1. Graph theory

→ What is Graph:- It is a pictorial representation of ~~vertices~~ vertices and edges.

Que: 5 workers have ability to do all the five work. Make some possible arrangement with graphical structure.

→ $n!$ possibilities.

Workers		Work
a	→	a_1
b	→	b_1
c	→	c_1
d	→	d_1
e	→	e_1

There are $n!$ possibilities to complete works by workers.

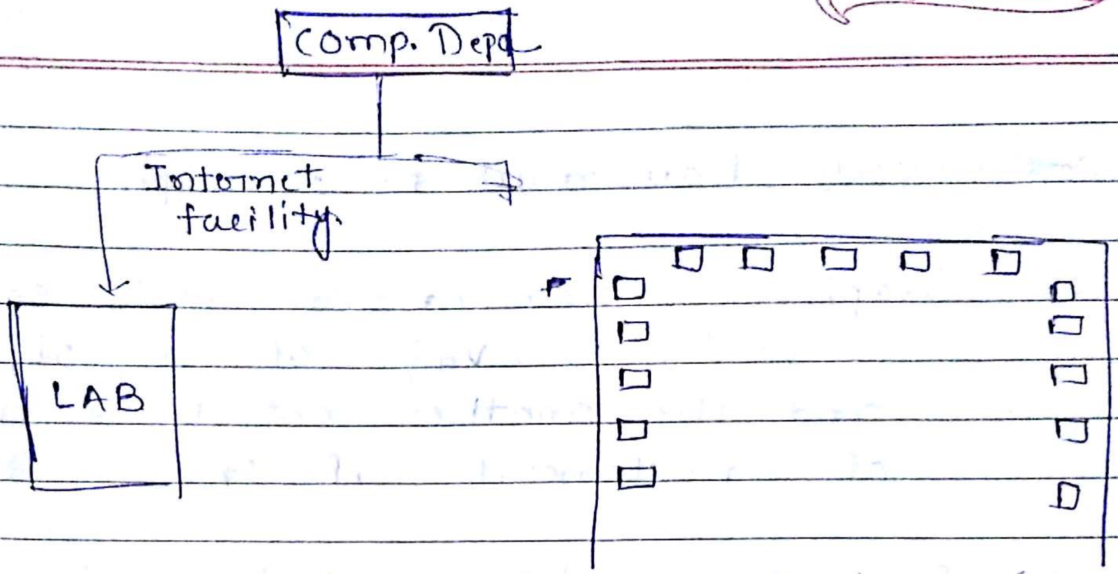
Que-1 Construct an Optimum ~~net~~ Internet network diagram for SVNIT.

Que-2 Construct a diagram to setup a lab of computer engineering with optimum cost, minimum risk and 15 PC.

Que-3 If you have 15 Origin and 10 destination then by using graph construct at least 5 possible diagram to sent the item from Origin to destination.

Que-4 What is the difference between graph and chart.

Ans-2



Ans-4

→ Graphs are meant to be focused on the data in question and how it trends. Graphs have exact numerical figures shown on axes, usually organized on the left and bottom of the graph.

→ Charts are designed to show differences in things like surveys and figures in a more aesthetically pleasing way.

→ every chart must be a graph.

⇒ Graph : consider a non-empty set, $V = \{v_1, v_2 \dots v_n\}$. of an objects and the another set $E = \{e_1, e_2 \dots e_n\}$ of an elements of e_n. $n = 1, 2 \dots$

→ Each e_n is determined by pair (v_i, v_j) of element of V, then system $G(V, E)$ is called a 'simply Graph' or 'Linear-graph'.

→ The elements $v_1, v_2 \dots v_n$ are known as vertices and the element $e_1, e_2 \dots e_n$ are called as edges.

<u>Note:</u> (1) When vertex v is an end-point of some edge e, then we say that e is '<u>Incident</u>' with vertex v.

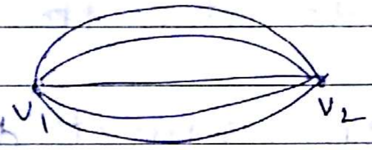
(2) Two vertices v_1, v_2 of a graph $G(V, E)$ are said be "<u>Adjecent</u>", if there exist an edge $e \in E$. Such that v_1 and v_2 are end vertices of e.

(3) Two edges are said to be "<u>Adjecent</u>" if they have common end <u>vertex</u>.

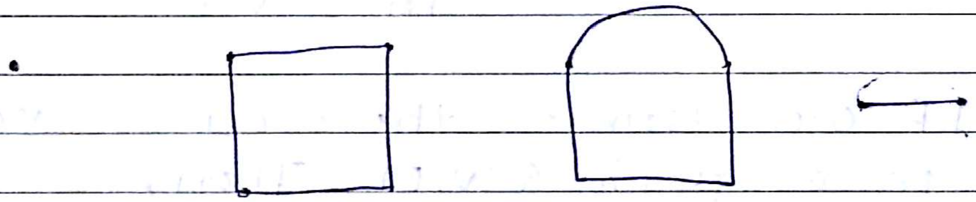
⇒ Self Loop – An edge having same vertex as both it's end vertex is called a self loop.

⇒ Parallel edges – More than one edges are associated with given pair of vertices then they are called as parallel edges.

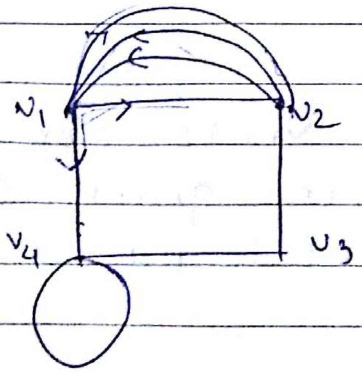
e.g.



⇒ Simple graph – A graph has neither a self-loop nor parallel edges is called simple graph.



⇒ Degree of a vertex – The no. of edges incident at a vertex (with self-loop counted twice) is called the degree of that vertex and is denoted as $d(v_i)$.



$d(v_1) = 5$
$d_{in}(v_1) = 2$
$d_{out}(v_1) = 3$.

Selfloop vertex will have two ~~edges~~ degrees.

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Page

Theorem-1 Prove that ~~sum~~ of degree of a given graph $G(V,E)$ is twice the no. of edges in it.

Proof Since each edge is associated with two vertices. also each edges contribute one degree to each vertex adjecent to it.

→ Therefore the sum of degree of vertices in $G(V,E)$ = 2X no. of edges in $G(V,E)$.

= 2 X e (where e = Total no. of edges in $G(V,E)$).

→ If we assume, there are n vertices in a graph $G(V,E)$. Then,

$$\sum_{i=1}^{n} d(v_i) = 2 \times e.$$

Theorem-2 Prove that the no. of vertices of odd degree in a graph $G(V,E)$ is always even.

Proof We know that, if $v_1, v_2 \ldots v_n$ are n vertices in a given graph $G(V,E)$ and e ~~ded~~ be the total no. of edges in $G(V,E)$.

→ Then by theorem-1 we have,

$$\sum_{i=1}^{n} d(v_i) = 2 \times e. \quad \text{--- (1)}$$

→ $$\sum_{odd} d(v_i) + \sum_{even} d(v_i) = 2 \times e. \quad \text{--- (2)}$$

→ Here, The Summation of $\sum_{even} d(v_i)$ stands for the sum of degree of all those vertices whose degrees are even that is Summation $\sum_{even} d(v_i)$ is "Even" --- (3)

→ From eq(2),

$$\sum_{odd} d(v_i) = 2 \times e - \sum_{even} d(v_i)$$

$$= \text{even} - \text{even}$$

$$= \text{even}$$

$$\therefore \sum_{odd} d(v_i) = \text{even} \quad \text{--- (4)}$$

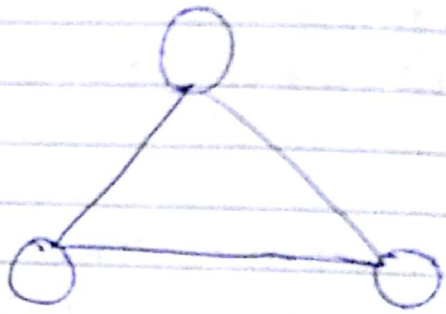
→ Eq. (4) shows that no. of vertices of odd degree in a graph is always even.

→ Because The sum of degrees of those vertices whose degrees are odd.

⇒ Regular graph — ~~If~~ A graph $G(V, E)$ is said to be regular if all the vertices in it have the same degree.

e.g.

$d(v_i) = 2$



$d(v_i) = 4$

for self loop $d(v)$=

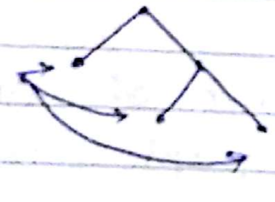
⇒ Isolated vertex —

A vertex having no any incident edge is called as isolated vertex.

⇒ Pendent vertex —

A vertex with degree 1 is called as pendent vertex

pendent vertex.



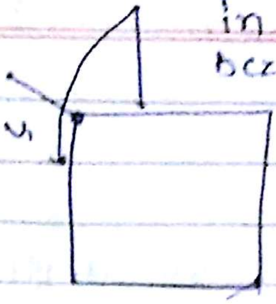
⇒ Edge in series —

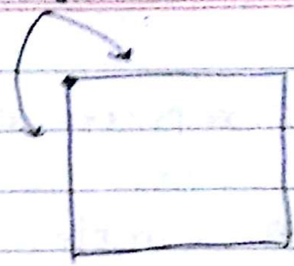
Two adjecent edges are said to be in series if their common vertex is of degree 2.

edges are not in series bcz $d(v_1) = 3$.

edges are in series.

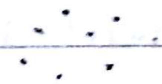
e.g. v_1





⇒ Null Graph —

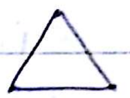
A Graph $G(V, E)$ is said to be null graph if $E = \phi$.

e.g. (i)  (ii) (iii) 

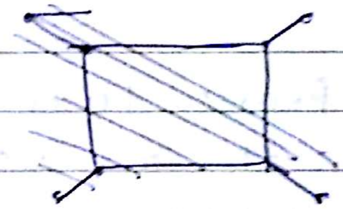
⇒ K- Regular Graph —

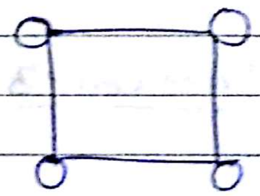
A graph $G(V, E)$ is said to be K-regular graph if all the vertices in it have K degree.

e.g.



2-regular graph



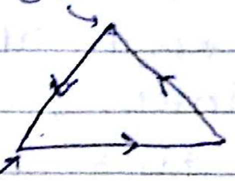


4-regular-gra

⇒ Directed graph —

A Graph $G(V, E)$ is called as directed graph if all its edges gives direction or graph contains directed edges.

outdegree



Indegree

Out → In

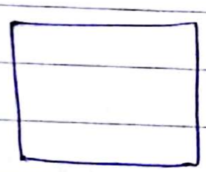
Complete graph is denoted as K_n.
$(n = 1, 2, 3, \ldots)$.

⇒ Complete graph –

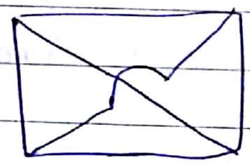
A simple graph with n vertices in which there is an edges b/w every pair of distinct vertices is called a complete graph of n vertices.

