

STAT 320: Principles of Probability

Unit 6 Part:A

Continuous Random Variables

United Arab Emirates University

Department of Statistics

Outline

- 1 Characterization of any CDF function
- 2 Continuous Random Variables
- 3 Expected Value, Variance & MGF of a Continuous Random Variable
- 4 A Few Examples

Reminder: The Cumulative Distribution Functions

Distribution Functions

Definition (Cumulative Distribution Function (cdf))

The **cumulative distribution function** or **cdf** of a *any* variable X , denoted by $F_X(x)$, is defined by

$$F_X(x) = P(X \leq x) \text{ for all } x \in \mathbb{R}.$$

CDF: Example

Consider the experiment of tossing three fair coins, and let X = number of heads observed. We have already seen that

x	0	1	2	3
$p_x(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

The cdf of X is:

$$F_x(x) = \begin{cases} 0 & \text{if } -\infty < x < 0 \\ \frac{1}{8} & \text{if } 0 \leq x < 1 \\ \frac{4}{8} & \text{if } 1 \leq x < 2 \\ \frac{7}{8} & \text{if } 2 \leq x < 3 \\ 1 & \text{if } 3 \leq x < \infty \end{cases}$$

Example of CDF

x	0	1	2	3
$P_X(X = x)$	$\frac{1}{8} = .125$	$\frac{3}{8} = .375$	$\frac{3}{8} = 0.375$	$\frac{1}{8} = .125$

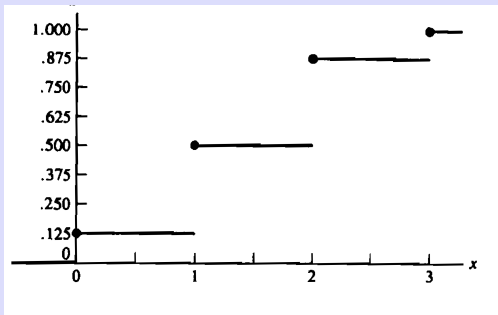


Figure: The plot of $F_X(x)$: CDF of the random variable X

Note that $F_X(\cdot)$ is defined for all values of $x \in \mathbb{R}$, not just for $x \in \mathbb{S}X := \{0, 1, 2, 3\}$. For example, $2.5 \notin \mathbb{S}X$, however

$$F_X(2.5) = P_X(x \leq 2.5) = P_X(X = 0) + P_X(X = 1) + P_X(X = 2) = \frac{7}{8}.$$

Characterization of *any* CDF Function

Characterization of a CDF

Theorem

*The function $F(x)$ is a cdf **if and only if** the following three conditions hold:*

- 1 $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow \infty} F(x) = 1$.
- 2 $F(x)$ is a nondecreasing function of x
- 3 $F(x)$ is right-continuous; that is, for every real number x_0 ,
 $\lim_{x \downarrow x_0} F(x) = F(x_0)$.

Comment: Let X be a random variable with the corresponding cdf $F_X(x)$ for $x \in \mathbb{R}$. Let $x_0 \in \mathbb{R}$ is arbitrary. Then

$$P(X = x_0) := P(X \in \{x_0\}) = \lim_{x \downarrow x_0} F_X(x) - \lim_{x \uparrow x_0} F_X(x).$$

Example: CDF continuous

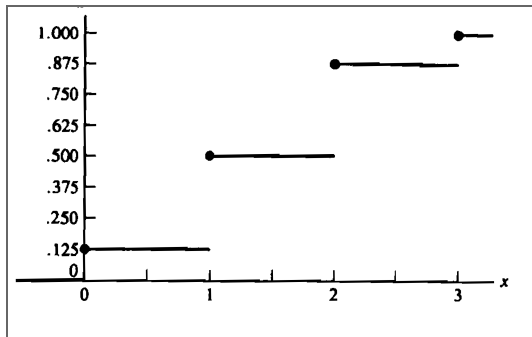


Figure: The plot of $F_X(x)$: CDF of the random variable X

Let $F_X(x)$ denotes the cdf function included in the above image. Therefore,

$$P(X = 2) = \lim_{x \downarrow 2} F_X(x) - \lim_{x \uparrow 2} F_X(x) = 0.5 - 0.125 = 0.375.$$

$$P(X = 1.5) = \lim_{x \downarrow 1.5} F_X(x) - \lim_{x \uparrow 1.5} F_X(x) = 0.5 - 0.5 = 0.$$

