STAT 320: Principles of Probability
Unit 7: Bivariate & Multivariate Random Vectors

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- In this part of the course we will focus the procedures to study multiple random variables together.

- Let $X_1, X_2, \ldots, X_n$ be random variables, to study their probabilistic propertis jointly we construct the Myltivariate Random Vector

$$\underset{\sim}{\boldsymbol{X}} := (X_1, X_2, \ldots, X_n)^T$$

- However, we will focus only the Bi-Variate case with two random variables that we will mostly denote by (X, Y)

Outline

Discrete Multivariate Random Variables

Bivariate Discrete Randmo Variable

- We call (X, Y) to be discrete bivariate random vector is both the random variables X and Y are discrete in nature. Also the corresponding support

$$\mathbb{S}_{X,Y} := \left\{(x, y) \in \mathbb{R}^2 : (x, y) \text{ is a possible value of } (X, Y)\right\}$$

- (X, Y) is a bovariate discrete random variable if the corresponding support $\mathbb{S}_{X,Y}$ is a discrete set.

Joint p.m.f/ Support/ Diagram of support

Probability Mass Function (pmf)

For a discrete random vector (X, Y), we define the probability mass function (pmf) $p_{X,Y}(x, y)$ of X by

$$\boxed{p_{X,Y}(x, y)} = P(X = x, Y = y) \text{ for all } (x, y) \in \mathbb{S}_{X,Y}$$

Joint Cumulative Distribution Function

Definition (Bivariate CDF)

Let X, Y be two discrete random variables. The joint cumulative distribution function is given by

$$F_{X,Y}(x, y) := P\left(X \leq x, Y \leq y\right).$$

Joint CDF (Discrete Random Variable)

Joint CDF from Joint p.m.f.

If the joint probability mass function of two random variables (X,Y) on the support $\mathbb{S}_{X,Y}$ is $p_{X,Y}(x,y) = P(X = x, Y = y)$. then

$$F_{X,Y}(x, y) = \sum_{\substack{s \le x,\ t \le y \\ \text{where } (s,t) \in \mathbb{S}_{X,Y}}} \sum p_{X,Y}(s, t)$$

Marginal Distributions for Discrete Random Vector

The marginal probability mass function of X is given by

$$p_X(x) = \sum_{\{t:(x,t)\in\mathbb{S}_{XY}\}} p_{X,Y}(x, t)$$

The marginal probability mass function of X is given by

$$p_Y(y) = \sum_{\{s:(s,y)\in\mathbb{S}_{XY}\}} p_{X,Y}(s, y)$$

Example

Example : Suppose two cards are drawn at random without replacement from a deck of 4 cards numbered 1, 2, 3, 4. Let X be the number on the first card and Y be the number of the second card.

a). Find the joint probability function of X and Y .

b). Find the marginal probability function of X.

c). Find the marginal probability function of Y .

:

Y \ X	1	2	3	4	Marginal of Y
1	0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$
2	$\frac{1}{12}$	0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$
3	$\frac{1}{12}$	$\frac{1}{12}$	0	$\frac{1}{12}$	$\frac{1}{4}$
4	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	0	$\frac{1}{4}$
Marginal of X	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	

Example

Example : A fair coin is flipped three times. Let X denotes the number of heads to occur in the first two flips, a and let Y denotes the number of heads to occur in the last two flips.

a). Find the joint probability function of (X, Y)

b). and the marginal probability functions of X, and Y.

c). Calculate $P(X = Y)$.

:

Y \ X	0	1	2	Marginal of Y
0	$\frac{1}{8}$	$\frac{1}{8}$	0	$\frac{1}{4}$
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{2}{4}$
2	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$
Marginal of X	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	

$$P(X = Y) = \sum_{\{(x, y) \in \mathbb{S}_{XY} : x = y\}} p_{X,Y}(x, y) = p_{X,Y}(0,0) + p_{X,Y}(1,1) + p_{X,Y}(2,2) = \frac{1}{2}.$$

Example

Example : Suppose that 3 balls are randomly selected from an urn containing 3 red, 4 white, and 5 blue balls. If we let X and Y denote respectively, the number of red and white balls chosen. Find the joint probability mass function of X and Y.

