STAT 380: Classification: Logistic Regression

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Outlook of the unit

- **Prediction and Classification Approaches**
  - Classification Techniques
    - Logistic regression
    - Discriminant analysis
  - Evaluating Performance of a Classification Technique
  - Tree-based methods: Decision trees
    - Classification trees
    - Regression trees

Classification Technique: Logistic Regression

A Few Examples of classification

Banking: determining whether a transaction is fraudulent.

E-commerce: forecasting whether a particular order will be paid for or which customer will buy a specific product.

National security: identifying whether a certain behavior indicates a possible thread.

Tourism: determining the rating a hotel should be awarded.

Medicine: diagnosing whether a particular disease is present.

Assumptions

- Extracting knowledge from massive data sets assumes that they contain non-random, useful information.
- Classification algorithms, especially predictions, do not have further assumptions about how the data was generated.
- Most classical statistical methods are not valid if their **assumptions** are not fulfilled.

This does not mean that big data is always good data.

Notations

n: number of observations in a sample

p: number of predictor variables (features)

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{np} \end{pmatrix}$$:matrix of predictor variables

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$ vector of response (dependent / target) variable

Data : $\{(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)\}$, where x_i is a vector of length p, represents the observed data from which a statistical model is built.

Classification Rule

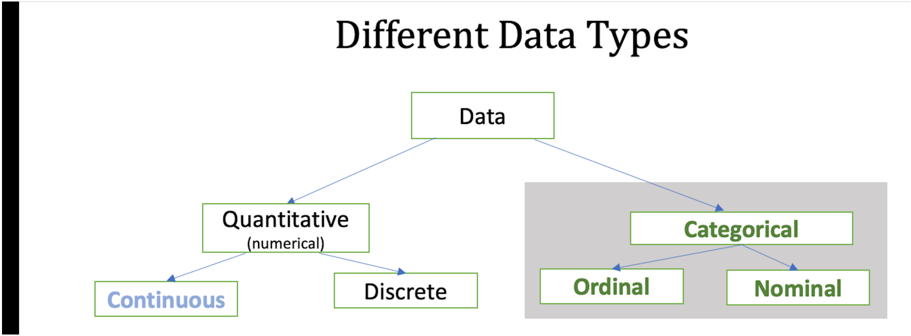
Classification employs a set of p inputs to explain a value of an outcome variable y.

- x is an object in an object space $\mathbb{X}$
- $y = y(x)$ is the response in a response space $\mathbb{Y}$
- Algorithm $a(x)$ is a function from $\mathbb{X}$ to $\mathbb{Y}$

In classification, $\mathbb{Y}$ is a set of discrete categories, $\{C_1, C_2, ..., C_K\}$, exclusively and exhaustively defining possible states of an element.
$\mathbb{Y} = \{C_1, C_2\}$ binary classification

$\mathbb{Y} = \{C_1, C_2, ..., C_K\}$ multiclass classification

Reminder (Types of Data)



Bernouli Distribution: A probability distribution for the Binary random Variables

$Y \sim$ Bernouli(π)

Support of the random variable: $Y \in \{0, 1\}$
Probabilities: $P(Y = 1) = \pi$ and $P(Y = 0) = 1 - \pi$

$E(Y) = \pi$ and $Var(Y) = \pi(1 - \pi)$.

Terminology: Often, in terms of notation we say:

"$Y = 1$" $\equiv$ "*Success*" / "*Survived*" / "*Accepted*" / "*Win*"

"$Y = 0$" $\equiv$ "*Failure*" / "*NotSurvived*" / "*NotAccepted*" / "*Loss*"

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.

Definition: Odds of “Success"

Odds: The odds of success are defined as the ratio of the probability of success (say, π) over the probability of failure $(1-\pi)$.

$$\text{Odds(“Success")} = \frac{\pi}{1-\pi}$$

Example: Odds

Example: In a Game of rolling a fair dice twice. A player wins if both throws result in same number.