Example: CDF continuous

Example: An example of a continuous cdf is the function

$$F_X(x) := \frac{1}{1 + e^{-x}} \text{ for all } x \in \mathbb{R}.$$

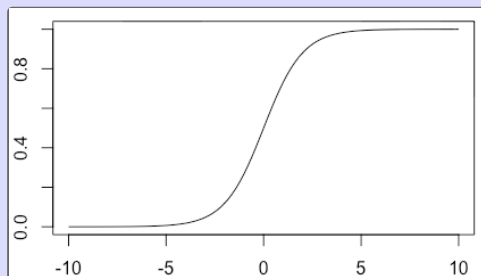


Figure: The plot of $F_X(x)$: CDF of the random variable X

Verify: The above function satisfies the three conditions required to be a CDF.

Example:

Question : : Prove that the following functions are valid cdfs.

1 $F(x) = e^{-e^{-x}}$ for all $x \in \mathbb{R}$.

2 $F(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}(x)$ for all $x \in \mathbb{R}$.

Definition (Discrete Random Variable)

A random variable X is discrete if its support \mathbb{S}_X is finite or countable infinite.

Alternative Characterization of Discrete Distributions: A random variable X is discrete if the corresponding cdf $F_X(x)$ is a step function of x . i.e. $F_X(x)$ increases only via jumps.

Outline

- 1 Characterization of any CDF function
- 2 Continuous Random Variables
- 3 Expected Value, Variance & MGF of a Continuous Random Variable
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Continuous Random Variables

Continuous and Discrete Random variable

Definition (Continuous Random Variable)

A random variable X is continuous if the corresponding cumulative distribution function (cdf) $F_X(x)$ is a continuous and continuously differentiable (at all points except for a finitely/countably many points) function of x .


Definition (Probability Density Function)

Let $F_X(x)$ be a cumulative distribution function (CDF) of a continuous random variable, then the corresponding **probability density function** or **pdf**, denoted as $f_X(x)$ is a function that satisfies the following criteria.

$$F_X(x) = \int_{-\infty}^x f_X(y) dy$$

Probability Density Function (pmf): For continuous RV

Comment: Using the Fundamental Theorem of Calculus, if $f_X(x)$ is continuous, we have the further relationship $f_X(x) = \frac{d}{dx} F_X(x)$.

 If X is a continuous random variable, then probabilities can be obtained by integrating its pdf over suitable region. Specifically, for $a, b \in \mathbb{R}$, $a < b$,

$$P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(x) dx.$$

Question: Is it true that a random variable must be continuous if its support is an interval ?

Question: Is it true that a random variable must be continuous if its support is \mathbb{R} ?

Question: Is it possible for a continuous random variable to have a finite support?

Example: CDF continuous

Example: An example of a continuous cdf is the function

$$F_X(x) := \frac{1}{1 + e^{-x}} \text{ for all } x \in \mathbb{R}.$$

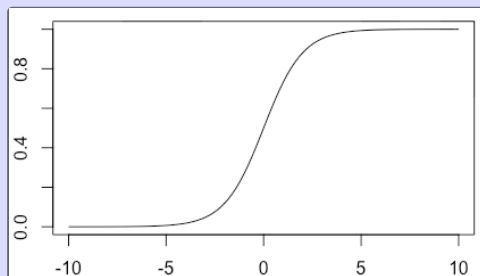


Figure: The plot of $F_X(x)$: CDF of the random variable X

Verify: The above function satisfies the three conditions required to be a CDF.