: :

Y \ X	0	1	2	3	Marginal of Y
0	$\frac{10}{220}$	$\frac{30}{220}$	$\frac{15}{220}$	$\frac{1}{220}$	$\frac{56}{220}$
1	$\frac{40}{220}$	$\frac{60}{220}$	$\frac{12}{220}$	0	$\frac{112}{220}$
2	$\frac{30}{220}$	$\frac{18}{220}$	0	0	$\frac{48}{220}$
3	$\frac{4}{220}$	0	0	0	$\frac{4}{220}$
Marginal of X	$\frac{84}{220}$	$\frac{108}{220}$	$\frac{27}{220}$	$\frac{1}{220}$	

Outline

Continuous Multivariate Random Variables

We say that X and Y are jointly continuous if there exists a function $f_{X,Y}(x, y)$, defined for all real x and y, having the property that, for every set C of pairs of real numbers (that is, C is a set in the two-dimensional plane),

$$P\Big((X, Y) \in C\Big) = \iint\limits_{\{(x,y)\in C\}} f_{X,Y}(s, t)\,ds\,dt$$

$f_{X,Y}(x, y)$ is called the joint probability density function of the random vector (X, Y).

Joint CDF for Continuous Random Vector

Definition (Bivariate CDF)

Let X, Y be two discrete random variables. The joint cumulative distribution function is given by

$$F_{X,Y}(x, y) := P\left(X \leq x, Y \leq y\right).$$

Joint CDF from Joint p.d.f.

If the joint probability density function of X and Y is $f_{X,Y}(x,y)$. then

$$F_{X,Y}(x, y) = \int\int_{\left\{\substack{s \leq x,\ t \leq y \\ \text{where } (s,t) \in \mathbb{S}_{X,Y}}\right\}} f_{X,Y}(s, t)\, ds\, dt$$

- If CDF of a bivariate continuous random variable is provided, then the corresponding p.d.f is obtained by following:

$$f_{X,Y}(x,y) = \frac{d^2 F(x,y)}{dx\ dy}$$

Marginal Distributions for Continuous Random Vector

The marginal probability mass function of X is given by

$$f_X(x) = \int_{\{t:(x,t)\in\mathbb{S}_{XY}\}} f_{X,Y}(x,t)dt$$

The marginal probability mass function of Y is given by

$$f_Y(y) = \int_{\{s:(s,y)\in\mathbb{S}_{XY}\}} f_{X,Y}(s,y)ds$$

Example

Example : The joint pdf of X; Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{x+y+1}{2} & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

a). Find the cumulative distribution function of (X, Y) .

b). Find the marginal density of X.

c). Find the marginal density of Y .

Example

Example : The joint pdf of X; Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{x+y+1}{2} & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

a). Find the cumulative distribution function of (X, Y) .

b). Find the marginal density of X.

c). Find the marginal density of Y .

$$F_{X,Y}(x,y) := \iint_{\{(s,t)\in\mathbb{S}_{XY}: s\le x, t\le y\}} f_{X,Y}(s,t)dtds = \int_0^s \int_0^t \frac{s+t+1}{2} dtds = \frac{xy(x+y+2)}{4} \text{ for } 0 < x < 1, 0 < y < 1.$$

$$f_X(x) := \int_{\{x:(x,y)\in\mathbb{S}_{XY}\}} f_{X,Y}(x,y)dy = \int_0^1 \frac{x+y+1}{2} dy = \frac{x}{2} + \frac{3}{4} \text{ for } 0 < x < 1.$$

$$f_Y(y) := \int_{\{x:(x,y)\in\mathbb{S}_{XY}\}} f_{X,Y}(x,y)dx = \int_0^1 \frac{x+y+1}{2} dx = \frac{y}{2} + \frac{3}{4} \text{ for } 0 < y < 1.$$

Example

Example : Let X, Y have joint cdf

$$F_{X,Y}(x,y) = \begin{cases} x^2y^3 & \text{for } 0 < x < 1, 0 < y < 1 \\ & \text{otherwise} \end{cases}$$

a). Find the joint density function of (X, Y) .

b). Find the marginal density of X.

c). Find the marginal density of Y .

