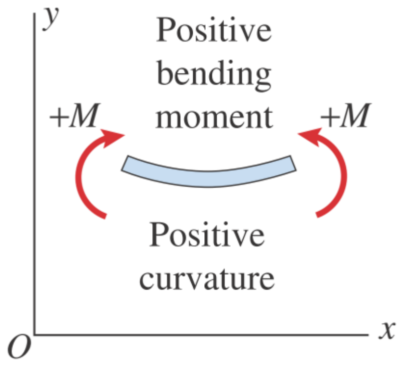
# DEFLECTION OF BEAMS

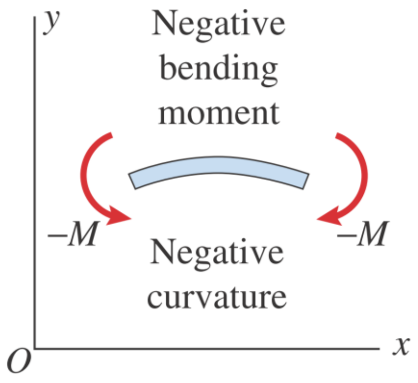
## MECHANICS OF MATERIAL (SOLID MECHANICS)

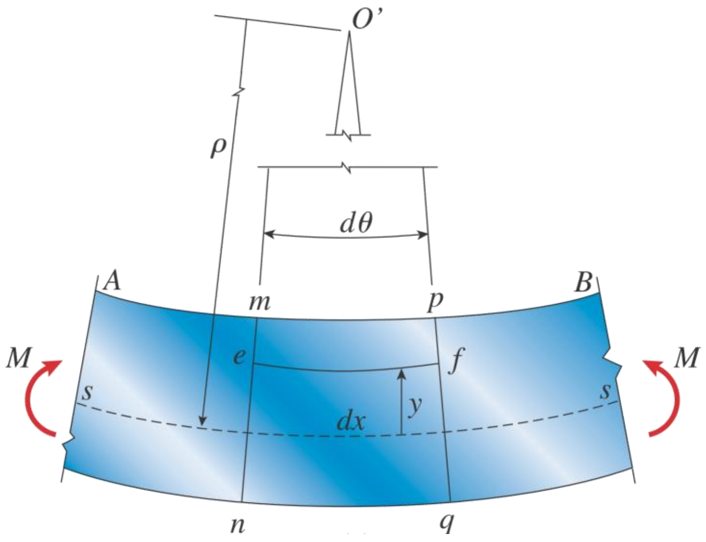
# PURE BENDING: FLEXURE FORMULA




$$\sigma_x = -E\kappa y$$

$$\boxed{\kappa = \frac{M}{EI}}$$




$$\sigma_x = -\frac{My}{I}$$

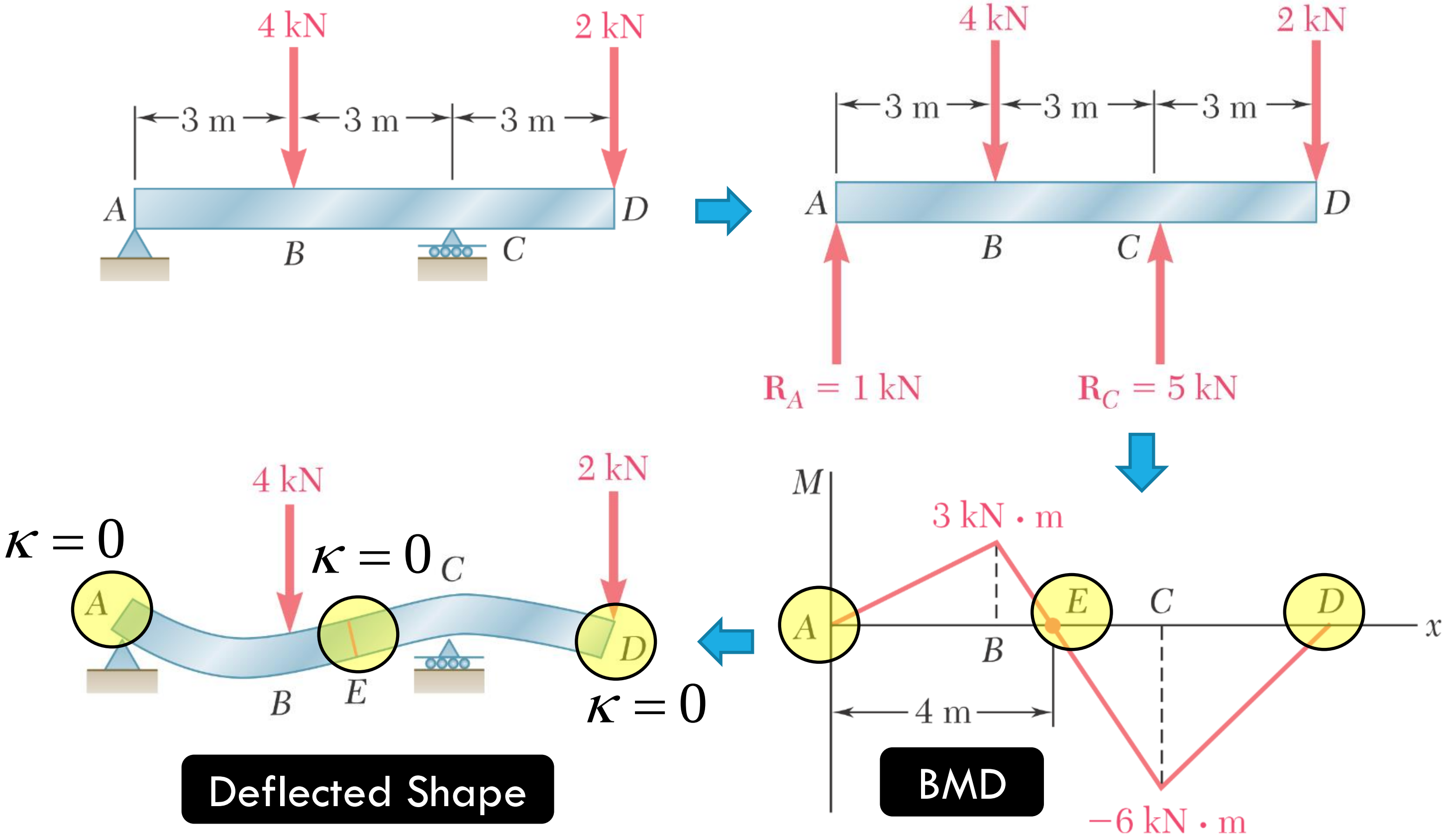
FLEXURE FORMULA!

