STAT 320: Principles of Probability
Unit 3: Introduction to Probability

United Arab Emirates University

Department of Statistics

Outline

Sample Space & Events

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Definition (Random Experiment)

A process of observation whose outcome is not known in advance with certainty.

Experiment : Single throw of a 6-sided die.

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An outcome is defined as any possible result of a random experiment.

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Sample Space: $S = \{B, G\}$ where the outcome G means that the child is a girl and B that it is a boy.

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Example : Consider a context of horse race where 7 horses have participated the race. They are marked as $1, 2, \ldots, 7$.

Experiment: Recording the order of the horse numbers of 7 horses according to their completion time. The positions for the horses can be 1, 2, 3, 4, 5, 6, and 7.

Sample Space: $S =$ All 7! permutations of (1, 2, 3, 4, 5,6,7)

An outcome (2,3, 1, 6, 5, 4, 7) means, for instance, that the number 2 horse comes in first, then the number 3 horse, then the number 1 horse, and so on.

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As there is no ambiguity in this case, for brevity of notation we often/will use the following notation:

Sample Space: $S = \{HH, HT, TH, TT\}$

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Experiment: Recording the outcome after rolling a dice **two times.**

Sample Space: The sample space consists of the 36 points

S ={1,2,3,4,5,6} × {1,2,3,4,5,6}

$S = \{(i, j) : i, j = 1, 2, 3, 4, 5, 6\}$ where the outcome (i, j) is said to occur if i appears on the first through and j on the second. other die.

Question : Let E be the event that the sum of the dice equals 7.Write down the explicit description of E.

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Sample Space: The sample space consists of all non-negative real numbers; that is, $S = \{x \in \mathbb{R} : x > 0\} = \mathbb{R}_+$.

Question : Let A be the event that the transistor does not last longer than 5 hours. Write down the event A in the notation of set theory.

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Reminder: Disjoint Events and Partition

Disjoint Events & Partition

Definition (Pairwise Disjoint Events)

Two events A and B are disjoint (or mutually exclusive) if $A \cap B = \emptyset$. A collection of events $\{A_i\}_{i=1}^{n}$ are pairwise disjoint (or mutually exclusive) if $A_i \cap A_j = \emptyset$ for all $i \neq j$.

Definition (Partition)

$A_1, A_2, \ldots, A_n$ are called partition of the sample space S if $\{A_i\}_{i=1}^{n}$ is pairwise disjoint and $\bigcup_{i=1}^{n} A_i = S$.

Comment : Any set A and it's complement, $\overline{A}$, creates a partition of S.

Comment : In the above definition, we may replace n by ∞ and the definition naturally extends.

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The Notion of Probability

Basic Notion of Probability

- **Probability** refers to the chance that a particular event will occur.
- The probability of an event is the proportion of times the event is expected to occur in repeated experiments.
- If we denote by $n(E)$ the number of times in the first n repetitions of the experiment that the event E occurs, the probability of the event E is defined by

$$P(E) = \lim_{n \mapsto 1} \frac{n(E)}{n}.$$

Axiomatic Definition of Probability

Definition (Probability)

Consider an experiment whose sample space is S. For each event E of the sample space S, we assume that a number $P(E)$ is defined and satisfies the following three axioms

1. $0 \leq P(E) \leq 1$
2. $P(S) = 1$
3. For any sequence of mutually exclusive events $E_1, E_2, E_3, \ldots,$ ((that is, events for which $E_i \cap E_j = \emptyset$ fpr all $i \neq j$)

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P\left(E_i\right)$$

We refer to $P(E)$ as the probability of the event E.

Comment :

Axiom 1 states that the probability of an event E is always between 0 and 1.

Axiom 2 Probability of the entire sample space is 1.

Axiom 3 states that, for any sequence of mutually exclusive events, the probability of at least one of these events occurring is just the sum of their respective probabilities.

Outline

1. Sample Space & Events
2. The Notion of Probability
3. A Few Properties of Probability
4. Examples

A Few Properties of Probability

Let (S, P) be a sample space along with the Probability measure. Let A, B are two events. Then

- $P(\emptyset) = 0$ where $\emptyset$ denotes the Null set.
- $P(A) \leq 1$.
- $P(\overline{A}) = 1 - P(A)$.
- If $A \subset B$ then $P(A) \leq P(B)$.
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Question : : Represent the probability of the following events using $P(A)$, $P(B)$ and $P(A \cap B)$.

- $P(A - B) = P(A \cap \overline{B}) = ?$
- $P(B - A) = ?$
- If A, B are disjoint, i.e. $A \cap B = \emptyset$ then what is $P(A \cup B) = ?$

Question : : Let A, B are two events such that $P(A) = \frac{1}{3}$ and $P(\overline{B}) = \frac{1}{4}$. Can A and B be disjoint? Explain.

Examples

Example : A smoke detector system uses two devices, A and B. If smoke is present, the probability that it will be detected by device A is 0.95; by device B,0.90; and by both devices, 0.88.

1. If smoke is present, find the probability that the smoke will be detected by either device A or B or both devices.
2. Find the probability that the smoke will be undetected.

