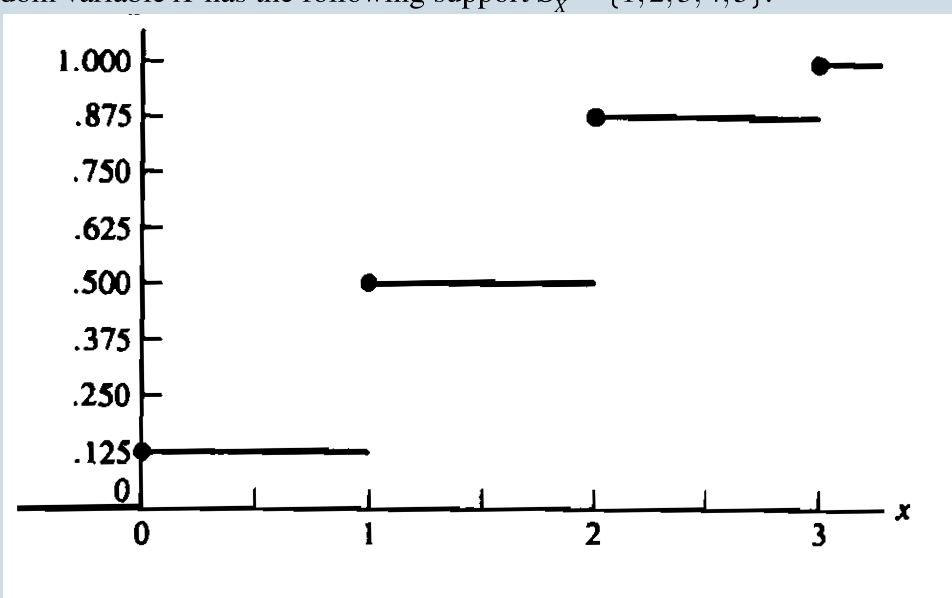
A Few Problems Aiming the Final Exam

1. If $X \sim \text{Binomial}(n = 50, \pi = 0.1)$ then
   (a) Obtain the value of $E(X)$, $\text{Var}(X)$, and $E(X^2)$.
   (b) What is the MGF of X?

2. If $X \sim \text{Geometric}(\pi = 0.2)$ then
   (a) Obtain the value of $E(X)$, $\text{Var}(X)$, and $E(X^2)$.
   (b) What is $E(4X + 10)$?
   (c) What is $\text{Var}(4X + 10)$?
   (d) What is the MGF of X?

3. If $X \sim \text{Poisson}(\lambda = 5)$, then
   (a) Obtain the value of $E(X)$, $\text{Var}(X)$, and $E(X^2)$
   (b) What is $E(3X + 50)$?
   (c) What is $\text{Var}(3X + 50)$?
   (d) What is the MGF of X?

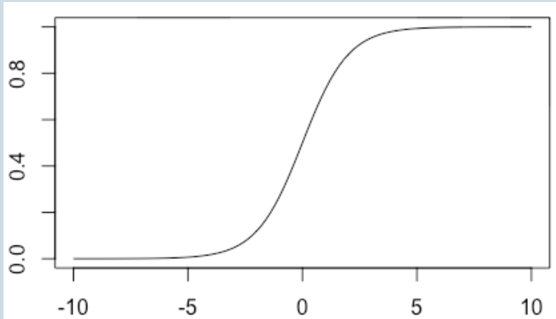
4. Suppose a random variable X has the following support $\mathbb{S}_X = \{1, 2, 3, 4, 5\}$.



   (a) What is the probability that $X = 2$?
   (b) What is the probability that $X = 1.5$
   (c) Obtain $P(1 < X \leq 3)$
   (d) Obtain $P(1 \geq X < 3)$

5. Consider the following CDF of the random variable

$$F_X(x) := \frac{1}{1+e^{-x}} \text{ for all } x \in \mathbb{R}.$$



(a) What is the probability that $X = 0$?
(b) What is the probability that $X = 2$
(c) Obtain $P(0 < X \leq 1)$
(d) Obtain $P(X < 2)$
(e) Identify the nature of the random variable (Discrete/Continuous/ Mixture of Discrete and Continuous.)

6. Example : Suppose that X is a continuous random variable whose probability density function is given by

$$f(x) := \begin{cases} C\,(4x - 2x^2) & \text{if } 0 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) For what value of C the provided function is a valid probability density function?
(b) Find $P(X > 1)$.
(c) Find $P(X \leq 1)$.
(d) Obtain *mean* (Expected value) of X.

