

STAT 320: Principles of Probability

Unit 4: Conditional Probability & Statistical Independence

United Arab Emirates University

Department of Statistics

Outline

- 1 Conditional Probability
- 2 Partition of an Event
- 3 Law of Total Probability
- 4 Bayes' Theorem
- 5 The Notion of Statistical Independence

Conditional Probability

Example0

Example :

Suppose that we toss 2 dice, and suppose that each of the 36 possible outcomes is equally likely to occur, hence has probability $\frac{1}{36}$. Suppose further that we observe that the first die is a 3. Then, what is the probability that the sum of the 2 dice equals 8 **given** the additional information that first die is a 3 .?

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Definition (Conditional Probability)

Let E , and F are two events such that $P(F) > 0$, then the conditional probability of E **given** F is defined to be,

$$P(E | F) := \frac{P(E \cap F)}{P(F)}.$$

Example

Question : Suppose that a fair die is rolled once. Find the conditional probability that *the number 1 appears*, **given** that *an odd number was obtained*.

$A := \{\text{Observe a 1.}\}$, $B := \{\text{Observe an odd number.}\}$

$$P(A | B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3}.$$

Example

Question : If two events, A and B, are such that $P(A) = 0.5$, $P(B) = 0.3$, and $P(A \cap B) = 0.1$, find the following

- 1 $P(A | B)$
- 2 $P(B | A)$
- 3 $P(A | A \cup B)$

Example

Example : A coin is flipped twice. Assuming that all four points in the sample space $\mathcal{S} = \{HH, HT, TH, TT\}$ are equally likely, what is the conditional probability that both flips land on heads, given that (a) the first flip lands on heads? (b) at least one flip lands on heads?

Solution:

Definition (Multiplication Rule of Probability)

Let E and F are two events, then

$$P(E \cap F) := P(E | F) \times P(F) .$$

Example

Example : Suppose that an urn contains 8 red balls and 4 white balls. We draw 2 balls from the urn without replacement. If we assume that at each draw each ball in the urn is equally likely to be chosen, what is the probability that both balls drawn are red?

Example

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Suppose that an urn contains 8 red balls and 4 white balls. We draw 2 balls from the urn without replacement. If we assume that at each draw each ball in the urn is equally likely to be chosen, what is the probability that both balls drawn are red?

Solution: Let R_1 and R_2 denote, respectively, the events that the first and second balls drawn are red. Now, given that the first ball selected is red, there are 7 remaining red balls and 4 white balls, so $P(R_2 | R_1) = \frac{7}{11}$. . As $P(R_1) = \frac{8}{12}$, the desired probability is

$$P(R_1 \cap R_2) = P(R_2 | R_1) \times P(R_1) = \frac{7}{11} \frac{8}{12} = \frac{14}{33}.$$

$$P(E_1 \cap E_2 \cap E_3) = P(E_1 | E_2 \cap E_3) \times P(E_2 | E_3) \times P(E_3) .$$

$$P(E_1 \cap E_2 \cap E_3) P(E_2 \cap E_1 \cap E_3) = P(E_2 | E_1 \cap E_3) \times P(E_1 | E_3) \times P(E_3) .$$

$$P(E_1 \cap E_2 \cap E_3) = P(E_3 \cap E_2 \cap E_1) = P(E_3 | E_2 \cap E_1) \times P(E_2 | E_1) \times P(E_1) .$$

$$P(A \cap B \cap C) = P(A \mid B \cap C) \times P(B \mid C) \times P(C).$$

$$P(A \cap B \cap C \cap D) = P(A \mid B \cap C \cap D) \times P(B \mid C \cap D) \times P(C \mid D) \times P(D).$$

$$P(A \cap B \cap C \cap D \cap E) = P(D \cap A \cap C \cap E \cap B) = P(D \mid A \cap C \cap E \cap B) \times P(A \cap C \cap E \cap B).$$

