STAT220: Probability

Unit 2: Combinatorial Methods

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Objectives

- Counting principles
- Multiplication principle
- Addition principle
- Permutations
- Combinations

Counting

- An efficient way of *counting* is necessary to handle large masses of statistical data and for an understanding of probability.
- In this lecture, we will learn a few counting techniques.
- Such techniques will enable us to count the following, without having to list all of the item:
  1. the number of ways,
  2. the number of samples, or
  3. the number of outcomes

- **Example 1:** Enrollment in the course Principles of probability consists of: 28 statistics majors, of whom 10 are **males**, and 53 math majors, of whom 4 are **males**. One of the enrolled students is selected at random. In how many results a **male** will be selected?
- **Solution:** $10 + 4 = 14.$
- **Addition Principle:** If a choice from Group I can be made in $\mathbf{n}$ ways and a choice from Group II can be made in $\mathbf{m}$ ways, then the number of choices possible from Group I or Group II is $\mathbf{n} + \mathbf{m}$
- **Example:** Say the only clean clothes you've got are 2 t-shirts and 4 pairs of jeans. How many different combinations can you choose?

- **Example 2:** Consider the previous example, where 2 students are selected at random. In how many ways can we select one math and one statistics student?
- **Solution:** $53 \star 28 = 1484$.
- **Multiplication Principle:** If a task involves two steps and the first step can be completed in $\mathbf{n}$ ways and the second step in $\mathbf{m}$ ways, then there are $\mathbf{n} \star \mathbf{m}$ ways to complete the task.
- **The generalized basic principle of counting:** If there are k steps in an operation of which first can be done in $\mathbf{n}_1$ ways, for each of these second can be done in $\mathbf{n}_2$ ways, for each of the first two the third step can be done in $\mathbf{n}_3$ ways, and so forth, then the whole operation can be done in $\mathbf{n}_1 \star \mathbf{n}_2 \star \mathbf{n}_3 \star \cdots \star \mathbf{n}_k$ ways.

Principles of Counting: Examples

- **Example 3:** A college planning committee consists of 3 freshmen, 4 sophomores, 5 juniors, and 2 seniors. A subcommittee of 4, consisting of 1 person from each class, is to be chosen. How many different subcommittees are possible?

- **Solution:** We may regard the choice of a subcommittee as the combined outcome of the four separate experiments of choosing a single representative from each of the classes. It then follows from the generalized version of the basic principle that there are $3 \times 4 \times 5 \times 2 = 120$ possible subcommittees.

- **Example 4:** How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

- **Solution:** By the generalized version of the basic principle, the answer is $26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 = 175,760,000.$

- **Example 5:** How many functions defined on n points are possible if each functional value is either 0 or 1?

- **Solution:** Let the points be $1, 2, \ldots, n$ Since $f(i)$ must be either 0 or 1 for each $i = 1, 2, \ldots, n$, it follows that there are 2^n possible functions.

Permutations

- How many different words can you make with the letters CAN?
- By direct enumeration we see that there are 6.
- A *permutation* is a linear arrangement of elements for which the order of the elements must be taken into account.
- The number of possible permutations of n distinct objects is given by: $n! = n(n-1)(n-2)\dots(2)(1)$.
- Note: The number of possible permutations of n distinct objects arranged in circle is given by $(n-1)!$.
- **Example 6:** How many different batting orders are possible for a baseball team consisting of 9 players?
- **Solution:** There are $9! = 362880$ possible batting orders.

Permutations: Examples

- **Example 7:** A class in probability theory consists of 6 men and 4 women. An examination is given, and the students are ranked according to their performance. Assume that no two students obtain the same score.
  (a) How many different rankings are possible?
  (b) If the men are ranked just among themselves and the women just among themselves, how many different rankings are possible?
- **Solution:** (a) Because each ranking corresponds to a particular ordered arrangement of the 10 people, the answer to this part is $10! = 3,628,800$.
  (b) Since there are 6! possible rankings of the men among themselves and 4! possible rankings of the women among themselves, it follows from the basic principle that there are $(6!)(4!) = (720)(24) = 17280$ possible rankings in this case.

- **Example 8:** Ms. Jones has 10 books that she is going to put on her bookshelf. Of these, 4 are mathematics books, 3 are chemistry books, 2 are history books, and 1 is a language book. Ms. Jones wants to arrange her books so that all the books dealing with the same subject are together on the shelf. How many different arrangements are possible?
- **Solution:** There are $4!\ 3!\ 2!\ 1!$ arrangements such that the mathematics books are first in line, then the chemistry books, then the history books, and then the language book. Similarly, for each possible ordering of the subjects, there are $4!\ 3!\ 2!\ 1!$ possible arrangements. Hence, as there are $4!$ possible orderings of the subjects, the desired answer is $4!\ 4!\ 3!\ 2!\ 1! = 6912$.

