STAT 320: Principles of Probability
Unit 4: Conditional Probability & Statistical Independence

United Arab Emirates University

Department of Statistics

Outline

Conditional Probability

Example

Example : Suppose that we toss 2 dice, and suppose that each of the 36 possible outcomes is equally likely to occur, hence has probability $\frac{1}{36}$. Suppose further that we observe that the first die is a 3. Then, what is the probability that the sum of the 2 dice equals 8 **given** the additional information that first die is a 3.?

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Definition (Conditional Probability)

Let E, and F are two events such that $P\left(F\right) > 0$, then the conditional probability of E **given** F is defined to be,

$$P\left(E \mid F\right) := \frac{P\left(E \cap F\right)}{P(F)}.$$

Example

Question : Suppose that a fair die is rolled once. Find the conditional probability that *the number 1 appears*, **given** that *an odd number was obtained.*

A:= { Observe a 1.}, B:={Observe an odd number.}

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3}.$$

Example

Question : If two events, A and B, are such that $P(A) = 0.5$, $P(B) = 0.3$, and $P(A \cap B) = 0.1$, find the following

1. $P(A \mid B)$
2. $P(B \mid A)$
3. $P(A \mid A \cup B)$

Example

Example : A coin is flipped twice. Assuming that all four points in the sample space $\mathscr{S} = \{HH, HT, TH, TT\}$ are

1. what is the conditional probability that both flips shows up Heads, given that the first flip is a Head?
2. what is the conditional probability that both flips results in Heads, given that at least one flip shows up Head?

Solution:

Definition (Multiplication Rule of Probability)

Let E and F are two events, then

$$P(E \cap F) := P(E \mid F) \times P(F).$$

$$P(E_1 \cap E_2 \cap E_3) = P(E_1 \mid E_2 \cap E_3) \times P(E_2 \mid E_3) \times P(E_3).$$

$$P(E_1 \cap E_2 \cap E_3)\, P(E_2 \cap E_1 \cap E_3) = P(E_2 \mid E_1 \cap E_3) \times P(E_1 \mid E_3) \times P(E_3).$$

$$P(E_1 \cap E_2 \cap E_3) = P(E_3 \cap E_2 \cap E_1) = P(E_3 \mid E_2 \cap E_1) \times P(E_2 \mid E_1) \times P(E_1).$$

$$P(A \cap B \cap C) = P(A \mid B \cap C) \times P(B \mid C) \times P(C).$$

$$P(A \cap B \cap C \cap D) = P(A \mid B \cap C \cap D) \times P(B \mid C \cap D) \times P(C \mid D) \times P(D).$$

$$P(A \cap B \cap C \cap D \cap E) = P(D \cap A \cap C \cap E \cap B) = P(D \mid A \cap C \cap E \cap B) \times P(A \cap C \cap E \cap B).$$

$$\begin{aligned} & \boxed{P(E_1 \cap E_2 \cap \cdots \cap E_k)} \\ := \; & \boxed{P(E_1 \mid E_2 \cap \cdots \cap E_k)} \times \boxed{P(E_2 \mid E_3 \cap \cdots \cap E_k)} \times \ldots \times \\ & \times \boxed{P(E_{k-1} \mid E_k)} \times \boxed{P(E_k)}. \end{aligned}$$

Outline

Partition of an Event Using Partion of Sample Space

Reminder from Unit1 Slides: Disjoint Sets

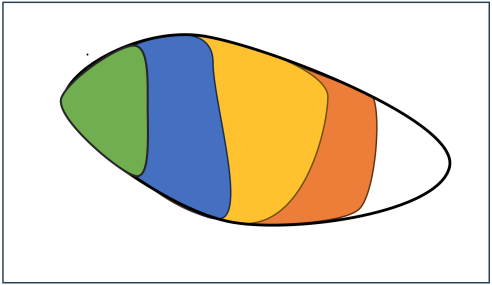
Disjoint Sets: Two sets A, and B are said to be **Disjoint Sets** or **mutually exclusive sets** if A and B does not have any elements in common.

A and B are Disjoint $\Leftrightarrow$ $A \cap B = \emptyset$.

Reminder from Unit1 Slides: Partition

Partition: A collection of sets $\{A_1, A_2, \cdots, A_k\}$ is called a **partition for a set** C if

1. $A_i \cap A_j = \emptyset$ for $1 \leq i \neq j \leq k$ (i.e. ,A_i, and A_j are Disjoint if $i \neq j$), and
2. $A_1 \cup A_2 \cup \cdots \cup A_k = C$.