$Prob(Win) = \pi = \frac{6}{36} = \frac{1}{6}$, $Prob(Loss) = 1 - \pi = 1 - \frac{1}{6}$

$$\text{Odds("Win")} = \frac{Prob(Win)}{Prob(Loss)} = \frac{\pi}{1-\pi} = \frac{\frac{1}{6}}{\frac{5}{6}} = \frac{1}{5}$$

Example: Odds

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Example: Odds

Example: Probability of 15 years “survival" of a patient after a critical medical procedure is “0.9”

$Prob(Survival) = \pi = 0.9$, $1 - \pi = 1 - 0.9 = 0.1$

$$\text{Odds(“Survival")} = \frac{Prob(Win)}{Prob(Loss)} = \frac{\pi}{1-\pi} = \frac{0.9}{0.1} = 9$$

Odds of 15 year survival of the patient after the medical procedure is 9 to 1

Example: Odds

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Odds of 15 year survival of the patient after the medical procedure is 9 to 1

Question: Odds

Question: In a game of tossing a fair coin Three times. A player wins if more number of Heads than that of the Tail. What is the odds of winning the game?

Probability of Win:

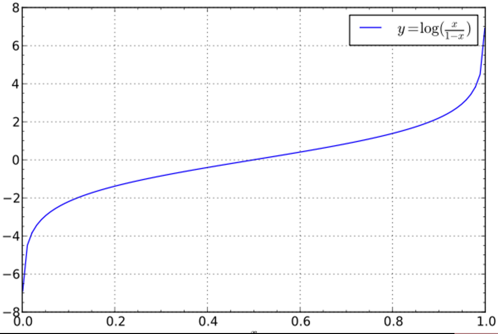
Odds of Win:

Definition: Logit Function

Logit: Logit is the natural log of an odds; That is, the logit of a number π between 0 and 1 is given by the formula:

$$\text{Logit}(\pi) = \log\left(\text{Odds}(\pi)\right) = \log\left(\frac{\pi}{1-\pi}\right)$$

The range of Logit(π) is $-\infty$ to ∞ when the range of π is between 0 and 1.

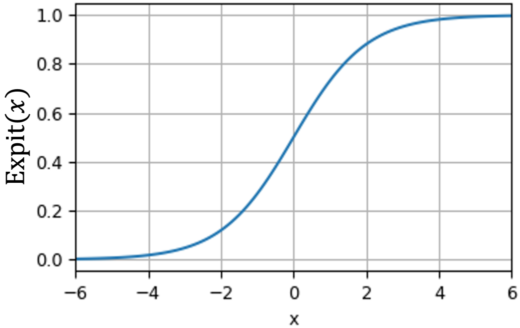


Definition: Expit Function

Expit: Expit of a number x between $-\infty$ and ∞ is given by:

$$\text{Expit}(x) = \frac{e^x}{1+e^x}$$

The range of Expit(x) is 0 to 1 when the range of x is between $-\infty$ and ∞.



$$\frac{e^V}{1+e^V}$$

$$\log\left(\frac{U}{1-U}\right)$$

$$\frac{W}{1-W}$$

$$\frac{e^{\beta_0+\beta_1 x}}{1+e^{\beta_0+\beta_1 x}}$$

$$\frac{e^V}{1+e^V}$$

$$\log(\frac{U}{1-U})$$

$$\frac{W}{1-W}$$

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Logit(0.6) $=$?

Odds(0.3) $=$?

Expit(0) $=$?

Expit(2) $=$?

'Expit' and 'Logit' are Inverse Function to each other

$$\text{Expit}(x) = \pi \implies x = \text{Logit}(\pi)$$

$$\text{Logit}(\pi) = x \implies \pi = \text{Expit}(x)$$

$$\text{Expit}(x) = 0.7 \implies x = \text{Logit}(0.7) = \log\left(\frac{0.7}{1-0.7}\right) = \log(\frac{0.7}{0.3})) = \log(2.333) = 0.368$$

$$\text{Logit}(\pi) = 2 \implies \pi = \text{Expit}(2) = \frac{e^2}{1+e^2} = 0.88$$

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$$\text{Logit}(\pi) = 2 \implies \pi = \text{Expit}(2) = \frac{e^2}{1+e^2} = 0.88$$

Question: ‘Expit’ and ’Logit’

Question: $\text{Expit}(x) = 0.8 \implies x = ?$

Question: $\text{Logit}(\pi) = -1 \implies \pi = ?$

Question: 'Logit' and 'Odds'

Logit and Odds: $\text{Logit}(\pi) = x \implies \text{Odds}(\pi) = \exp(x).$

Question: $\text{Logit}(\pi) = -1 \implies \text{Odds}(x) = ??$

Question: 'Logit' and 'Odds'

Assume $\text{Logit}(\pi) = \boxed{\beta_0 + \beta_1 x}$ The compute the following:

$\pi =$

$\text{Odds}(\pi) =$

‘Expit’ and ’Logit’ are Inverse Function to each other

In a logistic regression, Logit(π) is modelled instead of a direct modelling of π.

