

# STAT 320: Principles of Probability

## Unit 4: Conditional Probability & Statistical Independence

United Arab Emirates University

Department of Statistics

# Outline

- 1 Conditional Probability
- 2 Partition of an Event (Reminder From Unit1)
- 3 Law of Total Probability
- 4 Bayes' Theorem
- 5 The Notion of Statistical Independence
- 6 A Few Miscellaneous Examples

# Conditional Probability

# Example

## Example :

Suppose that we toss 2 dice, and suppose that each of the 36 possible outcomes is equally likely to occur, hence has probability  $\frac{1}{36}$ . Suppose further that we observe that the first die is a 3. Then, what is the probability that the sum of the 2 dice equals 8 **given** the additional information that first die is a 3 .?

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

## Definition (Conditional Probability)

Let  $E$ , and  $F$  are two events such that  $P(F) > 0$ , then the conditional probability of  $E$  **given**  $F$  is defined to be,

$$P(E | F) := \frac{P(E \cap F)}{P(F)}.$$

# Example

**Question :** Suppose that a fair die is rolled once. Find the conditional probability that *the number 1 appears*, **given** that *an odd number was obtained*.

$A := \{\text{Observe a 1.}\}$ ,  $B := \{\text{Observe an odd number.}\}$

$$P(A | B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3}.$$

# Example

**Question :** If two events, A and B, are such that  $P(A) = 0.5$ ,  $P(B) = 0.3$ , and  $P(A \cap B) = 0.1$ , find the following

- 1  $P(A | B)$
- 2  $P(B | A)$
- 3  $P(A | A \cup B)$

# Example

**Example :** A coin is flipped twice. Assuming that all four points in the sample space  $\mathcal{S} = \{HH, HT, TH, TT\}$  are

- 1 what is the conditional probability that both flips shows up Heads, given that the first flip is a Head?
- 2 what is the conditional probability that both flips results in Heads, given that at least one flip shows up Head?

Solution:



## Definition (Multiplication Rule of Probability )

Let  $E$  and  $F$  are two events, then

$$P(E \cap F) := P(E | F) \times P(F) .$$

$$P(E_1 \cap E_2 \cap E_3) = P(E_1 \mid E_2 \cap E_3) \times P(E_2 \mid E_3) \times P(E_3) .$$

$$P(E_1 \cap E_2 \cap E_3) P(E_2 \cap E_1 \cap E_3) = P(E_2 \mid E_1 \cap E_3) \times P(E_1 \mid E_3) \times P(E_3) .$$

$$P(E_1 \cap E_2 \cap E_3) = P(E_3 \cap E_2 \cap E_1) = P(E_3 \mid E_2 \cap E_1) \times P(E_2 \mid E_1) \times P(E_1) .$$

$$P(A \cap B \cap C) = P(A | B \cap C) \times P(B | C) \times P(C) .$$

$$P(A \cap B \cap C \cap D) = P(A | B \cap C \cap D) \times P(B | C \cap D) \times P(C | D) \times P(D) .$$

$$P(A \cap B \cap C \cap D \cap E) = P(D \cap A \cap C \cap E \cap B) = P(D | A \cap C \cap E \cap B) \times P(A \cap C \cap E \cap B) .$$

$$\begin{aligned} & P(E_1 \cap E_2 \cap \cdots \cap E_k) \\ := & P(E_1 \mid E_2 \cap \cdots \cap E_k) \times P(E_2 \mid E_3 \cap \cdots \cap E_k) \times \cdots \times \\ & \times P(E_{k-1} \mid E_k) \times P(E_k) . \end{aligned}$$

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## Partition of an Event Using Partition of Sample Space

# Reminder from Unit1 Slides: Disjoint Sets

**Disjoint Sets:**

Two sets  $A$ , and  $B$  are said to be **Disjoint Sets** or **mutually exclusive sets** if  $A$  and  $B$  does not have any elements in common.

