STAT 320: Principles of Probability
Unit 3: Introduction to Probability

United Arab Emirates University

Department of Statistics

Outline

1. Sample Space & Events
2. The Notion of Probability
3. A Few Properties of Probability
4. Examples

Sample Space & Events

Statistical ‘Experiment’ & Outcome

Definition (“Experiment”)

In the context of Statistics and Probability, the phrase **Experiment** refers to the process by which an observation is made.

Experiment : Single throw of a 6-sided die.

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Definition (Outcome)

An **Outcome** is the result from a Experiment.

Experiment : Single throw of a 6-sided die.

An Outcome: The number 5 appear in the die-throw example.

Sample Space

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Sample Space: $\mathscr{S} = \{1, 2, 3, 4, 5, 6\}$

Definition (Event)

An **event** is a collection of outcomes. It is simply a subset of the sample space, $\mathscr{S}$.

Event : If we consider, single throw of a 6-sided die. $A = \{2, 4, 6\}$, a subset of the sample space is an example of an event.

Example

Example : Experiment: Determination of and recording of the sex of a newborn child.

Example

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Sample Space: $\mathscr{S} = \{B, G\}$ where the outcome G means that the child is a girl and B that it is a boy.

Example

Example : Consider a context of horse race where 7 horses have participated the race. They are marked as $1, 2, \ldots, 7$.

Experiment: Recording the order of the horse numbers of 7 horses according to their completion time. The positions for the horses can be 1, 2, 3, 4, 5, 6, and 7.

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Experiment: Recording the order of the horse numbers of 7 horses according to their completion time. The positions for the horses can be 1, 2, 3, 4, 5, 6, and 7.

Sample Space: $\mathscr{S} =$ The Set of all 7! permutations of (1, 2, 3, 4, 5,6,7)

An outcome (2,3, 1, 6, 5, 4, 7) means, for instance, that the number 2 horse comes in first, then the number 3 horse, then the number 1 horse, and so on.

Question : Let A be the event that horse 3 wins the race. Write down the explicit description of A.

Example

Example : Consider the **single flip** of a coin.

Experiment: Recording the outcome after flipping a coin

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Sample Space: $\mathscr{S} = \{H, T\}$

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Sample Space: $\mathscr{S} = \{H, T\} \times \{H, T\} = \{(H, H), (H, T), (T, H), (T, T)\}$

As there is no ambiguity in this case, for brevity we often/will use the following notation: Sample Space: $\mathscr{S} = \{HH, HT, TH, TT\}$

Question : Let B be the event that the Head appears on the first throw.Write down the explicit description of B.

Example

Example : Consider the rolling of a dice **two times**

Experiment: Recording the outcome after rolling a dice **two times.**

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Experiment: Recording the outcome after rolling a dice **two times.**

Sample Space: The sample space consists of the 36 points

$\mathscr{S}$ ={1,2,3,4,5,6}×{1,2,3,4,5,6}

$\mathscr{S} = \{(i, j) : i, j = 1, 2, 3, 4, 5, 6\}$ where the outcome (i, j) is said to occur if i appears on the first through and j on the second. other die.

Question : Let E be the event that the sum of the dice equals 7.Write down the explicit description of E.

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(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Example

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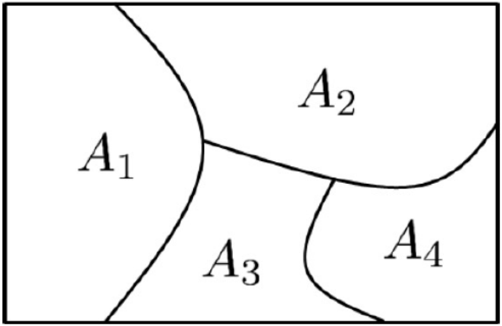
Question : Let A be the event that the transistor does not last longer than 5 hours. Write down the event A in the notation of set theory.

Reminder: Disjoint Events and Partition

Disjoint Events & Partition



Reminder from Unit1 Slides: Disjoint Sets

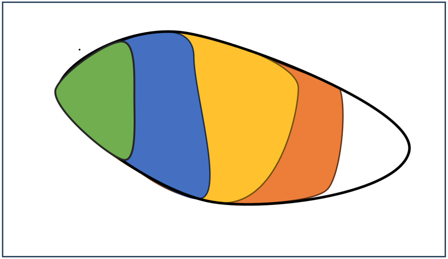
Disjoint Sets: Two sets A, and B are said to be **Disjoint Sets** or **mutually exclusive sets** if A and B does not have any elements in common.

A and B are Disjoint $\Leftrightarrow$ $A \cap B = \emptyset$.