Example

Example : Let X, Y have joint cdf

$$F_{X,Y}(x,y) = \begin{cases} x^2y^3 & \text{for } 0 < x < 1, 0 < y < 1 \\ & \text{otherwise} \end{cases}$$

a). Find the joint density function of (X, Y) .
b). Find the marginal density of X.
c). Find the marginal density of Y .

: :

$$f_{X,Y}(x,y) = \frac{d^2F_{X,Y}(x,y)}{dx\ dy} = 6xy^2 \text{ for } 0 \le x \le 1, 0 \le y \le 1.$$

$$f_X(x) := \int_{\{y:(x,y)\in\mathbb{S}_{XY}\}} f_{X,Y}(x,y)dy = \int_0^1 6xy^2dy = 2x \text{ for } 0 \le x \le 1.$$

$$f_Y(y) := \int_{\{x:(x,y)\in\mathbb{S}_{XY}\}} f_{X,Y}(x,y)dx = \int_0^1 6xy^2dx = 3y^2 \text{ for } 0 \le y \le 1.$$

Example

Example : The joint density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-x-2y} & \text{for } 0 < x < \infty, 0 < y < \infty \\ & \text{otherwise} \end{cases}$$

a). Find the marginal density of X.
b). Find the marginal density of Y .
c). Find $P(X > 1, Y < 1)$
d). Find $P(X < Y)$
e). Find $P(X < 4)$

Example : The joint density of X and Y is given by $f_{X,Y}(x,y) = \begin{cases} 2e^{-x-2y} & \text{for } 0 < x < \infty, 0 < y < \infty \\ & \text{otherwise} \end{cases}$

a). Find the marginal density of X.

b). Find the marginal density of Y .

c). Find $P(X > 1, Y < 1)$

d). Find $P(X < Y)$

e). Find $P(X < 4)$

$$f_X(x) := \int_{\{x:(x,y)\in\mathbb{S}_{XY}\}} f_{X,Y}(x,y)dy = \int_0^\infty 2e^{-x-2y}dy = e^{-x} \text{ for } 0 \le x < \infty.$$

$$f_Y(y) := \int_{\{y:(x,y)\in\mathbb{S}_{XY}\}} f_{X,Y}(x,y)dx = \int_0^\infty 2e^{-x-2y}dx = 2e^{-2y} \text{ for } 0 \le y \le \infty.$$

$$P(X > 1, Y < 1) = \iint_{\{(x,y)\in\mathbb{S}_{XY}:x>1,y<1\}} f_{X,Y}(x,y)dx = \int_1^\infty \int_0^1 2e^{-x-2y}dx = e^{-1} - e^{-3}.$$

$$P(X < Y) = \iint_{\{(x,y)\in\mathbb{S}_{XY}:x<y\}} f_{X,Y}(x,y)dx = \int_0^\infty \int_x^\infty 2e^{-x-2y}dydx = \frac{1}{3}.$$

$$P(X < 4) = \iint_{\{x\in\mathbb{S}_X:x<4\}} f_X(x)dx = \int_0^4 e^{-x}dx = 1 - e^{-4}.$$

Outline

Conditional Distributions

Definition (Conditional p.m.f)

If $p_{XY}(x, y)$ denotes the joint probability mass function (pmf) of two discrete random variables X and Y and if $p_X(x)$ and $p_Y(y)$ denote the marginal probability function of X, (Y respectively) then the conditional probability of X given $Y = y$ is given by

$$p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

The conditional probability of Y given X = x is given by

$$p_{Y|X}(y \mid x) = \frac{p_{X,Y}(x, y)}{p_X(x)}.$$

Example

Example : Suppose two cards are drawn at random without replacement from a deck of 4 cards numbered 1, 2, 3, 4. Let X be the number on the first card and Y be the number of the second card.

a). Find the conditional probability of X given Y = 2.

b). Use this to compute $P(X \leq 2 \mid Y = 2)$.

:

Example

Example : Suppose two cards are drawn at random without replacement from a deck of 4 cards numbered 1, 2, 3, 4. Let X be the number on the first card and Y be the number of the second card.

- a). Find the conditional probability of X given Y = 2.
- b). Use this to compute $P(X \leq 2 \mid Y = 2)$.

