STAT 320: Principles of Probability Unit 1 (Part-B): Review of a Few Mathemetical Functions

United Arab Emirates University

Department of Statistics

Outline

- A Few Mathemetical Function and Notation
- A Few Common Derivatives
- Exponential Series e^x or $(\exp(x))$
- Geometric Series
- Binomial Series $(1+x)^n$, $(a+b)^n$
- A Few Common Integrals
- A Few Import Mathemetical Functions

A Few Mathemetical Function and Notation

Absolute Value

Absolute value of a real number is the magnitude of the real number ignoring its sign. Formally, we have the following definition.

Definition (Absolute Value)

Let $x \in \mathbb{R}$ be any real number, then the **absolute value of** $x \in \mathbb{R}$ (denotes as |x|) is defined to be

$$\label{eq:continuous} \left| \begin{array}{ccc} \boldsymbol{x} & \text{if } \boldsymbol{x} \geq 0, \\ -\boldsymbol{x} & \text{if } \boldsymbol{x} < 0, \end{array} \right.$$

- |5| =
- |-7.6| =
- |0| =
- |1005.7| =
- |-200| =

Definition (Indicator Function)

Let A be a set. The **indicator function for the set** A, denoted by $\mathbb{I}_A(x)$, is defined to be

$$\mathbb{I}_{A}(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

$$\mathbb{I}_{[0.5]}(1.33) =$$

$$I_{[0,5]}(-9.12) =$$

$$\bullet \ \mathbb{I}_{_{\{HH,TT,HT\}}}(HH) =$$

•
$$\mathbb{I}_{\{HH,TT,HT\}}(HHHHH) =$$

•
$$\mathbb{I}_{\mathbb{R}_{+}}(-4.87) =$$

A Few Mathemetical Function and Notation

Definition (Factorial)

Let n be a **non-negative integer**, then the **factorial of n**, denoted as n! is defined to be

- 0! = 1
- 1! = 1,

Result:
$$n! = n \times \{(n-1)!\}$$
 for $n \ge 2$.

- 3! =
- 5! =
- 6! =

n choose r, $\binom{n}{r}$

Definition

Let n, r be two **non-negative integes**, such that $r \le n$, then the **n** choose **r**, denoted by $\binom{n}{r}$, is defined to be

$$\binom{n}{r} := \frac{n!}{(r!) \times ((n-r)!)}$$

If n, r be two **non-negative integes**, such that $r \le n$, then $\binom{n}{r} = \binom{n}{(n-r)}$

$$\bullet \ \ \, \binom{0}{0} = 1 \ , \ \, \binom{1}{0} = 1 \ , \ \, \binom{5}{3} = \frac{5!}{3! \times ((5-3)!)} = \frac{5!}{3! \times (2!)} = \frac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10.$$

- $(^{10}_{2}) =$

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$$\frac{dx^n}{dx} = nx^{n-1}$$
 for any integer n .

$$\frac{de^x}{dx} = e^x$$

 $rac{de^{mx}}{dx}=me^{mx}$ for any constant $m\in\mathbb{R}.$

$$\frac{d\log(x)}{dx} = \frac{1}{x}.$$

Assume
$$f'(x) := \frac{d f(x)}{dx}$$
 and $g'(x) := \frac{d g(x)}{dx}$ for the following formula

Product Rule:
$$\frac{d}{dx} \left\{ f(x)g(x) \right\} = f'(x) g(x) + f(x) g'(x)$$

$$\text{Addition Rule:} \ \, \frac{d}{dx} \left\{ c_1 \frac{f(x)}{f(x)} + c_2 \frac{g(x)}{g(x)} \right\} = c_1 \frac{f'(x)}{f'(x)} + c_2 \frac{g'(x)}{g'(x)} \, \text{ for any constant } c_1, c_2 \in \mathbb{R}.$$

Chain Rule:
$$\frac{d}{dx} \left\{ f\left(g(x)\right) \right\} = \frac{f'\left(g(x)\right) \times g'(x)}{f'\left(g(x)\right) \times g'(x)}$$

- $\frac{d}{dx}\left(\frac{1}{1-x}\right) =$ $\frac{d}{dx}\left((1-x)^n\right) =$

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The mathemetical constant "e"

Definition ("e")

The mathemetical constant *e* is an transcendental real number given by,

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots,$$

Approximately $e \approx 2.7183$

Exponential Series e^x or (exp(x))

Definition (Exponential Series)

For any real number $x \in \mathbb{R}$, the exponential series e^x (or sometimes denoted as $\exp(x)$) is defined as,

