MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Assignment 2 Report

Name: Subhalingam D Entry Number: 2018MT10770

1. Basic Probability

Question:

Suppose there are three possible languages - English (E), Hindi (H) and Tamil (T) - a person can speak in a geographical area. We have $\Omega = \{E, H, T\}$. Find all possible σ-field that can be defined on Ω. Let $\mathscr{F}_1$ be the second smallest (in terms of cardinality) σ-algebra defined on Ω. To maintain uniqueness for solving further questions, we impose $\{T\} \in \mathscr{F}_1$. Let P_1 be a probability function defined on $\mathscr{F}_1$ such that $P_1(\{T\}) = 1/3$. Complete the function definition for P_1, i.e., find $P_1(X) \forall X \in \mathscr{F}_1$, if possible.

Solution:

A σ-field $\mathscr{F}$ has three properties:

(a) $\emptyset \in \mathscr{F}$

(b) $A \in \mathscr{F} \implies A^C \in \mathscr{F}$

(c) For $A_i \in \mathscr{F}$, $\bigcup A_i \in \mathscr{F}$

An immediate corollary $\Omega \in \mathscr{F}$ follows from (a) and (b).

So we have a trivial σ-field $\{\emptyset, \Omega\}$.

Next, we try to add one singleton to the σ-field . W.l.o.g., let $\{T\}$ belong to the σ-field and this implies $\{T\}^C = \{E, H\}$ should also belong to this σ-field . Their union is $\{E, H, T\} = \Omega$ which belongs to it already. So the next smallest σ-field possible are $\{\emptyset, \{E\}, \{H, T\}, \Omega\}$, $\{\emptyset, \{H\}, \{E, T\}, \Omega\}$ and $\{\emptyset, \{T\}, \{E, H\}, \Omega\}$.

Next we try to include two singletons, w.l.o.g. let them be $\{E\}$ and $\{H\}$. But if $\{E\}$ and $\{H\}$ belong to a σ-field , then their union $\{E, H\}$ should also belong to it and hence. $\{E, H\}^C = \{T\}$ should also belong to it. Hence, two singleton is not possible in our case.

For three elements, we can observe that the σ-field should be the power set of Ω which is $\{\emptyset, \{E\}, \{H\}, \{T\}, \{E, H\}, \{E, T\}, \{H, T\}, \{E, H, T\}\}$

To summarize, the following five σ-field are possible:

(i) $\{\emptyset, \Omega\}$

(ii) $\{\emptyset, \{E\}, \{H, T\}, \Omega\}$

(iii) $\{\emptyset, \{H\}, \{E, T\}, \Omega\}$

(iv) $\{\emptyset, \{T\}, \{E, H\}, \Omega\}$

(v) $\{\emptyset, \{E\}, \{H\}, \{T\}, \{E, H\}, \{E, T\}, \{H, T\}, \Omega\}$

The second smallest σ-field with $\{T\}$ is $\mathscr{F}_1 = \{\emptyset, \{T\}, \{E, H\}, \Omega\}$. If P_1 is a probability function defined on $\mathscr{F}_1$, it should follow **Kolmogrov's axioms of probability**. So, $P_1(\Omega) = 1$ and $P_1(\emptyset) = 0$. We have $P_1(\{T\}) = 1/3$ (given). Since $\{T\}$ and $\{E, H\}$ are disjoint and $\{T\} \cup \{E, H\} = \Omega$ and from Kolmogrov's third axiom of probability, $P_1(\{T\}) + P_1(\{E, H\}) = P_1(\Omega)$ and hence $P_1(\{E, H\}) = 1 - 1/3 = 2/3$

Hence we have

$$P_1(X) = \begin{cases} 0 & X = \emptyset \\ 1/3 & X = \{T\} \\ 2/3 & X = \{E, H\} \\ 1 & X = \Omega \end{cases}$$

2. Random Variable/Function of a Random Variable

Question:

The cabs from a particular travels company arrive X minutes late than the expected time of arrival where X is a continuous rv uniformly distributed between 0 and 10 minutes. To attract customers, they add a discount of R paise (1 paise = 0.01 rupees) where $R = e^{X+1}$. Suppose a customer books a cab, find the pdf for the discount he/she can get due to late arrival and the expected discount for the same?