The conditional distribution of X given Y=2 is calculated for each possible value of X using

$$p_{X|Y=2}(x) = \frac{p_{X,Y}(x,2)}{p_Y(2)}.$$

The table below shows the results

x	$p_{X\|Y=2}(x)$
1	$\frac{1}{3}$
2	0
3	$\frac{1}{3}$
4	$\frac{1}{3}$

$P(X \leq 2 \mid Y = 2) = p_{X|Y=2}(1) + p_{X|Y=2}(2) = \frac{1}{3}.$

Example

Example : Suppose that 3 balls are randomly selected from an urn containing 3 red, 4 white, and 5 blue balls. If we let X and Y denote respectively, the number of red and white balls chosen.

a). Find the conditional probability of X given Y = 2.

b). Use this to compute $P(X \leq 2 \mid Y = 2)$.

:

Definition (Conditional p.d.f.)

If $f_{XY}(x, y)$ denotes the joint probability density function of two continuous random variables X and Y and if $f_X(x)$ and $f_Y(y)$ denote the marginal probability density function of X, (Y respectively) then the conditional probability density of X given $Y = y$ is given by

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

The conditional probability density of Y given X = x is given by

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x, y)}{f_X(x)}.$$

Example

Example : The joint pdf of X; Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{x+y+1}{2} & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

a). Find the conditional probability of X given Y = 0.5.

b). Use this to compute $P(X \leq 0.75 \mid Y = 0.5)$

Example

Example : The joint pdf of X; Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{x+y+1}{2} & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

a). Find the conditional probability of X given Y = 0.5.

b). Use this to compute $P(X \leq 0.75 \mid Y = 0.5)$

Note that, in an earlier example we have computed the marginal as follows

$$f_Y(y) := \int_{\{x:(x,y)\in\mathbb{S}_{XY}\}} f_{X,Y}(x,y)dx = \int_0^1 \frac{x+y+1}{2}dx = \frac{y}{2} + \frac{3}{4} \text{ for } 0 < y < 1.$$

a). The Conditional density of X given $Y = 0.5$ is $f_{X|Y=0.5}(x) = \frac{f_{X,Y}(x,0.5)}{f_Y(0.5)} = \frac{x}{2} + \frac{3}{4}$ for $0 < x < 1$.

b). $P(X \leq 0.75 \mid Y = 0.5) = \int_0^{0.75} f_{X|Y=0.5}(x)dx = \int_0^{0.75} (\frac{x}{2} + \frac{3}{4})dx = \frac{27}{32}$.

Example

Example : Let X, Y have joint cdf

$$F_{X,Y}(x,y) = \begin{cases} x^2y^3 & \text{for } 0 < x < 1, 0 < y < 1 \\ & \text{otherwise} \end{cases}$$

a). Find the conditional probability of X given Y = 0:5.

b). Use this to compute $P(X \geq 0.5 \mid Y = 0.5)$

Example

Example : The joint density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-x-2y} & \text{for } 0 < x < \infty, 0 < y < \infty \\ & \text{otherwise} \end{cases}$$

a). Find the conditional probability of X given Y = 1.

b). Find the marginal density of Y .

c). Use this to compute $P(X \leq 2 \mid Y = 1)$

Outline

Statistically Independent Random Variables

Definition (Independent Random Variables)

The random variables X and Y are said to be **statistically independent** random variables if, for any two events A and B,

$$P(X \in A, Y \in B) = P(X \in A) \times P(Y \in B)$$

Theorem

ndependent Discrete Random Variables Let (X, Y) *be bivariate discrete random vector with a probability density function* $p_{X,Y}(x, y)$ *on the support* $(x, y) \in \mathbb{S}_{X,Y}$.

Let $p_X(x)$ *be the marginal density of the random variable* X *on the support* $\mathbb{S}_X$

Let $p_Y(y)$ *be the marginal density of the random variable* X *on the support* $\mathbb{S}_Y$.

The continuous random variables X *and* Y *are* **statistically independent** *if the corresponding joint probability density function*

$$p_{X,Y}(x, y) = p_X(x) \times p_Y(y)$$

for all x *and* y, *and* $\mathbb{S}_{X,Y} = \mathbb{S}_X \times \mathbb{S}_Y$.

