3. Group Theory.

⇒ Group –

→ If G is a non-empty set and $*$ be a binary operation of G, then the algebric system $(G, *)$ is called a group if the following conditions are satisfied.

(1) For any $a, b, c \in G$.
$a * (b * c) = (a * b) * c$ (∵ Associativity)

(2) There exist an element $e \in G$ such that for any element $a \in G$,
$a * e = a = e * a$. (∵ Identity)

(3) For any element $a \in G$, there exist an element $\exists\, a^{-1} \in G$. Such that
$a * a^{-1} = e = a^{-1} * a$. (∵ Inverse)

→ If S is a non-empty set and $*$ be a binary operation on S then the Algebric System $\{S, *\}$ is called a semi-group. If the operation $*$ is associative.

→ If a Semi-Group $\{S, *\}$ has an identity element m with respect to the operation
doubt $*$ then $\{m, *\}$ is called as monoid

→ monoid is a Semi-group with identity element.

exa. Show that Q^+ of all positive rational number forms and abilian group under the operation $*$ defined by $a, b \in Q^+$,
$a * b = \frac{1}{2} ab$.

(i) $a*(b*c) = a*(\frac{1}{2}bc)$
$= \frac{1}{2} a (\frac{1}{2}bc)$
$= \frac{1}{2}(\frac{1}{2}ab) \cdot c$
$= (\frac{1}{2}ab) * c$
$= (a*b)*c$

(ii) $e \in Q^+$
$a \in Q^+$

$a * e = a$
$\frac{1}{2} ae = a$
$\therefore e = 2$

(iii) $a^{-1} \in Q^+$.
$a * a^{-1} = e$.
$\therefore \frac{1}{2} a \cdot a^{-1} = 2$
$\therefore a^{-1} = 4/a$

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If $*$ is a binary operation on the set R of real no. defined by $a*b = a+$
varify $\{R, *\}$ is a semi-group or not
Is it commutative?

$a * b = a + b + 2ab$

so, $a*(b*c)$

$= a * (b + c + 2bc)$.
$= a + b + c + 2bc + 2a(b + c + 2bc)$

$= a + b + c + 2bc + 2ab + 2ac + 4abc$

and $(a*b)*c$
$= (a + b + 2ab) * c$

$= a + b + 2ab + c + 2c(a + b + 2ab)$
$= a + b + 2ab + c + 2ac + 2bc + 4abc$

so, $a*b)*c = a*(b*c)$

so, It is associative. so, It is semi-group

$a*b = a + b + 2ab$
$b*a = b + a + 2ba$.

$\therefore a*b = b*a$.

so, It is commutative.

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$ad-bc \neq 0$.

Ex 9 If M_2 is a set of 2×2 non-singular Matrices over R. Prove that M_2 is a group under the operation of usual Matrix multiplication. Is it commutative.

+2ab.

(→) Here, $M_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} / \ ad-bc \neq 0, \ a, b, c, d \in R \right\}$

→ (i) For any $a, b, c \in M_2$. By Matrix theory we know that associative property is true. That is $A(BC) = (AB) \cdot C$.

→ (ii) There exist an element

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in M_2$, such that for any $A \in M_2$, $AI = IA = A$. That is identity exists.

→ (iii) for any matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2$.

there exist an element $\frac{1}{ad-bc} \text{adj } A \in M_2$

$A^{-1} = \frac{1}{|A|} \text{adj } A$.

That is A^{-1} exist for any $A \in M_2$.

So, We can say that M_2 satisfy all the properties of group w.r.t multiplication.

→ Matrices doesn't hold the property of commutivity.

⇒ When $\{G, *\}$ is a finite group (elements are fixed), then no of elements of G is called the order of G and is denoted by $o(G)$ or $|G|$.

⇒ A group $\{G, *\}$ in which binary operation $*$ is commutative is called a commutative group or abilian group.

⇒ If the element $a \in G$, where G is a group with identity element e, then the least positive integer m for which $a^m = e$ is called the order of the element a and is denoted as $o(a)$.

⇒ Properties of a Group —

(1) The identity element of group $\{G, *\}$ is Unique.

