A Few Discrete Random Variables

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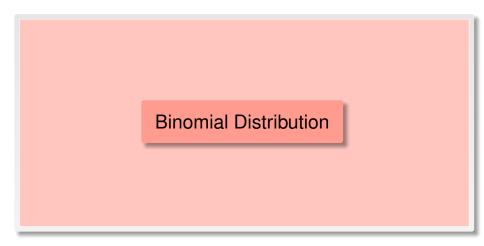
Department of Statistics



- **Binomial Distribution**



A Few



- A quality controlling team in TV manufacturer evaluates a products before they are send to market. They are interested in idensifying the number of defective products and the resources that should be optimally allocated to mitigate the defectives items.
- A broker in a stock exchange speculates how many shares out of 10 newly introduced shares will go up next day.
- A airline company is interested in identifying the number of last minute cancellations that may take place.
- A car insurance company have sold 2000 insurance policies on a specific new cars. They need to estimate the funts that should be made available/reserved to compensate the losses that may occur to the insured cars during the next one year.

Binomial Distribution Poisson Distribution Geometric Distribu

### A Bernoulli Trial/ Experiment

- The random experiment has only two outcomes. Namely SUCCESS, and FAILURE
- Events corresponding to the successive trials/experiemnts are statistically independent.
- such trials/experiements the have same chance/probability of success.

If a sequence of *n* independent Bernoulli trials is performed under the same condition, then the random variable that records the total number of successes is called the Binomial Random variable.

### Binomial Distribution

Binomial Distribution

#### Definition (Binomial Experiment)

An experiment is called a Binomial experiment if it satisfies the following 4 conditions:

- The experiment consists of n Bernoulli trials.
- Each trial results in a success (S) or a failure (F).
- The trials are independent.
- The probability of a success,  $\pi$ , is fixed throughout *n* trials.

# Binomial Distribution Binomial $(n, \pi)$

- Given a Binomial experiment consisting of n Bernoulli trials with success probability  $\pi$ , the Binomial random variable X associated with this experiment is defined as the number of successes among the n trials.
- The random variable X has the Binomial Distribution with parameters n and  $\pi$ ; denoted by  $X \sim Binomial(n, \pi)$ .
- The behavior of Binomial Distribution with different n and  $\pi$ .

# Binomial Distribution Binomial $(n, \pi)$

#### **Definition** (Binomial Distribution)

Let  $\pi \in (0,1)$ , then the probability mass function of Binomial $(n,\pi)$  is given by

$$p(x) := {n \choose x} \pi^x (1-\pi)^{n-x}$$
, for  $x \in \mathbb{S}_x$ , where  $\mathbb{S}_x = \{0, 1, \dots, n\}$ 

for 
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, where  $\mathbb{S}_{x} = \{0, 1, \dots, n\}$ 

Mean

$$E(X) = n\pi$$

Variance 
$$VAR(X) = n\pi(1 - \pi)$$

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### Expected Value of Binomial Distribution

$$E(X) := \sum_{y \in \mathbb{S}_{X}} y \, \rho_{X}(y)$$

$$= \sum_{y=0}^{n} y \binom{n}{y} \pi^{y} (1-\pi)^{n-y}$$

$$= (1-\pi)^{n} \sum_{y=0}^{n} y \binom{n}{y} \left(\frac{\pi}{1-\pi}\right)^{y}$$

$$= (1-\pi)^{n} \frac{np}{(1-\pi)^{n}}$$

$$= np \qquad (1)$$

$$E(X^{2}) := \sum_{y \in \mathbb{S}_{X}} y^{2} p_{X}(y)$$

$$= \sum_{y=0}^{n} y^{2} {n \choose y} \pi^{y} (1-\pi)^{n-y}$$

$$= (1-\pi)^{n} \sum_{y=0}^{n} y^{2} {n \choose y} \left(\frac{\pi}{1-\pi}\right)^{y}$$

$$= (1-\pi)^{n} \frac{mp + n(n-1)\pi^{2}}{(1-\pi)^{n}}$$

$$= np + n(n-1)\pi^{2}$$
 (2)