Definition (The Generalized Multiplication Rule:)

$$P(E_1 \cap E_2 \cap \dots \cap E_k) := P(E_1) \times P(E_2 | E_1) P(E_3 | E_2 \cap E_1) \times \dots \times P(E_k | E_1 \cap E_2 \cap \dots \cap E_{k-1}).$$

$$P(E_1 \cap E_2 \cap \dots \cap E_k)$$

$$:= P(E_1 | E_2 \cap \dots \cap E_k) \times P(E_2 | E_3 \cap \dots \cap E_k) \times \dots \times P(E_{k-1} | E_k) \times P(E_k).$$

$$P(E_1, E_2, \dots, E_k)$$

$$:= P(E_1 | E_2, \dots, E_k) \times P(E_2 | E_3, \dots, E_k) \times \dots \times P(E_{k-1} | E_k) \times P(E_k).$$

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Partition of an Event Using Partition of Sample Space

Reminder from Unit1 Slides: Disjoint Sets

Disjoint Sets:

Two sets A , and B are said to be **Disjoint Sets** or **mutually exclusive sets** if A and B does not have any elements in common.

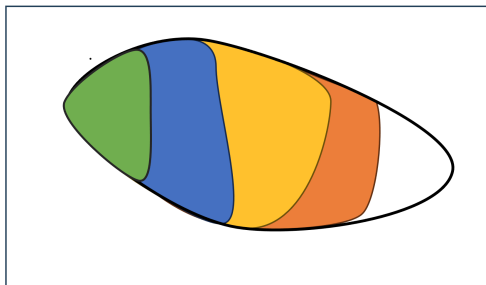
A and B are Disjoint $\Leftrightarrow A \cap B = \emptyset$.

Reminder from Unit1 Slides: Partition

Partition: A collection of sets $\{A_1, A_2, \dots, A_k\}$ is called a **partition** for a set C if

1 $A_i \cap A_j = \emptyset$ for $1 \leq i \neq j \leq k$ (i.e. A_i and A_j are Disjoint if $i \neq j$), and

2 $A_1 \cup A_2 \cup \dots \cup A_k = C$.



Partition of Sample Space

Partition of Sample Space:

A collection of sets $\{A_1, A_2, \dots, A_n\}$ is called a **partition** for a set \mathcal{S} if

1 the collection of events $\{A_i\}_{i=1}^n$ are **pairwise disjoint**, and

2
$$A_1 \cup A_2 \cup \dots \cup A_n = \mathcal{S}.$$

Comment : In the above definition, we may replace n by ∞ and the definition extends naturally.

Partition of Sample Space

Partition of Sample Space:

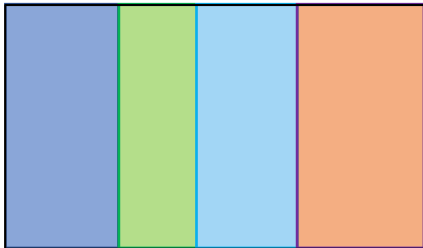
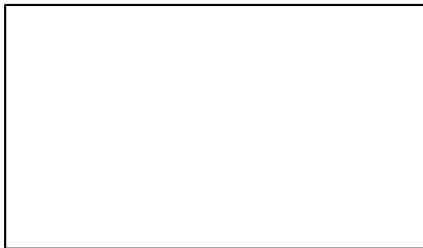
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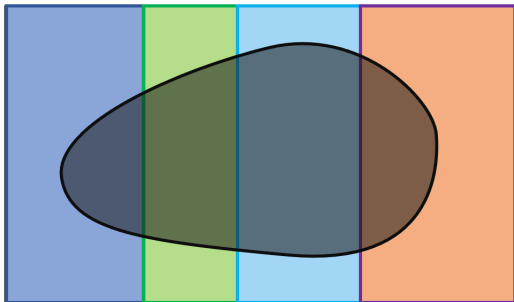
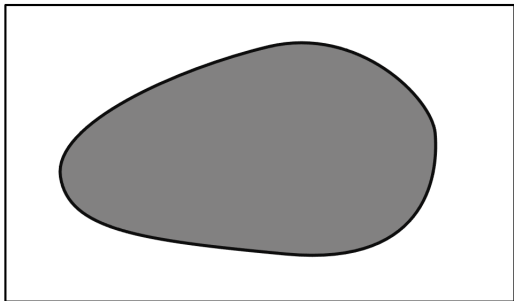
1 the collection of events $\{A_i\}_{i=1}^n$ are **pairwise disjoint**, and

2
$$A_1 \cup A_2 \cup \dots \cup A_n = \mathcal{S}.$$

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Comment : Any set A and its complement, \bar{A} , creates a partition of \mathcal{S} .







