STAT 320: Principles of Probability

Unit 6 Part:B

A Few Commonly Used Continuous Probability Distributions

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Reminder: Some Popular Integrals

$$\int x^n dx = \frac{x^{n+1}}{n+1} \text{ for any integer } n,\ n \neq -1.$$

$$\int \frac{1}{x} dx = \log(x)$$

$$\int e^{-x} dx = -e^{-x}.$$

$$\int e^{mx} dx = \frac{e^{mx}}{m} \text{ for any nonzero real number } m \in \mathbb{R},\ m \neq 0.$$

* Note: We have not included the constant term that appears as a constants while writing a indefinite integral. For the majority, if not all, of the integras in this course will be a definite integrals with a lower and upper limit.

Assume $f'(x) := \frac{d\ f(x)}{dx}$ and $g'(x) := \frac{d\ g(x)}{dx}$ for the following formula

Integral By Parts: $\int f(x)g(x)dx = f(x)\ (\int g(x)dx) - \int \left\{ f'(x)\ (\int g(x)dx) \right\} dx$

Addition Rule: $\int \left\{ c_1\ f(x) + c_2\ g(x) \right\} dx = c_1 \int f(x)dx + c_2 \int g(x)dx$ for any constant $c_1, c_2 \in \mathbb{R}$.

Example

Outline

Uniform Distribution

Uniform Distribution

The uniform random variable is used to model the behavior of a continuous random variable whose values are uniformly or evenly distributed over a given interval.

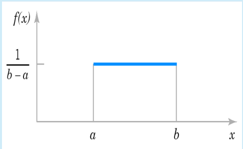
Definition (Uniform Distribution)

A random variable X is said to be uniformly distributed over the interval $[a, b]$, denoted by $X \sim \text{Uniform}(a, b)$, if its density function is

$$f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{Otherwise} \end{cases}$$

If $X \sim \text{Uniform}(a, b)$, then:

$\boxed{E(X) = \frac{a+b}{2}}$, and $\boxed{\text{Var}(X) = \frac{(b-a)^2}{12}}$



Let $X \sim \text{Uniform}(a, b)$ for $a < b$

Mean

$$E(X) = \frac{a+b}{2}$$

Variance

$$\text{VAR}(X) = \frac{(b-a)^2}{12}$$

MGF

$$M_X(t) = \begin{cases} \frac{e^{tb}-e^{ta}}{t(b-a)} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$$

Distribution	Support $\mathbb{S}_X$	pdf $f_X(x)$	Mean $E(X)$	Variance $\text{Var}(X)$	mgf $M_X(t)$
Uniform(a, b)	$[a, b]$	$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise .} \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$M_X(t) = \begin{cases} \frac{e^{tb}-e^{ta}}{t(b-a)} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$

Example : The time (in min) for a lab assistant to prepare the equipment for a certain experiment is believed to have a uniform distribution with a = 25 and b = 35.

a). Write the pdf of X and sketch its graph.

b). What is the probability that preparation time exceeds 33 min?

c). What is the probability that preparation time is within 2 minmutes of the **mean time**?

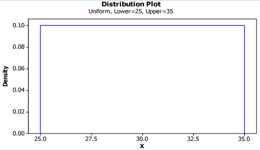
Example

Example : The time (in min) for a lab assistant to prepare the equipment for a certain experiment is believed to have a uniform distribution with a = 25 and b = 35.

- a). Write the pdf of X and sketch its graph.
- b). What is the probability that preparation time exceeds 33 min?
- c). What is the probability that preparation time is within 2 minmutes of the **mean time**?

Mean of the random variable is
$\boxed{E(X)} = \frac{25+35}{2} = \boxed{30}$.

$$f(x) := \begin{cases} \frac{1}{10} & \text{if } 25 \leq x \leq 35 \\ 0 & \text{otherwise.} \end{cases}$$



$$\begin{aligned} & P(X > 33) \\ = & \int_{33}^{35} f(x)dx \\ = & \left(\frac{x}{10}\right)\Big|_{33}^{35} \\ = & \frac{35 - 32}{10} \\ = & 0.2 \end{aligned}$$

$$\begin{aligned} & P\left(\boxed{E(X)} - 2 < X < \boxed{E(X)} + 2\right) \\ = & P(30 - 2 < X < 30 + 2) \\ = & P(28 < X < 32) \\ = & \int_{28}^{32} f(x)dx \\ = & \left(\frac{x}{10}\right)\Big|_{28}^{32} \\ = & \frac{32 - 28}{10} \\ = & 0.4 \end{aligned}$$

Exercise

Example : Upon studying low bids for shipping contracts, a micro-computer company finds that intrastate contracts have low bids that are uniformly distributed between 20 and 25, in units of thousands of dollars.

a). Find the probability that the low bid on the next intrastate shippingcontract is below $22,000.

b). Find the probability that the low bid on the next intrastate shipping contract is in excess of $24,000.

c). Find the expected value and standard deviation of low bids on contracts of the type described above.