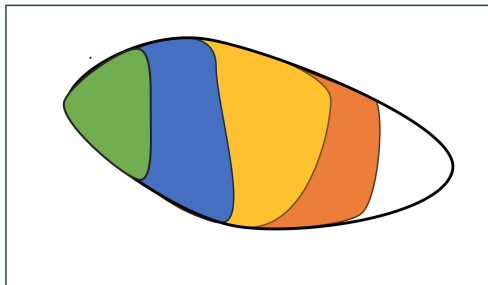
$A$  and  $B$  are Disjoint  $\Leftrightarrow A \cap B = \emptyset$ .

# Reminder from Unit1 Slides: Partition

**Partition:** A collection of sets  $\{A_1, A_2, \dots, A_k\}$  is called a **partition** for a set  $C$  if

1  $A_i \cap A_j = \emptyset$  for  $1 \leq i \neq j \leq k$  (i.e.  $A_i$  and  $A_j$  are Disjoint if  $i \neq j$ ), and

2  $A_1 \cup A_2 \cup \dots \cup A_k = C$ .





# Partition of Sample Space

## Partition of Sample Space:

A collection of sets  $\{A_1, A_2, \dots, A_n\}$  is called a **partition** for a set  $\mathcal{S}$  if

1 the collection of events  $\{A_i\}_{i=1}^n$  are **pairwise disjoint**, and

2 
$$A_1 \cup A_2 \cup \dots \cup A_n = \mathcal{S}.$$

*Comment* : In the above definition, we may replace  $n$  by  $\infty$  and the definition extends naturally.

# Partition of Sample Space

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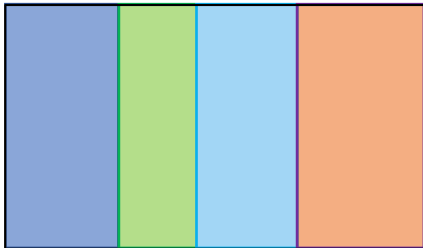
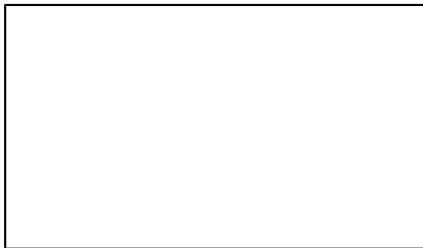
A collection of sets  $\{A_1, A_2, \dots, A_n\}$  is called a **partition** for a set  $\mathcal{S}$  if

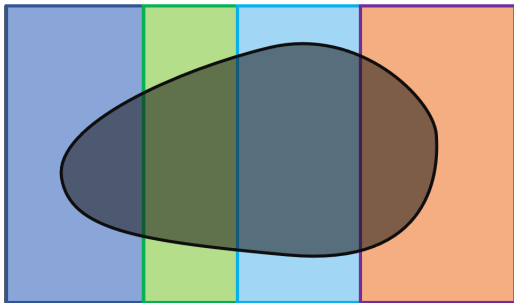
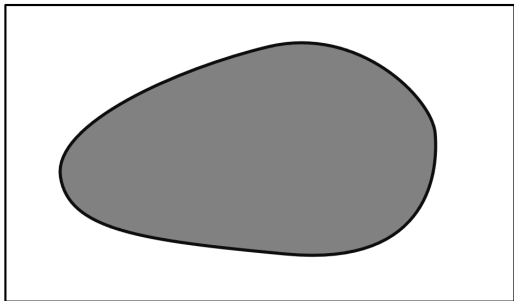
1 the collection of events  $\{A_i\}_{i=1}^n$  are **pairwise disjoint**, and

2 
$$A_1 \cup A_2 \cup \dots \cup A_n = \mathcal{S}.$$

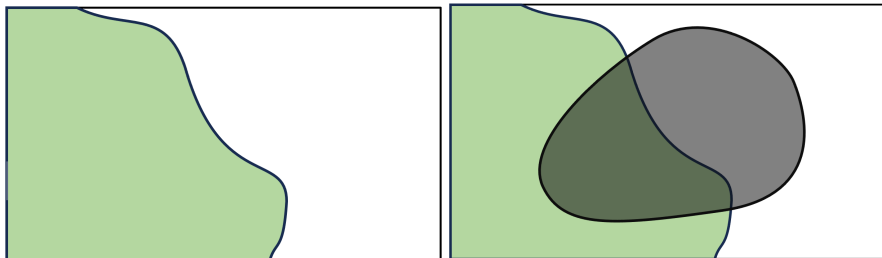
*Comment :* In the above definition, we may replace  $n$  by  $\infty$  and the definition extends naturally.