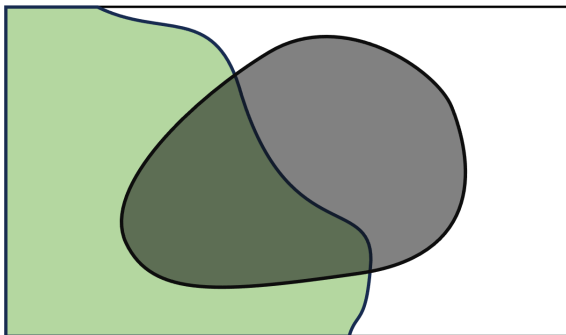


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Law of Total Probability

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$$E = (E \cap F) \cup (E \cap \bar{F})$$

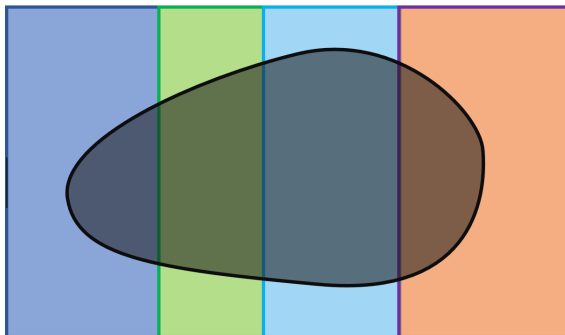
$$P(E) = P(E \cap F) + P(E \cap \bar{F})$$

Law of Total Probability

Let E and F be two events, then

$$P(E) = P(E | F)P(F) + P(E | \bar{F})(\bar{F})$$

Law of Total Probability (General)



Law of Total Probability (General)

Law of Total Probability (General):

Let E be an event. Assuming that the collection of sets $\{F_1, F_2, \dots, F_k\}$ forms a partition of \mathcal{S} , we have

$$P(E) = \sum_{j=1}^k P(E | F_j)P(F_j)$$

Example

Example :

An insurance company believes that people can be divided into two classes: those who are accident prone and those who are not. The company's statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability 0.4, whereas this probability is 0.2 for a person who is not accident prone. If we assume that 30% of the population is accident prone, what is the probability that a new policyholder will have an accident within a year of purchasing a policy?

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Solution: Let A_1 denote the event that the policyholder will have an accident within a year of purchasing the policy, and let A denote the event that the policyholder is accident prone. Hence, the desired probability is given by

$$P(A_1) = P(A_1 | A)P(A) + P(A_1 | \bar{A})P(\bar{A}) = (0.4)(0.3) + (0.2)(0.7) = 0.26$$

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Bayes' Theorem

Theorem

Let F_1, F_2, \dots, F_K be a set of mutually exclusive and exhaustive events (meaning that exactly one of these events must occur). Suppose now that E has occurred and we are interested in determining which one of the F_j also occurred. Then, we have the following theorem

$$P(F_i | E) = \frac{P(E | F_i)P(F_i)}{\sum_{j=1}^K P(E | F_j)P(F_j)}$$

- This theorem is well known as Bayes's theorem, after the English philosopher **Thomas Bayes**.
- We can use the following Applet to find the conditional probability using Bayes's theorem:
- http://www.thomsonedu.com/statistics/book_content/0495110817_wackerly/applets/seeingstats/Chpt2/bayesTree.html

Example

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Suppose that a new policyholder has an accident within a year of purchasing a policy. What is the probability that he or she is accident prone?

Solution: The desired probability is

$$P(A | A_1) = \frac{P(A \cap A_1)}{P(A_1)} = \frac{(0.4)(0.3)}{0.26} = \frac{6}{13}$$

Bayes' Theorem: Example

Question : A diagnostic test for a disease is such that it (correctly) detects the disease in 90% of the individuals who actually have the disease. Also, if a person does not have the disease, the test will report that he or she does not have it with probability 0.9. Only 1% of the population has the disease in question. If a person is chosen at random from the population and the diagnostic test indicates that she has the disease, what is the conditional probability that she does, in fact, have the disease? Are you surprised by the answer? Would you call this diagnostic test reliable?

