Lecture 8 & 9: Fuzzy Relation and Composition

(EC1710)

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Fuzzy Relation

- If A and B are two CRISP sets. The Cartesian Product $A \times B$ is a collection of ordered pairs such that

  $$A \times B = \{(a, b) \parallel a \in A \; and \; b \in B\}$$

  $A \times B$ provides a mapping from $a \in A \; to \; b \in$ B

- A particular mapping so mentioned is called a Relation.
- Example: Given two Crisp Sets A={1, 2, 3, 4}; B={3, 5, 7}

  Then $A \times B$ = {(1,3), (1,5), (1,7), (2,3), (2,5), (2,7), (3,3), (3,5), (3,7), (4,3), (4,5), (4,7)}

Let us define a Relation as $\boldsymbol{R} = \{(a, b) \mid b = a + 1, (a, b) \in A \times B\}$

In this case, R={(2,3), (4,5)}; we can represent the Relation R in a matrix form as

$$\begin{array}{cc} & \begin{array}{ccc} 3 & 5 & 7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Fuzzy Relation

- Fuzzy **Relation (R)** is a fuzzy set defined on the cartesian product of Crisp set χ_1, $\chi_2, \chi_3 \cdots\cdots \chi_n$
- Each element of a fuzzy **relation** has varying degrees of membership within the relationship
- The membership values indicates the strength of the **Relation** between the elements.

Example 1: Given two fuzzy sets
X={John, Mary, Preeti}   Y={ Ritu, Priya, Jack}
Define a relation R which represents closeness of the two fuzzy sets.

$$\boldsymbol{R} = \boldsymbol{X} \times \boldsymbol{Y} = \begin{array}{c c} & \begin{array}{ccc} \text{Ritu} & \text{Priya} & \text{Jack} \end{array} \\ \begin{array}{c} \text{John} \\ \text{Mary} \\ \text{Preeti} \end{array} & \begin{bmatrix} 0.9 & 0.8 & 0.5 \\ 1 & 0.3 & 0.1 \\ 0.7 & 0 & 0.6 \end{bmatrix} \end{array}$$

Fuzzy Relation

Example 2: Given two fuzzy sets
X={Typhoid, Viral, Cold}
Y={ Running nose, High temperature, Shivering}
Define a relation R which represents closeness of the two fuzzy sets.

$$\boldsymbol{R} = \boldsymbol{X} \times \boldsymbol{Y} = \begin{array}{r} \\ \text{Typhoid} \\ \text{Viral} \\ \text{Cold} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Running Nose} & \text{high temp.} & \text{shivering} \end{array} \\ \begin{bmatrix} 0.1 & 0.4 & 0.8 \\ 0.2 & 0.9 & 0.7 \\ 0.9 & 0.4 & 0.6 \end{bmatrix} \end{array}$$

Summary:

Fuzzy Relation – relates elements of one universe X to those of another universe

Y through the cartesian product of two universe.

$\mathrm{A} \in X$, $\mathrm{B} \in Y$, $\quad \mathrm{R} = A \times B \subset X * Y$

Fuzzy Relation

Fuzzy relation is defined as the Cartesian Product of two fuzzy sets.

A is a fuzzy set on the universe of Discourse X with $\mu_A(x) \parallel x \in X$

B is a fuzzy set on the universe Y with $\mu_B(y) \parallel y \in Y$

Then, $\boldsymbol{R} = A \times B \quad \subset \; X \times Y \quad ; \qquad \mu_R : A \times B \rightarrow [0, 1]$

Where R has its membership function given by

$$\boldsymbol{R} = \{((x, y), \mu_R(x, y)) \mid \mu_R(x, y) \geq 0 \,, \; x \in A, \; y \in B\}$$

