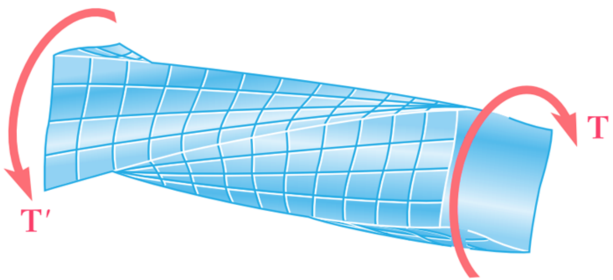
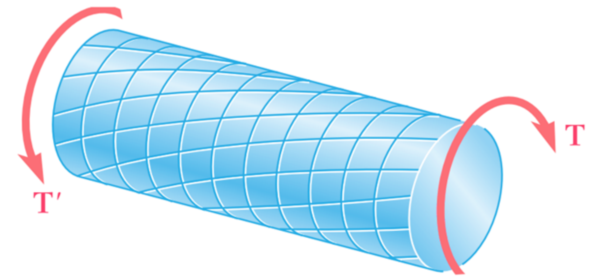
TOPIC # 4: TORSION

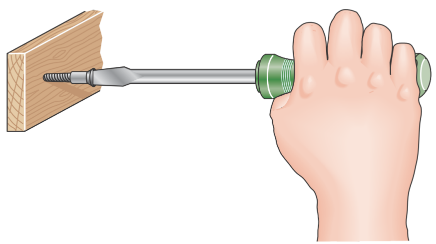


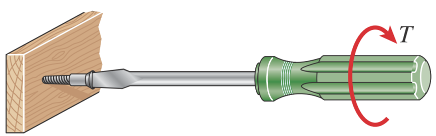
4.1 - Introduction

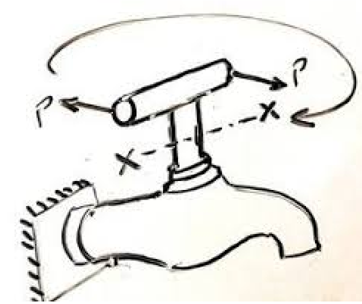


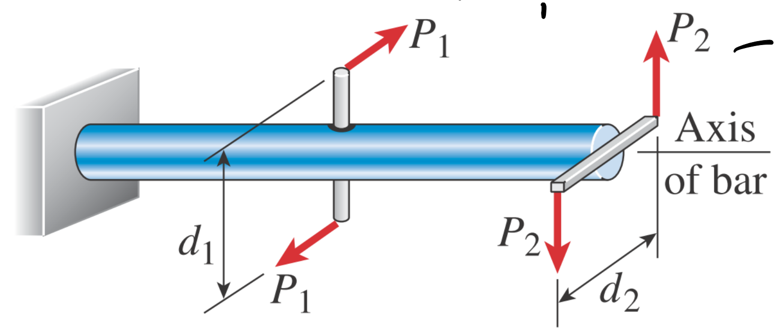
MECHANICS OF MATERIAL
(SOLID MECHANICS)

EXAMPLES OF TORSION (T)





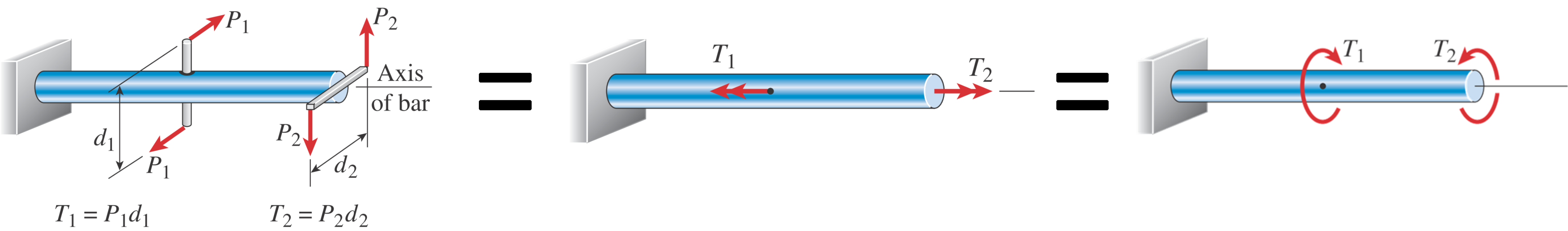




$T_1 = P_1 \times d_1$

$T_2 = P_2 \times d_2$

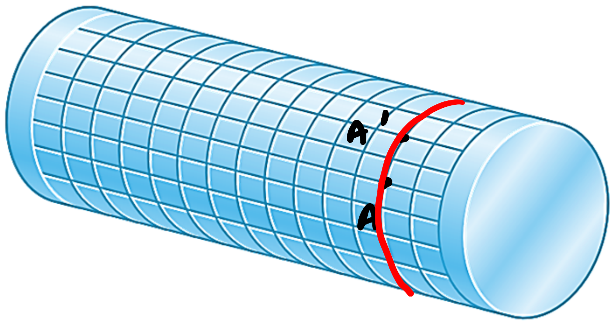
REPRESENTATION OF TORSION (T)

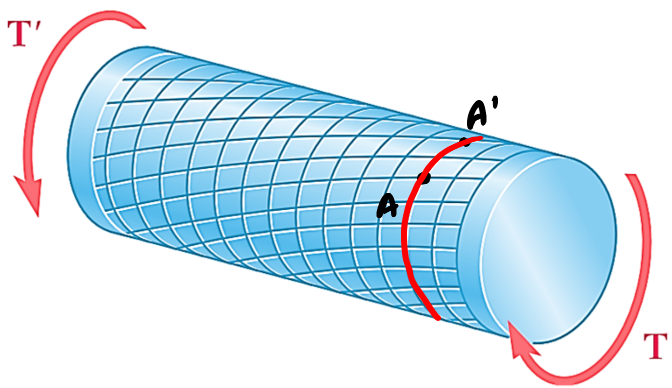


- Torque is a Vector quantity
- Right hand rule - Using your right hand, let your fingers curl in the direction of the moment, and then your thumb will point in the direction of the torsion vector

HOW DO STRUCTURES BEHAVE UNDER TORSION?

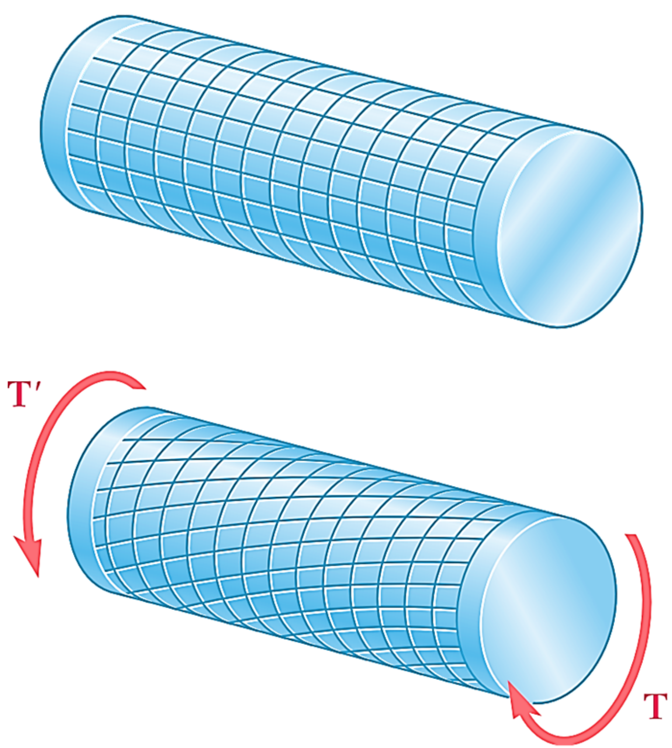
- Circular bars:

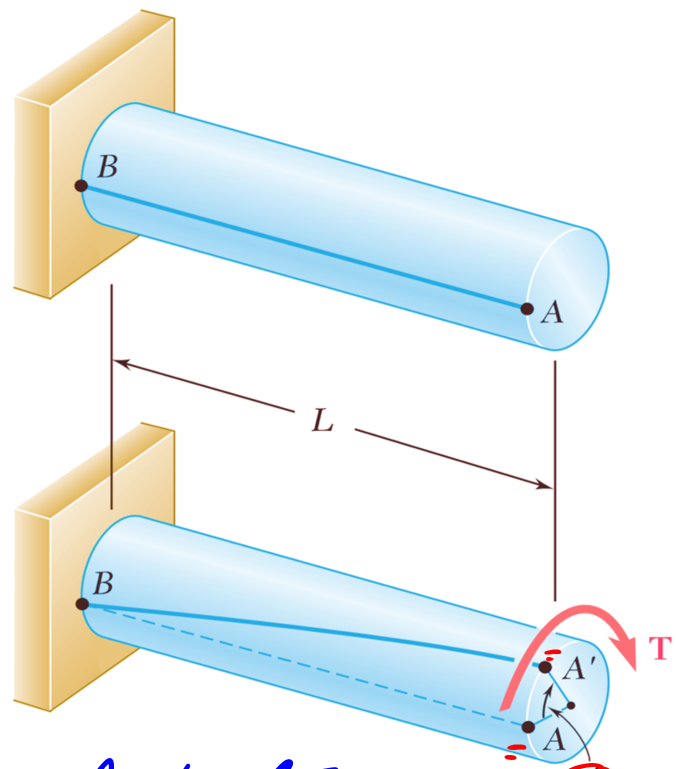




HOW DO STRUCTURES BEHAVE UNDER TORSION?

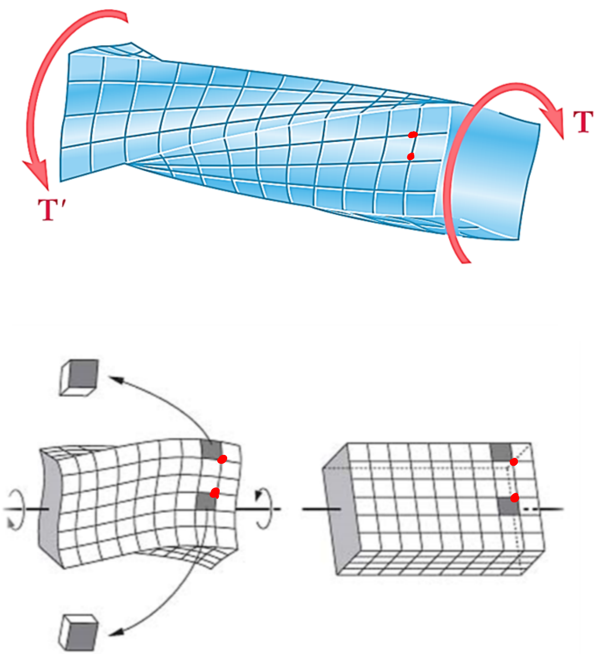
- Circular bars:



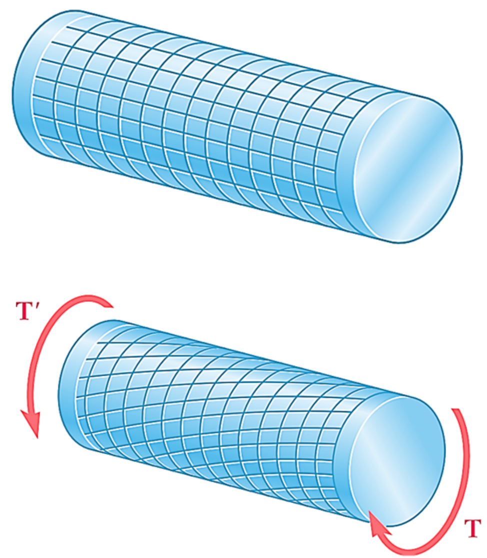


Angle of Twist ← ϕ

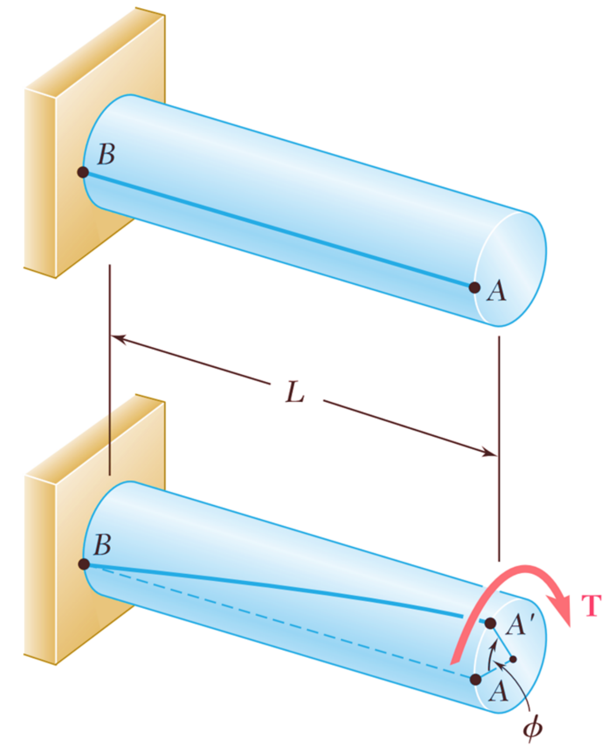
SPOT THE DIFFERENCE



WARPING!!



