STAT230: Principles of Probability

Unit 6: Multivariate Random Variables

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Discrete multivariate r.v.

We will just restrict the presentation to the bivariate case.

Definition

Let X, Y be two discrete random variable the joint probability mass function of X and Y is defined by $f(x, y) = P\{X = x,\ Y = y\}$.
The joint cumulative distribution function is given by

$$F(x, y) = P\{X \le x,\ Y \le y\} = \sum_{s \le x} \sum_{t \le y} f(s, t).$$

Definitions

The marginal probability function of X is given by

$$f_X(x) = \sum_y f(x, y).$$

The marginal probability function of Y is given by

$$f_Y(y) = \sum_x f(x, y).$$

Example 1

Example (1)

Suppose two cards are drawn at random without replacement from a deck of 4 cards numbered $1, 2, 3, 4$. Let X be the number on the first card and Y be the number of the second card.
Find the joint probability function of X and Y.
Find the marginal probability function of X.
Find the marginal probability function of Y.

Solution for Example 1

$Y \backslash X$	1	2	3	4	$f_Y(y)$
1	0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$
2	$\frac{1}{12}$	0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$
3	$\frac{1}{12}$	$\frac{1}{12}$	0	$\frac{1}{12}$	$\frac{1}{4}$
4	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	0	$\frac{1}{4}$
$f_X(x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	

Example 2

Example (2)

A fair coin is flipped three times. Let X denotes the number of heads to occur in the first two flips, a and let Y denotes the number of heads to occur in the last two flips. Find the joint probability function and the marginal probability functions of X and Y.
Evaluate $P\{X = Y\}$.

Solution for Example 2

$Y \backslash X$	0	1	2	$f_Y(y)$
0	$\frac{1}{8}$	$\frac{1}{8}$	0	$\frac{1}{4}$
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{2}{4}$
2	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$
$f_X(x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	

$P\{X = Y\} = f(0,0) + f(1,1) + f(2,2) = \frac{1}{2}.$

Example 3

Example (3)

Suppose that 3 balls are randomly selected from an urn containing 3 red, 4 white, and 5 blue balls. If we let X and Y denote respectively, the number of red and white balls chosen. Find the joint probability mass function of X and Y.

Solution for Example 3

$Y \backslash X$	0	1	2	3	$f_Y(y)$
0	$\frac{10}{220}$	$\frac{30}{220}$	$\frac{15}{220}$	$\frac{1}{220}$	$\frac{56}{220}$
1	$\frac{40}{220}$	$\frac{60}{220}$	$\frac{12}{220}$	0	$\frac{112}{220}$
2	$\frac{30}{220}$	$\frac{18}{220}$	0	0	$\frac{48}{220}$
3	$\frac{4}{220}$	0	0	0	$\frac{4}{220}$
$f_X(x)$	$\frac{84}{220}$	$\frac{108}{220}$	$\frac{27}{220}$	$\frac{1}{220}$	

Definition

Let X, Y be two continuous random variable, we define the joint (cumulative) probability distribution of X and Y as usual $F(x,y) = P\{X \le x,\ Y \le y\}$. The joint density function is given by

$$f(x,y) = \frac{d^2F(x,y)}{dxdy}.$$

Definitions

The Marginal density of X is $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$.
The Marginal density of Y is $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$.
The cumulative cdf is given by

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(s,t)dsdt.$$

Example 4

Example (4)

The joint pdf of X, Y is given by

$$f(x,y) = \frac{x+y+1}{2}$$

for $0 < x < 1$, $0 < y < 1$ and zero otherwise.
Find the cumulative distribution function of X, Y.
Find the marginal; density of X.
Find the marginal density of Y.

Solution for Example 4

- The the cumulative distribution function of X, Y is $F(x,y) = \int_0^x \int_0^y f(s,t)dtds = \int_0^x \int_0^y \frac{s+t+1}{2}dtds = \frac{xy(x+y+2)}{4}$ for $0 < x < 1$, $0 < y < 1$.
- The marginal density of X is

$$f_X(x) = \int_0^1 f(x,y)dy = \int_0^1 \frac{x+y+1}{2}dy = \frac{x}{2} + \frac{3}{4}$$

for $0 \le x \le 1$.
- The marginal density of Y is

$$f_Y(y) = \int_0^1 f(x,y)dx = \int_0^1 \frac{x+y+1}{2}dx = \frac{y}{2} + \frac{3}{4}$$

for $0 \le y \le 1$.