Partition of Sample Space

Partition of Sample Space: A collection of sets $\{A_1, A_2, \cdots, A_n\}$ is called a **partition** for a set $\mathscr{S}$ if

1. the collection of events $\{A_i\}_{i=1}^{n}$ are **pairwise disjoint**, and
2. $A_1 \cup A_2 \cup \cdots \cup A_n = \mathscr{S}$.

Comment : In the above definition, we may replace n by ∞ and the definition extends naturally.

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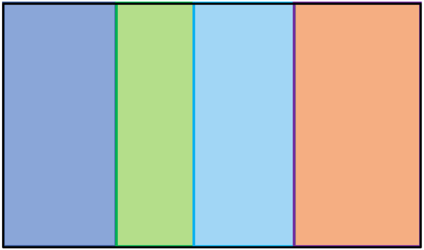
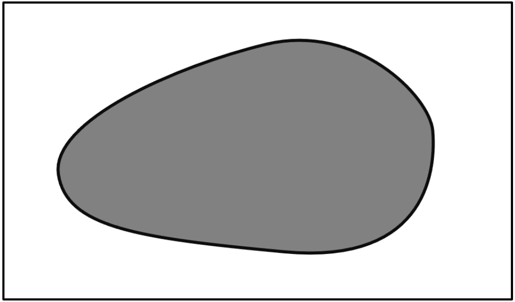
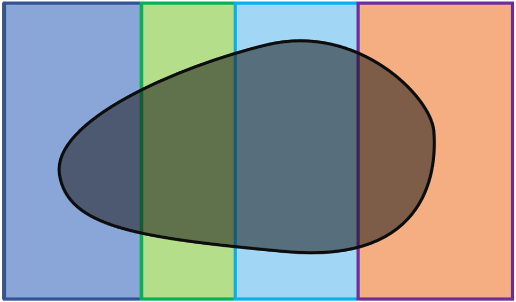
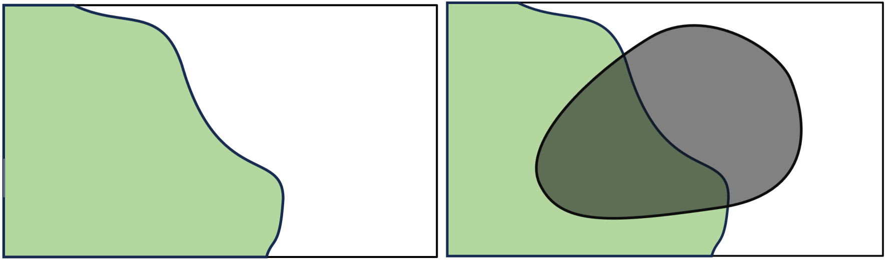
Comment : Any set A and it's complement, $\overline{A}$, creates a partition of $\mathscr{S}$.









Outline

Law of Total Probability

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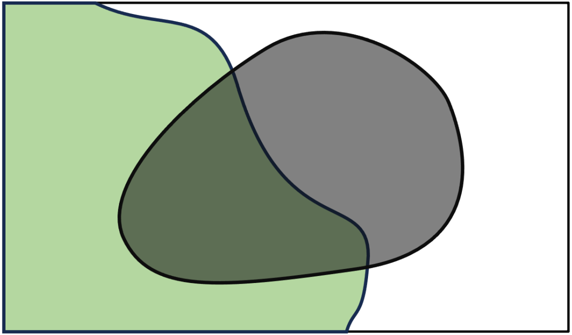


$$E = (E \cap F) \cup (E \cap \overline{F})$$

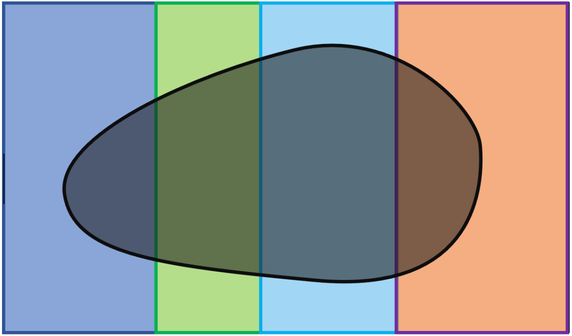
$$P(E) = P(E \cap F) + P(E \cap \overline{F})$$