Proof: If any group has two identity elements e_1 and e_2 for any $a \in G$

so, $e_1, e_2 \in G$

$$a * e_1 = e_1 * a = a.$$
$$a * e_2 = e_2 * a = a.$$

→ Now, Assume that first e_2 is an identity element, then we can write for e_1, as any element

$$e_1 * e_2 = e_2 * e_1 = e_1$$

→ Similarly, Assuming e_1 as identity element then for e_2,

$$e_2 * e_1 = e_1 * e_2 = e_2.$$

so, $e_1 = e_2$

so, for any $a \in G$,

$$a * e_1 = e_1 * a = a$$
$$a * e_2 = e_2 * a = a.$$

so, identity element will always be Unique.

(2) The inverse of $\{G, *\}$ is Unique.

Proof: For any $a \in G$ there is $\exists\, b, c \in G$.

→ Assuming that b and c are the inverse of a.

so, $a * b = b * a = e.$
$a * c = c * a = e.$

Now, $b = b * e$
$= b * (\cancel{b * a}\ a * c)$
$= (b * a) * c$ (∵ Group has associativity)

∴ $b = e * c$.

∴ $b = c$.

(3)

$a * b = a * c$

→ Now, Assuming that $a, b, c \in G$ and $\bar{a}^1$ also $\in G$.

so, $\bar{a}^1 * (a * b) = \bar{a}^1 * (a * c)$

$\therefore (\bar{a}^1 * a) * b = (\bar{a}^1 * a) * c$

$\therefore e * b = e * c$

$\therefore b = c.$

$\therefore a * b = a * c = b = c$

(ii) $b * a = c * a$

$(b * a) * \bar{a}^1 = (c * a) * \bar{a}^1.$

$\therefore b * (a * \bar{a}^1) = c * (a * \bar{a}^1)$

$\therefore b * e = c * e$

$\therefore b = c.$

Exa If $\{G, *\}$ is an abilian group show that $(a*b)^n = a^n * b^n \ \forall\, a, b \in G$ where n is a positive integer.

→ We will prove this by Mathematical Induction.

(i) for $n = 1 \Rightarrow (a*b)^1 = a^1 * b^1$

$$\therefore (a*b) = a*b$$

so. for $n=1$, it is proved.

(ii) for $n = 2 \Rightarrow (a*b)^2 = a^2 * b^2$.

$$L.H.S. = (a*b) * (a*b)$$

$$= a * (b*a) * b \quad (\because \text{Associativity})$$

$$= a * (a*b) * b \quad (\because \text{commutative})$$

$$= (a*a) * (b*b)$$

$$= a^2 * b^2$$

(iii) Now, Assuming that,

$$(a*b)^k = a^k * b^k.$$

(iv) for $n = k+1$.

$$(a*b)^{k+1} = (a*b)^k * (a*b)$$

$$= (a^k * b^k) * (a*b)$$

$$= (b^k * a^k) * a * b \quad (\because \text{Commut. Associati.})$$

$$= b^k * (a^k * a) * b$$

$$= b^k * (a^{k+1} * b)$$

$$= (b^k * b) * a^{k+1}$$

$$\therefore (a*b)^{k+1} = a^{k+1} * b^{k+1}$$

→ so, we can say that, $(a*b)^n = a^n * b^n$

(4) Prove that for any $a, b \in \{G, *\}$,
$(a*b)^{-1} = b^{-1} * a^{-1}$.

Proof: $a*b = x$ and $b^{-1} * a^{-1} = y$.

taking, $x * y$

$$= (a*b) * (b^{-1} * a^{-1})$$

$$= a*(b*b^{-1}) * a^{-1} \quad (\because \text{Associativity}).$$

$$= \underline{a*e} * a^{-1} = a * a^{-1} = e.$$

$$= \cancel{a * a^{-1} * e} = \cancel{a * a^{-1} * e}$$

$$= \cancel{e * a^{-1}} = e. = \cancel{e * e} = e$$

Similarly, $y * x$

$= (b^{-1} * a^{-1}) * (a * b)$

$= b^{-1} * (a^{-1} * a) * b$ ($\because$ Associativity)

$= b^{-1} * e * b$

$= \cancel{b^{-1} * b * e}$ $b^{-1} * b$

$= e$

so, $x * y = e$

$y * x = e$.

so, $x = y^{-1}$

and $y = x^{-1}$.

so, $(a * b)^{-1} = b^{-1} * a^{-1}$.

(5) If $a, b \in \{G, *\}$ then the eq. $a * x = b$ has a unique sol., $x = a^{-1} * b$

Proof: Suppose, we assume that eq. $a * x = b$ has two solutions,

$x = p$ and $x = q$.

so, $a * p = b$ and $a * q = b$.