<u>Note:</u> Every regular graph can not be a complete graph but every complete graph must be a regular graph.

e.g.



Regular



complete / Regular

<u>Theorem-3</u> Prove that in a simple graph with n vertices. The <u>max.</u> degree of any vertex is $n-1$ and the max. no. of edges $\frac{n(n-1)}{2}$.

<u>Proof:</u> Let $G(V,E)$ be a simple graph with n vertices, then there exist an edge ~~b/w~~ (may) b/w any two distinct vertices. That is for any vertex v of $G(V,E)$ may connect with each of the remaining $n-1$ vertex. Hence, The degree of each vertex in this graph $G(V,E)$ is $n-1$.

→ Here, graph $G(V,E)$ has n vertices.
The sum of degree of vertices =
$(n-1) + (n-1) + \dots$ (n times) $= 2 \times e$.
(By Theorem-1)

$$\therefore n(n-1) = 2 \times e$$

$$\therefore e = \frac{n(n-1)}{2} \quad (\text{Max.})$$

→ This shows that in a simple graph max. ~~no~~ degree of any vertex is $(n-1)$ and max. no. of edges in $\frac{n(n-1)}{2}$.

Theorem-4 Prove that in a complete graph with n vertices, the degree of any vertex is exactly $(n-1)$ and edges is exactly $\frac{n(n-1)}{2}$.

Proof: Let $G(V,E)$ be a complete graph and $G(V,E)$ contains n vertices and there exist edges b/w all the vertices any vertex v ~~may~~ will connect with each remaining $(n-1)$ vertex.

→ Here, graph $G(V,E)$ has n vertices.
The sum of degree of vertices =

$(n-1) + (n-1) + \dots$ (n times) $= 2 \times e$

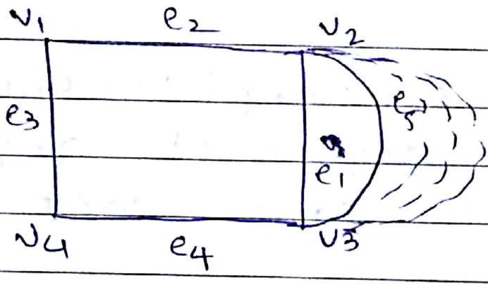
$$\therefore n(n-1) = 2 \times e.$$

$$\therefore e = \frac{n(n-1)}{2}$$

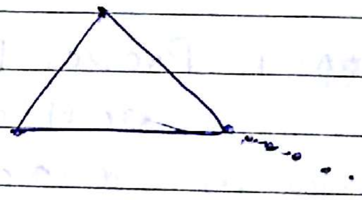
→ This shows that in a complete graph the ~~to~~ degree of any vertex is exactly $(n-1)$ and no. of edges is exactly $\frac{n(n-1)}{2}$.

⇒ Finite and Infinite graph –

→ A graph having finite no. of ~~Kgs.~~ edges and finite no. of vertices. This called as finite graph otherwise it is called as infinite graph.



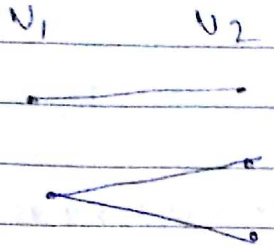
Infinite edges



Infinite vertices.

⇒ Bipartite graph –

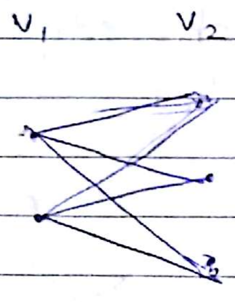
→ A Simple graph $G(V,E)$ is called a bipartite graph if a vertex set V of $G(V,E)$ can be partitioned into non-empty subset V_1 and V_2 such that each edge of $G(V,E)$ is incident with one vertex in V_1 and one vertex V_2. $V_1 \cup V_2$ is called a bipartation of $G(V,E)$.



Bipartite graph.

⇒ Complete bipartite graph –

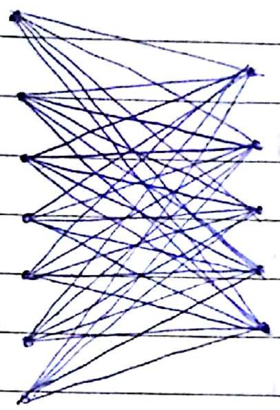
→ A Bipartite graph $G(V, E)$ with bipartation $V_1 \cup V_2$ is called a complete bipartite graph on m and n vertices. If the subset v_1 and v_2 contains m and n vertices respectively. Such that there is an edge b/w each pair of vertices v_i and v_j where $v_i \in V_1$ and $v_j \in V_2$ and It is denoted by $K_{m,n}$.



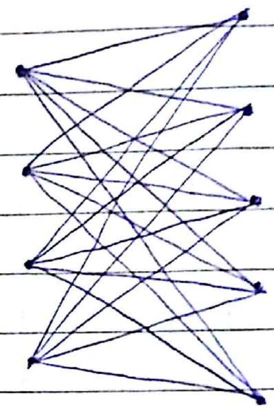
$K_{2,3}$.

Example 1

(1) Draw $K_{7,5}$

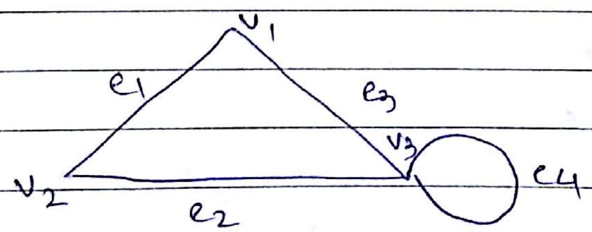


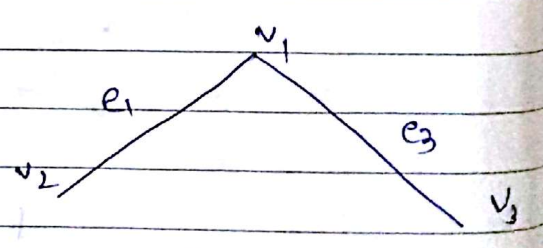
(2) Draw $K_{4,5}$.



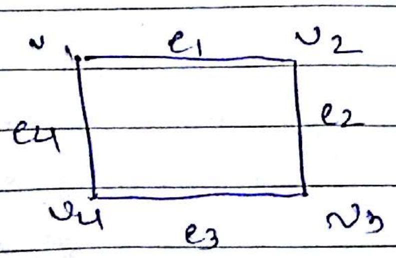
⇒ Sub-graph —

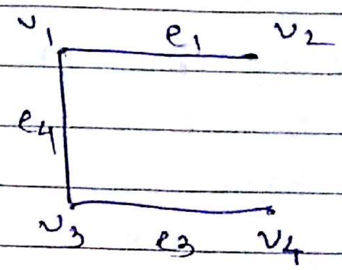
→ A graph $g(v_1, E_1)$ is called a sub-graph of a graph $G(v, E)$. if (i) All the vertices and edges of $g(v_1, E_1)$ are in $G(v, E)$. (ii) Each edge has same vertices in $G(v, E)$.





Sub graph.



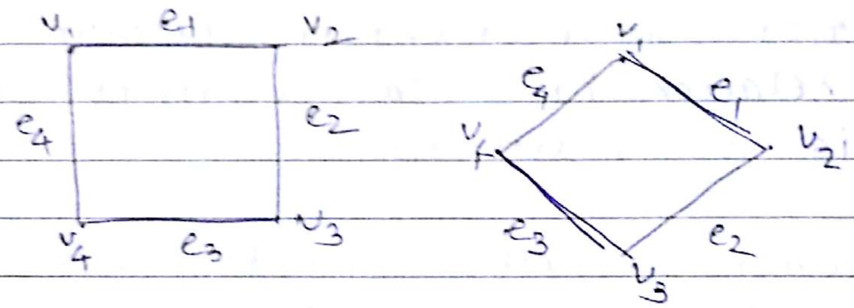


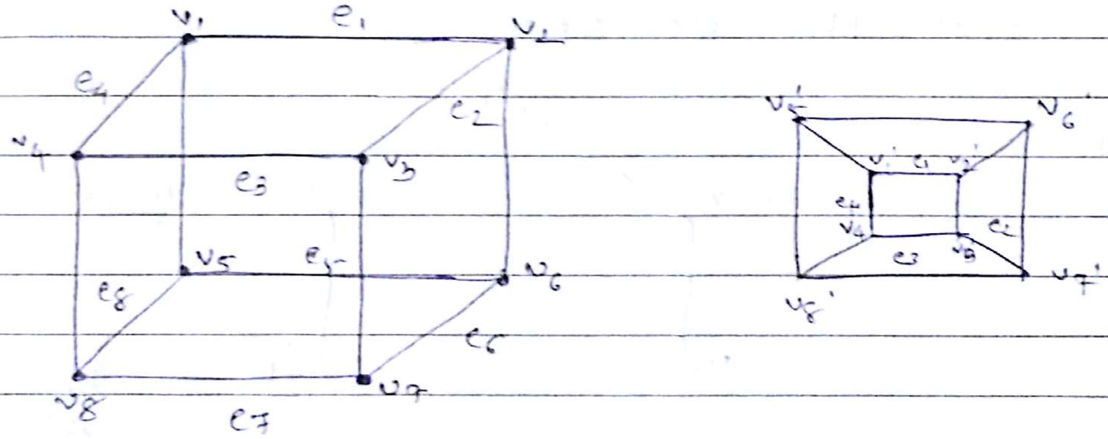
Sub graph.

⇒ Isomorphism ~~graph~~ of a graph –

→ A function ~~f:→~~ G → G' From a graph G → G' is said to be isomorphism if it is one to one and onto and incidence relationship b/w vertices and corresponding edges in G & G' is preserved.

e.g.





⇒ Necessary condition for isomorphic graph –

⇒ Two graph G and G' are said to be isomorphic if

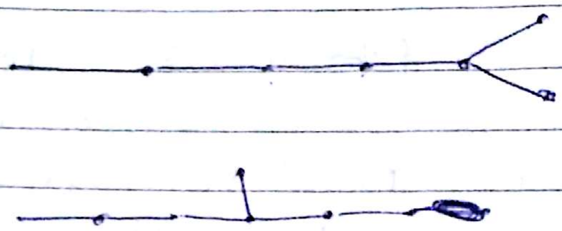
(i). G and G' have equal no. of vertices.

(ii) G and G' have equal no. of edges.

(iii) Equal no. of vertices with given degree.

but these conditions are not sufficient.

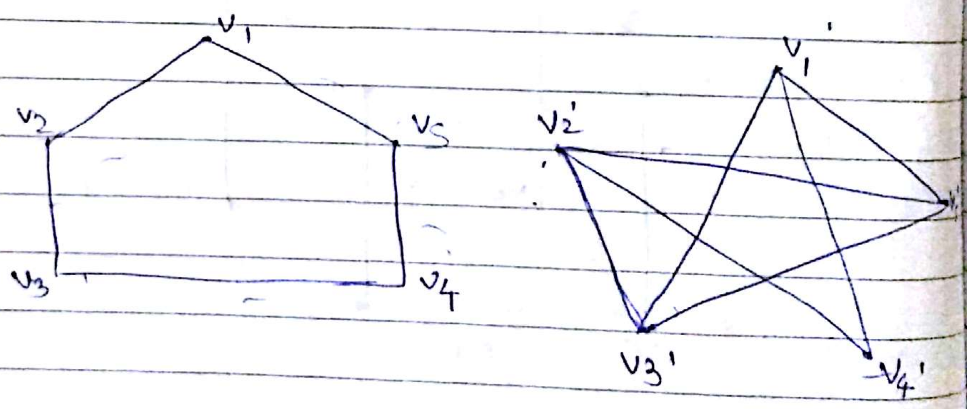
e.g.