$$\mu_{R(x,y)} = \mu_{A \times B}(x, y) = \min\{(\mu_A(x), \mu_B(y)\}$$

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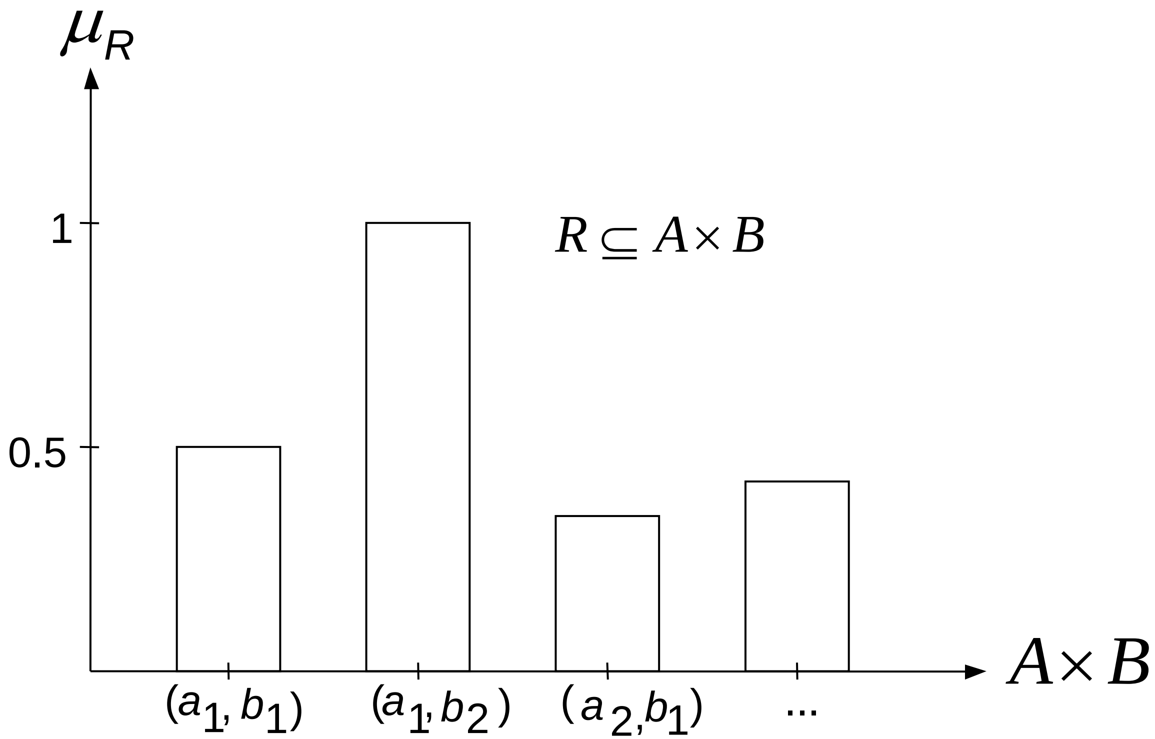
Fuzzy Relation



Fuzzy relation as a fuzzy set

Fuzzy Relation

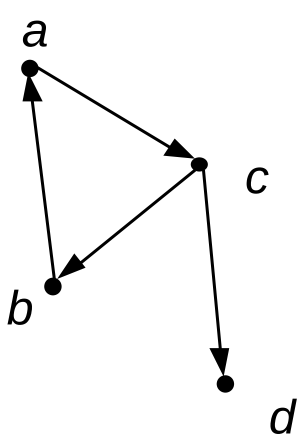
- **Example**

  Crisp relation R

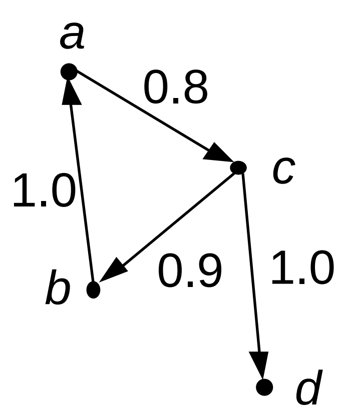
  $\mu_R(a, c) = 1$, $\mu_R(b, a) = 1$, $\mu_R(c, b) = 1$ and $\mu_R(c, d) = 1$.

