Practice Problems
Math 195

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Following are a set of practice problems for the Final exam, mostly on the topics done after the second midterm. They are not in any particular order of difficulty. Please make sure you know how to solve all of them before them final.

Problem 1 and 7 gives a lot of examples of convergence tests for series. Read section 11.7 for a list of strategy for choosing the correct test.

As usual, $\star$ marked problems are optional.

Problem 1

Find whether the following series converge:

1. $$\sum_{n=1}^{\infty} \frac{\sqrt{n^3+\sin(n)}}{\sqrt[3]{n^7+n^5}}$$

   [HINT: Squeeze theorem (comparison) + limit comparison]

2. $$\sum_{n=1}^{\infty} \frac{(1/n)^n}{(e^{2/n}-1)^n}$$

   [HINT: root test + L'Hospital]

3. $$\sum_{n=1}^{\infty} \frac{n^2}{(2n-1)!}$$

   [HINT: ratio test]

4. Show that the following series converges and find its sum.

   $$\sum_{n=1}^{\infty} \frac{5^{2n-1}}{3^{3n+2}}$$

   [HINT: Geometric series]

Problem 2

Consider the plane $\mathscr{P}$ that passes through the point $A = (3, 4, 5)$ and is perpendicular to the vector $\vec{n} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$. Let B be the point $(6, 6, 6)$ and let D be the foot of the perpendicular from B to $\mathscr{P}$.

1. Find the vector $\overrightarrow{AB}$.
2. Find the projection of $\overrightarrow{AB}$ onto $\vec{n}$. [HINT: Use the projection formula. This is the vector AD.]
3. What is the length of $\overline{BD}$? [HINT: $\triangle ABD$ is a right angled triangle. Use Pythagoras.]

Problem 3

1. Show that the curve $\vec{r}(t) = \frac{3}{2}(t^2 + 1)\hat{\imath} + (t^4 + 1)\hat{\jmath} + t^3\hat{k}$ is perpendicular to the ellipsoid $x^2 + 2y^2 + 3z^2 = 20$ at the point $(3, 2, 1)$.

   [HINT: At what value of t does the curve intersect the ellipsoid?]
2. Find the equation of the normal line to above ellipsoid at $(3, 2, 1)$.

[HINT: $r'(t)$ is the tangent to the curve. gradient (i.e. normal) is perpendicular to the surface. So if the curve and the surface are perpendicular, the tangent (to curve) and the normal (to surface) are parallel.]

Problem 3.5

Find all the points on the surface

$$x + y + z + xy - x^2 - y^2 = 0$$

where the tangent plane is parallel to the plane $8x - 4y - 2z = 7$.

[HINT: i.e. the gradient to the surface is normal to the plane.]

Problem 4

Consider the function

$$f(x, y, z) = \ln(\sin(xy)) + \cos(xz) + e^{yz}.$$

1. In which direction does f increase most rapidly at $(\pi/2, 1, 1)$?
2. Find the directional derivative of f at $(\pi/2, 1, 1)$ towards $(\pi/2, 5, 4)$.

Problem 5

Let

$$f(x, y) = \sin(x - y) + \cos(x + y), \quad x = \frac{s}{t}, \quad y = (\ln(s))^2$$

Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ when $s = 1, t = 1$.

Problem 6

1. Evaluate

$$\iint\limits_{\Omega} e^{x^2}\, dA$$

where Ω is the triangular region bounded by the $X-$axis, $2y = x$, and $x = 2$.

2. Evaluate

$$\int_0^{\pi/2} \int_z^{\pi/2} \int_0^{\sin z} 3x^2 \sin y \, dx \, dy \, dz$$

Problem 7

Determine whether the following series converge or diverge. In each case, clearly mention what test(s) or theorem you are using.

