Exam Assistance Note

Let n be a **non-negative integer,** then the **factorial of n**, denoted as $n!$ is defined to be

- $0! = 1$
- $1! = 1$
- $n! = n \times (n-1) \times \ldots \times 1$ for $n \geq 2$

Let n, r be two **non-negative integes,** such that $r \leq n$, then the **n choose r**, denoted by $\binom{n}{r}$, is defined to be

$$\binom{n}{r} := \frac{n!}{(r!) \times ((n-r)!)}$$

For any real number $x \in \mathbb{R}$, the exponential series e^x (or sometimes denoted as $\exp(x)$) is defined as,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Let $x \in \mathbb{R}$ be any real number, and $n \in \mathbb{Z}_+$ be any positive integer, then

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$$

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$$

Let $p \in \mathbb{R}$ be such that $|p| < 1$, then $\sum_{i=0}^{\infty} p^i = 1 + p + p^2 + p^3 + \cdots = \frac{1}{1-p}$.

(Ordered, without replacement) Let r, and n be two positive integers such that $r \leq n$. An ordered arrangement of r distinct objects is called a permutation. The number of ways of ordering n distinct objects taken r at a time, denoted by the symbol ${}^nP_r$, is given as

$${}^nP_r = n(n-1)(n-2)(n-r+1) = \frac{n!}{(n-r)!}.$$

Let $n \geq r$ be two non-negative integers. The number of different ways to select (/choose) r distinct objects from a list of n distinct (non-identical) objects is given as(or ${}^{n}C_{r}$),

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

The number of ways of partitioning n distinct objects into k distinct groups containing $n_1, n_2, \ldots, n_k$ objects, respectively, where each object appears exactly in one group and $\sum_{i=1}^{k} n_i = n$, is

$$\binom{n}{n_1, n_2, \ldots, n_k} := \frac{n!}{(n_1!)(n_2!)\ldots(n_k!)}$$

Number of ways n indistinguishable/identical objects can be organized into r different (ordered) groups is

$$\frac{(n+r-1)!}{n!(r-1)!} = \binom{n+r-1}{n}.$$

De-Arrangement probability with N distinct objects: $1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots + (-1)^N \frac{1}{N!}$.

Properties of Probability

Let $(\mathscr{S}, P)$ be a sample space along with the Probability measure. Let A, B be two events. Then,

- $P(\varnothing) = 0$ where $\varnothing$ denotes the Empty set (Null set).
- $P(A) \leq 1$.
- If $A \subseteq B$ then $P(A) \leq P(B)$.
- $P\left(\overline{A}\right) = 1 - P(A)$, where $\overline{A}$ denotes the complementary event to A.
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

$$A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C$$

Distributive laws of Union & Intersection

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

DeMorgan's laws

$$\overline{(A \cap B)} = \left(\overline{A} \cup \overline{B}\right)$$

$$\overline{(A \cup B)} = \left(\overline{A} \cap \overline{B}\right)$$

Let A_1, A_2, A_3 are three events. Then

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3) &= \left\{ P(A_1) + P(A_2) + P(A_3) \right\} \\ &\quad - \left\{ P(A_1 \cap A_2) + P(A_1 \cap A_3) + P(A_2 \cap A_3) \right\} + \left\{ P(A_1 \cap A_2 \cap A_3) \right\} \end{aligned}$$

Structure of Two × Two Table

	A	$\overline{A}$	Marginal of B
B	$P(A \cap B)$	$P\left(\overline{A} \cap B\right)$	$P(B)$
$\overline{B}$	$P\left(A \cap \overline{B}\right)$	$P\left(\overline{A} \cap \overline{B}\right)$	$P\left(\overline{B}\right)$
Marginal of A	$P(A)$	$P\left(\overline{A}\right)$	Total Probability = 1

Let E, and F are two events such that $P(F) > 0$, then the conditional probability of E **given** F is defined to be,

$$P(E \mid F) := \frac{P(E \cap F)}{P(F)}.$$

Let E and F are two events, then $P(E \cap F) := P(E \mid F) \times P(F)$.

Law of Total Probability

Let E and F be two events, then $P(E) = P(E \mid F)P(F) + P\left(E \mid \overline{F}\right)P\left(\overline{F}\right)$

Law of Total Probability (General):

Let E be an event. Assuming that the collection of sets $\{F_1, F_2, \ldots, F_k\}$ forms a partition of $\mathscr{S}$, we have

$$P(E) = \sum_{j=1}^{k} P(E \mid F_j)P(F_j)$$

Bayes' Theorem (General)

Let $F_1, F_2, \ldots, F_K$ be a set of mutually exclusive and exhaustive events (partition of the sample space $\mathscr{S}$). Suppose now that E be an event such that $P(E) > 0$, then

$$P(F_i \mid E) = \frac{P(E \mid F_i)P(F_i)}{\sum_{j=1}^{K} P(E \mid F_j)P(F_j)}$$

Bayes' Theorem

Let A, B are two events such that $P(A) > 0$, and $P(B) > 0$, then

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A \mid B)P(B) + P\left(A \mid \overline{B}\right)P\left(\overline{B}\right)}$$

Statistical Independence

Two events E and F are said to be statistically independent if $P(E \cap F) = P(E) \times P(F)$

Characterization of a pmf

Let $p(x)$ is **probability mass function** of a discrete random variable on the support $\mathbb{S}$, **if and only if** it satisfies the following conditions:

1. *Positivity:* $p(x) > 0$ for all $x \in \mathbb{S}$
2. *Total Probability:* $\sum_{\{x \in \mathbb{S}\}} p(x) = 1$.