Solution:

The cdf of discount for late arrival is:

$$\begin{aligned}
F_R(r) &= P(R \leq r) \\
&= P(e^{X+1} \leq r) \qquad \text{(Substituting } R\text{)} \\
&= P(X \leq \ln r - 1) \\
&= \int_0^{\ln r - 1} \tfrac{1}{10} dx \\
&= \tfrac{\ln r - 1}{10} \qquad \text{for } e^1 \leq r \leq e^{11}
\end{aligned}$$

The pdf can be found by differentiating the cdf and we obtain:

$$f_R(r) = \begin{cases} \frac{1}{10r} & e^1 \leq r \leq e^{11} \\ 0 & \text{otherwise} \end{cases}$$

The expected discount could have been found without using the pdf of R as $\int_0^{10} e^{x+1} f_X(x)dx$ but since we have found the pdf of R, we can directly write

$$\begin{aligned}
E(R) &= \int_{e^1}^{e^{11}} r \tfrac{1}{10r} dr \\
&= \tfrac{e^{11} - e^1}{10} \\
&\approx 59.87 \text{ rupees}
\end{aligned}$$

3. Stochastic Processes

Question:

Akash is determined to get an internship and starts mailing to companies continuously starting from time $t = 0$. The time the mails are sent is a Poisson process with rate λ_A per hour. Let X denote time at which Akash sends his second mail. Bala who has been doing his assignments got inspired and motivated by Akash and starts sending the mails from $t = 1$. The mails sent by Bala is also a Poisson process with rate λ_B per hour and is independent of Akash. *Note that all times mentioned in the time intervals are in hours.*

(a) What is the probability that Akash sends exactly 5 mails during the time interval $[1, 2]$?

(b) You come at $t = 1$ and got to know that Akash has sent one mail so far (and start doing some probability calculations). Find the conditional expectation of X given this information?

(c) What is the pmf of total number of mails sent by both Akash and Bala together during the time interval $[0, 2]$?

(d) It is known that total 10 mails we sent by both of them in the time interval $[0, 2]$. Find the probability that exactly 6 of them are sent by Akash?

Solution:

(a) The number of mails that Akash sends in time interval $[1, 2]$ is rv with $Poisson(\lambda_A)$. So the probability that Akash sends exactly 5 mails during the time interval $[1, 2]$ is $\frac{\lambda_A^5 e^{-\lambda_A}}{5!}$.

(b) Let A denote the event of exactly one arrival during the time interval $[0, 1]$. From $t = 1$, time until next arrival is an rv T that is exponentially distributed with parameter λ_A. So,

$$E(X|A) = 1 + E(T) = 1 + \frac{1}{\lambda_A}$$

(c) Let N be the total number of mails sent by Akash and Bala in $[0, 2]$. Let us split the intervals into $[0, 1]$ (the interval in which only Akash is sending mails) and $[1, 2]$ (the interval in which both of them send mails).

If N_1 is the total number of mails sent in $[0, 1]$, then N_1 is a $Poisson(\lambda_A)$ rv.

If N_2 is the total number of mails sent in $[1, 2]$, then N_2 is a $Poisson(\lambda_A + \lambda_B)$ rv which follows from the reproductive property of (independent) Poisson distribution.

We have $N = N_1 + N_2$. So $N \sim Poisson(2\lambda_A + \lambda_B)$ because N_1 and N_2 are independent and the sum follows from reproductive property. So the pmf of N is

$$p_N(n) = \begin{cases} \frac{(2\lambda_A+\lambda_B)^n e^{-(2\lambda_A+\lambda_B)}}{n!} & n = 0, 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

(d) For Akash sent 6 mails and 10 mails totally, we can claim that Bala has sent 4 mails in the interval. Also note that Bala starts sending mails only after $t = 1$.

$$\begin{aligned}
P(\text{Akash sends 6 mails}|\text{10 mails sent totally}) &= \frac{P(\text{Akash sends 6 mails, 10 mails sent totally})}{P(\text{10 mails sent totally})} \\
&= \frac{P(\text{Akash sends 6 mails, Bala sends 4 mails})}{P(\text{10 mails sent totally})} \\
&= \frac{\frac{(2\lambda_A)^6 e^{-2\lambda_A}}{6!} \cdot \frac{\lambda_B^4 e^{-\lambda_B}}{4!}}{\frac{(2\lambda_A+\lambda_B)^{10} e^{-(2\lambda_A+\lambda_B)}}{10!}} \\
&= {}^{10}C_6 \left(\frac{2\lambda_A}{2\lambda_A+\lambda_B}\right)^6 \left(\frac{\lambda_B}{2\lambda_A+\lambda_B}\right)^4
\end{aligned}$$

4. Stochastic Processes

Question:

Consider a random process $X(t), t \in \mathbb{R}$ defined as

$$X(t) = \sum_{j=-\infty}^{+\infty} A_j f_j(t)$$

where $j \in \mathbb{Z}$; A_js are iid rv with $E(A_j) = 1$ and $var(A_j) = 1$; $f_j(t)$ is defined as

$$f_j(t) = \begin{cases} 1 & j < t \leq j+1 \\ 0 & \text{otherwise} \end{cases}$$

Compute $E(X(t))$ and $E(X(t_1)X(t_2))$ for $t, t_1, t_2 \in \mathbb{R}$.