→ Here the no. of degrees are same and the no. of edges are same but it is not isomorphism, because incident relationship b/w vertices and edges is not preserved.

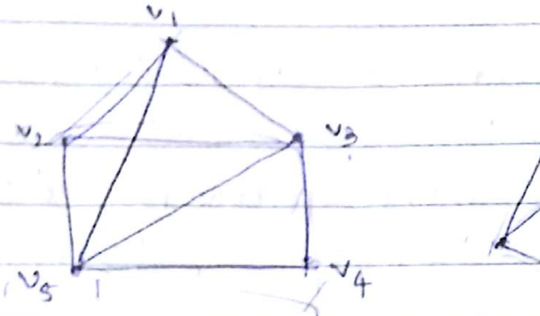
Exa. Examine whether the following pair of graph are isomorphic. If not isomorphic give the reasons.

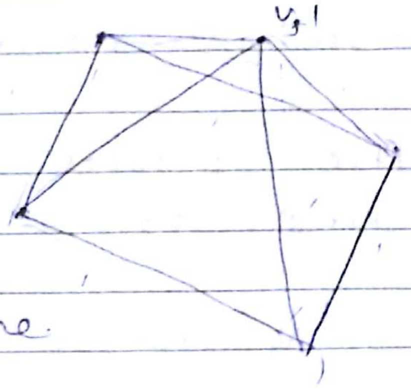
(1)



→ Here the no. of degrees are not same and the no. of edges are also not same so. There are not a pair of isomorphic.

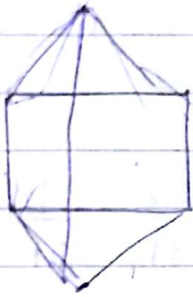
(2)

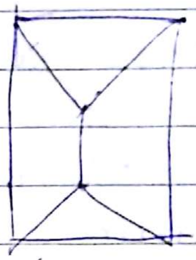




Degrees are not same.

(3)

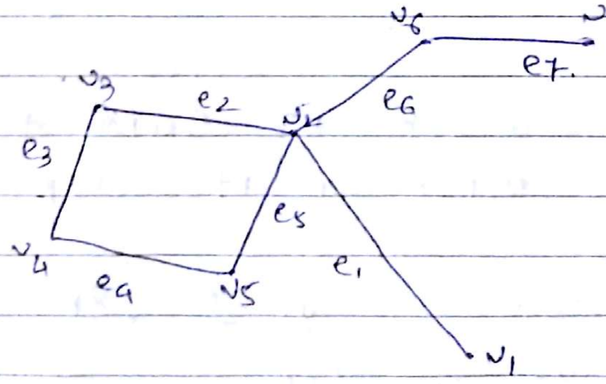




Isomorphic. (Degrees are same).

⇒ Walk, paths and circuits —

→ A walk is defined as finite alternative sequence of vertices and edges such that each edge is incident with the vertices preceding and following it.



$W_1 = v_1\ e_1\ v_2$

$W_2 = v_1\ e_1\ v_2\ e_2\ v_3\ e_3\ v_4\ e_4\ v_5\ e_5\ v_2$

$W_3 = v_2\ e_6\ v_6\ e_7\ v_7$

$W_4 = v_1\ e_1\ v_2\ e_6\ v_6\ e_7\ v_7$

$W_5 = v_3\ e_2\ v_2\ e_6\ v_6\ e_7\ v_7$

classmate
Date
Page

⇒ Terminal vertices –

→ The vertices with which walk begin and end called a terminal vertices

v_1 and v_2 are terminal vertices.

⇒ Close walk –

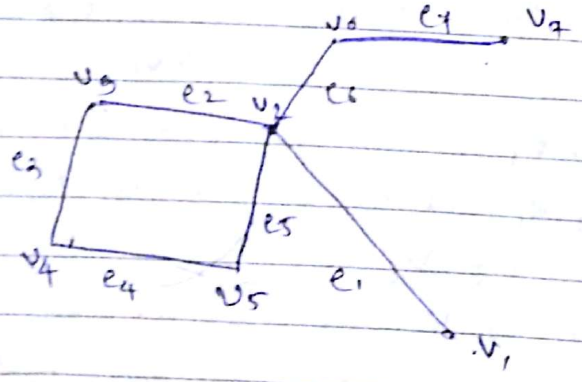
→ A walk is said to be close if it has same vertex as both of its terminal vertices.

⇒ ~~Terminal~~ open walk –

→ A walk is said to be open if it hasn't same vertex as both of its terminal vertices

⇒ Path –

→ An open walk is said to be path, if no vertex in it appears more than once.



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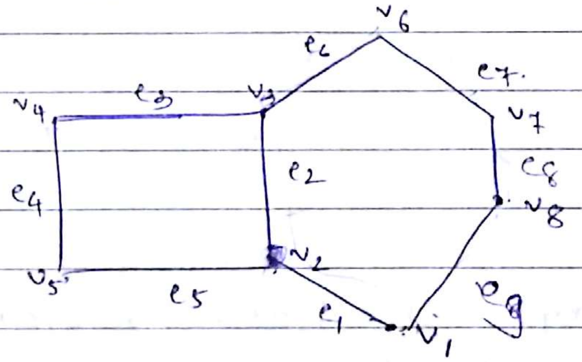
$P_1 = v_1 e_1 v_2 e_6 v_6 e_7 v_7. = 3$ (length)
$P_2 = v_1 e_1 v_2 e_5 v_5. = 2$
$P_3 = v_2 e_5 v_6 e_7 v_7. = 2$

⇒ Length of a path —

→ The no. of edges in a path, it is called a length of a path.

⇒ Circuit —

→ A close walk is said to be circuit if no vertex in it appear more than one except the terminal vertices.



$c_1 = v_2 e_2 v_3 e_3 v_4 e_4 v_5 e_5 v_2.$
$c_2 = v_1 e_1 v_2 e_2 v_3 e_6 v_6 e_7 v_7 e_8 v_8 e_9 v_1$

Note:-

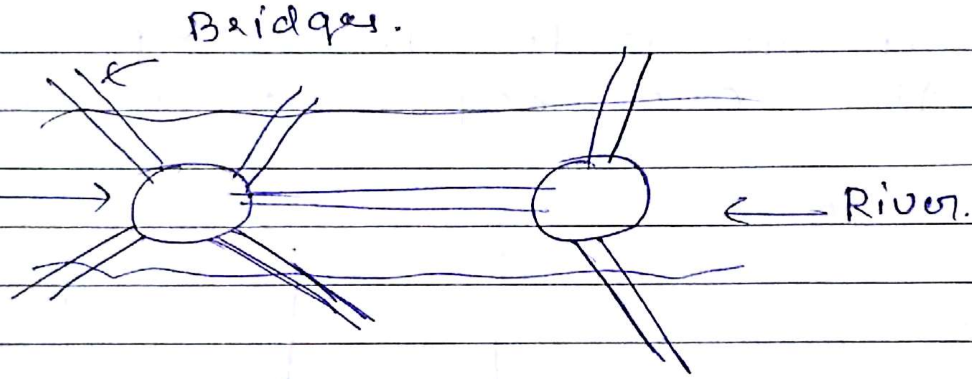
→ A self loop is a circuit. but it cannot be included in path or in other circuits.

→ Every vertex in a circuit is of degree 2

→ Degree of a vertex in a walk or in a path or in a circuit is counted only with respect to edges included in it.

→ In a path, the terminal vertices are of degree 1 and the ~~ver~~ other vertices are of degree 2.

Exa



Que Here, we have to start from any point (any bridge) and again we have to complete path at that point only but bridges shouldn't be repeated.

Ans: 'NO SOLUTION.'

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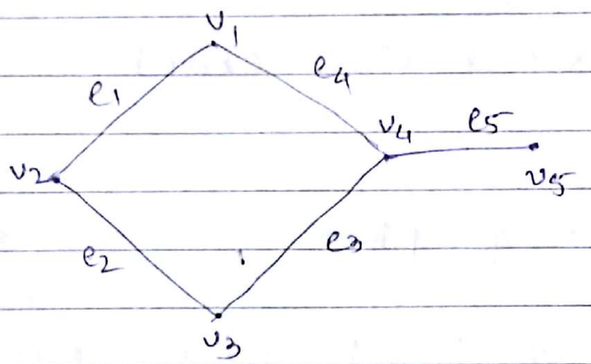
⇒ Edge disjoint subgraph —

→ Two subgraphs g_1 and g_2 of a graph $G(V, E)$ are said to be edge disjoint if they have no common edge.

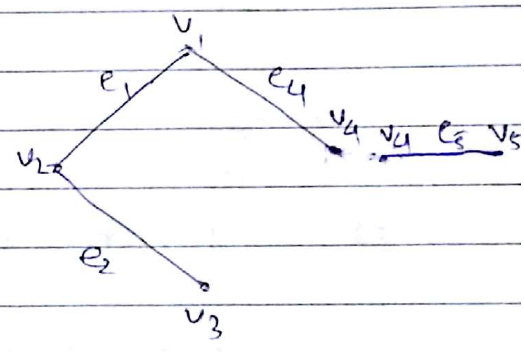
⇒ Vertex disjoint subgraph -

→ Two subgraphs g_1 and g_2 of a graph $G(V, E)$ are said to be vertex disjoint if they have no common vertex.

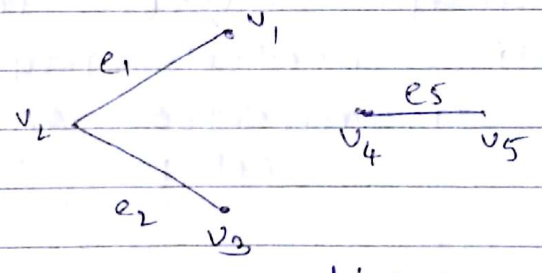
(1) e.g.



G(V, E).



edge disjoint subgraph.



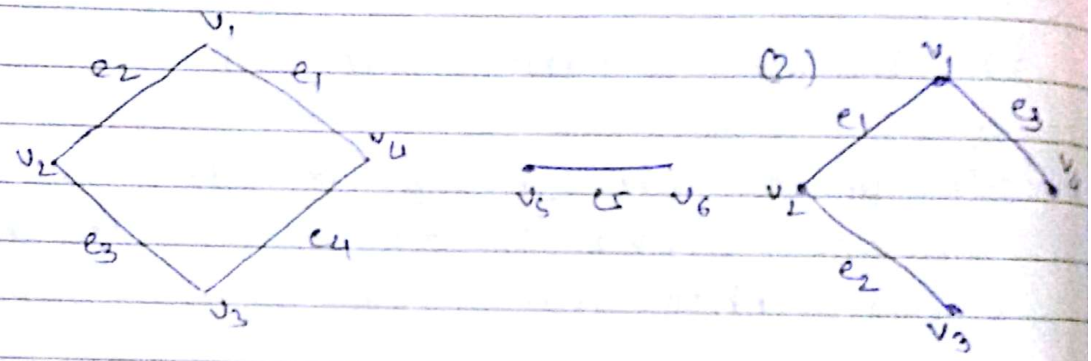
vertex disjoint subgraph.