Theorem

Independent Discrete Random Variables *Let (X, Y) be bivariate continuous random vector with a probability density function $f_{X,Y}(x, y)$ on the support $(x, y) \in \mathbb{S}_{X,Y}$.*

Let $f_X(x)$ be the marginal density of the random variable X on the support $\mathbb{S}_X$

Let $f_Y(y)$ be the marginal density of the random variable X on the support $\mathbb{S}_Y$.

The continuous random variables X and Y are **statistically independent** *if the corresponding joint probability density function*

$$f_{X,Y}(x, y) = f_X(x) \times f_Y(y)$$

for all x and y, and $\mathbb{S}_{X,Y} = \mathbb{S}_X \times \mathbb{S}_Y$.

Let X, Y be any two statistically independent random variables then then the following facts are true:

- For any two events A, B
  $P(X \in A, Y \in B) = P(X \in A) \times P(Y \in B)$
- For any two functions* $h(x)$ and $g(y)$
  $E\left(g(X)h(Y)\right) = E\left(g(X)\right)E\left(h(Y)\right)$
- If the X, Y has the marginal CDFs $F_X(x)$ and $F_Y(y)$ respectively, then the joint CDF
  $F_{X,Y}(x, y) = F_X(x) \times F_Y(y)$ for all x, y.

Example

Example : The joint pdf of X; Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{x+y+1}{2} & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

a). Are X and Y independent?

Example : The joint pdf of X; Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{x+y+1}{2} & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

a). Are X and Y independent?

We have already seen in a previous example that the marginals:

$$f_X(x) := \int_{\{x:(x,y)\in\mathbb{S}_{XY}\}} f_{X,Y}(x,y)dy = \int_0^1 \frac{x+y+1}{2}dy = \frac{x}{2} + \frac{3}{4} \text{ for } 0 < x < 1.$$

$$f_Y(y) := \int_{\{x:(x,y)\in\mathbb{S}_{XY}\}} f_{X,Y}(x,y)dx = \int_0^1 \frac{x+y+1}{2}dx = \frac{y}{2} + \frac{3}{4} \text{ for } 0 < y < 1.$$

Now observe that $f_X(x) \times f_Y(y) = (\frac{x}{2} + \frac{3}{4}) \times (\frac{y}{2} + \frac{3}{4}) \neq \frac{x+y+1}{2} = f_{X,Y}(x,y)$ Therefore, the random variables X and Y are **NOT statistically independent.**

Example

Example : Let X, Y have joint cdf

$$F_{X,Y}(x,y) = \begin{cases} x^2y^3 & \text{for } 0 < x < 1, 0 < y < 1 \\ & \text{otherwise} \end{cases}$$

a). Are X and Y independent?

Example : The joint density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-x-2y} & \text{for } 0 < x < \infty, 0 < y < \infty \\ & \text{otherwise} \end{cases}$$

a). Are X and Y independent?

:

Outline

Expectation for Different Functions of Multivariate Random Variables

Let X, Y be two discrete random variables with joint probability function $p_{X,Y}(x, y)$. Then the expected value of $g(X, Y)$ is given by

$$E\left(g(X, Y)\right) = \sum\sum_{(x,y)\in\mathbb{S}_{XY}} g(x, y)p_{X,Y}(x, y)$$

Let X, Y be two continuous random variables with joint probability density function $f_{X,Y}(x, y)$. Then the expected value of $g(X, Y)$ is given by

$$E\left(g(X, Y)\right) = \int\int_{(x,y)\in\mathbb{S}_{XY}} g(x, y)f_{X,Y}(x, y)dx\, dy$$

Example

Example : Let X, Y have joint cdf

$$f_{X,Y}(x,y) = \begin{cases} \frac{2}{7}(x+2y) & \text{for } 0 < x < 1, 1 < y < 2 \\ & \text{otherwise} \end{cases}$$

a). Find the expected value of $\frac{X}{Y^3}$

b). Find the expected value of XY

Outline

Reminder: Mean and Variance of a Random Variable

Mean: Let X be a random variable, then $E(X)$ denoted by μ_X is called the **mean** of the random variable.

