# STAT 320: Principles of Probability
# Unit 2: A Few Counting Principles & and Their Applications

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Department of Statistics

# Counting Principles: Objective

- We will consider a few basic notation of the Combinatorial analysis.

- An efficient way of counting is required in computing probabilities of events.

- Combinatorial analysis and counting principals is itself a higly nontrivial topic. Therefore, we will mostly keep our focus on the examples and procedures that would be reciprocal for our discussion on Probability.

# Outline

# Multiplication Principle

# Overview of Counting Principals

**Definition (Multiplication Principle of counting)**

*Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible outcomes and if, for each outcome of experiment 1, there are n possible outcomes of experiment 2, then together there are $m \times n$ possible outcomes of the two experiments.*

- If a task involves two steps and the first step can be completed in $m$ ways and the second step in $n$ ways, then there are $m \times n$ ways to complete the task.

# Example

**Example 1:** Enrollment in the course Principles of probability consists of: 28 statistics majors, and 53 math major students. If 2 students are selected at random. In how many ways can we select one math and one statistics student?

Solution: $53 \times 28 = 1484$.

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**Definition (Genralized Multiplication Principle of counting)**

*If there are k steps in an operation of which first can be done in $n_1$ ways, for each of these second can be done in $n_2$ ways, for each of the first two the third step can be done in $n_3$ ways, and so forth, then the whole operation can be done in* $n_1 \times n_2 \times n_3 \times \cdots \times n_k$ *ways.*

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# Example

**Example 3:** A college planning committee consists of 3 freshmen, 4 sophomores, 5 juniors, and 2 seniors. A subcommittee of 4, consisting of 1 person from each class, is to be chosen. How many different subcommittees are possible?

Solution: We may regard the choice of a subcommittee as the combined outcome of the four separate experiments of choosing a single representative from each of the classes. It then follows from the generalized version of the basic principle that there are $3 \times 4 \times 5 \times 2 = 120$ possible subcommittees.

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**Example 4:** A car manufacturer provides cars with the following different variations:

1. Manual or automatic transmission
2. With or without air conditioning
3. Three different stereo systems
4. Four possible exterior colors

How many different types of car the manufacturer sells?

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**Example :** How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

Solution: By the generalized version of the basic principle, the answer is $26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 = 175,760,000$.

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**Example 5:** An airline has six flights (6 flight numbers) from New York to California and seven flights (7 different flight numbers ) from California to Hawaii per day. If the flights are to be made on separate days, how many different flight arrangements can the airline offer from New York to Hawaii?

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# Outline

- The Notion of Order in Arrangements
- The Notion of With Replacement or Without Replacement

# **Ordered** Arrangements **with** Replacements

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## Definition (Ordered Arrangements with Replacement)

Let $n, r$ be positive integers. Number of different ways a sequence of $r$ symbols can be created from $n$ distinct symbols (when repetition of the symbols are allowed) is $n^r$.

You may also view the procedure as the following: Number of different ways $r-$tuples (a vector with $r$ coordinates) can be constructed where elements of each coordinates are chosen arbitrary from a set containing $n$ distinct objects.

Example : From the numbers {1, 2, . . . ,44}, a person may pick any six for her ticket. How many different groups of six numbers can be chosen from the forty-four **if the repeated selection of the numbers is allowed while the same numbers selected in a different order is considered different sequence?**

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# Permutations: **Ordered** Arrangements **Without** Replacements

## Definition (Factorial)

For a positive integer $n$, $n!$ (read $n$ factorial) is the product of all of the positive integers less than or equal to $n$. That is,

$\boxed{n!} = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1.$ Furthermore, we define $\boxed{0!} = 1$.

Example: $4! = 4 \times 3 \times 2 \times 1 = 24.$

Example: $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720.$

- Note that $n! = n \times (n-1)!$, for $n$ integer and $n \geq 1$.

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## Definition (Permutations (Ordered, without replacement))

Let $r$ be a positive integer such that $r \leq n$. An ordered arrangement of $r$ distinct objects is called a permutation. The number of ways of ordering $n$ distinct objects taken $r$ at a time will be designated by the symbol ${}^{n}P_{r}$. Note that

$$ {}^{n}P_{r} = n(n-1)(n-2)(n-r+1) = \frac{n!}{(n-r)!}. $$

This is also the different ways $r$ different objects can be chosen from n different objects when the order of the choice is important. However, there is no repetition in the choices of $r$ objects.

The number of possible permutations of n distinct objects is given by: $n! = n \times (n-1) \times \cdots \times 2 \times 1$.