Permutations

- We shall now determine the number of permutations of a set of n objects when certain of the objects are indistinguishable from each other.
- **Example 9:** How many different letter arrangements can be formed from the letters PEPPER?
- ....
- In general, the same reasoning as that used in Example 4 shows that there are

$$\frac{n!}{n_1!n_2!\dots n_r!}$$

  different permutations of n objects, of which n_1 are alike, n_2 are alike, . . . , n_r are alike.
- **Corollary:** The number of permutations of n objects taken r at a time is $n!/(n-r)!$.

- **Example 10:** A chess tournament has 10 competitors, of which 4 are Russian, 3 are from the United States, 2 are from Great Britain, and 1 is from Brazil. If the tournament result lists just the nationalities of the players in the order in which they placed, how many outcomes are possible?
- **Solution:** There are
$$\frac{10!}{4!3!2!1!}$$
possible outcomes
- **Example 11:** How many different signals, each consisting of 9 flags hung in a line, can be made from a set of 4 white flags, 3 red flags, and 2 blue flags if all flags of the same color are identical?
- **Solution:** There are
$$\frac{9!}{4!3!2!}$$
different signals.

Combinations

- We are often interested in determining the number of different groups of r objects that could be formed from a total of n objects. Order is NOT important.
- For instance, how many different groups of 3 could be selected from 5 items A, B, C, D, and E?
- ....
- **Notation and Definition:** We define $\binom{n}{r}$, for $r \leq n$, by $\binom{n}{r} = \frac{n!}{(n-r)!r!}$ and say that $\binom{n}{r}$ represents the number of possible combinations of n objects taken r at a time.
- Thus, $\binom{n}{r}$ represents the number of different groups of size r that could be selected from a set of n objects when the order of selection is not considered relevant.
- **Example 12:** A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?
- **Solution:** There are $\binom{20}{3} = 1140$ possible committees.

Combinations: Examples

- **Example 13:** From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed? What if 2 of the men are feuding and refuse to serve on the committee together?
- **Solution:** $\binom{5}{2} \times \binom{7}{3} = 350$. Because of total of $\binom{2}{2} \times \binom{5}{1} = 5$ out of $\binom{7}{3} = 35$ possible groups of 3 men contain both of the feuding men, it follows that there are $35 - 5 = 30$ that do not contain both of the feuding men. Then, the possible number of committees becomes $30 \times 10 = 300$.
- **Example 14:** Consider a set of n antennas of which m are defective and $n - m$ are functional and assume that all of the defectives and all of the functionals are considered indistinguishable. How many linear orderings are there in which no two defectives are consecutive?
- **Solution:** Imagine that the $n - m$ functional antennas are lined up among themselves. Now, if no two defectives are to be consecutive, then the spaces between the functional antennas must each contain at most one defective antenna. That is, in the $n - m + 1$ possible positions between the $n - m$ functional antennas, we must select m of these in which to put the defective antennas. Hence, there are $\binom{n-m+1}{m}$ possible orderings in which there is at least one functional antenna between any two defective ones.

- The values $\binom{n}{r}$ are often referred to as *binomial coefficients* because of their prominence in the binomial theorem.
- **The Binomial Theorem:**

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

- **Example 15:** Expand $(x+y)^3$.
- **Solution:** ......
- **Example 16:** How many subsets are there of a set consisting of n elements?
- **Solution:** ......

Multinomial Coefficients

- **Notation and Definition:** If $n_1 + n_2 + \ldots + n_r = n$, we define $\binom{n}{n_1, n_2, \ldots, n_r}$ by

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \ldots n_r!}$$

Thus, $\binom{n}{n_1, n_2, \ldots, n_r}$ represents the number of possible divisions of n distinct objects into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$.

- **Example 17:** A police department in a small city consists of 10 officers. If the department policy is to have 5 of the officers patrolling the streets, 2 of the officers working full time at the station, and 3 of the officers on reserve at the station, how many different divisions of the 10 officers into the 3 groups are possible?
- **Solution:** There are $\frac{10!}{5!2!3!} = 2520$ possible divisions.
- **Example 18:** Ten children are to be divided into an A team and a B team of 5 each. The A team will play in one league and the B team in another. How many different divisions are possible?
- **Solution:** There are $\frac{10!}{5!5!} = 252$ possible divisions.

- **Theorem 1:** There are

$$\binom{n-1}{r-1} = \frac{(n-1)!}{(r-1)!(n-r)!}$$

ways to distribute n similar balls on r urns where no urn remains empty.

- **Theorem 2:** There are

$$\binom{n+r-1}{r-1} = \frac{(n+r-1)!}{(r-1)!n!}$$

ways to distribute n similar balls on r urns.

- **Example 19:** In how many ways can we distribute 10 identical white boards among 4 schools, where each schools gets at least one white board?
- **Example 20:** Previous example. In how many ways can we distribute the white boards among the 4 schools?