$\therefore a * p = a * q$.

$\therefore p = q$

so, eq. has only unique solution.

Note: An element $a \in G$ is called as idempotent element, if $a * a = a$.

(6) A group $\{G, *\}$ can't have an idempotent element except identity element.

Proof: Suppose, a is an idempotent element,
$a = a * a$.

Now, suppose $a = e$. that we have to prove

$a * a' = e$ ∴ $a * e = a$
∴ $(a * a) * a' = e$ ∴ $(a * a) * e = a$
∴ $a * (a * a') = e$ ∴ $a * (a * e) = a$
∴ $a * e = e$ ∴ $a * a = a$
$\boxed{\therefore a = e}$

exa. If $*$ is defined on Z such that $a * b = a + b + 1$ is a group. Prove

(i) $a * (b * c) = a * (b + c + 1)$
$= a + b + c + 1 + 1$

$= a + b + c + 2.$

$(a * b) * c = (a + b + 1) * c$
$= a + b + 1 + c + 1$

$= a + b + c + 2.$
$\therefore a * (b * c) = (a * b) * c$

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(ii) Now, $a * e = a$.

$\therefore a + e + 1 = a$

$\therefore e = -1 \in Z$.

so, identity exists.

(iii) $a * \bar{a}^1 = e$.

$\therefore a + \bar{a}^1 + 1 = e$

$\therefore a + \bar{a}^1 = -2$

$\therefore \bar{a}^1 = -2 - a$.

a is an integer. so, $\bar{a}^1 \in Z$.

so, inverse exists.

Show that set of all polynomial of x under the operation of addition is a group.

$P(x) = \{ p(x) / p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots \}$.

where, $a_0, a_1, a_2 \dots a_n \in R$.

suppose, we have, $p(x)$, $q(x)$ and $r(x)$ $\in P(x)$

(i) $p(x) + [q(x) + r(x)]$

$= (a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots)$

$+ [(b_0 + c_0) + (b_1 + c_1)x + (b_2 + c_2)x^2 + \dots + (b_n + c_n)x^n]$

$= (a_0 + (b_0 + c_0)) + (a_1 + (b_1 + c_1))x + (a_2 + (b_2 + c_2))x^2$
$+ \dots + (a_n + (b_n + c_n))x^n + \dots$

$$= \left((a_0 + b_0) + c_0 \right) + \left[((a_1 + b_1) + c_1)x + ((a_2 + b_2) + c_2) \right.$$
$$\left. + \dots + ((a_n + b_n) + c_n)x^n + \dots \right]$$

($\because$ Associativity)

$$= [P(x) + q(x)] + r(x)$$

(ii) for any $p(x) \in P(x)$ $\exists$ $0(x) \in P(x)$

Such that $P(x) + 0(x) = P(x)$

where, $0(x) = 0 + 0x + 0x^2 + \dots + 0x^n + \dots$

(iii) $p(x) = a_0 + a_1 x + \dots + a_n x^n \in P(x)$

$p^{-1}(x) = -a_0 - a_1 x - \dots - a_n x^n \in (-P(x))$

so, $p(x) + p^{-1}(x) = 0(x) = p^{-1}(x) + p(x)$

exa. Prove that the sets $S = \{0, 1, 2, 3, 4\}$ is an finite abilion group of order 5 under addition % 5 as composition

$+_5$	0	1	2	3	4	~~5~~
0	0	1	2	3	4	~~0~~
1	1	2	3	4	0	~~1~~
2	2	3	4	0	1	~~2~~
3	3	4	0	1	2	~~3~~
4	4	0	1	2	3	~~4~~
~~5~~	~~0~~	~~1~~	~~2~~	~~3~~	~~4~~	~~0~~

(composition table)

From the table it is clear that set has Associative property

For any element $a \in$ set.
for identity,

$a * e = a$

here, $a +_5 e = a$

$\therefore e = 0$.

For inverse, there should exist,

$a * a^{-1} = e$.

here $1 +_5 4 = 0 = 4 +_5 1$.

$2 +_5 3 = 0 = 3 +_5 2$

so, $1, \cancel{\text{and}}\ 4$ and $2, 3$

Prove that set $= \{1, 3, 7, 9\}$ is ~~an~~ an abilion group under multiplication % 10.

$\times_{10}$	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

(i) It is close (All the elements belong to the set) so it is associative.

