STAT230: Principles of Probability

Unit 7: Product moments, covariance and Conditional Expectation

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Product moments

Definition

Let X and Y are two random variables then the rth and sth non-central product moment of X and Y is defined by $\mu'_{r,s} = E\{X^r Y^s\}$.
The rth and sth central product moment of X and Y is defined by $\mu_{r,s} = E\{(X - \mu_X)^r (Y - \mu_Y)^s\}$.
$\mu_{1,1} = E\{(X - \mu_X)(Y - \mu_Y)\}$ is called the covariance of X and Y.

Covariance

Note that $\mu_{1,1} = Cov(X,Y) = \mu'_{1,1} - \mu_X \mu_Y = E(XY) - E(X)E(Y)$.

Theorem

If X and Y are independent then $Cov(X,Y) = 0$ i.e. $E(XY) = E(X)E(Y)$. The converse is not true.

Note that $Var(X) = Cov(X,X)$.

Example 1

Example (1)

Suppose X and Y have the following joint distribution

		x		
		0	1	2
	0	1/6	1/3	1/12
y	1	2/9	1/6	0
	2	1/36	0	0

Find the covariance of X and Y.
Are X and Y independent?

Solution for Example 1

		x			
		0	1	2	$f_Y(y)$
	0	1/6	1/3	1/12	7/12
y	1	2/9	1/6	0	7/18
	2	1/36	0	0	1/36
$f_X(x)$		15/36	1/2	1/12	

$$E(XY) = \sum_x \sum_y xyf(x,y) = 0*0*1/6 + 1*0*1/3 + \ldots + 2*2*0 = 1/6.$$

$E(X) = \sum_x xf_X(x) = 0*15/36 + 1*1/2 + 2*1/12 = 2/3$
$E(Y) = \sum_y yf_Y(y) = 0*7/12 + 1*7/18 + 2*1/36 = 4/9$
$Cov(X,Y) = E(XY) - E(X)E(Y) = 1/6 - 2/3*4/9 = -7/54.$
X and Y are not independent since $Cov(X,Y) \neq 0$.

Example 2

Example (2)

Let X and Y have joint density

$$f(x,y) = \begin{cases} \frac{x+y+1}{2} & \text{for } 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Find the covariance of X and Y.
Are X and Y independent?

Solution for Example 2

$$E(XY) = \int_0^1 \int_0^1 xyf(x,y)dydx = \int_0^1 \int_0^1 xy\frac{x+y+1}{2}dydx = \frac{7}{24}.$$

$$E(X) = \int_0^1 \int_0^1 xf(x,y)dydx = \int_0^1 \int_0^1 x\frac{x+y+1}{2}dydx = \frac{13}{24}.$$

$$E(Y) = \int_0^1 \int_0^1 yf(x,y)dydx = \int_0^1 \int_0^1 y\frac{x+y+1}{2}dydx = \frac{13}{24}.$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{7}{24} - \frac{13}{24} \times \frac{13}{24} = \frac{-1}{576}.$$

X and Y are dependent since $Cov(X,Y) \neq 0$.

Example 3

Example (3)

Let X and Y have joint density

$$f(x,y) = \begin{cases} 2 & \text{for } x > 0,\ y > 0,\ x + y < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Find the covariance of X and Y.
Are X and Y independent?

Solution for Example 3

$$E(XY) = \int_0^1 \int_0^1 xyf(x,y)dydx = \int_0^1 \int_0^{1-x} 2xydydx = \frac{1}{12}.$$

$$E(X) = \int_0^1 \int_0^1 xf(x,y)dydx = \int_0^1 \int_0^{1-x} 2xdydx = \frac{1}{3}.$$

$$E(Y) = \int_0^1 \int_0^1 yf(x,y)dydx = \int_0^1 \int_0^{1-x} 2ydydx = \frac{1}{3}.$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{12} - \frac{1}{3} \times \frac{1}{3} = \frac{-1}{36}.$$

X and Y are dependent since $Cov(X,Y) \neq 0$.

Theorem

If $X_1, X_2, \ldots, X_n$ are random variables and $Y = \sum_{i=1}^{n} a_i X_i$ where a_i's are constants, then $E(Y) = \sum_{i=1}^{n} a_i E(X_i)$ and

$$Var(Y) = \sum_{i=1}^{n} a_i^2 Var(X_i) + 2 \sum_{i=1}^{n-1} \sum_{j>i} a_i a_j Cov(X_i, X_j).$$

If the X_i's are independent then

$$Var(Y) = \sum_{i=1}^{n} a_i^2 Var(X_i).$$

Note that

$$Var(Y) = \sum_{i=1}^{n} a_i^2 Var(X_i) + \sum_{i=1}^{n-1} \sum_{j \neq i} a_i a_j Cov(X_i, X_j).$$

and

$$Var(Y) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j Cov(X_i, X_j).$$

Moments of linear combinations of R.V.