Law of Total Probability

Let E and F be two events, then

$$P(E) = P(E \mid F)P(F) + P(E \mid \overline{F}))(\overline{F})$$

Law of Total Probability (General)



Law of Total Probability (General)

Law of Total Probability (General): Let E be an event. Assuming that the collection of sets $\{F_1, F_2, \ldots, F_k\}$ forms a partition of $\mathscr{S}$, we have

$$P(E) = \boxed{\sum_{j=1}^{k} P(E \mid F_j)P(F_j)}$$

Example

Example : An insurance company believes that people can be divided into two classes: those who are accident prone and those who are not. The companys statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability 0.4, whereas this probability is 0.2 for a person who is not accident prone. If we assume that 30% of the population is accident prone, what is the probability that a new policyholder will have an accident within a year of purchasing a policy?

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Solution: Let A_1 denote the event that the policyholder will have an accident within a year of purchasing the policy, and let A denote the event that the policyholder is accident prone. Hence, the desired probability is given by

$$P(A_1) = P(A_1 \mid A)P(A) + P(A1 \mid \overline{A})P(\overline{A}) = (0.4)(0.3) + (0.2)(0.7) = 0.26$$

Outline

Bayes' Theorem

Theorem

Let $F_1, F_2, \ldots, F_K$ be a set of mutually exclusive and exhaustive events (meaning that exactly one of these events must occur). Suppose now that E has occurred and we are interested in determining which one of the F_j also occurred. Then, we have the following theorem

$$\boxed{P(F_i \mid E) = \frac{P(E \mid F_i)P(F_i)}{\sum_{j=1}^{K} P(E \mid F_j)P(F_j)}}$$

- This theorem is well known as Bayes's theorem, after the English philosopher **Thomas Bayes**.
- We can use the following Applet to find the conditional probability using Bayes's theorem:
- `http://www.thomsonedu.com/statistics/book_content/0495110817_wackerly/applets/seeingstats/Chpt2/bayesTree.html`

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Suppose that a new policyholder has an accident within a year of purchasing a policy. What is the probability that he or she is accident prone?

Solution: The desired probability is

$$P(A \mid A_1) = \frac{P(A \cap A_1)}{P(A_1)} = \frac{(0.4)(0.3)}{0.26} = \frac{6}{13}$$

Outline

The Notion of Statistical Independence

Statistical Independence

Definition (Statistically Independent Event)

Two events E and F are said to be statistically independent if

$$\boxed{P(E \cap F) = P(E) \times P(F)}$$

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Corollary: Two events E and F are independent if and only if $P(E \mid F) = P(E)$ and $P(F \mid E) = P(F)$.

Generalized Definition of Statistical Independence

Definition (Statistically Independent Events)

Three events A_1, A_2, and A_3 are said to be statistically independent if

$$P(A_1 \cap A_1 \cap A_3) = P(A_1) \times P(A_2) \times P(A_3)$$

$$P(A_1 \cap A_2) = P(A_1) \times P(A_2)$$

$$P(A_1 \cap A_3) = P(A_1) \times P(A_3)$$

$$P(A_2 \cap A_3) = P(A_2) \times P(A_3)$$

Example

Example : A card is selected at random from an ordinary deck of 52 playing cards. If E is the event that the selected card is an ace and F is the event that it is a spade, then E and F are independent.

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Solution: This follows because $P(E \cap F) = \frac{1}{52}$ **whereas** $P(E) = \frac{4}{52}$ **and** $P(F) = \frac{13}{52}$.

Example

Example : Two coins are flipped, and all 4 outcomes are assumed to be equally likely. If E is the event that the first coin lands on heads and F the event that the second lands on tails, then E and F are independent, since $P(E \cap F) = P(\{HT\}) = \frac{1}{4}$. Where as $P(E) = P(\{HH, HT\}) = \frac{1}{2}$ and $P(F) = P(\{HT, TT\}) = \frac{1}{2}$

Example

Example : Suppose that we toss 2 fair dice. Let E_1 denote the event that the sum of the dice is 6 and F denote the event that the first die equals 4. Is E_1 statistically independent of F?

Answer: $P(E_1 \cap F) = P(\{(4,2)\}) = \frac{1}{36}$ where as $P(E_1) = \frac{5}{36}$ and $P(F) = \frac{1}{6}$. Therefore, E_1 and F are not statistically independet because, $P(E_1 \cap F) \neq P(E_1) \times P(F)$.

