

STAT 320: Principles of Probability

Unit 6 Part:B

A Few Commonly Used Continuous Probability Distributions

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Reminder: Some Popular Integrals

$$\int x^n dx = \frac{x^{n+1}}{n+1} \text{ for any integer } n, n \neq -1.$$

$$\int \frac{1}{x} dx = \log(x)$$

$$\int e^{-x} dx = -e^{-x}.$$

$$\int e^{mx} dx = \frac{e^{mx}}{m} \text{ for any nonzero real number } m \in \mathbb{R}, m \neq 0.$$

* Note: We have not included the constant term that appears as a constants while writing a indefinite integral. For the majority, if not all, of the integras in this course will be a definite integrals with a lower and upper limit.

Assume $f'(x) := \frac{d}{dx} f(x)$ and $g'(x) := \frac{d}{dx} g(x)$ for the following formula

Integral By Parts: $\int f(x)g(x)dx = f(x) (\int g(x)dx) - \int \left\{ f'(x) (\int g(x)dx) \right\} dx$

Addition Rule: $\int \left\{ c_1 f(x) + c_2 g(x) \right\} dx = c_1 \int f(x)dx + c_2 \int g(x)dx$ for any constant $c_1, c_2 \in \mathbb{R}$.

Example

Outline

1 Uniform Distribution

2 Exponential Distribution

3 Gamma Distribution

4 Beta Distribution

5 Normal Distribution

Uniform Distribution

Uniform Distribution

The uniform random variable is used to model the behavior of a continuous random variable whose values are uniformly or evenly distributed over a given interval.

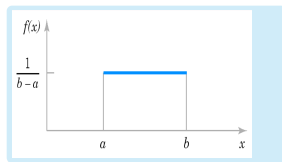
Definition (Uniform Distribution)

A random variable X is said to be uniformly distributed over the interval $[a, b]$, denoted by $X \sim \text{Uniform}(a, b)$, if its density function is

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{Otherwise} \end{cases}$$

If $X \sim \text{Uniform}(a, b)$, then:

$$E(X) = \frac{a+b}{2}, \text{ and } \text{Var}(X) = \frac{(b-a)^2}{12}$$



Let $X \sim \text{Uniform}(a, b)$ for $a < b$

Mean

$$E(X) = \frac{a+b}{2}$$

Variance

$$\text{VAR}(X) = \frac{(b-a)^2}{12}$$

MGF

$$M_X(t) = \begin{cases} \frac{e^{tb} - e^{ta}}{t(b-a)} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$$

Distribution	Support S_X	pdf $f_X(x)$	Mean $E(X)$	Variance $\text{Var}(X)$	mgf $M_X(t)$
Uniform(a, b)	$[a, b]$	$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$M_X(t) = \begin{cases} \frac{e^{tb} - e^{ta}}{t(b-a)} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$

Example :

The time (in min) for a lab assistant to prepare the equipment for a certain experiment is believed to have a uniform distribution with $a = 25$ and $b = 35$.

- a). Write the pdf of X and sketch its graph.
- b). What is the probability that preparation time exceeds 33 min?
- c). What is the probability that preparation time is within 2 minmutes of the **mean time**?

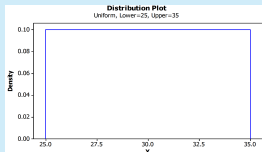
Example

Example :

The time (in min) for a lab assistant to prepare the equipment for a certain experiment is believed to have a uniform distribution with $a = 25$ and $b = 35$.

- Write the pdf of X and sketch its graph.
- What is the probability that preparation time exceeds 33 min?
- What is the probability that preparation time is within 2 min of the **mean time**?

$$f(x) := \begin{cases} \frac{1}{10} & \text{if } 25 \leq x \leq 35 \\ 0 & \text{otherwise.} \end{cases}$$



$$\begin{aligned} P(X > 33) &= \int_{33}^{35} f(x) dx \\ &= \left(\frac{x}{10} \right) \Big|_{33}^{35} \\ &= \frac{35 - 33}{10} \\ &= 0.2 \end{aligned}$$

Mean of the random variable is

$$E(X) = \frac{25+35}{2} = 30$$

$$\begin{aligned} P(E(X) - 2 < X < E(X) + 2) &= P(30 - 2 < X < 30 + 2) \\ &= P(28 < X < 32) \\ &= \int_{28}^{32} f(x) dx \\ &= \left(\frac{x}{10} \right) \Big|_{28}^{32} \\ &= \frac{32 - 28}{10} \\ &= 0.4 \end{aligned}$$

Exercise

Example :

Upon studying low bids for shipping contracts, a micro-computer company finds that intrastate contracts have low bids that are uniformly distributed between 20 and 25, in units of thousands of dollars.

- a). Find the probability that the low bid on the next intrastate shipping contract is below \$22,000.
- b). Find the probability that the low bid on the next intrastate shipping contract is in excess of \$24,000.
- c). Find the expected value and standard deviation of low bids on contracts of the type described above.

