STAT 320: Principles of Probability
Unit 1 (Part-B): Review of a Few Mathemetical Functions

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Outline

A Few Mathemetical Function and Notation

Absolute Value

Absolute value of a real number is the magnitude of the real number ignoring its sign. Formally, we have the following definition.

Definition (Absolute Value)

Let $x \in \mathbb{R}$ be any real number, then the **absolute value of** x (denotes as $|x|$) is defined to be

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0, \end{cases}$$

- $|5| =$
- $|-7.6| =$
- $|0| =$
- $|1005.7| =$
- $|-200| =$

Definition (Indicator Function)

Let A be a set. The **indicator function for the set** A, denoted by $\mathbb{I}_A(x)$, is defined to be

$$\mathbb{I}_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

- $\mathbb{I}_{[0,5]}(1.33) =$
- $\mathbb{I}_{[0,5]}(-9.12) =$
- $\mathbb{I}_{\{HH,TT,HT\}}(HH) =$
- $\mathbb{I}_{\{HH,TT,HT\}}(HHHH) =$
- $\mathbb{I}_{\mathbb{Z}}(1.87) =$
- $\mathbb{I}_{\mathbb{R}_+}(-4.87) =$
- $\mathbb{I}_{\mathbb{R}_+}(14) =$

A Few Mathematical Function and Notation

Definition (Factorial)

Let n be a **non-negative integer,** then the **factorial of n**, denoted as $n!$ is defined to be

1. $0! = 1$,
2. $1! = 1$,
3. $n! = n \times (n-1) \times \ldots \times 1$ for $n \geq 2$

Result: $n! = n \times \{(n-1)!\}$ for $n \geq 2$.

- $3! =$
- $5! =$
- $6! =$

n choose r, $\binom{n}{r}$

Definition

Let n, r be two **non-negative integes,** such that $r \leq n$, then the **n choose r**, denoted by $\binom{n}{r}$, is defined to be

$$\binom{n}{r} := \frac{n!}{(r!) \times ((n-r)!)}$$

Result: If n, r be two **non-negative integes,** such that $r \leq n$, then $\binom{n}{r} = \binom{n}{(n-r)}$

- $\binom{0}{0} = 1$, $\binom{1}{0} = 1$, $\binom{5}{3} = \frac{5!}{3! \times ((5-3)!)} = \frac{5!}{3! \times (2!)} = \frac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10.$
- $\binom{10}{2} =$
- $\binom{100}{95} =$

Outline

$\frac{dx^n}{dx} = nx^{n-1}$ for any integer n.

$\frac{de^x}{dx} = e^x$

$\frac{de^{mx}}{dx} = me^{mx}$ for any constant $m \in \mathbb{R}$.

$\frac{d \log(x)}{dx} = \frac{1}{x}$.

Assume $f'(x) := \frac{d\ f(x)}{dx}$ and $g'(x) := \frac{d\ g(x)}{dx}$ for the following formula

Product Rule: $\frac{d}{dx}\left\{ f(x)g(x) \right\} = f'(x)\ g(x) + f(x)\ g'(x)$

Addition Rule: $\frac{d}{dx}\left\{ c_1\ f(x) + c_2\ g(x) \right\} = c_1\ f'(x) + c_2\ g'(x)$ for any constant c_1, $c_2 \in \mathbb{R}$.

Chain Rule : $\frac{d}{dx}\left\{ f\left(g(x) \right) \right\} = f'\left(g(x) \right) \times g'(x)$.

- $\frac{d}{dx}\left(x^2 e^x\right) =$
- $\frac{d}{dx}\left(e^{30x^2}\right) =$
- $\frac{d}{dx}\left(xe^{x^2}\right) =$
- $\frac{d}{dx}\left(\frac{1}{1-x}\right) =$
- $\frac{d}{dx}\left((1-x)^n\right) =$

Outline

The mathematical constant "e"

Definition ("e")

The mathematical constant e is an transcendental real number given by,

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots,$$

Approxiimately $e \approx 2.7183$

Exponential Series e^x or $(\exp(x))$

Definition (Exponential Series)

For any real number $x \in \mathbb{R}$, the exponential series e^x (or sometimes denoted as $\exp(x)$) is defined as,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

- $$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots,$$

- $e^{x^2} =$

Let $\lambda > 0$, then

$$\boxed{\sum_{j=0}^{\infty} \frac{e^{jt}\lambda^j}{j!}} = \boxed{1 + \frac{e^t\lambda}{1!} + \frac{e^{2t}\lambda^2}{2!} + \frac{e^{3t}\lambda^3}{3!} + \cdots} = ?.$$

Let $\lambda > 0$, then

$$\sum_{j=1}^{\infty} \frac{\lambda^j}{(j-1)!} = \frac{\lambda}{0!} + \frac{\lambda^2}{1!} + \frac{\lambda^3}{2!} + \cdots = ?.$$

Let $\lambda > 0$, then $\displaystyle\sum_{m=4}^{\infty} \frac{\lambda^m}{(m-4)!}$ =?.

Let $\alpha \in \mathbb{R}$, then $\displaystyle\sum_{m=1}^{\infty} \frac{e^{\alpha m}}{(m+2)!}$ =?.

