STAT 320: Principles of Probability
Unit 1: A few Definitions and Notations from Set Theory

United Arab Emirates University

Department of Statistics

Outline

Definition of a Set

Definition of a Set

Definition (Set)

A Set is a well defined collection of **objects**.

- **Elements:** The objects inside a set are called the **elements** of the Set

- Usually we denote sets with upper-case letters, elements with lower-case letters.

- The following notation is used to show a set membership.
  - $x \in A$:means that x is a member of the set A.
  - $x \notin A$:means that x is NOT a member of the set A.

Venn Diagrams

Venn Diagrams are graphical representation of the sets that are often used to depict the relation between various sets

For the rest of the slides, we shall include ‘Venn Diagrams’ for visualization of different set-theoritic concepts.

Example

Example: Differnt Ways of Describing a Set

1. Listing all the elements: $A = \{1, 2, 3, 4, 5, 6\}$
2. Give a verbal description: A is the set of all integers from 1 to 6, inclusive.
3. Give a mathematical inclusion rule:

$$A = \{x \text{ is an Integer} : 1 \leq x \leq 6\}$$

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The Nulll Sets and the Universal Set

Definition (Null Set)

The **Null Set** *(or also called the* **Empty Set***) is a set with no elements in it.*

It is often denoted by $\emptyset$

Definition (Universal Set)

The **Universal Set** *is the set of all elements currently under consideration.*

In the context of probability theory, it is often denoted by $\mathbb{S}$ (or Ω)

The universal set contains all of the elements relevant to a given discussion.

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Relationship Between Sets

Subset: A set A is called a subset of a set B if all the elements of A are also the elements of B.

If $x \in A \implies x \in B$ then $A \subseteq B$.

Subsets: If A is a subset of B then we write $A \subseteq B$.

Note: The notation for subset is very similar to the notation for "less than or equal to" and means, "included in or equal to" in terms of the sets.

We write $A \nsubseteq B$ if A is not included in B

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- We write $A \not\subseteq B$ if A is not included in B

- We say **"A is a proper subset of B"** if all the elements of A are also the elements of B, but in addition there exists at least one element c such that $c \in B$ but $c \notin A$.

- **Proper Subset:** $A \subset B$, then we say **A is a proper subset of B.**

Comment: The notation for subset is very similar to the notation for "less than", and means, in terms of the sets, "included in but not equal to".

Subset: Example

Outline

Few Set Operations

Set Operation: Union

A Union B: A Union B, denoted by $A \cup B$ is the set of all elements that are either in A, or in B, or inside both the sets.

The logical operator is OR.

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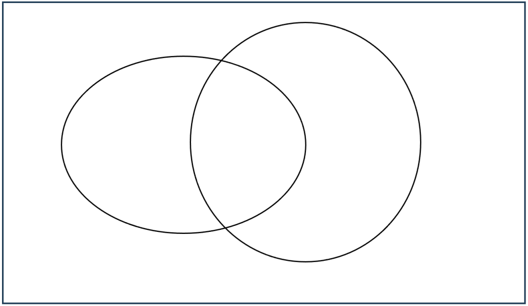

