STAT 320: Principles of Probability

Unit 5(PART:B)

A Few Discrete Random Variables

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Outline

Binomial Distribution

Binomial Distribution

- A **Bernoulli experiment** is a random experiment, the outcome of which can be classified in one of two mutually exclusive and exhaustive ways, say, “1=success" or “0=failure." Let Y be the number of success on a Bernoulli trial, then Y is called the Bernoulli random variable.

- If a sequence of n independent Bernoulli trials is performed under the same condition, we call it a set of n Bernoulli trials a Binomial experiment.

Binomial Distribution

Definition (Binomial Experiment)

An experiment is called a Binomial experiment if it satisfies the following 4 conditions:

- The experiment consists of n Bernoulli trials.
- Each trial results in a success (S) or a failure (F).
- The trials are independent.
- The probability of a success, p, is fixed throughout n trials.

Binomial Distribution Binomial(n, p)

1. Given a Binomial experiment consisting of n Bernoulli trials with success probability p, the Binomial random variable X associated with this experiment is defined as the number of successes among the n trials.
2. The random variable X has the Binomial Distribution with parameters n and p; denoted by$X \sim Binomial(n, p)$.
3. The behavior of Binomial Distribution with different n and p.

Binomial Distribution Binomial(n, p)

Definition (Binomial Distribution)

Let $p \in (0, 1)$, then the probability mass function of Binomial(n, p) is given by

$$\boxed{p(x) := \binom{n}{x} p^x (1-p)^{n-x}}, \text{ for } x \in \mathbb{S}_X, \text{ where } \mathbb{S}_X = \{0, 1, \ldots, n\}$$

Mean

$E(X) = np$

Variance

$VAR(X) = np(1-p)$

Expected Value of Binomial Distribution

$$
\begin{aligned}
E(X) &:= \sum_{y \in \mathbb{S}_X} y \, p_x(y) \\
&= \sum_{y=0}^{n} y \binom{n}{y} p^y (1-p)^{n-y} \\
&= (1-p)^n \sum_{y=0}^{n} y \binom{n}{y} \left(\frac{p}{1-p} \right)^y \\
&= (1-p)^n \frac{np}{(1-p)^n} \\
&= np
\end{aligned} \qquad (1)
$$

Expected Value of Binomial Distribution

$$
\begin{aligned}
E(X^2) &:= \sum_{y \in \mathbb{S}_X} y^2 \, p_X(y) \\
&= \sum_{y=0}^{n} y^2 \binom{n}{y} p^y (1-p)^{n-y} \\
&= (1-p)^n \sum_{y=0}^{n} y^2 \binom{n}{y} \left(\frac{p}{1-p}\right)^y \\
&= (1-p)^n \frac{mp + n(n-1)p^2}{(1-p)^n} \\
&= np + n(n-1)p^2
\end{aligned}
\tag{2}
$$

$Var(X) = E(x^2) - (E(X))^2 = np + n(n-1)p^2 - n^2p^2 = np - np^2 = np(1-p)$.

Expected Value of Binomial Distribution

$$
\begin{aligned}
M_X(t) &:= \sum_{y \in \mathbb{S}_X} e^{ty}\, p_x(y) \\
&= (1-p)^n \sum_{y=0}^{n} e^{ty} \binom{n}{y} \left(\frac{p}{1-p}\right)^y \\
&= (1-p)^n \sum_{y=0}^{n} \binom{n}{y} \left(\frac{pe^t}{1-p}\right)^y \\
&= (1-p)^n \sum_{y=0}^{n} e^{ty} \binom{n}{y} \left(\frac{p}{1-p}\right)^y \\
&= (1-p)^n \left(1 + \frac{pe^t}{1-p}\right)^n = \left(1 - p + pe^t\right)^n
\end{aligned} \tag{3}
$$

Example

Example : Five fair coins are flipped. If the outcomes are assumed independent.

1. Find the probability mass function of the number of heads obtained.
2. Find the probability that at least 3 heads are obtained.
3. Find the probability that at most 2 heads are obtained.