Solution:

The given process is a continuous-time random process.

It can be observed that for $j \in \mathbb{Z}$, $f_j(t) = 0 \; \forall t \in \mathbb{R} \setminus (j, j+1]$. So for $t \in (j, j+1]$, $X(t) = A_j$.

Hence, $E(X(t)) = E(A_j) = 1 \; \forall t \in \mathbb{R}$.

Consider $t_1, t_2 \in \mathbb{R}$. There are two cases: these numbers can belong to the same interval or different interval. *(Note that $A_j s$ are iid rv is given).*

Case (i): $j < t_1, t_2 \leq j+1$, then $E(X(t_1)X(t_2)) = E((A_j)(A_j)) = E((A_j)^2) = var(A_j) + E(A_j) = 1 + 1 = 2$.

Case (2): $j < t_1 \leq j+1$ and $k < t_2 \leq k+1$ where $j, k \in \mathbb{Z}$ and $j \neq k$, then $E(X(t_1)X(t_2)) = E((A_j)(A_k)) = E(A_j)E(A_k) = 1 \times 1 = 1$.

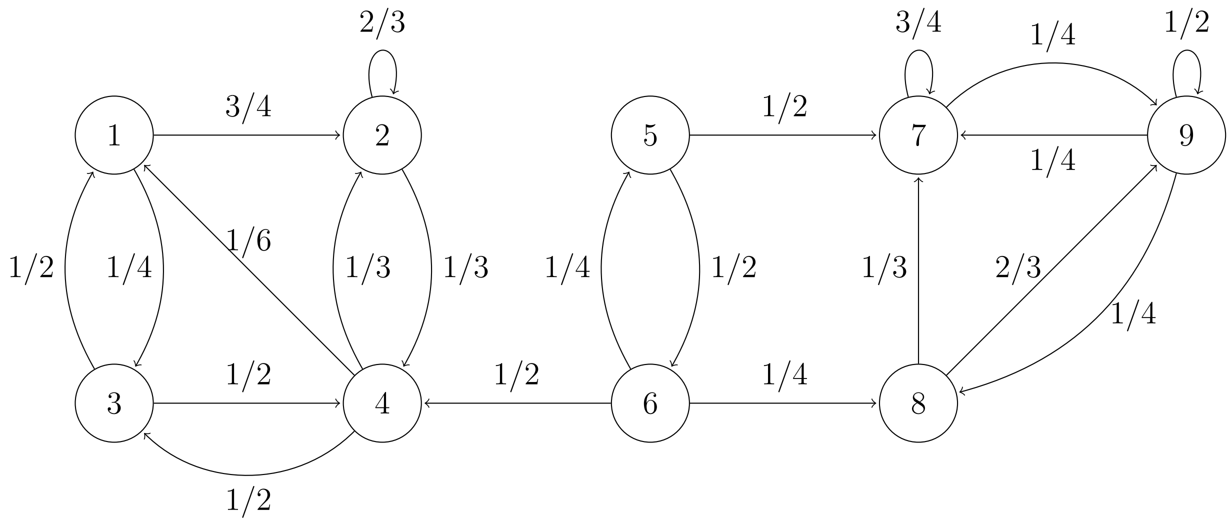
Hence, we have

$$E(X(t_1)X(t_2)) = \begin{cases} 2 & \text{if } t_1, t_2 \text{ belong to same sub-interval } (j, j+1] \text{ for } j \in \mathbb{Z} \\ 1 & \text{if } t_1, t_2 \text{ belong to different sub-intervals} \end{cases}$$

5. DTMC

Question:

Consider DTMC $\{X_n, n = 0, 1, 2, \dots\}$ having states $\{1, 2, 3, \dots, 9\}$ with the following state transition diagram given below. **Consider the initial state as** 6, i.e., $P(X_0 = 6) = 1$ and give its transition probability matrix.



(a) Find $P(X_2 = 7)$?

(b) What is the steady-state probability of being in state 5?

(c) Find all the recurrent classes. Suppose R_1 be the smallest (least number of states within it) such recurrent class. Find the probability that the chain gets absorbed in R_1, if it exists?

