Exam Assistance Note

Indicator Function

$$\mathbb{I}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}.$$

Properties of Probability:

Let $(\mathscr{F}, \mathbf{B}, P)$ be a sample space along with the corresponding Borel Sigma Algebra and a Probability measure. Let $A, B \in \mathscr{B}$. Then

- $P(\varnothing) = 0$ where $\varnothing$ denotes the Null set.
- $0 \leq P(A) \leq 1$.
- $P(A^c) = 1 - P(A)$.
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Bonferron's Inequality: If A, B are two events then $P(A \cap B) \geq P(A) + P(B) - 1$.

[Bayes' Theorem] Let $A_1, A_2, \ldots, n$ be a partition of the sample space, and let B be any event. Then, for each $i = 1, 2, ..., n$

$$P(A_i \mid B) = \frac{P(B \mid A_i)P(A_i)}{\sum_{j=1}^{n} P(B \mid A_j)P(A_j)}.$$

Cumulative Distribution Function (cdf)

The cumulative distribution function or cdf of a random variable X, denoted by $F_X(x)$, is defined by $F_X(x) = P(X \leq x)$ for all $x \in \mathbb{R}$.

[Properties of pdf] A function $f_x(x)$ is a pdf of a continuous random variable X if and only if

1. $f_x(x) \geq 0$ for all $x \in \mathbb{R}$.
2. $\int_{-\infty}^{\infty} f_X(x)dx = 1$

[Transformation of Single Variables] Let X have pdf $f_x(x)$ and let $Y = g(X)$, where $g(x)$ is a monotone function. Suppose that $f_X(x)$ is continuous on X and that $g^{-1}(y)$ has a continuous derivative on y. Then the pdf of Y is given by

$$f_Y(y) := f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|.$$

[Many to one transformations] Let X have pdf $f_X(x)$, let $Y = g(X)$. Suppose there exists a partition, $A_0, A_1, \ldots, A_k$, of S_X such that $P(X \in A_0) = 0$ and $f_X(x)$ is continuous on each $A_i's$. Suppose there exist junctions $g_1(X), \ldots, g_k(X)$, defined on $A_1, \ldots, A_k$, respectively, satisfying

1. $g(X) = g_i(X)$, for $x \in A_i$,
2. $g_i(x)$ is monotone on A_i,
3. the set $S_Y = \{y : y = g_i(x) \text{for some } x \in A_i\}$ is the same for each $i = 1, \ldots, k$, and
4. $g_i^{-1}(y)$ has a continuous derivative on y, for each i Then for $1, \ldots, k$.

$$f_Y(y) := \begin{cases} \sum_{i=1}^{k} f_X(g_i^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| & \text{for } y \in S_Y \\ 0 & \text{otherwise.} \end{cases}$$

Quantile function

Inverse of a cdf F, also known as the Quantile function of the distribution is defined as following $Q_F(y) := F^{-1}(y) := \inf\{x : F(X) \geq y\}$ for $0 < y < 1)$. **Note that the median** $m_Y := F^{-1}(\frac{1}{2})$.

Distribution	pdf	Mean	Variance	MGF: $M_X(t)$
Uniform(θ_1, θ_2)	$\frac{\mathbb{I}_{[\theta_1,\theta_2]}(x)}{\theta_2-\theta_1}$	$\frac{\theta_1+\theta_2}{2}$	$\frac{(\theta_2-\theta_1)^2}{12}$	$\frac{e^{t\theta_2}-e^{t\theta_1}}{t(\theta_2-\theta_1)}$
Gamma$(\alpha$,rate=$\beta)$	$\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}\,\mathbb{I}_{\mathbb{R}_+}(x)$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$	$\frac{1}{(1-\frac{t}{\beta})^\alpha}$
Exponential(rate=λ)	$\lambda e^{-\lambda x}\,\mathbb{I}_{\mathbb{R}_+}(x)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{1}{(1-\frac{t}{\lambda})}$
$\chi^2_{k\text{df}}$	$\chi^2_{k\text{df}} = \text{Gamma}(\frac{k}{2},\text{rate=}\frac{1}{2})$.			
Beta(α,β)	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}\mathbb{I}_{(0,1)}(x)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	${}_1F_1(a := \alpha, b := \alpha+\beta, t)$
Normal$(\mu$, Var=$\sigma^2)$	$\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2\sigma^2}(x-\mu)^2}\mathbb{I}_{\mathbb{R}}(x)$	μ	σ^2	$e^{\mu t+\frac{1}{2}\sigma^2 t^2}$

Result: If $Z \sim N(0,1)$ and $V \sim \chi^2_{v\text{df}}$ and X, V are **statistically independent** then the random variable $Y := \frac{Z}{\sqrt{V/v}} \sim t_{v\text{ df}}$.

Result: If $V_1 \sim \chi^2_{v_1\text{ df}}$ and $V_2 \sim \chi^2_{v_2\text{df}}$ and V_1, V_2 are **statistically independent** then the random variable $Y := \frac{V_1/v_1}{V_2/v_2} \sim F_{v_1,v_2}$.

