STAT 320: Principles of Probability

Unit 5(PART:A)

Introduction to Random Variables & Discrete Random Variables

United Arab Emirates University

Department of Statistics

Outline

1. Random Variables
2. Discrete Random Variables
3. Expected Value and Variance

Random Variables

Random Variables

1. Frequently, when an experiment is performed, we are interested mainly in some function of the outcome as opposed to the actual outcome itself.

2. For instance, in tossing dice, we are often interested in the sum of the two dice and are not really concerned about the separate values of each die.

3. These quantities of interest, or, more formally, these real-valued functions defined on the sample space, are known as random variables.

Random Variables

Events of major interest to the scientist, engineer, or businessperson are those identified by numbers, called numerical events. The research physician is interested in the event that ten of ten treated patients survive an illness; the businessperson is interested in the event that sales next year will reach $3 million. Let Y denote a variable to be measured in an experiment. The realized value of Y will vary depending on the outcome of the experiment. Thus it is called a random variable.

Definition (Random Variable)

A random variable is a function from a sample space $\mathscr{S}$ into the real numbers.

Example: Random Variable

Experiment	Random Variable
Toss two dice	X= sum of the numbers
Toss a coin 25 times	X number of heads in 25 tosses
Apply different amounts of fertilizer to corn plants	X = yield/acre

Notation: Random variables will always be denoted with uppercase letters and the realized values of the variable (or its range) will be denoted by the corresponding lowercase letters. Thus, the random variable X can take the value x.

Support/Range of a Random Variable

Definition (Support/Range of a Random Variable)

The set containing the all possible values of a random variable is called its **support** or **range**.

Notation: We will use the notation $\mathbb{S}_x$ (or simply $\mathbb{S}$ if there is no ambiguity) to denote the support of a random variable X.

Example: Consider the experiment of tossing a fair coin 3 times from. Define the random variable X to be the number of heads obtained in the 3 tosses. The Support of the random variable is $\mathbb{S}_x = \{0, 1, 2, 3\}$

Example

Example : Suppose that our experiment consists of tossing 3 fair coins. If we let Y denote the number of heads that appear, then Y is a random variable taking on one of the values $0, 1, 2$, and 3 with respective probabilities.

$$p_Y(0) = P(Y = 0) =$$
$$p_Y(1) = P(Y = 1) =$$
$$p_Y(2) = P(Y = 2) =$$
$$p_Y(3) = P(Y = 3) =$$

Example

Example : Suppose that our experiment consists of tossing 3 fair coins. If we let Y denote the number of heads that appear, then Y is a random variable taking on one of the values $0, 1, 2$, and 3 with respective probabilities.

$$p_Y(0) = P(Y = 0) =$$
$$p_Y(1) = P(Y = 1) =$$
$$p_Y(2) = P(Y = 2) =$$
$$p_Y(3) = P(Y = 3) =$$

Outline

1 Random Variables

2 Discrete Random Variables

3 Expected Value and Variance

Discrete Random Variables & its Probability Mass Function (pmf)

Definition (Discrete Random Variables)

A random variable that can take on at most a countable number of possible values is said to be discrete. *That is, if* $\mathbb{S}$*, the support of a random variable is finite or countable infinite then the corresponding random variable is discrete.*

Probability Mass Function (pmf)

For a discrete random variable X, we define the probability mass function (pmf) $p_x(x)$ of X by

$$p_x(x) = P(X = x) \text{ for all } x \in \mathbb{S}_x$$

Let X be a discrete random variable with probability mass function $p(x)$ defined on the support $\mathbb{S}$. Let $A \subset \mathbb{S}$ be an event, then

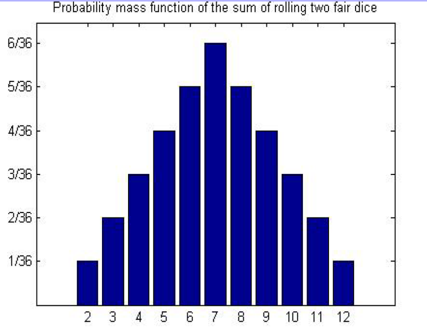
$$P\left(A\right) := P(X \in A) = \sum_{\left\{x \in A\right\}} p(x).$$

The pmf of the random variable representing the sum when two dice are rolled can be represented in multiple ways.