(ii) Here,
$1 \times_{10} 1 = 1$
$3 \times_{10} 1 = 3$
$7 \times_{10} 1 = 7$
$9 \times_{10} 1 = 9$.

so Identity element $e = 1$.

(iii) Here, 3 and 7 are the inverse of each other. Because,

$3 \times_{10} 7 = e = 1$
and $7 \times_{10} 3 = e = 1$.

⇒ Permutation Group – (The arrangement of elements of any set is called as a "Permutation")

(1) Bijective mapping –

A Bijective mapping of a non-empty set $S \to S$ is called a permutation S.

→ If $S = \{a, b\}$ then permutation set,

$$P_1 = \begin{pmatrix} a & b \\ a & b \end{pmatrix} \leftarrow \text{Permutation.}$$

$$P_2 = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

→ If $S = \{a, b, c\}$ then

$$P_1 = \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix} \qquad P_4 = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}$$

$$P_2 = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix} \qquad P_5 = \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}$$

$$P_3 = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix} \qquad P_6 = \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}$$

→ The composition operator $*$ can be defined by,

$$P_1 * P_2 = \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix} * \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}$$

$$= \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}$$

$$= P_2$$

⇒ Permutation Group -

→ The set G of all permutation on a non-empty set S under the binary operation $*$ of right composition of permutation is a group $\{G, *\}$ called the permutation group.

→ If $S = \{1, 2, 3 \ldots, N\}$, the permutation group is called the symmetric group of degree n and denoted by S_n.

The no. of elements of $S_n = n!$

$*$	P_1	P_2	P_3	P_4	P_5	P_6
P_1	P_1	P_2	P_3	P_4	P_5	P_6
P_2	P_2	P_1	P_4	P_3	P_6	P_5
P_3	P_3	P_6	P_5	P_2	P_1	P_4
P_4	P_4	P_5	P_6	P_1	P_2	P_3
P_5	P_5	P_4	P_1	P_6	P_3	P_2
P_6	P_6	P_3	P_4	P_5	P_4	P_1

$$P_1 * P_1 = \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix} * \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}$$

$$= \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}$$

$$= P_1$$

$$P_2 * P_1 = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix} * \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}$$

$$= \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}$$

$$= P_2$$

$$P_1 * P_2 = P_2.$$

$$P_1 * P_3 = \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix} * \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$$

$$= \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$$

$$= P_3.$$

$$P_2 * P_3 = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix} * \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$$

$$= \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}$$

$$= P_4.$$

$$P_2 * P_4 = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix} * \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}$$

$$= \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$$

$$= P_3.$$

$$P_2 * P_5 = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix} * \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}$$

$$= \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}$$

$$= P_6$$

$$P_2 * P_6 = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix} * \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}$$

$$= \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}$$

$$= P_5$$

$$P_3 * P_1 = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix} * \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}$$

$$= \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$$

$$= P_3.$$

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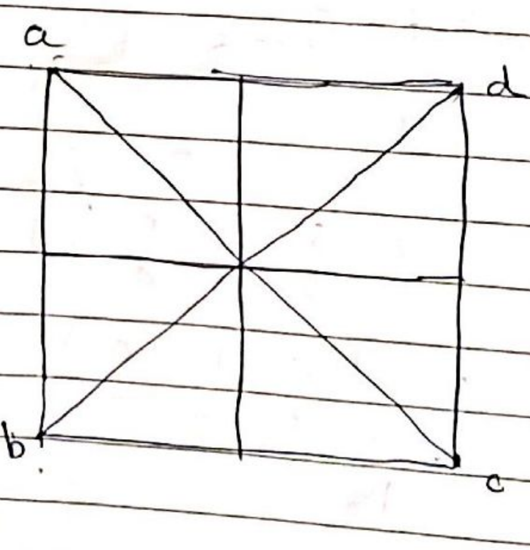
⇒ Dihedral group –

→ The set of transformation due to all rigid motion of a regular polygon of n sides resulting in identical polygons, but with different vertex names under the binary operation of right compotition $*$ is a group called "Dihedral Group". denoted by $\{D_n, *\}$. By rigid motion, we mean the rotation of regular polygon about its centre through angles $1 \times \frac{360}{n}, 2 \times \frac{360}{n}, \ldots n \times \frac{360}{n}$

in the anti-clock wise direction. and reflaction of a regular polygon about its line of symmetry.