  Fuzzy relation R

  $\mu_R(a, c) = 0.8$, $\mu_R(b, a) = 1.0$, $\mu_R(c, b) = 0.9$, $\mu_R(c, d) = 1.0$



(a) Crisp relation



(b) Fuzzy relation

A \ A	a	b	c	d
a	0.0	0.0	0.8	0.0
b	1.0	0.0	0.0	0.0
c	0.0	0.9	0.0	1.0
d	0.0	0.0	0.0	0.0

corresponding matrix

crisp and fuzzy relations

Fuzzy Relation

Operation of Fuzzy Relation

1) Union relation

$\forall\ (x, y) \in A \times B$

$\mu_{R \cup S}(x, y) = \text{Max}\ [\mu_R(x, y), \mu_S(x, y)] = \mu_R(x, y) \vee \mu_S(x, y)$

2) Intersection relation

$\mu_{R \cap S}(x) \quad = Min\ [\mu_R(x, y), \mu_S(x, y)] = \mu_R(x, y) \wedge \mu_S(x, y)$

3) Complement relation

$\forall\ (x, y) \in A \times B$

$\mu_R(x, y) = 1 - \mu_R(x, y)$

4) Inverse relation

For all $(x, y) \subseteq A \times B$, $\quad \mu_R^{-1}(y, x) = \mu_R(x, y)$

Fuzzy Relation

Examples

M_R	a	b	c
1	0.3	0.2	1.0
2	0.8	1.0	1.0
3	0.0	1.0	0.0

M_S	a	b	c
1	0.3	0.0	0.1
2	0.1	0.8	1.0
3	0.6	0.9	0.3

$M_{R \cup S}$	a	b	c
1	0.3	0.2	1.0
2	0.8	1.0	1.0
3	0.6	1.0	0.3

$M_{R \cap S}$	a	b	c
1	0.3	0.0	0.1
2	0.1	0.8	1.0
3	0.0	0.9	0.0

$M_{\bar{R}}$	a	b	c
1	0.7	0.8	0.0
2	0.2	0.0	0.0
3	1.0	0.0	1.0

Fuzzy Relation

Examples

#1

$$\boldsymbol{A = \{(a_1, 0.2), (a_2, 0.7), a_3, 0.4)\}}$$

$$\boldsymbol{B = \{(b_1, 0.5), (b_2, 0.6))\}}$$

$$\mathrm{R} = \mathrm{A} \times \mathrm{B} = \begin{array}{c} \\ \boldsymbol{a_1} \\ \boldsymbol{a_2} \\ \boldsymbol{a_3} \end{array} \begin{array}{c} \begin{array}{cc} \boldsymbol{b_1} & \boldsymbol{b_2} \end{array} \\ \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} \end{array}$$