Theorem

If $X_1, X_2, \ldots, X_n$ are random variables and $Y_1 = \sum_{i=1}^{n} a_i X_i$ and $Y_2 = \sum_{i=1}^{n} b_i X_i$ where a_i's and b_i's are constants, then

$$Cov(Y_1, Y_2) = \sum_{i=1}^{n} a_i b_i Var(X_i) + \sum_{i=1}^{n} \sum_{j \neq i} a_i b_j Cov(X_i, X_j).$$

If the X_i's are independent then

$$Cov(Y_1, Y_2) = \sum_{i=1}^{n} a_i b_i Var(X_i).$$

Example 4

Example (4)

For X and Y defined in Examples 1 and 2, let $Z_1 = 2X + 4Y$ and $Z_2 = X - 2Y$ find:
$E(Z_1)$, $E(Z_2)$, $Var(Z_1)$, $Var(Z_2)$ and $Cov(Z_1, Z_2)$.

Solution for Example 4

$Z_1 = 2X + 4Y$ and $Z_2 = X - 2Y$
$E(Z_1) = 2E(X) + 4E(Y)$,
$E(Z_2) = E(X) - 2E(Y)$,
$Var(Z_1) = 4Var(X) + 16Var(Y) + 16Cov(X,Y)$,
$Var(Z_2) = Var(X) + 4Var(Y) - 4Cov(X,Y)$ and
$Cov(Z_1, Z_2) = 2Var(X) - 4Cov(X,Y) + 4Cov(X,Y) - 8Var(Y)$.

	$E(X)$	$E(Y)$	$Var(X)$	$Var(Y)$	$Cov(X,Y)$
Example 1	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{7}{18}$	$\frac{49}{162}$	$\frac{-7}{54}$
Example 2	$\frac{13}{24}$	$\frac{13}{24}$	$\frac{47}{576}$	$\frac{47}{576}$	$\frac{-1}{576}$

	$E(Z_1)$	$E(Z_2)$	$Var(Z_1)$	$Var(Z_2)$	$Cov(Z_1,Z_2)$
Example 1	$\frac{28}{9}$	$\frac{-2}{9}$	$\frac{1015}{162}$	$\frac{343}{162}$	$\frac{-133}{81}$
Example 2	$\frac{13}{4}$	$\frac{-13}{24}$	$\frac{77}{48}$	$\frac{239}{576}$	$\frac{-47}{96}$

Example 5

Example (5)

Let X and Y be two independent random variables with $\mu_X = 2$, $\sigma_X = 4$, $\mu_Y = 3$ and $\sigma_Y = 2$. Let $Z_1 = X + 2Y + 3$ and $Z_2 = 3X - Y$. Find: $E(Z_1)$, $E(Z_2)$, $Var(Z_1)$, $Var(Z_2)$ and $Cov(Z_1, Z_2)$.

Solution for Example 5

We have $\mu_X = 2$, $\sigma_X = 4$, $\mu_Y = 3$ and $\sigma_Y = 2$, $Cov(X, Y) = 0$ and $Z_1 = X + 2Y + 3$ and $Z_2 = 3X - Y$.
$E(Z_1) = E(X) + 2E(Y) + 3 = 11$, $E(Z_2) = 3E(X) - E(Y) = 3$,
$Var(Z_1) = Var(X) + 4Var(Y) = 16 + 4 \times 4 = 32$,
$Var(Z_2) = 9Var(X) + Var(Y) = 9 \times 16 + 4 = 148$ and
$Cov(Z_1, Z_2) = 3Var(X) - 2Var(Y) = 3 \times 16 - 2 \times 4 = 40$.

Conditional Expectation

The conditional Expectation of $u(X)$ given $Y = y$ is given by

$$E(u(X)|y) = \sum_x u(x) f_{X|Y}(x|y)$$

for discrete random variables and

$$E(u(X)|y) = \int_{-\infty}^{\infty} u(x) f_{X|Y}(x|y) dx$$

for continuous random variables.

Example 6

Example (6)

For X and Y defined in Example 1 find $E(X|Y=1)$.
For X and Y in Example 2 find $E(Y|X=1/2)$.

Solution for Example 6

For Example 1 we have:

x	$f_{X\|Y=1}(x)$
0	4/7
1	3/7
2	0

$E(X|Y=1) = \sum_x x f_{X|Y=1}(x) = 3/7$.

For Example 2 we have $f_{X|Y=1/2}(x) = \frac{f(x,1/2)}{f_Y(1/2)} = \frac{x}{2} + \frac{3}{4}$ therefore

$$E(Y|X=1/2) = \int_0^1 x f_{X|Y=1/2}(x) = \int_0^1 x(\frac{x}{2} + \frac{3}{4}) = \frac{13}{24}.$$

Links

Virtual Library/Conditional Expectation
Virtual Library/Product moments and covariance