Example

Example : Now, suppose that we let E_2 be the event that the sum of the dice equals 7. Is E_2 statistically independent of F?

Answer: $P(E_2 \cap F) = P(\{(4,3)\}) = \frac{1}{36}$ where as $P(E_2) = \frac{1}{6}$ and $P(F) = \frac{1}{6}$. Therefore, E_2 and F are statistically independet because, $P(E_2 \cap F) = P(E_2) \times P(F)$.

Proposition : If two events A, and B are statistically independent, then

a). A and $\overline{B}$ are also statistically independent,

b). $\overline{A}$ and B are also statistically independent.

c). $\overline{A}$ and $\overline{B}$ are also statistically independent.

Solution:

Example

Example : If A and B are independent events with P(A) = 0.5 and P(B) = 0.2, find the following:

1. $P(A \cap B)$
2. $P(A \cup B)$
3. $P(\overline{A} \cap \overline{B})$
4. $P(\overline{A} \cup \overline{B})$

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2. $P(A \cup B)$
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4. $P(\overline{A} \cup \overline{B})$

Solution:

Outline

A Few Miscellaneous Examples

Question : : A diagnostic test for a disease is such that it (correctly) detects the disease in 90% of the individuals who actually have the disease. Also, if a person does not have the disease, the test will report that he or she does not have it with probability 0.9. Only 1% of the population has the disease in question. If a person is chosen at random from the population and the diagnostic test indicates that she has the disease, what is the conditional probability that she does, in fact, have the disease? Are you surprised by the answer? Would you call this diagnostic test reliable?

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$A = \{$The individial has the disease$\}$, $\overline{A} = \{$ does not have the disease$\}$, $B = \{$The test shows a POSITIVE result$\}$
$\overline{B} = \{$The test shows a NEGATIVE result$\}$

$P(B \mid A) = 0.9$, $P(\overline{B} \mid A^c) = 0.9$ and $P(A) = 0.01$

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid \overline{A})P(\overline{A})} = \frac{(0.9)(0.01)}{(0.9)(0.01) + (0.1)(0.99)} = 91\%$$

Example

Example : Suppose that we have 3 cards that are identical in form, except that both sides of the first card are colored red, both sides of the second card are colored black, and one side of the third card is colored red and the other side black. The 3 cards are mixed up in a hat, and 1 card is randomly selected and put down on the ground. If the upper side of the chosen card is colored red, what is the probability that the other side is colored black?

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Solution: Let RR, BB, and RB denote, respectively, the events that the chosen card is all red, all black, or the redblack card. Also, let R be the event that the upturned side of the chosen card is red. Then the desired probability is obtained

$$P(RB \mid R) = \frac{P(R \mid RB)P(RB)}{P(R \mid RR)P(RR) + P(R \mid RB)P(RB) + P(R \mid BB)P(BB)} = \frac{\frac{1}{2} \times \frac{1}{3}}{(1) \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} + (0) \times \frac{1}{3}} = \frac{1}{3}.$$

Example

Example : A new couple, known to have two children, has just moved into town. Suppose that the mother is encountered walking with one of her children. If this child is a girl, what is the probability that both children are girls?

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Solution:

Example

Example : The length, width, and height of a manufactured part are classified as being either within or outside specified tolerance limits. In a quality inspection 86% of the parts are found to be within the specified tolerance limits for width, but only 80% of the parts are within the specified tolerance limits for all three dimensions. However, 2% of the parts are within the specified tolerance limits for width and length but not for height, and 3% of the parts are within the specified tolerance limits for width and height but not for length. Moreover, 92% of the parts are within the specified tolerance limits for either width or height, or both of these dimensions.

1. If a part is within the specified tolerance limits for height, what is the probability that it will also be within the specified tolerance limits for width?
2. If a part is within the specified tolerance limits for length and width, what is the probability that it will be within the specified tolerance limits for all three dimensions?

Example

Example : A car repair is either on time or late and either satisfactory or unsatisfactory. If a repair is made on time, then there is a probability of 0.85 that it is satisfactory. There is a probability of 0.77 that a repair will be made on time. What is the probability that a repair is made on time and is satisfactory?

1.
2.

