STAT220: Probability

Unit 3: Basic Concepts of Probability

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Objectives

- Sample Space
- Probability Axioms
- Probability of Events
- Probability Properties

Sample Space

- Random Experiment:
  1. A process of observation whose outcome is not known in advance with certainty.
  2. Example: Single throw of a 6-sided die.
- Outcome:
  1. An outcome is defined as any possible result of a random experiment.
  2. Example: The number 5 appear in the die-throw example
- Sample Space:
  1. Set of all possible outcomes of a random experiment.
  2. Example: $S = \{1, 2, 3, 4, 5, 6\}$.
- Event:
  1. A collection of outcomes; a subset of S.
  2. Example: $A = \{2, 4, 6\}$, $B = \{3\}$.
  3. Note: All operations on events are similar to those on subsets studied in Unit 1.

Examples

- **Example 1:** If the outcome of an experiment consists in the determination of the sex of a newborn child, then $S = \{g, b\}$ where the outcome g means that the child is a girl and b that it is a boy.
- **Example 2:** If the outcome of an experiment is the order of finish in a race among the 7 horses having post positions 1, 2, 3, 4, 5, 6, and 7, then
  $S = \{\text{all 7! permutations of } (1, 2, 3, 4, 5, 6, 7)\}$
  The outcome $(2, 3, 1, 6, 5, 4, 7)$ means, for instance, that the number 2 horse comes in first, then the number 3 horse, then the number 1 horse, and so on.
- **Example 3:** If the experiment consists of flipping two coins, then the sample space consists of the following four points: $S = \{(H, H), (H, T), (T, H), (T, T)\}$ The outcome will be (H, H) if both coins are heads, (H, T) if the first coin is heads and the second tails, (T, H) if the first is tails and the second heads, and (T, T) if both coins are tails.

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Examples

- **Example 4:** If the experiment consists of tossing two dice, then the sample space consists of the 36 points $S = \{(i, j) : i, j = 1, 2, 3, 4, 5, 6\}$ where the outcome (i, j) is said to occur if i appears on the leftmost die and j on the other die.
- **Example 5:** If the experiment consists of measuring (in hours) the lifetime of a transistor, then the sample space consists of all nonnegative real numbers; that is, $S = \{x : 0, \ldots, x < q\}$.
- In Example 1, if $E = g$, then E is the event that the child is a girl.
- In Example 2, if $E = \{$all outcomes in S starting with a $3\}$ then E is the event that horse 3 wins the race.
- In Example 3, if $E = \{(H, H), (H, T)\}$, then E is the event that a head appears on the first coin.
- In Example 4, if $E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$, then E is the event that the sum of the dice equals 7.
- In Example 5, if $E = \{x : 0 \leq x \leq 5\}$, then E is the event that the transistor does not last longer than 5 hours.

Axioms of Probability

- Probability refers to the chance that a particular event will occur.
- The probability of an event is the proportion of times the event is expected to occur in repeated experiments
- If we denote by $n(E)$ the number of times in the first n repetitions of the experiment that the event E occurs, the probability of the event E is defined by $P(E) = \lim_{n\to\infty} \frac{n(E)}{n}$.

Axioms of Probability

- Consider an experiment whose sample space is S. For each event E of the sample space S, we assume that a number $P(E)$ is defined and satisfies the following three axioms
  1. **Axiom 1:** $0 \leq P(E) \leq 1$
  2. **Axiom 2:** $P(S) = 1$
  3. **Axiom 3:** For any sequence of mutually exclusive events $E1, E2, \ldots$ (that is, events for which $E_i \cap E_j = \emptyset$ when $i \neq j$),

$$P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$$

  We refer to P(E) as the probability of the event E.
- Note: Thus, Axiom 1 states that the probability that the outcome of the experiment is an outcome in E is some number between 0 and 1. Axiom 2 states that, with probability 1, the outcome will be a point in the sample space S. Axiom 3 states that, for any sequence of mutually exclusive events, the probability of at least one of these events occurring is just the sum of their respective probabilities.

Examples

- Example 6: If our experiment consists of tossing a coin and if we assume that a head is as likely to appear as a tail, then we would have $P(H) = P(T) = 1/2$. On the other hand, if the coin were biased and we felt that a head were twice as likely to appear as a tail, then we would have $P(H) = 2/3$ and $P(T) = 1/3$
- Example 7: If a die is rolled and we suppose that all six sides are equally likely to appear, then we would have $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$. From Axiom 3, it would thus follow that the probability of rolling an even number would equal $P(2, 4, 6) = P(2) + P(4) + P(6) = 1/2$

- Property 1: $P(\bar{E}) = 1 - P(E)$.
- Property 2: If $E \subset F$, then $P(E) \leq P(F)$.
- Property 3: $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.
- Property 4: $P(\bar{E} \cap F) = P(F) - P(E \cap F)$.
- Example 8: Aisha is taking two books along on her holiday vacation. With probability .5, she will like the first book; with probability .4, she will like the second book; and with probability .3, she will like both books. What is the probability that she likes neither book?
- Solution: ......
- Property 5: $P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E \cap F) - P(E \cap G) - P(F \cap G) + P(E \cap F \cap G)$.