→ Every vertex disjoint subgraph ~~is~~ must be edge disjoint subgraph.

⇒ Connected graph –

→ A graph G(V,E) is Said to be connected if there is at least one path between each pair of vertices.

(1) e.g.



Disconnected graph   Connected graph

⇒ Disconnected graph –

→ There exist at least one pair of vertices which is not joined by any path in a given graph G(V,E) Then it is Said to be disconnected graph.

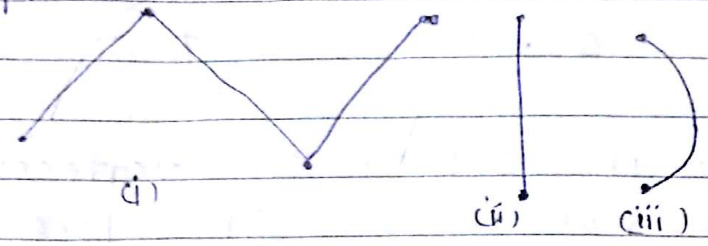
→ Null graph is disconnected graph

⇒ Component –   (if there exist only one vertex then it is connected)

→ If a graph G(V,E) is disconnected then it contains two or more than two connected subgraphs. Each of these connected subgraph is called a component of G(V,E).

$G(V, E)$

e.g.



→ Here (i), (ii), (iii) are the components of graph $G(V, E)$.

Theorem A graph $G(V, E)$ is disconnected if and only if it's vertex set V can be partitioned (components) into two non-empty disjoined subsets V_1 and V_2 such that there exist no edge in $G(V, E)$ whose one end vertex is in subset V_1 and other in subset V_2.

Proof: ~~Here A graph $G(V, E)$ is disconned. According to the definition of disconnected graph, there exist at least one pair of vertex~~

Part-1: Let, $G(V, E)$ be disconnected. Therefore there exist a pair (a, b) of vertices in $G(V, E)$ such that there is no path in $G(V, E)$ with terminal vertices as a and b.

→ Let, V_1 be the set of all vertices which are joined by a ~~pth~~ path to 'a'.

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Clearly v_1 doesnot include all the vertices of $G(V,E)$ as $b \notin v_1$

→ Let v_2 be the set of all vertices of $G(V,E)$ which are not in v_1. Therefore no vertex in v_1 is joined to a vertex in v_2 by an edge of $G(V,E)$.

Part 2: Let $G(V,E)$ be a graph and suppose v_1 and v_2 be the partitioned of vertex set V of graph $G(V,E)$ and there exist no edge in $G(V,E)$ whose one end vertex in v_1 and other is in v_2

→ To prove that, $G(V,E)$ is disconnected, suppose $G(V,E)$ is connected. Therefore by defination, there exist a path b/w each pair of vertices. That is there is a vertex $v_i \in v_1$ and $v_j \in v_2$ and (v_i, v_j) determine an edge. Clearly, This edge has one end vertex in v_1 and other is in v_2 which contradicts our assumptions that means Supposition is wrong.

→ Hence, A graph $G(V,E)$ is disconnected

Theorem: If A graph has exactly two vertices of odd degree. There must be a path joining these two vertices

Proof: Let, $G(V,E)$ be a graph with exactly two vertices a and b of odd degree.

Case-I :- If $G(V,E)$ is connected.
By definition of connected graph, There is a path b/w each pair of vertices in $G(V,E)$. That is there must be a path joining a and b.

Case-II :- If $G(V,E)$ is disconnected.
Let, g_1 and g_2 are connected subgraphs of $G(V,E)$.

→ If possible suppose, $a \in g_1$ and $b \in g_2$. That is g_1 and g_2 are connected sub graphs and contain only one vertex of odd degree but by theorem, we know that odd degree vertices in any graph are even in no. That is our supposition is wrong. That is either $a, b \in g_1$ or $a, b \in g_2$. as g_1 and g_2 are connected, there must be a path joining a & b.

Theorem: A simple graph with n vertices and k component can have almost $\frac{(n-k)(n-k+1)}{2}$ edges.

Proof: Let, $G(V,E)$ be a simple graph with n vertices and k-components and let $C_1, C_2 \ldots C_k$ be the component of a simple graph $G(V,E)$. Let no. of vertices in $C_1, C_2 \ldots C_k$ are $n_1, n_2 \ldots n_k$ respectively. Let n be the no. of vertices and e be the no. of edges in $G(V,E)$. Then we have $n_1 + n_2 \ldots + n_k = n$

$$\Rightarrow \sum_{i=1}^{k} n_i = n \quad \text{--- (1)}$$

→ Since $G(V,E)$ is Simple, all subgraphs of $G(V,E)$ are also simple. That is each C_i $(i = 1, 2 \ldots k)$ is simple. and by theorem, each C_i can have almost $\frac{n_i(n_i-1)}{2}$ no. of edges.

→ Therefore, $G(V,E)$ can have almost $\sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$ no. of edges. — (2)

→ From (1) we have, $\sum_{i=1}^{k} n_i = n$

$$\sum_{i=1}^{k} n_i - 1 = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1$$

$$= n - k$$

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$$\therefore (n-k) = \sum_{i=1}^{k} (n_i - 1)$$

$$\therefore (n-k)^2 = \left[\sum_{i=1}^{k} (n_i - 1)\right]^2$$

$$\therefore (n-k)^2 = \sum_{i=1}^{k} (n_i - 1)^2 + 2 \sum_{i \neq j}^{k} (n_i - 1)(n_j - 1)$$

$$\cancel{(\because (a+b)^2 = a^2 + b^2 + 2ab)}$$

$$\therefore (n-k)^2 \geq \sum_{i=1}^{k} (n_i - 1)^2.$$

$$\therefore (n-k)^2 \geq \sum_{i=1}^{k} \left[n_i^2 - 2n_i + 1\right]$$

$$\geq \sum_{i=1}^{k} n_i^2 - 2\sum_{i=1}^{k} n_i + \sum_{i=1}^{k} 1$$

$$\therefore (n-k)^2 \geq \sum_{i=1}^{k} n_i^2 - 2n + k$$

$$\therefore \sum_{i=1}^{k} n_i^2 \leq (n-k)^2 + 2n - k \longrightarrow (3)$$

so, from eq. (2),

$$\sum_{i=1}^{k} \frac{n_i(n_i - 1)}{2}$$

$$= \frac{1}{2} \sum_{i=1}^{k} n_i^2 - n_i$$

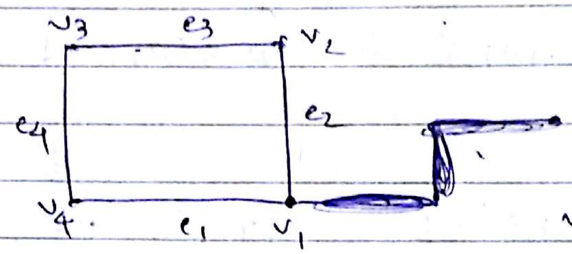
$$\leq \frac{1}{2} \left[(n-k)^2 + 2n - k - n\right]$$

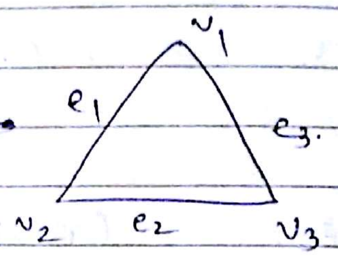
$$\leq \frac{1}{2} \left[(n-k)^2 + n - k\right]$$

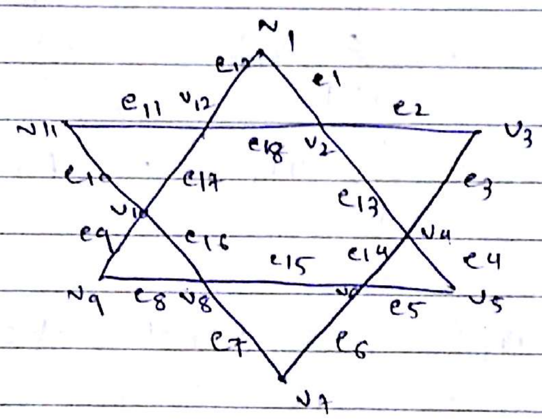
$$\leq \frac{1}{2} \left[(n-k)(n-k+1)\right]$$

⇒ Euler graph —

→ A close walk running through every edge of a graph exactly once is known a euler line and the graph consisting of a eular line is called a Euler Graph.







Theorem :- A given connected graph G(V,E) is eular graph if and only if all vertices of G(V,E) are of even degree

Proof :- (1) Let, G(V,E) be a Euler graph and connected, therefore G(V,E) contain a euler line which is a close walk containing all the edges of G exactly once.

→ That is at any vertex v of G, ~~we~~ we enter through one edge and exit through another edge while touching such vertex During each touch, we use two distinct edges incident to vertex V. so, v is of even degree.

→ If ~~we~~ v is the initial vertex, then also one edge incident to it. is used to start and other edge is used to finish as the walk is close, that is the initial vertex is also of even degree. That is each vertex of G is of even degree.

(2) Let, Each vertex in G is of even degree and G is connected graph. Let, v ~~with~~ be any vertex of G. construct a close walk ω_1. Starting from v and crossing every edge coming in the way exactly one and reaching at v again.

→ Since, each vertex' is of even degree this will be possible if all the edges of G are traced. then G is a euler graph.

→ If not, then let g_1 be the graph consisting of the close walk ω_1. Let g_1' be the graph consisting of those edges of

G which are not in ω_1. Clearly, g_1 and g_1' are subgraph of G and as G is connected, there exist at least one vertex common to g_1 and g_1'. Say u

Now, start from u, tracing a close walk ω_2 in g'. which is also possible as each vertex in g_1' are also of even degree.

But ω_1 and ω_2 can be combined to form a new close walk ω_3. If ω_3 is a graph G, then G is euler graph otherwise, we can proceed in similar way to form a new close walk in the Subgraph of G, which is consisting of those edges which are not in ω_3. This process terminates at certain stage and final close walk will be G.

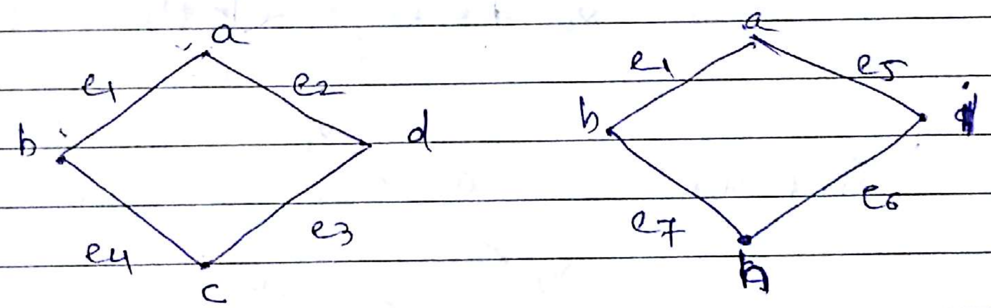
Hence G is euler graph.