Solution:

Example

Example : An island has three species of bird. Species 1 accounts for 45% of the birds, of which 10% have been tagged. Species 2 accounts for 38% of the birds, of which 15% have been tagged. Species 3 accounts for 17% of the birds, of which 50% have been tagged. If a tagged bird is observed, what are the probabilities that it is of species 1, of species 2, and of species 3?

Solution:

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Solution:

Example

Example : After production, an electrical circuit is given a quality score of A, B, C, or D. Over a certain period of time, 77% of the circuits were given a quality score A, 11% were given a quality score B, 7% were given a quality score C, and 5% were given a quality score D. Furthermore, it was found that 2% of the circuits given a quality score A eventually failed, and the failure rate was 10% for circuits given a quality score B, 14% for circuits given a quality score C, and 25% for circuits given a quality score D.

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Solution:

Example

Example : A bin contains 3 different types of disposable ashlights. The probability that a type 1 ashlight will give over 100 hours of use is 0.7, with the corresponding probabilities for type 2 and type 3 ashlights being 0.4 and 0.3, respectively. Suppose that 20 percent of the ashlights in the bin are type 1, 30 percent are type 2, and 50 percent are type 3.

1. What is the probability that a randomly chosen ashlight will give more than 100 hours of use?
2. Given that a ashlight lasted over 100 hours, what is the conditional probability that it was a type j ashlight, j = 1, 2, 3?

Solution:

Example

Example : We know the following about a colormetric method used to test lake water for nitrates. If water specimens contain nitrates, a solution dropped into the water will cause the specimen to turn red 95% of the time. When used on water specimens without nitrates, the solution causes the water to turn red 10% of the time (because chemicals other than nitrates are sometimes present and they also react to the agent). Past experience in a lab indicates that nitrates are contained in 30%of the water specimens that are sent to the lab for testing. If a water specimen is randomly selected

1. from among those sent to the lab, what is the probability that it will turn red when tested?
2. It it turns red when tested, what is the probability that it actually contains nitrates?

Solution:

Example

Example : When a organisation's website is accessed, there is a probability of 0.07 that the web address was typed in directly. In such a case, there is a probability of 0.08 that an online purchase will be made. On the other hand, when the website is accessed indirectly, which occurs with probability 0.93, then there is only a 0.01 chance that an online purchase will be made.

1. The probability that the website is accessed directly and that a purchase is made ?
2. What is the probability that the website is accessed indirectly and that a purchase is made?
3. What proportion of online purchases are from individuals who access the website directly?

Solution:

Example

Example : Three radar sets, operating independently, are set to detect any aircraft flying through a certain area. Each set has a probability of .02 of failing to detect a plane in its area. If an aircraft enters the area, what is the probability that it

1. goes undetected?
2. is detected by at least one radar set?
3. is detected by all three radar sets?

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Solution: This follows because $P(E \cap F) = \frac{1}{52}$ **whereas** $P(E) = \frac{4}{52}$ **and** $P(F) = \frac{13}{52}$.

Relation between: Statistical Independence & Disjointness

Result : If A and B are two events such that $P(A) > 0$ and $P(B > 0)$ then,

a). If the events A and B are Statistically Independent then they can not be disjoint./mutually exclushive.

b). If the events A and B are disjoint then they can not be statistically independent.

Example

Example : Suppose that an urn contains 8 red balls and 4 white balls. We draw 2 balls from the urn without replacement. If we assume that at each draw each ball in the urn is equally likely to be chosen, what is the probability that both balls drawn are red?

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Solution: Let R_1 and R_2 denote, respectively, the events that the first and second balls drawn are red. Now, given that the first ball selected is red, there are 7 remaining red balls and 4 white balls, so $P(R_2 \mid R_1) = \frac{7}{11}$. . As $P(R_1) = \frac{8}{12}$, the desired probability is

$$P(R_1 \mid R_2) = P(R_2 \mid R_1) \times P(R1) = \frac{7}{11}\frac{8}{12} = \frac{14}{33}.$$

Example

Example : A biased coin is tossed 3 tiimes. Assume that the tosses are independent to each other (Means events concerning only toss 1 is statistically independent to events concerning toss 2). Assume that the probability for Head for the toss is π (π is any number between 0 to 1, for example $\pi = 0.75$). Therefore all the outcomes in the sample space may not be equally likelly. Find the probability of the following events

1. $\{HHH\}$
2. $\{HHT\}$, $\{THH\}$, $\{HTH\}$
3. Exactly one of the toss results in a Head.
4. Exactly two of the tosses result in Head.

Questions?