Example 5

Example (5)

Let X, Y have joint cdf $F(x, y) = x^2y^3$ for $0 < x < 1$ and $0 < y < 1$.
Find the joint density function.
Find the marginal of X.
Find the marginal of Y.

Solution for Example 5

- $f(x,y) = \frac{d^2F(x,y)}{dxdy} = 6xy^2$ for $0 \le x \le 1$ and $0 \le y \le 1$.
- The marginal density of X is

$$f_X(x) = \int_0^1 f(x,y)dy = \int_0^1 6xy^2 dy = 2x$$

  for $0 \le x \le 1$.
- The marginal density of Y is

$$f_Y(y) = \int_0^1 f(x,y)dx = \int_0^1 6xy^2 dx = 3y^2$$

  for $0 \le y \le 1$.

Example 6

Example (6)

The joint density of X and Y is given by $f(x, y) = 2e^{-x-2y}$ for $0 < x < \infty$ and $0 < y < \infty$.
Find the marginal of X.
Find the marginal of Y.
Find $P\{X > 1, Y < 1\}$.
Find $P\{X < Y\}$.
Find $P\{X < 4\}$.

Solution for Example 6

- $f_X(x) = e^{-x}$ for $0 \leq x < \infty$ and $f_Y(y) = 2e^{-2y}$ for $0 \leq y < \infty$.
- $P\{X > 1, Y < 1\} = \int_1^{\infty} \int_0^1 f(x,y)dydx = \int_1^{\infty} \int_0^1 2e^{-x}e^{-2y}dydx = e^{-1} - e^{-3}$.
- $P\{X < Y\} = \int_0^{\infty} \int_x^{\infty} f(x,y)dydx = \int_0^{\infty} \int_x^{\infty} 2e^{-x}e^{-2y}dydx = \int_0^{\infty} e^{-3x}dx = \frac{1}{3}$.
- $P\{X < 4\} = \int_0^4 f_X(x)dx = \int_0^4 e^{-x}dx = 1 - e^{-4}$.

Conditional Distributions

- Discrete R.V.

Definition

If $f(x, y)$ denotes the joint probability function of two discrete random variables X and Y and if $f_X(x)$ and $f_Y(y)$ denote the marginal probability function of X, (Y respectively) then:
The conditional probability of X given $Y = y$ is given by $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$. The conditional probability of Y given $X = x$ is given by $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$.

Examples

Example (7)

Suppose two cards are drawn at random without replacement from a deck of 4 cards numbered $1, 2, 3, 4$. Let X be the number on the first card and Y be the number of the second card.
Find the conditional probability of X given $Y = 2$. Use this to compute $P\{X \leq 2|Y = 2\}$.

Example (8)

Suppose that 3 balls are randomly selected from an urn containing 3 red, 4 white, and 5 blue balls. If we let X and Y denote respectively, the number of red and white balls chosen.
Find the conditional probability of X given $Y = 2$. Use this to compute $P\{X \leq 2|Y = 2\}$.

Solution for Example 7

The conditional distribution of X given Y=2 is calculated for each possible value of X as $f_{X|Y=2}(x) = \frac{f(x,2)}{f_Y(2)}$. The table below shows the results

x	$f_{X\|Y=2}(x)$
1	1/3
2	0
3	1/3
4	1/3

$$P\{X \leq 2|Y = 2\} = f_{X|Y=2}(1) + f_{X|Y=2}(2) = 1/3.$$

- Continuous R.V.

Definition

If $f(x, y)$ denotes the joint probability density of two continuous random variables X and Y and if $f_X(x)$ and $f_Y(y)$ denote the marginal densities function of X, (Y respectively) then:
The conditional density of X given $Y = y$ is given by $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$.
The conditional probability of Y given $X = x$ is given by
$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$.

Examples

Example (9)

The joint pdf of X, Y is given by

$$f(x, y) = \frac{x + y + 1}{2}$$

for $0 < x < 1$, $0 < y < 1$ and zero otherwise.
Find the conditional probability of X given $Y = 0.5$. Use this to compute $P\{X \leq 0.75 | Y = 0.5\}$.

Example (10)

Let X, Y have joint cdf $F(x, y) = x^2 y^3$ for $0 < x < 1$ and $0 < y < 1$.
Find the conditional probability of X given $Y = 0.5$. Use this to compute $P\{X \geq 0.5 | Y = 0.5\}$.