Logistic Regression

Why There is a Need Logistic Regression ?

We have already discussed the **linear regression model and their estimation.**

Why do we need additional regression methods?

- In standard linear regression we assume that the **Response variable is continuous.**
- What if the Response variable is **categorical?** in nature?

Examples of Binary Variables

Let's imagine independent observations $y_1, \ldots, y_n$ of a variable of interest Y, which can take **only two values** (e.g. 0 and 1), e.g.:

- Workpiece is defective(1)/not defective(0)
- Banking: A customer is considered creditworthy: yes(1)/no(0)
- Health Science: Therapy is successful: yes(1)/no(0).

: Is the standard linear regression **appropriate** for modeling binary response/dependent variables?

Reminder: The SLR and Response Type

- The random sample of size n is denoted as $\{\underset{\sim}{\mathbf{y}}_i, bx_i\}_{i=1}^{n}$ where the respone $\underset{\sim}{\mathbf{y}}_i \in \mathbb{R}$.

- The simple linear regression line is given by

$$Y = \alpha + \beta X + \varepsilon. \qquad \varepsilon \sim \mathrm{N}(0, \sigma^2)$$

- The simple linear regression is not applicable when $\underset{\sim}{\mathbf{y}}_i$ NOT continuous.

- Specifically, we use the logistic regression when the data is of following type: random sample of size n is denoted as $\{\underset{\sim}{\mathbf{y}}_i, \underset{\sim}{\mathbf{x}}_i\}_{i=1}^{n}$ where $\underset{\sim}{\mathbf{y}}_i \in \{0, 1\}$ are responses and $\underset{\sim}{\mathbf{x}}_i$ pertains to numerical or continuous covariates.

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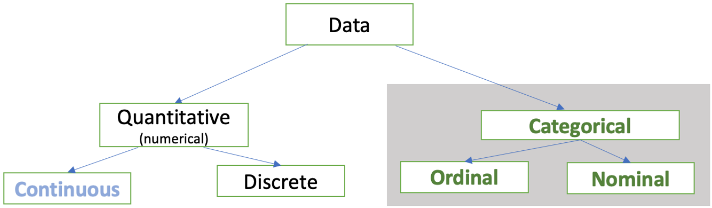
Different Data Types



- For standard Linear regression: Response variable is Continuous.
- Methods are available for Discrete random variables. Also, the variations of the continuous random variables are also possible.

- The Logistic regression is applicable when: Response variable is Categorical.
- Multicategory logistic regression are available when more than two categories are present in the Ordinal, Nominal Response variable.

- Standard Logistic regression is commonly used to analyze binary data.
- Other Binary regression models: Probit regression model, Complementary log-log model.

Reminder: Bernouli Distribution to model the Response

$Y \sim \text{Bernouli}(\pi)$

Support of the random variable: $Y \in \{0, 1\}$
Probabilities: $P(Y = 1) = \pi$ and $P(Y = 0) = 1 - \pi$

In-stead of modeling Y directly we model the probabilities π.

In particular: In a logistic regression, Logit(π) is assumed to be a function of the covariates.

The Logistic Regression

- The random sample of size n is denoted as $\{\underset{\sim}{\mathbf{y}}_i, bx_i\}_{i=1}^n$ where $\underset{\sim}{\mathbf{y}}_i \in \{0, 1\}$ and $\underset{\sim}{\mathbf{x}}_i$ can be of any type numerical or categorical.

$\underset{\sim}{\mathbf{y}}_i \sim \text{Bernouli}(\pi_i), 0 < \pi_i < 1.$

$\pi_i = Prob(Y_i = 1)$

Model: $\boxed{\text{Logit}\,(\pi_i) = \underset{\sim}{\mathbf{x}}_i^T \underset{\sim}{\beta}}$

A Few Results & Terminology in Logistic Regression

'Logit Link' and Probability of "Success" Logistic Regression

‘Logit’ and ’Expit’

$$\text{Logit}\left(\pi_i\right) = \underset{\sim}{\mathbf{x}}_i^T \underset{\sim}{\beta}$$

$$\Longrightarrow \quad \pi_i = \text{Expit}\left(\underset{\sim}{\mathbf{x}}_i^T \underset{\sim}{\beta}\right)$$

'Logit' and 'Expit'

$$\text{Logit}\left(\pi_i\right) = \underset{\sim}{\mathbf{x}}_i^T \underset{\sim}{\beta}$$

$$\Longrightarrow \quad \pi_i = \text{Expit}\left(\underset{\sim}{\mathbf{x}}_i^T \underset{\sim}{\beta}\right)$$