Solution: A: {device A detects smoke} B: {device B detects smoke}
$P(A \cup B) = P(A) + P(B) = P(A \cap B) = 0.95 + 0.90 - 0.88 = .97$
$P(\text{Smoke is undetected }) = 1 - P(\text{Smoke is detected by atleast one devices }) = 1 - P(A \cap B) = 1 - 0.97 = 0.03$

Inclusion Exclusion Principle

Inclusion Exclusion Principle: Specific Case $n = 3$

Let A_1, A_2, A_3 are three events. Then

$$\begin{aligned} P\left(A_1 \cup A_2 \cup A_3\right) &= \left\{ \boxed{P(A_1) + P(A_2) + P(A_3)} \right\} \\ &- \left\{ \boxed{P(A_1 \cap A_2) + P(A_1 \cap A_3) + P(A_2 \cap A_3)} \right\} \\ &+ \left\{ \boxed{P(A_1 \cap A_2 \cap A_3)} \right\} \end{aligned}$$

Inclusion Exclusion Principle

Lemma

Let $\{A_i\}_{i=1}^n$ *be a sequence of events, for all* $i = 1, 2, 3, \ldots, n$. *Then*

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n} \sum_{(i_1, i_2, \ldots, i_k) \in \mathbb{Q}_{n,k}} (-1)^{k+1} P\left(\bigcap_{m=1}^{k} A_{i_m}\right),$$

where $\mathbb{Q}_{n,k} := \{(i_1, i_2 \ldots, i_k) \in \mathbb{Z}_+^k : 1 \leq i_1 < i_2 < \ldots < i_k \leq n\}$.

$$\mathbb{Q}_{4,2} = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\},$$

$$\mathbb{Q}_{4,3} = \{(1,2,3),(1,3,4),(2,3,4)\},$$

$$\mathbb{Q}_{3,1} = ?, \mathbb{Q}_{3,2} = ?$$

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Examples

Finite Sample Spaces with Equally Likely Outcomes

- In many experiments, it is natural to assume that all outcomes in the sample space are equally likely to occur.

- That is, consider an experiment whose sample space S is a finite set, say, $S = \{1, 2, 3, \ldots, N\}$. Then it is often natural to assume that $P(\{1\}) = P(\{2\}) = \ldots = P(\{N\})$ which implies, from Axioms 2 and 3, that $P(\{i\}) = \frac{1}{N}$ for all $i = 1, \ldots N$.

- In equally-likely setup, it follows from Axiom 3 that, for any event E,

$$P(E) = \frac{\text{Number of outcomes in E}}{\text{Number of outcomes in S}}.$$

- In words, if we assume that all outcomes of an experiment are equally likely to occur, then the probability of any event E equals the proportion of outcomes in the sample space that are contained in E.

Example

Example : If two dice are rolled, what is the probability that the sum of the upturned faces will equal 7?

Solution: We shall solve this problem under the assumption that all of the 36 possible outcomes are equally likely. Since there are 6 possible outcomes namely, (1; 6); (2; 5); (3; 4), (4; 3); (5; 2), and (6; 1)that result in the sum of the dice being equal to 7, the

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Example : If 3 balls are randomly drawn from a bowl containing 6 white and 5 black balls, what is the probability that one of the balls is white and the other two black?

Solution: $n = 11 \times 10 \times 9 = 990$ and $n(E) = ?$ if $E = \{$one of the balls is white and the other two black$\}$ For the order WBB we have $6 \times 5 \times 4 = 120$ possibilities, for BWB, we have $5 \times 6 \times 4 = 120$ possibilities and for BBW we have $5 \times 4 \times 6 = 120$ possibilities. Then $P(E) = \frac{n(E)}{n} = \frac{120+120+120}{990} = \frac{4}{11}$.

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Example : A committee of 5 is to be selected from a group of 6 men and 9 women. If the selection is made randomly, what is the probability that the committee consists of 3 men and 2 women?

Solution: Because each of the $\binom{15}{5}$ possible committees is equally likely to be selected, the desired probability is

$$\frac{\binom{6}{3}\times\binom{9}{2}}{\binom{15}{5}} = 0.24$$

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Example : Suppose 5 people are to be randomly selected from a group of 20 individuals consisting of 10 married couples, and we want to determine $P(N)$, the probability that the 5 chosen are all unrelated. (That is, no two are married to each other.)

Solution: $P(N) = \frac{\binom{10}{5} \times 2^5}{\binom{20}{5}} \implies P(N) = \frac{20 \times 18 \times 16 \times 14 \times 12}{20 \times 19 \times 18 \times 17 \times 16}$

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Example : If n people are present in a room, what is the probability that no two of them celebrate their birthday on the same day of the year? How large need n be so that this probability is less than $\frac{1}{2}$?

Solution: $\frac{{}^{3}65P_n}{(365)^n} = \frac{(365)\times(364)\times\cdots\times(365-n+1)}{(365)^n}$

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Example : In the game of bridge, the entire deck of 52 cards is dealt out to 4 players. What is the probability that

a). one of the players receives all 13 spades?

b). each player receives 1 ace?

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Examples: On Sharing a Birthday

Question : : There are 15 students registered in the course STAT320. What is the probability that at least two of the students will share their Birthday? (Ignore the leap year and assume there is 365 days in a year)

Solution: $P(\{\text{At least two students have same birthday}\}) = 1 - P(\text{None of the students have same birthday}) =$

$1 - \frac{365 \times 364 \times \ldots \times 351}{365^{15}} \approx 1 - 0.7470987 = 0.25$

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Application of Inclusion Exclusion Principle

Question : : Suppose we turn over cards simultaneously from two well shuffled decks of ordinary playing cards. We say we obtain an exact match on a particular turn if the same card appears from each deck; for example, the queen of spades against the queen of spades. Let C_i denotes the event that there is an exact match at the i^{th} turn.

1. Argue that $P(C_i) = \frac{51!}{52!}$.
2. Argue that for $i \neq j$, $P(C_i \cap C_j) = \frac{50!}{52!}$.
3. Argue that for $i \neq j \neq k$, $P(C_i \cap C_j \cap C_k) = \frac{49!}{52!}$.
4. Let p_M equal the probability of at least one exact match. Show that $p_M := 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots - \frac{1}{52!}$.

Questions?