Example: CDF continuous

Example of CDF of a Continuous Random Variable:

$$F_X(x) := \begin{cases} 0 & \text{if } x \leq 0 \\ 1 - e^{-x} & \text{if } x > 0 \end{cases}$$

Verify: The above function satisfies the three conditions required to be a CDF.

Example:

Question : Prove that the following functions are valid cdfs.

1 $F(x) = e^{-e^{-x}}$ for all $x \in \mathbb{R}$.

2 $F(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}(x)$ for all $x \in \mathbb{R}$.

Characterization of Probability Density Function (pdf)

Definition (Continuous Random Variable)

A random variable X is said to be continuous if there is a function $f(x)$, called the probability density function (pdf), such that

1 $f(x) \geq 0$ for all x .

2
$$\int_{-\infty}^{\infty} f(x) dx = 1$$

3
$$P(a \leq X \leq b) = \int_a^b f(x) dx \text{ for all } a < b.$$

If X is a continuous random variable then,

• $P(X = c) = 0$ for any $c \in \mathbb{R}$

• $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b).$

Example

Example :

Suppose that X is a continuous random variable whose probability density function is given by

$$f(x) := \begin{cases} C(4x - 2x^2) & \text{if } 0 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

- a). What is the value of C ?
- b). Find $P(X > 1)$.

Example

Example :

Suppose that X is a continuous random variable whose probability density function is given by

$$f(x) := \begin{cases} C(4x - 2x^2) & \text{if } 0 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

- What is the value of C ?
- Find $P(X > 1)$.

According to the property of the pdf

$$\begin{aligned} \int f(x) dx &= 1 \\ \Rightarrow \int_0^2 C(4x - 2x^2) dx &= 1 \\ \Rightarrow C \left(2x^2 - \frac{2x^3}{3} \right) \Big|_0^2 &= 1 \\ \Rightarrow C \left(8 - \frac{16}{3} \right) &= 1 \\ \Rightarrow C &= \frac{3}{8} \end{aligned}$$

$$\begin{aligned} P(X > 1) &= \int_1^2 f(x) dx = \int_1^2 C(4x - 2x^2) dx \\ &= C \left(2x^2 - \frac{2x^3}{3} \right) \Big|_1^2 = 1 \\ &= C \left\{ \left(8 - \frac{16}{3} \right) - \left(2 - \frac{2}{3} \right) \right\} \\ &= C \left\{ \frac{8}{3} - \frac{4}{3} \right\} \\ &= \frac{3}{8} \times \frac{4}{3} \\ &= \frac{1}{2}. \end{aligned}$$

Example

Example :

For a given IT technician in a support center, let X denote the percentage of time, out of a 40-hour work week, that he is directly serving customers. Suppose that X has a probability density function given by

$$f(x) := \begin{cases} 3x^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

- a). Make a Graph of the above pdf.
- b). Find the probability that the technician will spend less than 30% of his workweek serving customers.
- c). Find the probability that the technician will spend 20% to 70% of his workweek serving customers.

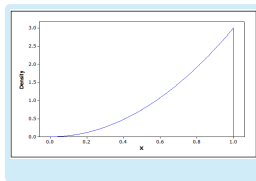
Example

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- Make a Graph of the above pdf.
- Find the probability that the technician will spend less than 30% of his workweek serving customers.
- Find the probability that the technician will spend 20% to 70% of his workweek serving customers.



$$\begin{aligned} P(X < 0.3) &= \int_0^{0.3} f(x) dx \\ &= \int_0^{0.3} 3x^2 dx \\ &= (x^3) \Big|_0^{0.3} \\ &= (0.3)^3 - (0)^3 \\ &= 0.027 \end{aligned}$$

$$\begin{aligned} P(0.2 < X < 0.7) &= \int_{0.2}^{0.7} f(x) dx \\ &= \int_{0.2}^{0.7} 3x^2 dx \\ &= (x^3) \Big|_{0.2}^{0.7} \\ &= (0.7)^3 - (0.2)^3 \\ &= 0.337 \end{aligned}$$

Exercise

Example :

The amount of time in hours that a computer functions before breaking down is a continuous random variable with probability density function given by

$$f(x) := \begin{cases} 100 e^{-\frac{x}{100}} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

- a). What is the probability that a computer will function between 50 and 150 hours before breaking down?
- b). What is the probability that it will function for fewer than 100 hours?