As a Tabular Format:

x	2	3	4	5	6	7	8	9	10	11	12
$p_x(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

As a Plot/Graph:



As a Function

$$p_x(x) = \begin{cases} \frac{x-1}{36} & \text{if } 2 \leq x \leq 7, \\ \frac{13-x}{36} & \text{if } 8 \leq x \leq 12 \end{cases}$$

Characterization of a pmf

Let $p(x)$ is **probability mass function** of a discrete random variable on the support $\mathbb{S}$, **if and only if** it satisfies the following conditions:

1. *Positivity:* $p(x) > 0$ for all $x \in \mathbb{S}$
2. *Total Probability:* $\sum_{\{x \in \mathbb{S}\}} p(x) = 1$.

Example

Example : A system consists of 2 components connected in parallel, then at least one must work correctly for the system to work correctly. Each component operates correctly with probability 0.8 and independent of the other. Let X be the number of components that work correctly. Find the probability distribution of X.

Example

Example : A system consists of 2 components connected in parallel, then at least one must work correctly for the system to work correctly. Each component operates correctly with probability 0.8 and independent of the other. Let X be the number of components that work correctly. Find the probability distribution of X.

Solution: X can take on only three possible values; 0, 1, or 2. Let E_i denote the event that component i works correctly. Then $P(E_i) = 0.8$. Thus, we have

- $p_X(0) = P(\overline{E_1} \cap \overline{E_2}) = P(\overline{E_1})P(\overline{E_2}) = (0.2)(0.2) = 0.04.$
- $p_X(1) = P(\overline{E_1} \cap E_2) + P(E_1 \cap \overline{E_2}) = (0.2)(0.8) + (0.8)(0.2) = 0.32.$
- $p_X(2) = P(E_1 \cap E_2) = P(E_1)P(E_2) = (0.8)(0.8) = 0.64.$

x	0	1	2
$p_X(x)$	0.04	0.32	0.64

Cumulative Distribution Function (CDF) of a discrete Random Variable

cumulative distribution function

Definition (cumulative distribution function)

Let X be a discrete random variable on the support $\mathbb{S}_X$ with the corresponding probability mass function

$$\boxed{P(X = x) = p_X(x) \text{ for } x \in \mathbb{S}_X}.$$

Then for any $a \in \mathbb{R}$, the cumulative distribution function (cdf), denoted by $F_X(\cdot)$ is the following quantity

$$\boxed{F_X(a) = P(X \leq a) = \sum_{\{x \leq a \,:\, x \in \mathbb{S}_X\}} p_X(x)}$$

A **pmf** of a discrete random variable is only positive/ revelant on the support of the random variable $\mathbb{S}$, However the **CDF** is defined for any real number.

Example

Example : If X be a discrete random variable on the support $\mathbb{S}_x = \{1, 2, 3, 4\}$ with the corresponding pmf specified as $p_x(1) = \frac{1}{4}, p_x(2) = \frac{1}{2}, p_x(3) = \frac{1}{8}$, and $p_x(4) = \frac{1}{8}$. Calculate the CDF function of X.

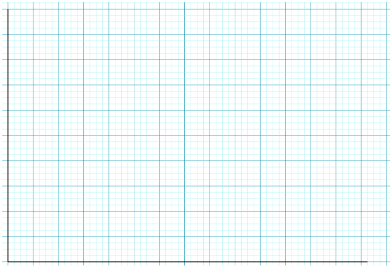
Example

Example : If X be a discrete random variable on the support $\mathbb{S}_X = \{1, 2, 3, 4\}$ with the corresponding pmf specified as $p_x(1) = \frac{1}{4}, p_x(2) = \frac{1}{2}, p_x(3) = \frac{1}{8}$, and $p_x(4) = \frac{1}{8}$. Calculate the CDF function of X.

