APPENDIX B

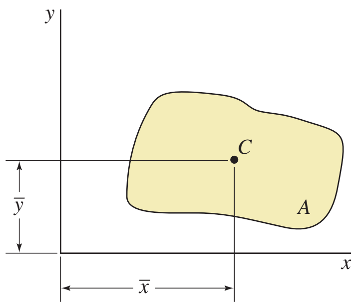
Theory of Properties of Areas

B-1 Centroid and Second Moment of Inertia

Here we define and explain geometric properties of areas that are relevant to *Mechanics of Materials*. The properties defined are tabulated for simple shapes in Appendix C.

1. Define the centroid.

The centroid C corresponds to the geometric center of an area A. For simple shapes like rectangles and circles, C is at the center. In general, the x- and y-coordinates of the centroid, $\bar{x}$ and $\bar{y}$, are equal to values of the x- and y-coordinates *averaged* over the area A. Since the average is over an infinite number of points, it is found, in general, by integration.



$$\bar{x} = \frac{1}{A}\int_A x\,dA$$

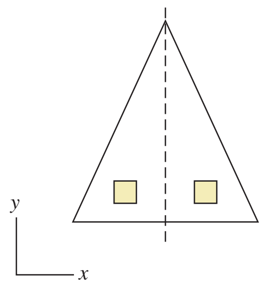
$$\bar{y} = \frac{1}{A}\int_A y\,dA$$

The centroid has a fixed position relative to the area. But the values $\bar{x}$ and $\bar{y}$ depend on the location of the area relative to the x-y origin.

2. Recognize that the centroid lies on a symmetry line, when present.

An area with a line of symmetry contains corresponding pairs of points, each of which has an average coordinate on the symmetry line. So $\bar{x}$ of the centroid is located on the symmetry line.

Whether an area is symmetric or not, if the x or y origin is located at the centroid, then the respective integral $\int_A x\,dA$ or $\int_A y\,dA$ is zero.



3. Define the second moment of inertia.

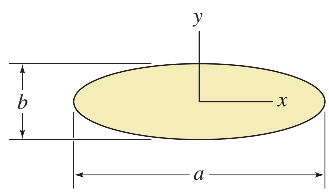
The second moment of inertia (sometimes referred to as a second moment of area) quantifies how spread out the area is from a given plane, often from planes that pass through the centroid. I_x captures the spread about the x plane or in the y-direction. I_y captures the spread about the y plane or in the x-direction. By integrating (averaging) the square of x (or y), the second moment of inertia captures the spread in both directions from the plane.

$$I_x = \int_A y^2\,dA$$

$$I_y = \int_A x^2\,dA$$

4. Observe the properties of the second moment of inertia.

As an example, here are the second moments of inertia of this ellipse about its center (centroid). Note the units are length to the fourth power.



$$I_x = \frac{\pi a b^3}{4}$$

$$I_y = \frac{\pi a^3 b}{4}$$

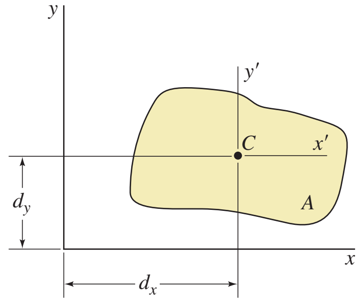
Since $a > b$ in the case drawn, the ellipse is spread more in the x-direction than the y. Correspondingly, $I_y > I_x$ since the larger dimension, a, is cubed.

5. Distinguish area from the moment of inertia.

Two areas can be equal in area but have very different moments of inertia. The left figure has the same area as the right, but it has a much larger I_x and I_y since the area is spread from the center. The bending and twisting resistance of a rod with the left cross-section would be greater because its moments of inertia are greater.

6. Calculate the second moments of inertia about other axes using the parallel axis theorem.

Sometimes one knows the second moment of inertia about a plane through the centroid, but wants the second moment of inertia about a different parallel plane. Take the origin of the axes x'-y' to be centered at the centroid. Say we want the moment of inertia about different axes, x-y, located at distances d_x and d_y away.



$$I_x = \int_A (y' + d_y)^2 dA = \int_A y'^2 dA + 2d_y \int_A y' dA + d_y^2 \int_A dA$$

The integral over y' is zero because y' is at the centroid. The integral over y'^2 is designated as I_{cx} because it is the second moment of inertia about the centroid. The last integral equals simply A. A similar formula can be found for the shift from x' to x.

The pair of formulas for finding moments of inertia for axes away from the centroid is called the *Parallel Axis Theorem.*

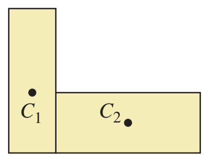
$$I_x = I_{cx} + Ad_y^2 \qquad I_y = I_{cy} + Ad_x^2$$

7. Recognize composite areas and use their individual centroids and moments of inertia appropriately to find those of the whole area.

Sometimes an area is a combination, or composite, of simpler areas. If the centroids and moments of inertia of the simpler areas are known, then those of the composite can be found.

This figure

can be recognized as a combination of two rectangles, each having an obvious centroid.



The centroid of the composite is the average of centroids, each weighted by its area.

$$\bar{x} = \frac{1}{A}\int_A x dA = \frac{1}{A_1 + A_2}\left[\int_{A_1} x dA + \int_{A_2} x dA\right] = \frac{A_1\bar{x}_1 + A_2\bar{x}_2}{A_1 + A_2}s \qquad \bar{y} = \frac{1}{A}\int_A y dA = \frac{A_1\bar{y}_1 + A_2\bar{y}_2}{A_1 + A_2}$$

Often one wants the moments of inertia about the centroid of the composite. It is the sum of the moments of inertia of the two areas.

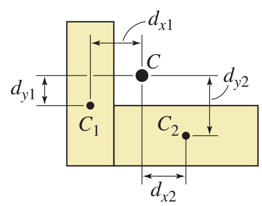
$$I_x = \int_A y^2 dA = \int_{A_1} y^2 dA + \int_{A_2} y^2 dA = I_{x1} + I_{x2}$$

But, the moments of inertia tabulated for simple areas (e.g., in Appendix C) are about the simple area's centroid. We can use the parallel axis theorem to convert the moments of inertia so they are about the composite centroid C:

$$I_{x1} = I_{cx1} + A_1 d_{y1}^2 \qquad I_{x2} = I_{cx2} + A_2 d_{y2}^2 \qquad I_{y1} = I_{cy1} + A_1 d_{x1}^2 \qquad I_{y2} = I_{cy2} + A_2 d_{x2}^2$$

$$I_x = I_{cx1} + A_1 d_{y1}^2 + I_{cx2} + A_1 d_{y2}^2$$

$$I_y = I_{cy1} + A_1 d_{x1}^2 + I_{cy2} + A_1 d_{x2}^2$$



As seen in Example B-1.1, composite areas can have holes of a simple shape. The contributions of the holes are subtracted from the summations for the centroids and moments of inertia.

>>End B-1