# CURVATURE AND MOMENT RELATIONSHIP

- For a general case of loading, bending moment *M* varies from one section to another

$$\therefore \kappa = \frac{1}{\rho} = \frac{M(x)}{EI}$$

- Zero curvature ($\kappa$) at points of zero bending moment



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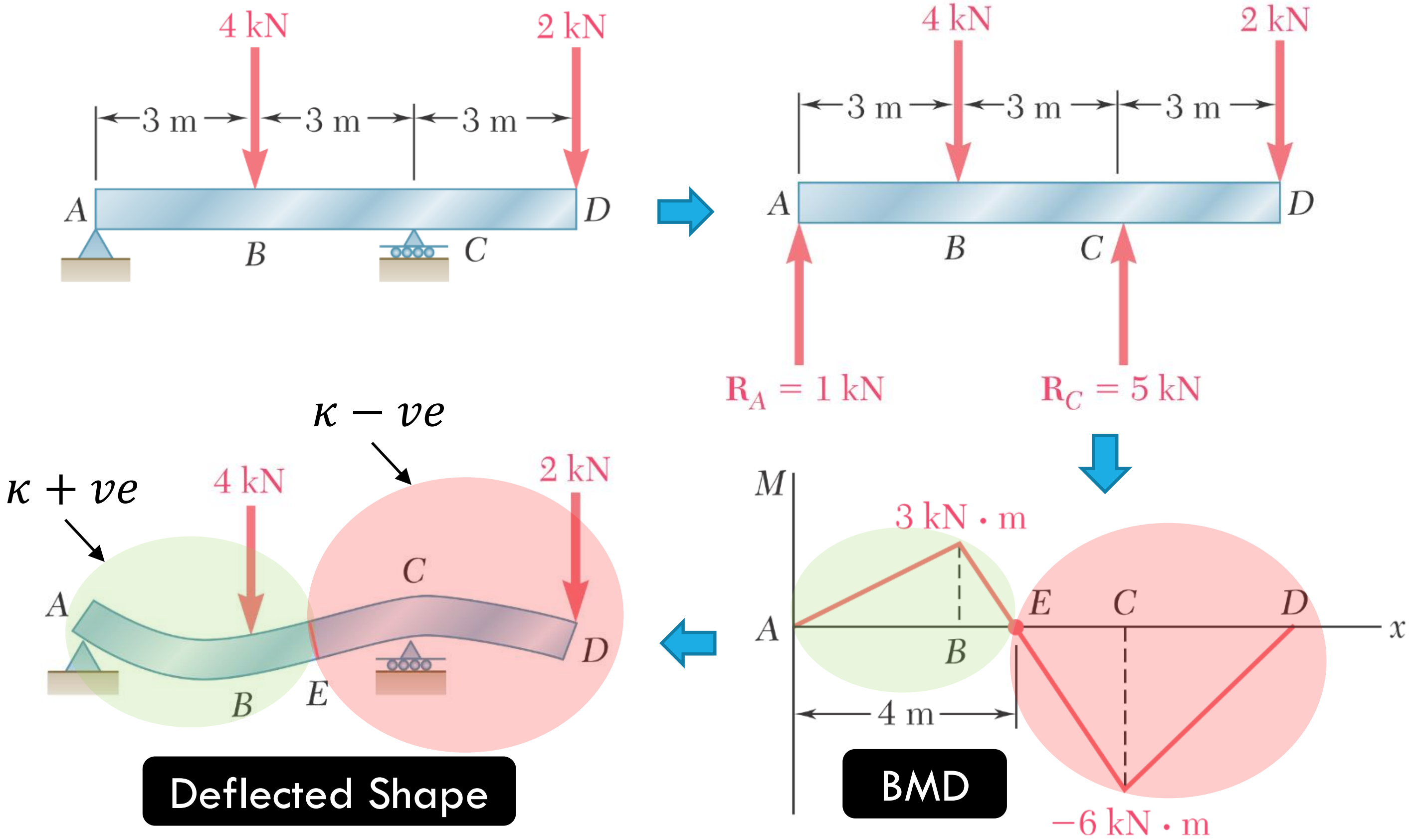
- Beam is concave upward where M and $\kappa$ is +ve
- Beam is concave downward where M and $\kappa$ is -ve



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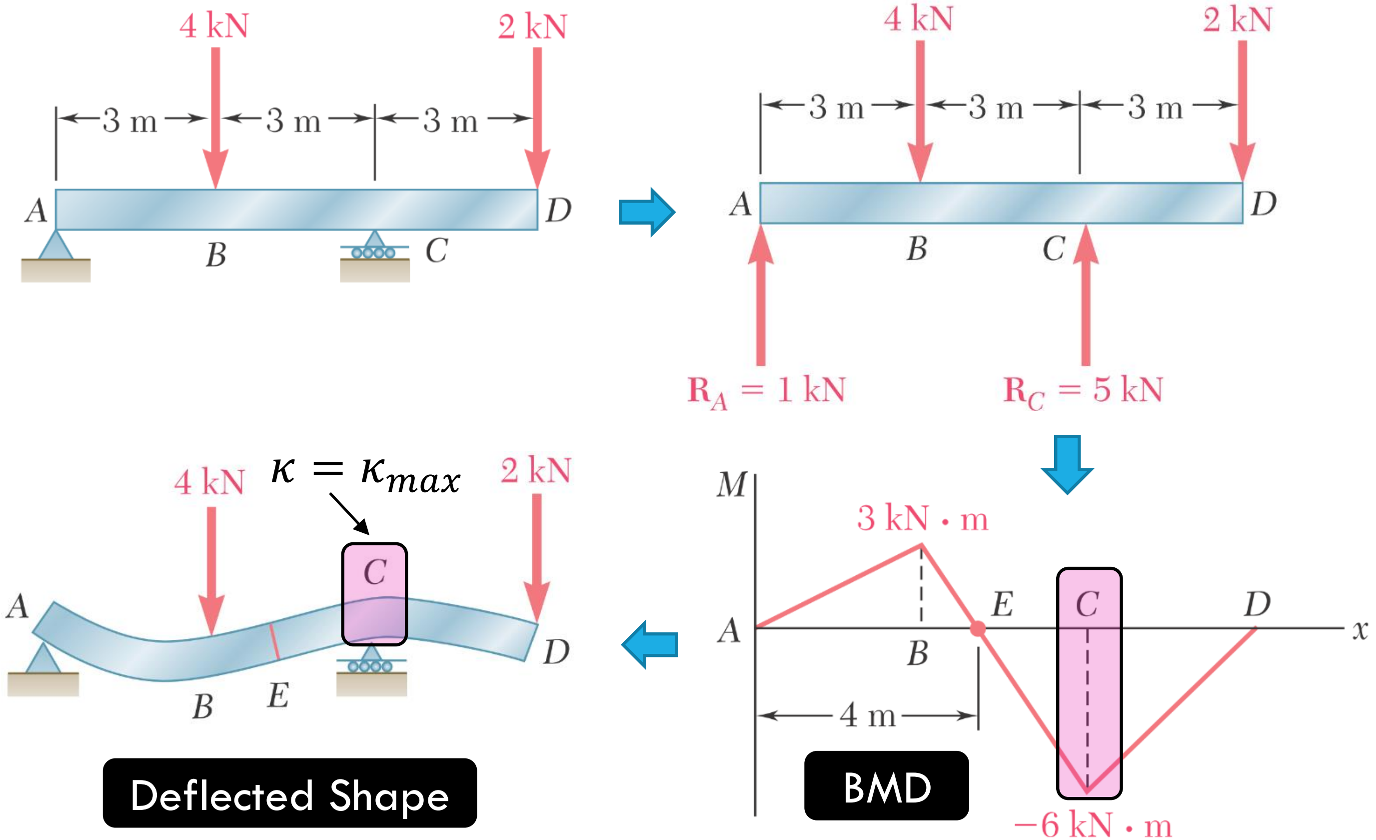
- Beam is concave downward where M and $\kappa$ is –ve
- Maximum curvature occurs where absolute M is max.
- Curvature helps establish a sense of shape of the beam