7. Example : For a given IT technician in a support center, let X denote the percentage of time, out of a 40-hour work week, that he is directly serving customers. Suppose that X has a probability density function given by

$$f(x) := \begin{cases} 3x^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the probability that the technician will spend 20% to 70% of hisworkweek serving customers.
(b) Obtain, $F(x)$, the CDF of X.
(c) Use $F(x)$ to compute $P(0.5 < X \leq 0.8)$.
(d) Obtain *mean* (Expected value) of X.
(e) find the *median* and First Quartile (Q_1) of the distribution

8. **Example :** The lifetime in hours of a certain kind of radio tube is a random variable having a probability density function given by

$$f(x) := \begin{cases} \frac{100}{x^2} & \text{if } x > 100 \\ 0 & \text{if } x \leq 100. \end{cases}$$

(a) What is the probability a randomly selected tube in a radio set will have to be replaced within the fist 150 hours of operation?
(b) Obtain the CDF function of the distribution
(c) Obtain *median* lifetime of a randomly selected radio tube.
(d) Does the *mean* / Expected value of the distribution exist?

9. **Example :** Find $E(e^X)$ and the Moment Generating Function for the continuous random variable with probability density function $f(x) := \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$

10. **Example :** Let X denote the resistance of a randomly chosen resistor, and suppose that its pdf is given by

$$f(x) := \begin{cases} \frac{x}{18} & \text{if } 8 \leq x \leq 10 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $P(8.6 < X \leq 9.8)$.
(b) Find the median of the resistance of such resistors.
(c) Find the mean and variance of X.

11. **Example :** The length of time required by students to complete a one-hour exam is a random variable with a density function given by

$$f(x) := \begin{cases} cy^2 + y & \text{for } 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find c that makes this function a valid probability density function.
(b) Find the F(y)
(c) Find the probability that a randomly selected student will finish in less than half an hour.
(d) Find the time that 95% of the students finish before it.
(e) Given that a particular student needs at least 15 minutes to complete the exam, find the probability that she will require at least 30 minutes to finish.

12. **Example :** The time (in min) for a lab assistant to prepare the equipment for a certain experiment is believed to have a uniform distribution with a = 25 and b = 35.
   (a) Write the probability density function of X.
   (b) What is the probability that preparation time exceeds 33 min?
   (c) What is the probability that preparation time is within 2 minmutes of the **mean time**?

13. **Example :** The failure rate for a type of electric light bulb is 0.002 per hour. Under the exponential model,
   (a) Find the probability that a randomly selected light bulb will fail in less than 1000 hours.
   (b) Find the probability that a randomly selected light bulb will last 2000 hours before failing.
   (c) Find the mean and the variance of time until failure.
   (d) Find the median time until failure.
   (e) Find the time where 95% of these bulbs are expected to fail before it.

14. **Example :** A gasoline wholesale distributor has bulk storage tanks that hold fixed supplies and are filled every Monday. Of interest to the wholesaler is the proportion of this supply that is sold during the week. Over many weeks of observation, the distributor found that this proportion could be modeled by a beta distribution with $\alpha = 4$ and $\beta = 2$.

   (a) Find the probability that the wholesaler will sell at least 90% of her stock in a given week.
   (b) What is the expected percentage of sell in a randomly selected week.

15. **Example :** The times of first failure of a unit of a brand of ink jet printers are approximately normally distributed with a mean of 1,500 hours and a standard deviation of 200 hours. Use the statistical calculator.
   (a) What fraction of these printers will fail before 1,000 hours?
   (b) What is the probability that the first failure time of a selected printer will fail be between 1,300 and 1700 hours?

16. **Example :** The times of first failure of a unit of a brand of ink jet printers are approximately normally distributed with a mean of 1,500 hours and a standard deviation of 200 hours. Use the statistical calculator.
   (a) what should be the guarantee time for these printers if the manufacturer wants only 5% to fail within the guarantee period.

17. **Example :** The SAT and ACT college entrance exams are taken by thousands of students each year. The mathematics portions of each of these exams produce scores that are approximately normally distributed. In recent years, SAT mathematics exam scores have averaged 480 with standard deviation 100. The average and standard deviation for ACT mathematics scores are 18 and 6, respectively.
    (a) An engineering school sets 550 as the minimum SAT math score for new students. What percentage of students will score below 550 in a typical year?
    (b) What score should the engineering school set as a comparable standard on the ACT math test?

18. **Example :** An engineer working for a manufacturer of electronic components takes a large number of measurements of a particular dimension of components from the production line. She finds that the distribution of dimensions is normal, with a mean of 2.340 cm and a standard deviation of 0.06 cm.
    (a) What percentage of measurements will be less than 2.45 cm?
    (b) What percentage of dimensions will be between 2.25 cm and 2.45 cm?
    (c) What value of the dimension will be exceeded by 98% of the components?

19. **Example :** Of the Type A electrical resistors produced by a factory, 85% have resistance greater than 41 ohms, and 3.7% of them have resistance greater than 45 ohms. The resistances follow a normal distribution.
    (a) What percentage of these resistors have resistance greater than 44 ohms?