1.

$$\sum \frac{\sqrt{(ln(n))^3 + \ln(n^3)}}{n^3 + 1}$$

2.

$$\sum \left(2 + \frac{1}{n}\right)^n$$

3.

$$\frac{1}{2} + \frac{2}{3^2} + \frac{4}{4^3} + \frac{8}{5^4} + \frac{16}{6^5} + \cdots$$

4.

$$\sum \sin\left(\frac{\pi}{4n^2}\right)$$

5.

$$\sum \frac{(2k+1)^{2k}}{(5k^2+1)^k}$$

6.

$$\sum \frac{1}{k}\left(\frac{1}{\ln k}\right)^{3/2}$$

7.

$$\sum \frac{(k!)^2}{(pk)!}, p \geq 2, p \in \mathbb{Z}$$

HINTS:

1. Comparison and Limit comparison test

2. $n-$th term divergence test

3. Geometric series

4. Comparison test
5. Root test and limit comparison test
6. integral test
7. Ratio test and limit comparison test

Problem 8

Suppose the three sides of an acute angled triangle $\triangle ABC$ are given by $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$, and $\overrightarrow{AB} = \vec{c}$ respectively. Express the following vectors in terms of $\vec{a}, \vec{b}$, and $\vec{c}$ only.

1. $\overrightarrow{AD}$ where D is the midpoint of $\overline{BC}$.
2. $\overrightarrow{AD}$ where D is the foot of the perpendicular from A to $\overline{BC}$.
3. $\star$ $\overrightarrow{AD}$ where D is the point in $\overline{BC}$ such that $\angle BAD = \angle DAC$.

   Part (3) is optional. Proceed as follows:

   (a) Show that if $\angle BAD = \angle DAC$, then $\frac{BD}{DC} = \frac{AB}{AC}$. Use sine law of traingles.

   (b) Use above relation to find $\overrightarrow{BD}$.

   (c) Find $\overrightarrow{AD}$ from $\overrightarrow{AB}$ and $\overrightarrow{BD}$.

Problem 9

Find

$$\int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} \int_x^y 2\, dz\, dy\, dx$$

Problem 10

Let

$$f(x,y) = \frac{x^3 + xy}{y\sqrt{4y - x^2}}$$

1. Find and describe the domain of the function $f(x,y)$. Also draw picture.
2. Show that

   $$\lim_{(x,y)\to(0,0)} f(x,y)$$

   does not exist by considering one path to the origin along the Y axis and another path along the parabola $y = x^2$.

[HINT: One of the limit is 0, another $2/\sqrt{3}$]

Problem 11⋆

1. Expand $f(x) = \ln(1+x)$ in powers of $(x-1)$ i.e. find the Taylor series at $x = 1$.
2. What is the radius of convergence of this series?

[HINT: It is somewhat hard to write down a closed form formula for the n-th term here. Compared to this, problem 12,13,14 are easier.]

Problem 12-15,17

Use Wolfram alpha to double check your answer.

Problem 12

Let $n \in \mathbb{N}$. Expand $(x-1)^n$ in powers of x. [HINT: i.e. find its Taylor series at 0.]

Problem 13

Expand $(1+2x)^{-4}$ in powers of x.

Problem 14

Expand $\sin(\pi x/2)$ in powers of $(x-1)$.

Problem 15

Find the radius of convergence for

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 3^k}(x+2)^k$$

[HINT: i.e. do Root/ratio test to find out for what values of x, this series is convergent.]

Problem 16

Suppose $\sum_{k=0}^{\infty} a_k(x+2)^k$ converges at $x = 4$. At what other values of x must the series converge? Does it necessarily converge at $x = -8$?

[HINT: This is not a calculation question. The answer here follows from the trichotomy of power series convergence discussed in class.]

Problem 17

Find the radius of convergence for the following series.

$$\sum \frac{e^k}{k^3}(x-1)^k$$

Problem 18

Find the area of the triangle $\triangle PQR$ where $P = (0, -2, 0), Q = (4, 1, -2)$, and $R = (5, 3, 1)$.

[HINT: Use cross products.]

Problem 19

Use scalar triple product to determine volume of the parallelepiped with adjacent edges AB, AC, AD where $A = (1, 0, 1), B = (2, 3, 0), C = (-1, 1, 4)$, and $D = (0, 3, 2)$.

In general if the triple product is zero, then the three vectors are coplanar.

Problem 20-27

Following problems are from the book. To double check your answer to the book problems use the link in the webpage to the solution manual or email me with specific question.