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^{n}}{n!} = 1 + \frac{x}{1!} + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \cdots,$$

•
$$e^{x^2} =$$

Let $\lambda > 0$, then

$$\sum_{j=0}^{\infty} \frac{e^{jt} \lambda^{j}}{j!} = 1 + \frac{e^{t} \lambda}{1!} + \frac{e^{2t} \lambda^{2}}{2!} + \frac{e^{3t} \lambda^{3}}{3!} + \cdots =?.$$

Let $\lambda > 0$, then

$$\sum_{j=1}^{\infty} \frac{\lambda^{j}}{(j-1)!} = 1 + \frac{\lambda}{0!} + \frac{\lambda^{2}}{1!} + \frac{\lambda^{3}}{2!} + \cdots =?.$$

Let $\lambda > 0$, then

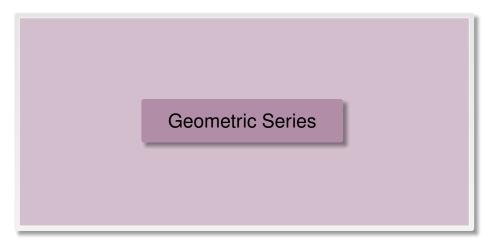
$$\sum_{m=4}^{\infty} \frac{\lambda^m}{(m-4)!} = ?.$$

Let $\alpha \in \mathbb{R}$, then

$$\sum_{m=1}^{\infty} \frac{e^{\alpha m}}{(m+2)!} = ?.$$

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Geometric Series

Let $x \in \mathbb{R}$ be such that |x| < 1, then

$$\sum_{i=0}^{\infty} x^i = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

- What is the value of $1 + 0.7 + (0.7)^2 + (0.7)^3 + \cdots =$
- ② What is the value of $1 0.7 + (0.7)^2 (0.7)^3 + \cdots =$

Let $q \in \mathbb{R}$ be such that |q| < 1, then

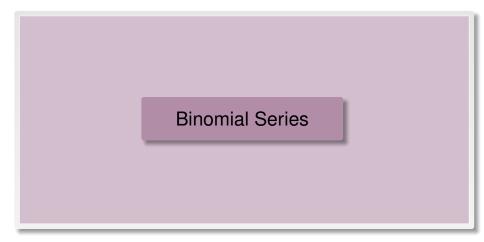
$$\sum_{j=0}^{\infty} j \, q^j = \left[q + 2q^2 + 3q^3 + 3q^4 \cdots \right] = ?.$$

Let
$$p \in \mathbb{R}$$
 be such that $|p| < 1$, then

$$\sum_{n=0}^{\infty} e^{nt} p^n = \cdots = ?.$$

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Let $x \in \mathbb{R}$ be any real number, and $n \in \mathbb{Z}_+$ be any positive integer, then

$$(1+x)^n = \sum_{i=0}^n \binom{n}{i} x^i$$

$$(1+x)^{n} = 1 + \binom{n}{1}x + \binom{n}{2}x^{2} + \dots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^{n}$$

Let $x \in \mathbb{R}$ be any real number, and $n \in \mathbb{Z}_+$ be any positive integer, then

$$(1+x)^n = \sum_{i=0}^n \binom{n}{i} x^i$$

$$(1+x)^{n} = 1 + \binom{n}{1}x + \binom{n}{2}x^{2} + \dots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^{n}$$

Let $x \in \mathbb{R}$ be any real number, and $n \in \mathbb{Z}_+$ be any positive integer, then

$$(a+b)^n = \sum_{i=0}^n \binom{n}{i} a^i b^{n-i}$$

$$(a+b)^n = b^n + \binom{n}{1}ab^{n-1} + \binom{n}{2}a^2b^{n-2} + \dots + \binom{n}{n-1}a^{n-1}b + \binom{n}{n}a^n$$

$$(1+y)^2 =$$

$$(1+z)^3 =$$

•
$$(p+q)^4 =$$

$$1 + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n-1} + \binom{n}{n} =$$

$$1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots + (-1)^{n-1} \binom{n}{n-1} + (-1)^n \binom{n}{n} =$$

$$\sum_{i=1}^{n} i \binom{n}{i} x^{i} =$$

$$\sum_{k=1}^{n} e^{kt} \binom{n}{k} x^{k} =$$

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$$\int x^n dx = \frac{x^{n+1}}{n+1} \text{ for any integer } n, n \neq -1.$$

$$\int \frac{1}{x} dx = \log(x)$$

$$\int e^{-x} dx = -e^{-x}.$$

$$\int e^{mx} dx = rac{e^{mx}}{m}$$
 for any nonzero real number $m \in \mathbb{R}, \, m
eq 0$.

