

Exam Assistance Note

Let n be a **non-negative integer**, then the **factorial of n** , denoted as $n!$ is defined to be

$$\square \quad 0! = 1 \quad \square \quad 1! = 1 \quad \square \quad n! = n \times (n-1) \times \dots \times 1 \text{ for } n \geq 2$$

Let n, r be two **non-negative integers**, such that $r \leq n$, then the **n choose r** , denoted by $\binom{n}{r}$, is defined to be

$$\binom{n}{r} := \frac{n!}{(r!) \times ((n-r)!)}$$

For any real number $x \in \mathbb{R}$, the exponential series e^x (or sometimes denoted as $\exp(x)$) is defined as,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Let $x \in \mathbb{R}$ be any real number, and $n \in \mathbb{Z}_+$ be any positive integer, then

$$(1+x)^n = \sum_{i=0}^n \binom{n}{i} x^i$$

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$$

Let $x \in \mathbb{R}$ be such that $|p| < 1$, then $\sum_{i=0}^{\infty} p^i = 1 + p + p^2 + p^3 + \dots = \frac{1}{1-p}$.

(Ordered, without replacement) Let r , and n be two positive integers such that $r \leq n$. An ordered arrangement of r distinct objects is called a permutation. The number of ways of ordering n distinct objects taken r at a time, denoted by the symbol nP_r , is given as

$${}^nP_r = n(n-1)(n-2)\dots(n-r+1) = \frac{n!}{(n-r)!}.$$

Let $n \geq r$ be two non-negative integers. The number of different ways to select (/choose) r distinct objects from a list of n distinct (non-identical) objects is given as (nC_r),

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

The number of ways of partitioning n distinct objects into k distinct groups containing n_1, n_2, \dots, n_k objects, respectively, where each object appears exactly in one group and $\sum_{i=1}^k n_i = n$, is

$$\binom{n}{n_1, n_2, \dots, n_k} := \frac{n!}{(n_1!)(n_2!) \dots (n_k!)}$$

Number of ways n indistinguishable/identical objects can be organized into r different (ordered) groups is

$$\frac{(n+r-1)!}{n!(r-1)!} = \binom{n+r-1}{n}.$$

De-Arrangement probability with N distinct objects: $1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \dots + (-1)^N \frac{1}{N!}.$

Properties of Probability

Let (\mathcal{S}, P) be a sample space along with the Probability measure. Let A, B be two events. Then,

- ☐ $P(\emptyset) = 0$ where \emptyset denotes the Null set.
- ☐ $P(A) \leq 1$.
- ☐ If $A \subseteq B$ then $P(A) \leq P(B)$.
- ☐ $P(\bar{A}) = 1 - P(A)$, where \bar{A} denotes the complementary event to A
- ☐ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Let A_1, A_2, A_3 are three events. Then

$$P(A_1 \cup A_2 \cup A_3) = \{ P(A_1) + P(A_2) + P(A_3) \} - \{ P(A_1 \cap A_2) + P(A_1 \cap A_3) + P(A_2 \cap A_3) \} + \{ P(A_1 \cap A_2 \cap A_3) \}$$

Structure of Two \times Two Table

	A	\bar{A}	Marginal of B
B	$P(A \cap B)$	$P(\bar{A} \cap B)$	$P(B)$
\bar{B}	$P(A \cap \bar{B})$	$P(\bar{A} \cap \bar{B})$	$P(\bar{B})$
Marginal of A	$P(A)$	$P(\bar{A})$	Total Probability = 1

Let E , and F are two events such that $P(F) > 0$, then the conditional probability of E given F is defined to be,

$$P(E | F) := \frac{P(E \cap F)}{P(F)}.$$

Let E and F are two events, then $P(E \cap F) := P(E | F) \times P(F)$.