Solution:

The transition probability matrix P is given by

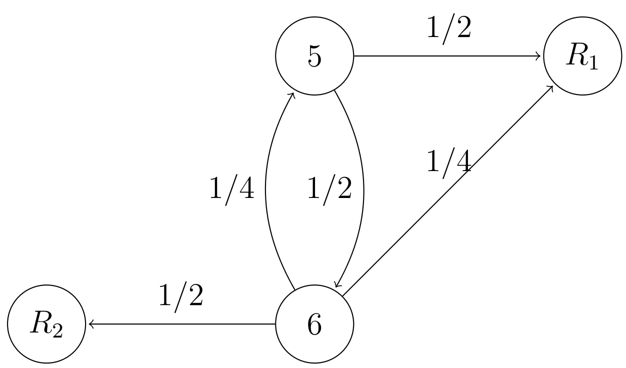
$$P = \begin{pmatrix} 0 & 3/4 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 1/6 & 1/3 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/4 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/2 \end{pmatrix}$$

(a) To reach 7 from 6 in two steps, there are two possible paths: $6 \to 8 \to 7$ and $6 \to 5 \to 7$ and hence,

$$\begin{aligned} P(X_2 = 7, X_0 = 6) &= P(X_2 = 7, X_1 = 8, X_0 = 6) + P(X_2 = 7, X_1 = 5, X_0 = 6) \\ &= \tfrac{1}{3} \cdot \tfrac{1}{4} + \tfrac{1}{2} \cdot \tfrac{1}{4} \\ &= \tfrac{5}{24} \end{aligned}$$

(b) State 5 is transient. So the steady-state probability of being in state 5 is 0.

(c) There are two recurrent classes. The smallest one among them R_1 is $\{7, 8, 9\}$. The other one is $\{1, 2, 3, 4\}$ (call it R_2). To obtain the probability that the chain gets absorbed in R_1, we will redraw the state diagram as:



Now define $a_i = P(\text{absorption by } R_1 | X_0 = i)\ \forall i \in S'$ where S' is the modified state space $\{R_1, R_2, 5, 6\}$.

We make use of this formula to obtain a_i:

$$a_i = \sum_{j \in S} a_j P_{i,j}$$

We set $a_{R_1} = 1$ and $a_{R_2} = 0$ according to requirement of the question. We obtain a_5 and a_6 by solving:

$$\begin{aligned} a_5 &= \tfrac{1}{2} \cdot a_{R_1} + \tfrac{1}{2} \cdot a_6 + 0 \cdot a_{R_2} \\ &= \tfrac{1}{2} + \tfrac{1}{2} \cdot a_6 \\ a_6 &= \tfrac{1}{4} \cdot a_{R_1} + \tfrac{1}{4} \cdot a_5 + \tfrac{1}{2} \cdot a_{R_2} \\ &= \tfrac{1}{4} + \tfrac{1}{4} \cdot a_5 \end{aligned}$$

Solving the equations we have $a_5 = \frac{5}{7}$ and $a_6 = \frac{3}{7}$

Hence, the probability that the chain gets absorbed in R_1 if initially started from state 6 is $a_6 = \frac{3}{7}$

6. DTMC

Question:

Consider a language in which a character can belong to any of the following three (exhaustive) types - consonants, vowels and white spaces. Assume that other symbols are included in one of vowels or consonants as we are not interested in them. In this language, the type of next character depends only on the present character (in short, it follows Markov Property). The white space are used to mark (differentiate between) two different words *and are the only markers to differentiate two words.* Also, there is no white space in the beginning and at the end of text and there are NO extra (two adjacent) white spaces in the corpus. Given a vowel, the probability that the next character is also vowel is 1/3 and the next character is a consonant is 1/2. Given a consonant, the probability that the next character is also a consonant is 2/5 and a vowel is 2/5. The probability that a word begins with a consonant is 2/3.

The above model can be modelled as DTMC. Complete the transition state diagram and answer the following questions:

(a) Find the expected number of **words** in a corpus with 115 million characters?

(b) Find the long-run average rate of transitions from a vowel to consonant in the same corpus with 115 million characters?

Solution:

Consider DTMC $\{X_n, n = 0, 1, 2, \dots\}$ having states $\{V, C, S\}$ where V, C, S represent vowel, consonant and white-space respectively.