Exercise

Example : A grocery store receives delivery each morning at a time that varies uniformly between 5:00 and 7:00 AM.

a). Write and sketch the pdf of the delivery arrival.

b). Find the probability that the delivery on a given morning will occur between 5:30 and 5:45 A.M.

c). Find the probability that the time of delivery will be within one-half standard deviation of the expected time.

Outline

Exponential Distribution

Exponential Distribution: Context

1. The exponential distribution is often used to model time (waiting time, interarrival time, hardware lifetime, failure time, etc.).
2. When the number of occurrences of an event follows Poisson distribution, the time between occurrences follows exponential distribution.

Exponential Distribution

Definition (Exponential Distribution)

The exponential probability distribution with parameter $\lambda > 0$ (called the rate parameter) is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

The cdf of the exponential distribution is

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-\lambda x} & \text{if } x > 0 \end{cases}$$

Let $X \sim$ Exponential(rate $= \lambda$) for $\lambda > 0$

Mean

$E(X) = \frac{1}{\lambda}$

Variance

$\text{VAR}(X) = \frac{1}{\lambda^2}$

MGF

$M_X(t) = \frac{\lambda}{\lambda - t}$ if $0 \leq t < \lambda$

Distribution	Support $\mathbb{S}_X$	pdf $f_X(x)$	Mean $E(X)$	Variance $\text{Var}(X)$	mgf $M_X(t)$
Exponential(rate $= \lambda$)	$(0, \infty)$	$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise .} \end{cases}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda - t}$ if $0 \leq t < \lambda$

Example

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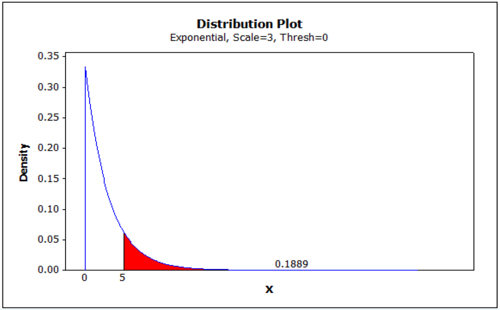

$$
\begin{aligned}
& P(X > 5) \\
= \;& 1 - P(X \le 5) \\
= \;& 1 - F(5) \\
= \;& 1 - \left(1 - e^{-\lambda 5}\right) \\
= \;& e^{-5\lambda} \\
= \;& e^{-5 \times \frac{1}{3}} \\
= \;& 0.1889
\end{aligned}
$$

Exercise

Example : The failure rate for a type of electric light bulb is 0.002 per hour. Under the exponential model,

a). Find the probability that a randomly selected light bulb will fail in less than 1000 hours.

b). Find the probability that a randomly selected light bulb will last 2000 hours before failing.

c). Find the mean and the variance of time until failure.

d). Find the median time until failure.

e). Find the time where 95% of these bulbs are expected to fail before it.

Exercise

Example : An engineer thinks that the best model for time between breakdowns of a generator is the exponential distribution with a mean of 15 days.

a). If the generator has just broken down, what is the probability that it will break down in the next 21 days?

b). What is the probability that the generator will operate for 30 days without a breakdown?

c). If the generator has been operating for the last 20 days, what is the probability that it will operate for another 30 days without a breakdown?

d). Comment on the results of parts (b) and (c).

Outline

$\Gamma(\alpha)$, for $\alpha > 0$

The Gamma Function

Gamma Function, $\Gamma(\alpha), \alpha > 0$

- $$\Gamma(\alpha) := \int_0^\infty x^{\alpha-1} e^{-x} dx$$ for $\alpha > 0$.

- $\Gamma(\alpha) > 0$ for all $\alpha > 0$.
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
- $\Gamma(1) = 1$
- $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$ for any $\alpha > 1$
- $\Gamma(n) = (n-1)!$ when n is a positive integer.

Gamma Function: Example

- $$\int_0^\infty x^7 e^{-x} dx =$$

- $$\int_0^\infty x^{\frac{5}{2}} e^{-3x} dx =$$

Gamma Function: Example

- $\dfrac{\Gamma(9.1)}{\Gamma(7.1)} =$

- Let $\alpha > 0$, $\dfrac{\Gamma(\alpha+3)}{\Gamma(\alpha)} =$

Gamma Distribution

Definition (Gamma Distribution)

The gamma random variable X describes waiting times between events. It can be thought of as a waiting time between Poisson distributed events, the pdf of a Gamma(α, λ) for $\alpha > 0, \lambda > 0$ is given as:

$$f(x) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \text{ for } 0 < x < \infty.$$

The parameter α is known as the shape parameter, while λ is called rate parameter.

Note that: The quantity λ is referred to as the rate parameter.