Exercise

Example :

A grocery store receives delivery each morning at a time that varies uniformly between 5:00 and 7:00 AM.

- a). Write and sketch the pdf of the delivery arrival.
- b). Find the probability that the delivery on a given morning will occur between 5:30 and 5:45 A.M.
- c). Find the probability that the time of delivery will be within one-half standard deviation of the expected time.

Outline

- 1 Uniform Distribution
- 2 Exponential Distribution
- 3 Gamma Distribution
- 4 Beta Distribution
- 5 Normal Distribution

Exponential Distribution

Exponential Distribution: Context

- 1 The exponential distribution is often used to model time (waiting time, interarrival time, hardware lifetime, failure time, etc.).
- 2 When the number of occurrences of an event follows Poisson distribution, the time between occurrences follows exponential distribution.

Exponential Distribution

Definition (Exponential Distribution)

The exponential probability distribution with parameter $\lambda > 0$ (called the rate parameter) is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

If $X \sim \text{Exponential}(\lambda)$ then $E(X) = \frac{1}{\lambda}$ and $\text{Var}(X) = \frac{1}{\lambda^2}$

The cdf of the exponential distribution is

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-\lambda x} & \text{if } x > 0 \end{cases}$$

Let $X \sim \text{Exponential}(\text{rate} = \lambda)$ for $\lambda > 0$

Mean

$$E(X) = \frac{1}{\lambda}$$

Variance

$$\text{VAR}(X) = \frac{1}{\lambda^2}$$

MGF

$$M_X(t) = \frac{\lambda}{\lambda - t} \text{ if } 0 \leq t < \lambda$$

Distribution	Support S_X	pdf $f_X(x)$	Mean $E(X)$	Variance $\text{Var}(X)$	mgf $M_X(t)$
Exponential(rate = λ)	$(0, \infty)$	$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda - t} \text{ if } 0 \leq t < \lambda$

Example

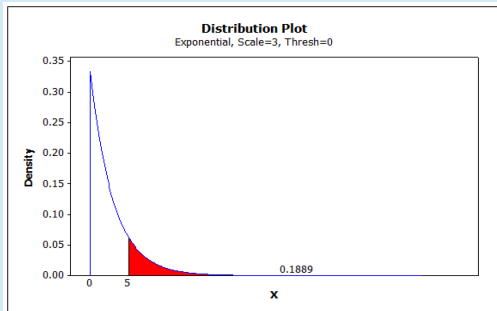
Example :

Suppose that a study of a certain computer system reveals that the response time, in seconds, has an exponential distribution with a mean of 3 seconds. What is the probability that response time exceeds 5 seconds?

Example

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Suppose that a study of a certain computer system reveals that the response time, in seconds, has an exponential distribution with a mean of 3 seconds. What is the probability that response time exceeds 5 seconds?



$$\begin{aligned}
 &P(X > 5) \\
 &= 1 - P(X \leq 5) \\
 &= 1 - F(5) \\
 &= 1 - (1 - e^{-\lambda 5}) \\
 &= e^{-5\lambda} \\
 &= e^{-5 \times \frac{1}{3}} \\
 &= 0.1889
 \end{aligned}$$

Exercise

Example :

The failure rate for a type of electric light bulb is 0.002 per hour. Under the exponential model,

- a). Find the probability that a randomly selected light bulb will fail in less than 1000 hours.
- b). Find the probability that a randomly selected light bulb will last 2000 hours before failing.
- c). Find the mean and the variance of time until failure.
- d). Find the median time until failure.
- e). Find the time where 95% of these bulbs are expected to fail before it.

Exercise

Example :

An engineer thinks that the best model for time between breakdowns of a generator is the exponential distribution with a mean of 15 days.

- a). If the generator has just broken down, what is the probability that it will break down in the next 21 days?
- b). What is the probability that the generator will operate for 30 days without a breakdown?
- c). If the generator has been operating for the last 20 days, what is the probability that it will operate for another 30 days without a breakdown?
- d). Comment on the results of parts (b) and (c).

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$$\Gamma(\alpha), \text{ for } \alpha > 0$$

The Gamma Function

Gamma Function, $\Gamma(\alpha), \alpha > 0$



$$\Gamma(\alpha) := \int_0^{\infty} x^{\alpha-1} e^{-x} dx \text{ for } \alpha > 0.$$



$$\Gamma(\alpha) > 0 \text{ for all } \alpha > 0.$$



$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$



$$\Gamma(1) = 1$$



$$\Gamma(\alpha) = (\alpha - 1) \Gamma(\alpha - 1) \text{ for any } \alpha > 1$$



$$\Gamma(n) = (n - 1)! \text{ when } n \text{ is a positive integer.}$$

Gamma Function: Example



$$\int_0^{\infty} x^7 e^{-x} dx =$$



$$\int_0^{\infty} x^{\frac{5}{2}} e^{-3x} dx =$$

Gamma Function: Example



$$\frac{\Gamma(9.1)}{\Gamma(7.1)} =$$



Let $\alpha > 0$, $\frac{\Gamma(\alpha + 3)}{\Gamma(\alpha)} =$

Gamma Distribution

Definition (Gamma Distribution)

The gamma random variable X describes waiting times between events. It can be thought of as a waiting time between Poisson distributed events, the pdf of a $\text{Gamma}(\alpha, \lambda)$ for $\alpha > 0, \lambda > 0$ is given as:

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \text{ for } 0 < x < \infty.$$

The parameter α is known as the shape parameter, while λ is called rate parameter.