Problem 20

(Chapter 14 review problem 35) If $u = x^2y^3 + z^4$, where $p + 3p^2, y = pe^p$, and $z = p\sin p$, find $\frac{\partial u}{\partial p}$.

Problem 21⋆

(Chapter 14 review problem 41) If $z = f(u, v)$, where $u = xy$ and $v = y/x$, and f has continuous second partial derivatives, show that

$$x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} = -4uv\frac{\partial^2 z}{\partial u \partial v} + 2v\frac{\partial z}{\partial v}$$

[HINT: Use chain rule, multiple times.]

Problem 22

(Chapter 14 review problem 50) Find parametric equations of the tangent line at the point $(-2, 2, 4)$ to the curve of intersection of the surface $z = 2x^2 - y^2$ and the plane $z = 4$.

[HINT: The main thing to note is that the tangent line is parallel to XY plane (why?) and so it is enough to find the tangent vetor to the projection of the intersection curve to XY plane at $(-2,2)$]

[ALTERNATE SOLUTION: The tangent line that we are trying to find is perpendicular to the gradient of both the surface and the plane at (-2,2,4). So it's their cross product.]

Problem 23

(Chapter 14 review problem 52) Find local max/min/saddle points of

$$f(x,y) = x^3 - 6xy + 8y^3$$

Problem 24

(Chapter 14 review problem 60) Use Lagrange multipliers to find maximum and minimum of $f(x,y) = 1/x + 1/y$ subject to the constraint $1/x^2 + 1/y^2 = 1$.

Problem 25

Chapter 14 review problem 65.

[HINT: Take a and $b\,(b > a)$ as sides of the rectangle. Then sum of the top two sides is $b \sec\theta$. Use Lagrangian multipliers.]

Problem 26

Chapter 15 review problem $27, 45, 55, 57$.

[HINT: For 57, the 4 sides of the square tell you what the change of variable should be.]

Problem 27

Problems 15.9.$(24, 27)$.

[HINT: For 27, the 4 sides of the domain tell you what the change of variable should be.]

Problem 28-30

In all of the following LPP, first try to draw a picture. That tells you what vertices you need to consider. Use the algorithm only in case you don't want to or can't (maybe it's 3D) draw the picture. Please do NOT use Pivot method.

Problem 28

A calculator company produces a scientific calculator and a graphing calculator. Long-term projections indicate an expected demand of at least 100 scientific and 80 graphing calculators each day. Because of limitations on production capacity, no more than 200 scientific and 170 graphing calculators can be made daily. To satisfy a shipping contract, a total of at least 200 calculators much be shipped each day.

If each scientific calculator sold results in a \$ 2 loss, but each graphing calculator produces a \$ 5 profit, how many of each type should be made daily to maximize net profits?

[HINT: First write this as a LPP. The answer is 100 scientific calculators and 170 graphing calculators.]

Problem 29

A gold processor has two sources of gold ore, source A and source B. In order to keep his plant running, at least three tons of ore must be processed each day. Ore from source A costs \$20 per ton to process, and ore from source B costs \$10 per ton to process. Costs must be kept to less than \$80 per day. Moreover, Federal Regulations require that the amount of ore from source B cannot exceed twice the amount of ore from source A. If ore from source A yields 2 oz. of gold per ton, and ore from source B yields 3 oz. of gold per ton, how many tons of ore from both sources must be processed each day to maximize the amount of gold extracted subject to the above constraints?

[HINT: First write this as a LPP. Answer: the maximum yield of gold is 16oz. by processing 2 tons of ore from source A and 4 tons from source B.]

Problem 30

A publisher has orders for 600 copies of a certain text from San Francisco and 400 copies from Sacramento. The company has 700 copies in a warehouse in Novato and 800 copies in a warehouse in Lodi. It costs \$5 to ship a text from Novato to San Francisco, but it costs \$10 to ship it to Sacramento. It costs \$15 to ship a text from Lodi to San Francisco, but it costs \$4 to ship it from Lodi to Sacramento. How many copies should the company ship from each warehouse to San Francisco and Sacramento to fill the order at the least cost?

[HINT: The least cost of \$4600.]