*Comment :* Any set  $A$  and its complement,  $\bar{A}$ , creates a partition of  $\mathcal{S}$ .









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## Law of Total Probability



# Law of Total Probability



$$E = (E \cap F) \cup (E \cap \bar{F})$$

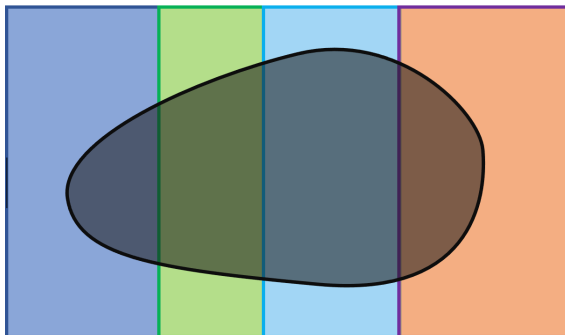
$$P(E) = P(E \cap F) + P(E \cap \bar{F})$$

# Law of Total Probability

Let  $E$  and  $F$  be two events, then

$$P(E) = P(E | F)P(F) + P(E | \bar{F})(\bar{F})$$

# Law of Total Probability (General)



# Law of Total Probability (General)

Law of Total Probability (General):

Let  $E$  be an event. Assuming that the collection of sets  $\{F_1, F_2, \dots, F_k\}$  forms a partition of  $\mathcal{S}$ , we have

$$P(E) = \sum_{j=1}^k P(E | F_j)P(F_j)$$

# Example

**Example :**

An insurance company believes that people can be divided into two classes: those who are accident prone and those who are not. The company's statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability 0.4, whereas this probability is 0.2 for a person who is not accident prone. If we assume that 30% of the population is accident prone, what is the probability that a new policyholder will have an accident within a year of purchasing a policy?

# Example

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**Solution:** Let  $A_1$  denote the event that the policyholder will have an accident within a year of purchasing the policy, and let  $A$  denote the event that the policyholder is accident prone. Hence, the desired probability is given by

$$P(A_1) = P(A_1 | A)P(A) + P(A_1 | \bar{A})P(\bar{A}) = (0.4)(0.3) + (0.2)(0.7) = 0.26$$

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# Bayes' Theorem



## Theorem

Let  $F_1, F_2, \dots, F_K$  be a set of mutually exclusive and exhaustive events (meaning that exactly one of these events must occur). Suppose now that  $E$  has occurred and we are interested in determining which one of the  $F_j$  also occurred. Then, we have the following theorem

$$P(F_i | E) = \frac{P(E | F_i)P(F_i)}{\sum_{j=1}^K P(E | F_j)P(F_j)}$$

- This theorem is well known as Bayes's theorem, after the English philosopher **Thomas Bayes**.
- We can use the following Applet to find the conditional probability using Bayes's theorem:
- [http://www.thomsonedu.com/statistics/book\\_content/0495110817\\_wackerly/applets/seeingstats/Chpt2/bayesTree.html](http://www.thomsonedu.com/statistics/book_content/0495110817_wackerly/applets/seeingstats/Chpt2/bayesTree.html)

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**Solution:** The desired probability is

$$P(A | A_1) = \frac{P(A \cap A_1)}{P(A_1)} = \frac{(0.4)(0.3)}{0.26} = \frac{6}{13}$$

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# The Notion of Statistical Independence

# Statistical Independence

## Definition (Statistically Independent Event)

Two events  $E$  and  $F$  are said to be statistically independent if

$$P(E \cap F) = P(E) \times P(F)$$

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Corollary: Two events  $E$  and  $F$  are independent if and only if  $P(E | F) = P(E)$  and  $P(F | E) = P(F)$ .



