STAT 320: Principles of Probability
Unit 7: Multivariate Random Variables

United Arab Emirates University

Department of Statistics

Outline

Discrete Multivariate Random Variables

Joint Probability Mass Function

We will just restrict the presentation to the bivariate case.

Joint Cumulative Distribution Function

Definition (Bivariate CDF)

Let X, Y be two discrete random variables. The joint cumulative distribution function is given by

$$F_{X,Y}(x, y) := P\left(X \leq x, Y \leq y\right).$$

If the joint probability mass function of X and Y is $p_{X,Y}(x, y) = P(X = x; Y = y)$. then

$$P\left(X \leq x, Y \leq y\right) = \sum_{\{s \leq x,\ t \leq y \text{ where } (s,t) \in \mathcal{S}_{X,Y}\}} \sum p_{X,Y}(s, t)$$

Marginal Distributions

The marginal probability mass function of X is given by

$$p_X(x) = \sum_{(x,t)\in\mathcal{S}_{XY}} p_{X,Y}(x,t)$$

The marginal probability mass function of X is given by

$$p_Y(y) = \sum_{(s,y)\in\mathcal{S}_{XY}} p_{X,Y}(s,y)$$

Example

Example : Suppose two cards are drawn at random without replacement from a deck of 4 cards numbered 1, 2, 3, 4. Let X be the number on the first card and Y be the number of the second card.

a). Find the joint probability function of X and Y .

b). Find the marginal probability function of X.

c). Find the marginal probability function of Y .

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Y \ X	1	2	3	4	Marginal of Y
1	0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$
2	$\frac{1}{12}$	0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$
3	$\frac{1}{12}$	$\frac{1}{12}$	0	$\frac{1}{12}$	$\frac{1}{4}$
4	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	0	$\frac{1}{4}$
Marginal of X	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	

Example

Example : A fair coin is flipped three times. Let X denotes the number of heads to occur in the first two flips, a and let Y denotes the number of heads to occur in the last two flips.

a). Find the joint probability function of (X, Y)

b). and the marginal probability functions of X, and Y.

c). Calculate $P(X = Y$.

:

Y \ X	0	1	2	Marginal of Y
0	$\frac{1}{8}$	$\frac{1}{8}$	0	$\frac{1}{4}$
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{2}{4}$
2	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$
Marginal of X	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	

$$P(X = Y) = \sum_{(x, y) \in \mathcal{S}_{XY}} p_{X,Y}(x, y) = p_{X,Y}(0,0) + p_{X,Y}(1,1) + p_{X,Y}(2,2) = \frac{1}{2}.$$

Example

Example : Suppose that 3 balls are randomly selected from an urn containing 3 red, 4 white, and 5 blue balls. If we let X and Y denote respectively, the number of red and white balls chosen. Find the joint probability mass function of X and Y.

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Y \ X	0	1	2	3	Marginal of Y
0	$\frac{10}{220}$	$\frac{30}{220}$	$\frac{15}{220}$	$\frac{1}{220}$	$\frac{56}{220}$
1	$\frac{40}{220}$	$\frac{60}{220}$	$\frac{12}{220}$	0	$\frac{112}{220}$
2	$\frac{30}{220}$	$\frac{18}{220}$	0	0	$\frac{48}{220}$
3	$\frac{4}{220}$	0	0	0	$\frac{4}{220}$
Marginal of X	$\frac{84}{220}$	$\frac{108}{220}$	$\frac{27}{220}$	$\frac{1}{220}$	

Outline

Continuous Multivariate Random Variables

Joint Cumulative Distribution Function

Definition (Bivariate CDF)

Let X, Y be two discrete random variables. The joint cumulative distribution function is given by

$$F_{X,Y}(x, y) := P\left(X \leq x, Y \leq y\right).$$

If the joint probability density function of X and Y is $f_{X,Y}(x, y)$. then

$$P\left(X \leq x, Y \leq y\right) = \iint_{\left\{s \leq x,\, t \leq y \text{ where } (s,t) \in \mathcal{S}_{X,Y}\right\}} f_{X,Y}(s, t)\, ds\, dt$$

$$f(x;y) = \frac{d^2 F(x,y)}{dx\,dy}$$

$$F(x;y) = \int_{\{(s \le x, t \le y) : (s, t) \in \mathcal{S}_{XY}\}} f_{X,Y}(s, t)\,ds\,dt :$$

Marginal Distributions

The marginal probability mass function of X is given by

$$f_X(x) = \int_{\{t:(x,t)\in\mathcal{S}_{XY}\}} f_{X,Y}(x,t)dt$$

The marginal probability mass function of X is given by

$$f_Y(y) = \int_{\{s:(s,y)\in\mathcal{S}_{XY}\}} f_{X,Y}(s,y)ds$$

Example

Example : The joint pdf of X; Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{x+y+1}{2} & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

a). Find the cumulative distribution function of (X, Y) .

b). Find the marginal density of X.

c). Find the marginal density of Y .