Reminder from Unit1 Slides: Partition

Partition: A collection of sets $\{A_1, A_2, \cdots, A_k\}$ is called a **partition for a set** C if

1. $A_i \cap A_j = \emptyset$ for $1 \leq i \neq j \leq k$ (i.e. ,A_i, and A_j are Disjoint if $i \neq j$), and
2. $A_1 \cup A_2 \cup \cdots \cup A_k = C$.



Pairwise Disjoint Events

Definition (Pairwise Disjoint Events)

A collection of events $\{A_i\}_{i=1}^{k}$ are pairwise disjoint (or mutually exclusive) if $A_i \cap A_j = \emptyset$ for all $1 \leq i \neq j \leq k$.

Partition of Sample Space

Partition of Sample Space: A collection of sets $\{A_1, A_2, \cdots, A_n\}$ is called a **partition** for a set $\mathscr{S}$ if

1. the collection of events $\{A_i\}_{i=1}^n$ are **pairwise disjoint**, and
2. $A_1 \cup A_2 \cup \cdots \cup A_n = \mathscr{S}$.

Comment : In the above definition, we may replace n by ∞ and the definition extends naturally.

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Comment : Any set A and it's complement, $\bar{A}$, creates a partition of $\mathscr{S}$.

Outline

Notion of Probability

Basic Notion of Probability

- **Probability** refers to the chance that a particular event will occur.
- The probability of an event is the proportion of times the event is expected to occur in repeated experiments.
- If we denote by n(E) the number of times in the first n repetitions of the experiment that the event E occurs, the probability of the event E is defined by

$$\boxed{P(E) = \lim_{n \mapsto \infty} \frac{\mathrm{n(E)}}{n}.}$$

Axiomatic Definition of Probability

Definition (Probability)

Consider an experiment whose sample space is $\mathscr{S}$. A set-function $P(E)$, defined for each event E of $\mathscr{S}$, is said to be a **probability** if, it satisfies the following three axioms

1. $\boxed{0 \leq P(E) \leq 1}$ for any event E
2. $\boxed{P(\mathscr{S}) = 1}$
3. For any sequence of mutually disjoint (mutually exclusive) events $E_1, E_2, E_3, \ldots,$ (that is, events for which $E_i \cap E_j = \emptyset$ for all $i \neq j$)

$$\boxed{P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)}$$

We refer to $P(E)$ as the **probability** of the event E.

Comment :

Axiom 1 states that the probability of an event E is always between 0 and 1.

Axiom 2 Probability of the entire sample space $\mathscr{S}$ is 1.

Axiom 3 states that, for any sequence of **mutually exclusive events (i.e. disjoint events)**, the probability of at least one of these events occurring is just the sum of their respective probabilities.

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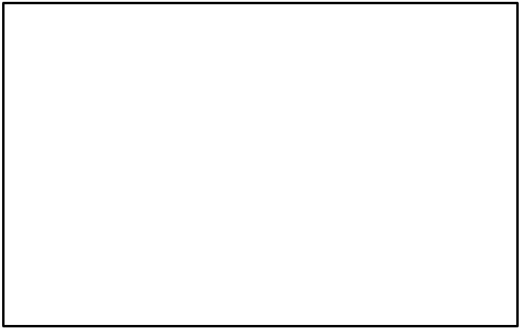
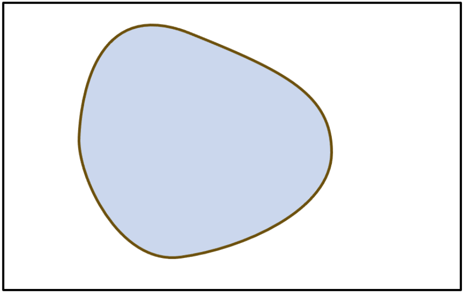
A Few Properties of Probability

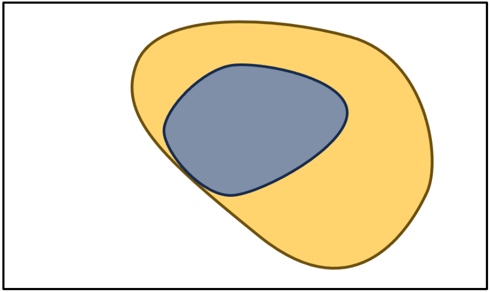
$P(\emptyset) = 0$, and $P(\mathscr{S}) = 1$