Distribution	pdf	Mean: $E(X)$	Variance :Var(X)	MGF: $M_X(t)$
Binomial(n,p)	$\binom{n}{x}p^x(1-p)^{(n-x)}, x \in \{0,1,2,\dots,n\}$	np	$np(1-p)$	$(1-p+pe^t)^n$
Geometric(p)	$(1-p)^{x-1}p, x \in \{1,2,3,\dots\}$.	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$
Poisson(λ)	$\frac{e^{-\lambda}\lambda^x}{(x!)}$ for $x \in \{0,1,2,\dots\}$	λ	λ	$e^{\lambda(e^t-1)}$

Conditional Distribution

Let (X,Y) be a continuous bivariate random vector with joint pdf $f(x,y)$ and marginal pdfs $f_X(x)$ and $f_Y(y)$. For any x such that $f_X(x) > 0$, the conditional pdf of Y given that $X = x$ is the function of y denoted by $f(Y \mid x)$ and defined by

$$f_{Y|X}(y \mid x) := \frac{f_{X,Y}(x,y)}{f_X(x)} \text{ for all } x \in S_{Y|X=x},$$

where $S_{Y|X=x}$ denotes the set of all possible values of y when the random variable X takes the value x.

Independent Random Variables

Let (X,Y) be a bivariate random vector with joint pdf (or pmf) $f_{X,Y}(x,y)$ (or $p_{X,Y}(x,y)$) and marginal pdfs (or pmfs) $f_X(x)$ and $f_Y(y)$ (or $p_X(x)$ and $p_X(x)$ respectively). Then X and Y are called statistically independent random variables if,

$$f_{X,Y}(x,y) = f_X(x)f_Y(y),$$

for all $(x,y) \in \mathbb{R}^2$ and $S_{X,Y} = S_X \times S_Y$, where $S_X := \{x \in R : f_X(x) > 0\}$, $S_X := \{y \in R : f_Y(y) > 0\}$.

Let (X,Y) be a bivariate random vector with joint pdf (or pmf) $f(x,y)$ on the support $S_{X,Y}$. Then X and Y are independent random variables if and only if there exist functions $g(x)$ and $h(y)$, such that the joint pdf can be represented as

$$f_{X,Y}(x,y) := g(x)h(y)$$

for all $(x,y) \in \mathbb{R}^2$ and $S_{X,Y} = S_X \times S_Y$, where $S_X := \{x \in R : g(x) > 0\}$, $S_X := \{y \in R : h(y) > 0\}$.

Theorem:
Let X and Y be independent random variables.

1. For any $A \in \mathbb{R}$ and $B \in \mathbb{R}$, $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$; that is, the events $X \in A$ and $Y \in B$ are independent events.
2. Let $g(x)$ be a function only of x and $h(y)$ be a function only of y. Then
$$E\left(g(X)h(Y)\right) = \left[E\left(g(X)\right)\right]\left[E\left(h(Y)\right)\right].$$
3. Assuming that the corresponding mgf's exist, $M_{X+Y}(t) = M_X(t)M_Y(t)$, where $M_X(t)$, $M_Y(t)$, and $M_{X+Y}(t)$ denote the mgf of X, Y and $X+Y$ respectively.

Transformation of the Bivariate Continuous RV

If (X, Y) is a continuous random vector with joint pdf $f_{X,Y}(x,y)$. If the random vector (U,V) be defined by the **one-to-one** transformation $(U,V) := \left(g_1(X,Y), g_2(X,Y)\right)$ from $S_{X,Y}$ to $S_{U,V}$, then a version of the joint pdf for (U,V) can be obtained as

$$f_{U,V}(u,v) := f_{X,Y}\left(h_1(u,v), h_2(u,v)\right)|J(u,v)|,$$

where $|J(u,v)|$, denotes the abosolute value of the diterminat of the matrix

$$J(u,v) := \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial h_1(u,v)}{\partial u} & \frac{\partial h_1(u,v)}{\partial v} \\ \frac{\partial h_2(u,v)}{\partial u} & \frac{\partial h_2(u,v)}{\partial v} \end{bmatrix}.$$

Gamma Function

Let $\alpha > 0$ then function $\Gamma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is defined as the following integral, $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and if $\alpha > 0$ then $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. Specifically, if n is a positive integer then $\Gamma(n) = (n-1)!$.

Chebychev's Inequality: Let X be a random variable with $E(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$. If $0 < \sigma^2 < \infty$ then

$$P(|X-\mu| < t\sigma) \geq 1 - \frac{1}{t^2} \quad \text{for any } t > 0.$$

Convex Function

A function $g(x)$ is **convex** if

$$g\left(\lambda x+(1-\lambda)y\right) \leq \lambda g(x)+(1-\lambda)g(y),$$

for all x and y, and $0<\lambda<1$.

If a function $g(x)$ is twice differentiable then it is **convex** if $g''(x) := \frac{d^2(g(x))}{dx^2} > 0$ for all x

Jensen's Inequality: For any random variable X, if g(x) is a convex junction, then

$$E\left(g(X)\right) \geq g\left(E(X)\right).$$