Composition of fuzzy relation

If R $\Longrightarrow$Relation on universe X and Y

S $\Longrightarrow$ Relation on universe Y and Z

Then Composition T of fuzzy relation R and S is denoted as

$$\mathbf{T = R \circ S}$$

It gives direct mapping between X and Z

There are two methods of finding out Composition

- **Max –min composition**

$$\boldsymbol{\mu_{RoS} = max_{y \in Y}[min\,(\mu_R(x,y),\ (\mu_S(y,z)]}$$

- **Max-product composition**

$$\boldsymbol{\mu_{RoS} = max_{y \in Y}[\,(\mu_R(x,y)\,.\,(\mu_S(y,z)]}$$

Composition of fuzzy relations

- Max-min composition

$$\forall(x, y) \in A \times B,\ \forall(y, z) \in B \times C$$

$$\mu_{S \circ R}(x,\ z) = \max_y[\min(\mu_R(x,\ y),\ \mu_S(y,\ z))]$$

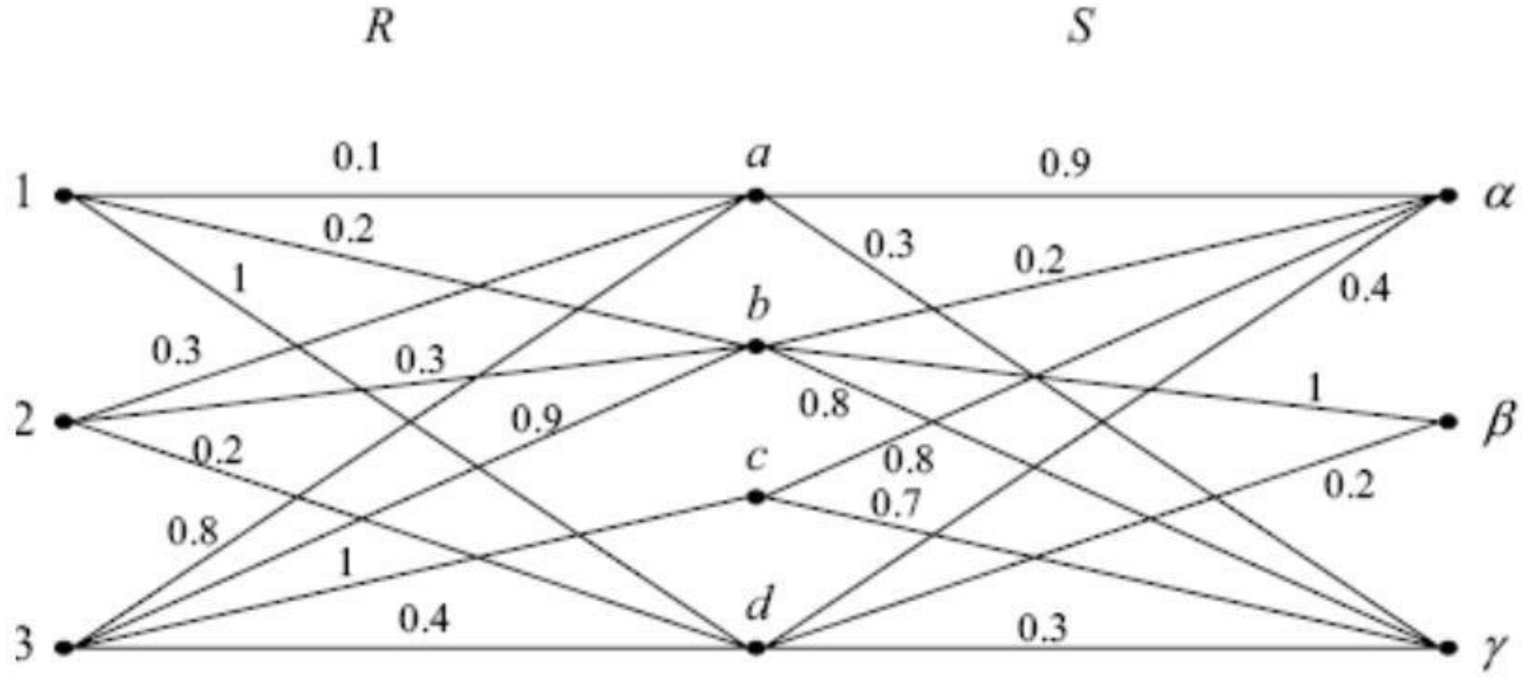
$$= \vee_y[\mu_R(x,\ y) \wedge \mu_S(y,\ z)]$$

- Example

R	a	b	c	d
1	0.1	0.2	0.0	1.0
2	0.3	0.3	0.0	0.2
3	0.8	0.9	1.0	0.4

S	α	β	γ
a	0.9	0.0	0.3
b	0.2	1.0	0.8
c	0.8	0.0	0.7
d	0.4	0.2	0.3

Composition of fuzzy relations



Composition of fuzzy relations

- Example

R	a	b	c	d
1	0.1	0.2	0.0	1.0
2	0.3	0.3	0.0	0.2
3	0.8	0.9	1.0	0.4

S	α	β	γ
a	0.9	0.0	0.3
b	0.2	1.0	0.8
c	0.8	0.0	0.7
d	0.4	0.2	0.3

$$\mu_{S\bullet R}(1,\ \alpha) = \max[\min(0.1, 0.9), \min(0.2, 0.2), \min(0.0, 0.8), \min(1.0, 0.4)]$$
$$= \max[0.1, 0.2, 0.0, 0.4] = 0.4$$