 $Var(X) = E(x^2) - (E(X))^2 = np + n(n-1)\pi^2 - n^2p^2 = np - np^2 = np - np^2$  $np(1-\pi).$ 

### Expected Value of Binomial Distribution

$$M_{X}(t) := \sum_{y \in \mathbb{S}_{X}} e^{ty} \, \rho_{X}(y)$$

$$= (1 - \pi)^{n} \sum_{y=0}^{n} e^{ty} \binom{n}{y} \left(\frac{\pi}{1 - \pi}\right)^{y}$$

$$= (1 - \pi)^{n} \sum_{y=0}^{n} \binom{n}{y} \left(\frac{pe^{t}}{1 - \pi}\right)^{y}$$

$$= (1 - \pi)^{n} \sum_{y=0}^{n} e^{ty} \binom{n}{y} \left(\frac{\pi}{1 - \pi}\right)^{y}$$

$$= (1 - \pi)^{n} \left(1 + \frac{pe^{t}}{1 - \pi}\right)^{n} = (1 - \pi + pe^{t})^{n}$$

(3)

A Few

Poisson Distribution Geometric Distribu

### Example

Example: Five fair coins are flipped. If the outcomes are assumed independent.

- Find the probability mass function of the number of heads obtained.
- Find the probability that at least 3 heads are obtained.
- Find the probability that at most 2 heads are obtained.

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**Solution:** Let  $X = \text{The number of heads in 5 tossed coins. } X \sim Binomial(n = 5, \pi = 0.5).$ 

- $P(X=0) = 0.5^5 = 0.0313$
- $P(X = 1) = {5 \choose 1} 0.5^5 = 0.1563$
- $P(X=2) = \binom{5}{2} 0.5^5 = 0.3125$
- $P(X=3) = {5 \choose 3} 0.5^5 = 0.3125$
- $P(X = 4) = {5 \choose 4} 0.5^5 = 0.1563$
- $P(X=0) = {5 \choose 5} 0.5^5 = 0.0313$



Binomial Distribution Poisson Distribution

### Example

Example: It is known that screws produced by a certain company will be defective with probability .01, independently of each other. The company sells the screws in packages of 10 and offers a money-back guarantee that at most 1 of the 10 screws is defective. What proportion of packages sold must the company replace? Use the Binomial Calculator or Statistical Tables.

# Example

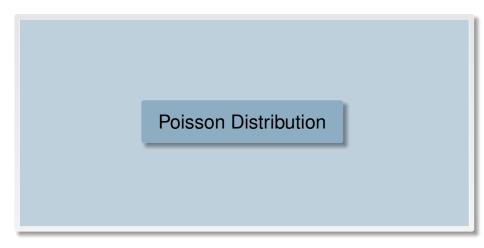
Example: The following gambling game, known as the wheel of fortune (or chuck-a-luck), is quite popular at many carnivals and gambling casinos: A player bets on one of the numbers 1 through 6. Three dice are then rolled, and if the number bet by the player appears i times, i = 1; 2; 3, then the player wins i units; if the number bet by the player does not appear on any of the dice, then the player loses 1 unit. Is this game fair to the player?

### Outline

- Poisson Distribution



A Few



### Poisson Distribution

Binomial Distribution

The Poisson distribution models the number of occurrences of an event when there is a known average rate per unit time or space  $\lambda$ .

#### Definition (Poisson Distribution)

The requirements for a Poisson distribution are that:

- no two events can occur simultaneously,
- events occur independently in different intervals, and
- the expected number of events in each time interval remain constant.

# Poisson Distribution: pmf, Expected Value

The Poisson distribution models the number of occurrences of an event when there is a known average rate per unit time or space  $\lambda$ .