Law of Total Probability

Let E and F be two events, then

$$P(E) = P(E | F)P(F) + P(E | \bar{F})P(\bar{F})$$

Law of Total Probability (General):

Let E be an event. Assuming that the collection of sets $\{F_1, F_2, \dots, F_k\}$ forms a partition of \mathcal{S} , we have

$$P(E) = \sum_{j=1}^k P(E | F_j)P(F_j)$$

Bayes' Theorem (General)

Let F_1, F_2, \dots, F_K be a set of mutually exclusive and exhaustive events (partition of the sample space \mathcal{S}). Suppose now that E be an event such that $P(E) > 0$, then

$$P(F_i | E) = \frac{P(E | F_i)P(F_i)}{\sum_{j=1}^K P(E | F_j)P(F_j)}$$

Bayes' Theorem

Let A, B are two events such that $P(A) > 0$, and $P(B) > 0$, then

$$P(B | A) = \frac{P(A | B)P(B)}{P(A | B)P(B) + P(A | \bar{B})P(\bar{B})}$$

Statistical Independence

Two events E and F are said to be statistically independent if $P(E \cap F) = P(E) \times P(F)$

Characterization of a pmf

Let $p(x)$ is **probability mass function** of a discrete random variable on the support \mathbb{S} , **if and only if** it satisfies the following conditions:

1. *Positivity*: $p(x) > 0$ for all $x \in \mathbb{S}$

2. *Total Probability*: $\sum_{\{x \in \mathbb{S}\}} p(x) = 1$.

“CDF” of a Discrete Random Variable

Let X be a discrete random variable on the support \mathbb{S}_X with the corresponding probability mass function

$$P(X = x) = p_X(x) \text{ for } x \in \mathbb{S}_X.$$

Then for any $a \in \mathbb{R}$, the cumulative distribution function (cdf), denoted by $F_X(\cdot)$ is the following quantity

$$F_X(a) = P(X \leq a) = \sum_{\{x \leq a : x \in \mathbb{S}_X\}} p_X(x)$$

“Expected Value” or “Mean” of a Discrete Random Variable

If X is a random variable with pmf $p_X(x)$ on the support \mathbb{S}_X , then the expected value (the mean) of X denoted by $E(X)$ (or μ_X) is given by

$$E(X) = \sum_{\{x \in \mathbb{S}_X\}} x p_X(x),$$

assuming the above summation/series exists /well-defined.

“Variance & Standard Deviation (SD) ” of a Random Variable

The variance of X , denoted by $\text{Var}(X)$ is defined as $\text{Var}(X) := E(X^2) - (E(X))^2$

$$\sigma_X = \text{SD}(X) := \sqrt{\text{Var}(X)}$$

“Moment Generating Function (MGF) ” of a Discrete Random Variable

$$M_X(t) := E\left(e^{tX}\right) = \sum_{\{x \in \mathbb{S}_X\}} e^{tx} p_X(x).$$

Standard Properties of a few Discrete Distributions

Distribution	Support \mathbb{S}_X	pmf $p_X(x)$	Mean $E(X)$	Variance $\text{Var}(X)$	mgf $M_X(t)$
Binomial(n, π)	$\{0, 1, \dots, n\}$	$\binom{n}{x} \pi^x (1 - \pi)^{n-x}$	$n\pi$	$n\pi(1 - \pi)$	$(1 - \pi + \pi e^t)^n$
Poisson(λ)	$\{0, 1, 2, \dots\}$	$\frac{e^{-\lambda} \lambda^x}{x!}$	λ	λ	$e^{\lambda e^t - \lambda}$
Geometric(π)	$\{1, 2, \dots\}$	$(1 - \pi)^{x-1} \pi$	$\frac{1}{\pi}$	$\frac{1 - \pi}{\pi^2}$	$\frac{\pi e^t}{1 - (1 - \pi)e^t}$
Negative-Binomial(r, π)	$\{r + 1, r + 2, \dots\}$	$\binom{x-1}{r-1} (1 - \pi)^{x-r} \pi^r$	$\frac{r}{\pi}$	$\frac{r(1 - \pi)}{\pi^2}$	$\left(\frac{\pi e^t}{1 - (1 - \pi)e^t} \right)^r$

A Standard Deck of Cards