Example

Example : Let X, Y have joint cdf

$$F_{X,Y}(x,y) = \begin{cases} x^2y^3 & \text{for } 0 < x < 1, 0 < y < 1 \\ & \text{otherwise} \end{cases}$$

- a). Find the joint density function of (X, Y) .
- b). Find the marginal density of X.
- c). Find the marginal density of Y .

Example

Example : The joint density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-x-2y} & \text{for } 0 < x < \infty, 0 < y < \infty \\ & \text{otherwise} \end{cases}$$

a). Find the marginal density of X.
b). Find the marginal density of Y .
c). Find $P(X > 1, Y < 1)$
d). Find $P(X < Y)$
e). Find $P(X < 4)$

Outline

Conditional Distributions

Definition

If $p_{XY}(x, y)$ denotes the joint probability function of two discrete random variables X and Y and if $p_X(x)$ and $p_Y(y)$ denote the marginal probability function of X, (Y respectively) then the conditional probability of X given $Y = y$ is given by

$$p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

The conditional probability of Y given X = x is given by

$$p_{Y|X}(y \mid x) = \frac{p_{X,Y}(x, y)}{p_X(x)}.$$

Example

Example : Suppose two cards are drawn at random without replacement from a deck of 4 cards numbered 1, 2, 3, 4. Let X be the number on the first card and Y be the number of the second card.

a). Find the conditional probability of X given Y = 2.

b). Use this to compute $P(X \leq 2 \mid Y = 2)$.

Example

Example : Suppose that 3 balls are randomly selected from an urn containing 3 red, 4 white, and 5 blue balls. If we let X and Y denote respectively, the number of red and white balls chosen.

a). Find the conditional probability of X given Y = 2.

b). Use this to compute $P(X \leq 2 \mid Y = 2)$.

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Definition

If $f_{XY}(x, y)$ denotes the joint probability density function of two continuous random variables X and Y and if $f_X(x)$ and $f_Y(y)$ denote the marginal probability density function of X, (Y respectively) then the conditional probability density of X given $Y = y$ is given by

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

The conditional probability density of Y given X = x is given by

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x, y)}{f_X(x)}.$$

Example

Example : The joint pdf of X; Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{x+y+1}{2} & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

a). Find the conditional probability of X given Y = 0.5.

b). Use this to compute $P(X \leq 0.75 \mid Y = 0.5)$

Example

Example : Let X, Y have joint cdf

$$F_{X,Y}(x,y) = \begin{cases} x^2y^3 & \text{for } 0 < x < 1, 0 < y < 1 \\ & \text{otherwise} \end{cases}$$

a). Find the conditional probability of X given Y = 0:5.

b). Use this to compute $P(X \geq 0.5 \mid Y = 0.5)$

Example

Example : The joint density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-x-2y} & \text{for } 0 < x < \infty, 0 < y < \infty \\ & \text{otherwise} \end{cases}$$

a). Find the conditional probability of X given Y = 1.

b). Find the marginal density of Y .

c). Use this to compute $P(X \leq 2 \mid Y = 1)$

Outline

Statistically Independent Random Variables

Definition (Independent Random Variables)

Two random variables X and Y are said to be independent if $f_{X,Y}(x, y) = f_X(x)f_Y(y)$ for all x and y's,

Example

Example : The joint pdf of X; Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{x+y+1}{2} & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

a). Are X and Y independent?

Example

Example : Let X, Y have joint cdf

$$F_{X,Y}(x,y) = \begin{cases} x^2y^3 & \text{for } 0 < x < 1, 0 < y < 1 \\ & \text{otherwise} \end{cases}$$

a). Are X and Y independent?

Example

Example : The joint density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-x-2y} & \text{for } 0 < x < \infty, 0 < y < \infty \\ & \text{otherwise} \end{cases}$$

a). Are X and Y independent?