⇒ Operation on graph —

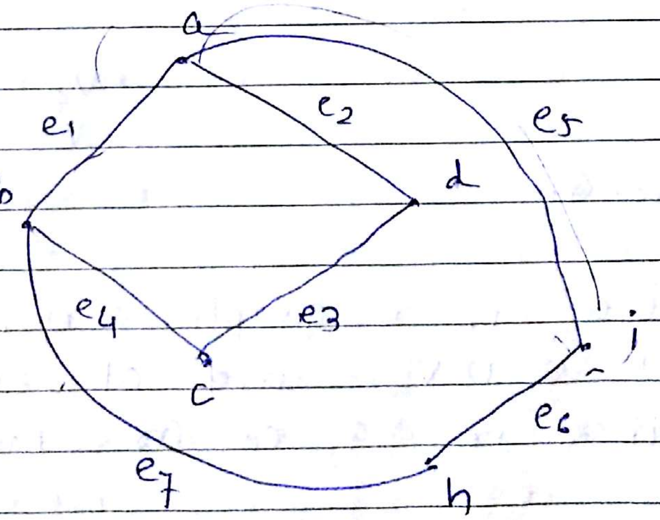
(1) Union of two graphs $G_1(V_1,E_1)$ and $G_2(V_2, E_2)$ and another Graph $G_3(V_3, E_3)$ whose vertex set $V_3 = V_1 \cup V_2$ and edges set $E_3 = E_1 \cup E_2$.

(2) Intersection - Similarly, the intersection of graph G_1 and G_2 is a graph $G_4(V_4, E_4)$ consisting only those vertices and edges that are in both.

e.g.

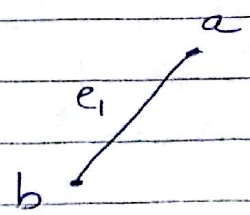


Union -

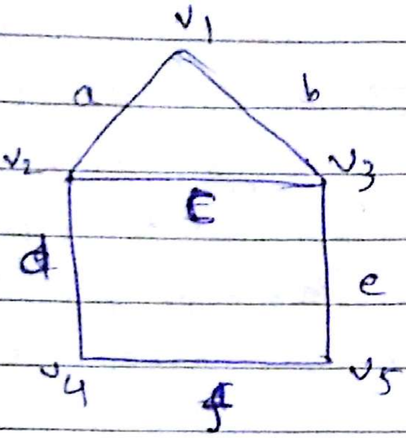


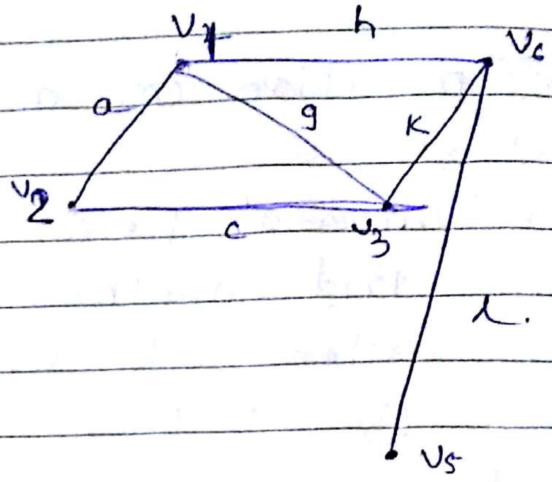
$G_1 \cup G_2$.

Intersection -

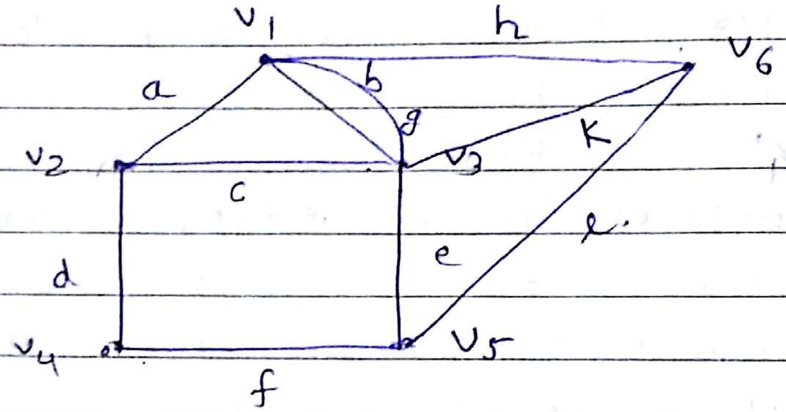


$G_1 \cap G_2$

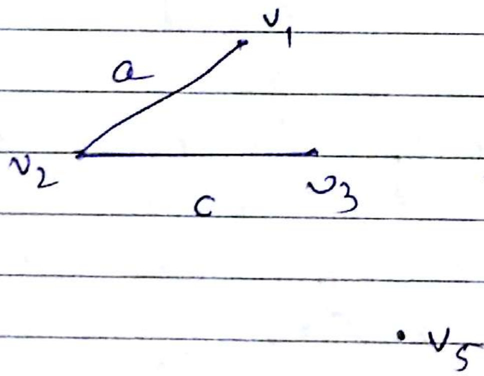




$G_1 \cup G_2$

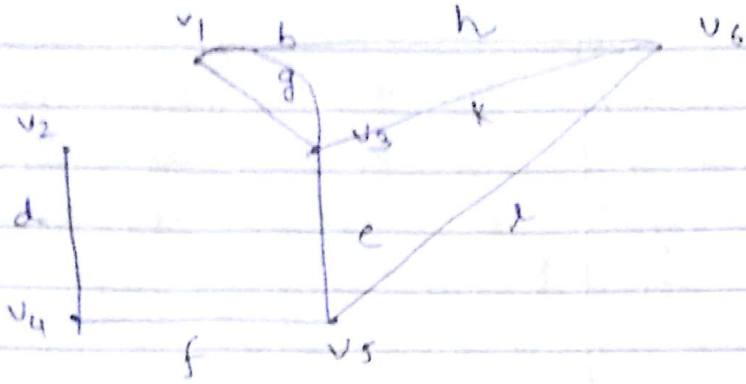


$G_1 \cap G_2$



$\Rightarrow$ $G_1 \oplus G_2$ —

$G_1 \oplus G_2$, is a graph consisting of the vertex set $V_1 \cup V_2$ and of edges that are either in G_1 or G_2 but not in both



$G_1 \oplus G_2$.

⇒ Decomposition –

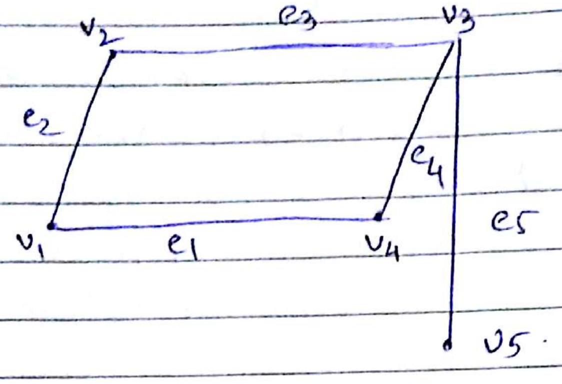
→ A graph G(V, E) is Said to how been decomposed into subgraphs $g_1(V_1, E_1)$ and $g_2(V_2, E_2)$ if $g_1 \cup g_2 = G$ and $g_1 \cap g_2 =$ null graph.

⇒ ~~Delay~~ Deletion –

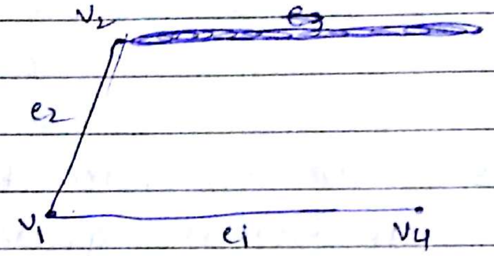
→ If v_i is a vertex in a Graph G(V, E), then $G - v_i$ denotes a Subgraph of G obtain by removing v_i from G.

Note:- Deletion of vertex always implies the deletion of all edges incident on that vertex

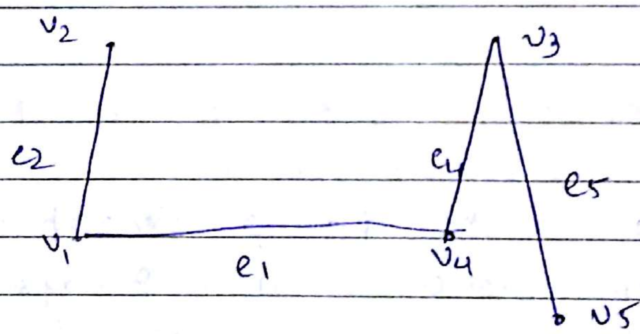
→ If e_j is an edge in G Then $G - e_j$ is a subgraph of G obtain by deleting e_j from G. Deletion of an edge doesn't implies deletion of its n vertices.



G(V, E).



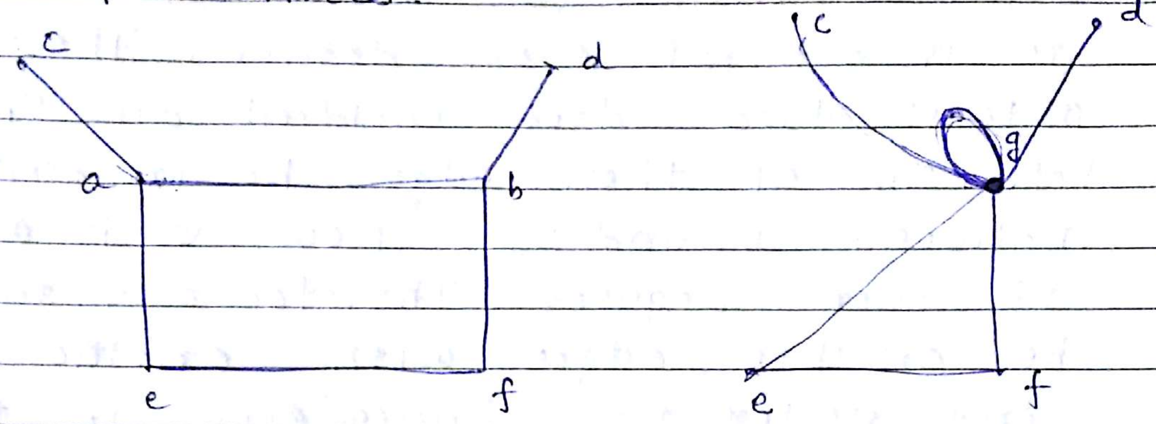
G − v3 (Deletion of vertex).



G − e3 (Deletion of edge).

⇒ Fuse − A pair of vertices a, b in a graph are said to be fused if two vertices are replaced by a single new vertex such that every edge, that was incident on either a or b or on both is incident on the new vertex.

→ Thus, fusion of two vertices doesn't alter the no. of edges, but it reduce the no. of vertices.



⇒ Theorem: A connected graph G is euler graph if and only if it can be decompose into circuits.

<u>Proof:</u> Let, G(V, E) be a connected graph which can be decomposed into circuits. That is G is the union of edge disjoint circuits. Since, the degree of each vertex in a circuit is two that the degree of every vertices in this system G is even and by theorem we know that, if G is connected graph with all the vertices are of even degree, then G is euler. There fore G is euler.