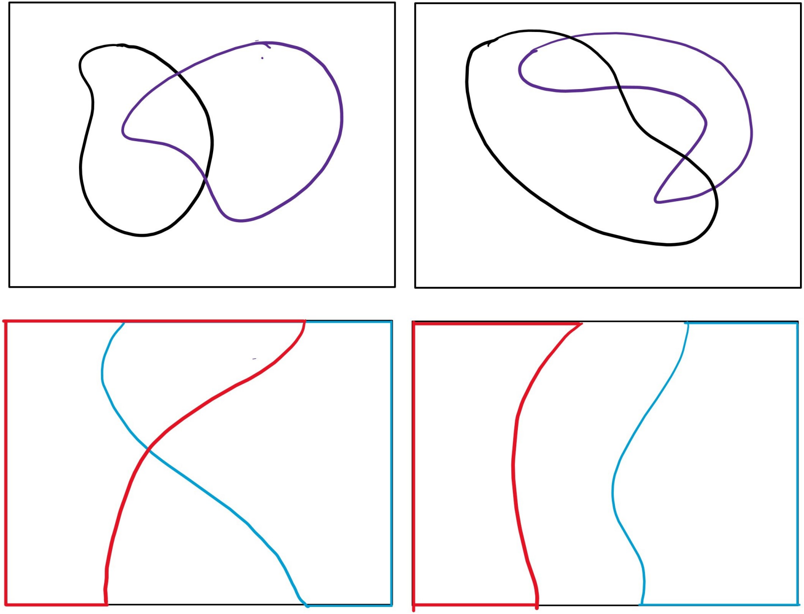

Set Operation: Intersection

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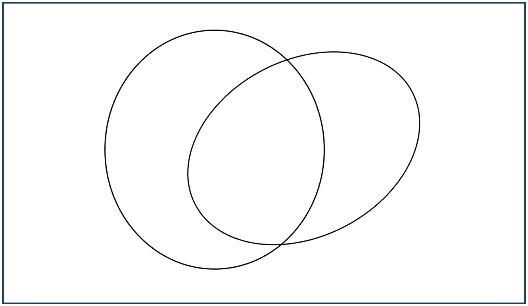

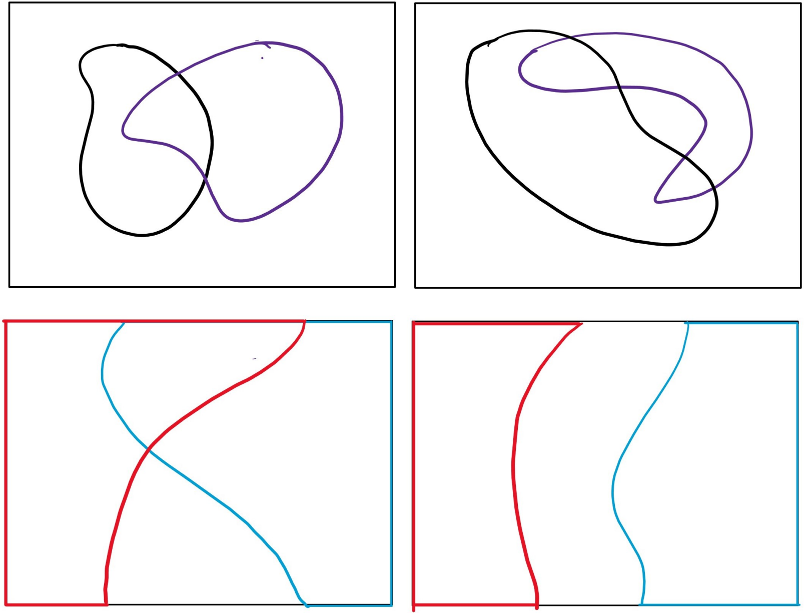

Set Operation: Complement

A Complement : A Complement, denoted by $\overline{A}$ is the set of all elements that are not in A. Sometimes it is also denoted by A^c

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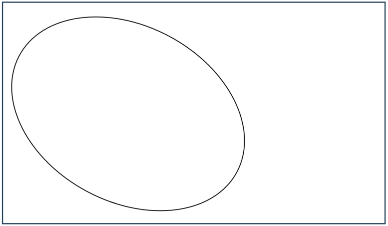

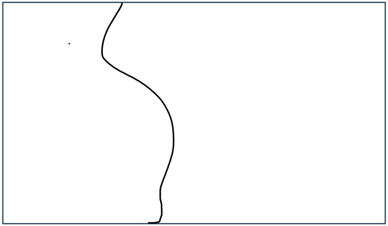

Set Operation: Set Difference

A Minus B: A Minus B, denoted by $A - B$ is the set of all elements that are only an element of A, but not an element of the set B.

$A - B$ is same as: $A \cap \overline{B}$.

Comment: $A - B$ is not same as $B - A$

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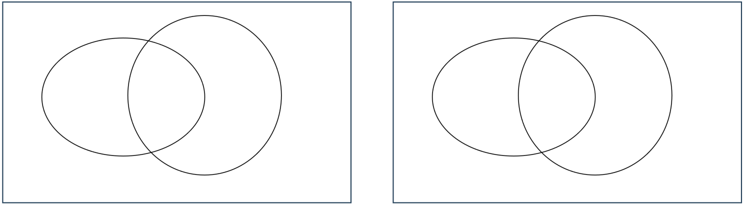