NO WARPING !!

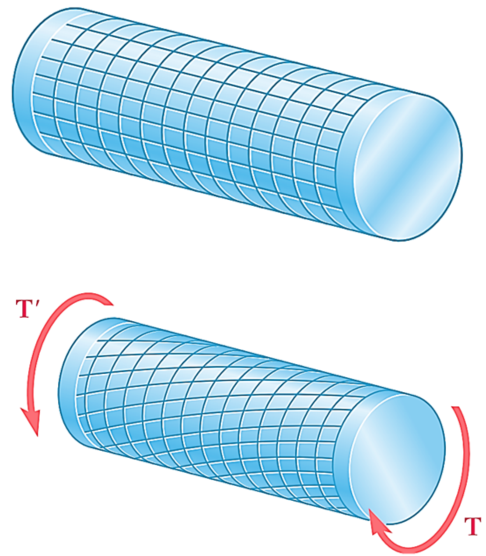


CIRCULAR SHAFTS UNDERGO:
PURE TORSION

- Important Observation:

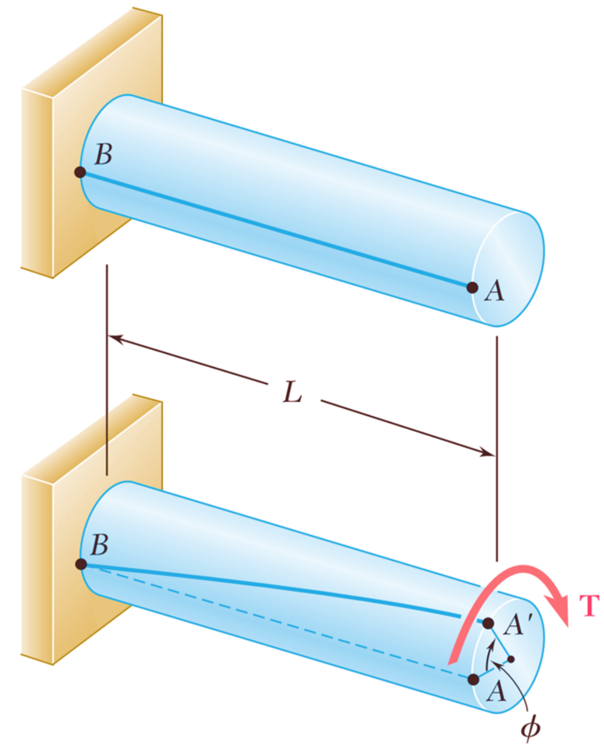
No Warping

Pure Torsion

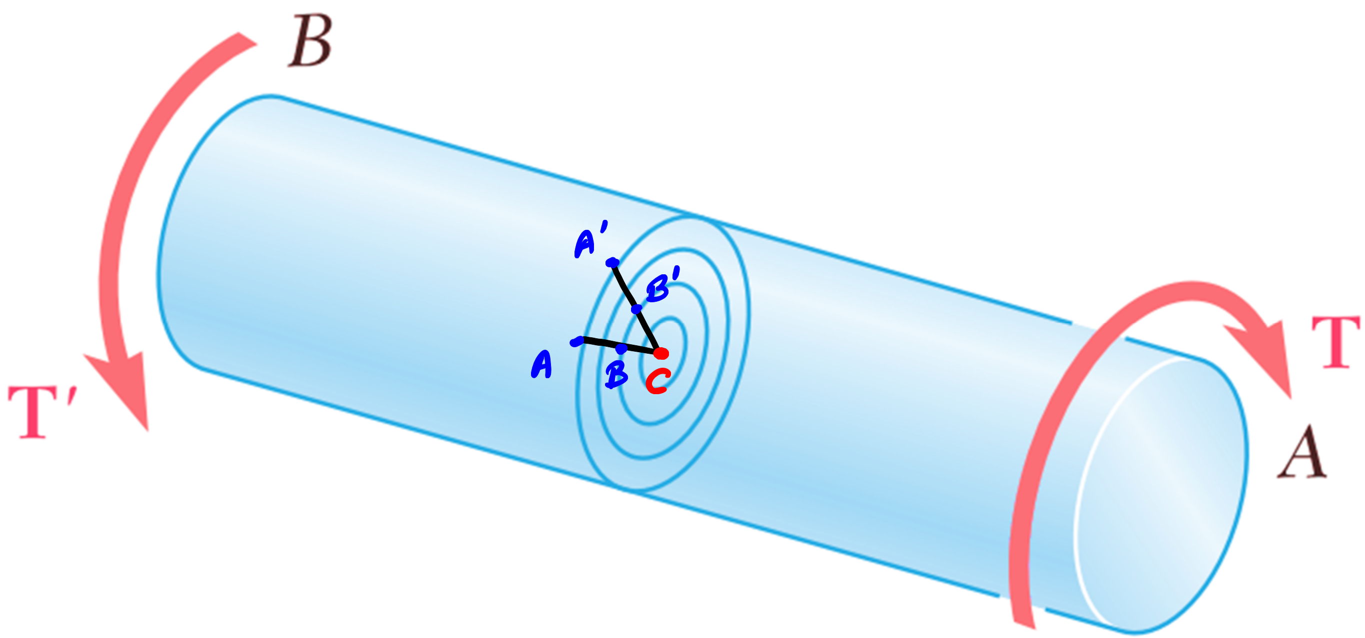


Points on a circular cross-section **remain** on the same cross section

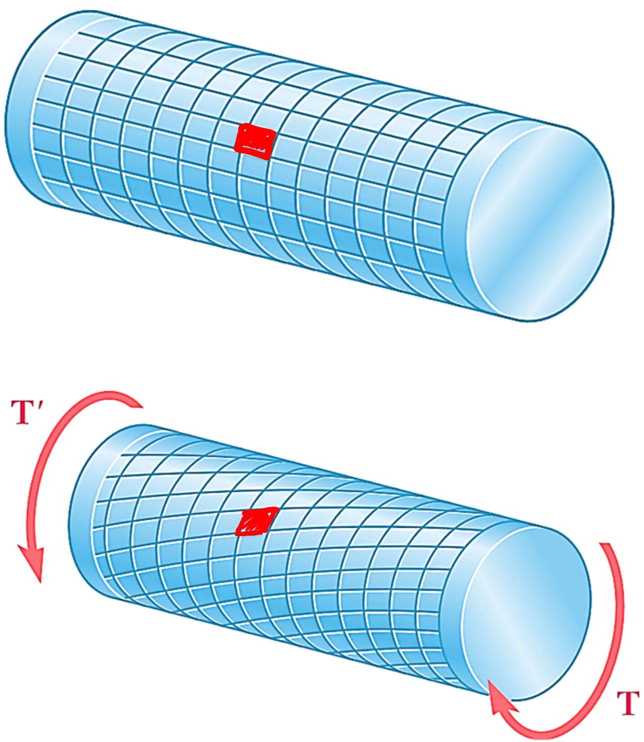
NO WARPING!



PURE TORSION



PURE SHEAR CONDITION



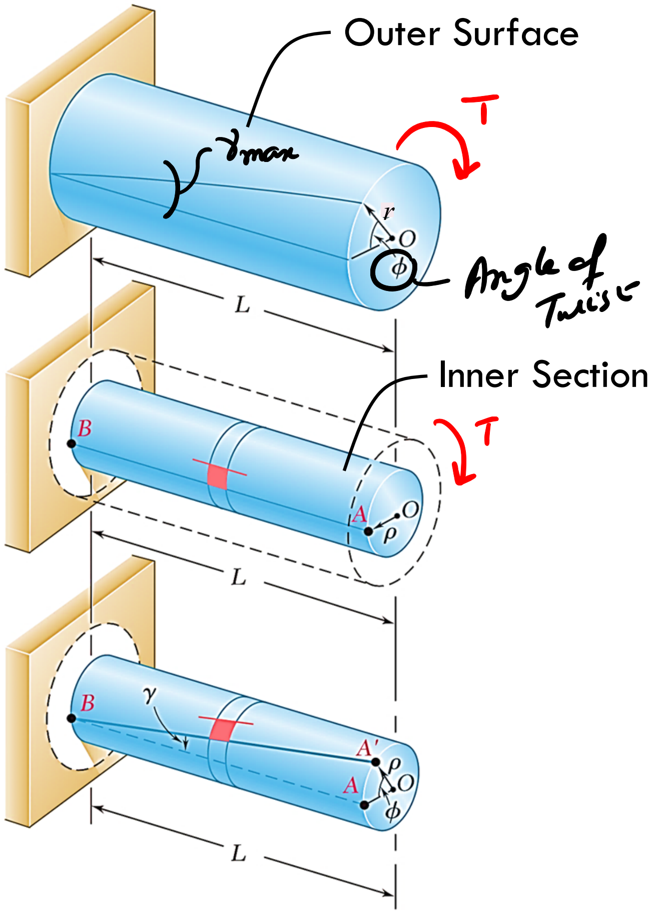
Elements only undergo a change in shape, and minor contraction or elongation

All elements are in a state of "PURE SHEAR"

TOPIC # 4.2: TORSION – SHEARING STRAINS & STRESSES; TORSION FORMULA

MECHANICS OF MATERIAL
(SOLID MECHANICS)

SHEARING STRAINS



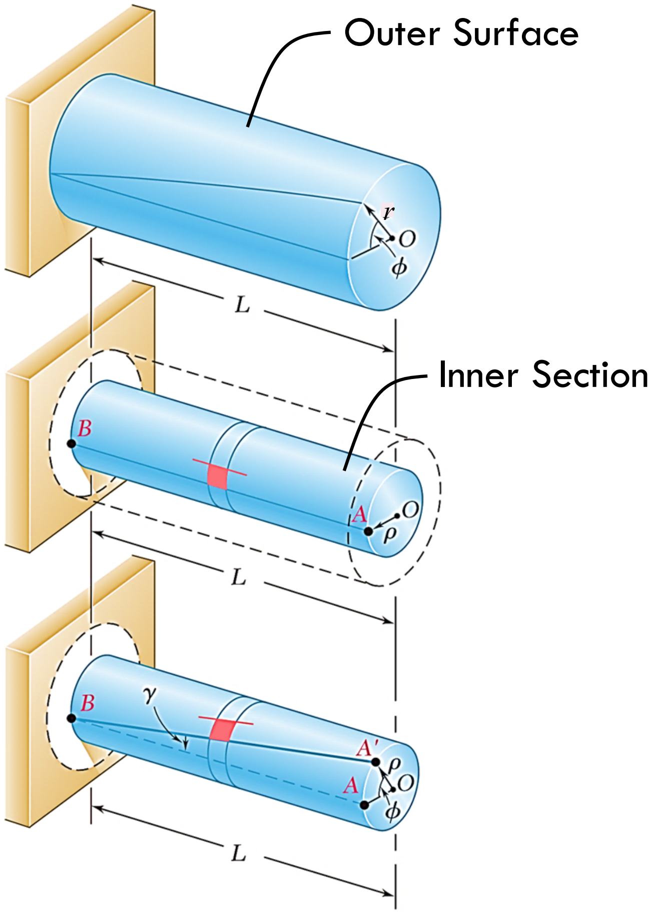
$\Delta ABA'$

$\tan\gamma = \frac{AA'}{L} \Rightarrow \gamma = \frac{AA'}{L}$ — ①

$AA' = \rho\phi$ — ②

$$\boxed{\begin{aligned} \gamma &= \frac{\rho\phi}{L} \\ \gamma_{max} &= \frac{r\phi}{L} \end{aligned}} \quad \Bigg\} \quad \boxed{\gamma = \frac{\rho}{r}\gamma_{max}}$$