Exercise

Example :

The lifetime in hours of a certain kind of radio tube is a random variable having a probability density function given by

$$f(x) := \begin{cases} \frac{100}{x^2} & \text{if } x > 100 \\ 0 & \text{if } x \leq 100. \end{cases}$$

What is the probability that exactly 2 of 5 such tubes in a radio set will have to be replaced within the first 150 hours of operation? Assume

that the events $E_i, i = 1, 2, 3, 4, 5$, that the i th such tube will have to be replaced within this time are independent.

Cumulative Distribution Function (cdf)

Definition (The Cumulative Distribution Function)

The cumulative distribution function (cdf) $F(x)$ of a continuous random variable X with pdf $f(x)$ is defined for every number x by

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(x)dx.$$

Few Properties of $F(x)$

- 1 $P(a < X \leq b) = F(b) - F(a)$.
- 2 $P(X > b) = 1 - F(b)$
- 3 If X is a continuous random variable with cdf $F(x)$ then at every x at which $\frac{dF(x)}{dx}$ exists: $f(x) = \frac{dF(x)}{dx}$

Example

Example :

For a given IT technician in a support center, let X denote the percentage of time, out of a 40-hour work week, that he is directly serving customers. Suppose that X has a probability density function given by

$$f(x) := \begin{cases} 3x^2 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

- a). Obtain, $F(x)$, the CDF of X .
- b). Use $F(x)$ to compute $P(0.5 < X \leq 0.8)$.

Example

Example :

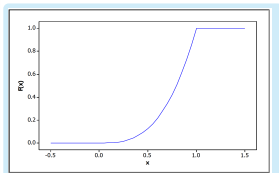
For a given IT technician in a support center, let X denote the percentage of time, out of a 40-hour work week, that he is directly serving customers. Suppose that X has a probability density function given by

$$f(x) := \begin{cases} 3x^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

- Obtain, $F(x)$, the CDF of X and Graph it.
- Use $F(x)$ to compute $P(0.5 < X \leq 0.8)$.

$$\begin{aligned} F(x) = P(X \leq x) &= \int_0^x f(y) dy \\ &= \int_0^x 3y^2 dy \\ &= (y^3) \Big|_0^x \\ &= x^3 \end{aligned}$$

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1. \end{cases}$$



$$\begin{aligned} &P(0.5 < X < 0.8) \\ &= F(0.8) - F(0.5) \\ &= (0.8)^3 - (0.5)^3 \\ &= 0.387 \end{aligned}$$

Example

Example : Let X be a continuous random variable with Cumulative Distribution Function $F(x)$, and density function $f(x)$.

- 1 Obtain the cumulative distribution function of $Y = 2X$.
- 2 Obtain the probability density function of $Y = 2X$.

Percentiles, Quantiles, and Median

Definition (Percentiles)

Let p be a number between 0 and 1. The $(100)^{\text{th}}$ percentile of the distribution of a continuous random variable X , we shall denote by c , is that value for which

$$F(c) = p$$

i.e. $c = F^{-1}(p)$. where $F^{-1}(\cdot)$ is the inverse cumulative distribution function.