20. If $X \sim \text{Exponential}(\lambda = 5)$, then
    (a) Obtain the value of $E(X)$, $\text{Var}(X)$, and $E(X^2)$
    (b) What is $E(3X + 50)$?
    (c) What is $\text{Var}(3X + 50)$?
    (d) What is the MGF of X?

21. If $X \sim \text{Gamma}(\alpha = 10, \lambda = 5)$, then
    (a) Obtain the value of $E(X)$, $\text{Var}(X)$, and $E(X^2)$
    (b) What is $E(3X + 10)$?
    (c) What is $\text{Var}(3X - 5)$?
    (d) What is the MGF of X?

22. If $X \sim \text{Normal}(\mu = 100, \sigma = 5)$, then
    (a) What is $E(X^2)$
    (b) Obtain $P(X \leq 110)$
    (c) Obtain $P(95 \leq X \leq 110)$
    (d) What is the MGF of X?

23. **Example :** The joint pdf of X; Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{x+y+1}{2} & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the cumulative distribution function of (X,Y) .
(b) Find the marginal density of X.
(c) Find the marginal density of Y .
(d) Find the conditional probability of X given Y = 0.5.
(e) Use this to compute $P(X \le 0.75 \mid Y = 0.5)$

24. **Example :** Let X, Y have joint cdf

$$F_{X,Y}(x,y) = \begin{cases} x^2y^3 & \text{for } 0 < x < 1, 0 < y < 1 \\ & \text{otherwise} \end{cases}$$

(a) Find the joint density function of (X,Y) .
(b) Find the marginal density of X.
(c) Find the marginal density of Y .
(d) Find the conditional probability of X given Y = 0:5.
(e) Use this to compute $P(X \ge 0.5 \mid Y = 0.5)$

25. **Example :** The joint density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-x-2y} & \text{for } 0 < x < \infty, 0 < y < \infty \\ & \text{otherwise} \end{cases}$$

(a) Find the marginal density of X.
(b) Find the marginal density of Y .
(c) Find $P(X > 1, Y < 1)$
(d) Find $P(X < Y)$
(e) Find $P(X < 4)$
(f) Find the conditional probability of X given Y = 1.
(g) Find the marginal density of Y .
(h) Use this to compute $P(X \le 2 \mid Y = 1)$

26. **Example :** The joint pdf of X; Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{x+y+1}{2} & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Are X and Y statistically independent?

27. **Example :** Let X, Y have joint cdf

$$F_{X,Y}(x,y) = \begin{cases} x^2y^3 & \text{for } 0 < x < 1, 0 < y < 1 \\ & \text{otherwise} \end{cases}$$

(a) Are X and Y independent?

28. **Example :** The joint density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-x-2y} & \text{for } 0 < x < \infty, 0 < y < \infty \\ & \text{otherwise} \end{cases}$$

(a) Are X and Y independent?

29. **Example :** Let X, Y have joint cdf

$$f_{X,Y}(x,y) = \begin{cases} \frac{2}{7}(x+2y) & \text{for } 0 < x < 1, 1 < y < 2 \\ & \text{otherwise} \end{cases}$$

(a) Find the expected value of $\frac{X}{Y^3}$
(b) Find the expected value of XY

30. **Example :** Let X and Y have joint density

$$f_{X,Y}(x,y) = \begin{cases} 2 & \text{for } x > 0, y > 0, x + y < 1 \\ & \text{otherwise} \end{cases}$$

(a) Find the covariance of X and Y .
(b) Are the random variables X, and Y statistically independent?

31. **Example :** Given here is the joint probability function associated with data obtained in a study of automobile accidents in which a child (under age 5 years) was in the car and at least one fatality occurred. Specifically, the study focused on whether or not the child survived and what type of seatbelt (if any) he or she used. Define

$$Y_1 = \begin{cases} 0 & \text{if the child survived} \\ 1 & \text{if not,} \end{cases} \quad \text{and, } Y_2 = \begin{cases} 0 & \text{if no belt used,} \\ 1 & \text{if adult belt used} \\ 2 & \text{if car-seat belt used} \end{cases}$$

Notice that Y_1 is the number of fatalities per child and, since children's car seats usually utilize two belts, Y_2 is the number of seatbelts in use at the time of the accident

y_2 \ y_1	0	1
0	0.38	0.17
1	0.14	0.02
2	0.24	0.05

(a) What is the Marginal distribution of Y_1?
(b) What is the Marginal distribution of Y_2?
(c) Obtain Covariance between Y_1 and Y_2.