Part-2 → Let, $G(V,E)$ be a euler graph and is connected. Therefore each vertex in G is of even degree consider a vertex v_1 of G as v_1 is of even degree, There are atleast two edges incident on v_1. Let, One of these edges be incident between v_1 and v_2. Now, v_2 is also of even degree. Therefore v_2 must be another edge with another end vertex v_3 Proceeding in this way, we will arrive a vertex which is already in this walk because G is a closed walk. This will form a circuit say C_1.

→ The component $G - C_1$ will be a subgraph of G with every vertices in it are of even degree as G has every vertices of even degree. We can construct another circuit say C_2 in $G - C_1$. As above, $(G-C_1)-C_2$ will be a subgraph of G with every vertices in it of even degree continuing this process until no edge is left. Since, G is finite, this process will terminate and we get set of edge disjoint circuits whose union is G and intersection is null graph.

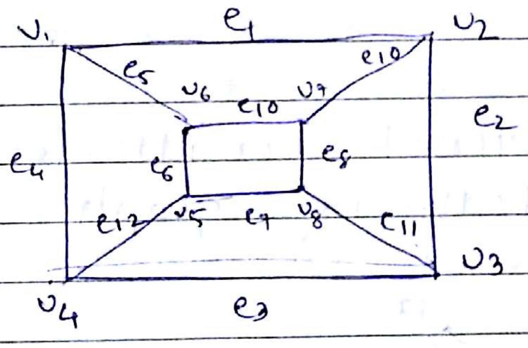
→ So, by definition we can say G is decomposed into circuits.

⇒ Hamiltonian Path –

An open walk in a connected graph G is called a hamiltonian Path if it traverses every vertices of G exactly once.

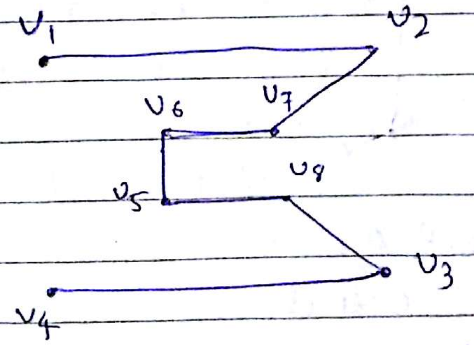
⇒ Hamiltonian circuit –

A close walk in a connected graph G is called a hamiltonian circuit if it traverses every vertices of G exactly once except that vertex from which it starts.

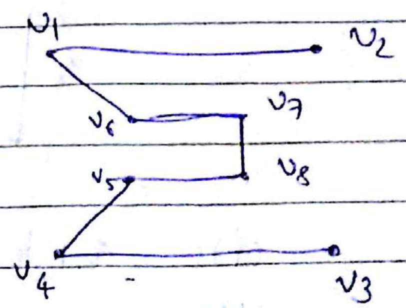


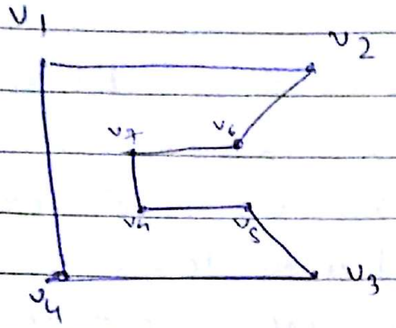
→ hamiltonian Path –

(i)



(ii)





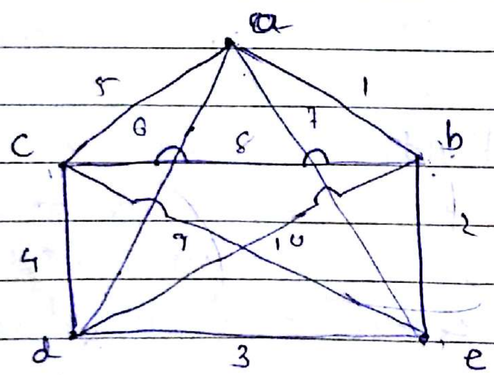
hamiltonian circuit

Note: A hamiltonial circuit is a connected graph with n vertices consist of exactly n edges.

→ The length of a hamiltonian path in a graph with n vertices is $n-1$

→ The removal of an edge from a hamiltonian circuit gives a hamiltonian path.

Que (A) Write 5 distinct hamiltonian circuit from the following graph.



(i) a 1 b 2 e 3 d 4 c 5 a.

(ii) d 3 e 2 b ~~e~~ 1 a 5 c 4 d

(iii) e 3 d 4 c 5 a 1 b 2 e.

(iv) c 4 d 3 e 2 b 1 a 5 c.

(v) b 1 a 5 c 4 d 3 e 2 b.

(8) Find possible edge disjoint hamiltonian circuit - (edge shouldn't be common)

(i) a 1 b 2 e 3 d 4 c 5 a

(ii) a 7 e 9 c 8 b 10 d 6 a.

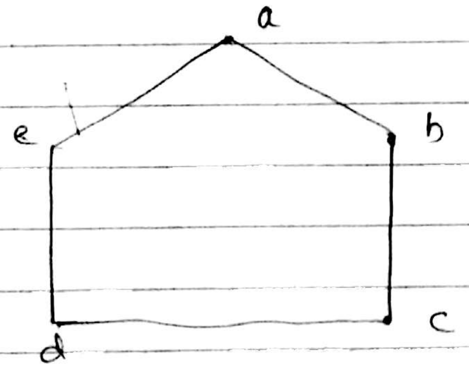
~~Note~~:

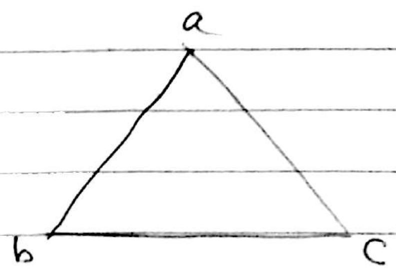
Result: (1) A hamiltonian circuit can be constructed in a complete graph

(2) In a complete graph with n vertices there are $\frac{n-1}{2}$ edge disjoint hamiltonian circuit if n is an odd number and $n \geq 3$.

n

Que (3) Draw a graph which has hamiltonian circuit and which is a euler line





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Que Prove that a simple graph with n vertices must be connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.

Proof: Let G be a simple graph with n vertices. If possible, suppose G is not connected therefore G must have atleast two components.

→ Again by theorem we know that the max. no. of edges in a simple graph G with k components is $\frac{(n-k)(n-k+1)}{2}$ by taking $k=2$, we get

the no. of edges in $G \leq \frac{(n-2)(n-1)}{2}$

but it is given that the no. of edges in G is $> \frac{(n-2)(n-1)}{2}$. Therefore our supposition is wrong. G must be connected.

Theorem: Prove that if a connected graph G is decompose into two subgraph g_1 and g_2 then there must be atleast one vertex common between g_1 and g_2.

Proof→ Here If we assume that there is no any vertex is common between g_1 and g_2.

→ so, $g_1 \cap g_2 = \phi$ but $g_1 \cup g_2 \neq G$.

so, It doesn't satisfy the definition of decomposition.

→ So, our supposition is wrong and there must be atleast one vertex common between g_1 and g_2.

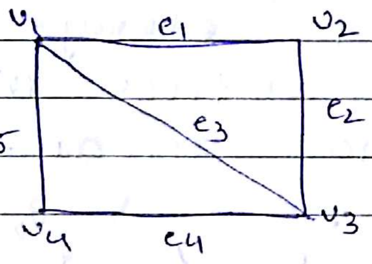
⇒ Matrix representation of a graph —

⇒ Adjacency matrix — When G is a simple graph with n vertices $v_1, v_2 \dots v_n$. Then the matrix $A_G = [a_{ij}]$ where $a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \text{ is edge in } G \\ 0 & \text{otherwise.} \end{cases}$

~~an edge edge in G~~

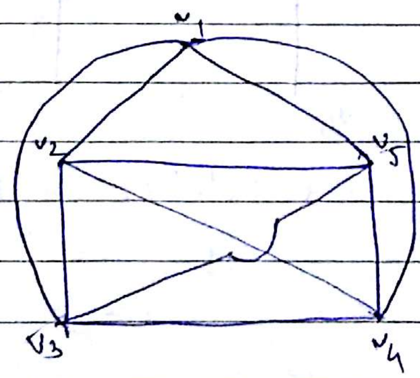
is called the adjacency matrix of G.

e.g. (i)



$$\begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 1 & 1 \\ v_2 & 1 & 0 & 1 & 0 \\ v_3 & 1 & 1 & 0 & 1 \\ v_4 & 1 & 0 & 1 & 0 \end{array}$$

(ii)



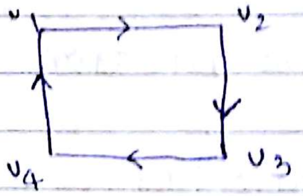
$$\begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0 & 1 & 1 & 1 & 1 \\ v_2 & 1 & 0 & 1 & 1 & 1 \\ v_3 & 1 & 1 & 0 & 1 & 1 \\ v_4 & 1 & 1 & 1 & 0 & 1 \\ v_5 & 1 & 1 & 1 & 1 & 0 \end{array}$$

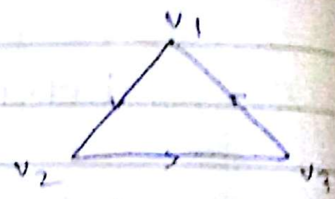
connectedness
strongly | weakly | uni

⇒ connectedness —

→ (i) A directed graph is said to be strongly connected if there is a path from v_i to v_j and from v_j to v_i where v_i and v_j are any pair of vertices of the graph.

e.g.

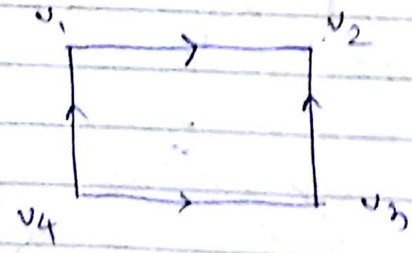




strongly connected.

(ii) A directed graph is said to be weakly connected if there is a path between every pair of vertices in the underlying undirected graph. In other words, A directed graph is weakly connected iff there is always a path between every two vertices when the direction of the edge are disregarded

e.g.

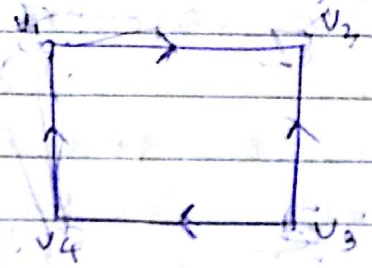


weekly connected.

(iii) Unilaterly connected —

A simple directed graph is said to be Unilaterly connected if for any pair of the vertices of the graph atleast one of the pair of vertices of graph is reachable from the other vertex.

e.g.



2. Trees.

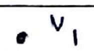
⇒ Tree – A tree is a connected graph without any circuit.

Note: (i) A graph consisting of only one vertex is a tree.

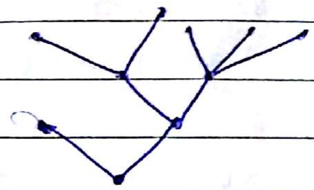
(ii) A tree is a simple graph.

(iii) A path is a tree.

e.g. (i)



(ii)



⇒ Properties of a tree –

(i) There is one and only path between every pair of vertices in a tree

Proof: Let T be a tree. Therefore T is connected graph without any circuit. Since, T is connected there exist at least one path between every pair of vertices in T.

