Exam 1

Probability and Statistics 2022

Indian Institute of Management, Udaipur

16th August, 2022

Name:

- There are a total of 115 points in this Question Paper. Answer as much as you can. If your acquired score is greater than equal to 100, it will be counted as 100%.
- There are three parts in this Exam. Part-I involves TRUE/FALSE type, Part-II pertains to the short type and Part-III contains descriptive type questions.
- The Exam is scheduled for 3 hours and 15 minutes.
- There are three $\star$ marked problems that are more involved than the rest. In case you are stuck in one of those, you may consider solving other problems first and then continue with the $\star$ marked problems.
- You may take help from the "Exam Assistance Note" containing a few required definitions and formula.

For instructor's use only

Problem Number	Obtained Score	Total Score
Problem 1		
Problem 2		
Problem 3		
Problem 4		
Problem 5		
Problem 6		
Problem 7		
Problem 8		
TOTAL		

Part I

Identify whether the following statements are True or False. You do not need any further justification to your answers.

1. (a) **Statement:** Let X be a continuous random variable on the support $\mathbb{S}_X := [0,1]$ with the following probability density function $f_X(x) := 3x^2$ if $0 \le x \le 1$. Then $P(X = 0.1) = 0.03$

Score:
Total Score: 5

Ans: TURE FALSE

(b) **Statement:** If the sample space (i.e. support) of a random variable is the interval $[0,\infty)$, then it must be a continuous random variable.

Score:
Total Score: 5

Ans: TURE FALSE

(c) **Statement:** Let A,B are two events such that $P(A) = \frac{1}{3}$ and $P(B) = \frac{4}{5}$. **Then the events A and B can not be disjoint.**

Score:
Total Score: 5

Ans: TURE FALSE

(d) **Statement:** The following function $f_X(x) := x\,\mathbb{I}_{[-1,2]}(x)$ is a valid pdf.

Score:
Total Score: 5

Ans: TURE FALSE

(e) **Statement:** Let the joint pdf of the random vector (X,Y) is given as

$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2+y^2 \le 1, \\ 0 & \text{otherwise.} \end{cases}$ Then X and Y are not statistically independent.

Score:
Total Score: 5

Ans: TURE FALSE

(f) **Statement:** If X and Y are two random variables such that $\mathrm{Cov}(X,Y) = 0$, then X and Y are statistically independent.

Score:
Total Score: 5

Ans: TRUE FALSE

Part II

Answer the following short type questions. You do not need any further justification to your answers.

2. (a) Let the Cumulative Distribution Function of a continuous random variable X be $F_X(x) := 1 - e^{-3x}$ for $x > 0$. Find the median of the random variable.

Score: ____
Total Score: 5

(b) Let X be a continuous random variable with the probability density function $f_X(x) := \frac{x}{50}\mathbb{I}_{[0,10]}(x)$. Find $P(0 < X < 5)$.

Score: ____
Total Score: 5

(c) Let X, Y are independent and identically distributed random variables with mean 0, variance σ^2. Represent the $\mathbf{Var}(\mathbf{3X}+\mathbf{Y})$ in terms of σ^2.

Score: ______
Total Score: 5

(d) If $V_1 \sim \chi^2_{5\text{ df}}$ and $V_2 \sim \chi^2_{20\text{df}}$ and V_1, V_2 are **statistically independent** then what is the distribution of the random variable $\mathbf{Y} := \frac{\mathbf{4V_1}}{\mathbf{V_2}}$?

Score: ______
Total Score: 5

Part III

Answer the following descriptive type questions. You need to show your work to get the full credit.

3. A smoke detector system uses two devices, A and B. If smoke is present, the probability that it will be detected by **device A is 0.92, by device B, 0.87; and by both devices, 0.82.**

(a) If smoke is present, find the probability that the smoke will be detected by either device A or B or both.

Score: ____
Total Score: 7

(b) Find the probability that the smoke will be undetected by both the devices.

Score: ____
Total Score: 3

4. A company that manufactures and bottles apple juice uses a machine that automatically fills 15-ounce bottles. There is some variation, however, in the amounts of liquid dispensed into the bottles that are filled. The amount dispensed has been observed to be approximately normally distributed with mean 15 ounces and standard deviation 1 ounce. Determine the proportion of bottles that will have more than 16 ounces dispensed into them.

Score: ____
Total Score: 10

5*. Let the random variable $\mathbf{X}$ be Uniformly distributed on the interval $[-1, 1]$. What is the probability density function of the random variable $\mathbf{Y}$ where $\mathbf{Y} = \mathbf{X^2}$. (Clearly write the sample space of the random variable $\mathbf{Y}$ and show steps when deriving its density.)

Score: ____
Total Score: 10

6. Females and Males observed to react differently to a given set of circumstances. It has been observed that 75% of the females react positively to these circumstances, whereas only 50% of males react positively. A group of 150 people, 100 female and 50 male, was subjected to these circumstances, and the subjects were asked to describe their reactions on a written questionnaire. A response picked at random from the all the 150 individuals was negative. What is the probability that it was that of a male?

Score: ____
Total Score: 10

7*. Let the total number of items a specific Amazon seller sells in a week, denoted by the random variable Y, follows a Poisson distribution with mean $\lambda = 200$. From experience, the seller knows that proportion of items that are returned is 10%. Let X denotes the number of items returned in a randomly selected week.

(a) What is the expected number of items that are returned during a week?

Score: ____
Total Score: 5

(b) Let us assume that the seller makes a 25% profit on each items that are sold but are not returned. However, if a product is returned then the seller needs to issue a refund of entire amount to the specific customer. Additionally the seller needs to pay for the cost of return shipping of amount **D**:=₹300 per items returned. If the price at which the seller sells each item is **S:**=₹5000, then what is expected net profit for the seller over a period of a randomly selected week?

Score: ____

Total Score: 10

8*. Suppose X and Y have joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} 6x^2y & \text{if } 0 \leq x \leq y, \quad x+y \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Show that the marginal distribution of X is Beta($\alpha = 3, \beta = 2$).

(Hint: Note that for a fixed value $x \in S_X = [0,1]$ the possible values of y is $x \leq y \leq 2 - x$.)

Score: ____
Total Score: 10