SHEARING STRAINS

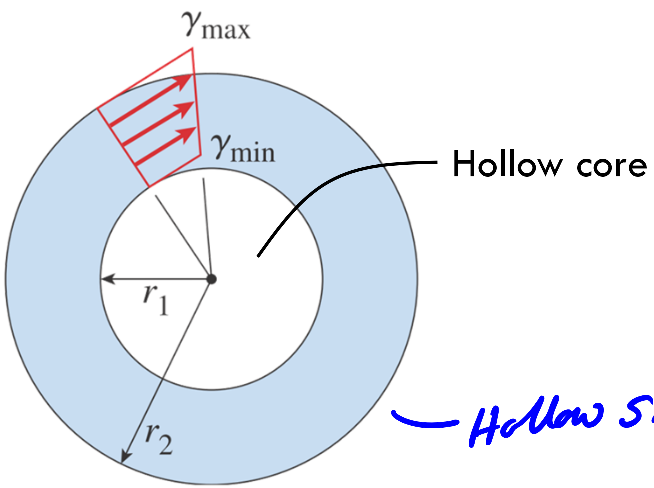


$$\gamma = \frac{\rho\phi}{L}$$

$$\gamma_{\max} = \frac{r\phi}{L}$$

$$\gamma = \frac{\rho\phi}{L} = \frac{\rho}{r}\gamma_{\max}$$

CIRCULAR TUBES



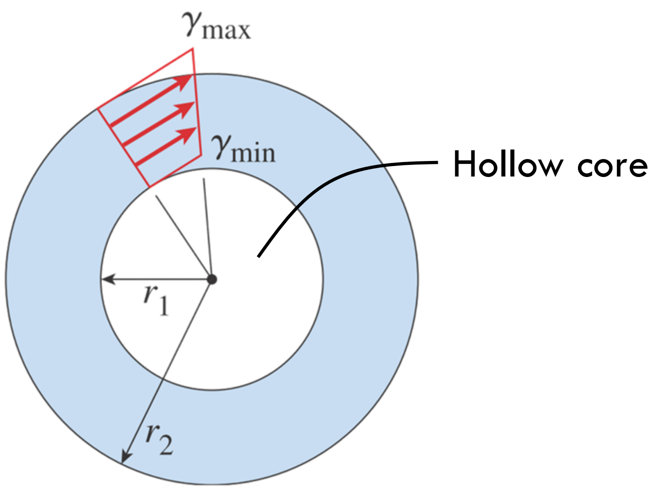
Hollow shaft

$$\boxed{\begin{aligned}\gamma_{min} &= \frac{r_1 \phi}{L} \\ \gamma_{max} &= \frac{r_2 \phi}{L}\end{aligned}} \Bigg\} \quad \boxed{\gamma_{min} = \frac{r_1}{r_2}\gamma_{max}}$$

Solid Shaft

$$\left.\begin{aligned}\gamma &= \frac{\rho \phi}{L} \\ \gamma_{max} &= \frac{r \phi}{L}\end{aligned}\right\} \quad \gamma = \frac{\rho}{r}\gamma_{max}$$

CIRCULAR TUBES



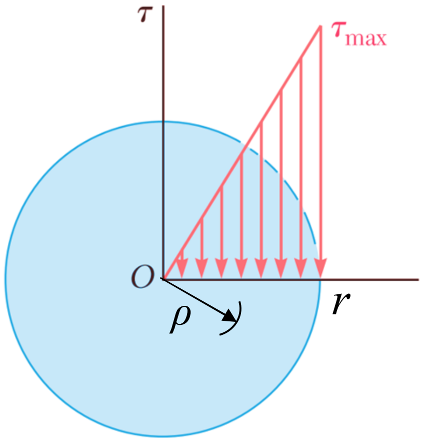
$$\gamma_{min} = \frac{r_1}{L}\phi$$

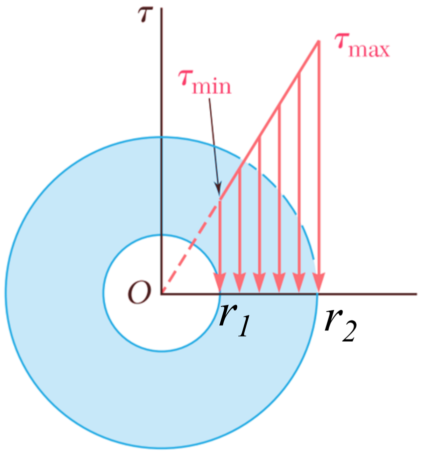
$$\gamma_{max} = \frac{r_2}{L}\phi$$

$$\gamma_{min} = \frac{r_1}{L}\phi = \frac{r_1}{r_2}\gamma_{max}$$

SHEAR STRESSES

$$\boxed{\tau = G\gamma}$$





$$\tau_{max} = \frac{G r_2 \phi}{L}$$

$$\tau_{min} = \frac{G r_1 \phi}{L}$$

$$\boxed{\tau_{min} = \frac{r_1}{r_2}\tau_{max}}$$

HOLLOW SHAFT

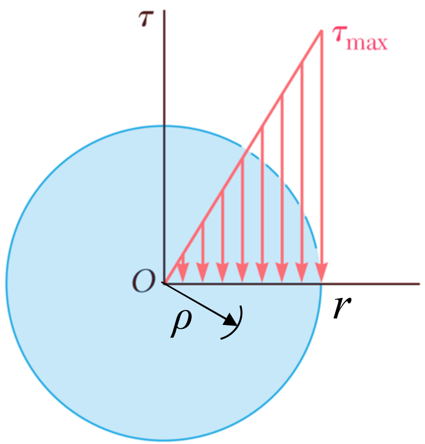
$$\gamma_{max} = \frac{r\phi}{L} \rightarrow \tau_{max} = \frac{G r \phi}{L}$$

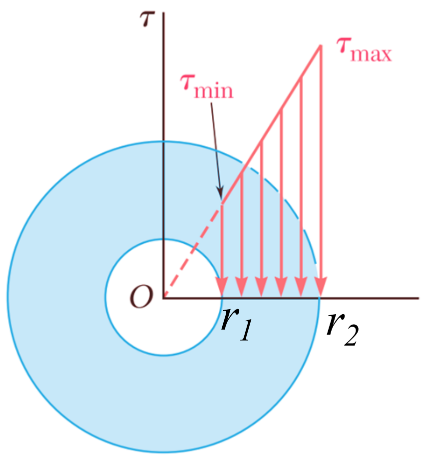
$$\tau = \frac{G\rho\phi}{L}$$

$$\boxed{\tau = \frac{\rho}{r}\tau_{max}}$$

Solid Shaft

SHEAR STRESSES



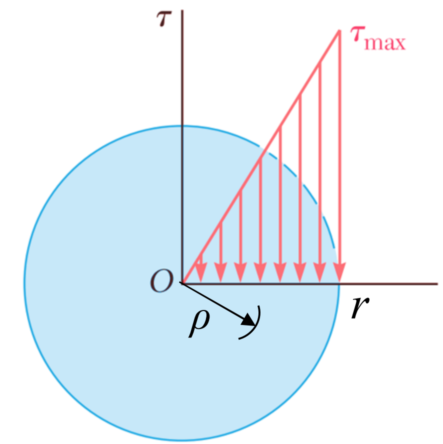


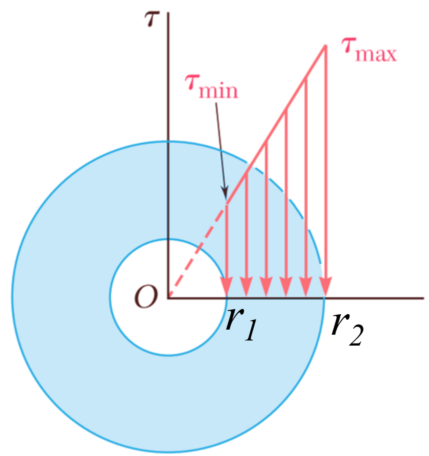
$$\tau_{max} = G\frac{r\phi}{L}$$

$$\tau = G\frac{\rho\phi}{L} = \frac{\rho}{r}\tau_{max}$$

$$\tau_{min} = \frac{r_1}{r_2}\tau_{max}$$

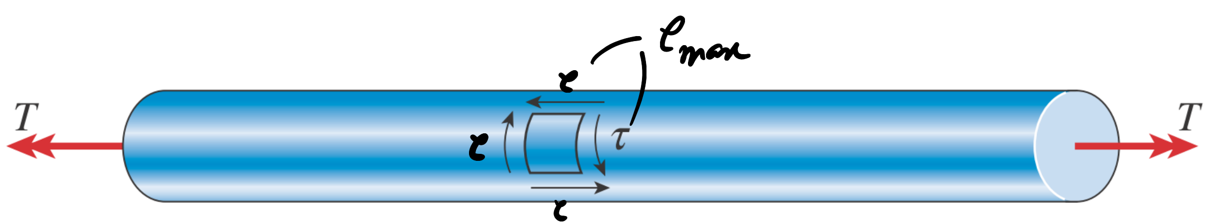
SHEAR STRESSES





$$\tau_{max} = G\frac{r\phi}{L}$$

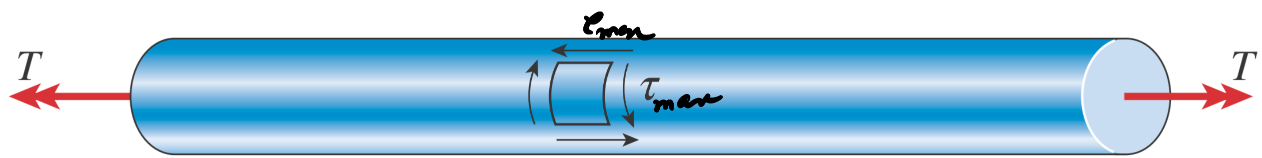
$$\tau = G\frac{\rho\phi}{L} = \frac{\rho}{r}\tau_{max}$$

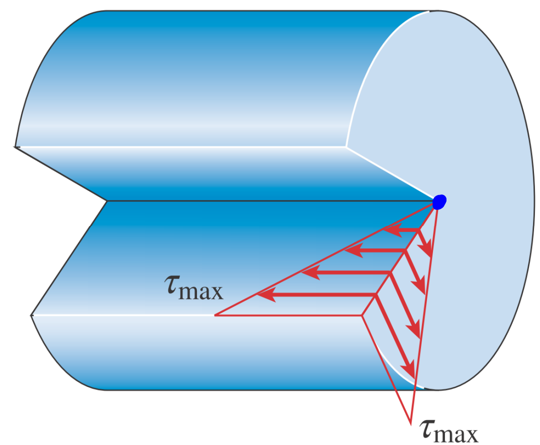


$$\tau_{min} = \frac{r_1}{r_2}\tau_{max}$$

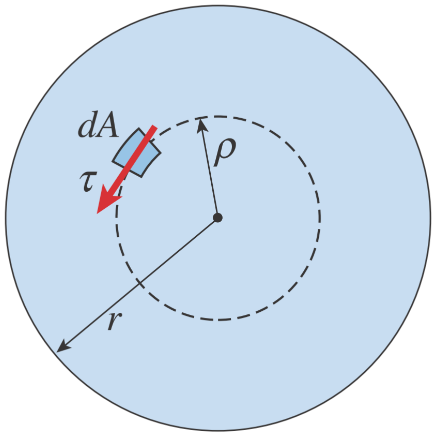
SHEAR STRESSES

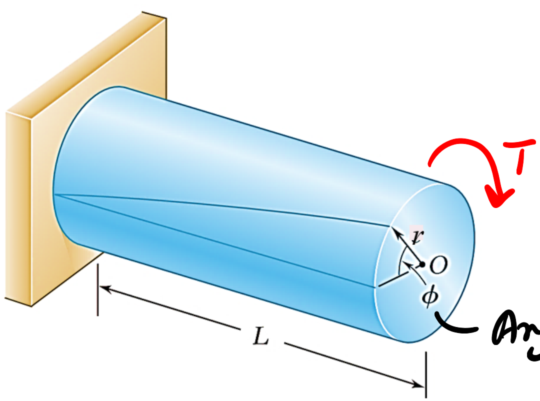
- Internal shear stresses





THE TORSION FORMULA





J → Solid circular shafts

$$\boxed{J = \frac{\pi r^4}{2}};$$

Hollow shaft (r_2, r_1)

$$\boxed{J = \frac{\pi r_2^4}{2} - \frac{\pi r_1^4}{2}}$$

$$dM = (\tau dA) \times \rho$$

$$T = \int_A dM = \int \tau \rho dA$$

$$T = \int \frac{\rho}{r} \tau_{max} \rho dA = \frac{\tau_{max}}{r} \int \rho^2 dA = \frac{\tau_{max} J}{r}$$

Polar moment of inertia (J)