Solution:

$$F_X\left(\boxed{a}\right) = \begin{cases} 0 & \text{if } \boxed{a} < 1 \\ \frac{1}{4} & \text{if } 1 \leq \boxed{a} < 2 \\ \frac{3}{4} & \text{if } 2 \leq \boxed{a} < 3 \\ \frac{7}{8} & \text{if } 3 \leq \boxed{a} < 4 \\ \frac{7}{8} & \text{if } 4 \leq \boxed{a} \end{cases}$$

Graph of CDF of X



Let the pmf of a discrete random varibale X is given as

x	0	1	2
$p_X(x)$	0.04	0.32	0.64

Find the corresponding CDF.

Let the pmf of a discrete random varibale X is given as

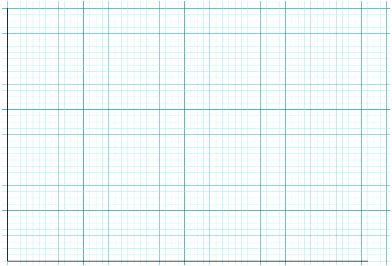
x	0	1	2
$p_X(x)$	0.04	0.32	0.64

Find the corresponding CDF.

Solution:

$$F_X\left(a\right)=\begin{cases}0 & \text{if } a<0\\ 0.04 & \text{if } 0\leq a<1\\ 0.36 & \text{if } 1\leq a<2\\ 1 & \text{if } 2\leq a\end{cases}$$

Graph of CDF of X



Outline

1. Random Variables
2. Discrete Random Variables
3. Expected Value and Variance

Expected Value & Variance of a Discrete Random Variable

The "**Expected Value**" or "**Mean**" of a Discrete Random Variable

Definition (The "**Expected Value**" or "**Mean**" of a Discrete Random Variable)

If X is a random variable with pmf $p_X(x)$ on the support $\mathbb{S}_X$, then the expected value (the mean) of X denoted by $E(X)$ (or μ_X) is given by

$$\boxed{E\left(X\right) = \sum_{\left\{x \in \mathbb{S}_X\right\}} x\, p_X(x),}$$

assuming the above summation/series exists /well-defined.

Definition (The Expected Value of a Function of a Discrete Random Variable)

Let the random variable X has the probability mass funciton $p_X(x)$ for all $x \in \mathbb{S}_X$, the support of X. Let $h(x)$ be any* function, then the expected value of $h(X)$ is defined as

$$\boxed{E\left(h(X)\right) = \sum_{\{x \in \mathbb{S}_X\}} h(x)\, p_X(x),}$$

assuming the above summation/series exists /well-defined.

Variance

Variance The variance of X, denoted by $\mathrm{Var}(X)$ is deifined as

$$\boxed{\mathrm{Var}(X) := E\left(X - \mu_X\right)^2},$$

where $\mu_X = E(X)$, the mean of the random variable.

Definition (Variance)

The variance of X, denoted by $\mathrm{Var}(X)$ is deifined as

$$\boxed{\mathrm{Var}(X) := E(X^2) - \left(E(X)\right)^2}$$

Standard Deviation

Definition (Variance)

The variance of X, denoted by Var(X) is deifined as

$$\boxed{\sigma_X = \mathrm{SD}(X) := \sqrt{\mathrm{Var}(X)}}$$

Var(X) is often denoted by σ^2.

Properties of Expected Value and Variance

1. $E(a + bX) = a + bE(X)$
2. $\text{Var}(a + bX) = b^2\text{Var}(X)$

Moment Generating Function (mgf)

Moment Generating Function (mgf)

Definition (Moment Generating Function)

The Moment Generating Function (mgf) of X, denoted by $\mathrm{M}_X(t)$ is deifined as

$$\boxed{\mathrm{M}_X(t) := E\left(e^{tX}\right)},$$

whenever it exists.