Special percentiles:

- 1 The median of a continuous distribution, denoted by m , is the 50th percentile. So m satisfies $m = F^{-1}(0.5)$.
- 2 The first and the third quartiles can be computed as $Q_1 = F^{-1}(0.25)$
- 3 The third and the third quartiles can be computed as $Q_3 = F^{-1}(0.75)$

Example

Example :

For a given IT technician in a support center, let X denote the percentage of time, out of a 40-hour work week, that he is directly serving customers. Suppose that X has a probability density function given by

$$f(x) := \begin{cases} 3x^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

- a). find the median, and
- b). the interquartile range of the distribution.

Example

Example :

For a given IT technician in a support center, let X denote the percentage of time, out of a 40-hour work week, that he is directly serving customers. Suppose that X has a probability density function given by

$$f(x) := \begin{cases} 3x^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

- a). find the median, and
- b). the interquartile range of the distribution.

We have already Shown

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^3 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1. \end{cases}$$

Note that if $F(x) = y \implies x^3 = y \implies x = y^{\frac{1}{3}} \implies F^{-1}(y) = y^{\frac{1}{3}}$.

$$m = F^{-1}(0.5) = (0.5)^{\frac{1}{3}} = 0.794$$

IQR

$$\begin{aligned} &= Q_3 - Q_1 \\ &= F^{-1}(0.75) - F^{-1}(0.25) \\ &= (0.75)^{\frac{1}{3}} - (0.25)^{\frac{1}{3}} \\ &= 0.909 - 0.630 \\ &= 0.279 \end{aligned}$$

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Expected Value, Variance , & MGF of a Continuous Random Variable

Expected Value, or **mean** of a Continuous Random VariableDefinition (Expected Value or **mean** of a Continuous Random Variable)

If X is a continuous random variable with pdf $f(x)$ on the support \mathbb{S}_X , then the expected value (the mean) of X denoted by $E(X)$ is given by

$$E(X) = \int_{\mathbb{S}_X} x f(x) dx,$$

assuming the above integral exists.

$E(X)$ is sometimes also denoted by μ_X

Definition (Expected Value of a function of a Continuous Random Variable)

Let $h(x)$ be any* function. If X is a continuous random variable with pdf $f(x)$ on the support \mathbb{S}_X , then the expected value $h(X)$ denoted by $E(h(X))$ is given by

$$E(h(X)) = \int_{\mathbb{S}_X} h(x)f(x)dx,$$

assuming the above integral exists.

Variance of a Random Variable

Definition (Variance a Random Variable)

Variance of a random variable X is defined to be

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

A Few Properties of Expected Value and Variance of a Random Variable

Let a and b be constants, then

- 1 $E(a + bX) = a + bE(X)$
- 2 $\text{Var}(a + bX) = b^2\text{Var}(X)$

Moment Generating Function (mgf)

Definition (Moment Generating Function)

The Moment Generating Function (mgf) of X , denoted by $M_X(t)$ is defined as

$$M_X(t) := E \left(e^{tX} \right),$$

whenever it exists.

If a continuous random variable X has the probability density function $f_X(x)$ for all $x \in \mathbb{S}_X$, the support of X , then assuming it exists

$$M_X(t) := E \left(e^{tX} \right) = \int_{\{x \in \mathbb{S}_X\}} e^{tx} f_X(x) dx.$$

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A Few Examples

Example

Example :

For a given IT technician in a support center, let X denote the percentage of time, out of a 40-hour work week, that he is directly serving customers. Suppose that X has a probability density function given by

$$f(x) := \begin{cases} 3x^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

- a). Find the expected value of percentage of time the technician spends serving customers.
- b). variance of percentage of time the technician spends serving customers.

Example

Example :

For a given IT technician in a support center, let X denote the percentage of time, out of a 40-hour work week, that he is directly serving customers. Suppose that X has a probability density function given by

$$f(x) := \begin{cases} 3x^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

- a). Find the expected value of percentage of time the technician spends serving customers.
- b). variance of percentage of time the technician spends serving customers.