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$A = \{\text{The individual has the disease}\}$, $\bar{A} = \{\text{does not have the disease}\}$, $B = \{\text{The test shows a POSITIVE result}\}$
 $\bar{B} = \{\text{The test shows a NEGATIVE result}\}$

$$P(B | A) = 0.9, P(\bar{B} | A^c) = 0.9 \text{ and } P(A) = 0.01$$

$$P(A | B) = \frac{P(B | A)P(A)}{P(B | A)P(A) + P(B | \bar{A})P(\bar{A})} = \frac{(0.9)(0.01)}{(0.9)(0.01) + (0.1)(0.99)} = 91\%$$

Example

Example :

Suppose that we have 3 cards that are identical in form, except that both sides of the first card are colored red, both sides of the second card are colored black, and one side of the third card is colored red and the other side black. The 3 cards are mixed up in a hat, and 1 card is randomly selected and put down on the ground. If the upper side of the chosen card is colored red, what is the probability that the other side is colored black?

Example

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Suppose that we have 3 cards that are identical in form, except that both sides of the first card are colored red, both sides of the second card are colored black, and one side of the third card is colored red and the other side black. The 3 cards are mixed up in a hat, and 1 card is randomly selected and put down on the ground. If the upper side of the chosen card is colored red, what is the probability that the other side is colored black?

Solution: Let RR, BB, and RB denote, respectively, the events that the chosen card is all red, all black, or the redblack card. Also, let R be the event that the upturned side of the chosen card is red. Then the desired probability is obtained

$$P(RB | R) = \frac{P(R | RB)P(RB)}{P(R | RR)P(RR) + P(R | RB)P(RB) + P(R | BB)P(BB)} = \frac{\frac{1}{2} \times \frac{1}{3}}{(1) \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} + (0) \times \frac{1}{3}} = \frac{1}{3}.$$

Example

Example :

A new couple, known to have two children, has just moved into town. Suppose that the mother is encountered walking with one of her children. If this child is a girl, what is the probability that both children are girls?

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Solution:

Example

Example :

The length, width, and height of a manufactured part are classified as being either within or outside specified tolerance limits. In a quality inspection 86% of the parts are found to be within the specified tolerance limits for width, but only 80% of the parts are within the specified tolerance limits for all three dimensions. However, 2% of the parts are within the specified tolerance limits for width and length but not for height, and 3% of the parts are within the specified tolerance limits for width and height but not for length. Moreover, 92% of the parts are within the specified tolerance limits for either width or height, or both of these dimensions.

- 1 If a part is within the specified tolerance limits for height, what is the probability that it will also be within the specified tolerance limits for width?
- 2 If a part is within the specified tolerance limits for length and width, what is the probability that it will be within the specified tolerance limits for all three dimensions?

Example

Example :

A car repair is either on time or late and either satisfactory or unsatisfactory. If a repair is made on time, then there is a probability of 0.85 that it is satisfactory. There is a probability of 0.77 that a repair will be made on time. What is the probability that a repair is made on time and is satisfactory?

1

2

Solution:

Example

Example :

An island has three species of bird. Species 1 accounts for 45% of the birds, of which 10% have been tagged. Species 2 accounts for 38% of the birds, of which 15% have been tagged. Species 3 accounts for 17% of the birds, of which 50% have been tagged. If a tagged bird is observed, what are the probabilities that it is of species 1, of species 2, and of species 3?

Solution:

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Example :

After production, an electrical circuit is given a quality score of A, B, C, or D. Over a certain period of time, 77% of the circuits were given a quality score A, 11% were given a quality score B, 7% were given a quality score C, and 5% were given a quality score D. Furthermore, it was found that 2% of the circuits given a quality score A eventually failed, and the failure rate was 10% for circuits given a quality score B, 14% for circuits given a quality score C, and 25% for circuits given a quality score D.