# Generalized Definition of Statistical Independence

## Definition (Statistically Independent Events)

Three events  $A_1$ ,  $A_2$ , and  $A_3$  are said to be statistically independent if

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \times P(A_2) \times P(A_3)$$

$$P(A_1 \cap A_2) = P(A_1) \times P(A_2)$$

$$P(A_1 \cap A_3) = P(A_1) \times P(A_3)$$

$$P(A_2 \cap A_3) = P(A_2) \times P(A_3)$$

# Example

**Example :** A card is selected at random from an ordinary deck of 52 playing cards. If  $E$  is the event that the selected card is an ace and  $F$  is the event that it is a spade, then  $E$  and  $F$  are independent.

# Example

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A card is selected at random from an ordinary deck of 52 playing cards. If  $E$  is the event that the selected card is an ace and  $F$  is the event that it is a spade, then  $E$  and  $F$  are independent.

**Solution:** This follows because  $P(E \cap F) = \frac{1}{52}$  whereas  $P(E) = \frac{4}{52}$  and  $P(F) = \frac{13}{52}$ .

# Example

**Example :** Two coins are flipped, and all 4 outcomes are assumed to be equally likely. If  $E$  is the event that the first coin lands on heads and  $F$  the event that the second lands on tails, then  $E$  and  $F$  are independent, since  $P(E \cap F) = P(\{HT\}) = \frac{1}{4}$ . Where as  $P(E) = P(\{HH, HT\}) = \frac{1}{2}$  and  $P(F) = P(\{HT, TT\}) = \frac{1}{2}$

# Example

Example :

Suppose that we toss 2 fair dice. Let  $E_1$  denote the event that the sum of the dice is 6 and  $F$  denote the event that the first die equals 4. Is  $E_1$  statistically independent of  $F$ ?

Answer:  $P(E_1 \cap F) = P(\{(4, 2)\}) = \frac{1}{36}$  where as  $P(E_1) = \frac{5}{36}$  and  $P(F) = \frac{1}{6}$ . Therefore,  $E_1$  and  $F$  are not statistically independent because,  $P(E_1 \cap F) \neq P(E_1) \times P(F)$ .

# Example

**Example :** Now, suppose that we let  $E_2$  be the event that the sum of the dice equals 7. Is  $E_2$  statistically independent of  $F$ ?

Answer:  $P(E_2 \cap F) = P(\{(4, 3)\}) = \frac{1}{36}$  where as  $P(E_2) = \frac{1}{6}$  and  $P(F) = \frac{1}{6}$ . Therefore,  $E_2$  and  $F$  are statistically independent because,  $P(E_2 \cap F) = P(E_2) \times P(F)$ .

**Proposition :** If two events  $A$ , and  $B$  are statistically independent, then

- a).  $A$  and  $\bar{B}$  are also statistically independent,
- b).  $\bar{A}$  and  $B$  are also statistically independent.
- c).  $\bar{A}$  and  $\bar{B}$  are also statistically independent.

Solution:

# Example

**Example :** If A and B are independent events with  $P(A) = 0.5$  and  $P(B) = 0.2$ , find the following:

1  $P(A \cap B)$

2  $P(A \cup B)$

3  $P(\bar{A} \cap \bar{B})$

4  $P(\bar{A} \cup \bar{B})$



# Example

**Example :** If A and B are independent events with  $P(A) = 0.5$  and  $P(B) = 0.2$ , find the following:

- 1  $P(A \cap B)$
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- 3  $P(\bar{A} \cap \bar{B})$
- 4  $P(\bar{A} \cup \bar{B})$

**Solution:**

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## A Few Miscellaneous Examples

**Question :** A diagnostic test for a disease is such that it (correctly) detects the disease in 90% of the individuals who actually have the disease. Also, if a person does not have the disease, the test will report that he or she does not have it with probability 0.9. Only 1% of the population has the disease in question. If a person is chosen at random from the population and the diagnostic test indicates that she has the disease, what is the conditional probability that she does, in fact, have the disease? Are you surprised by the answer? Would you call this diagnostic test reliable?