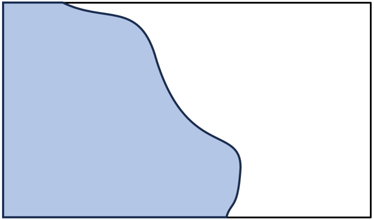
$P(A) \leq 1$



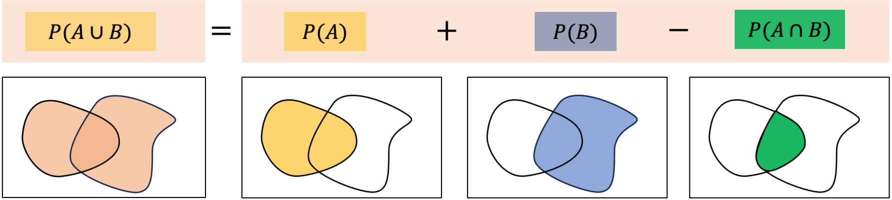
If $A \subset B$ then $P(A) \leq P(B)$



$P(\overline{A}) = 1 - P(A)$



$P(A \cup B) = P(A) + P(B) - P(A \cap B)$



Summary: A Few Properties of Probability

Let $(\mathscr{S}, P)$ be a sample space along with the Probability measure. Let A, B be two events. Then

- $P(\emptyset) = 0$ where $\emptyset$ denotes the Null set.
- $P(A) \leq 1$.
- If $A \subset B$ then $P(A) \leq P(B)$.
- $P(\overline{A}) = 1 - P(A)$, where $\overline{A}$ denotes the complementary event to A
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Question : Represent the probability of the following events using $P(A), P(B)$ and $P(A \cap B)$.

1. $P(A - B) = P(A \cap \overline{B}) = ?$
2. $P(B - A) = ?$
3. If A, B are disjoint, i.e. $A \cap B = \emptyset$ then what is $P(A \cup B) = ?$

Question : Let A, B be two events such that

$$P(A) = \frac{1}{3} \text{ and } P(\overline{B}) = \frac{1}{4}.$$

Can A and B be disjoint? Explain.

Examples

Example : A smoke detector system uses two devices, A and B. If smoke is present, the probability that it will be detected by device A is 0.95; by device B,0.90; and by both devices, 0.88.

1. If smoke is present, find the probability that the smoke will be detected by either device A or B or both devices.
2. Find the probability that the smoke will be undetected.

Solution: A: {device A detects smoke} B: {device B detects smoke}
$P(A \cup B) = P(A) + P(B) = P(A \cap B) = 0.95 + 0.90 - 0.88 = .97$
$P(\text{Smoke is undetected }) = 1 - P(\text{Smoke is detected by atleast one devices }) = 1 - P(A \cap B) = 1 - 0.97 = 0.03$

Inclusion Exclusion Principle

Inclusion Exclusion Principle: Specific Case $n = 3$

Let A_1, A_2, A_3 are three events. Then

$$
\begin{aligned}
P\left(A_1 \cup A_2 \cup A_3\right) &= \left\{ P(A_1) + P(A_2) + P(A_3) \right\} \\
&\quad - \left\{ P(A_1 \cap A_2) + P(A_1 \cap A_3) + P(A_2 \cap A_3) \right\} \\
&\quad + \left\{ P(A_1 \cap A_2 \cap A_3) \right\}
\end{aligned}
$$

Inclusion Exclusion Principle

Lemma

Let $\{A_i\}_{i=1}^n$ *be a sequence of events, for all* $i = 1, 2, 3, \ldots, n$*. Then*

$$P\left(\bigcup_{i=1}^n A_i\right) = \sum_{k=1}^n \sum_{(i_1, i_2, \ldots, i_k) \in \mathbb{Q}_{n,k}} (-1)^{k+1} P\left(\bigcap_{m=1}^k A_{i_m}\right),$$

where $\mathbb{Q}_{n,k} := \{(i_1, i_2 \ldots, i_k) \in \mathbb{Z}_+^k : 1 \le i_1 < i_2 < \ldots < i_k \le n\}$.

$$\mathbb{Q}_{4,2} = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\},$$

$$\mathbb{Q}_{4,3} = \{(1,2,3), (1,3,4), (2,3,4)\},$$

$$\boxed{\mathbb{Q}_{3,1} = ?, \mathbb{Q}_{3,2} = ?}$$

Outline

A Few Examples:

when $\mathscr{S}$ is a finite set.

Finite Sample Spaces with Equally Likely Outcomes

In many experiments, it is natural to assume that all outcomes in the sample space are equally likely to occur.