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_0^1 x(3x^2)dx \\ &= \int_0^1 (3x^3)dx \\ &= \left. \frac{3x^4}{4} \right|_0^1 \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)dx \\ &= \int_0^1 x^2 (3x^2)dx \\ &= \int_0^1 (3x^4)dx \\ &= \left. \frac{3x^5}{5} \right|_0^1 \\ &= \frac{3}{5} \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= E(X^2) - (E(X))^2 \\ &= \frac{3}{5} - \left(\frac{3}{4}\right)^2 \\ &= 0.6 - (0.75)^2 \\ &= 0.0375 \end{aligned}$$

Example

Example : Find $E(X)$ and $\text{Var}(X)$ when the density function of X is

$$f(x) := \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Example

Example :

Find $E(X)$ and $\text{Var}(X)$ when the density function of X is

$$f(x) := \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x)dx \\ &= \int_0^1 x(2x)dx \\ &= \int_0^1 (2x^2)dx \\ &= \left. \frac{2x^3}{3} \right|_0^1 \\ &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)dx \\ &= \int_0^1 x^2 (2x)dx \\ &= \int_0^1 (2x^3)dx \\ &= \left. \frac{2x^4}{4} \right|_0^1 \\ &= \frac{2}{4} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= E(X^2) - (E(X))^2 \\ &= \frac{1}{2} - \left(\frac{2}{3}\right)^2 \\ &= \frac{1}{2} - \frac{4}{9} \\ &= \frac{10}{9} \end{aligned}$$

Example

Example : Find $E(e^X)$ when the density function of X is

$$f(x) := \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Example

Example : Find $E(e^X)$ when the density function of X is

$$f(x) := \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} E(e^X) &= \int_{-\infty}^{\infty} e^x f(x) dx \\ &= \int_0^1 e^x (1) dx \\ &= e^x \Big|_0^1 \\ &= e^1 - e^0 \\ &= e - 1 \end{aligned}$$

Exercise

Example :

Let X denote the resistance of a randomly chosen resistor, and suppose that its pdf is given by

$$f(x) := \begin{cases} \frac{x}{18} & \text{if } 8 \leq x \leq 10 \\ 0 & \text{otherwise.} \end{cases}$$

- 1 Find and graph the cdf of X .
- 2 Find $P(8.6 < X \leq 9 : 8)$.
- 3 Find the median of the resistance of such resistors.
- 4 Find the mean and variance of X .

Exercise

Example :

The length of time to failure (in hundreds of hours) for a transistor is a random variable X with cumulative distribution function given by

$$F(x) := \begin{cases} 1 - e^{-x^2} & \text{for } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

- 1 Find a pdf of X $f(x)$.
- 2 Find the probability that the transistor operates for at least 200 hours.
- 3 Find the 30th percentile of X .

Exercise

Example :

Weekly CPU time used by an accounting firm has probability density function (measured in hours) given by

$$f(x) := \begin{cases} \frac{3}{64}x^2(4 - x) & \text{for } 0 \leq x \leq 4 \\ 0 & \text{otherwise.} \end{cases}$$

- 1 Find the $F(x)$ for weekly CPU time.
- 2 Find the probability that the of weekly CPU time will exceed two hours for a selected week.
- 3 Find the expected value and variance of weekly CPU time.
- 4 Find the probability that the of weekly CPU time will be within half an hour of the expected weekly CPU time.
- 5 The CPU time costs the firm \$200 per hour. Find the expected value and variance of the weekly cost for CPU time.

Exercise

Example :

The length of time required by students to complete a one-hour exam is a random variable with a density function given by

$$f(x) := \begin{cases} cy^2 + y & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

- 1 Find c that makes this function a valid probability density function.
- 2 Find the F(y)
- 3 Find the probability that a randomly selected student will finish in less than half an hour.
- 4 Find the time that 95% of the students finish before it.
- 5 Given that a particular student needs at least 15 minutes to complete the exam, find the probability that she will require at least 30 minutes to finish.

Questions?