Explanation of ‘Logit’ Link

$$\pi_{\boldsymbol{i}} = \text{Expit}\left(\underset{\sim}{\mathbf{x}}_i^T \underset{\sim}{\beta}\right) = \frac{\exp\left(\underset{\sim}{\mathbf{x}}_i^T \underset{\sim}{\beta}\right)}{1 + \exp\left(\underset{\sim}{\mathbf{x}}_i^T \underset{\sim}{\beta}\right)}$$

‘Logit’ and ’Odds’ in Logistic Regression

Reminder: Odds of "Success"

Odds: The odds of success are defined as the ratio of the probability of success (say, π) over the probability of failure $(1-\pi)$.

$$\text{Odds(``Success'')} = \frac{\pi}{1-\pi}$$

'Logit' and 'Oddds'

$$\text{Logit}\,(\pi_i) = \underset{\sim}{\mathbf{x}}_i^T \underset{\sim}{\beta} \implies \log\left(\frac{\pi_i}{1-(\pi_i)}\right) = \underset{\sim}{\mathbf{x}}_i^T \underset{\sim}{\beta}$$

$$\implies \log\left(\text{Odds}(\pi_i)\right) = \underset{\sim}{\mathbf{x}}_i^T \underset{\sim}{\beta} \implies \text{Odds}(\pi_i) = e^{\underset{\sim}{\mathbf{x}}_i^T \underset{\sim}{\beta}}$$

'Odds' and the 'Logit' Link

$$\boxed{\mathrm{Odds}(\pi_i)} = \exp\left(\underset{\sim}{\mathbf{x}}_i^T \underset{\sim}{\boldsymbol{\beta}}\right)$$

Interpretation of a Regression Coefficient: Intercept

Interpretation of the Intercept

Let the fitted model be

$$\text{Logit}(\pi_i) = \boxed{\beta_0} + \beta_1 x_{i,1} + \ldots + \beta_p x_{i,p}$$

$$\Longrightarrow \quad \text{Odds}(\pi_i) = \exp\left(\boxed{\beta_0} + \beta_1 x_{i,1} + \ldots + \beta_p x_{i,p}\right)$$

$$\Longrightarrow \quad \text{Odds}(\pi_i) = \exp\left(\boxed{\beta_0} + \beta_1 \times 0 + \ldots + \beta_p \times 0\right) = \exp\left(\boxed{\beta_0}\right)$$

when $x_{i,1} = 0, x_{i,2} = 0, \ldots, x_{i,p} = 0$.

e^{β_0} : Odds that the corresponding Response is 1 when all the covariate values are zero.

Interpretation of the Intercept

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when $x_{i,1} = 0, x_{i,2} = 0, \ldots, x_{i,p} = 0$.

e^{β_0} : Odds that the corresponding Response is 1 when all the covariate values are zero.

Interpretation of a Regression Coef.: A Numerical Covariate

Interpretation of a regression coefficient: numerical variable

$$\text{Logit}(\pi_i) = \beta_0 + \beta_1 x_{i,1} + \ldots + \beta_p x_{i,p}$$

Assuming: $x_{i,1}$ to be a numerical covariate.

$$\text{Odds}(\pi_i)\,|_{x_{i,1}=x} = \exp\left(\beta_0 + \beta_1 x + \beta_2 x_{i,2} \ldots + \beta_p x_{i,p}\right)$$

$$\text{Odds}(\pi_i)\,|_{x_{i,1}=x+1} = \exp\left(\beta_0 + \beta_1 (x+1) + \beta_2 x_{i,2} \ldots + \beta_p x_{i,p}\right)$$

$$\frac{\text{Odds}(\pi_i)\,|_{x_{i,1}=x+1}}{\text{Odds}(\pi_i)\,|_{x_{i,1}=x}} = \exp(\beta_1).$$

Interpretation of a regression coefficient: numerical variable

$$\text{Logit}(\pi_i) = \beta_0 + \beta_1 x_{i,1} + \ldots + \beta_p x_{i,p}$$

Assuming: $x_{i,1}$ to be a numerical covariate.