**Example :** How many different words can you make with the letters "CAN" ?

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**Example :** A class in probability theory consists of 6 men and 4 women. An examination is given, and the students are ranked according to their performance. Assume that no two students obtain the same score.

(a) How many different rankings are possible?

(b) If the men are ranked just among themselves and the women just among themselves, how many different rankings are possible?

Solution:

(a) Because each ranking corresponds to a particular ordered arrangement of the 10 people, the answer to this part is $10! = 3,628,800$.

(b) Since there are 6! possible rankings of the men among themselves and 4! possible rankings of the women among themselves, it follows from the basic principle that there are $(6!)(4!) = (720)(24) = 17280$ possible rankings in this case.

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**Example :** Ms. Jones has 10 books that she is going to put on her bookshelf. Of these, 4 are mathematics books, 3 are chemistry books, 2 are history books, and 1 is a language book. Ms. Jones wants to arrange her books so that all the books dealing with the same subject are together on the shelf. How many different arrangements are possible?

Solution: There are 4!3!2!1! arrangements such that the mathematics books are first in line, then the chemistry books, then the history books, and then the language book. Similarly, for each possible ordering of the subjects, there are 4!3!2!1! possible arrangements. Hence, as there are 4! possible orderings of the subjects, the desired answer is $4!(4!3!2!1!) = 6912$.

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# Outline

# Combinations:

Arrangements **without replacement** when **order is not important**

# Combinations (Un-ordered, without replacement)

## Definition (Combinations)

The number of combinations of $n$ objects taken $r$ at a time is the number of subsets, each of size $r$ , that can be formed from the $n$ objects. This number will be denoted by $^{n}C_{r}$ or

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

This is also the different ways $r$ different objects can be chosen from n different objects when the order of the choice is not important. However, there is no repetition in the choices of $r$ objects.

**Example :** A motherboard has 8 slots, where we want to place 4 **identical** cards. How many possibilities are there?

Ans: $\binom{8}{4} = \frac{8!}{(8-4)!(4!)}$

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**Example :** From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed? What if 2 of the men are feuding and refuse to serve on the committee together?

Solution: $\binom{5}{2} \times \binom{7}{3} = 350$

Now a total of $\binom{2}{2} \times \binom{5}{1} = 5$ out of $\binom{7}{3} = 35$ possible groups of 3 men contain both of the feuding men, it follows that there are 35 -5 = 30 groups that do not contain both of the feuding men. Then, the possible number of committees becomes

$30 \times 10 = 300$

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## Definition (Multinomial Probability)

The number of ways of partitioning $n$ distinct objects into $k$ distinct groups containing $n_1, n_2, \ldots, n_k$ objects, respectively, where each object appears in exactly one group and $\sum_{i=1}^{k} n_i = n$, is

$$\binom{n}{n_1, n_2, \ldots, n_k} := \frac{n!}{(n_1!)(n_2!)\ldots(n_k!)}$$

The above procedure is also utilized to determine the number of permutations of a set of n objects when certain of the objects are indistinguishable from each other.

**Example :** How many different words can you make with the letters "PEPPER" ?

Solution: $\frac{6!}{(3!)(2!)(1!)}$

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**Example :** A chess tournament has 10 competitors, of which 4 are Russian, 3 are from the United States, 2 are from Great Britain, and 1 is from Brazil. If the tournament result lists just the nationalities of the players in the order in which they placed, how many outcomes are possible?

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**Example :** How many different signals, each consisting of 9 flags hung in a line, can be made from a set of 4 white flags, 3 red flags, and 2 blue flags if all flags of the same color are identical?

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**Example :** A police department in a small city consists of 10 officers. If the department policy is to have 5 of the officers patrolling the streets, 2 of the officers working full time at the station, and 3 of the officers on reserve at the station, how many different divisions of the 10 officers into the 3 groups are possible?

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# Outline

## Arrangements **With Replacement** When **Order is Not Important**

# Unordered Arrangements with Replacement

## Definition (Unordered Arrangements with Replacement)

Number of ways $n$ indistinguishable objects can be organized into r different groups is

$$\frac{(n+r-1)!}{n!(r-1)!} = \binom{n+r-1}{n}.$$

The processes can viewed as: How many different ways $n$ identical balls can be placed in $r$ different urns. Ordering of the groups are not important.

It is also referred to as the occupancy problem

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Solution: $\frac{(6+44-1)!}{(43!)(6!)} = \binom{49}{43}$

# Binomial and Multinomial Coefficient

Questions?