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Solution: Let X = The number of heads in 5 tossed coins. $X \sim Binomial(n = 5, p = 0.5)$.

1. $P(X = 0) = 0.5^5 = 0.0313$
2. $P(X = 1) = \binom{5}{1}0.5^5 = 0.1563$
3. $P(X = 2) = \binom{5}{2}0.5^5 = 0.3125$
4. $P(X = 3) = \binom{5}{3}0.5^5 = 0.3125$
5. $P(X = 4) = \binom{5}{4}0.5^5 = 0.1563$
6. $P(X = 0) = \binom{5}{5}0.5^5 = 0.0313$

Example

Example : It is known that screws produced by a certain company will be defective with probability .01, independently of each other. The company sells the screws in packages of 10 and offers a money-back guarantee that at most 1 of the 10 screws is defective. What proportion of packages sold must the company replace? Use the Binomial Calculator or Statistical Tables.

Example

Example : The following gambling game, known as the wheel of fortune (or chuck-a-luck), is quite popular at many carnivals and gambling casinos: A player bets on one of the numbers 1 through 6. Three dice are then rolled, and if the number bet by the player appears i times, i = 1; 2; 3, then the player wins i units; if the number bet by the player does not appear on any of the dice, then the player loses 1 unit. Is this game fair to the player?

Outline

Poisson Distribution

Poisson Distribution

The Poisson distribution models the number of occurrences of an event when there is a known average rate per unit time or space λ.

Definition (Poisson Distribution)

The requirements for a Poisson distribution are that:

1. no two events can occur simultaneously,
2. events occur independently in different intervals, and
3. the expected number of events in each time interval remain constant.

Poisson Distribution: pmf, Expected Value

The Poisson distribution models the number of occurrences of an event when there is a known average rate per unit time or space λ.

Definition (Poisson Distribution: pmf, Expected Value)

The requirements for a Poisson distribution are that:

1. The probability mass function of Poisson(λ) is given by

$$p(x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \text{ for } x = 0, 1, 2, 3, \ldots$$

2. If $X \sim Poisson(\lambda)$, then $E(X) = \lambda$, and $\text{Var}(X) = \lambda$.

Expected Value of Binomial Distribution

$$
\begin{aligned}
M_X(t) &:= \sum_{y \in \mathbb{S}_X} e^{ty} \, p_x(y) \\
&= \sum_{y=0}^{\infty} e^{ty} \frac{e^{-\lambda} \lambda^y}{y!} \\
&= e^{-\lambda} \sum_{y=0}^{\infty} \frac{\left(\lambda e^t\right)^y}{y!} \\
&= e^{\lambda e^t - \lambda}
\end{aligned} \tag{4}
$$

A few Examples of Poisson Distribution

Example : The number of customers arriving at a service counter within one-hour period.

Example : The number of typographical errors in a book counted per page.

Example : The number of email messages received at the technical support center daily.

Example : The number of traffic accidents that occur on a specific road during a month.

A Few Examples of Poisson Distribution

Example : Messages arrive at an electronic message center at random times, with an average of 9 messages per hour.

1. What is the probability of receiving exactly five messages during the next hour?
2. What is the probability that more than 10 messages will be received within the next two hours?

1. The number of messages received in an hour, X is modeled by Poisson distribution with $\lambda = 9$, i.e. $X \sim \text{Poisson}(9)$. $P(X = 5) = \frac{9^5 \exp(-9)}{5!}$
2. The number of messages received within a 2-hour period, Y is another Poisson distribution with $Y = (2)(9) = 18$, i.e. $Y \sim \text{Poisson}(18)$. $P(Y > 10) = 1 - P(Y \leq 10) = \ldots = 0.9696$

Group Work

1. Develop a real life example in which you can easily apply:
    1. Group 1: Poisson distribution.
    2. Group 2: Binomial distribution.
    3. Group 3: Poisson distribution
2. In each case, propose two problems which can be solved using the Statistical Calculator.
3. Can you propose an idea in which you can mix both distributions? (extra)

Outline

Geometric Distribution

Geometric Distribution

1. Suppose that independent trials, each having a probability p, $0 < p < 1$, of being a success, are performed until a success occurs.
2. Example: The first head in tossing coin several times.
3. Then, Geometric distribution models the number of trials performed until a success occurs.

