# Multivariate Random Variables

UAEU

# Discrete multivariate r.v.

We will just restrict the presentation to the bivariate case. Let $X$, $Y$ be two discrete random variable the joint probability function of $X$ and $Y$ is defined by $f(x,y) = P\{X = x,\ Y = y\}$.
The joint cumulative distribution function is given by

$$F(x,y) = P\{X \leq x,\ Y \leq y\} = \sum_{s \leq x} \sum_{t \leq y} f(s,t).$$

# Definitions

The marginal probability function of $X$ is given by

$$f_X(x) = \sum_y f(x, y).$$

The marginal probability function of $Y$ is given by

$$f_Y(y) = \sum_x f(x, y).$$

# Examples

**Example 1** *Suppose two cards are drawn at random without replacement from a deck of* $4$ *cards numbered* $1, 2, 3, 4$. *Let* $X$ *be the number on the first card and* $Y$ *be the number of the second card.*
*Find the joint probability function of* $X$ *and* $Y$.
*Find the marginal probability function of* $X$.
*Find the marginal probability function of* $Y$.

# Examples

**Example 2** *A fair coin is flipped three times. Let* $X$ *denotes the number of heads to occur in the first two flips, a and let* $Y$ *denotes the number of heads to occur in the last two flips. Find the joint probability function and the marginal probability functions of* $X$ *and* $Y$*. Evaluate* $P\{X = Y\}$*.*

# Continuous multivariate r.v.

Let $X, Y$ be two continuous random variable, we define the joint (cumulative) probability distribution of $X$ and $Y$ as usual $F(x,y) = P\{X \leq x,\ Y \leq y\}$. The joint density function is given by

$$f(x,y) = \frac{d^2F(x,y)}{dxdy}.$$

# Definitions

The Marginal density of $X$ is $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$.
The Marginal density of $Y$ is $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$.
The cumulative cdf is given by

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(s,t)dsdt.$$

# Examples

**Example 3** *The joint pdf of* $X, Y$ *is given by*

$$f(x, y) = \frac{x + y + 1}{2}$$

*for* $0 < x < 1$*,* $0 < y < 1$ *and zero otherwise.*
*Find the cumulative distribution function of* $X, Y$*.*
*Find the marginal; density of* $X$*.*
*Find the marginal density of* $Y$*.*

# Examples

**Example 4** *Let* $X, Y$ *have joint cdf* $F(x, y) = x^2y^3$ *for* $0 < x < 1$ *and* $0 < y < 1$.
*Find the joint density function.*
*Find the marginal of* $X$.
*Find the marginal of* $Y$.

# Conditional Distributions

- Discrete R.V.

**Definition 1** *If $f(x, y)$ denotes the joint probability function of two discrete random variables $X$ and $Y$ and if $f_X(x)$ and $f_Y(y)$ denote the marginal probability function of $X$, ($Y$ respectively) then:*
*The conditional probability of $X$ given $Y = y$ is given by $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$. The conditional probability of $Y$ given $X = x$ is given by $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$.*

# Example

**Example 5** *Go to examples 1 and 2 and find the conditional probability of* $X$ *given* $Y = 2$*. Use this to compute* $P\{X \leq 2|Y = 2\}$*.*

# Conditional Distributions

- Continuous R.V.

**Definition 2** *If $f(x,y)$ denotes the joint probability density of two continuous random variables $X$ and $Y$ and if $f_X(x)$ and $f_Y(y)$ denote the marginal densities function of $X$, ($Y$ respectively) then:*
*The conditional density of $X$ given $Y = y$ is given by*
$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$.
*The conditional probability of $Y$ given $X = x$ is given by*
$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$.

# Examples

**Example 6** *Go to example 3 and 4 and find the conditional probability of* $X$ *given* $Y = 0.5$. *Use this to compute* $P\{X \leq 0.75|Y = 0.5\}$.

# Independence

**Definition 3** *Two random variable $X$ and $Y$ are said to be independent iff*

- $f(x, y) = f_X(x)f_Y(y)$ *for all $x$ and $y$'s,*

Note that independence is also equivalent to:

- $f_{X|Y}(x|y) = f_X(x)$ for all $x$ and all $f_Y(y) > 0$, or
- $f_{Y|X}(y|x) = f_Y(y)$ for all $y$ and $f_X(x) > 0$.

# Examples

**Example 7** *For examples 1,2,3 and 4 check if* $X$ *and* $Y$ *are independent.*

# Expectation for multivariate R.V.

Let $X$, $Y$ be two discrete random variables with joint probability function $f(x, y)$. Then the expected value of $g(X, Y)$ is given by

$$E(g(X,Y)) = \sum_x \sum_y g(x,y) f(x,y).$$

Let $X$, $Y$ be two continuous random variables with joint density function $f(x, y)$. Then the expected value of $g(X, Y)$ is given by

$$E(g(X,Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f(x,y) dx dy.$$

# Properties

If $c_1, c_2, \ldots, c_n$ then

$$E\left(\sum_{i=1}^{n} c_i g_i(X_1, X_2, \ldots, X_k)\right) = \sum_{i=1}^{n} c_i E(g_i(X_1, X_2, \ldots, X_k)).$$

# Examples

**Example 8** *The joint density f $X$ and $Y$ is given by*

$$f(x,y) = \begin{cases} \frac{2}{7}(x+2y) & \textit{for } 0 < x < 1,\ 1 < y < 2 \\ 0 & \textit{elsewhere.} \end{cases}$$

*Find the expected value of $X/Y^3$.*
*Find the expected value of $(X+Y)^2$.*

# Links

Virtual Library/Joint Distributions

Virtual Library/Conditional Distributions