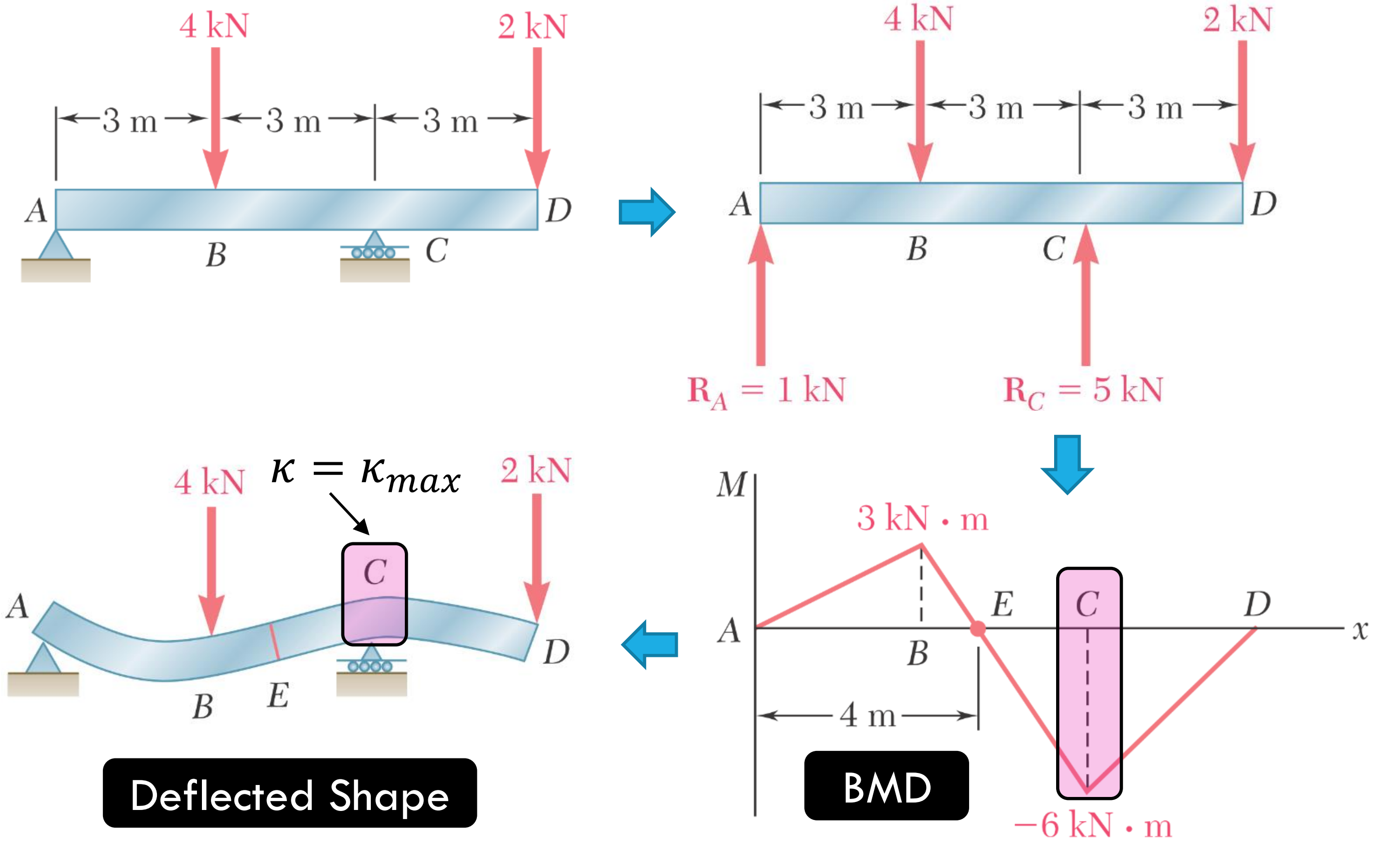


# CURVATURE AND MOMENT RELATIONSHIP

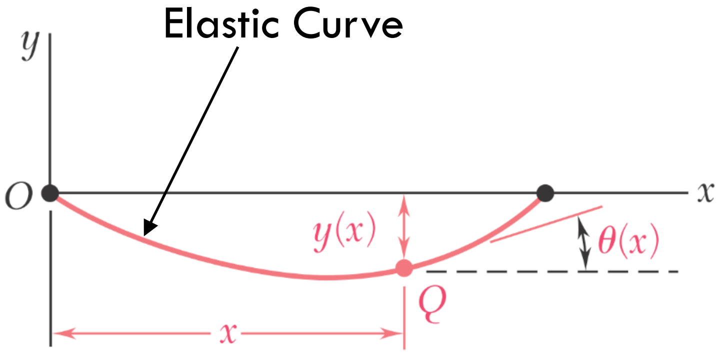
- For a general case of loading, bending moment M varies from one section to another

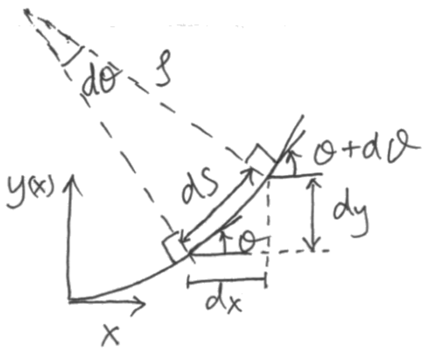
$$\therefore \kappa = \frac{1}{\rho} = \frac{M(x)}{EI}$$

- Zero curvature ($\kappa$) at points of zero bending moment
- Beam smiles where M and $\kappa$ is +ve
- Beam cries where M and $\kappa$ is –ve
- Maximum curvature occurs where absolute M is max.
- Curvature helps establish a sense of shape of the beam
- We need an equation of the beam shape (elastic curve) to find deflections and slope!