Outline

Expectation for Different Functions of Multivariate Random Variables

Let X, Y be two discrete random variables with joint probability function $p_{X,Y}(x,y)$. Then the expected value of $g(X,Y)$ is given by

$$E\left(g(X,Y)\right) = \sum_{(x,y)\in\mathcal{S}_{XY}} g(x,y)p_{X,Y}(x,y)$$

Let X, Y be two continuous random variables with joint probability density function $f_{X,Y}(x,y)$. Then the expected value of $g(X,Y)$ is given by

$$E\left(g(X,Y)\right) = \int_{(x,y)\in\mathcal{S}_{XY}} g(x,y)f_{X,Y}(x,y)dx\,dy$$

Example

Example : Let X, Y have joint cdf

$$f_{X,Y}(x,y) = \begin{cases} \frac{2}{7}(x+2y) & \text{for } 0 < x < 1, 1 < y < 2 \\ & \text{otherwise} \end{cases}$$

a). Find the expected value of $\frac{X}{Y^3}$

b). Find the expected value of XY

Outline

Reminder: Mean and Variance of a Random Variable

Mean: Let X be a random variable, then $E(X)$ denoted by μ_X is called the **mean** of the random variable.

Variance: Let X be a random variable, then $E\left(X-\mu_X\right)^2$ denoted by $\mathrm{Var}(X)$ is called the **Variance** of the random variable. Note that, the alternative formula for variance is:

$$\mathrm{Var}(X) := E(X^2) - (E(X))^2 .$$

Covariance

Definition (Covariance)

Let X, and Y be two random variables with a joint distribution. Then

$$\text{Cov}(X, Y) = E\left((X - \mu_X)(Y - \mu_Y)\right),$$

where μ_X and μ_Y denotes the mean of the random variables X, and Y respectively.

An Alternative Formulation for the covariance is the following:

Cov(X,Y)= E(XY)- E(X)E(Y)

Statistically Independent Random Variables and Covariance

Theorem

If X, and Y are two **statistically independent** *random variables, then*

$$Cov(X, Y) = 0$$

. However, the converse of the result is not true in general.

Example

Example : Suppose X and Y have the following joint distribution :

Y \ X	0	1	2
0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$
1	$\frac{2}{9}$	$\frac{1}{6}$	0
2	$\frac{1}{36}$	0	0

1. Find the covariance of X and Y .
2. Show that X, and Y are not statistically independent?

Example

Example : Let X and Y have joint density

$$f_{X,Y}(x,y) = \begin{cases} 2 & \text{for } x > 0, y > 0, x + y < 1 \\ & \text{otherwise} \end{cases}$$

a). Find the covariance of X and Y .

b). Are the random variables X, and Y statistically independent?

Expected Value of Linear Combination

Let $X_1, X_2, \ldots X_n$ are random variables and $Y = a_0 + \sum_{i=1}^{n} a_i X_i$, where $a_i's$ are constants then

1. $E(Y) = a_0 + \sum_{i=1}^{n} a_i E(X_i)$
2. $\text{Var}(Y) = \sum_{i=1}^{n} a_i^2 \text{Var}(X_i) + 2\sum\sum_{1 \le i < j \le n} a_i a_j \text{Cov}(X_i, X_j)$

If $X_1, X_2, \ldots X_n$ are mutually statistically independent then,

$$\text{Var}(Y) = \sum_{i=1}^{n} a_i^2 \text{Var}(X_i)$$

Let $X_1, X_2, \ldots X_n$are random variables and $Y_1 = a_0 + \sum_{i=1}^{n} a_i X_i$, and $Y_2 = b_0 + \sum_{i=1}^{n} b_i X_i$, where $a_i's$ abd $b_i's$ are constants then

$\mathrm{Cov}(Y_1, Y_2) = \sum_{i=1}^{n} a_i b_i \mathrm{Var}(X_i) + 2\sum\sum_{1 \le i < j \le n} a_i b_j \mathrm{Cov}(X_i, X_j)$

If $X_1, X_2, \ldots X_n$are mutually statistically independent then,

$$\mathrm{Cov}(Y) = \sum_{i=1}^{n} a_i b_j \mathrm{Var}(X_i)$$

Example

Example : Let X and Y have joint distribution. For X and Y defined in the previous two examples, Let $Z_1 = 2X + 4Y$ and $Z_2 = X - 2Y$

a). Find $E(Z_1)$, $E(Z_2)$

b). Find Var(Z_1), Var(Z_2)

c). Find Cov(Z_1, Z_2).

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Example

Example : Let X and Y be two independent random variables with means 2, 3 respectively. , The variances of X, Y is provided as 4 and 2. Let $Z_1 = X + 2Y + 3$ and $Z_2 = 3X - Y$. Find: $E(Z_1)$, $E(Z_2)$, Var(Z_1), Var(Z_2) and Cov(Z_1,Z_2).

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Questions?