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Example

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The length, width, and height of a manufactured part are classified as being either within or outside specified tolerance limits. In a quality inspection 86% of the parts are found to be within the specified tolerance limits for width, but only 80% of the parts are within the specified tolerance limits for all three dimensions. However, 2% of the parts are within the specified tolerance limits for width and length but not for height, and 3% of the parts are within the specified tolerance limits for width and

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Solution:

Example

Example :

A bin contains 3 different types of disposable ashlights. The probability that a type 1 ashlight will give over 100 hours of use is 0.7, with the corresponding probabilities for type 2 and type 3 ashlights being 0.4 and 0.3, respectively. Suppose that 20 percent of the ashlights in the bin are type 1, 30 percent are type 2, and 50 percent are type 3.

- 1 What is the probability that a randomly chosen ashlight will give more than 100 hours of use?
- 2 Given that a ashlight lasted over 100 hours, what is the conditional probability that it was a type j ashlight, $j = 1, 2, 3$?

Solution:

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The Notion of Statistical Independence

Statistical Independence

Definition (Statistically Independent Event)

Two events E and F are said to be statistically independent if

$$P(E \cap F) = P(E) \times P(F)$$

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Two events E and F are said to be statistically independent if

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Corollary: Two events E and F are independent if and only if $P(E | F) = P(E)$ and $P(F | E) = P(F)$.

Example

Example : A card is selected at random from an ordinary deck of 52 playing cards. If E is the event that the selected card is an ace and F is the event that it is a spade, then E and F are independent.

Example

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A card is selected at random from an ordinary deck of 52 playing cards. If E is the event that the selected card is an ace and F is the event that it is a spade, then E and F are independent.

Solution: This follows because $P(E \cap F) = \frac{1}{52}$ whereas $P(E) = \frac{4}{52}$ and $P(F) = \frac{13}{52}$.

Example

Example : Two coins are flipped, and all 4 outcomes are assumed to be equally likely. If E is the event that the first coin lands on heads and F the event that the second lands on tails, then E and F are independent, since $P(E \cap F) = P(\{HT\}) = \frac{1}{4}$. Where as $P(E) = P(\{HH, HT\}) = \frac{1}{2}$ and $P(F) = P(\{HT, TT\}) = \frac{1}{2}$

Example

Example :

Suppose that we toss 2 fair dice. Let E_1 denote the event that the sum of the dice is 6 and F denote the event that the first die equals 4. Is E_1 statistically independent of F ?

Answer: $P(E_1 \cap F) = P(\{(4, 2)\}) = \frac{1}{36}$ where as $P(E_1) = \frac{5}{36}$ and $P(F) = \frac{1}{6}$. Therefore, E_1 and F are not statistically independent because, $P(E_1 \cap F) \neq P(E_1) \times P(F)$.

Example :

Now, suppose that we let E_2 be the event that the sum of the dice equals 7. Is E_2 statistically independent of F ?

Answer: $P(E_2 \cap F) = P(\{(4, 3)\}) = \frac{1}{36}$ where as $P(E_2) = \frac{1}{6}$ and $P(F) = \frac{1}{6}$. Therefore, E_2 and F are statistically independent because, $P(E_2 \cap F) = P(E_2) \times P(F)$.

Proposition : If E and F are statistically independent, then so are E and \bar{F} .

Solution:

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A card is selected at random from an ordinary deck of 52 playing cards. If E is the event that the selected card is an ace and F is the event that it is a spade, then E and F are independent.

Solution: This follows because $P(E \cap F) = \frac{1}{52}$ whereas $P(E) = \frac{4}{52}$ and $P(F) = \frac{13}{52}$.

Relation between: Statistical Independence & Disjointness

Example :

Generalized Definition of Statistical Independence

Definition (Statistically Independent Events)

Three events E , F , and G are said to be statistically independent if

$$P(E \cap F \cap G) = P(E) \times P(F) \times P(G)$$

$$P(E \cap F) = P(E) \times P(F)$$

$$P(E \cap G) = P(E) \times P(G)$$

$$P(F \cap G) = P(F) \times P(G)$$

Questions?