<u>Que.</u> Prove that $\{D_4, *\}$. is a group.



$$P_1 = \begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix}$$

$$P_2 = \begin{pmatrix} a & b & c & d \\ d & a & b & c \end{pmatrix}$$

$$P_3 = \begin{pmatrix} a & b & c & d \\ c & d & a & b \end{pmatrix}$$

$$P_4 = \begin{pmatrix} a & b & c & d \\ b & c & d & a \end{pmatrix}$$

$$P_5 = \begin{pmatrix} a & b & c & d \\ d & c & b & a \end{pmatrix}$$

$$P_6 = \begin{pmatrix} a & b & c & d \\ a & d & c & b \end{pmatrix}$$

$$P_7 = \begin{pmatrix} a & b & c & d \\ b & a & d & c \end{pmatrix}$$

$$P_8 = \begin{pmatrix} a & b & c & d \\ c & b & a & d \end{pmatrix}$$

*	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
P_1	P_1	P_5	P_3	P_4				
P_2								
P_3								
P_4								
P_5								
P_6								
P_7								
P_8								

<u>example</u> If $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 5 & 4 & 6 & 2 \end{pmatrix}$

$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 6 & 2 & 4 \end{pmatrix}$

then find $\alpha\beta$, $\beta\alpha$, α^2, β^2, α^{-1}, β^{-1}

$$\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 5 & 4 & 6 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 6 & 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 2 & 6 & 4 & 3 \end{pmatrix}$$

$$\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 6 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 5 & 4 & 6 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 3 & 2 & 1 & 4 \end{pmatrix}$$

$$\alpha^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 5 & 4 & 6 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 5 & 4 & 6 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 6 & 4 & 2 & 1 \end{pmatrix}$$

$$\beta^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 6 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 6 & 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 4 & 3 & 6 \end{pmatrix}$$

$$\alpha^{-1} = \begin{pmatrix} 3 & 1 & 5 & 4 & 6 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

$$\beta^{-1} = \begin{pmatrix} 5 & 3 & 1 & 6 & 2 & 4 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

Exa. If α and β are elements of symmetric group S_4,

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

$\alpha\beta$, $\beta\alpha$, α^2, α^{-1} and also find the Order of α.

$$\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$$

$$\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

$$\alpha^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\alpha^{-1} = \begin{pmatrix} 3 & 4 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$\alpha\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = e.$$

$$\alpha^4 = \alpha * \alpha * \alpha * \alpha$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

so, Order is $\underline{4}$.

In symmetric group S_3 find all those elements a and b such that,

(1) $(a*b)^2 \neq a^2 * b^2$

(2) $a^2 = e$, $a^3 = e$

~~(1) $P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$~~

~~$P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & c & b \end{pmatrix}$~~

(1) $P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

$P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

$P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

$P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, $P_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

⇒ Cyclic group –

A group {G, *} is said to be cyclic, if there exist an element $a \in G$, such that every element x of G can be expressed as $x = a^n$ for some integer in such a case the cyclic group is said to be generated by a or a is a generator of g.

e.g. $S = \{(1, -1, i, -i), \times\}$.
w.r.t Multiplication

*	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

Table is closed. so, Associativity is true

→ and identity element is 1.

and inverse also exists

i and -i are inverse of each other.

so, S is a group.

$i^1 = i$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$

so, i is a generator of G.

⇒ properties of a cyclic group

(1) Cyclic group is abilian.

Proof: Let $b, c \in G$ where G is a cyclic group and a is a generator of G for any integer m and n, $b = a^m$ and $c = a^n$.

Now, $b * c = a^m * a^n$
$= \underbrace{(a * a * a \ldots * a)}_{m \text{ times}} * \underbrace{(a * a * \ldots * a)}_{n}$
$= a^{m+n}$
$= a^{n+m}$
$= c * b.$

so, Cyclic group is abilian.

(2) If a is a generator of a cyclic group $\{G, *\}$. then $\bar{a}^{1}$ is also a generator of $\{G, *\}$.

Proof: Let. $b \in G$, then $b = a^m$ where m is an integer.

$$b = a^m \leftarrow \text{integer.}$$
$$= (\bar{a}^{1})^{-m}$$

This shows that $\bar{a}^{1}$ is also a generat of $\{G, *\}$.

exa. Show that a group $S = \{(1, 2, 3, 4, 5, 6), \times_7\}$ is a cyclic group. How many generators are there for this group? what are they?

exa. Show that every group of order 3 is cyclic and every group of order 4 abilian.