From the description, we get the transition probability matrix P (in the order (C, V, S) as:

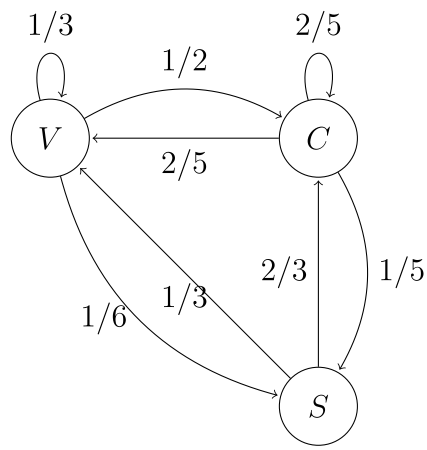
$$P = \begin{pmatrix} 1/3 & 1/2 & p_{13} \\ 2/5 & 2/5 & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

We make the following observations:

- $p_{33} = 0$ as two white-spaces can't occur be adjacent to each other *(given)*
- $p_{32} = 2/3$ as the probability that a word begins with a consonant *(which means transition from white-space to consonant)* is $2/3$ *(given)*
- $p_{13} = 1/6$, $p_{23} = 1/5$ and $p_{31} = 1/3$ as the sum of probabilities in a row should add up to one in P

Now, we complete the transition probability matrix P as

$$P = \begin{pmatrix} 1/3 & 1/2 & 1/6 \\ 2/5 & 2/5 & 1/5 \\ 1/3 & 2/3 & 0 \end{pmatrix}$$



And hence, we can draw the following state transition diagram:

Moreover, we have the initial distribution as $X_0 = \begin{pmatrix} 1/3 & 2/3 & 0 \end{pmatrix}$.

To find the stationary distribution $\pi = \begin{pmatrix} \pi_V & \pi_C & \pi_S \end{pmatrix}$, we set up:

$$\pi = \begin{pmatrix} \pi_V & \pi_C & \pi_S \end{pmatrix} = \begin{pmatrix} \pi_V & \pi_C & \pi_S \end{pmatrix} \begin{pmatrix} 1/3 & 1/2 & 1/6 \\ 2/5 & 2/5 & 1/5 \\ 1/3 & 2/3 & 0 \end{pmatrix}$$

where we get the following equations:

$$\begin{aligned} \pi_V &= \tfrac{\pi_V}{3} + \tfrac{2\pi_C}{5} + \tfrac{\pi_S}{3} \\ \pi_C &= \tfrac{\pi_V}{2} + \tfrac{2\pi_C}{5} + \tfrac{2\pi_S}{3} \\ \pi_S &= \tfrac{\pi_V}{6} + \tfrac{\pi_C}{5} + 0 \\ 1 &= \pi_V + \pi_C + \pi_S \end{aligned}$$

Upon solving, we get $\pi = \begin{pmatrix} 42/115 & 55/115 & 18/115 \end{pmatrix}$

We can make the following observations on the chain:

- **irreducible** as we can go from one state to any other state in finite number of steps
- **aperiodic** as there is a self-transition (e.g. $p_{11} > 0$)
- all states are **+ve recurrent** as it is a finite state space irreducible DTMC

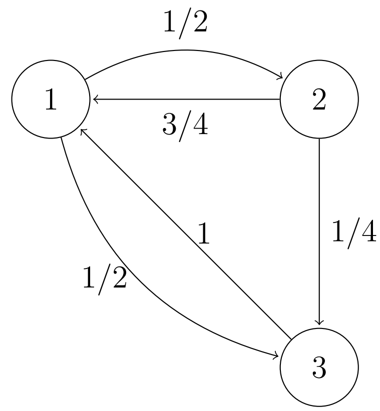
Hence, the limiting distribution exists and is equal to the stationary distribution.

(a) The average fraction of characters that are white-spaces is $\pi_S = 18/115$. So the expected number of white-spaces is $\pi_S \cdot 115,000,000 = 18,000,000$. Expected number of words is one more than expected number of white-spaces and so that is $18,000,001$.

(b) The long-run average rate of transitions from a vowel to consonant is $\pi_V \cdot p_{12} = \frac{42}{115} \cdot \frac{1}{2} = \frac{21}{115}$. So the expected number of such transitions will be $\frac{21}{115} \cdot 115,000,000 = 21,000,000$

7. CTMC

Question: Consider a CTMC $X(t)$ whose jump chain is given below:



Take $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$. Find the generator matrix for the chain and *hence* find the limiting distribution for $X(t)$.