# EQUATION OF ELASTIC CURVE




Recall from elementary calculus

$$\frac{dy}{dx} = \tan\theta, \quad \frac{1}{\rho} = \frac{d\theta}{ds} = \frac{d\theta}{dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

$$\frac{d(\tan\theta)}{dx} = \frac{d^2y}{dx^2} = \frac{d(\tan\theta)}{d\theta}\frac{d\theta}{dx} = \left(1+\tan^2\theta\right)\frac{d\theta}{dx} = \left(1+\left(\frac{dy}{dx}\right)^2\right)\frac{d\theta}{dx}$$

$$\frac{d\theta}{dx} = \frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2}, \quad \boxed{\kappa = \frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}}$$

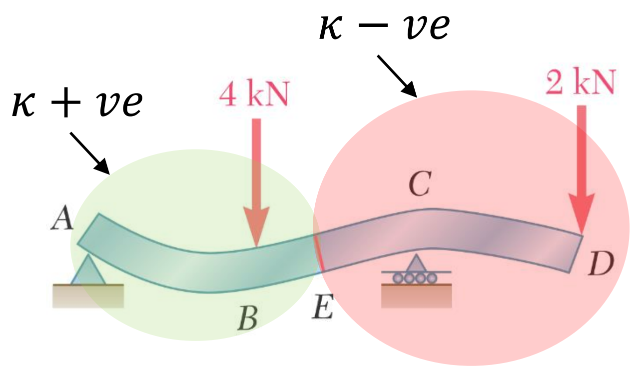
This is the **exact expression** for curvature-deflection relationship

[Holds good for small or large deflections/rotations]

- However, for small angles of rotation:

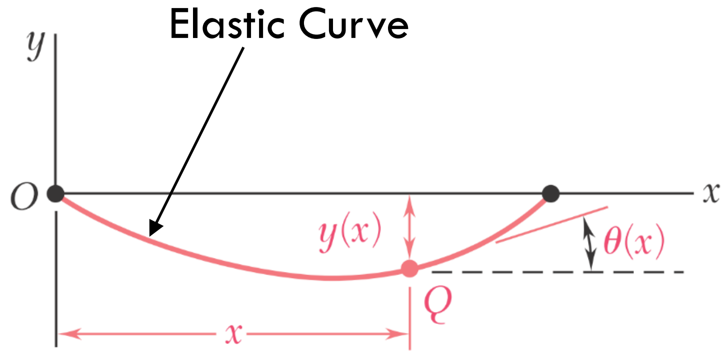
$$\kappa = \frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}} \approx \frac{d^2y}{dx^2}$$

# DEFLECTION OF BEAMS




$$\kappa = \frac{M(x)}{EI}$$

From "Pure Bending"

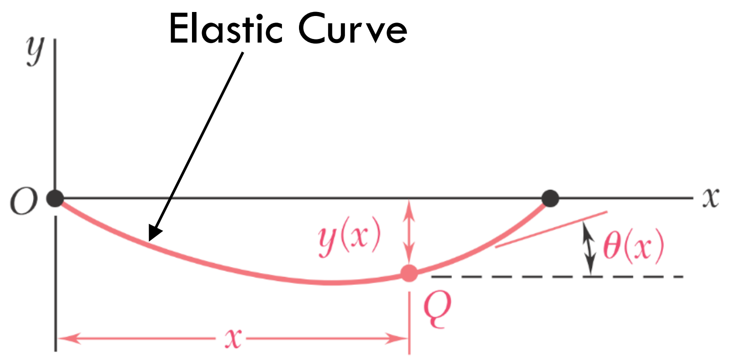



$$\kappa = \frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}} \approx \frac{d^2y}{dx^2}$$

From Calculus

$$\boxed{\frac{d^2y}{dx^2} = \frac{M(x)}{EI}}$$

# EQUATION OF ELASTIC CURVE




$$EI\frac{d^2y}{dx^2} = M(x)$$

$$\Rightarrow EI\frac{dy}{dx} = \int_0^x M(x)\,dx + C_1$$

↳ beam is prismatic → $I \neq I(x)$

For small angles, $\frac{dy}{dx} = \tan\theta \simeq \theta(x)$

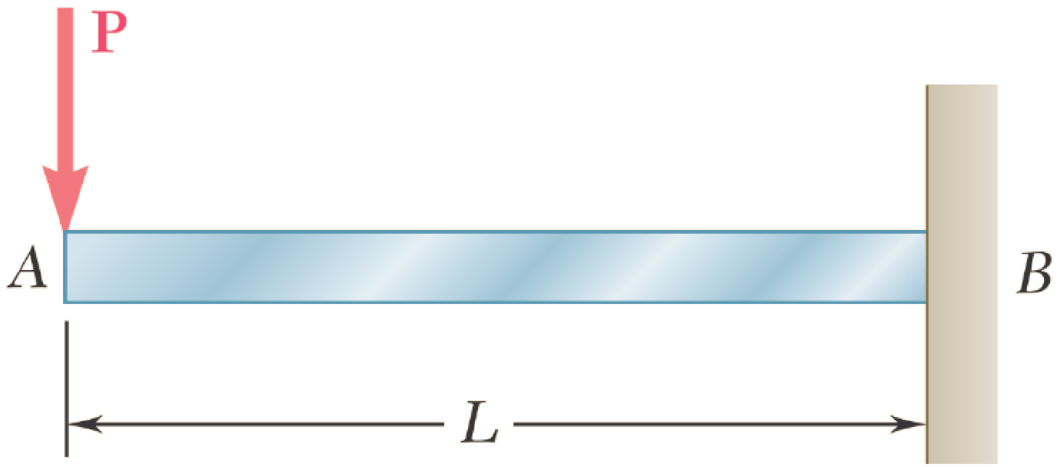
$$\Rightarrow EI\,\theta(x) = \int_0^x M(x)\,dx + C_1$$ — Gives slope!

$$\Rightarrow EI\,y = \int_0^x \left[\int_0^x M(x)\,dx + C_1\right]dx + C_2$$ — Gives deflection!

$\underbrace{C_1\ ,\ C_2}$ → Constants of integration

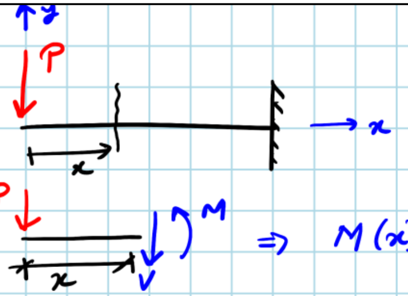
↳ Determined from boundary conditions.