(1)

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	6	1

so, ~~gp~~

$3^1 = 3$

$3^2 = 3 \times_7 3 = 2$

$3^3 = 2 \times_7 3 = 6$

$3^4 = 6 \times_7 3 = 4$

$3^5 = 4 \times_7 3 = 5$

$3^6 = 5 \times_7 3 = 1$

$3^7 = 1 \times_7 3 = 3.$

so, 3 is a generator of a group.

(5)

⇒ Sub-group –

If $\{G, *\}$ is a group and H subset of (H ⊆ G) is a non-empty subset that satisfy the following condition.

(1) for $a, b \in H$, $a * b \in H$.

(2) $e \in H$, where e is an identity element of group $\{G, *\}$.

(3) $\forall a \in H$ then a^{-1} also $\in H$.

for any.

Example $G = \{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}$.

$H = \{\ldots -4, -2, 0, 2, 4 \ldots\}$.

Here, $\{H, *\}$ is a Subgroup w.r.t. $*$ operation.

but $I = \{\ldots -3, -1, 0, 1, 3 \ldots\}$ is not a Subgroup because for '+' operation $a + b \notin I$. ($\because$ first property is wrong)

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n:

The Necessary and Sufficient condition for a non-emty Subset H of a group $\{G, *\}$ to be a subgroup is $a, b \in H$ then $a * b^{-1} \in H$.

G is a group and H is a Sub-group of H

Let, $a, b \in H$.

$\Rightarrow b \in H$

$\Rightarrow b^{-1} \in H$ ($\because$ H is a Sub-group).

$\Rightarrow a, b^{-1} \in H$.

$\Rightarrow a * b^{-1} \in H$ ($\because$ H is close).

2: Let, $a * b^{-1} \in H$, where $a, b \in H$

~~so, H is a close.~~

~~and $a, b^{-1} \in H$~~

~~$\therefore a, b \in H$~~

~~$\therefore b$~~

As, H is non-empty subset of G. If $b = a$, the given condition gives $a * a^{-1} = e \in H$

Using the given condition for pair $e, a \in H$, we have $e * a^{-1} \in H \Rightarrow a^{-1} \in H$.

Similarly we can prove $b^{-1} \in H$. ($\because b = a$)

Using the condition for the pair $a, b^{-1} \in H$, $a * b^{-1} \in H$

we get $a * (b^{-1})^{-1} \in H$

$\Rightarrow a * b \in H$.

⇒ Group Homomorphism –

If $\{G, *\}$ and $\{G', \Delta\}$ are two Sub-groups then a mapping $f: G \to G'$ is called a group homomorphism if for any $a, b \in$ $f(a * b) = f(a) \Delta f(b)$.

→ A group homo-morphism f is called group isomorphism if f is one-one and onto (bijective).

→ If $f: G \to G'$ is a group homo-morphism from $\{G, *\}$ to $\{G', \Delta\}$. Then (1) $f(e) = e'$ where e and e' are identity element G and G' respectively.

(2) for any $a \in G$, $f(a^{-1}) = [f(a)]^{-1}$.

(3) If H is a Sub-group of G then $F(h) = \{f(h) / h \in H\}$ is a Subgroup of G

(1) $f(e * e) = f(e) \Delta f(e)$ ($\because$ by difinition of homo-morphism)

$\therefore$ $f(e) = f(e) \Delta f(e)$

means $f(e)$ is idempotent element.

but the idempotent element of group is ident

Therefore $f(e) = e'$.

(2) $a \in G$, $\bar{a}^{1} \in G$.

$f(a * b) = f(a) \Delta f(b)$

$\therefore f(a * \bar{a}^{1}) = f(a) \Delta f(\bar{a}^{1})$

$\therefore f(e) = f(a) \Delta f(\bar{a}^{1})$

$\therefore e' = f(a) \Delta f(\bar{a}^{1})$

and $f(\bar{a}^{1} * a) = f(\bar{a}^{1}) \Delta f(a)$.

$\therefore f(e) = f(\bar{a}^{1}) \Delta f(a)$

$\therefore e' = f(\bar{a}^{1}) \Delta f(a)$.

so, $f(\bar{a}^{1}) = [f(a)]^{-1}$.

(3) Here, $h_1, h_2 \in H \subseteq G$.

$h_1' = f(h_1) \in F(H)$

$h_2' = f(h_2) \in f(h)$

and $f(h_1 * h_2) = f(h_1) \Delta f(H_2)$.