A Few Rules for Various Set Operations

$$A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C$$

Distributive laws of Union & Intersection

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

DeMorgan's laws

$$\overline{(A \cap B)} = (\bar{A} \cup \bar{B})$$

$$\overline{(A \cup B)} = (\bar{A} \cap \bar{B})$$

Set Operation: Examples

Outline

A Few Examples Using Venn Diagrams

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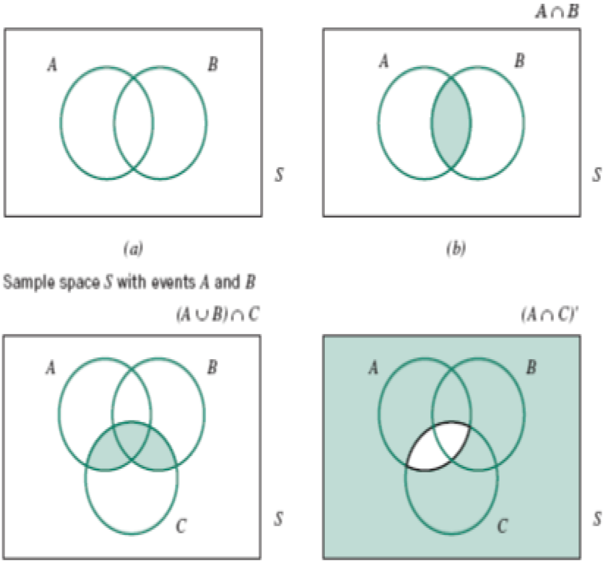
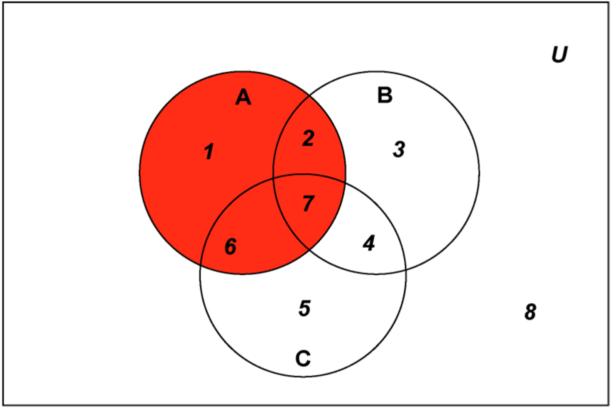
Venn Diagrams: Examples



Venn Diagrams: Examples

Let $\mathbb{S} = \{1, 2, 3, 4, 5, 6, 7, 8\}$,
$A = \{1, 2, 6, 7\}$, $B = \{2, 3, 4, 7\}$, and $C = \{4, 5, 6, 7\}$



Outline

Disjoint Sets & Partition

Disjoint Sets

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A and B are Disjoint $\Leftrightarrow$ $A \cap B = \emptyset$.

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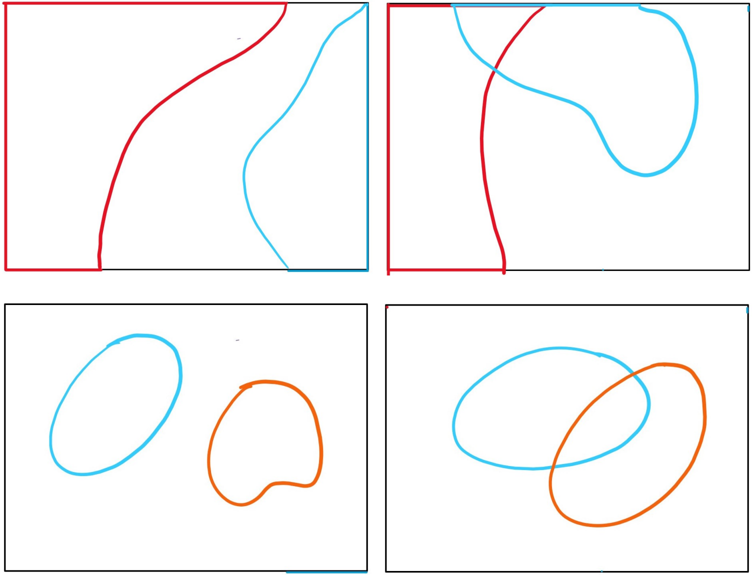
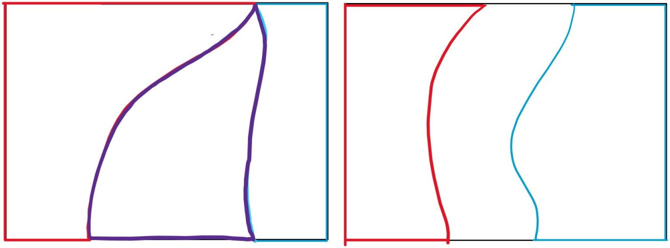


Exhaustive

Exhaustive: A collection of sets $A_1, A_2, \cdots, A_k$ are exhaustive for the set C if $A_1 \cup A_2 \cup \cdots \cup A_k = C$.