# EXAMPLE PROBLEM




Find deflection and slope at A

(Solution – next page)




$\Rightarrow \quad M(x) = -Px$

$$EI\frac{d^2y}{dx^2} = M(x)$$

$$\therefore EI\frac{d^2y}{dx^2} = -Px$$

$$\Rightarrow EI\frac{dy}{dx} = -\frac{Px^2}{2} + C_1 \leftarrow \text{Integrating once}$$

$$\Rightarrow EIy = -\frac{Px^3}{6} + C_1x + C_2 \leftarrow \text{Integrating twice.}$$

**Boundary Conditions**

$@\ x = L,\ \theta_B = 0 \ \left( \left.\frac{dy}{dx}\right|_B = 0 \right)$

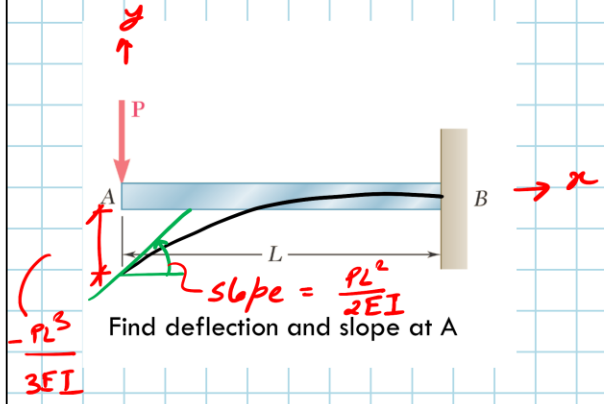
$@\ x = L,\ y_B = 0$

**After substitution:**

$$C_1 = \frac{PL^2}{2}; \quad C_2 = -\frac{PL^3}{3}$$

$$\therefore \text{Slope} \rightarrow EI\frac{dy}{dx} = -\frac{Px^2}{2} + \underbrace{\frac{PL^2}{2}}_{C_1}$$

$$\therefore \text{Deflection} \rightarrow EIy = -\frac{Px^3}{6} + \underbrace{\frac{PL^2}{2}}_{C_1}x \underbrace{- \frac{PL^3}{3}}_{C_2}$$



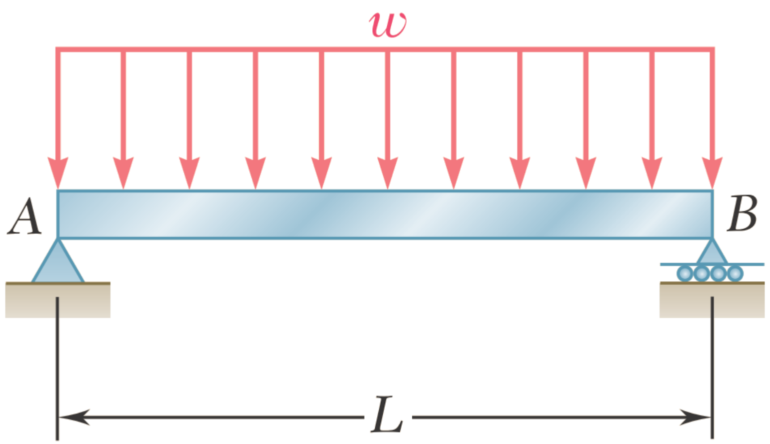

Find deflection and slope at A

**At A : $x = 0$**

$$\therefore \left.\frac{dy}{dx}\right|_{x=0} = \boxed{\left.\frac{dy}{dx}\right|_A = \frac{PL^2}{2EI}}$$

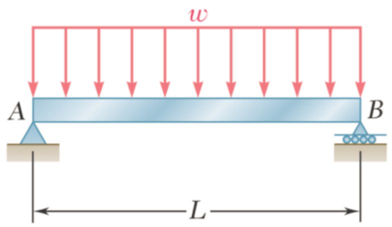
$$\therefore \boxed{y_A = -\frac{PL^3}{3EI}}$$

# EXAMPLE PROBLEM




Find maximum deflection in the beam

(Solution – next page)




Find elastic curve equation and maximum deflection of the beam

Using method of sections, M(x) at any distance x from left end:

$$M(x) = -\frac{wx^2}{2} + \frac{wlx}{2}$$

$$\therefore EI\frac{d^2y}{dx^2} = -\frac{wx^2}{2} + \frac{wlx}{2}$$

Integrating twice again

$$EIy = -\frac{wx^4}{24} + \frac{wlx^3}{12} + C_3 x + C_4$$

$@A: x = 0 ; y_A = 0$

$@B: x = L ; y_B = 0$

$\hookrightarrow C_3 = -\frac{wl^3}{24} , C_4 = 0$

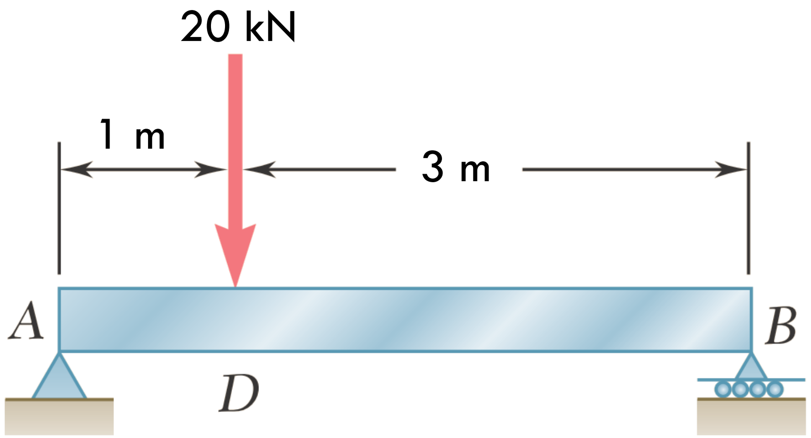
Equation of elastic curve

$$y = \frac{w}{24EI}\left(-x^4 + 2Lx^3 - L^3x\right)$$

Max deflection @ $x = \frac{L}{2}$

$$\boxed{y_{max} = -\frac{5}{384}\frac{wL^4}{EI}}$$

# EXAMPLE PROBLEM



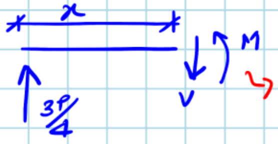

Find slope and deflection at D

modulus of elasticity of $E = 10\,\text{GPa}$ and a rectangular cross section of $b = 60\,\text{mm}$ and $h = 125\,\text{mm}$.