→ If possible suppose, there exist a pair of vertices (a, b) in T joined by two path say P_1 and P_2. Then $P_1 \cup P_2$ will be a circuit which is a contradiction as T has no circuits.

Therefore there is one and only one path b/w every pair of vertices in a tree

(ii) If in a graph G, there is one and only one path b/w every pair of vertices then G is a tree.

Proof: Let G be a graph such that there is one and only path between every pair of vertices That is there exist a path b/w every pair of vertices in G. That is G is connected

→ If possible suppose, there is a circuit c in G containing two vertices a, b. clearly there exist two distinct path joining a and b which is a contradiction to the hypothesis (Given statement). Therefore G is without circuit. That is G is connected and without circuit. Hence G is a tree.

(iii) A tree with n vertices has n-1 edges - (Use of Mathematical Induction).

Proof: ~~Here~~ We prove this property by induction on the no. of vertices n for n=1 a tree with one vertex has no edges is true. For n=2 tree with 2 vertices clearly contain 1 edge.

→ Assume that, it is true for $n = k$ that is a tree with k vertices has $k-1$ edges.

→ Let, T be a tree with $k+1$ vertices, since T is a tree it has at least two pendent vertices say a and b. and let e be the edge incident with vertex a in T. If we delete vertex a from T, then we get a tree $T-a$ with k vertices. Therefore by our assumption, $T-a$ has $k-1$ edges.

→ We know that deletion of a vertex from a graph will also remove all the edge incident on that vertex. Therefore deletion of a vertex a from T will also remove the edge e from T. clearly, $\{T-a\} \cup \{e\}$ will be a tree with $k+1$ vertices and $(k-1)+1 = k$ edges. Therefore the property is true for $n = k+1$. Hence, a tree with n vertices has $n-1$ edges.

⇒ Minimally connected graph —

A connected graph is said to be Minimally connected if removal of any edge from it disconnects the graph.

Result-1: Minimally connected graph can't have a circuits.

Result-2: A graph is a tree if and only if it is minimally connected.

Proof: Let T be a tree. Since T is a tree, it is connected and without circuits. That is removal of any edge from T will disconnect T. Therefore T is minimally connected.

→ Suppose, G is minimally connected graph. By definition, G is connected and by result there is no circuit in G which satisfy definition of tree. Hence, G is a tree.

Theorem: A Graph with n vertices, $n-1$ edges and no circuit is connected.

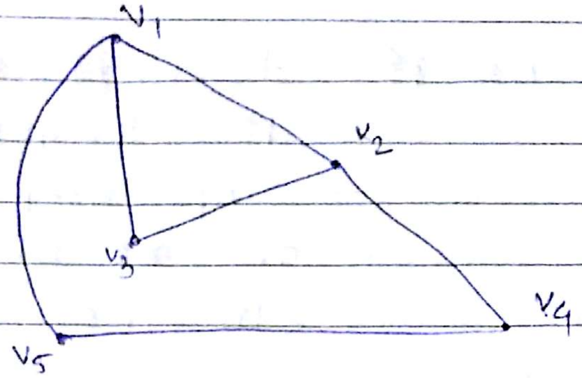
Proof: Let G be a graph with n vertices $n-1$ edges and no circuit. If possible suppose G is not connected. Then G must have components say c_1 and c_2 (at least two).

→ Since G is without circuit, c_1 and c_2 also without circuit. Let, $v_1 \in c_1$ and $v_2 \in c_2$. Then there exist no path between v_1 and v_2. Now if we add an edge e b/w v_1 and v_2 we get a new connected graph G' with n vertices, ~~and~~ n edges and without circuits. Which is a contradiction as a circuitless connected graph with n vertices has $n-1$ edges. Therefore Our Supposition is wrong. G must be connected.

⇒ Distance in a Connected graph –

In a connected graph G, the distance b/w two of its vertices v_i and v_j is the length of the shortest path b/w them and it is denoted by $d(v_i, v_j)$.

e.g.



$d(v_1, v_5) = 1$
$d(v_1, v_4) = 2$.
$d(v_4, v_3) = 2$.

Theorem: The distance b/w vertices of a connected graph is a metric.

⇒ Metric – If M is a non-empty set and ρ is a function such that $\rho : M \times M \rightarrow R^+ \cup \{0\}$ then ρ is said to be metric if,

(i) $\rho(x,y) \geq 0$ & if $\rho(x,y) = 0 \iff x = y$.
(ii) $\rho(x,y) = \rho(y,x)$.
(iii) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$

Proof: Let G be a connected graph and $v_1, v_2, v_3 \in G$ be arbitrary vertices.

(i) We have $d(v_1, v_2)$ = the length of shortest path b/w v_1 and v_2
= the no of edges in the shortest path b/w v_1 and v_2.

$\therefore\ d(v_1, v_2) \geq 0$.

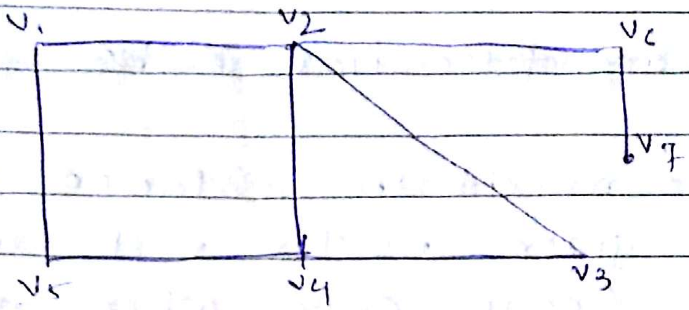
and if $d(v_1, v_2) = 0$.
= the length of shortest path b/w v_1 and v_2
= the no. of edges b/w v_1 and in the shortest path.

$\Leftrightarrow v_1 = v_2$.

(ii) $d(v_1, v_2)$ = the length of shortest path b/w v_1 and v_2.
= the no. of edges in the shortest path b/w v_1 and v_2
= the no of edges in the shortest b/w v_2 and v_2
= the length of shortest path b/w v_2 and v_1.
$\therefore d(v_1, v_2) = d(v_2, v_1)$.

(iii) Since, $d(v_1, v_2)$ = the length of shortest path b/w v_1 and v_2
that doesn't exist any other part in G which is shorter than this
Therefore $d(v_1, v_2) \leq d(v_1, v_3) + d(v_3, \ldots$

$\Rightarrow$ Eccentricity — The essentricity $E(v)$ of any vertex v in a graph G is the distance from v to the vertex farthest from it in a G.



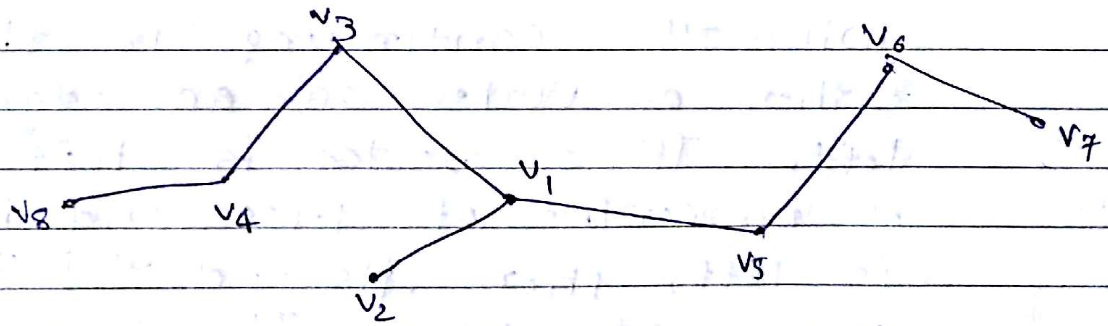
$E(v_1) = 3.$
$E(v_2) = 2.$
$E(v_3) = 3.$

⇒ Centre – A vertex with minimum eccentricity in a graph G is called a centre of G.

⇒ radius – The eccentricity of the centre in a tree is defined as the radius of a tree.

⇒ Diameter – The Diameter of the tree T is the length of the longest path in T.

e.g.



Here, centre = v_1
$E(v_1) = 3.$

radius = 3.
Diameter = 6.

Scanned by CamScanner

Theorem: Every tree has either one or two centres

Proof: The maximum distance d(v, vi) from a given vertex v to any other vertex vi occur only when vi is a pendent vertex. Clearly, A tree with one vertex and a tree with two vertices have one and two centres respectively.

→ Let. T be a tree with more than two vertices. Therefore T has atleast two pendent vertices. Delete all the pendent vertices from T and let T' will be the resultant graph then T' is also a tree.

→ The centre of T will still remain the centre of T' (∵ By definition of centre) Again remove all the pendent vertices from T' and let the resultant tree will T''. Continuing in this way, either a vertex or an edge will left. If a vertex is left, then it is a centre of tree and if an edge is left, then its end vertices are two centre of tree. Therefore every tree has either one or two centres classmate Date Page

Note: If a tree has two centres, then they must be adjecent.

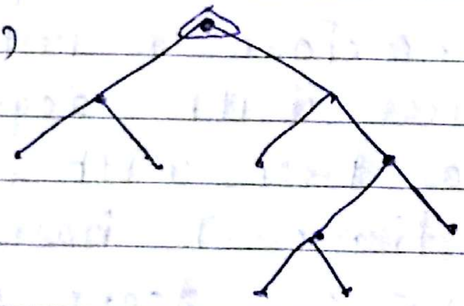
⇒ Rooted and Binary Tree –

(1)→ A tree in which one vertex is distinguish from all the other vertices is called a rooted tree. Such a distinguish vertex is called a root of a tree and it is generally enclosed in a triangle.

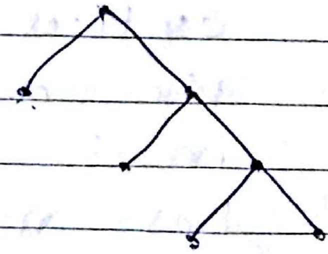
(2) A tree in which there is exactly one vertex of degree 2 and each of the remaining vertices are of degree 1 or 3 is called a binary tree.

e.g. of binary tree –

(i)



(ii)



2-ary tree (Binary Tree)

⇒ Properties of binary tree —

(1) The no. of vertices in a binary tree is always odd.

Proof: Let, T be a binary tree with n vertices therefore, by definition One vertex of T is of degree two and remaining $n-1$ vertices are of degree 1 or 3. That is $n-1$ odd ~~qu~~ degree vertices are in T. But the no. of odd degree vertices in any graph is even that is $n-1$ is ~~odd~~ even and $n-1+1$ is ~~even~~ odd so, n is odd.

(2) No. of pendent vertices in a binary tree with n vertices is $(n+1)/2$.

→ Let, T be a binary tree with n vertices and let p be the no. of pendent vertices in T. Therefore $n-p-1$ will be the no. of vertices with degree 3 in T. Since, a tree with n vertices has $n-1$ edges. Hence T has $n-1$ edges. Therefore the sum of degree of all vertices in T will be $2(n-1)$. Therefore

$$2\cdot 1 + 1\cdot p + 3\cdot(n-p-1) = 2(n-1)$$

$$\therefore 2 + p + 3n - 3p - 3 = 2(n-1)$$

$$\therefore 3n - 2p - 1 = 2(n-1)$$

$$\therefore 3n - 2p - 1 = 2n - 2$$

$$\therefore n + 1 = 2p \Rightarrow p = n+1/2.$$

⇒ Internal vertices – A non-pendent vertex in a tree is called an internal vertex.