Now, $h_1' \Delta h_2'^{-1} = f(h_1) \Delta [f(h_2)]^{-1}$

$= f(h_1) \Delta F(h_2^{-1})$ ($\because$ Result-2).

$= f(h_1 * h_2^{-1})$

$= f(h_3)$ ($\because h_3 = h_1 * h_2^{-1} \in H$)

$= F(H)$.

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Prove that the intersection of two Sub-group of a group G is also a Sub-group of G but Union of two sub groups of group G need not be a Sub-group of

$H_1 = \{\ldots -6, -4, -2, 0, 2, 4, 6 \ldots\}$

$H_2 = \{\ldots -9, -6, -3, 0, 3, 6, 9 \ldots\}$

Here, $H_1 \cap H_2$ ~~H_2~~ is a Subgroup.

but $H_1 \cup H_2 = \{\ldots -9, -6, -4, -3, -2, 0, 2, 3, 4, 6 \ldots\}$

Here, for $\forall a, b \in (H_1 \cup H_2)$,

$a * b^{-1} \notin (H_1 \cup H_2)$.

Kernal of Homomorphism—

If $f: G \to G'$ is a group homomorphism from $\{G, *\}$ $\{G', \Delta\}$ then set of elements which are mapped to e'. That is an identity element G'. That is called the kernal of homomorphism f and it is denoted by $Ker(f)$.

Theorem: The Kernal of homomorphism f from group $\{G, *\} \to \{G', \Delta\}$ is a subgroup of $\{G, *\}$.

Proof: Assuming that, $a, b \in Ker(f)$.

and $f(a) = e'$
$f(b) = e'$.

By Definition of homomorphism,

$$f(a * b^{-1}) = f(a) \Delta f(b^{-1})$$
$$= f(a) \Delta [f(b)]^{-1}$$
$$= e' \Delta (e')^{-1}$$
$$= e'$$

$\therefore f(a * b^{-1}) = e'$

$\therefore a * b^{-1} \in Ker(f)$.

→ so, $Ker(f)$ is a Sub-group of $\{G, *\}$.

If G is a set of all pair (a, b) where $a \neq 0$ and b are real and binary operation $*$ on G defined by $(a, b) * (c, d) = (ac, bc + d)$ show that $\{G, *\}$ is a non-abilian group.
Also show that the subset H of all those elements of G which are of the form $(1, b)$ is ~~the~~ a sub-group of

G is defined by $(a, b) * (c, d) = (ac, bc + d)$

Associativity: (a_1, b_1), (a_2, b_2) and $(a_3, b_3) \in$

$(a_1, b_1) * [(a_2, b_2) * (a_3, b_3)]$

$= (a_1, b_1) * [(a_2 a_3, b_2 a_3 + b_3)]$

$= (a_1, b_1) * (a_2 a_3, b_2 a_3 + b_3)$

$= a_1 a_2 a_3, b_1 a_2 a_3 + b_2 a_3 + b_3.$

and $[(a_1, b_1) * (a_2, b_2)] * (a_3, b_3)$

$= [(a_1 a_2, b_1 a_2 + b_2)] * (a_3, b_3)$

$= a_1 a_2 a_3, a_3(b_1 a_2 + b_2) + b_3$

$= a_1 a_2 a_3, b_1 a_2 a_3 + b_2 a_3 + b_3.$

So, Associativity is proved.

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ii) Identity -

$$(a_1, b_1) * (e_1, e_2) = (a_1, b_1)$$

$$\therefore (a_1 e_1, b_1 e_1 + e_2) = (a_1, b_1)$$

$\therefore a_1 e_1 = a_1$
$\Rightarrow e_1 = 1$

$b_1 e_1 + e_2 = b_1$
$\therefore b_1 + e_2 = b_1$
$\therefore e_2 = 0$

$\therefore$ so, Identity elements pair $(e_1, e_2) = (1, 0)$.

iii) Inverse: Suppose, (c, d) is a inverse of (a, b).

$$(a, b) * (c, d) = (e_1, e_2)$$

$$\therefore (ac, bc + d) = (1, 0)$$

$\therefore ac = 1$ and $bc + d = 0$

$\therefore c = 1/a$ $\qquad b\left(\frac{1}{a}\right) + d = 0$

$$\therefore d = -b/a$$

$$\therefore (c, d) = (1/a, -b/a).$$

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$\Rightarrow$ Non-Abilian -

$(a_1, b_1) * (a_2, b_2) \neq (a_2, b_2) * (a_1, b_1)$.