Definition (Geometric Distribution)

The probability mass function of *Geometric*(p) is given by

$$p(x) = (1-p)^{x-1} p \text{ for } x = 1, 2, 3, \ldots,$$

- If $X \sim \text{Geometric}(p)$ then $E(X) = \frac{1}{p}$, and $\text{Var}(X) = \frac{1-p}{p^2}$

Expected Value of Binomial Distribution

$$
\begin{aligned}
M_X(t) &:= \sum_{y \in \mathbb{S}_X} e^{ty}\, p_x(y) \\
&= \sum_{y=1}^{\infty} e^{ty} (1-p)^{y-1} p \\
&= p \sum_{z=0}^{\infty} e^{tz+t} (1-p)^z \\
&= pe^t \sum_{z=0}^{\infty} \left((1-p)e^t\right)^z \\
&= \frac{pe^t}{1-(1-p)e^t} \qquad (5)
\end{aligned}
$$

Geometric Distribution: Example

Example : Suppose that the probability of engine malfunction during any one-hour period is $p = 0.02$. Find the probability that a given engine will survive two hours.

Geometric Distribution: Example

Example : Suppose that the probability of engine malfunction during any one-hour period is p = 0.02. Find the probability that a given engine will survive two hours.

Solution:
Letting Y denote the number of one-hour intervals until the first malfunction, we have

$$
\begin{aligned}
& P(\text{Survival for Next Two Hours}) \\
= \ & P(Y \geq 3) \\
= \ & 1 - P(Y \leq 2) \\
= \ & 1 - \sum_{y=1}^{2} p(y) \\
= \ & 1 - \{p(1) + p(2)\} \\
= \ & 1 - 0.02 - 0.98 \times 0.02 \\
= \ & 0.9604
\end{aligned}
$$

Exercise Find the mean and standard deviation of Y.

Outline

Negative Binomial Distribution

1. Suppose that independent trials, each having probability p, $0 < p < 1$, of being a success are performed until a total of r successes is accumulated.
2. Example: The third head in tossing coin several times.
3. Then, Negative Binomial distribution models the number of trials performed until a the rth success occurs.

Definition (Negative Binomial Distribution)

The probability mass function of Negative Binomial RV, denoted by Negative-Binomial(r, p) is given by

$$p(x) = \binom{x-1}{r-1} p^{r-1}(1-p)^{x-r} \text{ for } x = r+1, r+2, r+3, \ldots,$$

- If $X \sim$ Negative-Binomial(r, p) then $E(X) = \frac{r}{p}$, and $\text{Var}(X) = \frac{r(1-p)}{p^2}$

Geometric Distribution: Example

Example : A machine produces 1% defective parts. Using the statistical calculator, calculate the probability that

1. 10 parts have to be selected until to get 2 defective parts.
2. Between 20 to 25 parts have to be selected to get 2 defective parts.

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Solution:
Letting Y denote the number of

$$AA \tag{6}$$

Exercise Find the mean and standard deviation of Y.

Outline

Miscellaneous Problems

Example : Approximately 10% of the glass bottles coming off a production line have serious flaws in the glass. If two bottles are randomly selected, find the mean and variance of the number of bottles that have serious flaws.

Example : Suppose that a lot of electrical fuses contains 5% defectives. If a sample of 5 fuses is tested, find the probability of observing at least one defective.

Example : An oil exploration firm is formed with enough capital to finance ten explorations. The probability of a particular exploration being successful is 0.10 . Assume the explorations are independent.

1. Find the mean and variance of the number of successful explorations.
2. Suppose the firm has a fixed cost of \$20,000 in preparing equipment prior to doing its first exploration. If each successful exploration costs \$30,000 and each unsuccessful exploration costs \$15,000, find the expected total cost to the firm for its ten explorations.

Questions?