* Note: We have not included the constant term that appears as a constants while writing a indefinite integral. For the majority,

if not all, of the integras in this course will be a definite integrals with a lower and upper limit.

Assume
$$f'(x) := \frac{d}{dx} \frac{f(x)}{dx}$$
 and $g'(x) := \frac{d}{dx} \frac{g(x)}{dx}$ for the following formula

Integral By Parts:
$$\int f(x)g(x)dx = f(x) \left(\int g(x)dx \right) - \int \left\{ f'(x) \left(\int g(x)dx \right) \right\} dx$$

Addition Rule:
$$\int \left\{ c_1 \frac{f(x)}{f(x)} + c_2 \frac{g(x)}{g(x)} \right\} dx = c_1 \int f(x) dx + c_2 \int g(x) dx \text{ for any constant } c_1, c_2 \in \mathbb{R}.$$

Some Non Trivial Integrals

$$\int x^{\alpha-1} e^{-x} dx \text{ for } \alpha > 0.$$

$$\int x^{\alpha-1} (1-x)^{\beta-1} dx \text{ for } \alpha > 0, \beta > 0.$$

$$\int e^{-x^2} dx$$

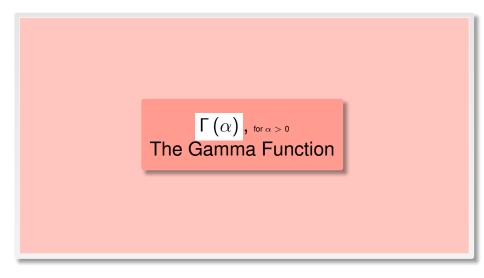
$$\int \frac{\sin(x)}{x} dx.$$

if not all, of the integras in this course will be a definite integrals with a lower and upper limit.

^{*} Note: We have not included the constant term that appears as a constants while writing a indefinite integral. For the majority,

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Gamma Function, $\Gamma(\alpha)$, $\alpha > 0$

$$\Gamma(\alpha) := \int_{0}^{\infty} x^{\alpha-1} e^{-x} dx \text{ for } \alpha > 0.$$

- $\Gamma(\alpha) > 0$ for all $\alpha > 0$.
- $\Gamma(1) = 1$

- $\Gamma(\alpha) = (\alpha 1) \Gamma(\alpha 1)$ for any $\alpha > 1$
- $\Gamma(n) = (n-1)!$ when *n* is a positive integer.

Gamma Function: Example

$$\int_{0}^{\infty} x^{7} e^{-x} dx =$$

$$\int_{0}^{\infty} x^{\frac{5}{2}} e^{-3x} dx =$$

Gamma Function: Example

$$\frac{\Gamma(9.1)}{\Gamma(7.1)} =$$

Let
$$\alpha > 0$$
, $\frac{\Gamma(\alpha + 3)}{\Gamma(\alpha)} =$

$$\mathscr{B}(\alpha,\beta)$$
 $\alpha > 0, \beta > 0$
The Beta Function

Beta Function, $\mathcal{B}(\alpha, \beta), \alpha > 0, \beta > 0$

$$\mathscr{B}(\alpha,\beta) := \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx \text{ for } \alpha > 0, \beta > 0.$$

 $\mathscr{B}(\alpha,\beta)$ is often calculated using the following equation:

$$\mathscr{B}(\alpha,\beta) := \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Beta Function, $\mathcal{B}(\alpha, \beta), \alpha > 0, \beta > 0$

$$\int\limits_{0}^{1}x^{3}(1-x)^{9}dx=$$

$$\int_{0}^{1} x^{30} (1-x)^{1.2} dx =$$

Let
$$\alpha > 0, \beta > 0$$
, then compute

$$\frac{\mathscr{B}(\alpha+1,\beta)}{\mathscr{B}(\alpha,\beta)} =$$

 $\Phi(x)$ Function, $x \in \mathbb{R}$ The Standard Normal CDF

The Standard Normal CDF (PHI function) $\Phi(x)$,

 $x \in \mathbb{R}$.

$$\Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \text{ for all } x \in \mathbb{R}.$$

•
$$\Phi(0) = \frac{1}{2}$$

$$\bullet$$
 $0 \leq \Phi(x) \leq 1$

Discussion on Various Concepts

Log (function) Equation of Line, Circles

Log (function) Gamma, Beta, Phi function Equation of Line and Regions Circles and Regions

