STAT230: Principles of Probability

Unit 4: Conditional Probability and Independence

Department of Statistics

United Arab Emirates University

September 17, 2019

Objectives

- Conditional Probability
- Bayes' Theorem
- Independent Events

Conditional Probability

- **Example 0:** Suppose that we toss 2 dice, and suppose that each of the 36 possible outcomes is equally likely to occur and hence has probability $\frac{1}{36}$. Suppose further that we observe that the first die is a 3. Then, given this information, what is the probability that the sum of the 2 dice equals 8?
- Solution: ....
- If we let E and F denote, respectively, the event that the sum of the dice is 8 and the event that the first die is a 3, then the probability just obtained is called the conditional probability that E occurs given that F has occurred and is denoted by

$$P(E|F)$$

- **Definition:** If $P(F) > 0$, Then

$$P(E|F) = \frac{P(E \cap F)}{P(F)}. \tag{1}$$

- **Example 1:** A student is taking a one-hour-time-limit makeup examination. Suppose the probability that the student will finish the exam in less than x hours is $x/2$, for all $0 \le x \le 1$. Then, given that the student is still working after $.75$ hour, what is the conditional probability that the full hour is used?
- **Solution:** Let L_x denote the event that the student finishes the exam in less than x hours, $0 \le x \le 1$, and let F be the event that the student uses the full hour. Because F is the event that the student is not finished in less than 1 hour,

$$P(F) = P(L_1^c) = 1 - P(L_1) = .5$$

- **Solution (Cont.):** Now, the event that the student is still working at time $.75$ is the complement of the event $L_{.75}$, so the desired probability is obtained from

$$P(F|L^c_{.75}) = \frac{P(F \cap L^c_{.75})}{P(L^c_{.75})} = \frac{P(F)}{1 - P(L^c_{.75})} = \frac{.5}{.625} = .8.$$

- **Example 2:** A coin is flipped twice. Assuming that all four points in the sample space $S = \{(h,h),(h,t),(t,h),(t,t)\}$ are equally likely, what is the conditional probability that both flips land on heads, given that (a) the first flip lands on heads? (b) at least one flip lands on heads?

- **Solution:............**

- **Note: If each outcome of a finite sample space S is equally likely, then, conditional on the event that the outcome lies in a subset $F \subset S$, all outcomes in F become equally likely. In such cases, it is often convenient to compute conditional probabilities of the form $P(E|F)$ by using F as the sample space. Indeed, working with this reduced sample space often results in an easier and better understood solution**

- Multiplying both sides of Equation (1) by P(F), we obtain

$$P(E \cap F) \;=\; P(E|F) \times P(F). \qquad (2)$$

- **Example 3:** Celine is undecided as to whether to take a French course or a chemistry course. She estimates that her probability of receiving an A grade would be $1/2$ in a French course and $2/3$ in a chemistry course. If Celine decides to base her decision on the flip of a fair coin, what is the probability that she gets an A in chemistry?
- **Solution:**.................

- **Example 4:** Suppose that an urn contains 8 red balls and 4 white balls. We draw 2 balls from the urn without replacement. If we assume that at each draw each ball in the urn is equally likely to be chosen, what is the probability that both balls drawn are red?
- **Solution:** Let R_1 and R_2 denote, respectively, the events that the first and second balls drawn are red. Now, given that the first ball selected is red, there are 7 remaining red balls and 4 white balls, so $P(R_2|R_1) = \frac{7}{11}$. As $P(R_1) = \frac{8}{12}$, the desired probability is

$$P(R_1 \cap R_2) = P(R_2|R_1) \times P(R_1) = \frac{7}{11}\frac{8}{12} = \frac{14}{33}$$

- A generalization of Equation (2), which provides an expression for the probability of the intersection of an arbitrary number of events, is sometimes referred to as *the multiplication rule*.

- **The multiplication rule**:

  $P(E_1E_2\ldots E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)\ldots P(E_n|E_1E_2\ldots E_{n-1})$.

