist, can we be certain that it is the only one?

Luckily, theres a theorem that answers these questions for us.

Theorem 2.1: Existence of a unique solution

Let $\mathcal{R}$ be a rectangular region in the ty-plane defined by

$$\mathcal{R} = \{(t,y) \mid a \le t \le b, c \le y \le d\}$$

that contains the point (t_0, y_0) in its interior. IF $f(t,y)$ and $\partial f/\partial y$ are continuous on $\mathcal{R}$, THEN there exists some interval $\mathbf{I}_0$ defined as $(t_0 - \epsilon, t_0 + \epsilon)$, for some $\epsilon > 0$, contained in (a,b) and a unique function $y(t)$ defined on $\mathbf{I}_0$ that is a solution of the initial value problem $y' = f(t,y), \quad y(t_0) = y_0$.

§3. Theorems Have Hypotheses and Conclusions

The existence and uniqueness theorem is actually two different theorems with different hypotheses and conclusions.

1. **Existence:** IF $f(t,y)$ is continuous on a square containing (t_0, y_0), THEN there exists a solution on an interval $(t_0 - \epsilon, t_0 + \epsilon)$ for some $\epsilon > 0$.
2. **Uniqueness:** IF $f(t,y)$ and $\frac{\partial f}{\partial y}$ are *both* continuous on a square containing (t_0, y_0), THEN there exists a unique solution on an interval $(t_0 - \epsilon, t_0 + \epsilon)$ for some $\epsilon > 0$.

Question 1.

Show that the initial value problem

$$\frac{dy}{dt} = t\sqrt{y}, \quad y(0) = 0$$

has at least two solutions since the equilibrium solution $y(t) = 0$ and the solution $y(t) = \frac{1}{16}t^4$ both satisfy the IVP.

Question 2.

Using the Existence and Uniqueness Theorem, we look at the functions $f(t,y) = t\sqrt{y}$ and $\frac{\partial f}{\partial y} = \frac{t}{2\sqrt{y}}$. At the origin $(0,0)$ what can we say about $f(t,y)$ and $f_y(t,y)$?

Question 3.

What can we say about $f(t,y)$ and $f_y(t,y)$ at $(2,4)$? What does this imply about existence and uniqueness of the corresponding IVP $y' = ty^{1/2}, y(2) = 4$?

§4. Implications of the Existence & Uniqueness Theorem

Extendability: Consider the IVP

$$\frac{dy}{dt} = 1 + y^2, \quad y(0) = 0$$

What is $y(2)$? Discuss!

Role of equilibrium solutions:

Lemma 4.1: When solution curves do not intersect

IF $y' = f(t,y)$ is a first-order differential equation with f and $\partial f/\partial y$ both continuous for all values of t and y in some region **S** in the ty-plane, THEN inside the region **S**, the solution curves of the differential equation will form a non-intersecting space-filling family of curves.

Uniqueness and Numerical Approximation:

Question 4.

a) Use the Euler's method in `dfield` to plot the solution to the IVP

$$\frac{dy}{dt} = e^t \sin y, \quad y(0) = 5$$

b) Check that the constant function $y(t) = n\pi$ is a solution for any integer n.

c) Explain why we should not believe the numerical results.

Question 5.

a) Show that $y_1(t) = t^2$ and $y_2(t) = t^2 + 1$ are both solutions of $\frac{dy}{dt} = -y^2 + y + 2yt^2 + 2t - t^2 - t^4$.

b) Show that if $y(t)$ is another solution to the given ODE with initial condition $0 < y(0) < 1$ then $t^2 < y(t) < t^2 + 1$ for all t.

c) Illustrate your answer by using technology to explore the slope field.