Math 1800-C Handout 9: Summary of Chapter 8 - How to calculate line integrals

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§1. Summary of Chapter 8

We learned the following theorems in chapter 8.

Parametrized Curves: If the curve C can be parametrized as $\vec{\mathbf{r}}(t)$, $a \leq t \leq b$, then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\,dt$$

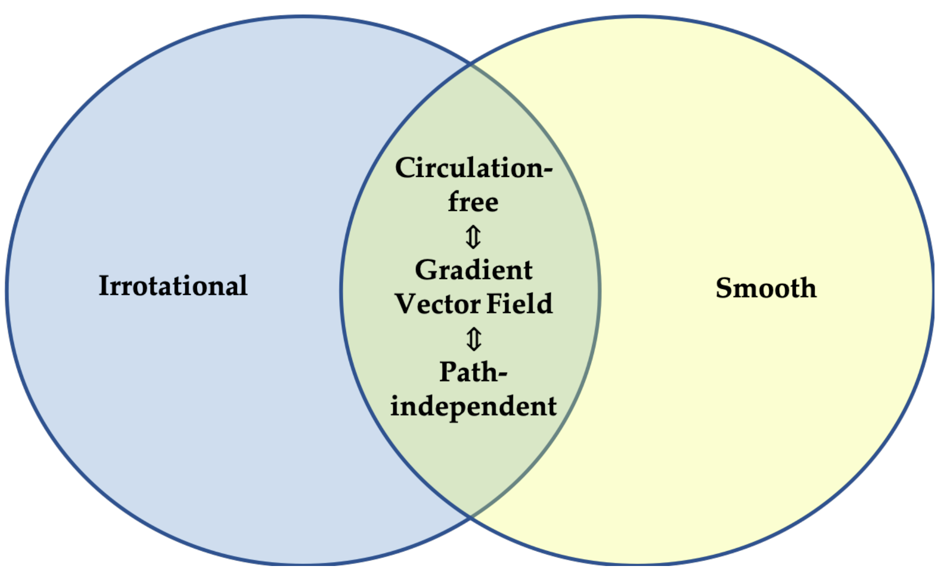
Fundamental Theorem of Line Integrals: If the vector field $\vec{\mathbf{F}}$ is a gradient vector field i.e. $\vec{\mathbf{F}} = \nabla f$, and the curve C starts at P and ends at Q, then

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \nabla f \cdot d\vec{\mathbf{r}} = f(Q) - f(P)$$