#### Definition (Poisson Distribution: pmf, Expected Value)

The requirements for a Poisson distribution are that:

The probability mass function of Poisson( $\lambda$ ) is given by

$$p(x) = \frac{e^{-\lambda} \lambda^x}{x!}$$
 for  $x = 0, 1, 2, 3, ...$ 

If  $X \sim Poisson(\lambda)$ , then  $E(X) = \lambda$ , and  $Var(X) = \lambda$ .

### **Expected Value of Binomial Distribution**

$$M_{X}(t) := \sum_{y \in \mathbb{S}_{X}} e^{ty} \, \rho_{X}(y)$$

$$= \sum_{y=0}^{\infty} e^{ty} \frac{e^{-\lambda} \lambda^{y}}{y!}$$

$$= e^{-\lambda} \sum_{y=0}^{\infty} \frac{(\lambda e^{t})^{y}}{y!}$$

$$= e^{\lambda e^{t} - \lambda}$$
(4)

Example: The number of customers arriving at a service counter within one-hour period.

Example: The number of typographical errors in a book counted per page.

Example: The number of email messages received at the technical support center daily.

Example: The number of traffic accidents that occur on a specific road during a month.

## A Few Examples of Poisson Distribution

Example: Messages arrive at an electronic message center at random times, with an average of 9 messages per hour.

- What is the probability of receiving exactly five messages during the next hour?
- What is the probability that more than 10 messages will be received within the next two hours?

- The number of messages received in an hour, X is modeled by Poisson distribution with  $\lambda = 9$ , i.e.  $X \sim \text{Poisson}(9)$ .  $P(X = 5) = \frac{9^5 \exp(-9)}{10^{10}}$
- The number of messages received within a 2-hour period, Y is another Poisson distribution with Y=(2)(9)=18, i.e.  $Y\sim \text{Poisson}(18)$ .  $P(Y>10)=1-P(Y\leq 10)=\ldots=0.9696$



# Group Work

- Develop a real life example in which you can easily apply:
  - Group 1: Poisson distribution.
    - Group 2: Binomial distribution.
    - Group 3: Poisson distribution
- In each case, propose two problems which can be solved using the Statistical Calculator.
- Can you propose an idea in which you can mix both distributions? (extra)

#### Outline

- Geometric Distribution



A Few

#### $\bigcirc$ Suppose that independent trials, each having a probability $\pi$ , $0 < \pi < 1$ , of being a success, are performed until a success occurs.

- Example: The first head in tossing coin several times.
- Then. Geometric distribution models the number of trials performed until a success occurs.

#### Definition (Geometric Distribution)

The probability mass function of  $Geometric(\pi)$  is given by

$$p(x) = (1 - \pi)^{x-1}\pi$$
 for  $x = 1, 2, 3, ...,$ 



$$M_X(t) := \sum_{y \in \mathbb{S}_X} e^{ty} \, \rho_X(y)$$

$$= \sum_{y=1}^{\infty} e^{ty} (1-\pi)^{y-1} \pi$$

$$= \pi \sum_{z=0}^{\infty} e^{tz+t} (1-\pi)^z$$

$$= \rho e^t \sum_{z=0}^{\infty} ((1-\pi)e^t)^z$$

$$= \frac{\rho e^t}{1-(1-\pi)e^t}$$
 (5)

### Geometric Distribution: Example

Example: Suppose that the probability of engine malfunction during any one-hour period is  $\pi = 0.02$ . Find the probability that a given engine will survive two hours.

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#### Solution:

Letting Y denote the number of one-hour intervals until the first malfunction, we have

$$P(\text{Survival for Next Two Hours}) \\ = P(Y \ge 3) \\ = 1 - P(Y \le 2) \\ = 1 - \sum_{y=1}^{2} p(y) \\ = 1 - \{p(1) + p(2)\} \\ = 1 - 0.02 - 0.98 \times 0.02 \\ = 0.9604$$

Exercise Find the mean and standard deviation of Y.