Recap

$$\tau = \frac{\rho}{r} \tau_{max}$$

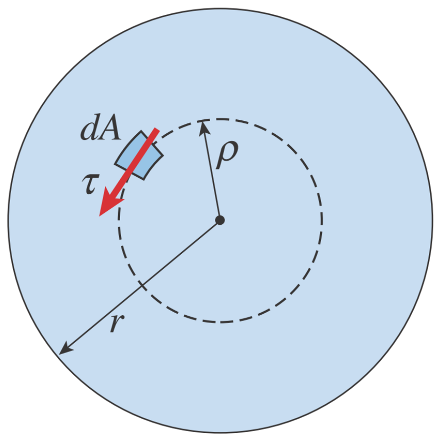
$$\tau_{max} = G \frac{r\phi}{L}$$

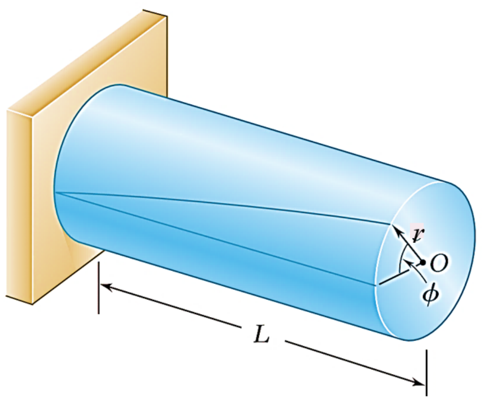
$$\Rightarrow \tau_{max} = \frac{Tr}{J} \rightarrow \tau = \frac{T\rho}{J}$$

$$\Rightarrow \frac{G \cancel{r} \phi}{L} = \frac{T\cancel{r}}{J} \Rightarrow \boxed{\phi = \frac{TL}{JG}}$$

Torsion Formula

THE TORSION FORMULA



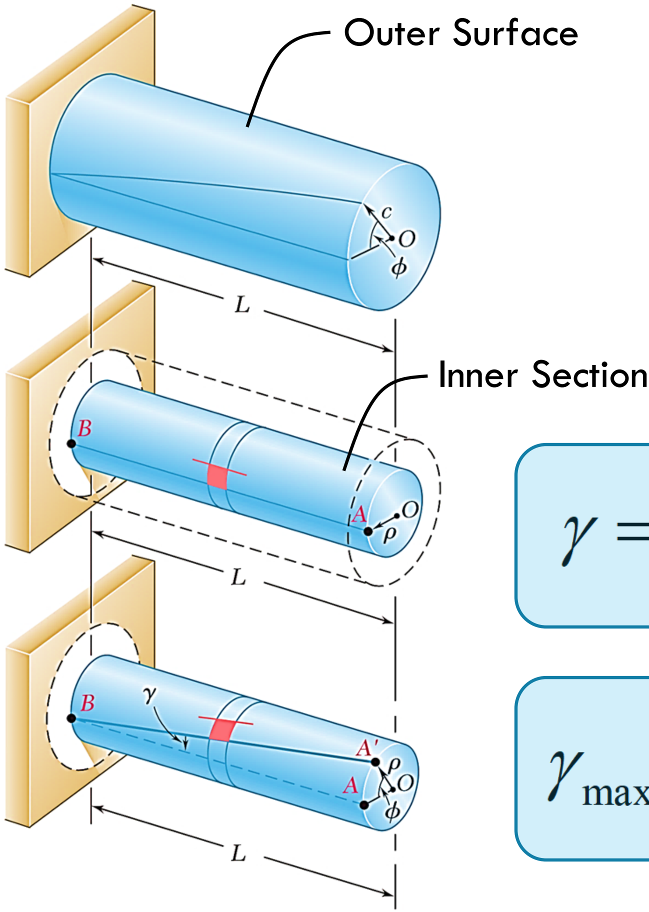


$$\tau_{\max} = \frac{Tr}{J}$$

$$\tau = \frac{T\rho}{J} = \frac{\rho}{r}\tau_{\max}$$

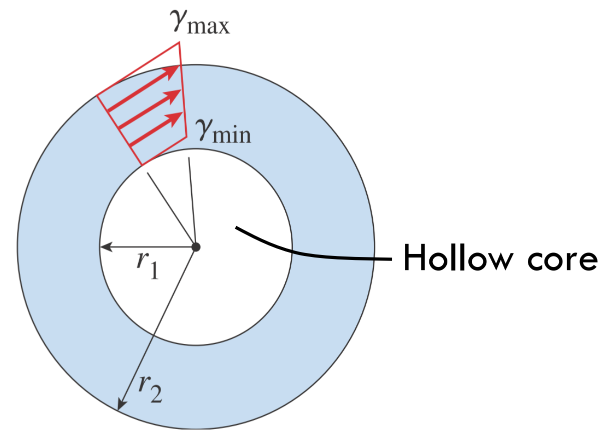
$$\phi = \frac{TL}{JG}$$

OVERALL SUMMARY



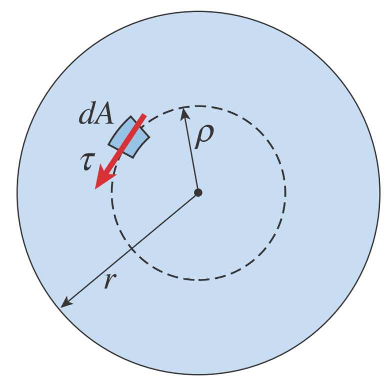
$$\gamma = \frac{\rho\phi}{L}$$

$$\gamma_{\max} = \frac{r\phi}{L}$$



$$\gamma_{\min} = \frac{r_1}{L}\phi = \frac{r_1}{r_2}\gamma_{\max}$$

$$\tau_{\min} = \frac{r_1}{r_2}\tau_{\max}$$



$$\tau_{\max} = \frac{Tr}{J}$$

$$\tau = \frac{T\rho}{J} = \frac{\rho}{r}\tau_{\max}$$

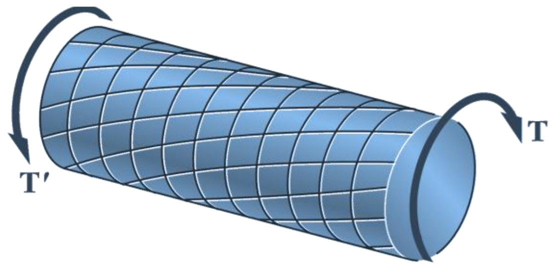
$$\phi = \frac{TL}{JG}$$

ANALOGIES WITH AXIAL LOADING

- Axial

$$\delta = \frac{PL}{AE}$$

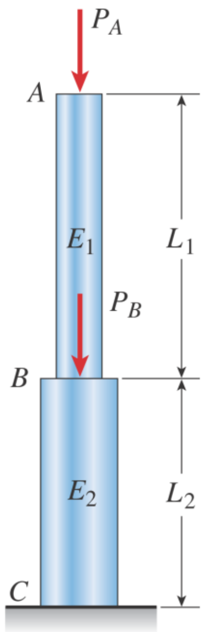
- Torsional (Pure Shear)



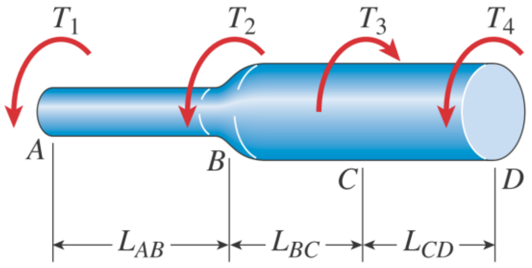
$$\phi = \frac{TL}{JG}$$

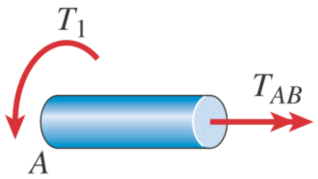
NON-UNIFORM TORSION

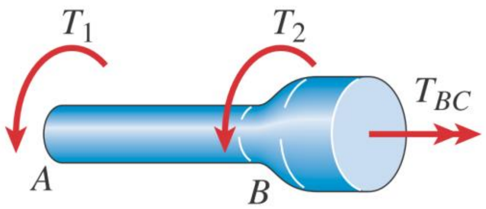
Remember the Axial Case ?

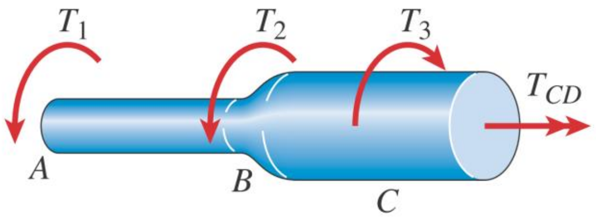


$$\delta = \sum_{i=1}^{n} \delta_i = \sum_{i=1}^{n} \frac{N_i L_i}{A_i E_i}$$





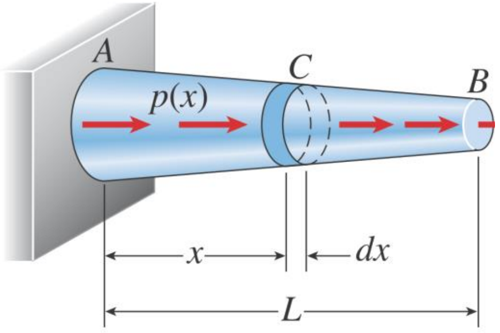




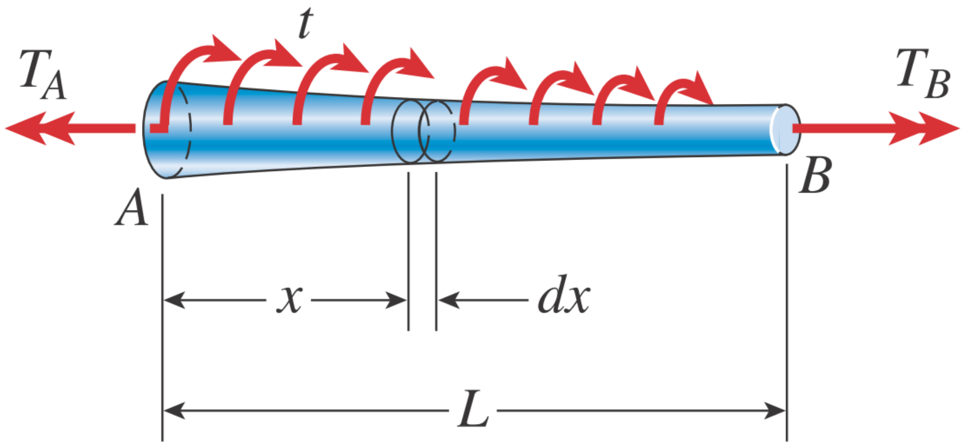
$$\phi = \sum_{i=1}^{n} \phi_i = \sum_{i=1}^{n} \frac{T_i L_i}{J_i G_i}$$

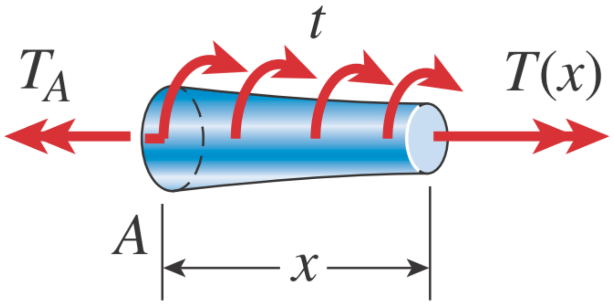
NON-UNIFORM TORSION

Remember the Axial Case ?



$$\delta = \int_0^L d\delta = \int_0^L \frac{N(x)dx}{EA(x)}$$

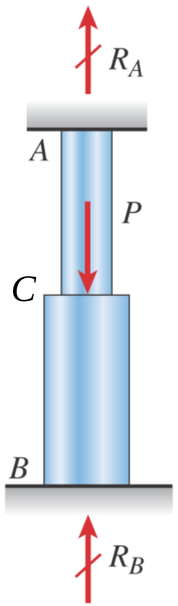




$$\phi = \int_0^L d\phi = \int_0^L \frac{T(x)dx}{GJ(x)}$$

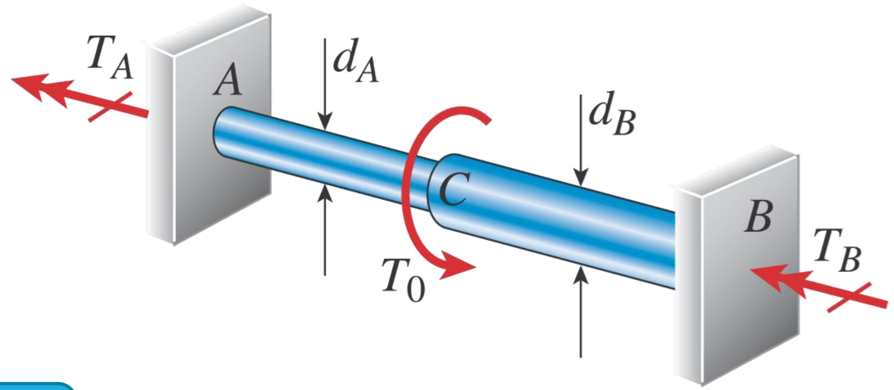
STATICALLY INDETERMINATE MEMBERS

Remember the Axial Case ?