$$\text{Odds}(\pi_i) \,\Big|_{x_{i,1} = x} = \exp\left(\beta_0 + \beta_1 x + \beta_2 x_{i,2} \ldots + \beta_p x_{i,p}\right)$$

$$\text{Odds}(\pi_i) \,\Big|_{x_{i,1} = x+1} = \exp\left(\beta_0 + \beta_1 (x + 1) + \beta_2 x_{i,2} \ldots + \beta_p x_{i,p}\right)$$

$$\frac{\text{Odds}(\pi_i) \,\Big|_{x_{i,1} = x+1}}{\text{Odds}(\pi_i) \,\Big|_{x_{i,1} = x}} = \exp(\beta_1).$$

Interpretation of a regression coefficient corresponding to a numerical variable

$$\frac{\text{Odds}(\pi_i) \,|_{x_{i,1}=\, x+1}}{\text{Odds}(\pi_i) \,|_{x_{i,1}=\, x}} = \exp(\boxed{\beta_1}).$$

$$\text{Odds}(\pi_i) \,|_{x_{i,1}=\, x+1} = \exp(\boxed{\beta_1})\, \text{Odds}(\pi_i) \,|_{x_{i,1}=\, x}.$$

e^{β_1} : If all other variables remain fixed, the Odds of "Succcess ($Y_i = 1$)" is changed by the factor e^{β_1} when the $x_{1,i}$ is **increased by** 1.

Interpretation of a regression coefficient corresponding to a numerical variable

$$\frac{\text{Odds}(\pi_i)\,\big|_{x_{i,1}=\,x+1}}{\text{Odds}(\pi_i)\,\big|_{x_{i,1}=\,x}} = \exp(\beta_1).$$

$$\text{Odds}(\pi_i)\,\big|_{x_{i,1}=\,x+1} = \exp(\beta_1)\,\text{Odds}(\pi_i)\,\big|_{x_{i,1}=\,x}.$$

e^{β_1} : If all other variables remain fixed, the Odds of "Succcess ($Y_i =$ 1)" is changed by the factor e^{β_1} when the $x_{1,i}$ is **increased by** 1.

Finally: Interpretation of a regression coefficient corresponding to a numerical variable

$$\frac{\text{Odds}(\pi_i)\mid_{x_{i,1}=\ x+1}}{\text{Odds}(\pi_i)\mid_{x_{i,1}=\ x}} = \exp(\boxed{\beta_1}).$$

$$\frac{\text{Odds}(\pi_i)\mid_{x_{i,1}=\ x+1} - \text{Odds}(\pi_i)\mid_{x_{i,1}=\ x}}{\text{Odds}(\pi_i)\mid_{x_{i,1}=\ x}} \times 100\% = \left(\boxed{e^{\beta_1}} - 1\right) \times 100\%.$$

$(e^{\beta_1} - 1) \times 100\%$: If all other variables remain fixed, the Odds of "Success ($Y_i = 1$)" is increased (decreased if it is a negative number) by $(e^{\beta_1} - 1) \times 100\%$ when the variable $x_{1,i}$ is **increased by** 1.

Finally: Interpretation of a regression coefficient corresponding to a numerical variable

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$(e^{\beta_1} - 1) \times 100\%$: If all other variables remain fixed, the Odds of "Success ($Y_i = 1$)" is increased (decreased if it is a negative number) by $(e^{\beta_1} - 1) \times 100\%$ when the variable $x_{1,i}$ is **increased by** 1.

Interpretation of Regression Coefficient: A Categorical Covariate

Interpretation of a reg. coeff. : Categorical Covariate

$$\text{Logit}(\pi_i) = \beta_0 + \beta_1 x_{i,1} + \beta_2 \; x_{i,2} + \ldots + \beta_p x_{i,p}$$

Assume: $x_{i,2}$ to be a Categorical Gender variable. $x_{i,2} = 1$ for Female and $x_{i,2} = 0$ for Male.

$$\text{Odds}(\pi_i) \mid_{'Male'} = \exp\left(\beta_0 + \beta_1 x_{i,1} + \beta_2 \; 0 + \beta_2 x_{i,2} \ldots + \beta_p x_{i,p}\right)$$

$$\text{Odds}(\pi_i) \mid_{'Female'} = \exp\left(\beta_0 + \beta_1 x_{i,1} + \beta_2 (1) + \beta_2 x_{i,2} \ldots + \beta_p x_{i,p}\right)$$

$$\frac{\text{Odds}(\pi_i) \mid_{'Female'}}{\text{Odds}(\pi_i) \mid_{'Male'}} = \exp(\beta_2).$$

Interpretation of a reg. coeff. : Categorical Covariate

$$\text{Logit}\,(\pi_i) = \beta_0 + \beta_1 x_{i,1} + \beta_2\, x_{i,2} + \ldots + \beta_p x_{i,p}$$