If $X \sim$ Gamma(α, λ) then $E(X) = \frac{\alpha}{\lambda}$ and , Var$(X) = \frac{\alpha}{\lambda^2}$

Let $X \sim$ Gamma(shape $= \alpha$, rate $= \lambda$) for $\alpha > 0, \lambda > 0$

Mean

$E(X) = \frac{\alpha}{\lambda}$

Variance

$\text{VAR}(X) = \frac{\alpha}{\lambda^2}$

MGF

$M_X(t) = \frac{1}{\left(1-\frac{t}{\lambda}\right)^{\alpha}}$ if $0 \leq t < \lambda$

Distribution	Support $\mathbb{S}_X$	pdf $f_X(x)$	Mean $E(X)$	Variance $\text{Var}(X)$	mgf $M_X(t)$
Gamma(α, λ) shape $= \alpha$, rate $= \lambda$	$(0, \infty)$	$\frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$ if $x > 0$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\frac{1}{\left(1-\frac{t}{\lambda}\right)^{\alpha}}$ if $0 \leq t < \lambda$

Exercise

Example : Suppose the time spent by a randomly selected student who uses a terminal connected to a local time-sharing computer facility has a gamma distribution with mean 20 min and variance 80 min^2.

a). What are the values of α and λ?

b). What is the probability that a student uses the terminal for at most 24 min?

c). What is the probability that a student spends between 20 and 40 min using the terminal?

Exercise

Example : A pumping station operator observes that the demand for water at a certain hour of the day can be modeled as an exponential random variable with a mean of 100 cfs (cubic feet per second).

a). 1 Find the probability that the demand will exceed 200 cfs on a randomly selected day.

b). What is the maximum water producing capacity that the station should keep on line for this hour so that the demand will have a probability of only 0.01 of exceeding this production capacity?

Outline

Beta Function, $\mathscr{B}(\alpha, \beta), \alpha > 0, \beta > 0$

- $$\int_0^1 x^3(1-x)^9 dx =$$

$\mathscr{B}(\alpha, \beta)$ $\alpha > 0, \beta > 0$

The Beta Function

Beta Function, $\mathscr{B}(\alpha, \beta), \alpha > 0, \beta > 0$

$$\mathscr{B}(\alpha, \beta) := \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$$ for $\alpha > 0, \beta > 0$.

$\mathscr{B}(\alpha, \beta)$ is often calculated using the following equation:

$$\mathscr{B}(\alpha, \beta) := \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Beta Function, $\mathscr{B}(\alpha, \beta), \alpha > 0, \beta > 0$

- $\displaystyle\int_0^1 x^3(1-x)^9 dx =$

- $\displaystyle\int_0^1 x^{30}(1-x)^{1.2} dx =$

Beta Distribution

Beta Distribution

The beta random variable X represents the proportion or probability outcomes. For example, the beta distribution might be used to find how likely it is that the preferred candidate for mayor will receive 70% of the vote.

Definition (Beta Distribution)

Probability Density Function of the Beta(α, β), $\alpha > 0, \beta > 0$ is given as

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \text{ for } 0 \le x \le 1.,$$

where $\Gamma(\alpha)$ is defined by $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$.

If $X \sim \text{Beta}(\alpha, \beta)$ then $E(X) = \frac{\alpha}{\alpha+\beta}$ and , $\text{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Let $X \sim \text{Beta}(\alpha, \beta)$ for $\alpha > 0, \beta > 0.$

Mean

$$E(X) = \frac{\alpha}{\alpha+\beta}$$

Variance

$$\text{VAR}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

MGF

$M_X(t) =$ A Complicated Series

Distribution	Support $\mathbb{S}_X$	pdf $f_X(x)$	Mean $E(X)$	Variance $\text{Var}(X)$	mgf $M_X(t)$
$\text{Beta}(\alpha, \beta)$ shape1 $= \alpha$, shape2 $= \beta$	$(0, 1)$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} x^{\beta-1}$ if $0 < x < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	$- - -$

Exercise

Example : A gasoline wholesale distributor has bulk storage tanks that hold fixed supplies and are filled every Monday. Of interest to the wholesaler is the proportion of this supply that is sold during the week. Over many weeks of observation, the distributor found that this proportion could be modeled by a beta distribution with $\alpha = 4$ and $\beta = 2$.

1. Find the probability that the wholesaler will sell at least 90% of her stock in a given week.
2. What is the expected percentage of sell in a randomly selected week.

Distribution	Support $\mathbb{S}_X$	pdf $f_X(x)$	Mean $E(X)$	Variance $\mathrm{Var}(X)$	mgf $M_X(t)$
Uniform(a, b)	$[a, b]$	$\begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise .} \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$M_X(t) = \begin{cases} \frac{e^{tb}-e^{ta}}{t(b-a)} \\ 1 \end{cases}$
Exponential(λ)	$(0, \infty)$	$\begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise .} \end{cases}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}$ if $0 \leq t <$
Gamma(α, λ) shape $= \alpha$, rate $= \lambda$	$(0, \infty)$	$\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$ if $x > 0$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\frac{1}{\left(1-\frac{t}{\lambda}\right)^\alpha}$ if $0 \leq$
Beta(α, β)	$(0, 1)$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} x^{\beta-1}$ if $0 < x < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	$- - -$
Normal(μ, σ^2) mean $= \mu$, Var $= \sigma^2$	$(-\infty, \infty)$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ for $x \in \mathbb{R}$	μ	σ^2	$e^{\mu t + \frac{t^2\sigma^2}{2}}$

Outline

Normal Distribution

Questions?