- **Negative Binomial Distribution**



A Few





- $\bigcirc$  Suppose that independent trials, each having probability  $\pi$ ,  $0 < \pi < 1$ , of being a success are performed until a total of r successes is accumulated.
- Example: The third head in tossing coin several times.
- Then, Negative Binomial distribution models the number of trials performed until a the rth success occurs.

#### Definition (Negative Binomial Distribution)

The probability mass function of Negative Binomial RV, denoted by Negative-Binomial $(r, \pi)$  is given by

$$p(x) = {x-1 \choose r-1} \pi^{r-1} (1-\pi)^{x-r} \text{ for } x = r+1, r+2, r+3, \dots,$$

If  $X \sim \text{Negative-Binomial}(r, \pi)$  then  $E(X) = \frac{r}{\pi}$ , and  $Var(X) = \frac{r}{\pi}$ 

### Geometric Distribution: Example

Example: A machine produces 1% defective parts. Using the statistical calculator, calculate the probability that

- 10 parts have to be selected until to get 2 defective parts.
- Between 20 to 25 parts have to be selected to get 2 defective parts.

### Geometric Distribution: Example

Example: A machine produces 1% defective parts. Using the statistical calculator, calculate the probability that

- 10 parts have to be selected until to get 2 defective parts.
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Solution: Letting Y denote the number of

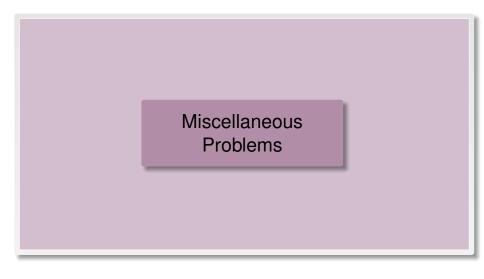
> AA(6)

Exercise Find the mean and standard deviation of Y.

- Miscellaneous Problems



A Few





Example: Approximately 10% of the glass bottles coming off a production line have serious flaws in the glass. If two bottles are randomly selected, find the mean and variance of the number of bottles that have serious flaws.

Example: Cars arrive at a toll both according to a Poisson process with mean 80 cars per hour. If the attendant makes a oneminute phone call, what is the probability that at least 1 car arrives during the call?

Example: Suppose that a lot of electrical fuses contains 5% defectives. If a sample of 5 fuses is tested, find the probability of observing at least one defective.

Example: An oil exploration firm is formed with enough capital to finance ten explorations. The probability of a particular exploration being successful is 0.10. Assume the explorations are independent.

- Find the mean and variance of the number of successful explorations.
- Suppose the firm has a fixed cost of \$20,000 in preparing equipment prior to doing its first exploration. If each successful exploration costs \$30,000 and each unsuccessful exploration costs \$15,000, find the expected total cost to the firm for its ten explorations.

Example: A food manufacturer uses an extruder (a machine that produces bite-size cookies and snack food) that yields revenue for the firm at a rate of \$200 per hour when in operation. However, the extruder breaks down an average of two times every day it operates. If Y denotes the number of breakdowns per day, the daily revenue generated by the machine is  $R = 1600 - 50Y^2$ . Find the expected daily revenue for the extruder.

Example: A particular sale involves four items randomly selected from a large lot that is known to contain 10% defectives. Let Y denote the number of defectives among the 20 sold. The purchaser of the items will return the defectives for repair, and the repair cost is given by  $Cost = 3Y^2 + Y + 2$ . Find the expected repair cost.

Example: In a certain population, it is known that 80% of the individuals have the Rhesus (Rh) factor present in their blood.

- If 5 volunteers are randomly selected from the population, what is the probability that at least one does not have the Rh factor?
- If five volunteers are randomly selected, what is the probability that at most four have the Rh factor?

### Example: appears.

Consider rolling a fair dice multiple times untill the first 6

- Find the expected number of throws required to get the first 6.
- What is the probability that more then 8 throws are required to obtain the first 6?