If the random variable X has the probability mass funciton $p_X(x)$ for all $x \in \mathbb{S}_X$, the support of X, then assuming it exists

$$M_X(t) := E\left(e^{tX}\right) = \sum_{\{x \in \mathbb{S}_X\}} e^{tx} \, p_X(x).$$

Properties of a Moment Generating Function

- If it exists, the moment generating function is unique for a random variable. It means, no two random variable/distribution can have same moment geneting function. Therefore, a distribution can be identified by the form of its moment generating function.

- If it exists, the mgf can be used to obtain the moments of a random variable in the following way:

Assuming it exists

$$\frac{d}{dt}\{M_X(t)\}\bigg|_{t=0} = E(X)$$

Assuming it exists

$$\frac{d^k}{dt^k}\{M_X(t)\}\bigg|_{t=0} = E(X^k) \text{ for } k = 1, 2, \ldots$$

Example

The probability distribution of X, the number of daily network black-outs is given by

x	0	1	2
$p_X(x)$	0.7	0.2	0.1

Find the Expected value and variance of the random variable **X**.

Example

The probability distribution of X, the number of daily network black-outs is given by

x	0	1	2
$p_X(x)$	0.7	0.2	0.1

Find the Expected value and variance of the random variable **X.**

Solution:

$$\begin{aligned}\mu_X = E(X) &= \sum_{x\in\{0,1,2\}} x p_X(x)\\ &= 0\times p_X(0)+1\times p_X(1)+2\times p_X(2)\\ &= 0\times 0.7+1\times 0.2+2\times 0.1\\ &= 0.4\end{aligned}$$

$$\begin{aligned}E(X^2) &= \sum_{x\in\{0,1,2\}} x^2 p_X(x)\\ &= 0^2\times p_X(0)+1^2\times p_X(1)+2^2\times p_X(2)\\ &= 0\times 0.7+1\times 0.2+4\times 0.1\\ &= 0.6\end{aligned}$$

Hence $\mathrm{Var}(X) := E(X^2)-(E(X))^2 = 0.6-(0.4)^2 = 0.6-0.16 = 0.44$

The Standard Deviation $\sigma = \sqrt{\sigma^2} = \sqrt{\mathrm{Var}(X)} = \sqrt{0.44} = 0.6633$

Example

The probability distribution of X, the number of daily network blackouts is given by

x	0	1	2
$p_X(x)$	0.7	0.2	0.1

A small internet trading company estimates that each network blackout results in a \$500 loss. Compute expectation and variance of this company's daily loss due to blackouts.

Example

The probability distribution of X, the number of daily network blackouts is given by

x	0	1	2
$p_X(x)$	0.7	0.2	0.1

A small internet trading company estimates that each network blackout results in a $500 loss. Compute expectation and variance of this company's daily loss due to blackouts.

The daily loss due to blackouts is given by $h(X) = 500X$. We need to find $E(h(X))$ and Varinace of $Var(h(X))$.

Solution:

$$\begin{aligned}\mu_X = E(X) &= \sum_{x\in\{0,1,2\}} x p_X(x) \\ &= 0 \times p_X(0) + 1 \times p_X(1) + 2 \times p_X(2) \\ &= 0 \times 0.7 + 1 \times 0.2 + 2 \times 0.1 \\ &= 0.4\end{aligned}$$

$$\begin{aligned}E(X^2) &= \sum_{x\in\{0,1,2\}} x^2 p_X(x) \\ &= 0^2 \times p_X(0) + 1^2 \times p_X(1) + 2^2 \times p_X(2) \\ &= 0 \times 0.7 + 1 \times 0.2 + 4 \times 0.1 \\ &= 0.6\end{aligned}$$

Let the pmf of a discrete is given as

$$p_x(x) := \frac{1}{2^x} \text{ for } x = 1, 2, 3, \ldots$$

Exercises on Computing $E(X)$, $Var(X)$, MGF

Find $E(X)$ and $Var(X)$, where X is the outcome when we roll a fair die.

Exercises on Computing $E(X)$, $Var(X)$, MGF

We say that $\mathbb{I}_A(x)$ is an indicator function for the event A if

$$\mathbb{I}_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Obtain $E(\mathbb{I}_A(X))$ and $\text{Var}\left(\mathbb{I}_A(x)\right)$.

Questions?