- **Example 5**: An ordinary deck of 52 playing cards is randomly divided into 4 piles of 13 cards each. Compute the probability that each pile has exactly 1 ace.
- **Solution**: Define events E_i, $i = 1, 2, 3, 4$, as follows:
  $E_1 =$ {the ace of spades is in any one of the piles},
  $E_2 =$ {the ace of spades and the ace of hearts are in different piles}
  $E_3 =$
  {the aces of spades, hearts, and diamonds are all in different piles}
  $E_4 =$ {all 4 aces are in different piles}
  The desired probability is $P(E_1E_2E_3E_4)$, and by the multiplication rule,
  $P(E_1E_2E_3E_4) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)P(E_4|E_1E_2E_3)$.
  $P(E_1) = 1$, E_1 is sample space.
  $P(E_2|E_1) = 39/51$, $P(E_3|E_1E_2) = 26/50$ and
  $P(E_4|E_1E_2E_3) = 13/49$. Then $P(E_1E_2E_3E_4) = \frac{39\times26\times13}{51\times50\times49} = .105$.

Total Law of Probability

- Let E and F be two events. First we may express E as $E = EF \cup EF^c$. The one can write

$$P(E) = P(E|F)P(F) + P(E|F^c)P(F^c) \tag{3}$$

- Now, assume that the suite $F_1, F_2, \ldots, F_n$ of events forms a partition of $\mathcal{S}$. It's straightforward to generalize result (3) to the following result:
- **Proposition:** For any given event E in $\mathcal{S}$, we have

$$P(E) = \sum_{i=1}^{n} P(E|F_i)P(F_i). \tag{4}$$

- Note: Equation (4) is called *total law of probability*.

Total Law of Probability: Examples

- **Example 5**: An insurance company believes that people can be divided into two classes: those who are accident prone and those who are not. The companys statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability .4, whereas this probability decreases to .2 for a person who is not accident prone. If we assume that 30 percent of the population is accident prone, what is the probability that a new policyholder will have an accident within a year of purchasing a policy?

- **Solution**: Let A_1 denote the event that the policyholder will have an accident within a year of purchasing the policy, and let A denote the event that the policyholder is accident prone. Hence, the desired probability is given by

$$P(A) = P(A|A_1)P(A_1) + P(A|A_1^c)P(A_1^c) = (.4) \times (.3) + (.2) \times (.7) = .26$$

.

BAYES'S Theorem

- Again, let $F_1, \ldots, F_n$ be a set of mutually exclusive and exhaustive events (meaning that exactly one of these events must occur). Suppose now that E has occurred and we are interested in determining which one of the F_j also occurred. Then, by Equation (4), we have the following theorem
- **Theorem**:

$$P(F_j|E) = \frac{P(E|F_j)P(F_j)}{\sum_{i=1}^{n} P(E|F_i)P(F_i)} \tag{5}$$

- This theorem is well known as Bayes's theorem, after the English philosopher *Thomas Bayes*.
- We can use the following Applet to find the conditional probability using Bayes's theorem:

http://www.thomsonedu.com/statistics/book_content/0495110817_wackerly/applets/seeingstats/Chpt2/bayesTree.html

- **Example 6**: Suppose that a new policyholder has an accident within a year of purchasing a policy. What is the probability that he or she is accident prone?
- **Solution**: The desired probability is $P(A_1|A) = \frac{P(A|A_1)P(A_1)}{P(A)} = \frac{(.4)(.3)}{.26} = \frac{6}{13}$.

BAYES'S Theorem: Examples

- **Example 7**: In answering a question on a multiple-choice test, a student either knows the answer or guesses. Let p be the probability that the student knows the answer and $1-p$ be the probability that the student guesses. Assume that a student who guesses at the answer will be correct with probability $1/m$, where m is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that he or she answered it correctly?
- **Solution**: Let C and K denote, respectively, the events that the student answers the question correctly and the event that he or she actually knows the answer. Now,

$$\begin{aligned} P(K|C) &= \frac{P(C|K)P(K)}{P(C|K)P(K)+P(C|K^c)P(K^c)} = \frac{p}{p+(1/m)(1-p)} \\ &= \frac{mp}{1+(m-1)p}. \end{aligned}$$

BAYES'S Theorem: Examples

- **Example 8**: A laboratory blood test is 95 percent effective in detecting a certain disease when it is, in fact, present. However, the test also yields a false positive result for 1 percent of the healthy persons tested. (That is, if a healthy person is tested, then, with probability .01, the test result will imply that he or she has the disease.) If .5 percent of the population actually has the disease, what is the probability that a person has the disease given that the test result is positive?
- **Solution**: Let D be the event that the person tested has the disease and E the event that the test result is positive. Then the desired probability is

$$\begin{aligned} P(D|E) &= \frac{P(E|D)P(D)}{P(E|D)P(D) + P(E|D^c)P(D^c)} = \frac{(.95)(.005)}{(.95)(.005) + (.01)(.99} \\ &= \frac{95}{294} = .323. \end{aligned}$$

- **Example 9**: Suppose that we have 3 cards that are identical in form, except that both sides of the first card are colored red, both sides of the second card are colored black, and one side of the third card is colored red and the other side black. The 3 cards are mixed up in a hat, and 1 card is randomly selected and put down on the ground. If the upper side of the chosen card is colored red, what is the probability that the other side is colored black?