Solution: The jump chain is irreducible and transition matrix for the jump chain is given by

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 3/4 & 0 & 1/4 \\ 1 & 0 & 0 \end{pmatrix}$$

If the generator matrix $Q = (q_{ij})$, then

$$q_{ij} = \begin{cases} \lambda_i p_{ij} & i \neq j \\ -\lambda_i & i = j \end{cases}$$

We have $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$. Hence we can obtain the generator matrix as

$$Q = \begin{pmatrix} -1 & 1/2 & 1/2 \\ 3/2 & -2 & 1/2 \\ 3 & 0 & -3 \end{pmatrix}$$

We can obtain the limiting distribution $\pi = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix}$ using the relation $\pi\mathbf{Q} = \mathbf{0}$ and having $\pi_1 + \pi_2 + \pi_3 = 1$ additionally.

We obtain the following equations:

$$\begin{aligned} -\pi_1 + \tfrac{3}{2}\pi_2 + 3\pi_3 &= 0 \\ \tfrac{1}{2}\pi_1 - 2\pi_2 &= 0 \\ \tfrac{1}{2}\pi_1 + \tfrac{1}{2}\pi_2 - 3\pi_3 &= 0 \\ \pi_1 + \pi_2 + \pi_3 &= 1 \end{aligned}$$

Solving the above equations, we get the limiting distribution for $X(t)$ as $\begin{pmatrix} \frac{24}{35} & \frac{6}{35} & \frac{5}{35} \end{pmatrix}$.

8. CTMC

Question:

The attendance for a course has to be marked through an online system. An app can be installed in the phones of the students to mark it directly from their phones and one tablet is installed in the room to assist students without a smartphone or for marking attendance in case of technical problems. One day, there is an update to the app which leads to a bug and not everyone is able to mark the attendance through their phones at once (in one attempt). So everyone decide to form a queue to mark the attendance in the tablet and standing in the queue, they also try to mark their attendance through their phones simultaneously to try their luck. Assume that all the students in the class have a smartphone (and the reason for queue is only because of technical difficulties) and for the sake of simplicity, also assume that the total number of students is not known and very large.

Students arrive to the tablet at rate of 15 per minute. On average, it takes about 30 seconds on average for a student to mark the attendance in the tablet which is exponentially distributed. *However, if the student has successfully marked the attendance through mobile app while standing in line, he/she leaves the line immediately.* Suppose that the success for marking attendance in mobile app is exponentially distributed with average 2 minutes.

Now, answer the following questions:

(a) Draw the transition diagram and set up the equations for computing π_is. *It is not necessary to solve them and further questions can be answered in terms of π_is.*

(b) What is the long-run fraction of time that the tablet is occupied?

(c) What fraction of students standing in the queue end up marking the attendance in their own phones, without ever reaching the tablet?

(d) What fraction of students come for marking in the tablet and immediately mark their attendance in the tablet (without waiting in a queue)?

Solution:

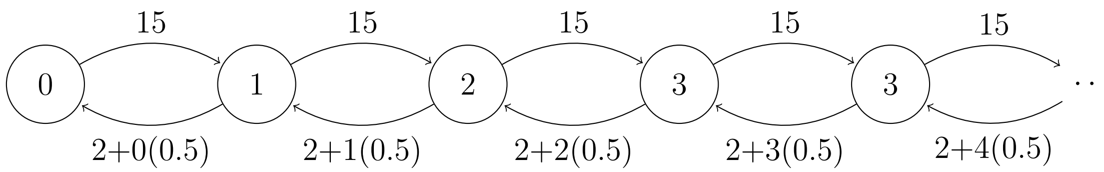
(a) Let $X(t)$ be the number of students in the queue (both waiting and using the tablet). It can be claimed that $X(t)$ is a CTMC with state space $S = \{0, 1, 2, 3, ...\}$.

The arrival time to the system is $\lambda = 15$ per minute. However, for the departure, there are two possible cases: a student marking the attendance in the tablet (obtaining service) leaves after marking in the tablet, which happens with parameter $\mu_1 = 2$ per minute (as mean time taken is 30 seconds); or, a student waiting in the queue has successfully marked the attendance through the mobile app with parameter $\mu_2 = 0.5$ per minute (as mean time taken is 2 minutes).

For $i > 1$ number of students in system, transition to next state $(i + 1)$ happens at a rate of $\lambda = 15$ per minute and the total rate for transition to previous state is $\mu = \mu_1 + (i - 1) \cdot \mu_2$ as each of $(i - 1)$ students in the queue (without service) has rate

μ_2 for leaving the queue and the person in service has rate μ_1. Hence, $\mu = 2+(i-1)0.5$ per minute.