(Solution – next page)

$R_1 : 0 < x < L/4$

$(y_1, \theta_1)$

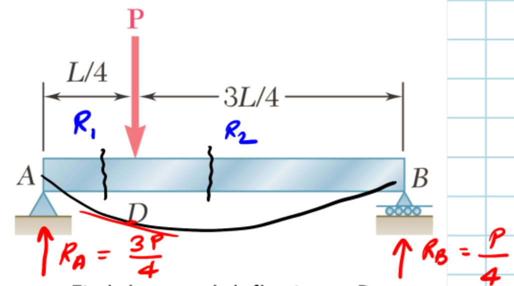


$M = \frac{3Px}{4}$

$R_2 : x > L/4$

$(y_2, \theta_2)$

$M = \frac{3Px}{4} - P\left(x - \frac{L}{4}\right)$




Find slope and deflection at D

**Region #1 ($R_1$)**

$$EI\frac{d^2y}{dx^2} = \frac{3Px}{4}$$

Integrating once & twice:

$$EI\,\theta_1 = \frac{3Px^2}{8} + C_1 \quad \text{--- (1)}$$

$$EI\,y_1 = \frac{Px^3}{8} + C_1x + C_2 \quad \text{--- (2)}$$

**Region #2 ($R_2$)**

$$EI\frac{d^2y}{dx^2} = \frac{3Px}{4} - P\left(x - \frac{L}{4}\right)$$

Integrating

$$EI\,\theta_2 = -\frac{Px^2}{8} + \frac{PLx}{4} + C_3 \quad \text{--- (3)}$$

$$EI\,y_2 = -\frac{Px^3}{24} + \frac{PLx^2}{8} + C_3x + C_4 \quad \text{--- (4)}$$

**Boundary Conditions:**

$[x = 0,\ y_1 = 0]$

$[x = L,\ y_2 = 0]$

$[x = \frac{L}{4},\ \theta_1 = \theta_2]$ — Slope Compatibility

$[x = \frac{L}{4},\ y_1 = y_2]$ — Displacement Compatibility

$$C_1 = -\frac{7PL^2}{128};\ C_2 = 0;\ C_3 = -\frac{11PL^2}{128};\ C_4 = \frac{PL^3}{384}$$

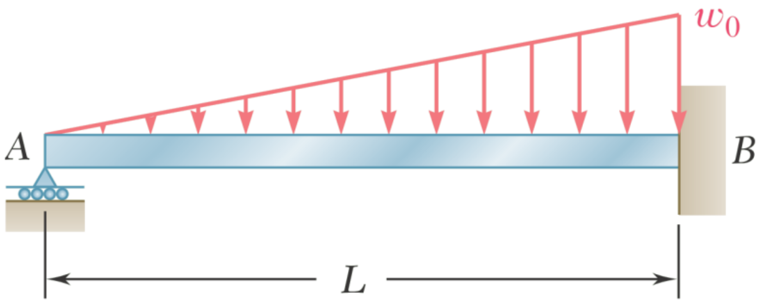
$$\boxed{\theta_D = -\frac{PL^2}{32EI}}\ ;\ \boxed{y_D = -\frac{3PL^3}{256EI}}$$

For given values, we get:

$\theta_D = -0.10$

$y_D = 153\ mm$

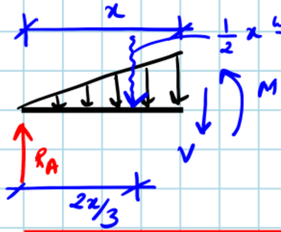
# STATICALLY INDETERMINATE BEAMS!




[Beam Statically Indeterminate to First Degree]

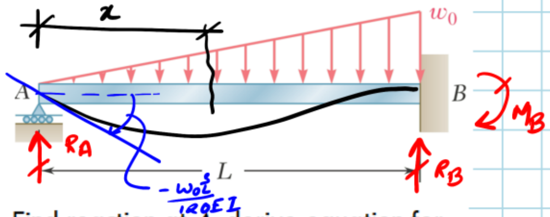
Find reaction at *A*, derive equation for elastic curve, and find slope at *A*.

(Solution – next page)




$\frac{1}{2} x \frac{w_0 x}{L} = \frac{w_0 x^2}{2L}$

$$\boxed{M = R_A x - \frac{w_0 x^3}{6L}}$$




Find reaction at A, derive equation for elastic curve, and find slope at A.

We know

$$EI \frac{d^2y}{dx^2} = M(x) = R_A x - \frac{w_0 x^3}{6L}$$

Integrating twice

$$EI \frac{dy}{dx} = \frac{R_A x^2}{2} - \frac{w_0 x^4}{24L} + C_1 \leftarrow \text{First integration}$$

$$EIy = \frac{R_A x^3}{6} - \frac{w_0 x^5}{120L} + C_1 x + C_2$$

Boundary Conditions

① At A: $x = 0, y = 0$

② At B: $x = L, y = 0$

③ At B: $x = L, \theta = \frac{dy}{dx} = 0$

On solving

$$R_A = \frac{w_0 L}{10}; \quad C_1 = -\frac{w_0 L^3}{120}; \quad C_2 = 0$$

∴ Equation of Elastic Curve:

$$EIy = \frac{1}{6}\left(\frac{w_0 L}{10}\right)x^3 - \frac{w_0 x^5}{120L} - \frac{w_0 L^3 x}{120}$$

∴ Slope @ A: $x = 0$

$$EI \left.\frac{dy}{dx}\right|_{x=0} \rightarrow \theta_A = -\frac{w_0 L^3}{120EI}$$

# DEFLECTION OF BEAMS

[Using Singularity Functions]

← Amazing Method

## MECHANICS OF MATERIAL (SOLID MECHANICS)

# Singularity Functions or Macaulay Functions

## (Mathematical Rules)

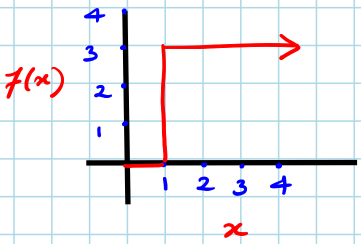
$\langle\ \rangle$ – Macaulay Brackets

A singularity function is defined as:

$$f(x) = \langle x-a\rangle^n = \left\{\begin{array}{ll}(x-a)^n & \text{for } x \geq a \\ 0 & \text{for } x < a\end{array}\right\} \text{ Valid for } n \geq 0$$

Example:

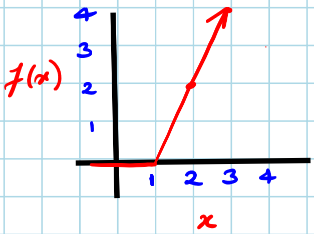
$$f(x) = 3\langle x-1\rangle^0$$




$$f(x) = 2\langle x-1\rangle^1$$

$x=2$: $2(2-1)^1 = 2$

$x=3$: $2(3-1)^1 = 2\times 2 = 4$




Integration Rules for Macaulay Functions

$$\int \langle x-a\rangle^n\,dx = \frac{\langle x-a\rangle^{n+1}}{n+1} \quad \text{for } n \geq 0$$

$$= \langle x-a\rangle^{n+1} \quad \text{for } n < 0$$

# SINGULARITY FUNCTIONS

<u>Sign Convention</u>

P, w positive upwards,
M positive clockwise

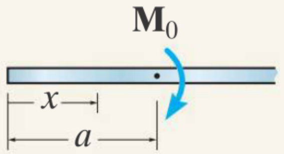
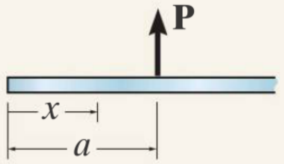
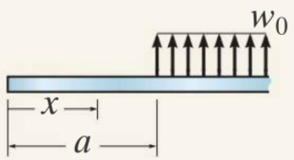
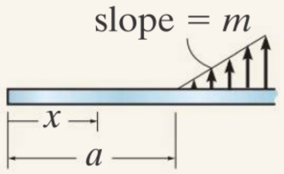
<u>Macaulay Functions</u>

$$\langle x-a\rangle^n = \begin{cases} 0 & \text{for } x < a \\ (x-a)^n & \text{for } x \geq a \end{cases}$$

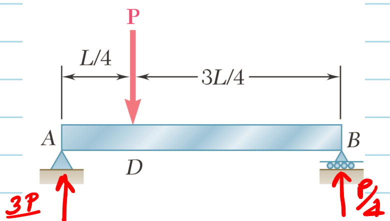
$$n \geq 0$$

<u>Integration rules</u>

$$\int \langle x-a\rangle^n dx = \begin{cases} \dfrac{\langle x-a\rangle^{n+1}}{n+1} & \text{for } n \geq 0 \\ \langle x-a\rangle^{n+1} & \text{for } n < 0 \end{cases}$$

| Loading | Loading Function $w = w(x)$ | Shear $V = \int w(x)dx$ | Moment $M = \int V dx$ |
|---|---|---|---|
|  | $w = M_0\langle x-a\rangle^{-2}$ | $V = M_0\langle x-a\rangle^{-1}$ | $M = M_0\langle x-a\rangle^0$ |
|  | $w = P\langle x-a\rangle^{-1}$ | $V = P\langle x-a\rangle^0$ | $M = P\langle x-a\rangle^1$ |
|  | $w = w_0\langle x-a\rangle^0$ | $V = w_0\langle x-a\rangle^1$ | $M = \frac{w_0}{2}\langle x-a\rangle^2$ |
|  | $w = m\langle x-a\rangle^1$ | $V = \frac{m}{2}\langle x-a\rangle^2$ | $M = \frac{m}{6}\langle x-a\rangle^3$ |

| Loading | Loading Function $w = w(x)$ | Shear $V = \int w(x)dx$ | Moment $M = \int V dx$ |
|---|---|---|---|
| $\mathbf{M}_0$; $x$; $a$ | $w = M_0\langle x-a\rangle^{-2}$ | $V = M_0\langle x-a\rangle^{-1}$ | $M = M_0\langle x-a\rangle^{0}$ |
| $\mathbf{P}$; $x$; $a$ | $w = P\langle x-a\rangle^{-1}$ | $V = P\langle x-a\rangle^{0}$ | $M = P\langle x-a\rangle^{1}$ |
| $w_0$; $x$; $a$ | $w = w_0\langle x-a\rangle^{0}$ | $V = w_0\langle x-a\rangle^{1}$ | $M = \frac{w_0}{2}\langle x-a\rangle^{2}$ |
| slope $= m$; $x$; $a$ | $w = m\langle x-a\rangle^{1}$ | $V = \frac{m}{2}\langle x-a\rangle^{2}$ | $M = \frac{m}{6}\langle x-a\rangle^{3}$ |



Find slope and deflection at D

$$M(x) = \frac{3P}{4}\langle x-0\rangle^1 - P\langle x-\tfrac{L}{4}\rangle^1$$

$$EI\frac{d^2y}{dx^2} = M(x)$$

$$\Rightarrow EI\frac{d^2y}{dx^2} = \frac{3P}{4}\langle x-0\rangle^1 - P\langle x-\tfrac{L}{4}\rangle^1$$

$$\Rightarrow EI\frac{dy}{dx} = \frac{3P}{4\times 2}\langle x-0\rangle^2 - \frac{P}{2}\langle x-\tfrac{L}{4}\rangle^2 + C_1$$

$$\Rightarrow EIy = \frac{3P}{8\times 3}\langle x-0\rangle^3 - \frac{P}{2\times 3}\langle x-\tfrac{L}{4}\rangle^3 + C_1x + C_2$$

**BOUNDARY CONDITIONS**

$[x=0,\ y=0\ ;\ x=L,\ y=0]$

$$EI\times 0 = \frac{3P}{24}\cancel{\langle 0-0\rangle^3}^{\,0} - \frac{P}{6}\cancel{\langle 0-\tfrac{L}{4}\rangle^3}^{\,0} + \cancel{C_1\times 0}^{\,0} + C_2$$

$$\therefore\ C_2 = 0$$

$$EI\times 0 = \frac{3P}{24}\langle L-0\rangle^3 - \frac{P}{6}\langle L-\tfrac{L}{4}\rangle^3 + C_1L + 0$$

$$\therefore\ C_1 = -\frac{7PL^2}{128}$$

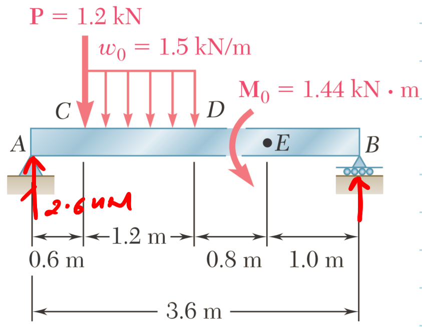
**Equation of elastic curve:**

$$EIy = \frac{3P}{24}\langle x-0\rangle^3 - \frac{P}{6}\langle x-\tfrac{L}{4}\rangle^3 - \frac{7PL^2x}{128}$$

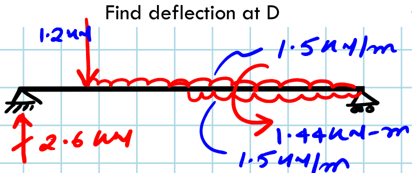
Substituting $x = L/4$ i.e. at D

$$\boxed{y_D = -\frac{3PL^3}{256EI}}\ ;\ \boxed{\theta_D = -\frac{PL^2}{32EI}}$$

| Loading | Loading Function $w = w(x)$ | Shear $V = \int w(x)dx$ | Moment $M = \int V dx$ |
|---|---|---|---|
| $M_0$ | $w = M_0\langle x-a\rangle^{-2}$ | $V = M_0\langle x-a\rangle^{-1}$ | $M = M_0\langle x-a\rangle^{0}$ |
| $P$ | $w = P\langle x-a\rangle^{-1}$ | $V = P\langle x-a\rangle^{0}$ | $M = P\langle x-a\rangle^{1}$ |
| $w_0$ | $w = w_0\langle x-a\rangle^{0}$ | $V = w_0\langle x-a\rangle^{1}$ | $M = \frac{w_0}{2}\langle x-a\rangle^{2}$ |
| slope $= m$ | $w = m\langle x-a\rangle^{1}$ | $V = \frac{m}{2}\langle x-a\rangle^{2}$ | $M = \frac{m}{6}\langle x-a\rangle^{3}$ |