Solution for Example 9

The conditional density of X given $Y = 0.5$ if computed as $f_{X|Y=0.5}(x) = \frac{f(x,0.5)}{f_Y(0.5)}$. In this case we already computed $f_Y(y) = y/2 + 3/4$ for $0 \le y \le 1$ therefore

$$f_{X|Y=0.5}(x) = \frac{f(x, 0.5)}{f_Y(0.5)} = x/2 + 3/4; \text{ for } 0 \le x \le 1.$$

$P\{X \le 0.75|Y = 0.5\} = \int_0^{0.75} f_{X|Y=0.5}(x)dx = \int_0^{0.75} x/2 + 3/4dx = 27/32.$

Example (11)

The joint density of X and Y is given by $f(x,y) = 2e^{-x-2y}$ for $0 < x < \infty$ and $0 < y < \infty$.
Find the conditional probability of X given $Y = 1$. Use this to compute $P\{X \leq 2|Y = 1\}$.

Independence

Definition

Two random variable X and Y are said to be independent iff

- $f(x,y) = f_X(x)f_Y(y)$ for all x and y's,

Note that independence is also equivalent to:

- $f_{X|Y}(x|y) = f_X(x)$ for all x and all $f_Y(y) > 0$, or
- $f_{Y|X}(y|x) = f_Y(y)$ for all y and $f_X(x) > 0$.

Example

Example (12)

Suppose two cards are drawn at random without replacement from a deck of 4 cards numbered $1, 2, 3, 4$. Let X be the number on the first card and Y be the number of the second card.
Are X and Y independent?

Example (13)

Suppose that 3 balls are randomly selected from an urn containing 3 red, 4 white, and 5 blue balls. If we let X and Y denote respectively, the number of red and white balls chosen.
Are X and Y independent?

Solution for Example 12

Note that in particular $f(1,1) = 0$ is not equal to $f_X(1) * f_Y(1) = \frac{1}{4} * \frac{1}{4} = \frac{1}{16}$ therefore X and Y are not independent we say that X and Y are dependent.

Examples

Example (14)

The joint pdf of X, Y is given by

$$f(x,y) = \frac{x+y+1}{2}$$

for $0 < x < 1$, $0 < y < 1$ and zero otherwise.
Are X and Y independent?

Example (15)

Let X, Y have joint cdf $F(x,y) = x^2y^3$ for $0 < x < 1$ and $0 < y < 1$.
Are X and Y independent?

Example (16)

The joint density of X and Y is given by $f(x,y) = 2e^{-x-2y}$ for $0 < x < \infty$ and $0 < y < \infty$.
Are X and Y independent?

Solution for Example 14

Note that $f(x, y) = \frac{x+y+1}{2}$ is not equal to $f_X(x) * f_Y(y) = (\frac{x}{2} + \frac{3}{4}) * (\frac{y}{2} + \frac{3}{4}) = \frac{xy}{4} + \frac{3x}{8} + \frac{3y}{8} + \frac{9}{16}$ therefore X and Y are not independent we say that X and Y are dependent.

Expectation for multivariate R.V.

Let X, Y be two discrete random variables with joint probability function $f(x,y)$. Then the expected value of $g(X,Y)$ is given by

$$E(g(X,Y)) = \sum_x \sum_y g(x,y) f(x,y).$$

Let X, Y be two continuous random variables with joint density function $f(x,y)$. Then the expected value of $g(X,Y)$ is given by

$$E(g(X,Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f(x,y) dx dy.$$

Properties

If $c_1, c_2, \ldots, c_n$ then

$$E\left(\sum_{i=1}^{n} c_i g_i(X_1, X_2, \ldots, X_k)\right)$$

$$= \sum_{i=1}^{n} c_i E(g_i(X_1, X_2, \ldots, X_k)).$$

Examples

Example (17)

Suppose two cards are drawn at random without replacement from a deck of 4 cards numbered $1, 2, 3, 4$. Let X be the number on the first card and Y be the number of the second card.
Find $E(X+Y)$.

Solution $E(X+Y) = \sum_x \sum_y (x+y) f(x,y) = 5$.

Example (18)

The joint density f X and Y is given by

$$f(x,y) = \begin{cases} \frac{2}{7}(x+2y) & \text{for } 0 < x < 1,\ 1 < y < 2 \\ 0 & \text{elsewhere.} \end{cases}$$

Find the expected value of X/Y^3.
Find the expected value of $(X+Y)^2$.

Solution $E(X/Y^3) = \int_0^1 \int_1^2 \frac{x}{y^3} f(x,y) dydx = \frac{5}{28}$.

Links

Virtual Library/Joint Distributions
Virtual Library/Conditional Distributions