A and B are exhaustive for C $\Leftrightarrow \bigcup_{i=1}^{k} A_i = C$.



Exhaustive: Special Case

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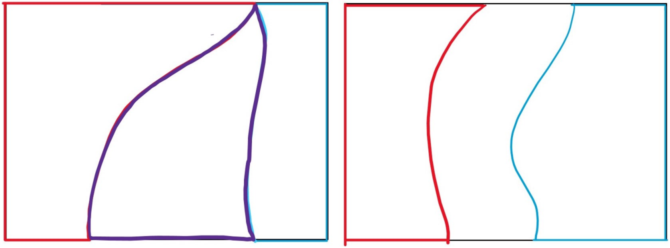

Partition

Partition: A collection of sets $\{A_1, A_2, \cdots, A_k\}$ is called a **partition for a set** C if

1. $A_i \cap A_j = \emptyset$ for $1 \leq i \neq j \leq k$ (i.e. ,A_i, and A_j are Disjoint if $i \neq j$), and
2. $A_1 \cup A_2 \cup \cdots \cup A_k = C$ (i.e. , A and B is exhaustive for C).

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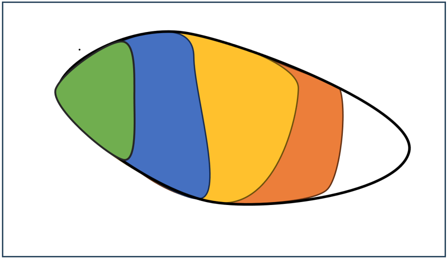

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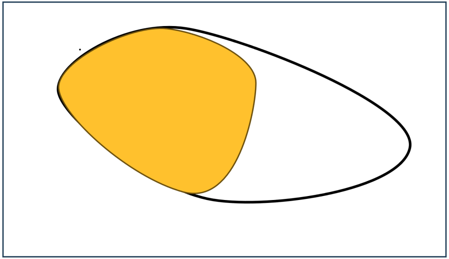
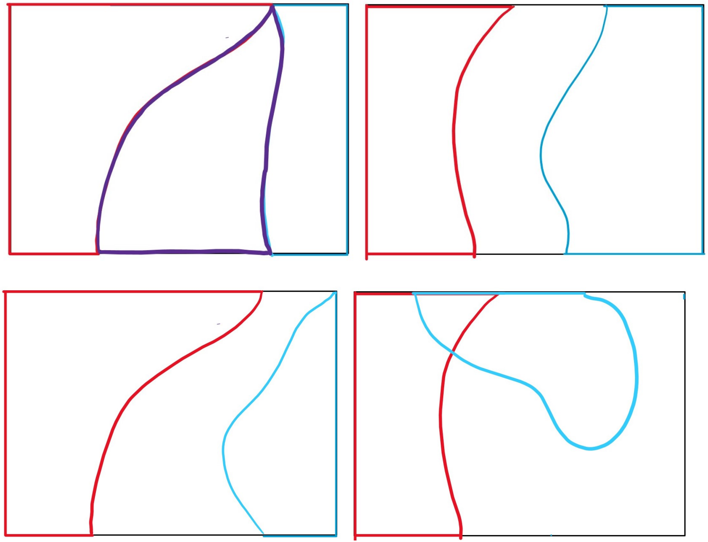
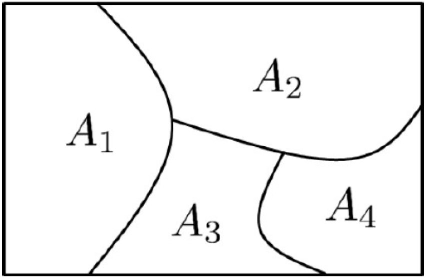




Example

$$S = \{1, 2, 3, 4, 5, 6\}$$



$$A_1 = \{1, 2\}, A_2 = \{3\}, A_3 = \{4\}, A_4 = \{5, 6\}$$

Is $A_1 = \{1, 2, 3\}, A_2 = \{3\}\ A_3 = \{4\}, A_4 = \{4, 5, 6\}$ a partition of S?