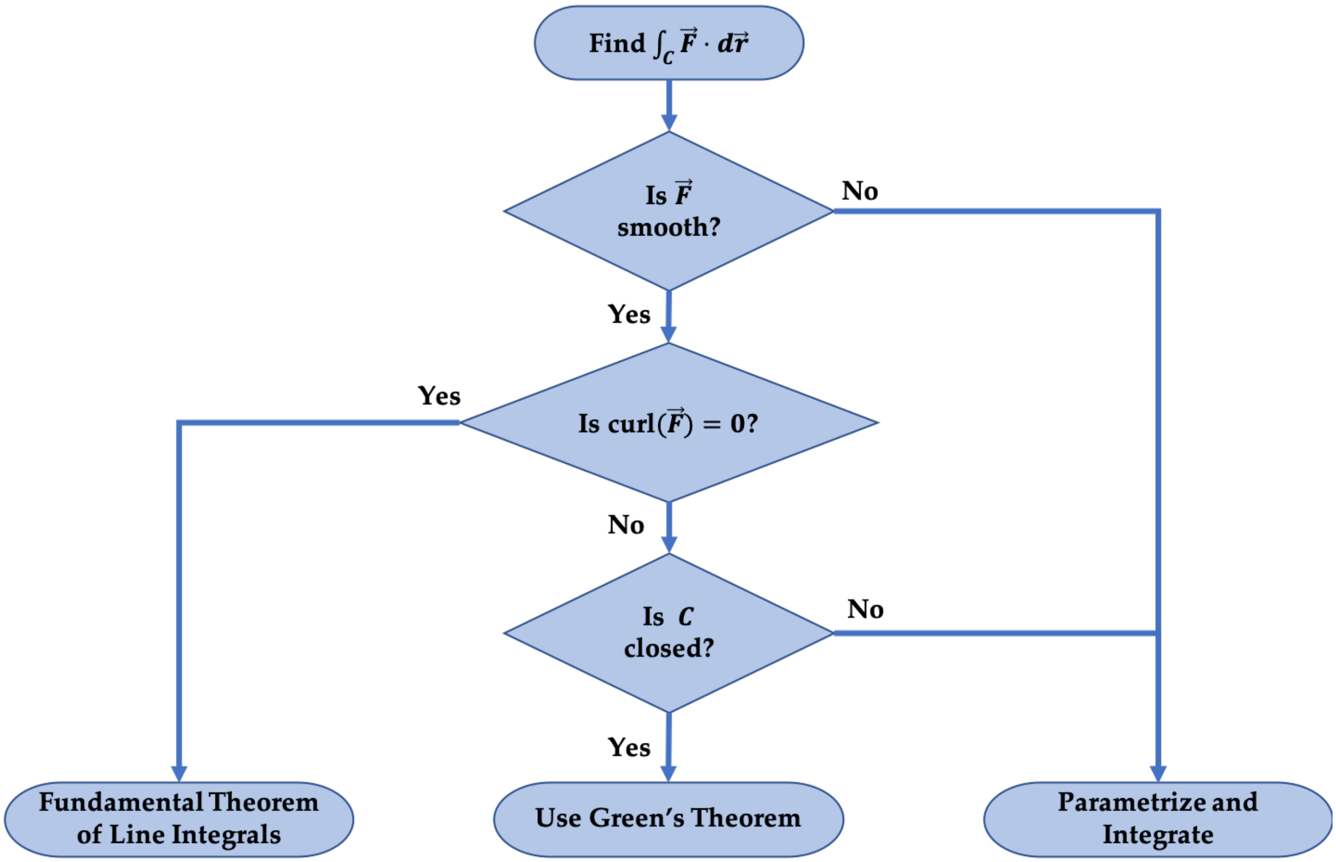
Green's Theorem: If C is a *simple, closed, oriented* curve and the vector field $\vec{\mathbf{F}}$ is *smooth* over the simply-connected region R enclosed by C (oriented so that R is always to the left of C), then

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_R \mathbf{curl}\,\vec{\mathbf{F}}\,dA$$

§2. Vector Fields Venn Diagram



§3. Calculating Line Integral - a flowchart



§4. Practice Problems

■ **Exercise 1.**

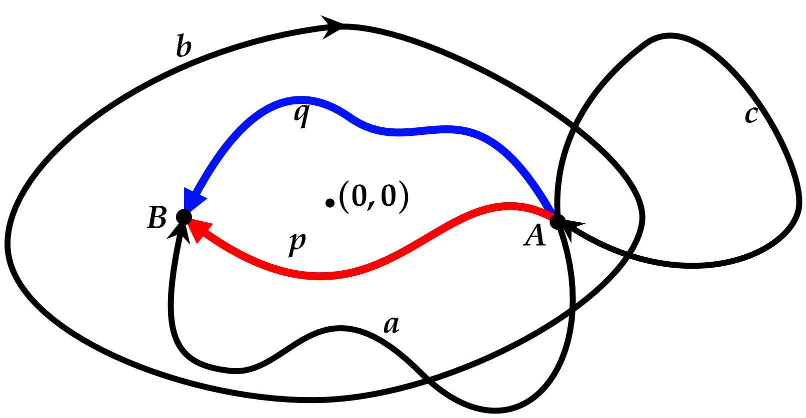


Figure 1

Suppose $\vec{\mathbf{F}}$ is an irrotational vector field in the plane (that is, its curl is everywhere zero) that is not defined at the origin $O = (0, 0)$. See figure 1. Suppose the line integral of $\vec{\mathbf{F}}$ along the path p from A to B is 5 and the line integral of $\vec{\mathbf{F}}$ along the path q from A to B is -4. Find the line integral of F along the paths a, b and c.

■ Exercise 2.

Evaluate the following line integrals.

(a) $\oint_C y\,dx + \left(x + y^2\right) dy$ where C is the ellipse $4x^2 + 9y^2 = 36$ with counterclockwise orientation.