Inclusion-Exclusion Identity

- Proposition:

$$
\begin{aligned}
P(E_1 \cup E_2 \cup \ldots E_n) &= \sum_{i=1}^{n} P(E_i) - \sum_{i_1<i_2} P(E_{i_1} \cap E_{i_2}) + \ldots \\
&+ (-1)^{r+1} \sum_{i_1<i_2<\ldots<i_r} P(E_{i_1} \cap \ldots \cap E_{i_r}) \\
&+ \ldots + (-1)^{n+1} P(E_{i_1} \cap \ldots \cap E_{i_n})
\end{aligned}
$$

  The summation $\sum_{i_1<i_2<\ldots<i_r} P(E_{i_1} \cap \ldots \cap E_{i_r})$ is taken over all the $\binom{n}{r}$ possible subsets of size r of the set $\{1, 2, \ldots, n\}$.

- In words, this Proposition states that the probability of the union of n events equals the sum of the probabilities of these events taken one at a time, minus the sum of the probabilities of these events taken two at a time, plus the sum of the probabilities of these events taken three at a time, and so on.

Sample Spaces Having Equally Likely Outcomes

- In many experiments, it is natural to assume that all outcomes in the sample space are equally likely to occur.
- That is, consider an experiment whose sample space S is a finite set, say, $S = \{1, 2, ..., N\}$. Then it is often natural to assume that $P(1) = P(2) = \; = P(N)$ which implies, from Axioms 2 and 3, that $P(i) = 1/N, \; i = 1, 2, \ldots, N$.
- From this equation, it follows from Axiom 3 that, for any event E,

$$P(E) \quad = \quad \frac{\text{number of outcomes in } \; E}{\text{number of outcomes in } \; S}.$$

- In words, if we assume that all outcomes of an experiment are equally likely to occur, then the probability of any event E equals the proportion of outcomes in the sample space that are contained in E.

Examples

- Example 9: If two dice are rolled, what is the probability that the sum of the upturned faces will equal 7?
- Solution: We shall solve this problem under the assumption that all of the 36 possible outcomes are equally likely. Since there are 6 possible outcomesnamely, $(1, 6)$, $(2, 5)$, $(3, 4)$, $(4, 3)$, $(5, 2)$, and $(6, 1)$that result in the sum of the dice being equal to 7, the desired probability is $6/36 = 1/6$.
- Example 10: If 3 balls are randomly drawn from a bowl containing 6 white and 5 black balls, what is the probability that one of the balls is white and the other two black?
- Solution: $n = 11 \times 10 \times 9 = 990$ and $n(E) =?$ if $E = \{\text{one of the balls is white and the other two black}\}$. For the order WBB we have $6 \times 5 \times 4 = 120$ possibilities, for BWB, we have $5 \times 6 \times 4 = 120$ possibilities and for BBW we have $5 \times 4 \times 6 = 120$ possibilities. Then $P(E) = n(E)/n = (3 \times 120)/990 = 4/11$.

Examples

- **Example 10:** A committee of 5 is to be selected from a group of 6men and 9 women. If the selection is made randomly, what is the probability that the committee consists of 3 men and 2 women?
- **Solution:** Because each of the $\binom{15}{5}$ possible committees is equally likely to be selected, the desired probability is $\frac{\binom{6}{3}\times\binom{9}{2}}{\binom{15}{5}} = .24.$
- **Example 11:** Suppose 5 people are to be randomly selected from a group of 20 individuals consisting of 10 married couples, and we want to determine P(N), the probability that the 5 chosen are all unrelated. (That is, no two are married to each other.)
- **Solution:** $P(N) = \frac{\binom{10}{5}\times 2^5}{\binom{20}{5}}$ or $P(N) = \frac{20.18.16.14.12}{20.19.18.17.16}$.
- **Example 12:** If n people are present in a room, what is the probability that no two of them celebrate their birthday on the same day of the year? How large need n be so that this probability is less than $1/2$?
- **Solution:** $\frac{365\times 364\times 363\ldots(365-n+1)}{(365)^n}$.

Examples

- **Example 13:** A poker hand consists of 5 cards. If the cards have distinct consecutive values and are not all of the same suit, we say that the hand is a straight. For instance, a hand consisting of the five of spades, six of spades, seven of spades, eight of spades, and nine of hearts is a straight. What is the probability that one is dealt a straight?
- **Solution:** ....
- **Example 14:** A 5-card poker hand is said to be a full house if it consists of 3 cards of the same denomination and 2 other cards of the same denomination (of course, different from the first denomination). Thus, one kind of full house is three of a kind plus a pair. What is the probability that one is dealt a full house?
- **Solution:** ....
- **Example 15:** In the game of bridge, the entire deck of 52 cards is dealt out to 4 players. What is the probability that
(a) one of the players receives all 13 spades;
(b) each player receives 1 ace?
- **Solution:** ....

- **Example 16:** A football team consists of 20 offensive and 20 defensive players. The players are to be paired in groups of 2 for the purpose of determining roommates. If the pairing is done at random, what is the probability that there are no offensive/defensive roommate pairs?
- **Solution:** ....
- **Example 17:** A total of 36 members of a club play tennis, 28 play squash, and 18 play badminton. Furthermore, 22 of the members play both tennis and squash, 12 play both tennis and badminton, 9 play both squash and badminton, and 4 play all three sports. How many members of this club play at least one of three sports?
- **Solution:** ....
- **Example 18:** Compute the probability that 10 married couples are seated at random at a round table, then no wife sits next to her husband.
- **Solution:** ....