A Few Discrete Random Variables

United Arab Emirates University

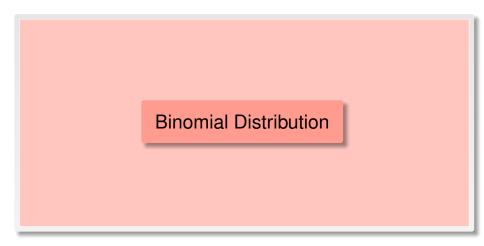
Department of Statistics



- **Binomial Distribution**



A Few



#### **Binomial Distribution**

- A Bernoulli experiment is a random experiment, the outcome of which can be classified in one of two mutually exclusive and exhaustive ways, say, "1=success" or "0=failure." Let Y be the number of success on a Bernoulli trial, then Y is called the Bernoulli random variable.
- If a sequence of n independent Bernoulli trials is performed under the same condition, we call it a set of n Bernoulli trials a Binomial experiment.

#### Binomial Distribution

#### Definition (Binomial Experiment)

An experiment is called a Binomial experiment if it satisfies the following 4 conditions:

- The experiment consists of n Bernoulli trials.
- Each trial results in a success (S) or a failure (F).
- The trials are independent.
- The probability of a success, p, is fixed throughout n trials.

## Binomial Distribution Binomial (n, p)

- Given a Binomial experiment consisting of n Bernoulli trials with success probability p, the Binomial random variable X associated with this experiment is defined as the number of successes among the n trials.
- The random variable X has the Binomial Distribution with parameters n and p; denoted by  $X \sim Binomial(n, p)$ .
- The behavior of Binomial Distribution with different n and p.

## Binomial Distribution Binomial (n, p)

#### **Definition** (Binomial Distribution)

Let  $p \in (0, 1)$ , then the probability mass function of Binomial(n, p) is given by

$$p(x) := \binom{n}{x} p^x (1-p)^{n-x}$$
, for  $x \in \mathbb{S}_x$ , where  $\mathbb{S}_x = \{0, 1, \dots, n\}$ 

Mean

$$E(X) = np$$

Variance VAR(X) = np(1 - p)

## Expected Value of Binomial Distribution

$$E(X) := \sum_{y \in \mathbb{S}_{X}} y \, \rho_{X}(y)$$

$$= \sum_{y=0}^{n} y \binom{n}{y} \rho^{y} (1-\rho)^{n-y}$$

$$= (1-\rho)^{n} \sum_{y=0}^{n} y \binom{n}{y} \left(\frac{\rho}{1-\rho}\right)^{y}$$

$$= (1-\rho)^{n} \frac{n\rho}{(1-\rho)^{n}}$$

$$= n\rho$$

$$(1)$$

$$E(X^{2}) := \sum_{y \in \mathbb{S}_{X}} y^{2} p_{x}(y)$$

$$= \sum_{y=0}^{n} y^{2} {n \choose y} p^{y} (1-p)^{n-y}$$

$$= (1-p)^{n} \sum_{y=0}^{n} y^{2} {n \choose y} \left(\frac{p}{1-p}\right)^{y}$$

$$= (1-p)^{n} \frac{mp + n(n-1)p^{2}}{(1-p)^{n}}$$

$$= np + n(n-1)p^{2}$$
 (2)

 $Var(X) = E(x^2) - (E(X))^2 = np + n(n-1)p^2 - n^2p^2 = np - np^2 = np - np^2$ np(1-p).

## Expected Value of Binomial Distribution

$$M_{X}(t) := \sum_{y \in \mathbb{S}_{X}} e^{ty} p_{X}(y)$$

$$= (1-p)^{n} \sum_{y=0}^{n} e^{ty} \binom{n}{y} \left(\frac{p}{1-p}\right)^{y}$$

$$= (1-p)^{n} \sum_{y=0}^{n} \binom{n}{y} \left(\frac{pe^{t}}{1-p}\right)^{y}$$

$$= (1-p)^{n} \sum_{y=0}^{n} e^{ty} \binom{n}{y} \left(\frac{p}{1-p}\right)^{y}$$

$$= (1-p)^{n} \left(1 + \frac{pe^{t}}{1-p}\right)^{n} = (1-p+pe^{t})^{n}$$

(3)

Poisson Distribution Geometric Distribu

#### Example

Example: Five fair coins are flipped. If the outcomes are assumed independent.

- Find the probability mass function of the number of heads obtained.
- Find the probability that at least 3 heads are obtained.
- Find the probability that at most 2 heads are obtained.

# Example

Example: Five fair coins are flipped. If the outcomes are assumed independent.