(3) The no. of internal vertices in a binary tree with p pendent vertices is $p-1$.

proof: Let, T be a binary tree with n vertices. and let p, the no. of pendent vertices in T. Therefore $n-p$ will be the no. of internal vertices in T.

$$n-p$$

$$= n - \left(\frac{n+1}{2}\right)$$

$$= \frac{2n-n-1}{2}$$

$$= \frac{n-1}{2}$$

$$= \frac{n+1-2}{2}$$

$$= \frac{n+1}{2} - 1$$

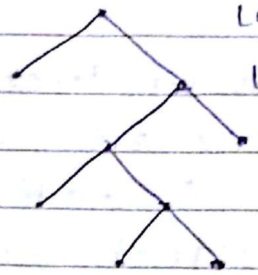
$$= p-1.$$

⇒ Level – In a Binary tree, a vertex v_i is said to be at level l_i. If v_i is at distance l_i from the root.

⇒ Height of a binary tree –

→ Maximum level of any vertex [l_{max}] in a binary tree is called the height of the binary tree.

e.g.



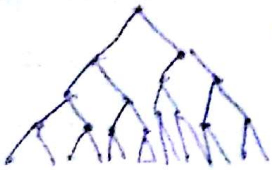
Height of a binary tree is $\underline{4}$

Result-1 In usual notation prove that Max $l_{max} = \frac{n-1}{2}$. Or maximum possible height of a binary tree with n vertices is given by $\frac{n-1}{2}$.

Proof: The height of a binary tree is given by the max. level of any vertex. Again, by definition of a binary tree and l_{max}, there should be minimum possible vertices at each level. Clearly at level 0, only one vertex and for remaining level, there are minimum 2 vertices.

→ Let, T be a binary tree with n vertices and k level, then we have,

$$1 + 2 + 2 + 2 \ldots (k \text{ times}) = n$$



Min level

Max. level

$\therefore 1 + 2k = n$

$\therefore k = \frac{n-1}{2}$

→ That is $\text{Max}\ l_{max} = \frac{n-1}{2}$.

Result. 2 In usual notation prove that $\text{Min}(l_{max}) = [\log_2(n+1) - 1]$ or min possible height of a binary tree with n vertices is given by $[\log_2(n+1) - 1]$.

Proof: The height of a binary tree is given by the max level of any vertex. For minimum of l_{max}, there should be max. possible vertices at each level.
Clearly, The max. no. of vertices at level k is 2^k. Let T be a binary tree with n vertices at k level. Then we have,

$$2^0 + 2^1 + 2^2 + \dots + 2^k \geq n.$$

$$\therefore \frac{2^{k+1} - 1}{\cancel{2}_2 - 1} \geq n \qquad a\left(\frac{r^n - 1}{r-1}\right)$$

n = no. of terms

$$\therefore \frac{2^{k+1} - 1}{2-1} \geq n.$$

$$\therefore 2^{k+1} - 1 \geq n$$

$$\therefore 2^{k+1} \geq n + 1$$

taking log at both sides.

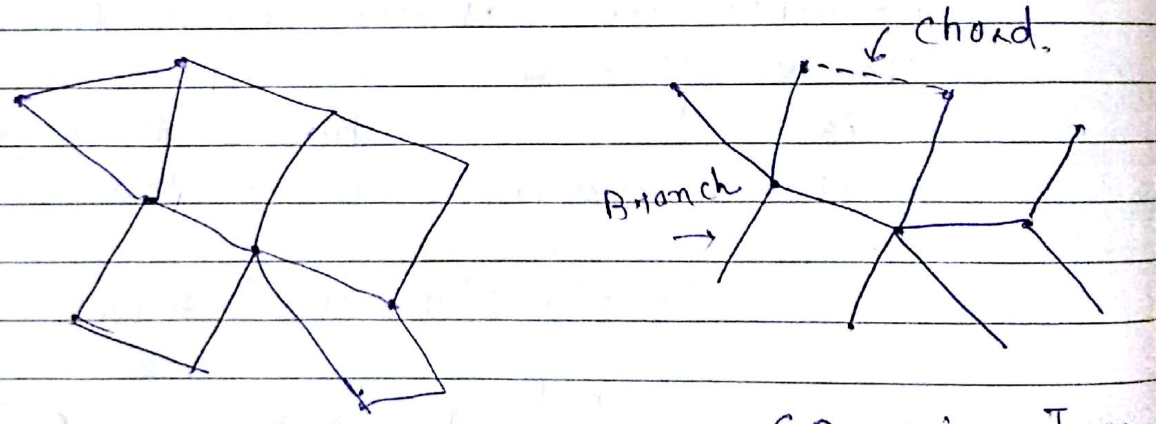
$\therefore k+1 \log 2 \geq \log(n+1)$

$\therefore k+1 \geq \log_2(n+1)$

$\therefore k \geq \lceil \log_2(n+1) - 1 \rceil$.

→ Since level is always an integer. Therefor $\min[l_{max}] = \lceil \log_2(n+1) - 1 \rceil$.

⇒ Spanning Tree - A tree T is said to be spanning tree of a connected graph G if T is a subgraph of G and it contains all the vertices of G



graph   Spanning Tree

⇒ Branch - An edge in a Spanning Tree T is called a branch of tree T.

⇒ Chord - An edge of G, which is not in spanning tree is called chord or $\boxed{\text{link}}$ with respect to that spanning tree.

Theorem: Every connected graph has atleast one spanning tree —

proof: Let G be a connected graph with n vertices.

→ Case-1: If G has no circuits, then Clearly, G itself is a spanning tree in G.

→ Case-2: If G has circuits, and let c be a circuit in G. Delete an edge e from c. Then G will remain connected with all vertices.

→ If there is not circuit left in G, then G-{e} is a connected graph without circuits with n vertices. Therefore G-{e} will be a spanning tree of G. If there is a circuit left in G-{e} then repeat the operation till an edge from the last circuit is deleted. Then last remaining graph is a spanning tree. Therefore every connected graph has atleast one spanning tree.

Theorem: With respect to any of its spanning tree, a connected graph of n vertices and e edges has n-1 tree branches and (e-n+1) chords.

⇒ Rank and Nullity of a graph –

→ Rank of a graph is denoted by r that is defined as $r = n - k$.

→ Nullity of a graph is denoted as μ that is defined as $\mu = e - n + k = e - r$

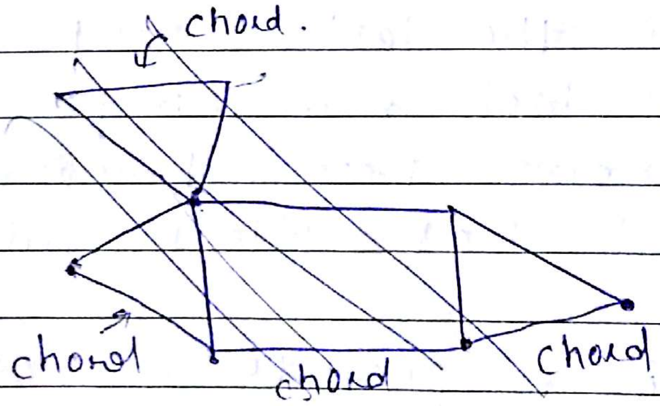
where n = No. of vertices in G.
e = No. of edges in G
k = No. of components in G.

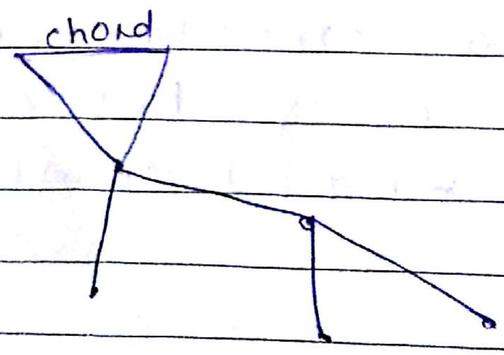
Note: (1) If G is connected, then rank of G is $r = n - 1$ and nullity of G, $\mu = e - n + 1$

⇒ Fundamental circuit –

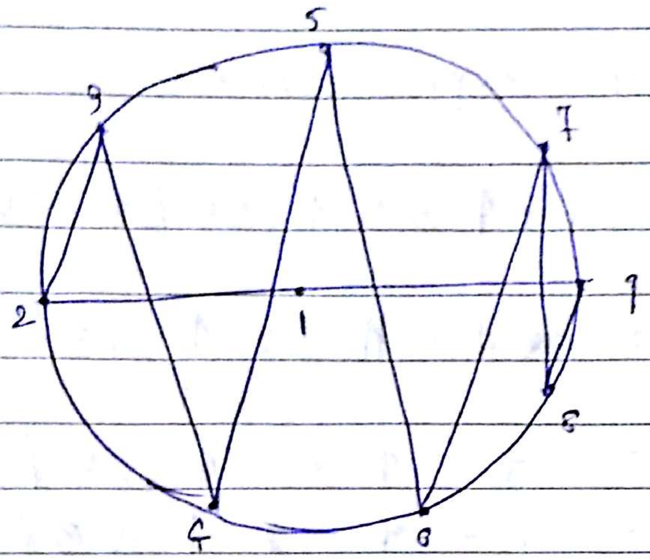
A circuit formed by adding one chord to a spanning tree is called a fundamental circuit.

e.g.





⇒ ~~City~~ Seating Arrangement problem —



→ Adjacent vertices should not remain same —

$\uparrow$ = exponencial.

⇒ Find the value of

(1) Prefix expression –

e.g. $+ - \uparrow 3\ 2 \uparrow 2\ 3 / 8 - 4\ 2$

$= + - \uparrow 3\ 2 \uparrow 2\ 3 / 8\ (4-2)$

$= + - \uparrow 3\ 2 \uparrow 2\ 3 / 8\ 2$

$= + - \uparrow 3\ 2 \uparrow 2\ 3\ (8/2)$

$= + - \uparrow 3\ 2 \uparrow 2\ 3\ 4$

$= + - 3^2\ 2^3\ 4$

$= + (9-8)\ 4$

$= + 1\ 4$

$= 5$

(2) postfix expression –

$7\ 2 - 3 + 2\ 3\ 2 + - 1\ 3 - * /$

$= (7-2)\ 3 + 2\ 3\ 2 + - 1\ 3 - * /$

$= (5+3)\ 2\ 3\ 2 + - 1\ 3 - * /$

$= 8\ 2\ 5 - 1\ 3 - * /$

$= 8\ {-3}\ {-2} * /$

$= 8\ 6 /$

$= 8/6$

$= 4/3$

Examples

(1) Prefix: + - * 2 3 5 / ↑ 2 3 8

= + - * 2 3 5 / 2^3 8

= + - * 2 3 5 8 8

= + - * 2 3 5 1

= + - 6 5 1

= + 1 1

= 2.

(2) Postfix: 3 2 * 2 ↑ 5 3 - 8 4 / * -

= 6 2 ↑ 5 3 - 8 4 / * -

= 36 5 3 - 8 4 / * -

= 36 2 8 4 / * -

= 36 2 2 * -

= 36 4 -

= 32.