Consider the context when the sample space $\mathscr{S}$ is a finite set. To be specific, let $\mathscr{S} = \{1, 2, 3, \ldots, N\}$. If we assume that all the outcome is equally likely, then $P(\{1\}) = P(\{2\}) = \cdots = P(\{N\})$. It is intuitive and also easy to derive using Axioms 2 and Axiom3 of Probability that

$$P(\{i\}) = \frac{1}{N} \text{ for all } i = 1, \ldots N.$$

Finite Sample Spaces with Equally Likely Outcomes

If the sample space $\mathscr{S}$ is finite and all the corresponding outcomes (elements in $\mathscr{S}$) are equally-likely, then it follows from Axiom 3 that, for any event E,

$$P(E) = \frac{\text{Number of elements in E}}{\text{Number of elements in } \mathscr{S}}.$$

Assuming the finite sample space, if all the outcomes are equally likely to occur, then the probability of any event E equals the proportion of outcomes in the sample space that are contained in E.

- To refer to the “Equally Likely Setup” we typically use the phrases such as the ‘Fair Coin’ or ‘Fair Dice’

- For problems with Dice and Coin, if it is not Explicitly mentioned, we assume it to be a "Fair Coin" or "Fair Dice".

Example

Example : If two dice are rolled, what is the probability that the sum of the upturned faces will equal 7?

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Solution: We shall solve this problem under the assumption that all of the 36 possible outcomes are equally likely. Since there are 6 possible outcomes namely, (1; 6); (2; 5); (3; 4), (4; 3); (5; 2), and (6; 1)that result in the sum of the dice being equal to 7, the

Example

Example : If 3 balls are randomly drawn from a bowl containing 6 white and 5 black balls, what is the probability that one of the balls is white and the other two black?

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Solution: $n = 11 \times 10 \times 9 = 990$ and $n(E) = ?$ if $E = \{$one of the balls is white and the other two black$\}$ For the order WBB we have $6 \times 5 \times 4 = 120$ possibilities, for BWB, we have $5 \times 6 \times 4 = 120$ possibilities and for BBW we have $5 \times 4 \times 6 = 120$ possibilities. Then $P(E) = \frac{n(E)}{n} = \frac{120+120+120}{990} = \frac{4}{11}$.

Example

Example : A committee of 5 is to be selected from a group of 6 men and 9 women. If the selection is made randomly, what is the probability that the committee consists of 3 men and 2 women?

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Solution: Because each of the $\binom{15}{5}$ possible committees is equally likely to be selected, the desired probability is

$$\frac{\binom{6}{3}\times\binom{9}{2}}{\binom{15}{5}} = 0.24$$

Example

Example : Suppose 5 people are to be randomly selected from a group of 20 individuals consisting of 10 married couples, and we want to determine $P(N)$, the probability that the 5 chosen are all unrelated. (That is, no two are married to each other.)

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Solution: $P(N) = \dfrac{\binom{10}{5} \times 2^5}{\binom{20}{5}} \implies P(N) = \frac{20 \times 18 \times 16 \times 14 \times 12}{20 \times 19 \times 18 \times 17 \times 16}$

Example

Example : If n people are present in a room, what is the probability that no two of them celebrate their birthday on the same day of the year? How large need n be so that this probability is less than $\frac{1}{2}$?

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Solution: $\frac{^{365}P_n}{(365)^n} = \frac{(365)\times(364)\times\cdots\times(365-n+1)}{(365)^n}$

Example

Example : A poker hand consists of 5 cards. If the cards have distinct consecutive values and are not all of the same suit, we say that the hand is a straight. For instance, a hand consisting of the five of spades, six of spades, seven of spades, eight of spades, and nine of hearts is a straight. What is the probability that one is dealt a straight?

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Solution: ?

Example

Example : In the game of bridge, the entire deck of 52 cards is dealt out to 4 players. What is the probability that

a). one of the players receives all 13 spades?

b). each player receives 1 ace?

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Solution: ?

Examples: On Sharing a Birthday

Question : : There are 15 students registered in the course STAT320. What is the probability that at least two of the students will share their Birthday? (Ignore the leap year and assume there is 365 days in a year)

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Solution: $P(\{\text{At least two students have same birthday}\}) = 1 - P(\text{None of the students have same birthday}) =$

$1 - \frac{365 \times 364 \times \ldots \times 351}{365^{15}} \approx 1 - 0.7470987 = 0.25$

Application of Inclusion Exclusion Principle

Question : : Suppose we turn over cards simultaneously from two well shuffled decks of ordinary playing cards. We say we obtain an exact match on a particular turn if the same card appears from each deck; for example, the queen of spades against the queen of spades. Let C_i denotes the event that there is an exact match at the i^{th} turn.

1. Argue that $P(C_i) = \frac{51!}{52!}$.
2. Argue that for $i \neq j$, $P(C_i \cap C_j) = \frac{50!}{52!}$.
3. Argue that for $i \neq j \neq k$, $P(C_i \cap C_j \cap C_k) = \frac{49!}{52!}$.
4. Let p_M equal the probability of at least one exact match. Show that $p_M := 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots - \frac{1}{52!}$.

Questions?