- Find the probability mass function of the number of heads obtained.
- Find the probability that at least 3 heads are obtained.
- Find the probability that at most 2 heads are obtained.

**Solution:** Let  $X = \text{The number of heads in 5 tossed coins. } X \sim Binomial(n = 5, p = 0.5).$ 

- $P(X=0) = 0.5^5 = 0.0313$
- $P(X = 1) = {5 \choose 1} 0.5^5 = 0.1563$
- $P(X=2) = {5 \choose 2} 0.5^5 = 0.3125$
- $P(X=3) = {5 \choose 3} 0.5^5 = 0.3125$
- $P(X = 4) = {5 \choose 4} 0.5^5 = 0.1563$
- $P(X=0) = {5 \choose 5} 0.5^5 = 0.0313$



Binomial Distribution Poisson Distribution

## Example

Example: It is known that screws produced by a certain company will be defective with probability .01, independently of each other. The company sells the screws in packages of 10 and offers a money-back guarantee that at most 1 of the 10 screws is defective. What proportion of packages sold must the company replace? Use the Binomial Calculator or Statistical Tables.

## Example

Example: The following gambling game, known as the wheel of fortune (or chuck-a-luck), is quite popular at many carnivals and gambling casinos: A player bets on one of the numbers 1 through 6. Three dice are then rolled, and if the number bet by the player appears i times, i = 1; 2; 3, then the player wins i units; if the number bet by the player does not appear on any of the dice, then the player loses 1 unit. Is this game fair to the player?

#### Outline

- Poisson Distribution



A Few

#### Poisson Distribution

The Poisson distribution models the number of occurrences of an event when there is a known average rate per unit time or space  $\lambda$ .

#### Definition (Poisson Distribution)

The requirements for a Poisson distribution are that:

- no two events can occur simultaneously,
- events occur independently in different intervals, and
- the expected number of events in each time interval remain constant.

Binomial Distribution

## Poisson Distribution: pmf, Expected Value

The Poisson distribution models the number of occurrences of an event when there is a known average rate per unit time or space  $\lambda$ .

#### Definition (Poisson Distribution: pmf, Expected Value)

The requirements for a Poisson distribution are that:

The probability mass function of Poisson( $\lambda$ ) is given by

$$p(x) = \frac{e^{-\lambda} \lambda^x}{x!}$$
 for  $x = 0, 1, 2, 3, ...$ 

(a) If  $X \sim Poisson(\lambda)$ , then  $E(X) = \lambda$ , and  $Var(X) = \lambda$ .

#### **Expected Value of Binomial Distribution**

$$M_X(t) := \sum_{y \in \mathbb{S}_X} e^{ty} \, \rho_X(y)$$

$$= \sum_{y=0}^{\infty} e^{ty} \frac{e^{-\lambda} \lambda^y}{y!}$$

$$= e^{-\lambda} \sum_{y=0}^{\infty} \frac{(\lambda e^t)^y}{y!}$$

$$= e^{\lambda e^t - \lambda}$$
(4)

## A few Examples of Poisson Distribution

Example: The number of customers arriving at a service counter within one-hour period.

Example: The number of typographical errors in a book counted per page.

Example: The number of email messages received at the technical support center daily.

Example: The number of traffic accidents that occur on a specific road during a month.

## A Few Examples of Poisson Distribution

Example: Messages arrive at an electronic message center at random times, with an average of 9 messages per hour.

- What is the probability of receiving exactly five messages during the next hour?
- What is the probability that more than 10 messages will be received within the next two hours?

- The number of messages received in an hour, X is modeled by Poisson distribution with  $\lambda = 9$ , i.e.  $X \sim \text{Poisson}(9)$ .  $P(X = 5) = \frac{9^5 \exp(-9)}{5!}$
- The number of messages received within a 2-hour period, Y is another Poisson distribution with Y = (2)(9) = 18, i.e.  $Y \sim Poisson(18)$ . P(Y > 10) = 1 - P(Y < 10) = ... = 0.9696



## Group Work

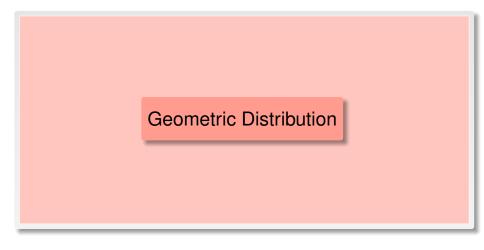
- Develop a real life example in which you can easily apply:
  - Group 1: Poisson distribution.
  - Group 2: Binomial distribution.
  - Group 3: Poisson distribution
- In each case, propose two problems which can be solved using the Statistical Calculator.
- Can you propose an idea in which you can mix both distributions? (extra)

#### Outline

- Geometric Distribution



A Few





#### Geometric Distribution

- Suppose that independent trials, each having a probability p, 0 , of being a success, are performed until a successoccurs.
- Example: The first head in tossing coin several times.
- Then. Geometric distribution models the number of trials performed until a success occurs.