Examples

A Few Questions:

Let Ω be the universal set and $\emptyset$ denotes the emptyset.

- $A \cup \emptyset =$
- $A \cup \overline{A} =$
- $A \cap \emptyset =$
- $A - \overline{A} =$
- $A \cap \overline{A} =$
- $A \cup \Omega =$
- $A \cap \Omega =$
- If $A \subset B$ then $A \cap B =$
- If $A \subset B$ then $A \cup B =$

Outline

Real Numbers ($\mathbb{R}$) & Intervals

Real Numbers, Intervals

For the definitions below, assume $a \in \mathbb{R}$, $b \in \mathbb{R}$, and $a < b$.

- $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$
- $(a, b] := \{x \in \mathbb{R} : a < x \leq b\}$
- $[a, b) := \{x \in \mathbb{R} : a \leq x < b\}$
- $(a, b) := \{x \in \mathbb{R} : a < x < b\}$

Examples of Intervals

$[0,1] := \{x \in \mathbb{R} : 0 \leq x \leq 1\}$= Set of all real numbers from 0 to 1, including the numbers 0 and 1.

$(0,1) := \{x \in \mathbb{R} : 0 < x < 1\}$=Set of all real numbers between 0 to 1. It does not include the numbers 0, and 1.

$(0,\infty) := \{x \in \mathbb{R} : 0 < x\}$=Set of all positive real numbers. The interval **does not include 0**

$[0,\infty) := \{x \in \mathbb{R} : 0 < x\}$=Set of all non-negative real numbers. The interval **does include 0**

Examples: Union & Intersection of Intervals

Outline

The Notion of Cartesian Product

The Notion of 'Tuple'

Tuple: Let k be an integer. A k **-tuple** is the a the ordered sequence of values often written inside parenthesis while different elements are separated by comma.

Example:

- $(1.5, 2)$ is a **2-tuple**.
- $(1.5, 2, 6, 2)$ is a **4-tuple**.
- (H, T, H, H, H) is a **5-tuple**

Set Operation: Cartesian Product

Cartesian Product of A and B: Cartesian Product of A and B, denoted by $A \times B$ is the set of all "two-tuple" objects where the first element in the tuple is from the set A and the second element is taken from the set B.

$$A \times B = \{(x, y) : x \in A, y \in B\}.$$

Comment: In general $A \times B$ is not same as $B \times A$. However, $A \times A$ is denoted by A^2. Note that A is a set and not a number.

Cartesian Product of Multiple Sets Let A_1, A_2, ..., A_n are non empty sets. Then their Cartesian product is defined as following: $A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) : a_i \in A_i \text{ for } i = 1, \ldots n\}$.

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A Few Standard Notation

$\mathbb{Z}$: Set of all Integers. i.e. $\mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3, \ldots\}$.

$\mathbb{Z}_+$: Set of all non-negative Integers. i.e. $\mathbb{Z}_+ = \{1, 2, 3, \ldots\}$.

$\mathbb{R}$: Set of all real numbers.

$\mathbb{R}_+$: Set of all positive real numbers.

- $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\}$
- $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x_1, x_2, x_3) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}, x_3 \in \mathbb{R}\}$
- $\mathbb{Z}_+^2 = \mathbb{Z}_+ \times \mathbb{Z}_+ = \{(z_1, z_2) : z_1 \in \mathbb{Z}_+, z_2 \in \mathbb{Z}_+\}$

A Few Standard Notation

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Discussion on Cardinality of Sets

A Discussion on Cardinality of Various Sets

Finite Set

Examples:
$\{1, 2, 3, 4, 5, 6\}$,
$\{HH, HT, TH, TT\}$

The Notion of Infinity (∞)

Infinite Set: Countably Infinite Set

Examples: $\mathbb{Z}_+ := \{1, 2, 3, \ldots,\}$, $\mathbb{Z}_+^2$

Infinite Set: Uncountable Set

Examples: $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}_+$

Questions?