Outline

Geometric Series

Geometric Series

Let $x \in \mathbb{R}$ be such that $|x| < 1$, then

$$\sum_{i=0}^{\infty} x^i = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

1. What is the value of $1 + 0.7 + (0.7)^2 + (0.7)^3 + \cdots =$
2. What is the value of $1 - 0.7 + (0.7)^2 - (0.7)^3 + \cdots =$

Let $q \in \mathbb{R}$ be such that $|q| < 1$, then

$$\sum_{j=0}^{\infty} j\, q^j = q + 2q^2 + 3q^3 + 3q^4 \cdots = ?.$$

Let $p \in \mathbb{R}$ be such that $|p| < 1$, then

$$\sum_{n=0}^{\infty} e^{nt}\, p^n = \cdots =?.$$

Outline

Binomial Series

Binomial Series

Let $x \in \mathbb{R}$ be any real number, and $n \in \mathbb{Z}_+$ be any positive integer, then

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$$

$$(1+x)^n = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n$$

Binomial Series

Let $x \in \mathbb{R}$ be any real number, and $n \in \mathbb{Z}_+$ be any positive integer, then

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$$

$$(1+x)^n = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n$$

Let $x \in \mathbb{R}$ be any real number, and $n \in \mathbb{Z}_+$ be any positive integer, then

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

$$(a+b)^n = b^n + \binom{n}{1} ab^{n-1} + \binom{n}{2} a^2 b^{n-2} + \cdots + \binom{n}{n-1} a^{n-1} b + \binom{n}{n} a^n$$

- $(1 + y)^2 =$

- $(1 + z)^3 =$

- $(p + q)^4 =$

Binomial Series

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$$

$$1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} =$$

$$1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^{n-1}\binom{n}{n-1} + (-1)^n\binom{n}{n} =$$

Binomial Series

$$\boxed{\sum_{i=1}^{n} i\binom{n}{i}x^{i}} =$$

$$\boxed{\sum_{k=0}^{n} e^{kt} \binom{n}{k} x^k} =$$

Outline

$$\int x^n dx = \frac{x^{n+1}}{n+1} \text{ for any integer } n,\ n \neq -1.$$

$$\int \frac{1}{x} dx = \log(x)$$

$$\int e^{-x} dx = -e^{-x}.$$

$$\int e^{mx} dx = \frac{e^{mx}}{m} \text{ for any nonzero real number } m \in \mathbb{R},\ m \neq 0.$$

* Note: We have not included the constant term that appears as a constants while writing a indefinite integral. For the majority, if not all, of the integras in this course will be a definite integrals with a lower and upper limit.

Assume $f'(x) := \frac{d \ f(x)}{dx}$ and $g'(x) := \frac{d \ g(x)}{dx}$ for the following formula

Integral By Parts: $\int f(x)g(x)dx = f(x) \left(\int g(x)dx\right) - \int \left\{ f'(x) \left(\int g(x)dx\right) \right\} dx$

Addition Rule: $\int \left\{ c_1 \ f(x) + c_2 \ g(x) \right\} dx = c_1 \int f(x)dx + c_2 \int g(x)dx$ for any constant $c_1, c_2 \in \mathbb{R}$.

Some Non Trivial Integrals

$$\int x^{\alpha-1}e^{-x}dx \text{ for } \alpha > 0.$$

$$\int x^{\alpha-1}(1-x)^{\beta-1}dx \text{ for } \alpha > 0, \beta > 0.$$

$$\int e^{-x^2}dx$$

$$\int \frac{\sin(x)}{x}dx.$$

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Outline

$\Gamma(\alpha)$, for $\alpha > 0$

The Gamma Function

Gamma Function, $\Gamma(\alpha), \alpha > 0$

$$\Gamma(\alpha) := \int_0^{\infty} x^{\alpha-1} e^{-x} dx$$ for $\alpha > 0$.

- $\Gamma(\alpha) > 0$ for all $\alpha > 0$.
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
- $\Gamma(1) = 1$

- $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$ for any $\alpha > 1$
- $\Gamma(n) = (n - 1)!$ when n is a positive integer.

Gamma Function: Example

$$\int_0^\infty x^7 e^{-x} dx =$$

$$\int_0^\infty x^{\frac{5}{2}} e^{-3x} dx =$$

Gamma Function: Example

$$\frac{\Gamma(9.1)}{\Gamma(7.1)} =$$

Let $\alpha > 0$, $\dfrac{\Gamma(\alpha+3)}{\Gamma(\alpha)} =$

$\mathscr{B}(\alpha, \beta)_{\alpha > 0, \beta > 0}$

The Beta Function

Beta Function, $\mathscr{B}(\alpha, \beta), \alpha > 0, \beta > 0$

$$\mathscr{B}(\alpha, \beta) := \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$$ for $\alpha > 0, \beta > 0$.

$\mathscr{B}(\alpha, \beta)$ is often calculated using the following equation:

$$\mathscr{B}(\alpha, \beta) := \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Beta Function, $\mathscr{B}(\alpha, \beta), \alpha > 0, \beta > 0$

$$\int_0^1 x^3(1-x)^9 dx =$$

$$\int_0^1 x^{30}(1-x)^{1.2} dx =$$

Let $\alpha > 0, \beta > 0$, then compute

$$\frac{\mathscr{B}(\alpha+1,\beta)}{\mathscr{B}(\alpha,\beta)} =$$

$\Phi(x)$ Function, $x \in \mathbb{R}$ The Standard Normal CDF

The Standard Normal CDF (PHI function) $\Phi(x)$, $x \in \mathbb{R}$.

$$\Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \text{ for all } x \in \mathbb{R}.$$

- $\Phi(0) = \frac{1}{2}$
- $\Phi(-x) + \Phi(x) = 1 \implies \Phi(-x) = 1 - \Phi(x)$ for all $x \in \mathbb{R}$
- $0 \leq \Phi(x) \leq 1$
- $\lim_{x \to -\infty} \Phi(x) = 0$, and $\lim_{x \to \infty} \Phi(x) = 1$

Discussion on Various Concepts

Log (function) Equation of Line, Circles

Log (function) Gamma, Beta, Phi function Equation of Line and Regions Circles and Regions

Questions?