(b) $\int \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x, y) = xy\,\vec{\mathbf{i}} + x^2\,\vec{\mathbf{j}}$ and C is given by $\vec{\mathbf{r}}(t) = \sin t\,\vec{\mathbf{i}} + (1 + t)\,\vec{\mathbf{j}}$, $0 \le t \le \pi$

(c) $\int \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x, y) = \left(4x^3y^2 - 2xy^3\right)\vec{\mathbf{i}} + \left(2x^4y - 3x^2y^2 + 4y^3\right)\vec{\mathbf{j}}$ and C is given by $\vec{\mathbf{r}}(t) = (t + \sin \pi t)\,\vec{\mathbf{i}} + (2t + \cos \pi t)\,\vec{\mathbf{j}}$, $0 \le t \le 1$

(d) $\int \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x, y, z) = \sin y\,\vec{\mathbf{i}} + x \cos y\,\vec{\mathbf{j}} - \sin z\,\vec{\mathbf{k}}$, and C is the helix $x = 3\cos t, y = t, z = 3\sin t$ from $(3, 0, 0)$ to $(0, \pi/2, 3)$

(e) $\oint_C \sqrt{1 + x^3}dx + 2xydy$ where C is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 3)$

(f) $\int \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x, y, z) = \left(3x^2yz - 3y\right)\vec{\mathbf{i}} + \left(x^3z - 3x\right)\vec{\mathbf{j}} + \left(x^3y + 2z\right)\vec{\mathbf{k}}$ and C is the curve shown in figure 2.

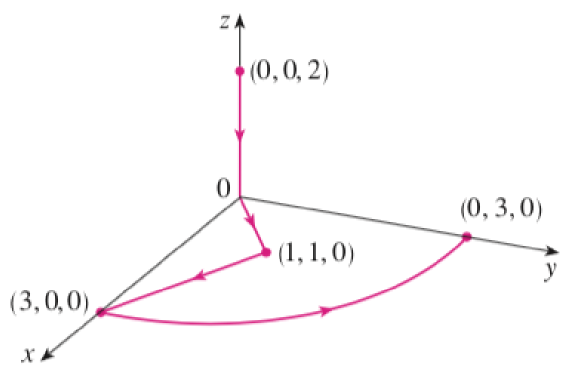


Figure 2

(g) $\int \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where

$$\vec{\mathbf{F}}(x, y, z) = \left\langle 4xe^{2x^2+3y^2+4z^2}, 6ye^{2x^2+3y^2+4z^2}, 8ze^{2x^2+3y^2+4z^2} \right\rangle$$

and C is

$$\vec{\mathbf{r}}(t) = \langle (2 + \cos(7t))\cos(t), (2 + \cos(7t))\sin(t), \sin(7t) \rangle$$

parametrized by $0 \le t \le \pi$ starting at $t = 0$ and ending at $t = \pi$.

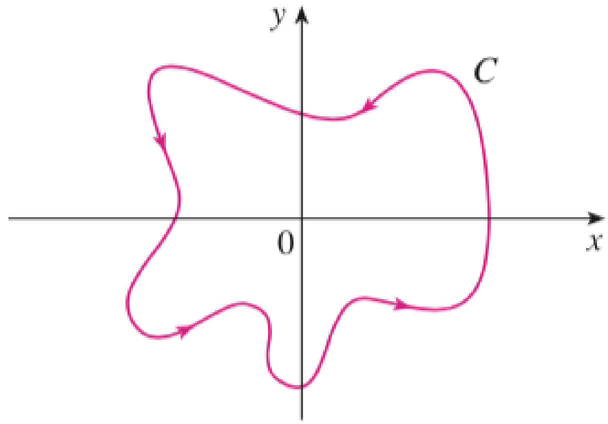


Figure 3

(h) $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y) = \frac{(2x^3+2xy^2-2y)\,\vec{\mathbf{i}}+(2y^3+2x^2y+2x)\,\vec{\mathbf{j}}}{x^2+y^2}$ and C is the curve shown in figure 3.

(i) $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y,z) = \langle y\cos x - xy\sin x, xy + x\cos x\rangle$, and C is the triangle from $(0,0)$ to $(0,4)$ to $(2,0)$ to $(0,0)$.