Note that: The quantity λ is referred to as the rate parameter.

If $X \sim \text{Gamma}(\alpha, \lambda)$ then $E(X) = \frac{\alpha}{\lambda}$ and , $\text{Var}(X) = \frac{\alpha}{\lambda^2}$

Let $X \sim \text{Gamma}(\text{shape} = \alpha, \text{rate} = \lambda)$ for $\alpha > 0, \lambda > 0$

Mean

$$E(X) = \frac{\alpha}{\lambda}$$

Variance

$$\text{VAR}(X) = \frac{\alpha}{\lambda^2}$$

MGF

$$M_X(t) = \frac{1}{(1 - \frac{t}{\lambda})^\alpha} \text{ if } 0 \leq t < \lambda$$

Distribution	Support \mathbb{S}_X	pdf $f_X(x)$	Mean $E(X)$	Variance $\text{Var}(X)$	mgf $M_X(t)$
Gamma(α, λ) shape = α , rate = λ	$(0, \infty)$	$\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$ if $x > 0$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\frac{1}{(1 - \frac{t}{\lambda})^\alpha}$ if $0 \leq t < \lambda$

Exercise

Example :

Suppose the time spent by a randomly selected student who uses a terminal connected to a local time-sharing computer facility has a gamma distribution with mean 20 min and variance 80 min^2 .

- a). What are the values of α and λ ?
- b). What is the probability that a student uses the terminal for at most 24 min?
- c). What is the probability that a student spends between 20 and 40 min using the terminal?

Exercise

Example :

A pumping station operator observes that the demand for water at a certain hour of the day can be modeled as an exponential random variable with a mean of 100 cfs (cubic feet per second).

- a). 1 Find the probability that the demand will exceed 200 cfs on a randomly selected day.
- b). What is the maximum water producing capacity that the station should keep on line for this hour so that the demand will have a probability of only 0.01 of exceeding this production capacity?

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Beta Function, $\mathcal{B}(\alpha, \beta), \alpha > 0, \beta > 0$



$$\int_0^1 x^3(1-x)^9 dx =$$

$$\mathcal{B}(\alpha, \beta)_{\alpha > 0, \beta > 0}$$

The Beta Function

Beta Function, $\mathcal{B}(\alpha, \beta), \alpha > 0, \beta > 0$

$$\mathcal{B}(\alpha, \beta) := \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx \quad \text{for } \alpha > 0, \beta > 0.$$

$\mathcal{B}(\alpha, \beta)$ is often calculated using the following equation:

$$\mathcal{B}(\alpha, \beta) := \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Beta Function, $\mathcal{B}(\alpha, \beta), \alpha > 0, \beta > 0$



$$\int_0^1 x^3(1-x)^9 dx =$$



$$\int_0^1 x^{30}(1-x)^{1.2} dx =$$

Beta Distribution

Beta Distribution

The beta random variable X represents the proportion or probability outcomes. For example, the beta distribution might be used to find how likely it is that the preferred candidate for mayor will receive 70% of the vote.

Definition (Beta Distribution)

Probability Density Function of the $\text{Beta}(\alpha, \beta)$, $\alpha > 0, \beta > 0$ is given as

$$f(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} \quad \text{for } 0 \leq x \leq 1.,$$

where $\Gamma(\alpha)$ is defined by $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$.

If $X \sim \text{Beta}(\alpha, \beta)$ then $E(X) = \frac{\alpha}{\alpha+\beta}$ and , $\text{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Let $X \sim \text{Beta}(\alpha, \beta)$ for $\alpha > 0, \beta > 0$.

Mean

$$E(X) = \frac{\alpha}{\alpha + \beta}$$

Variance

$$\text{VAR}(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

MGF

$$M_X(t) = \text{A Complicated Series}$$

Distribution	Support \mathbb{S}_X	pdf $f_X(x)$	Mean $E(X)$	Variance $\text{Var}(X)$	mgf $M_X(t)$
$\text{Beta}(\alpha, \beta)$ shape1 = α , shape2 = β	(0, 1)	$\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} x^{\beta-1}$ if $0 < x < 1$	$\frac{\alpha}{\alpha + \beta}$	$\frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$	---

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Normal Distribution

Questions?