We used Equations of Compatibility!

$$\delta_{AB} = \delta_{AC} + \delta_{CB} = 0$$



Equilibrium $\Rightarrow$ $T_0 = T_A + T_B$

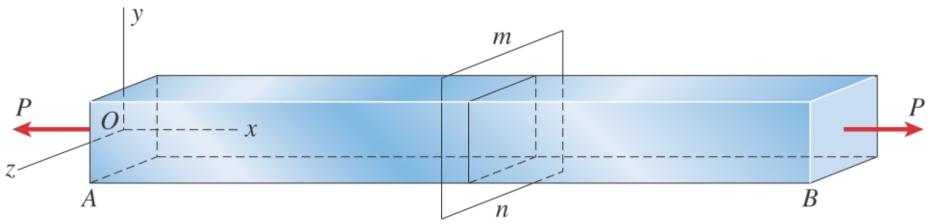
Compatibility $\Rightarrow$ $\phi_{AB} = \phi_{AC} + \phi_{CB} = 0$

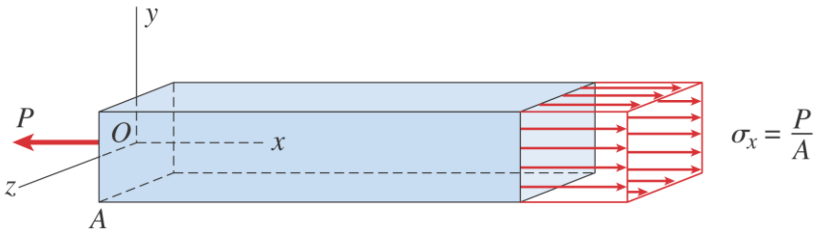
$$\phi_{AC} = \frac{T_A L_{AC}}{J_{AC} G} \text{ and } \phi_{CB} = -\frac{T_B L_{CB}}{J_{CB} G}$$

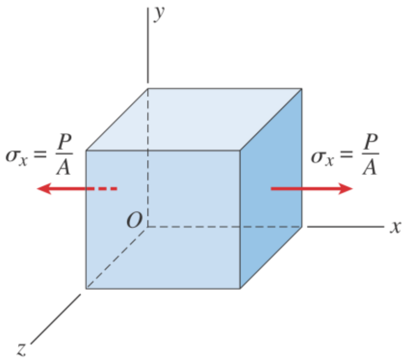
Use all equations to solve for T_A and T_B

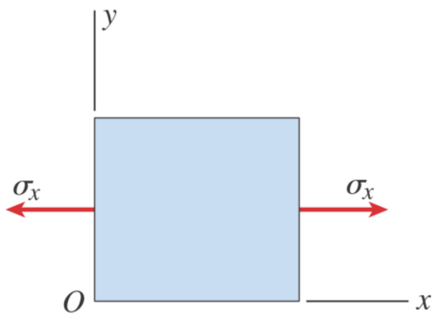
STRESSES ON INCLINED SECTIONS

Axial Loading Case



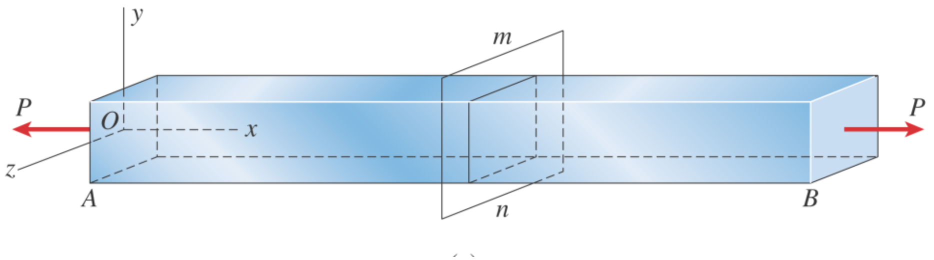


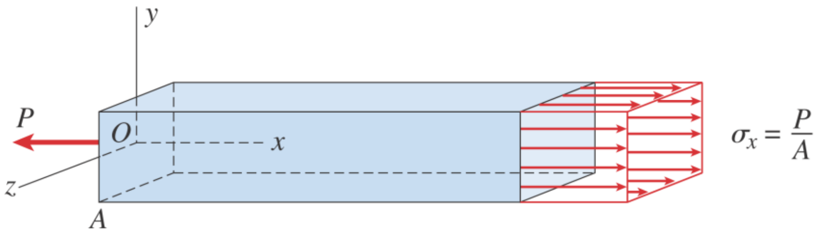




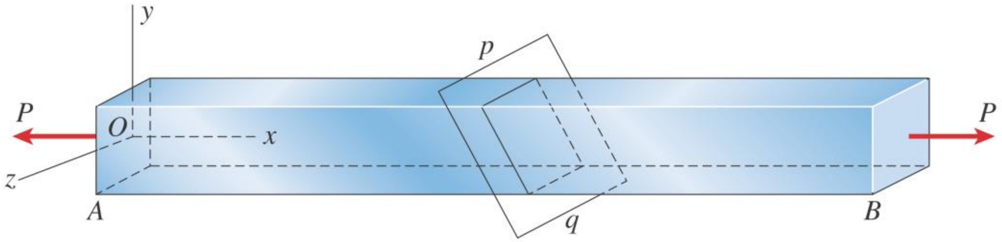
STRESSES ON INCLINED SECTIONS

<u>Axial Loading Case</u>



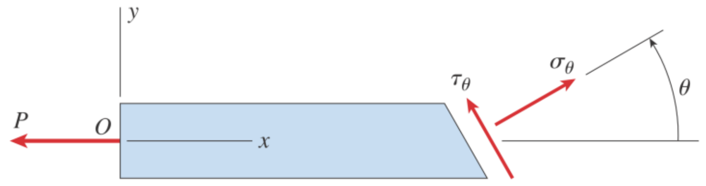


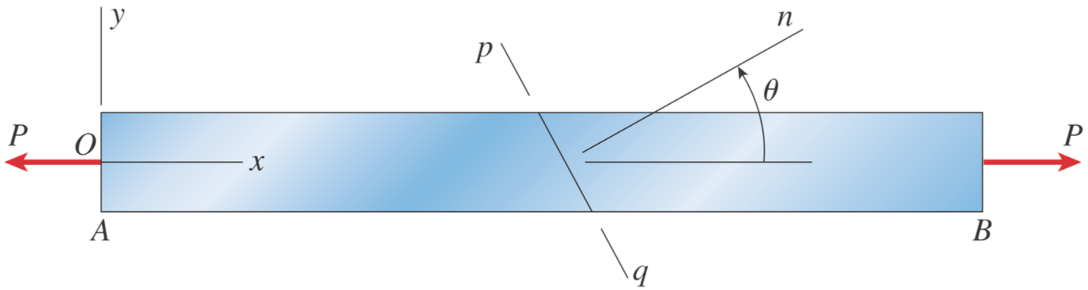
What if?



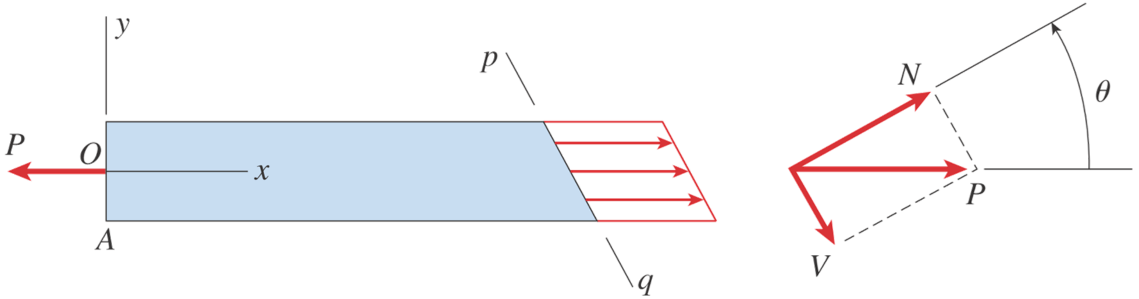
STRESSES ON INCLINED SECTIONS

+ve sign convention





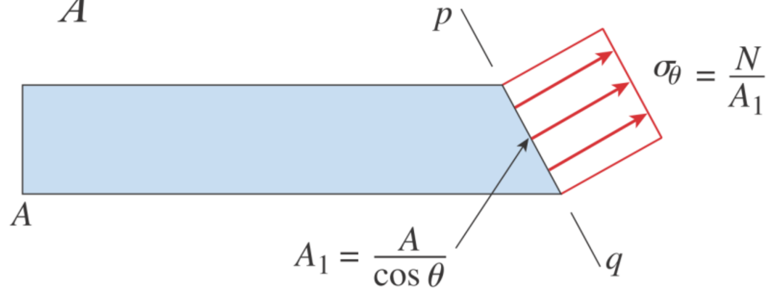
(a)



(b)

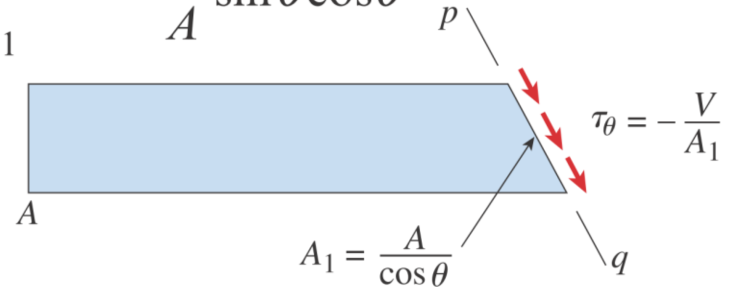
$$N = P\cos\theta \qquad V = P\sin\theta$$

$$\sigma_\theta = \frac{N}{A_1} = \frac{P}{A}\cos^2\theta$$



(c)

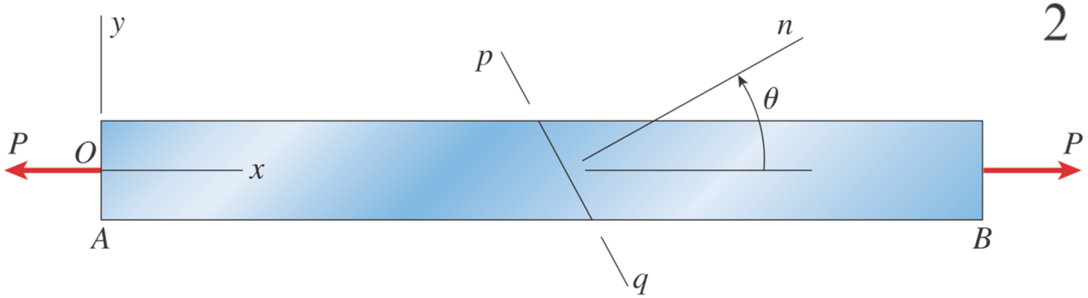
$$\tau_\theta = -\frac{V}{A_1} = -\frac{P}{A}\sin\theta\cos\theta$$



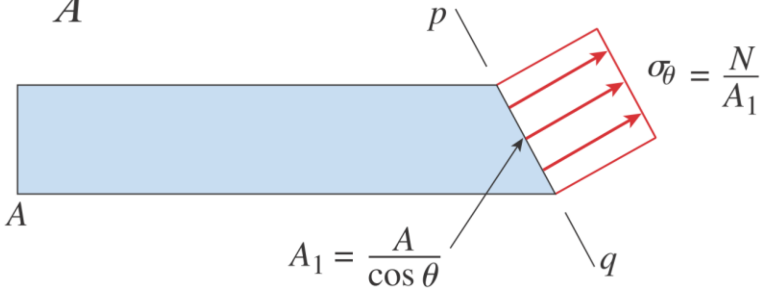
(d)

STRESSES ON INCLINED SECTIONS

$$\frac{\sigma_x}{2}(1 + \cos 2\theta) \Leftarrow \sigma_\theta = \frac{N}{A_1} = \frac{P}{A}\cos^2\theta$$

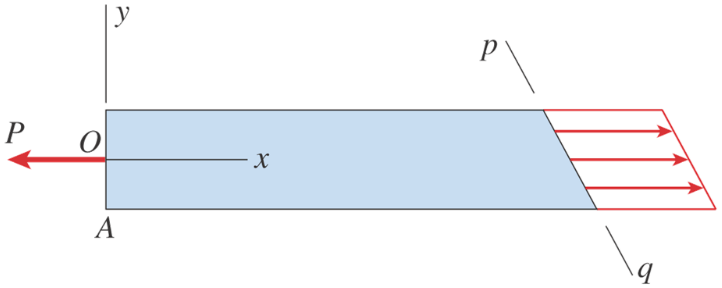


(a)