Hence, we obtain the following state diagram:



With flow out equals flow in, we use balance equations to obtain π_is.

$$
\begin{aligned}
15\pi_0 &= 2\pi_1 \\
(15+2)\pi_1 &= 15\pi_0 + (2+0.5)\pi_2 \\
(15+(2+1\times 0.5))\pi_2 &= 15\pi_1 + (2+2\times 0.5)\pi_3 \\
(15+(2+2\times 0.5))\pi_3 &= 15\pi_2 + (2+3\times 0.5)\pi_4 \\
(15+(2+3\times 0.5))\pi_4 &= 15\pi_3 + (2+4\times 0.5)\pi_5 \\
\dots &= \dots \\
(15+(2+(i-1)\times 0.5))\pi_i &= 15\pi_{i-1} + (2+i\times 0.5)\pi_{i+1} \\
\dots &= \dots \\
1 &= \pi_0 + \pi_1 + \pi_2 \dots
\end{aligned}
$$

(b) The tablet is occupied if $X(t) \geq 1$ or $X(t) \neq 0$. Thus the the fraction of time that the tablet is occupied is $1-\pi_0$.

(c) Students leave the queue after marking their attendance in their own phone at rate $(0)\pi_0 + (0)\pi_1 + (0.5)\pi_2 + (2\times 0.5)\pi_3 + (3\times 0.5)\pi_4 + \dots$ per minute and students arrive at rate of 15 per minute. So fraction of students end up marking the attendance in their own phones, without ever reaching the tablet is $\frac{\pi_2+2\pi_3+3\pi_4+4\pi_5+\dots}{30}$

(d) Student coming at state $X(0) = 0$ can directly mark the attendance in the tablet without waiting. Thus the fraction of students come for marking in the tablet and immediately mark their attendance in the tablet (without waiting in a queue) is π_0.

9. Queueing Models

Question:

(a) Consider a communication node in which packets arrive according to Poisson process. It has only one output link. Consider the following three cases and give Kendall notation for each of them:

   (i) Packets are exponentially distributed. The node has infinite buffer capacity.

   (ii) Packet length can be l_1, l_2, l_3, l_4, l_5 with probability P_{l_1}, P_{l_2}, P_{l_3}, P_{l_4}, P_{l_5} respectively. The node has NO buffer capacity.

   (iii) Packet length is fixed, say L. The node has buffer capacity as n, i.e., only n packets can be stored in the buffer.

(b) Consider M/M/106/4040/2020 and M/M/106/2020/2020 queuing system. Which one has better performance? Which one might you NOT prefer? Why/Why not?

(c) Consider a pure delay process where customers arrive according to Poisson process with parameter $\lambda = 5$. The service time is an exponential distribution with mean 1/5 minutes. There are two servers but only one of them is always available. The other server would, however, start the service only when the queue length becomes two. When there are no more customers in the queue, the server that becomes idle (complete service) first becomes unavailable for further service and remain unavailable until the queue length becomes two again. Take the discipline to be FCFS. Sketch the state diagram for this system?

Solution:

(a) In all the cases,
   Arrival of packets is Poisson distribution - [M]
   Only one output link, meaning single server - [1]

   (i) Packets are exponentially distributed - [M]
   Infinite buffer capacity
   So it is **M/M/1** system.

   (ii) Packet length can be l_1, l_2, l_3, l_4, l_5 with probability P_{l_1}, P_{l_2}, P_{l_3}, P_{l_4}, P_{l_5} respectively which means general service - [G]
   No buffer capacity - [1]
   So it is **M/G/1/1** system.

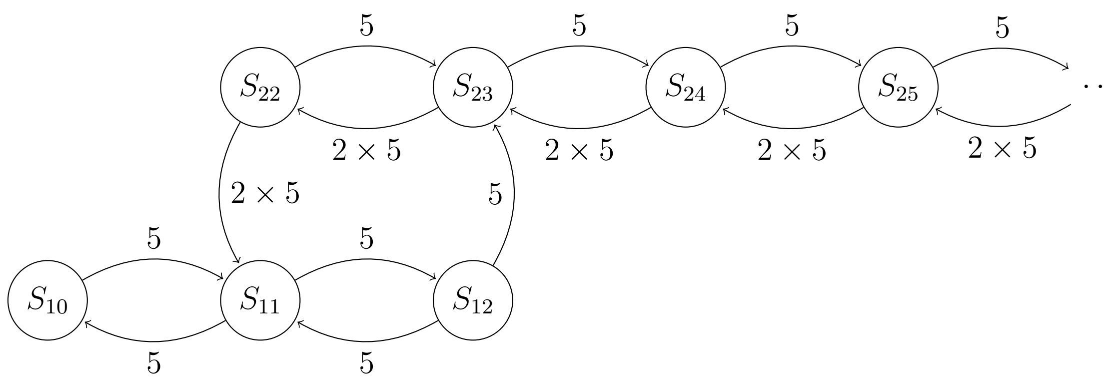
   (iii) Packet length is fixed, meaning deterministic service - [D]
   Buffer capacity is $n + 1$ can be in service - [n+1]
   So it is **M/D/1/n+1** system.