- **Example 9**: Suppose that we have 3 cards that are identical in form, except that both sides of the first card are colored red, both sides of the second card are colored black, and one side of the third card is colored red and the other side black. The 3 cards are mixed up in a hat, and 1 card is randomly selected and put down on the ground. If the upper side of the chosen card is colored red, what is the probability that the other side is colored black?
- **Solution**: Let RR, BB, and RB denote, respectively, the events that the chosen card is all red, all black, or the redblack card. Also, let R be the event that the upturned side of the chosen card is red. Then the desired probability is obtained by

$$
\begin{aligned}
P(RB|R) &= \frac{P(R|RB)P(RB)}{P(R|RR)P(RR) + P(R|RB)P(RB) + P(R|BB)P(BB)} \\
&= \frac{(1/2)(1/3)}{(1)(1/3) + (1/2)(1/3) + (0)(1/3)} = 1/3
\end{aligned}
$$

- **Example 10:** A total of n balls are sequentially and randomly chosen, without replacement, from an urn containing r red and b blue balls ($n \leq r + b$). Given that k of the n balls are blue, what is the conditional probability that the first ball chosen is blue?
- **Solution:** Show that this probability is $\frac{k}{n}$.

BAYES'S Theorem: Examples

- **Example 11**: A new couple, known to have two children, has just moved into town. Suppose that the mother is encountered walking with one of her children. If this child is a girl, what is the probability that both children are girls?
- **Solution**: ....
- **Example 12**: A bin contains 3 different types of disposable flashlights. The probability that a type 1 flashlight will give over 100 hours of use is .7, with the corresponding probabilities for type 2 and type 3 flashlights being .4 and .3, respectively. Suppose that 20 percent of the flashlights in the bin are type 1, 30 percent are type 2, and 50 percent are type 3.
  (a) What is the probability that a randomly chosen flashlight will give more than 100 hours of use?
  (b) Given that a flashlight lasted over 100 hours, what is the conditional probability that it was a type j flashlight, $j = 1, 2, 3$?
- **Solution**: ....

Independent Events

- **Definition:** Two events E and F are said to be *independent* if and only if

$$P(E \cap F) \;=\; P(E) \times P(F). \tag{6}$$

- **Corollary:** Two events E and F are independent if and only if $P(E|F) = P(E)$ and $P(F|E) = P(F)$.
- **Example 13**: A card is selected at random from an ordinary deck of 52 playing cards. If E is the event that the selected card is an ace and F is the event that it is a spade, then E and F are independent. This follows because $P(EF) = 1/52$ whereas $P(E) = 4/52$ and $P(F) = 13/52$.

Independent Events: Examples

- **Example 14:** Two coins are flipped, and all 4 outcomes are assumed to be equally likely. If E is the event that the first coin lands on heads and F the event that the second lands on tails, then E and F are independent, since $P(EF) = P((H,T)) = 1/4$, whereas $P(E) = P((H,H),(H,T)) = 1/2$ and $P(F) = P((H,T),(T,T)) = 1/2$.
- **Example 15:** Suppose that we toss 2 fair dice. Let E_1 denote the event that the sum of the dice is 6 and F denote the event that the first die equals 4. Then $P(E_1F) = P((4,2)) = 1/36$ whereas $P(E1)P(F) = \frac{5}{36} \times \frac{1}{6} = \frac{5}{216}$. Hence, E_1 and F are not independent.
- **Example 16:** Now, suppose that we let E2 be the event that the sum of the dice equals 7. Is E_2 independent of F? Answer: Yes. Since $P(E_2F) = P((4,3)) = 1/36$ and $P(E_2)P(F) = \frac{1}{6}.\frac{1}{6} = \frac{1}{36}$.

Independent Events:

- **Proposition:** If E and F are independent, then so are E and F^c.
- **Proof:**............
- **Definition:** Three events E, F, and G are said to be independent if

$$\begin{aligned} P(EFG) &= P(E)P(F)P(G) \\ P(EF) &= P(E)P(F) \\ P(EG) &= P(E)P(G) \\ P(FG) &= P(F)P(G) \end{aligned}$$