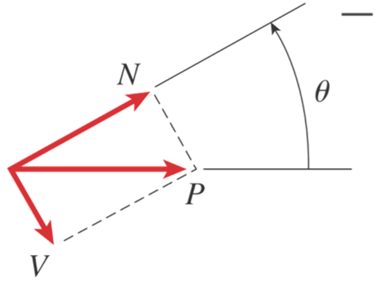


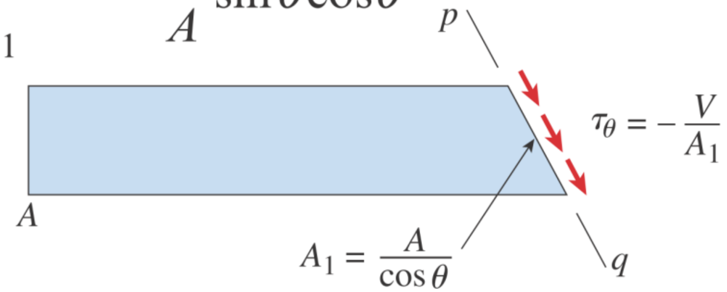
(c)

$$-\frac{\sigma_x}{2}(\sin 2\theta) \Leftarrow \tau_\theta = -\frac{V}{A_1} = -\frac{P}{A}\sin\theta\cos\theta$$



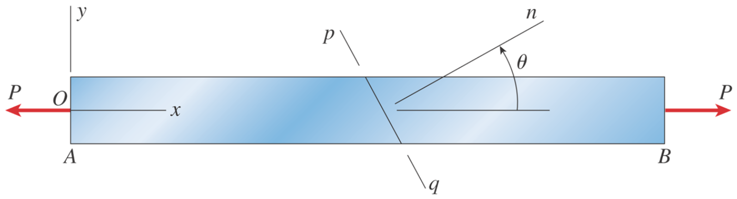
(b)



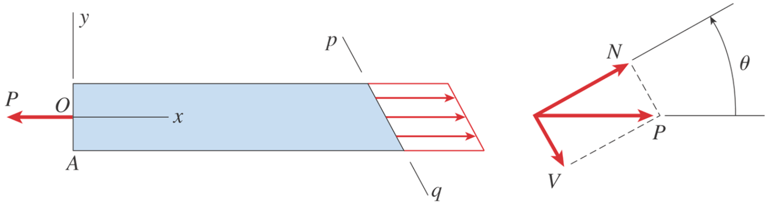


(d)

STRESSES ON INCLINED SECTIONS



(a)

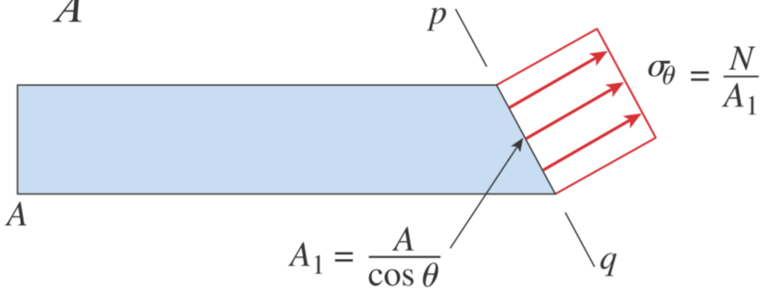


(b)

$$\frac{\sigma_x}{2}(1 + \cos 2\theta) \Leftarrow \sigma_\theta = \frac{N}{A_1} = \frac{P}{A}\cos^2\theta$$

Maximum normal stress occurs at $\theta = 0°$

$$\sigma_{max} = \sigma_x$$

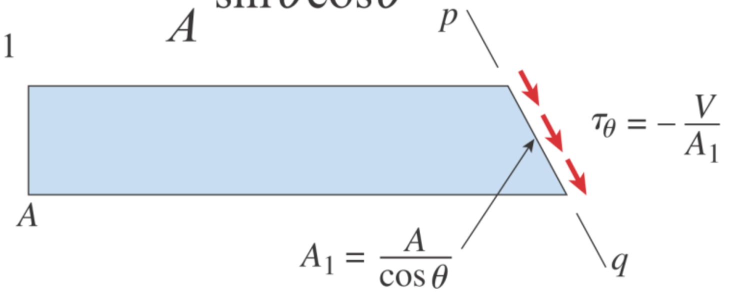


(c)

$$-\frac{\sigma_x}{2}(\sin 2\theta) \Leftarrow \tau_\theta = -\frac{V}{A_1} = -\frac{P}{A}\sin\theta\cos\theta$$

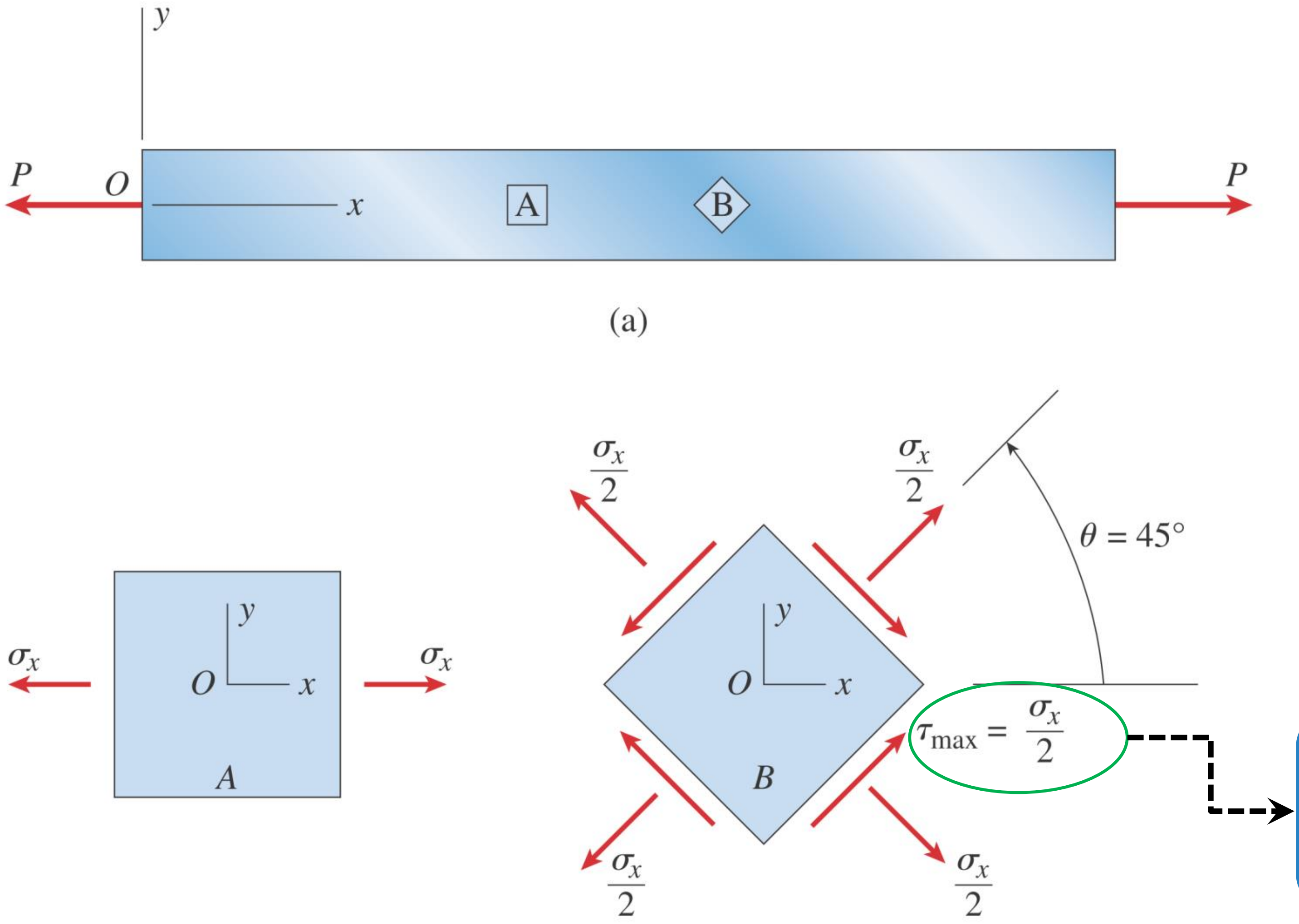
Maximum shear stress occurs at $\theta = 45°$

$$\tau_{max} = \frac{\sigma_x}{2}$$



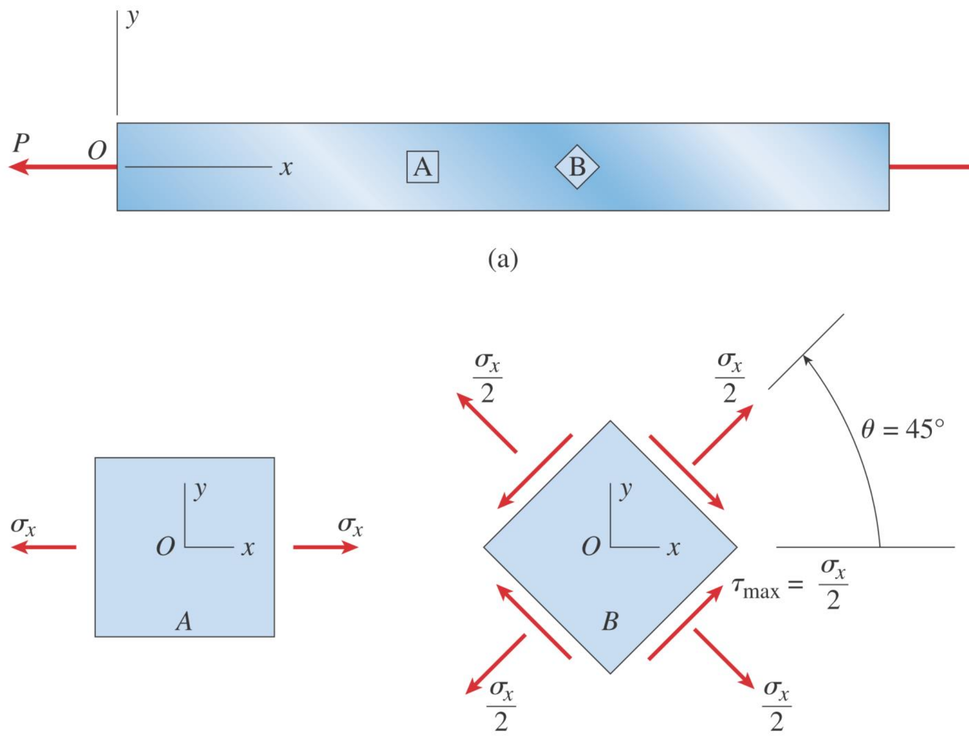
(d)

STRESSES ON INCLINED SECTIONS

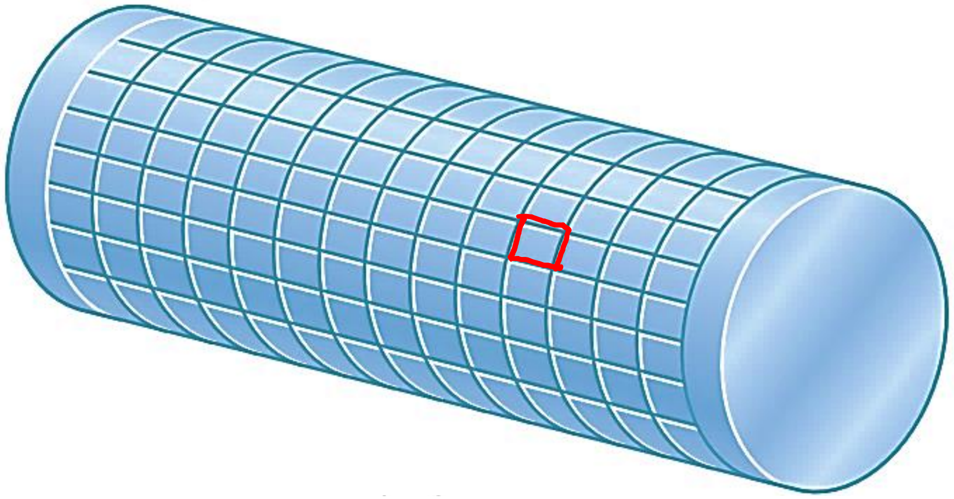


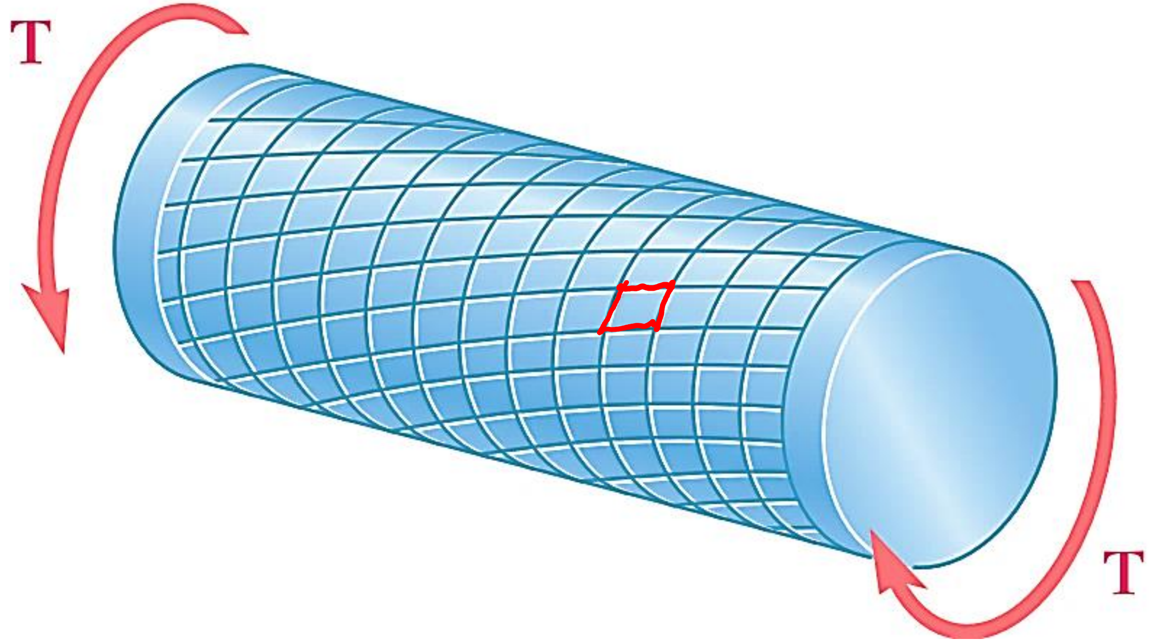
May lead to failure in elements weak in shear, when loaded axially!