Assume: $x_{i,2}$ to be a Categorical Gender variable. $x_{i,2} = 1$ for Female and $x_{i,2} = 0$ for Male.

$$\text{Odds}(\pi_i)\,|_{\text{'Male'}} = \exp\left(\beta_0 + \beta_1 x_{i,1} + \beta_2\, 0 + \beta_2 x_{i,2} \ldots + \beta_p x_{i,p}\right)$$

$$\text{Odds}(\pi_i)\,|_{\text{'Female'}} = \exp\left(\beta_0 + \beta_1 x_{i,1} + \beta_2\,(1) + \beta_2 x_{i,2} \ldots + \beta_p x_{i,p}\right)$$

$$\frac{\text{Odds}(\pi_i)\,|_{\text{'Female'}}}{\text{Odds}(\pi_i)\,|_{\text{'Male'}}} = \exp(\beta_2).$$

Interpretation: Regression Coefficient for a Categorical Covariate

$$\frac{\text{Odds}(\pi_i)\,\big|_{\text{'}Female\text{'}}}{\text{Odds}(\pi_i)\,\big|_{\text{'}Male\text{'}}} = \exp(\beta_2).$$

$$\text{Odds}(\pi_i)\,\big|_{\text{'}Female\text{'}} = \exp(\beta_2)\,\text{Odds}(\pi_i)\,\big|_{\text{'}Male\text{'}}.$$

e^{β_2} : Assuming all other variables remain fixed, the Odds of "Success ($Y_i = 1$)" for a datapoint corresponding to the category '*Female*' is e^{β_2} times the Odds of a data point with the category '*Male*'

Interpretation: Regression Coefficient for a Categorical Covariate

$$\frac{\text{Odds}(\pi_i)\mid_{\text{`}Female\text{'}}}{\text{Odds}(\pi_i)\mid_{\text{`}Male\text{'}}} = \exp(\beta_2).$$

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Finally: Interpretation: Regression Coefficient for a Categorical Covariate

$$\frac{\text{Odds}(\pi_i)\ |_{\text{'}Female\text{'}}}{\text{Odds}(\pi_i)\ |_{\text{'}Male\text{'}}} = \exp(\beta_2).$$

$$\frac{\text{Odds}(\pi_i)\ |_{\text{'}Female\text{'}} - \text{Odds}(\pi_i)\ |_{\text{'}Male\text{'}}}{\text{Odds}(\pi_i)\ |_{\text{'}Male\text{'}}} \times 100\% = \left(e^{\beta_1} - 1\right) \times 100\%.$$

$(e^{\beta_1} - 1) \times 100\%$: If all other variables remain fixed, the Odds/Chance of "Success ($Y_i = 1$)" for a *'Female'* is $(e^{\beta_1} - 1) \times 100\%$ **more** (**less** if it is a negative number) compared to the corresponding Odds/Chance for a *'Male'*.

Finally: Interpretation: Regression Coefficient for a Categorical Covariate

$$\frac{\text{Odds}(\pi_i)\,\big|_{\text{'}\mathit{Female}\text{'}}}{\text{Odds}(\pi_i)\,\big|_{\text{'}\mathit{Male}\text{'}}} = \exp(\beta_2).$$

$$\frac{\text{Odds}(\pi_i)\,\big|_{\text{'}\mathit{Female}\text{'}} - \text{Odds}(\pi_i)\,\big|_{\text{'}\mathit{Male}\text{'}}}{\text{Odds}(\pi_i)\,\big|_{\text{'}\mathit{Male}\text{'}}} \times 100\% = \left(e^{\beta_1} - 1\right) \times 100\%.$$

$(e^{\beta_1} - 1) \times 100\%$: If all other variables remain fixed, the Odds/Chance of "Success ($Y_i = 1$)" for a *'Female'* is $(e^{\beta_1} - 1) \times 100\%$ **more** (**less** if it is a negative number) compared to the corresponding Odds/Chance for a *'Male'*.

Interpretation of coefficients: Summary

- The direct interpretation of the $\hat{\beta}$ is highly nontrivial and often non-intuitive.
- Therefore, in Logistic regression, it is common practice to interpret the $\exp(\hat{\beta})$ which has a straightforward interpretation with Odds.