Variance: Let X be a random variable, then $E\left(X-\mu_X\right)^2$ denoted by $\mathrm{Var}(X)$ is called the **Variance** of the random variable. Note that, the alternative formula for variance is:

$$\mathrm{Var}(X) := E(X^2) - (E(X))^2 .$$

Covariance

Definition (Covariance)

Let X, and Y be two random variables with a joint distribution. Then

$$\mathrm{Cov}(X,Y) = E\left((X-\mu_X)(Y-\mu_Y)\right),$$

where μ_X and μ_Y denotes the mean of the random variables X, and Y respectively.

An Alternative Formulation for the covariance is the following:

$$\boxed{\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)}$$

Statistically Independent Random Variables and Covariance

Theorem

If X, and Y are two **statistically independent** *random variables, then*

$$\boxed{Cov(X, Y) = 0}.$$

However, the converse of the result is not true in general.

Example

Example : Suppose X and Y have the following joint distribution :

Y \ X	0	1	2
0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$
1	$\frac{2}{9}$	$\frac{1}{6}$	0
2	$\frac{1}{36}$	0	0

1. Find the covariance of X and Y .
2. Show that X, and Y are not statistically independent?

Example

Example : Let X and Y have joint density

$$f_{X,Y}(x,y) = \begin{cases} 2 & \text{for } x > 0, y > 0, x + y < 1 \\ & \text{otherwise} \end{cases}$$

a). Find the covariance of X and Y .

b). Are the random variables X, and Y statistically independent?

Expected Value of Linear Combination

Let $X_1, X_2, \ldots X_n$ are random variables and $Y = a_0 + \sum_{i=1}^{n} a_i X_i$, where $a_i's$ are constants then

- $$E(Y) = a_0 + \sum_{i=1}^{n} a_i E(X_i)$$

$$\text{Var}(Y) = \sum_{i=1}^{n} a_i^2 \text{Var}(X_i) + 2 \sum\sum_{1 \le i < j \le n} a_i a_j \text{Cov}(X_i, X_j)$$

If $X_1, X_2, \ldots X_n$ are mutually statistically independent then,

$$\text{Var}(Y) = \sum_{i=1}^{n} a_i^2 \text{Var}(X_i)$$

Let $X_1, X_2, \ldots X_n$ are random variables.

$$Y_1 = a_0 + \sum_{i=1}^{n} a_i X_i, \quad \text{and} \quad Y_2 = b_0 + \sum_{i=1}^{n} b_i X_i,$$

where $a_i's$ abd $b_i's$ are constants then

$$\text{Cov}(Y_1, Y_2) = \sum_{i=1}^{n} a_i b_i \text{Var}(X_i) + 2 \sum\sum_{1 \le i < j \le n} a_i b_j \text{Cov}(X_i, X_j)$$

If $X_1, X_2, \ldots X_n$are mutually statistically independent then,

$$\text{Cov}(Y) = \sum_{i=1}^{n} a_i b_j \text{Var}(X_i)$$

Example

Example : Let X and Y have joint distribution. For X and Y defined in the previous two examples, Let $Z_1 = 2X + 4Y$ and $Z_2 = X - 2Y$

a). Find $E(Z_1)$, $E(Z_2)$

b). Find $\mathrm{Var}(Z_1)$, $\mathrm{Var}(Z_2)$

c). Find $\mathrm{Cov}(Z_1, Z_2)$.

Example

Example : Let X and Y be two independent random variables with means 2, 3 respectively. , The variances of X, Y is provided as 4 and 2. Let $Z_1 = X + 2Y + 3$ and $Z_2 = 3X - Y$. Find:

a). Find $E(Z_1)$, $E(Z_2)$

b). Find $\mathrm{Var}(Z_1)$, $\mathrm{Var}(Z_2)$

c). Find $\mathrm{Cov}(Z_1, Z_2)$.