Composition of fuzzy relations

- Example

R	a	b	c	d
1	0.1	0.2	0.0	1.0
2	0.3	0.3	0.0	0.2
3	0.8	0.9	1.0	0.4

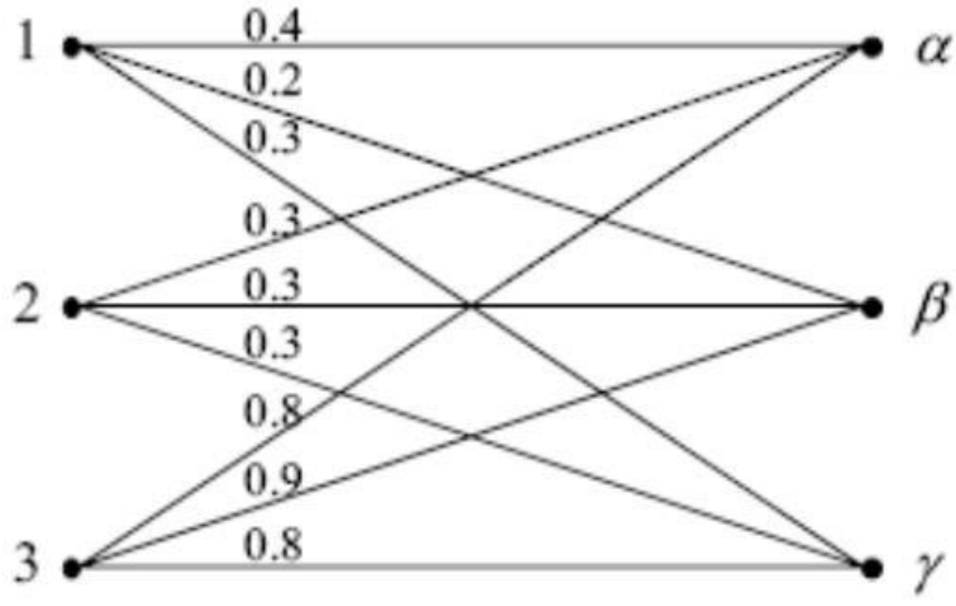
S	α	β	γ
a	0.9	0.0	0.3
b	0.2	1.0	0.8
c	0.8	0.0	0.7
d	0.4	0.2	0.3

$$\mu_{S \bullet R}(1,\ \beta) = \max[\min(0.1, 0.0), \min(0.2, 1.0), \min(0.0, 0.0), \min(1.0, 0.2)]$$
$$= \max[0.0, 0.2, 0.0, 0.2] = 0.2$$

Composition of fuzzy relations

S•R	α	β	γ
1	0.4	0.2	0.3
2	0.3	0.3	0.3
3	0.8	0.9	0.8

S•R



Composition of fuzzy relation

Example:

Obtain fuzzy relation T as a composition between fuzzy Relation R and S.

$$\mathbf{R} = \begin{array}{c c} & \begin{array}{cc} Y_1 & Y_2 \end{array} \\ \begin{array}{c} X_1 \\ X_2 \end{array} & \begin{bmatrix} \mathbf{0.6} & \mathbf{0.3} \\ \mathbf{0.2} & \mathbf{0.9} \end{bmatrix} \end{array} \qquad \mathbf{S} = \begin{array}{c c} & \begin{array}{ccc} Z_1 & Z_2 & Z_3 \end{array} \\ \begin{array}{c} Y_1 \\ Y_2 \end{array} & \begin{bmatrix} \mathbf{1} & \mathbf{0.5} & \mathbf{0.3} \\ \mathbf{0.8} & \mathbf{0.4} & \mathbf{0.7} \end{bmatrix} \end{array}$$

Solution:

1. Max-min composition

$$T_{2\times3} = \begin{bmatrix} \max(0.6,0.3) & \max(0.5,0.3) & \max(0.3,0.3) \\ \max(0.2,0.8) & \max(0.2,0.4) & \max(0.2,0.7) \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & 0.5 & 0.3 \\ 0.8 & 0.4 & 0.7 \end{bmatrix}$$

Composition of fuzzy relation

Example:

Obtain fuzzy relation T as a composition between fuzzy Relation R and S.

$$\mathbf{R} = \begin{array}{c|cc} & Y_1 & Y_2 \\ \hline X_1 & \mathbf{0.6} & \mathbf{0.3} \\ X_2 & \mathbf{0.2} & \mathbf{0.9} \end{array} \qquad \mathbf{S} = \begin{array}{c|ccc} & Z_1 & Z_2 & Z_3 \\ \hline Y_1 & \mathbf{1} & \mathbf{0.5} & \mathbf{0.3} \\ Y_2 & \mathbf{0.8} & \mathbf{0.4} & \mathbf{0.7} \end{array}$$

Solution:

1. Max-product composition

$$T_{2\times3} = \begin{bmatrix} \max(0.6,0.24) & \max(0.3,0.12) & \max(0.18,0.21) \\ \max(0.2,0.72) & \max(0.1,0.36) & \max(0.06,0.63) \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & 0.3 & 0.21 \\ 0.72 & 0.36 & 0.63 \end{bmatrix}$$

Example 2

Let $R_1(x, y)$ and $R_2(y, z)$, be defined as the following relational matrices

$$R_1 = \begin{matrix} 0.6 & 0.5 \\ 1 & 0.1 \\ 0 & 0.7 \end{matrix} \qquad R_2 = \begin{matrix} 0.7 & 0.3 & 0.4 \\ 0.9 & 0.1 & 0.6 \end{matrix}$$

Calculate the max-min composition $R_1 \circ R_2$

$$R_1 \circ R_2 = \begin{matrix} 0.6 & 0.5 \\ 1 & 0.1 \\ 0 & 0.7 \end{matrix} \quad \circ \quad \begin{matrix} 0.7 & 0.3 & 0.4 \\ 0.9 & 0.1 & 0.6 \end{matrix}$$

$\mu_{R_1 \circ R_2}(x_1, x_2)$ =max [min (0.6,0.7) ,min (0.5,0.9)]= max (0.6,0.5)= 0.6

Similarly we can calculate the other entries. The relational matrix for max-min composition in fuzzy relation is thus

$$R_1 \circ R_2 = \begin{matrix} \mathbf{0.6} & \mathbf{0.3} & \mathbf{0.5} \\ \mathbf{0.7} & \mathbf{0.3} & \mathbf{0.4} \\ \mathbf{0.7} & \mathbf{0.1} & \mathbf{0.6} \end{matrix}$$

Example 3

Let $R_1(x, y)$ and $R_2(y, z)$, be defined as the following relational matrices

$$R_1 = \begin{bmatrix} 0.1 & 0.2 & 0 & 1 & 0.7 \\ 0.3 & 0.5 & 0 & 0.2 & 1 \\ 0.8 & 0 & 1 & 0.4 & 0.3 \end{bmatrix} \qquad R_2(y, z) = \begin{bmatrix} 0.9 & 0 & 0.3 & 0.4 \\ 0.2 & 1 & 0.8 & 0 \\ 0.8 & 0 & 0.7 & 1 \\ 0.4 & 0.2 & 0.3 & 0 \\ 0 & 1 & 0 & 0.8 \end{bmatrix}$$

Calculate the max-min composition $\boldsymbol{R_1 \circ R_2\ (x, z)}$

$\mu_{R_1 \circ R_2}(x_1, z_1) =$ max[min(0.1,0.9),min(0.2,0.2) ,min (0,0.8) ,min(1,0.4),
min (0.7,0)]

= max [0.1,0.2,0,0.4,0]=0.4

Similarly we can determine the grades of membership for all pairs (x_i, z_i) ; i=1,2,3 & j=1,2…

The relational matrix for max-min composition in fuzzy relation is thus

$$R_1 \circ R_2 = \begin{bmatrix} 0.4 & 0.7 & 0.3 & 0.7 \\ 0.3 & 1 & 0.5 & 0.8 \\ 0.8 & 0.3 & 0.7 & 1 \end{bmatrix}$$

Example 3

Calculate by max product composition

$$R_1 = \begin{bmatrix} 0.1 & 0.2 & 0 & 1 & 0.7 \\ 0.3 & 0.5 & 0 & 0.2 & 1 \\ 0.8 & 0 & 1 & 0.4 & 0.3 \end{bmatrix} \quad R_2(y,z) = \begin{bmatrix} 0.9 & 0 & 0.3 & 0.4 \\ 0.2 & 1 & 0.8 & 0 \\ 0.8 & 0 & 0.7 & 1 \\ 0.4 & 0.2 & 0.3 & 0 \\ 0 & 1 & 0 & 0.8 \end{bmatrix}$$