$\hat{\beta}$	$\text{Odds}(\pi_i) = \frac{P(Y_i=1)}{P(Y_i=0)}$	%change in Odds	$P(Y_i = 1)$
$\hat{\beta} > 0$	increases by $\exp(\hat{\beta})$	$(\exp(\hat{\beta}) - 1) \times 100\%$	increases
$\hat{\beta} < 0$	decreases by $\exp(\hat{\beta})$	$(\exp(\hat{\beta}) - 1) \times 100\%$	decreases
$\hat{\beta} = 0$	remains the same	0%	same

A Data Example

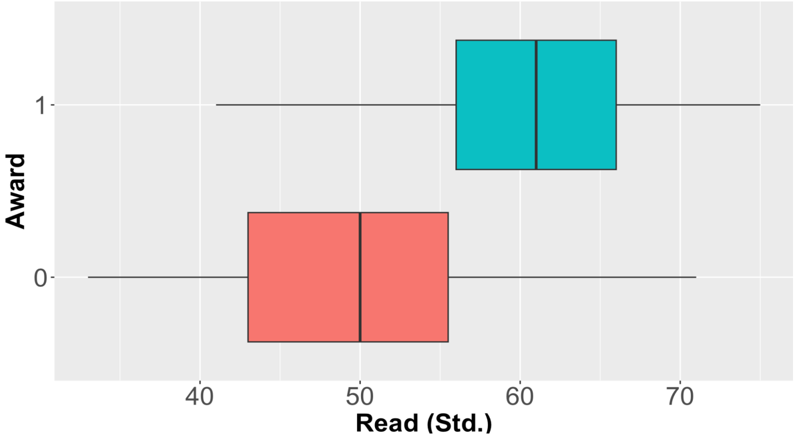
Awards Data

The Dataset have two columns. 'Award' and 'Read'

- *'Award'* : Indicates whether a student obtained the Award or not.

- *'Read'* : Hours the student have spend in additional literature study.

- We want to model the response 'Award' based on the numerical covariate 'Read'.


Award
1
0
40
50
60
70
Read (Std.)

$model.lr$<-$glm(Award \sim Read, data = performance, family = "binomial")$

$summary(model.lr)$

```
Call:
glm(formula = Award ~ Read, family = "binomial", data = performance)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-2.0332  -0.6785  -0.3506  -0.1565   2.6143

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -9.79394    1.48174  -6.610 3.85e-11 ***
Read         0.15634    0.02561   6.105 1.03e-09 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 222.71  on 199  degrees of freedom
Residual deviance: 167.07  on 198  degrees of freedom
AIC: 171.07

Number of Fisher Scoring iterations: 5
```

Reminder: Interpretation of coefficients

- The direct interpretation of the $\hat{\beta}$ is highly nontrivial and often non-intuitive.
- Therefore, in Logistic regression, it is common practice to interpret the $\exp(\hat{\beta})$ which has a straightforward interpretation with Odds.

$\hat{\beta}$	$\text{Odds}(\pi_i) = \frac{P(Y_i=1)}{P(Y_i=0)}$	%change in Odds	$P(Y_i = 1)$
$\hat{\beta} > 0$	increases by $\exp(\hat{\beta})$	$(\exp(\hat{\beta}) - 1) \times 100\%$	increases
$\hat{\beta} < 0$	decreases by $\exp(\hat{\beta})$	$(\exp(\hat{\beta}) - 1) \times 100\%$	decreases
$\hat{\beta} = 0$	remains the same	0%	same

Illustration: interpretation of results

- Model to be estimated:
  $P(AWARD = 1 \mid READ) = \frac{\exp(\beta_0 + \beta_1 * Read)}{1 + \exp(\beta_0 + \beta_1 * Read)}$
- Estimated parameter: $\hat{\beta}_1 = 0.15634$ (Regression coefficient corresponding to the variable 'Read')

$e^{\hat{\beta}_1} = \exp(0.15634) = 1.1692$: For each hour increment in the variable 'Read' (Study time aiming the award), on average, the Odds/chance for a student to win the prize in increased by a factor of 1.1692 times.

$(e^{\hat{\beta}_1 - 1}) \times 100\% = 16.92\%$: For each hour increment in the variable 'Read' (Study time aiming the award), on average, a student have 16.92 % more Odds/chance to win the prize.

Illustration: interpretation of results

- Model to be estimated:
  $P(AWARD = 1 \mid READ) = \frac{\exp(\beta_0 + \beta_1 * Read)}{1 + \exp(\beta_0 + \beta_1 * Read)}$
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$e^{\hat{\beta}_1} = \exp(0.15634) = 1.1692$: For each hour increment in the variable 'Read' (Study time aiming the award), on average, the Odds/chance for a student to win the prize in increased by a factor of 1.1692 times.