(b) Both M/M/106/4040/2020 and M/M/106/2020/2020 have same performance as customers fit into each system. However, M/M/106/4040/2020 might not be preferred because of wastage of buffer capacity.

(c) To summarize, we have Poisson arrivals, Exponential service time, 2 servers and infinite buffer. Suppose we denote the states by S_{ij} where i is the number of available (active) servers and j is the total number of customers in system (waiting and in service).

By understanding the working of the system, we can note that there would be a transition from S_{12} to S_{23} as this is a case of arrival in which queue length becomes two, the server who was idle also starts working. Also, S_{22} makes a transition to S_{11} when a customer leaves and the other server has no more customers to serve and remains unavailable until the queue length becomes two again.

We can now draw the following state transition diagram:



10. Queueing Models

Question:

Consider a queuing system in front of counter of shop in which inter-arrival times of customers are modelled as Exponential Distribution with mean 50s. Consider the following three cases:

(A) There is only one single-server queue. The service time is exponentially distributed with mean 40s. There is infinite buffer space.

(B) The owner hires two severs with average time of service for each server double that of case A (i.e., service time of each server is exponentially distributed with expectation 80s) to reduce the amount paid to servers. Two single-server queues are set up each with infinite buffer space.

(C) Consider the same two set of servers in Case (B) (i.e., service time of each server is exponentially distributed with expectation 80s). Instead of two single-server queue, one two-server queue is now set up with infinite buffer space.

With the above three cases, answer the following questions:

(i) Give the Kendall notations and the parameters of the distributions (inter-arrival and service) for each of the three cases?

(ii) What is the average number of customers in the system in cases (A) and (B)?

(iii) How long on average does each customer spend in the line **waiting to get served** in cases (A) and (B)?

(You can assume that a customer choosing a particular queue is equally likely in case (B))

Solution:

Let the parameter for inter-arrival time distribution of customers be λ and the parameter for service time distribution be μ.

(i) Case (A) is M/M/1 queue with $\lambda_A = 1/50 = 0.02$ and $\mu_A = 1/40 = 0.025$
Case (B) is effectively two M/M/1 queues. Since a customer joining a particular line is equally likely, the average inter-arrival time of customers would be double for a particular queue. So $\lambda_B = 1/100 = 0.01$ and $\mu_B = 1/80 = 0.0125$
Case (C) is M/M/2 queue. $\lambda_C = 1/50 = 0.02$ and $\mu_C = 1/80 = 0.0125$.

(ii) Suppose N represent number of customers in system, R represent time spent and Q be time spent in the line excluding the time spent for service.

For M/M/1 queue,

$$
\begin{aligned}
E(N) &= \tfrac{\lambda}{\mu-\lambda} \\
\lambda \cdot E(R) &= E(N) \qquad \text{(by Little's formula)} \\
E(R) &= \tfrac{1}{\lambda}\tfrac{\lambda}{\mu-\lambda} \\
&= \tfrac{1}{\mu-\lambda} \\
E(Q) &= E(R) - \tfrac{1}{\mu} \\
&= \tfrac{1}{\mu-\lambda} - \tfrac{1}{\mu}
\end{aligned}
$$

Using the above results we have the average number of customers in the system in case (A) as $\frac{\lambda_A}{\mu_A-\lambda_A} = \frac{0.02}{0.025-0.02} = 4$. For case (B), the average number of customers in each system is $\frac{\lambda_B}{\mu_B-\lambda_B} = \frac{0.01}{0.0125-0.01} = 4$. There are two such systems and totally, 8 customers are expected in the both the systems for the counter.

(iii) The average time spent in the line excluding the time spent for service by each customers in case (A) is $\frac{1}{\mu_A-\lambda_A} - \frac{1}{\mu_A} = \frac{1}{0.025-0.02} - \frac{1}{0.025} = 160s$ and in case (B), it is $\frac{1}{\mu_B-\lambda_B} - \frac{1}{\mu_B} = \frac{1}{0.0125-0.01} - \frac{1}{0.0125} = 320s$.