$\mu_{R_1}(x_1, y_1) \cdot \mu_{R_2}(y_1, z_1) = 0.1*0.9 = 0.09$

$\mu_{R_1}(x_1, y_2) \cdot \mu_{R_2}(y_2, z_1) = 0.2*0.2 = 0.04$

$\mu_{R_1}(x_1, y_3) \cdot \mu_{R_2}(y_3, z_1) = 0*0.8 = 0$

$\mu_{R_1}(x_1, y_4) \cdot \mu_{R_2}(y_4, z_1) = 1*0.4 = 0.4$

$\mu_{R_1}(x_1, y_5) \cdot \mu_{R_2}(y_5, z_1) = 0.7*0 = 0$

$\boldsymbol{\mu_{R_1 \circ R_2}(x_1, z_1)}$ **= max (0.09,0.04,0,0.4,0)= 0.4**

Similarly, by performing all calculation, we get $\boldsymbol{\mu_{R_1 \circ R_2}} =$

$$\begin{bmatrix} 0.4 & 0.7 & 0.3 & 0.56 \\ 0.27 & 1 & 0.4 & 0.8 \\ 0.8 & 0.3 & 0.7 & 1 \end{bmatrix}$$

Example of Crisp composition

$X = \{x_1, x_2\}$ Y = $\{y_1, y_2\}$ Z = $\{z_1, z_2\}$

R = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and Q = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Find the max-min composition and max product composition

$S = R \circ Q$

$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the max-min composition.

And

$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the max-product composition.

It is clear from the example that max-min composition and max product composition of crisp relations will yield the same result, but in fuzzy max-min composition and max product composition have different result.

Composition of fuzzy relation

Exercise:

Find Max –min composition and

Max-product composition

For given fuzzy relation.

$$\boldsymbol{R} = \begin{array}{c|c} & \begin{array}{cc} Y_1 & Y_2 \end{array} \\ \begin{array}{c} X_1 \\ X_2 \\ X_3 \end{array} & \begin{bmatrix} \mathbf{0.5} & \mathbf{0.1} \\ \mathbf{0.2} & \mathbf{0.9} \\ \mathbf{0.8} & \mathbf{0.6} \end{bmatrix} \end{array} \qquad \boldsymbol{S} = \begin{array}{c|c} & \begin{array}{ccc} Z_1 & Z_2 & Z_3 \end{array} \\ \begin{array}{c} Y_1 \\ Y_2 \end{array} & \begin{bmatrix} \mathbf{0.6} & \mathbf{0.4} & \mathbf{0.7} \\ \mathbf{0.5} & \mathbf{0.8} & \mathbf{0.9} \end{bmatrix} \end{array}$$

Alpha-Cut/ Alpha Level or Cut Worthy Set of Fuzzy Relation

Let X={1,2,3,4} and Let A={(1,0.2), (2,0.4), (3,0.6), (4,0.9)}

Let $\alpha = 0.3$

Then $A_{\alpha=0.3} = \{2,3,4\}$ as

$\mu_A(1) = 0.2 < 0.3$
$\mu_A(2) = 0.4 > 0.3$
$\mu_A(3) = 0.6 > 0.3$
$\mu_A(4) = 0.9 > 0.3$

α-cut of fuzzy relation

- Assume $R \subseteq A \times B$, and R_α is a α-cut relation. Then

$$R_\alpha = \{(x, y) \mid \mu_R(x, y) \geq \alpha, \quad x \in A, y \in B\}$$

Note that R_α is a crisp relation. □

- Example

$$M_R = \begin{array}{c|ccc} & & & \\ \hline & 0.9 & 0.4 & 0.0 \\ & 0.2 & 1.0 & 0.4 \\ & 0.0 & 0.7 & 1.0 \\ & 0.4 & 0.2 & 0.0 \end{array}$$

$$\Lambda = \{0, 0.2, 0.4, 0.7, 0.9, 1.0\}$$

α-cut of fuzzy relation

0.9	0.4	0.0
0.2	1.0	0.4
0.0	0.7	1.0
0.4	0.2	0.0

$M_{R\,0.4} =$

1	1	0
0	1	1
0	1	1
1	0	0

$M_{R\,0.9} =$

1	0	0
0	1	0
0	0	1
0	0	0

$M_{R\,0.7} =$

1	0	0
0	1	0
0	1	1
0	0	0

$M_{R\,1.0} =$

0	0	0
0	1	0
0	0	1
0	0	0