Exam Assistance Note

Let n be a **non-negative integer,** then the **factorial of n**, denoted as $n!$ is defined to be

- $0! = 1$
- $1! = 1$
- $n! = n \times (n-1) \times \ldots \times 1$ for $n \geq 2$

Let n, r be two **non-negative integes,** such that $r \leq n$, then the **n choose r**, denoted by $\binom{n}{r}$, is defined to be

$$\binom{n}{r} := \frac{n!}{(r!) \times ((n-r)!)}$$

For any real number $x \in \mathbb{R}$, the exponential series e^x (or sometimes denoted as $\exp(x)$) is defined as,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Let $x \in \mathbb{R}$ be any real number, and $n \in \mathbb{Z}_+$ be any positive integer, then

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$$

$$(1+x)^n = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n$$

Let $x \in \mathbb{R}$ be such that $|p| < 1$, then $\sum_{i=0}^{\infty} p^i = 1 + p + p^2 + p^3 + \cdots = \frac{1}{1-p}$.

Ordered, without replacement) Let r, and n be two positive integers such that $r \leq n$. An ordered arrangement of r distinct objects is called a permutation. The number of ways of ordering n distinct objects taken r at a time, denoted by the symbol ${}^nP_r$, is given as

$${}^nP_r = n(n-1)(n-2)(n-r+1) = \frac{n!}{(n-r)!}.$$

Let $n \geq r$ be two non-negative integers. The number of different ways to select (/choose) r distinct objects from a list of n distinct (non-identical) objects is given as(or ${}^nC_r$),

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

The number of ways of partitioning n distinct objects into k distinct groups containing $n_1, n_2, \ldots, n_k$ objects, respectively, where each object appears exactly in one group and $\sum_{i=1}^{k} n_i = n$, is

$$\binom{n}{n_1, n_2, \ldots, n_k} := \frac{n!}{(n_1!)(n_2!)\ldots(n_k!)}$$

Number of ways n indistinguishable/identical objects can be organized into r different (ordered) groups is

$$\frac{(n+r-1)!}{n!(r-1)!} = \binom{n+r-1}{n}.$$

Let $(\mathscr{S}, P)$ be a sample space along with the Probability measure. Let A, B be two events. Then,

- $P(\varnothing) = 0$ where $\varnothing$ denotes the Null set.
- $P(A) \leq 1$.
- If $A \subseteq B$ then $P(A) \leq P(B)$.
- $P(\overline{A}) = 1 - P(A)$, where $\overline{A}$ denotes the complementary event to A
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Let A_1, A_2, A_3 are three events. Then

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3) &= \left\{ P(A_1) + P(A_2) + P(A_3) \right\} \\ &\quad - \left\{ P(A_1 \cap A_2) + P(A_1 \cap A_3) + P(A_2 \cap A_3) \right\} + \left\{ P(A_1 \cap A_2 \cap A_3) \right\} \end{aligned}$$

De-Arrangement probability with N distinct objects: $1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots + (-1)^N \frac{1}{N!}$.

Let E, and F are two events such that $P(F) > 0$, then the conditional probability of E **given** F is defined to be,

$$P(E \mid F) := \frac{P(E \cap F)}{P(F)}.$$

Let E and F are two events, then $P(E \cap F) := P(E \mid F) \times P(F)$.

Let E and F be two events, then $P(E) = P(E \mid F)P(F) + P(E \mid \overline{F}))(\overline{F})$

Law of Total Probability (General): Let E be an event. Assuming that the collection of sets $\{F_1, F_2, \ldots, F_k\}$ forms a partition of $\mathscr{S}$, we have

$$P(E) = \sum_{j=1}^{k} P(E \mid F_j)P(F_j)$$

Bayes' Theorem

Let $F_1, F_2, \ldots, F_K$ be a set of mutually exclusive and exhaustive events (meaning that exactly one of these events must occur). Suppose now that E has occurred and we are interested in determining which one of the F_j also occurred. Then, we have the following theorem

$$P(F_i \mid E) = \frac{P(E \mid F_i)P(F_i)}{\sum_{j=1}^{K} P(E \mid F_j)P(F_j)}$$

Statistical Independence

Two events E and F are said to be statistically independent if $P(E \cap F) = P(E) \times P(F)$

Characterization of a pmf

Let $p(x)$ is **probability mass function** of a discrete random variable on the support $\mathcal{S}$, **if and only if** it satisfies the following conditions:

1. *Positivity:* $p(x) > 0$ for all $x \in \mathcal{S}$
2. *Total Probability:* $\sum_{\{x \in \mathcal{S}\}} p(x) = 1$.

"CDF" of a Discrete Random Variable

Let X be a discrete random variable on the support $\mathcal{S}[X]$ with the corresponding probability mass function

$$P(X = x) = p_X(x) \text{ for } x \in \mathcal{S}_X .$$

Then for any $a \in \mathbb{R}$, the cumulative distribution function (cdf), denoted by $F_X(\cdot)$ is the following quantity

$$F_X(a) = P(X \le a) = \sum_{\{x \le a \,:\, x \in \mathcal{S}[X]\}} p_X(x)$$

"Expected Value" or "Mean" of a Discrete Random Variable

If X is a random variable with pmf $p_X(x)$ on the support $\mathcal{S}[X]$, then the expected value (the mean) of X denoted by $E(X)$ (or μ_X) is given by

$$E(X) = \sum_{\{x \in \mathcal{S}[X]\}} x \, p_X(x),$$

assuming the above summation/series exists /well-defined.

"Variance & Standard Deviation (SD)" of a Random Variable

The variance of X, denoted by $\mathrm{Var}(X)$ is deifined as $\mathrm{Var}(X) := E(X^2) - \left(E(X)\right)^2$

$$\sigma_X = \mathrm{SD}(X) := \sqrt{\mathrm{Var}(X)}$$

"Moment Generating Function (MGF)" of a Discrete Random Variable

$$M_X(t) := E\left(e^{tX}\right) = \sum_{\{x \in \mathcal{S}_X\}} e^{tx} \, p_X(x) .$$

Distribution	Support $\mathcal{S}_X$	pmf $p_X(x)$	Mean $E(X)$	Variance $\mathrm{Var}(X)$	mgf $M_X(t)$
Binomial(n,π)	$\{0,1,\ldots,n\}$	$\binom{n}{x}\pi^x(1-\pi)^{n-x}$	$n\pi$	$n\pi(1-\pi)$	$(1-\pi+\pi e^t)^n$
Poisson(λ)	$\{0,1,2,\ldots\}$	$\frac{e^{-\lambda}\lambda^x}{x!}$	λ	λ	$e^{\lambda e^t-\lambda}$
Geometric(π)	$\{1,2,\ldots\}$	$(1-\pi)^{x-1}\pi$	$\frac{1}{\pi}$	$\frac{1-\pi}{\pi}$	$\frac{\pi e^t}{1-(1-\pi)e^t}$
Negative-Binomial(r,π)	$\{r+1,r+2,\ldots\}$	$\binom{x-1}{r-1}(1-\pi)^{x-r}\pi^r$	$\frac{r}{\pi}$	$\frac{r(1-\pi)}{\pi}$	$\left(\frac{\pi e^t}{1-(1-\pi)e^t}\right)^r$