#### Definition (Geometric Distribution)

The probability mass function of Geometric(p) is given by

$$p(x) = (1-p)^{x-1}p$$
 for  $x = 1, 2, 3, ...,$ 



## Expected Value of Binomial Distribution

$$M_X(t) := \sum_{y \in \mathbb{S}_X} e^{ty} \, \rho_X(y)$$

$$= \sum_{y=1}^{\infty} e^{ty} (1-p)^{y-1} p$$

$$= p \sum_{z=0}^{\infty} e^{tz+t} (1-p)^z$$

$$= p e^t \sum_{z=0}^{\infty} ((1-p)e^t)^z$$

$$= \frac{p e^t}{1-(1-p)e^t}$$
(5)

## Geometric Distribution: Example

Example: Suppose that the probability of engine malfunction during any one-hour period is p = 0.02. Find the probability that a given engine will survive two hours.

#### Geometric Distribution: Example

Example: Suppose that the probability of engine malfunction during any one-hour period is p = 0.02. Find the probability that a given engine will survive two hours.

#### Solution:

Letting Y denote the number of one-hour intervals until the first malfunction, we have

$$P(\text{Survival for Next Two Hours}) \\ = P(Y \ge 3) \\ = 1 - P(Y \le 2) \\ = 1 - \sum_{y=1}^{2} p(y) \\ = 1 - \{p(1) + p(2)\} \\ = 1 - 0.02 - 0.98 \times 0.02 \\ = 0.9604$$

Exercise Find the mean and standard deviation of Y.

#### Outline

- **Negative Binomial Distribution**



A Few

#### **Negative Binomial** Distribution



- Suppose that independent trials, each having probability p, 0 , of being a success are performed until a total of rsuccesses is accumulated.
- Example: The third head in tossing coin several times.
- Then, Negative Binomial distribution models the number of trials performed until a the rth success occurs.

#### Definition (Negative Binomial Distribution)

The probability mass function of Negative Binomial RV, denoted by Negative-Binomial(r, p) is given by

$$p(x) = {x-1 \choose r-1} p^{r-1} (1-p)^{x-r} \text{ for } x = r+1, r+2, r+3, \dots,$$

If  $X \sim \text{Negative-Binomial}(r, p)$  then  $E(X) = \frac{r}{p}$ , and  $Var(X) = \frac{r}{p}$ 

## Geometric Distribution: Example

Example: A machine produces 1% defective parts. Using the statistical calculator, calculate the probability that

- 10 parts have to be selected until to get 2 defective parts.
- Between 20 to 25 parts have to be selected to get 2 defective parts.

#### Geometric Distribution: Example

Example: A machine produces 1% defective parts. Using the statistical calculator, calculate the probability that

- 10 parts have to be selected until to get 2 defective parts.
- Between 20 to 25 parts have to be selected to get 2 defective parts.

Solution: Letting Y denote the number of

> AA(6)

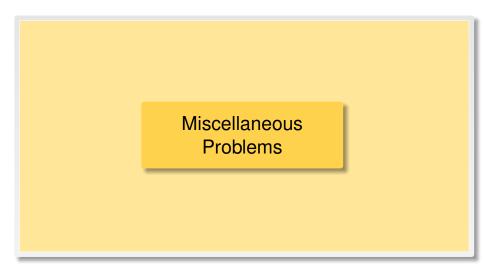
Exercise Find the mean and standard deviation of Y.

#### Outline

- Miscellaneous Problems



A Few





Example: Approximately 10% of the glass bottles coming off a production line have serious flaws in the glass. If two bottles are randomly selected, find the mean and variance of the number of bottles that have serious flaws.

Example: Suppose that a lot of electrical fuses contains 5% defectives. If a sample of 5 fuses is tested, find the probability of observing at least one defective.

Example: An oil exploration firm is formed with enough capital to finance ten explorations. The probability of a particular exploration being successful is 0.10. Assume the explorations are independent.

- Find the mean and variance of the number of successful explorations.
- Suppose the firm has a fixed cost of \$20,000 in preparing equipment prior to doing its first exploration. If each successful exploration costs \$30,000 and each unsuccessful exploration costs \$15,000, find the expected total cost to the firm for its ten explorations.