(j) $\int \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}}(x,y) = xy^2\,\vec{\mathbf{i}} + x^2y\,\vec{\mathbf{j}}$, and C is $\vec{\mathbf{r}}(t) = \cos t\,\vec{\mathbf{i}} + 2\sin t\,\vec{\mathbf{j}}, \quad 0 \le t \le \pi/2$

(k) $\int \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, where $\vec{F}(x,y,z) = \langle yz + x^2, xz + y^2 + \sin(y), xy + \cos(z)\rangle$, and $\vec{\mathbf{r}}(t) = \langle\cos(20t)\sin(t), \sin(20t)\sin(t), \cos(t)\rangle$ with $0 \le t \le \pi$

■ Exercise 3.

(a) A 160 lb man carries a 25 lb can of paint up a helical staircase that encircles a silo with a radius of 20 ft. If the silo is 90 ft high and the man makes exactly three complete revolutions climbing to the top, how much work is done by the man against gravity?

(b) Suppose there is a hole in the can of paint and 9 lb of paint leaks steadily out of the can during the mans ascent. How much work is done?

■ Exercise 4.

Compute the line integral of the vector field

$$\vec{\mathbf{F}}(x,y,z) = \left\langle \cos(x), 2 + \cos(y), e^z + x\left(y^2 + z^2\right)\right\rangle$$

along the curve

$$\vec{\mathbf{r}}(t) = \langle t, \cos(t), \sin(t)\rangle \text{ with } 0 \le t \le 3\pi.$$

HINT: Write $\vec{\mathbf{F}}$ as $\vec{\mathbf{G}} + \vec{\mathbf{H}}$ where $\vec{\mathbf{G}}$ is a gradient vector field. Then do the two integrals separately.

■ Exercise 5.

Look at the shaded region G bounded by a circle of radius 2 and an inner *figure eight lemniscate* (see figure 4) with parametric equation

$$\vec{\mathbf{r}}(t) = \langle \sin(t), \sin(t)\cos(t) \rangle$$

with $0 \le t \le 2\pi$. The picture shows the curve and the arrows indicate some of the velocity vectors of the curve. Find the area of this region G.

[Hint: Use Green's theorem and the vector field $x\hat{j}$.]

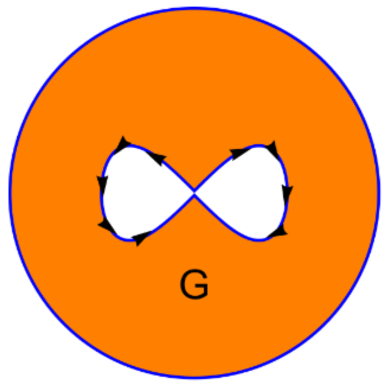


Figure 4

■ Exercise 6.

What is the line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ of the vector field

$$\vec{\mathbf{F}}(x, y) = \langle 1 + y + 2xy, y^2 + x^2 \rangle$$

along the boundary C of the planar castle region shown in the picture 5. Each of the 5 windows is a unit square and the base of the castle has length 9. The boundary consists of 6 curves which are all oriented so that the region is to the left.

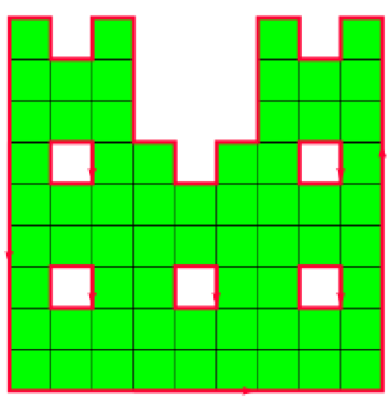


Figure 5

■ Exercise 7.

Let C be the boundary curve of the white Yang part of the Ying-Yang symbol in the disc of radius $\mathbf{6}$. You can see in figure 6 that the curve C has three parts, and that the orientation of each part is given. Find the line integral of the vector field

$$\vec{F}(x, y) = \langle -y + \sin\left(e^x\right), x \rangle$$

around C. Notice that the Ying and the Yang have the same area.

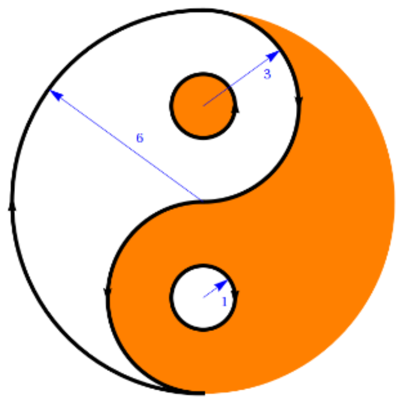


Figure 6