“CDF” of a Discrete Random Variable

Let X be a discrete random variable on the support $\mathbb{S}_X$ with the corresponding probability mass function

$$P(X = x) = p_X(x) \text{ for } x \in \mathbb{S}_X .$$

Then for any $a \in \mathbb{R}$, the cumulative distribution function (cdf), denoted by $F_X(\cdot)$ is the following quantity

$$F_X(a) = P(X \leq a) = \sum_{\{x \leq a \,:\, x \in \mathbb{S}_X\}} p_X(x)$$

“Expected Value” or “Mean” of a Discrete Random Variable

If X is a random variable with pmf $p_X(x)$ on the support $\mathbb{S}_X$, then the expected value (the mean) of X denoted by $E(X)$ (or μ_X) is given by

$$E(X) = \sum_{\{x \in \mathbb{S}_X\}} x \, p_X(x),$$

assuming the above summation/series exists /well-defined. Additionally, assuming it exists, for any* funciton $h(x)$,

$$E\big(h(X)\big) = \sum_{\{x \in \mathbb{S}_X\}} h(x) \, p_X(x),$$

“Variance & Standard Deviation (SD) ” of a Random Variable

The variance of X, denoted by $\mathrm{Var}(X)$ is deifined as

$$\mathrm{Var}(X) := E(X^2) - \big(E(X)\big)^2$$

$$\mathrm{E}(X^2) := \mathrm{Var}(X) + \big(E(X)\big)^2$$

$$\sigma_X = \mathrm{SD}(X) := \sqrt{\mathrm{Var}(X)}$$

"**Moment Generating Function** (MGF) " of a Discrete Random Variable

$$M_X(t) := E\left(e^{tX} \right) = \sum_{\left\{ x \in \mathbb{S}_X \right\}} e^{tx} \, p_X(x) \, .$$

Standard Properties of a few Discrete Distributions

Distribution	Support $\mathbb{S}_X$	pmf $p_X(x)$	Mean $E(X)$	Variance $\mathrm{Var}(X)$	mgf $M_X(t)$
Binomial(n,π)	$\{0,1,\ldots,n\}$	$\binom{n}{x}\pi^x(1-\pi)^{n-x}$	$n\pi$	$n\pi(1-\pi)$	$(1-\pi+\pi e^t)^n$
Poisson(λ)	$\{0,1,2,\ldots\}$	$\frac{e^{-\lambda}\lambda^x}{x!}$	λ	λ	$e^{\lambda e^t-\lambda}$
Geometric(π)	$\{1,2,\ldots\}$	$(1-\pi)^{x-1}\pi$	$\frac{1}{\pi}$	$\frac{1-\pi}{\pi^2}$	$\frac{\pi e^t}{1-(1-\pi)e^t}$
Negative-Binomial(r,π)	$\{r,r+1,r+2,\ldots\}$	$\binom{x-1}{r-1}(1-\pi)^{x-r}\pi^r$	$\frac{r}{\pi}$	$\frac{r(1-\pi)}{\pi^2}$	$\left(\frac{\pi e^t}{1-(1-\pi)e^t}\right)^r$

Standard Properties of a few Continuous Distributions

Distribution	Support $\mathbb{S}_X$	pdf $f_X(x)$	Mean $E(X)$	Variance $\mathrm{Var}(X)$	mgf $M_X(t)$
Uniform(a,b)	$[a,b]$	$\begin{cases}\frac{1}{b-a} & \text{if } x\in[a,b]\\ 0 & \text{otherwise .}\end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$M_X(t)=\begin{cases}\frac{e^{tb}-e^{ta}}{t(b-a)} & \text{if } t\neq 0\\ 1 & \text{if } t=0\end{cases}$
Exponential(λ)	$(0,\infty)$	$\begin{cases}\lambda e^{-\lambda x} & \text{if } x>0\\ 0 & \text{otherwise .}\end{cases}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}$ if $0\le t<\lambda$
Gamma(α,λ) shape $=\alpha$, rate $=\lambda$	$(0,\infty)$	$\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}$ if $x>0$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\frac{1}{\left(1-\frac{t}{\lambda}\right)^\alpha}$ if $0\le t<\lambda$
Beta(α,β)	$(0,1)$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}x^{\beta-1}$ if $0<x<1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	$---$
Normal(μ,σ^2) mean $=\mu$, Var $=\sigma^2$	$(-\infty,\infty)$	$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ for $x\in\mathbb{R}$	μ	σ^2	$e^{\mu t+\frac{t^2\sigma^2}{2}}$

A Standard Deck of Cards