Example

Example : Gasoline is to be stocked in a bulk tank once at the beginning of each week and then sold to individual customers. Let Y_1 denote the proportion of the capacity of the bulk tank that is available after the tank is stocked at the beginning of the week. Because of the limited supplies, Y_1 varies from week to week. Let Y_2 denote the proportion of the capacity of the bulk tank that is sold during the week. Because Y_1 and Y_2 are both proportions, both variables take on values between 0 and 1. Further, the amount sold, Y_2, cannot exceed the amount available, Y_1. Suppose that the joint density function for Y_1 and Y_2 is given by

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} 3y_1 & \text{for } 0 < x < \infty, 0 \leq y_2 \leq y_1 \leq 1 \\ & \text{otherwise} \end{cases}$$

a). Find the probability that less than one-half of the tank will be stocked and more than one-quarter of the tank will be sold. i.e. $P(0 \leq Y_1 \leq 0.5, Y_2 > 0.25)$.

b). What is the probability that less than one-half of the tank will be stocked given that more than one-quarter of the tank will be sold. $P(0 \leq Y_1 \leq 0.5 \mid Y_2 > 0.25)$.

c). Find the marginal density of Y_1

d). Find the marginal density of Y_2

e). Find $E(Y_2)$

f). Find the conditional density of Y_2 given $Y_1 = 0.25$.

Example

Example : Given here is the joint probability function associated with data obtained in a study of automobile accidents in which a child (under age 5 years) was in the car and at least one fatality occurred. Specifically, the study focused on whether or not the child survived and what type of seatbelt (if any) he or she used. Define

$$Y_1 = \begin{cases} 0 & \text{if the child survived} \\ 1 & \text{if not,} \end{cases} \quad \text{and, } Y_2 = \begin{cases} 0 & \text{if no belt used,} \\ 1 & \text{if adult belt used} \\ 2 & \text{if car-seat belt used} \end{cases}$$

Notice that Y_1 is the number of fatalities per child and, since children's car seats usually utilize two belts, Y_2 is the number of seatbelts in use at the time of the accident

y_2 \ y_1	0	1
0	0.38	0.17
1	0.14	0.02
2	0.24	0.05

1. Find F(1, 2). What is the interpretation of this value?
2. What is the Marginal distribution of Y_1?
3. What is the Marginal distribution of Y_2?

Example

Example : The management at a fast-food outlet is interested in the joint behavior of the random variables Y_1, defined as the total time between a customer's arrival at the store and departure from the service window, and Y_2, the time a customer waits in line before reaching the service window. Because Y_1 includes the time a customer waits in line, we must have $Y_1 \geq Y_2$. The relative frequency distribution of observed values of Y_1 and Y_2 can be modeled by the probability density function

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} e^{-y_1} & \text{for } 0 \leq y_2 \leq y_1 \leq \infty \\ 0 & \text{otherwise} \end{cases},$$

with time measured in minutes. Find

- a). $P(Y - X \leq 1)$.
- b). $P(Y_1 \geq 2Y_2)$.
- c). $P(Y_1 - Y_2 \geq 1)$.
- d). Find $E(Y_1 - Y_2)$, expected service time.
- e). Find Marginal Density of Y_1.
- f). Find Marginal Density of Y_2.
- g). Find Conditional Density of Y_2 given $Y_1 = 2$.
- h). Find $E(Y_2 \mid Y_1 = 2)$

Outline

Moment Generating Function

Definition (Moment Generating Function)

The **moment generating function** of a random variable X is given by

$$M_X(t) = E\left(e^{tX}\right)$$

- If X is a discrete random variable with a probability mass funciton (pmf) $p_x(x)$ on the support of the random variable $\mathbb{S}_x$, then assuming the existance/finiteness of the quantity

$$M_X(t) := E\left(e^{tX}\right) = \sum_{x \in \mathbb{S}_X} e^{tx} p_x(x).$$

- If X is a continuous random variable with a probability density funciton (pdf) $f_x(x)$ on the support of the random variable $\mathbb{S}_x$, then assuming the existance/finiteness of the quantity

$$M_X(t) := E\left(e^{tX}\right) = \int_{x \in \mathbb{S}_X} e^{tx} p_x(x) dx.$$

Definition (Raw Moments of a Random Variable)

Let r be a positive integer, then the r^{th} raw moments (non-centered) of a random variable X is defined as $\mu'_{r:X} = E(X^r)$.

Theorem

Let X be a r.v. with the moment generating function $M_x(t)$, then, assuming existence, the r^{th} raw moments (non-centered) for the random variable can be obtained as

$$\mu'_{r:X} = \left.\frac{d^r M_X(t)}{dt^r}\right|_{t=0}$$

Discuss Uniqueness of MGF

Example

Questions?