32. **Example :** Gasoline is to be stocked in a bulk tank once at the beginning of each week and then sold to individual customers. Let Y_1 denote the proportion of the capacity of the bulk tank that is available after the tank is stocked at the beginning of the week. Because of the limited supplies, Y_1 varies from week to week. Let Y_2 denote the proportion of the capacity of the bulk tank that is sold during the week. Because Y_1 and Y_2 are both proportions, both variables take on values between 0 and 1. Further, the amount sold, Y_2, cannot exceed the amount available, Y_1. Suppose that the joint density function for Y_1 and Y_2 is given by

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 3y_1 & \text{for } 0 < x < \infty, 0 \le y_2 \le y_1 \le 1 \\ & \text{otherwise} \end{cases}$$

(a) Find the probability that less than one-half of the tank will be stocked and more than one-quarter of the tank will be sold. i.e. $P(0 \le Y_1 \le 0.5, Y_2 > 0.25)$.
(b) What is the probability that less than one-half of the tank will be stocked given that more than one-quarter of the tank will be sold. $P(0 \le Y_1 \le 0.5 \mid Y_2 > 0.25)$.
(c) Find the marginal density of Y_1
(d) Find the marginal density of Y_2
(e) Find $E(Y_2)$
(f) Find the conditional density of Y_2 given $Y_1 = 0.25$.

33. Example : The management at a fast-food outlet is interested in the joint behavior of the random variables Y_1, defined as the total time between a customer's arrival at the store and departure from the service window, and Y_2, the time a customer waits in line before reaching the service window. Because Y_1 includes the time a customer waits in line, we must have $Y_1 \geq Y_2$. The relative frequency distribution of observed values of Y_1 and Y_2 can be modeled by the probability density function

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} e^{-y_1} & \text{for } 0 \leq y_2 \leq y_1 \leq \infty \\ 0 & \text{otherwise} \end{cases},$$

with time measured in minutes. Find

(a) Find Marginal Density of Y_1.
(b) Find Marginal Density of Y_2.
(c) $P(Y_1 < 2, Y_2 > 1)$.
(d) $P(Y_1 \geq 2Y_2)$.
(e) $P(Y_1 - Y_2 \geq 1)$.
(f) Find $E(Y_1 - Y_2)$, expected service time.
(g) Find Conditional Density of Y_2 given $Y_1 = 2$.
(h) Find $E(Y_2 \mid Y_1 = 2)$

34. Example : Let X and Y have joint distribution. For X and Y defined in the previous two examples, Let $Z_1 = 2X + 4Y$ and $Z_2 = X - 2Y$

(a) Find $E(Z_1)$, $E(Z_2)$
(b) Find $\text{Var}(Z_1)$, $\text{Var}(Z_2)$
(c) Find $\text{Cov}(Z_1, Z_2)$.

35. Example : Let X and Y be two statistically independent random variables with means 2, 3 respectively. , The variances of X, Y is provided as 4 and 2. Let $Z_1 = X + 2Y + 3$ and $Z_2 = 3X - Y$. Find:

(a) Find $E(Z_1)$, $E(Z_2)$
(b) Find $\text{Var}(Z_1)$, $\text{Var}(Z_2)$
(c) Find $\text{Cov}(Z_1, Z_2)$.

36. Identify the Distributions along with parameters for the following moment generating functions.

(a) $M_X(t) = (0.7 + 0.3e^t)^{10}$
(b) $M_X(t) = \exp\left(t^2\right)$

37. Identify the Distributions along with parameters for the following moment generating functions.

(a) $M_X(t) = \frac{1}{\left(1 - \frac{t}{5}\right)^{0.5}}$
(b) $M_X(t) = e^{5e^t - 5}$

38. Identify the Distributions along with parameters for the following moment generating functions.

(a) $M_X(t) = \frac{0.2e^t}{1-0.8e^t}$

(b) $M_X(t) = \frac{1}{\left(1-\frac{t}{10}\right)}$

(c) $M_X(t) = \exp\left(-2+5t^2\right)$

39. A soft-drink machine has a random amount Y_2 in supply at the beginning of a given dayand dispenses a random amount Y_1 during the day (with measurements in gallons). It is not resupplied during the day, and hence $Y_1 \leq Y_2$. It has been observed that Y_1 and Y_2 have a joint density given by

$$f(y_1, y_2) := \begin{cases} \frac{1}{2} & 0 \leq y_1 \leq y_2 \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that the marginal distribution of Y_2 is given as

$$f_{y_2}(y_2) := \begin{cases} \frac{y_2}{2} & 0 \leq y_2 \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

(b) Obtain the conditional density of Y_1 given $Y_2 = \frac{3}{2}$.

(c) $P(Y_1 \leq \frac{1}{2} \mid Y_2 = \frac{3}{2})$

(d) $P(Y_1 \leq \frac{1}{2} \mid Y_2 \leq \frac{3}{2})$