$(e^{\hat{\beta}_1 - 1}) \times 100\% = 16.92\%$: For each hour increment in the variable 'Read' (Study time aiming the award), on average, a student have 16.92 % more Odds/chance to win the prize.

Illustration: interpretation of results

- Model to be estimated:
  $P(AWARD = 1 \mid READ) = \frac{\exp(\beta_0 + \beta_1 * Read)}{1 + \exp(\beta_0 + \beta_1 * Read)}$
- Estimated parameter: $\hat{\beta}_0 = -9.79394$ (Regression coefficient corresponding to the variable 'Read')

$e^{\hat{\beta}_0} = \exp(-9.79394) < 0.00006$: On an average, the Odds/Chance for a student to win the Award is less than 0.00006 if he/she spends 0 hours of Reading.

Illustration: interpretation of results

What is the estimated probability of wining the prize for a student, if she/she spends 70 hours of Read?

$\frac{\exp(-9.8+0.16*70)}{1+\exp(-9.8+0.16*70)} = 0.8$: The estimated probability of wining the prize for a student, if she/she spends 70 hours of Read is 0.80.

THANK YOU

Outline

Example: Analysis of the O ring Failure Data

- The Space Shuttle Challenger crashed 73 seconds after liftoff on January 28th, 1986. The disaster claimed the lives of all seven astronauts on board. The details surrounding this disaster were very involved.

- If you are interested in learning more, watch this 18-minute video documentary on PBS.org.

- "The engineers who manufactured the large boosters that launched the rocket had data on the possible failures that could happen during cold temperatures. They tried to prevent the launch, but were ultimately ignored and disaster ensued."

Context: Analysis of the O ring Failure Data

- It was known that there is an association between the O-Ring seal failure and the low atmosphere temparature temperature.

- The “fail" column in the data set below records how many O-rings experienced failures during that particular launch. The “temp" column lists the outside temperature at the time of launch.

Model

The seven rows of the Data is:

	temperature	oringFail
1	53	1
2	56	1
3	57	1
4	63	0
5	66	0
6	67	0
7	67	0

Model:

$$\pi_i = Prob(\text{oringFail}_i = 1)$$

$$\text{Logit}\,(\pi_i) = \boxed{\beta_0} + \beta_1 \text{temparature}_i$$

Code

```{r }
fit_oring<-glm(oringFail~temperature, data=oring1, family="binomial")
summary(fit_oring)
```

Output

```
Call:
glm(formula = oringFail ~ temperature, family = "binomial", data = oring1)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-1.2125  -0.8253  -0.4706   0.5907   2.0512

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) 10.87535    5.70291   1.907   0.0565 .
temperature -0.17132    0.08344  -2.053   0.0400 *
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 28.975  on 23  degrees of freedom
Residual deviance: 23.030  on 22  degrees of freedom
AIC: 27.03

Number of Fisher Scoring iterations: 4
```

Model

Fitted Model:

$$\pi_i = Prob(\text{oringFail}_i = 1)$$

$$\text{Logit}\,(\pi_i) = \hat{\beta}_0 + \hat{\beta}_1 \text{temparature}_i$$

Interpret the Intercept

Interpret the regression coefficient corresponding to the temparature

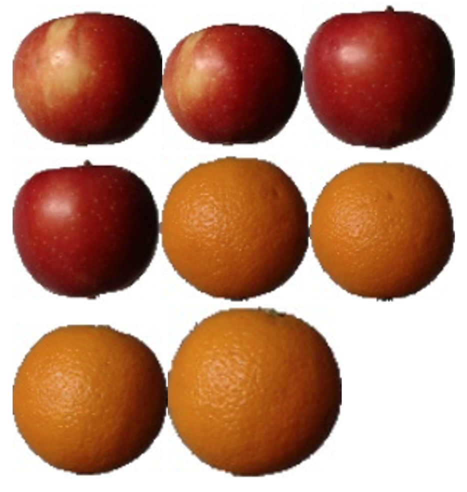
Prediction

The actual temperature at the Challenger launch was 31 F.
Predict the Probability of Failure.

Comment

It is interesting to note that all of these data were available prior to launch. However, engineers were unable to effectively analyze the data and use them to provide a convincing argument against launching Challenger to NASA managers.

Example: Can Logistic Regression separates "Apples" from the "Oranges"?



How about this?