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$A = \{\text{The individual has the disease}\}$ ,  $\bar{A} = \{\text{does not have the disease}\}$ ,  $B = \{\text{The test shows a POSITIVE result}\}$   
 $\bar{B} = \{\text{The test shows a NEGATIVE result}\}$

$P(B | A) = 0.9$ ,  $P(\bar{B} | A^c) = 0.9$  and  $P(A) = 0.01$

$$P(A | B) = \frac{P(B | A)P(A)}{P(B | A)P(A) + P(B | \bar{A})P(\bar{A})} = \frac{(0.9)(0.01)}{(0.9)(0.01) + (0.1)(0.99)} = 8.33\%$$

**Example :**

An island has three species of bird. Species 1 accounts for 45% of the birds, of which 10% have been tagged. Species 2 accounts for 38% of the birds, of which 15% have been tagged. Species 3 accounts for 17% of the birds, of which 50% have been tagged. If a tagged bird is observed, what are the probabilities that it is of species 1, of species 2, and of species 3?

**Solution:**

# Example

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Solution:

**Example :**

After production, an electrical circuit is given a quality score of A, B, C, or D. Over a certain period of time, 77% of the circuits were given a quality score A, 11% were given a quality score B, 7% were given a quality score C, and 5% were given a quality score D. Furthermore, it was found that 2% of the circuits given a quality score A eventually failed, and the failure rate was 10% for circuits given a quality score B, 14% for circuits given a quality score C, and 25% for circuits given a quality score D.

- 1 If a circuit did not fail, what is the probability that it had received a quality score A?
- 2 If a circuit failed, what is the probability that it had received a quality score either C or D?

**Solution:**



# Example

## Example :

A bin contains 3 different types of disposable ashlights. The probability that a type 1 ashlight will give over 100 hours of use is 0.7, with the corresponding probabilities for type 2 and type 3 ashlights being 0.4 and 0.3, respectively. Suppose that 20 percent of the ashlights in the bin are type 1, 30 percent are type 2, and 50 percent are type 3.

- 1 What is the probability that a randomly chosen ashlight will give more than 100 hours of use?
- 2 Given that a ashlight lasted over 100 hours, what is the conditional probability that it was a type  $j$  ashlight,  $j = 1, 2, 3$ ?

Solution:

# Example

## Example :

We know the following about a colormetric method used to test lake water for nitrates. If water specimens contain nitrates, a solution dropped into the water will cause the specimen to turn red 95% of the time. When used on water specimens without nitrates, the solution causes the water to turn red 10% of the time (because chemicals other than nitrates are sometimes present and they also react to the agent). Past experience in a lab indicates that nitrates are contained in 30% of the water specimens that are sent to the lab for testing. If a water specimen is randomly selected

- 1 from among those sent to the lab, what is the probability that it will turn red when tested?
- 2 If it turns red when tested, what is the probability that it actually contains nitrates?

Solution:

**Example :**

Three radar sets, operating independently, are set to detect any aircraft flying through a certain area. Each set has a probability of .02 of failing to detect a plane in its area. If an aircraft enters the area, what is the probability that it

- 1 goes undetected?
- 2 is detected by at least one radar set?
- 3 is detected by all three radar sets?

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**Solution:**

# Example

**Example :** A card is selected at random from an ordinary deck of 52 playing cards. If  $E$  is the event that the selected card is an ace and  $F$  is the event that it is a spade, then  $E$  and  $F$  are statistically independent.

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**Solution:** This follows because  $P(E \cap F) = \frac{1}{52}$  whereas  $P(E) = \frac{4}{52}$  and  $P(F) = \frac{13}{52}$ .

# Relation between: Statistical Independence & Disjointness

**Result** : If  $A$  and  $B$  are two events such that  $P(A) > 0$  and  $P(B > 0)$  then,

- a). If the events  $A$  and  $B$  are Statistically Independent then they can not be disjoint./mutually exclusive.
- b). If the events  $A$  and  $B$  are disjoint then they can not be statistically independent.

# Example

## Example :

A biased coin is tossed 3 times. Assume that the tosses are independent to each other (Means events concerning only toss 1 is statistically independent to events concerning toss 2). Assume that the probability for Head for the toss is  $\pi$  ( $\pi$  is any number between 0 to 1, for example  $\pi = 0.75$ ). Therefore all the outcomes in the sample space may not be equally likely. Find the probability of the following events

- 1  $\{HHH\}$
- 2  $\{HHT\}, \{THH\}, \{HTH\}$
- 3 Exactly one of the toss results in a Head.
- 4 Exactly two of the tosses result in Head.



Questions?