STRESSES ON INCLINED SECTIONS

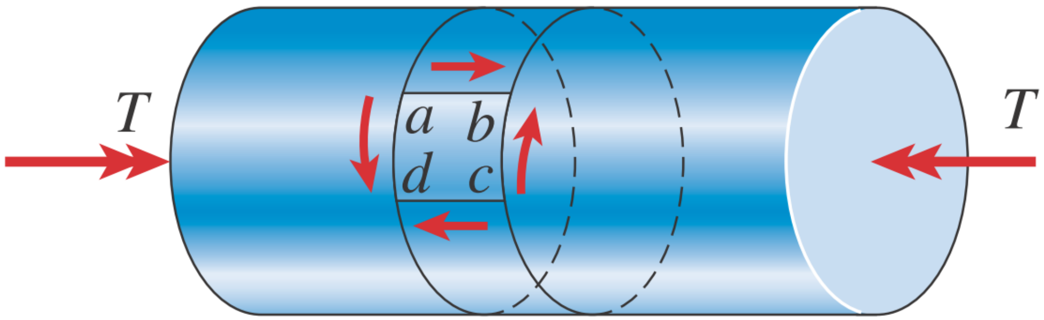


CASE OF TORSION

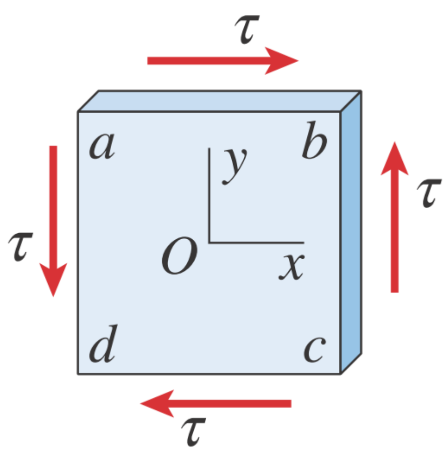




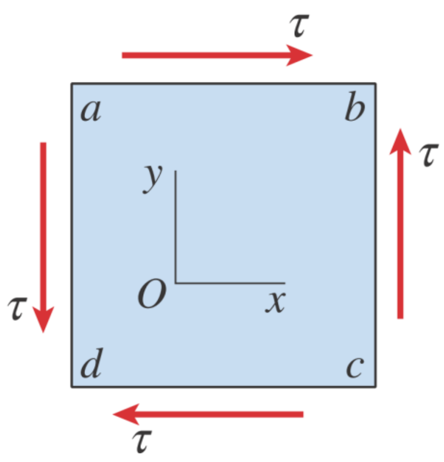
CASE OF TORSION



Pure Torsion



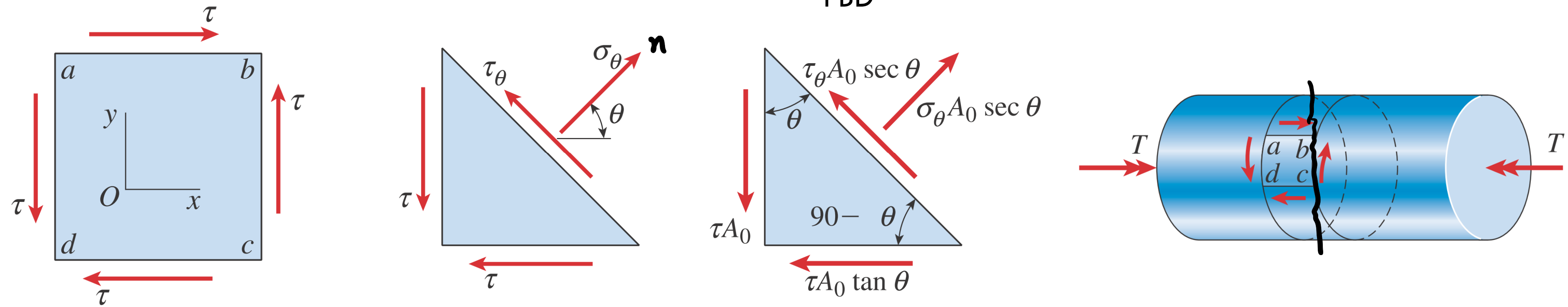
Element in Pure Shear (3D)



Element in Pure Shear (2D)

STRESSES ON INCLINED SECTIONS

FBD



<u>Plane with Normal at angle θ</u>

$\sigma_\theta = \sigma_{max}$ @ $\theta = 45^\circ$

$\hookrightarrow \sigma_{max} = \tau$

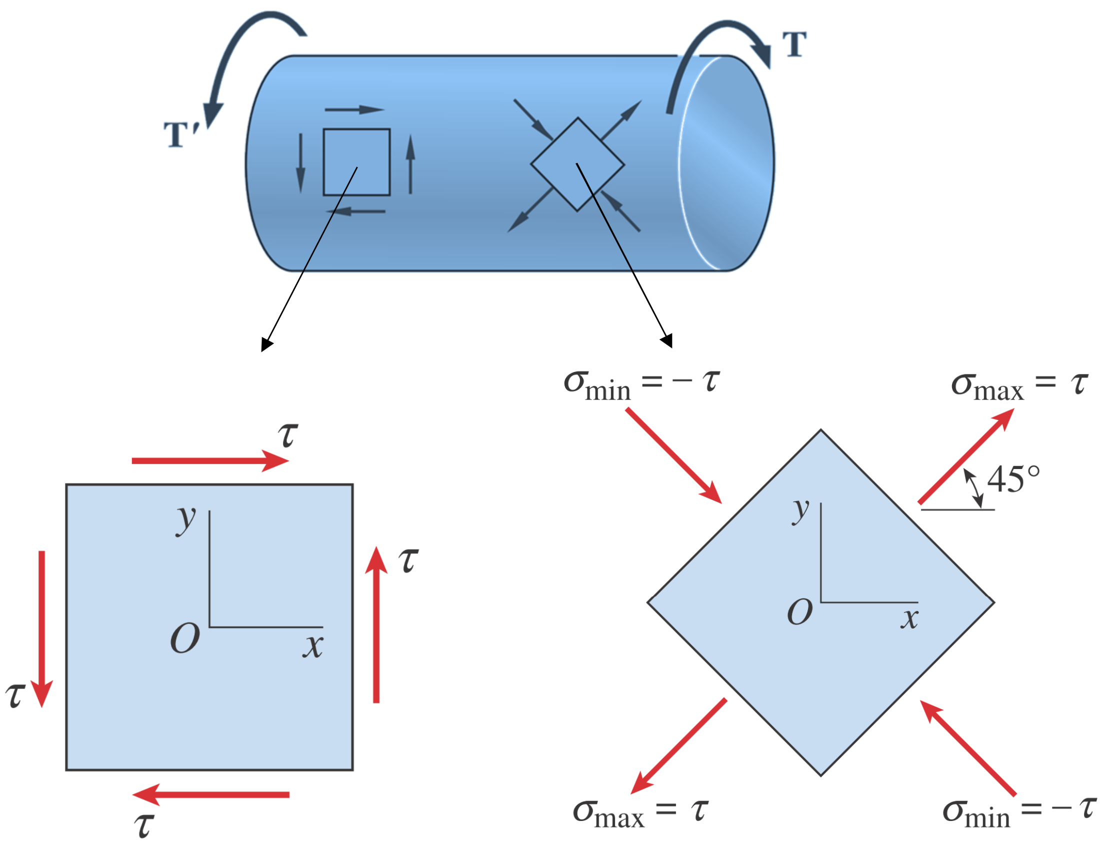
Following force balance equilibrium equations:

$$\sigma_\theta = 2\tau \sin\theta \cos\theta \qquad \boxed{\sigma_\theta = \tau \sin 2\theta}$$

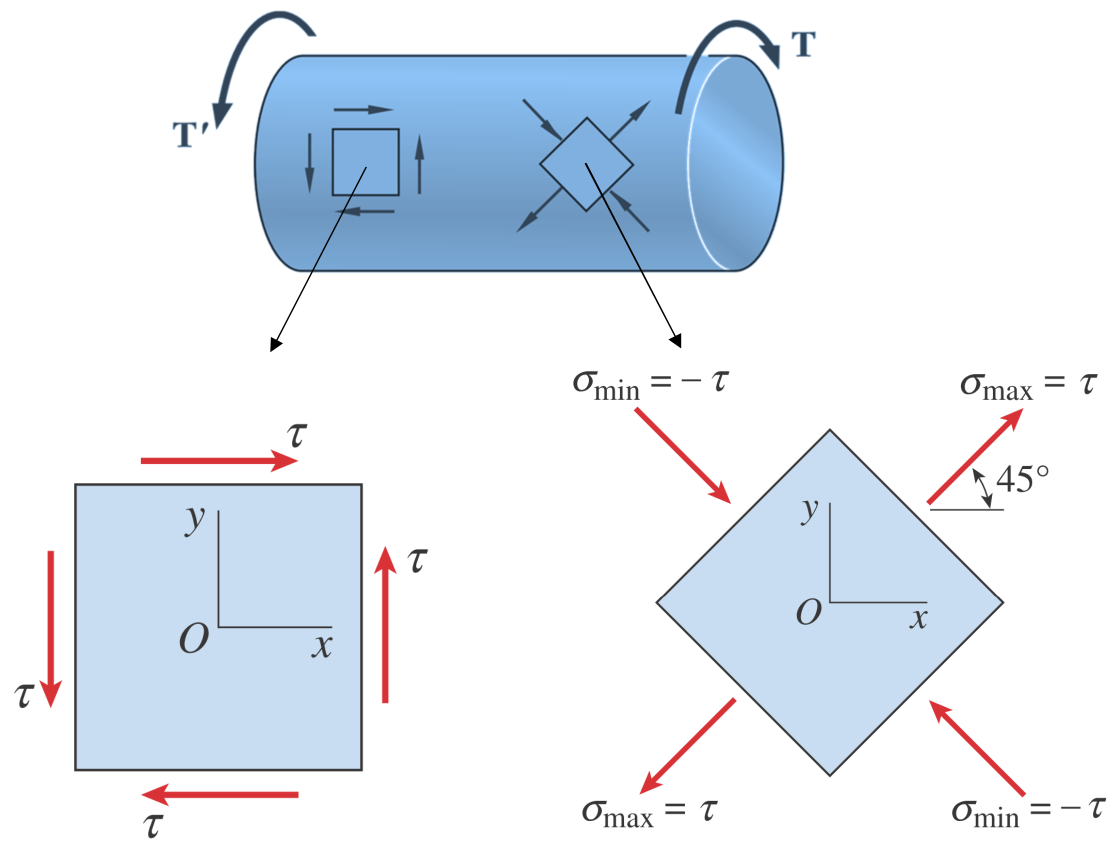
$$\tau_\theta = \tau(\cos^2\theta - \sin^2\theta) \qquad \boxed{\tau_\theta = \tau \cos 2\theta}$$