Find deflection at D



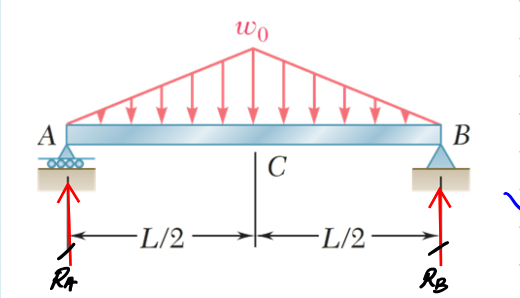

$$EI\frac{d^2y}{dx^2} = M(x) = 2.6\langle x-0\rangle^{1} - 1.2\langle x-0.6\rangle^{1} - \frac{1.5}{2}\langle x-0.6\rangle^{2} + \frac{1.5}{2}\langle x-1.8\rangle^{2} - 1.44\langle x-2.6\rangle^{0}$$

Integrating twice: $$EIy = \frac{2.6}{6}\langle x-0\rangle^{3} - \frac{1.2}{6}\langle x-0.6\rangle^{3} - \frac{1.5}{24}\langle x-0.6\rangle^{4} + \frac{1.5}{24}\langle x-1.8\rangle^{4} - \frac{1.44}{2}\langle x-2.6\rangle^{2} + C_1 x + C_2$$

Boundary Conditions:

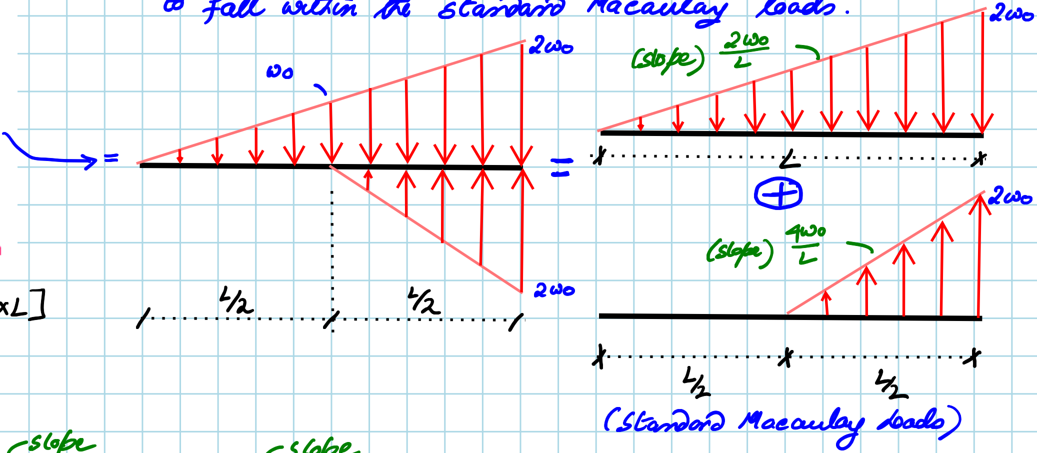
$x=0, y=0$ ; $x=L, y=0$ $\rightarrow$ $C_2 = 0$ ; $C_1 = -2.692$

$\therefore$ At D $(x = 1.8m)$ $\rightarrow$ $\boxed{y_D = \frac{-2.794}{EI}}$




Find slope at A and maximum deflection in the beam

To use Macaulay Functions, we must adjust the loading to fall within the standard Macaulay loads.




From symmetry: $R_A = R_B = \frac{1}{2}\left[\frac{1}{2} \times w_0 \times L\right]$

$= \frac{w_0 L}{4}$

$$EI\frac{d^2y}{dx^2} = R_A\langle x-0\rangle^1 - \frac{\overbrace{\frac{2w_0}{L}}^{\text{slope}}}{6}\langle x-0\rangle^3 + \frac{\overbrace{\frac{4w_0}{L}}^{\text{slope}}}{6}\left\langle x-\frac{L}{2}\right\rangle^3$$

$$\Rightarrow EI\frac{dy}{dx} = \frac{w_0 L}{4\times 2}\langle x-0\rangle^2 - \frac{w_0}{3\times 4L}\langle x-0\rangle^4 + \frac{\cancel{2}w_0}{3L\times \cancel{4}_2}\left\langle x-\frac{L}{2}\right\rangle^4 + C_1 \rightsquigarrow \text{slope}$$

$$\Rightarrow EIy = \frac{w_0 L}{24}\langle x-0\rangle^3 - \frac{w_0}{60L}\langle x-0\rangle^5 + \frac{w_0}{30L}\left\langle x-\frac{L}{2}\right\rangle^5 + C_1 x + C_2 \rightsquigarrow \text{Deflection}$$

Boundary Conditions : $\left\{\begin{array}{l} @\, x=0,\ y=0 \\ @\, x=L,\ y=0 \end{array}\right\}$ Substitute in deflection equation.

$$EI\times 0 = 0 - 0 + 0 + C_1\times 0 + C_2 \Rightarrow C_2 = 0$$

and $$EI\times 0 = \frac{w_0 L}{24}(L)^3 - \frac{w_0}{60L}(L)^5 + \frac{w_0}{30L}(L/2)^5 + C_1\times L$$

$$\Rightarrow C_1 = \frac{w_0}{60}L^3 - \frac{w_0 L^3}{960} - \frac{w_0 L^3}{24} = -\frac{5}{192}w_0 L^3 \quad \therefore C_1 = -\frac{5}{192}w_0 L^3$$

∴ Slope @A : $EI\frac{dy}{dx} = 0 - 0 + 0 + C_1 = -\frac{5}{192}w_0 L^3$

$$\therefore \boxed{\left.\frac{dy}{dx}\right|_{x=A} = -\frac{5}{192EI}w_0 L^3}$$

∴ Maximum deflection (by observation @ $x = \frac{L}{2}$)

$$\therefore EI\left. y\right|_{x=L/2} = \frac{w_0 L}{24}(L/2)^3 - \frac{w_0}{60L}\times\left(\frac{L}{2}\right)^5 - \frac{w_0}{30L}\times 0 - \frac{5}{192\times 2}w_0 L^4$$

$$= -\frac{w_0 L^4}{120} \Rightarrow \boxed{y @ x=\frac{L}{2} = -\frac{w_0 L^4}{120EI}}$$