Product moments

Definition 1 *Let X and Y ne two random variables then the rth and sth non-central product moment of X and Y is defined by $\mu'_{r,s} = E\{X^r Y^s\}$.*
The rth and sth central product moment of X and Y is defined by $\mu_{r,s} = E\{(X - \mu_X)^r (Y - \mu_Y)^s\}$.
$\mu_{1,1} = E\{(X - \mu_X)(Y - \mu_Y)\}$ is called the covariance of X and Y.

Note the
$\mu_{1,1} = Cov(X,Y) = \mu'_{1,1} - \mu_X \mu_Y = E(XY) - E(X)E(Y)$.

Theorem 1 *If X and Y are independent then $Cov(X,Y) = 0$ i.e. $E(XY) = E(X)E(Y)$. The converse is not true.*

Examples

Example 1 *Suppose X and Y have the following joint distribution*

		x		
		0	1	2
	0	1/6	1/3	1/12
y	1	2/9	1/6	0
	2	1/36	0	0

Find the covariance of X and Y.
Are X and Y independent?

Example

Example 2 *Let* X *and* Y *have joint density*

$$f(x,y) = \begin{cases} 2 & \textit{for } x > 0,\ y > 0,\ x + y < 1 \\ 0 & \textit{elsewhere.} \end{cases}$$

Find the covariance of X *and* Y*.*
Are X *and* Y *independent?*

Moments of linear combinations of

Theorem 2 *If $X_1, X_2, \ldots, X_n$ are random variables and $Y = \sum_{i=1}^{n} a_i X_i$ where a_i's are constants, then $E(Y) = \sum_{i=1}^{n} a_i E(X_i)$ and*

$$Var(Y) = \sum_{i=1}^{n} a_i^2 Var(X_i) + 2 \sum_{i=1}^{n-1} \sum_{j>i} a_i a_j Cov(X_i, X_j).$$

If the X_i's are independent then

$$Var(Y) = \sum_{i=1}^{n} a_i^2 Var(X_i).$$

Moments of linear combinations of

Theorem 3 *If $X_1, X_2, \ldots, X_n$ are random variables and $Y_1 = \sum_{i=1}^{n} a_i X_i$ and $Y_2 = \sum_{i=1}^{n} b_i X_i$ where a_i's and b_i's are constants, then*

$$Cov(Y_1, Y_2) = \sum_{i=1}^{n} a_i b_i Var(X_i) + \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j Cov(X_i, X_j).$$

If the X_i's are independent then

$$Cov(Y_1, Y_2) = \sum_{i=1}^{n} a_i b_i Var(X_i).$$

Examples

Example 3 *For X and Y defined in Examples 1 and 2, let $Z_1 = 2X + 4Y$ and $Z_2 = X - 2Y$ find:*
$E(Z_1)$, $E(Z_2)$, $Var(Z_1)$, $Var(Z_2)$ *and* $Cov(Z_1, Z_2)$.

Examples

Example 4 *Let X and Y be two independent random variables with $\mu_X = 2$, $\sigma_X = 4$, $\mu_Y = 3$ and $\sigma_Y = 2$. Let $Z_1 = X + 2Y + 3$ and $Z_2 = 3X - Y$. Find: $E(Z_1)$, $E(Z_2)$, $Var(Z_1)$, $Var(Z_2)$ and $Cov(Z_1, Z_2)$.*

Conditional Expectation

The conditional Expectation of $u(X)$ given $Y = y$ is given by

$$E(u(X)|y) = \sum_x u(x) f_{X|Y}(x|y)$$

for discrete random variables and

$$E(u(X)|y) = \int_{-\infty}^{\infty} u(x) f_{X|Y}(x|y) dx$$

for continuous random variables.

Examples

Example 5 *For X and Y defined in example 1 find $E(X|Y=1)$.*
For X and Y in Example 2 find $E(Y|X=1/2)$.

Links

Virtual Library/Conditional Expectation

Virtual Library/Product moments and covariance