τ_θ @ $\theta = 45^\circ \Rightarrow \tau_\theta = 0$

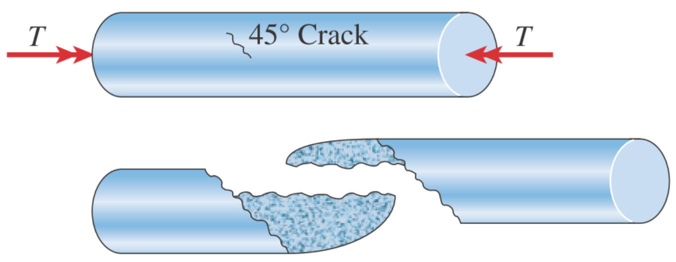
STRESSES ON INCLINED SECTIONS



STRESSES ON INCLINED SECTIONS

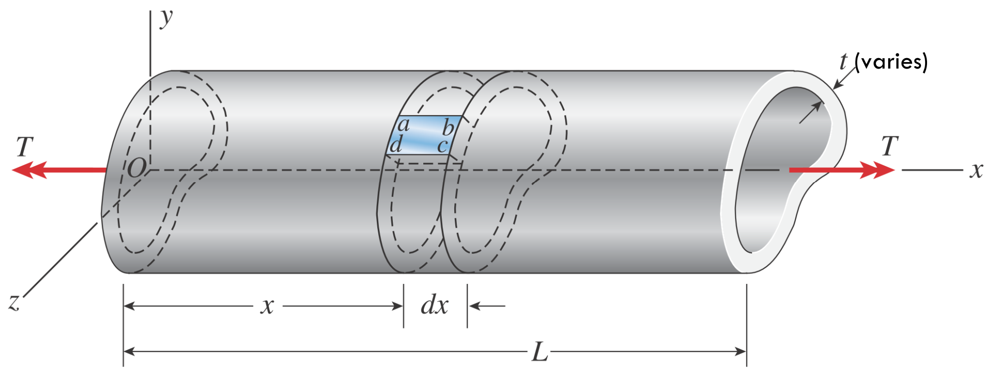


Brittle materials (eg. concrete and chalk) are weak in tension. Hence under pure torsion, it tends to fail along the 45° plane

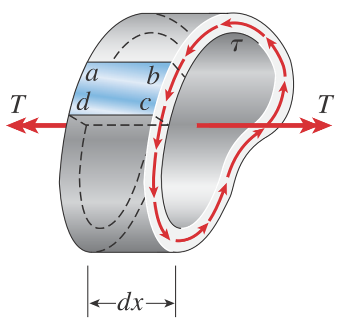


THIN WALLED TUBES

- Shear Stress and Shear Flow

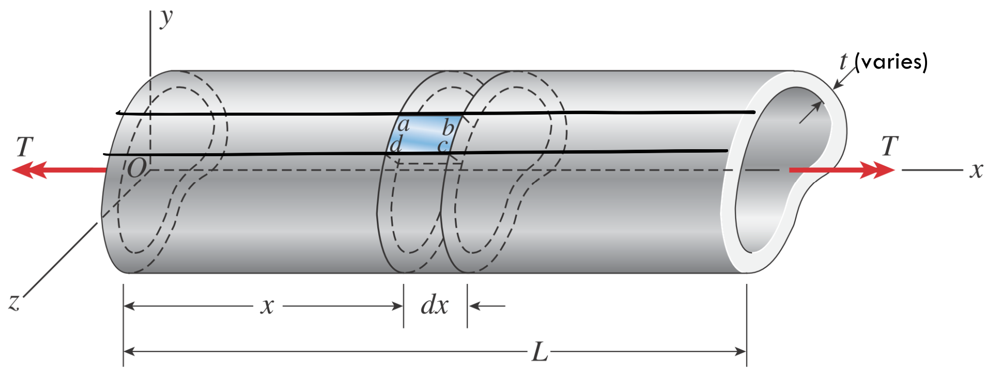


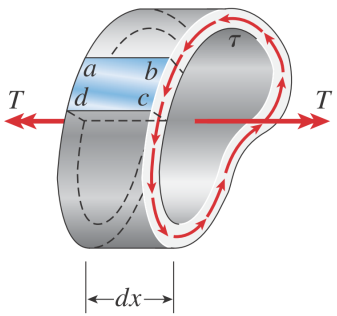
Wall is thin – Assume shear stress is constant through the thickness

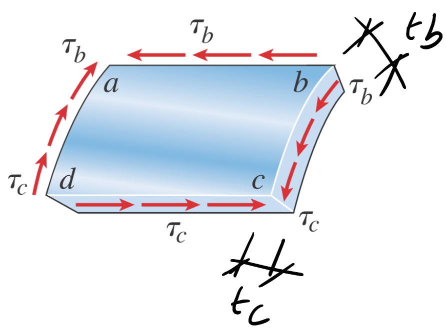


THIN WALLED TUBES

- Shear Stress and <u>Shear Flow</u>



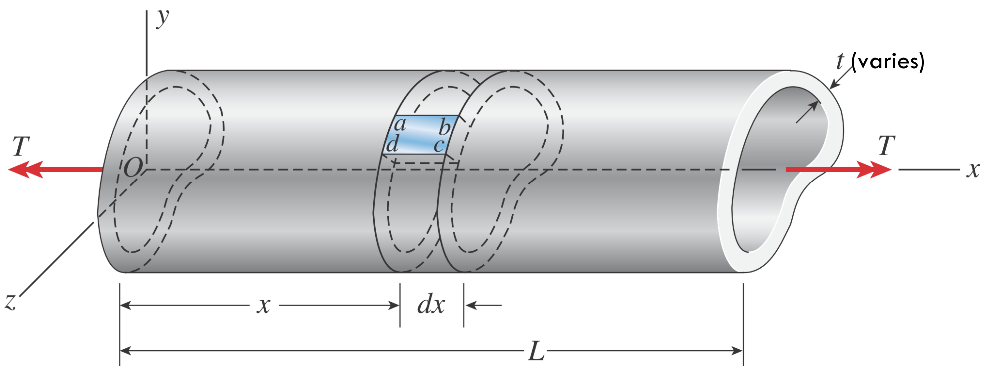




Wall is thin – Assume shear stress is constant through the thickness

THIN WALLED TUBES

- Shear Stress and <u>Shear Flow</u>



Wall is thin – Assume shear stress is constant through the thickness

$$F_b = \tau_b \times t_b \times dx \quad \text{——} \quad (1)$$

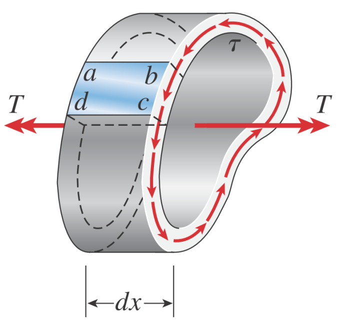
$$F_c = \tau_c \times t_c \times dx \quad \text{——} \quad (2)$$

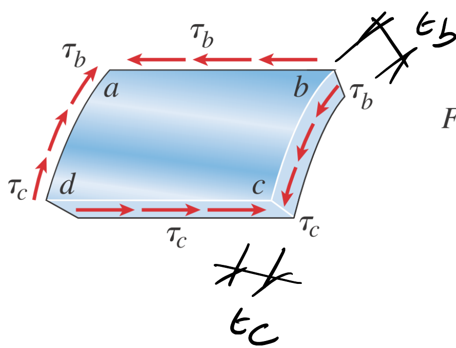
$$\Sigma F_x = 0$$

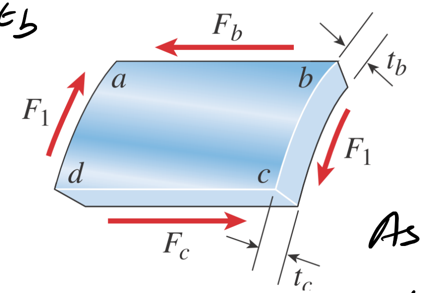
$$\Rightarrow \tau_b \times t_b \times \cancel{dx} = \tau_c \times t_c \times \cancel{dx}$$

$$\Rightarrow \boxed{\tau_b t_b = \tau_c t_c = q}$$

Shear Flow



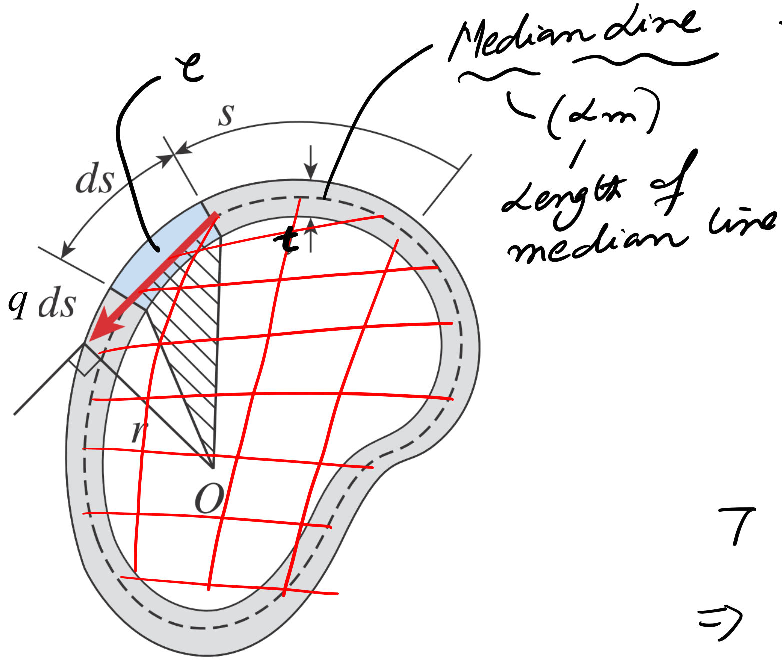




As long as the thickness is small, product of τ & $t \rightarrow$ constant

THIN WALLED TUBES

- Torsion Formula



$$dT = \tau t\, ds\, r = q r\, ds$$

$$\Rightarrow T = \int_0^{L_m} q r\, ds = q \int_0^{L_m} r\, ds$$

$A_s \rightarrow$ Area of shaded Δ

$$\Rightarrow A_s = \frac{1}{2} \times ds \times r$$

$$\Rightarrow r\, ds = 2A_s$$

$$\Rightarrow \int_0^{L_m} r\, ds = 2A_m$$

Area enclosed by the median line

(NOT the cross sectional area of tube)

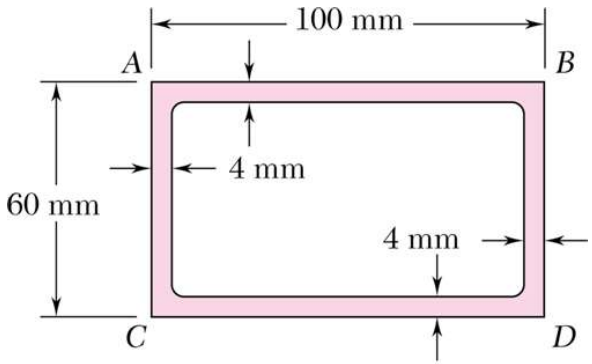
$$T = q \times 2A_m$$

$$\Rightarrow q = \frac{T}{2A_m} \Rightarrow \tau t = \frac{T}{2A_m} \Rightarrow \boxed{\tau = \frac{T}{2tA_m}}$$

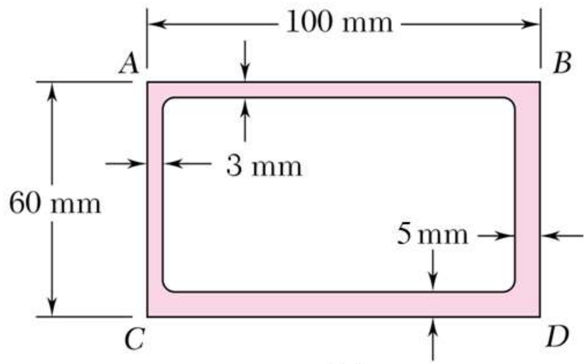
Assuming 't' remains same throughout

PROBLEM

Extruded aluminum tubing with rectangular cross-section has torque loading 2.7 kN-m. Find shearing stress in each of four walls considering (a) uniform wall thickness of 4 mm and (b) wall thicknesses of 3 mm on *AB* and *AC* and 5 mm on *CD* and *BD*.



(a)



(b)

① $A_m = (100-4)(60-4) = 5376\,mm^2.$

$$\tau = \frac{T}{2 t A_m}$$

$$= \frac{2.7\times10^3\times10^3}{2\times4\times5376}\ N/mm^2$$

$$= 62.77\ MPa.$$

② $A_m = (100-1.5-2.5)(60-1.5-2.5)$

$= 5376\ mm^2$

$$\tau_{AB} = \tau_{AC} = \frac{T}{2 t_1 A_m}$$

$$= \frac{2.7\times10^6}{2\times3\times5376} = 83.7\ MPa.$$

$$\tau_{BD} = \tau_{CD} = \frac{2.7\times10^6}{2\times5\times5376} = 50.2\ MPa$$