$\Rightarrow$

$H = \{ (1, b) / b \in G \}$

$H \subseteq G$.

so, $(1, a_1), (1, a_2) \in H$.

Now, $(1, a_1) * [(1, a_2)]^{-1}$

$= (1, a_1) * (1, -a_1)$

$= (1, 0). \in H$.

ple Show that $\{H, *\}$ is a subgroup of a symmetric group $\{S_3, *\}$ of degree 3. where $H = \{P_1, P_2\}$

Here, $H = \{P_1, P_2\}$.

and $P_1 * P_2^{-1}$ ~~$\in H$~~

$= P_1 * P_2$
$= P_2 \in H$

$\therefore P_1 * P_2^{-1} \in H$

and $P_2^{-1} * P_1$

$= P_2 * P_1$
$= P_2 \in H$

$\therefore P_1 * P_2^{-1} \in H$

and $P_1^{-1} * P_2$

$= P_1 * P2$
$= P_2 \in H$

$\therefore P_1 * P_2^{-1} \in H$

and $P_2 * P_1^{-1}$
$= P_2 * P_1$
$= P_2 \in H$

$\therefore P_1 * P_2^{-1} \in H$. $\therefore$ So, H is a Subgroup

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<u>Exa.</u> If G is a set of all pair (a, b), a, b are real numbers and $*$ is a binary operation defined by $(a, b) * (c, d) = (a+$
prove that $\{G, *\}$ is a group, if G' is ~~an~~ the group of all real numbers. Prove that the mapping $f: G \to G'$ defined by $f((a, b)) = a$ for $\forall a, b \in G$ is a homomorphism

→ first, we have to prove ~~$(a, b) * (c, d)$~~.
G is a group.

(i) Associativity: Suppose, (a_1, a_2), (b_1, b_2), $(c$

so, $(a_1, a_2) * [(b_1, b_2) * (c_1, c_2)]$

$= (a_1, a_2) * [(b_1 + c_1, b_2 + c_2)]$.

$= (a_1 + b_1 + c_1, a_2 + b_2 + c_2)$.

and $[(a_1, a_2) * (b_1, b_2)] * (c_1, c_2)$

$= [(a_1 + b_1, a_2 + b_2)] * (c_1, c_2)$

$= (a_1 + b_1 + c_1, a_2 + b_2 + c_2)$.

so, Associativity is proved.

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(ii) Identity :

td)

$$(a_1, a_2) * (e_1, e_2) = (a_1, a_2)$$

$$\therefore (a_1 + e_1, a_2 + e_2) = (a_1, -a_2)$$

$$\therefore a_1 + e_1 = a_1$$
$$\therefore e_1 = e_2 = 0.$$

(iii) Inverse: Suppose, (c,d) is a inverse of (a,b)

$$\text{so, } (a,b) * (c,d) = (e_1, e_2)$$

$\in G$.

$$\therefore (a+c, b+d) = (0, 0)$$

$$\therefore c = -a, \; d = -b.$$

→ so, G is a <u>GROUP</u>.

→ Now Mapping, $f : G \to G'$ defined by
$f(a,b) = a$.

$$\text{so, } f(A * B) = f(A) \, \Delta \, f(B)$$

where, ~~A,B~~ = $A = (a_1, a_2)$, $B = (b_1, b_2) \in G$.

$$\therefore f(A * B) = f((a_1, a_2) * (b_1, b_2))$$
$$= f(a_1 + b_1, a_2 + b_2)$$
$$= a_1 + b_1 \quad (\because f(a,b) = a)$$
$$= f(a_1, a_2) + f(b_1, b_2)$$
$$= f(A) \, \Delta \, f(B)$$

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30. ~~If~~ $f : G \to G'$ is a homomorphism

Example. show that $\{H, *\}$ is a sub-group of a symmetric group $\{S_3, *\}$ of degree where $H = \{P_1, P_3, P_5\}$

Here $P_1, P_3, P_5 \in H$.

$P_1 * P_3^{-1} = P_1 * P_5 = P_5 \in H$

~~$P_3^{-1} * P_1 = P_5 * P_1$~~

$P_3 * P_1^{-1} = P_3 * P_1 = P_3 \in H$

$P_3 * P_5^{-1} = P_3 * P_3 = P_5 \in H.$

$P_5 * P_3^{-1} = P_5 * P_5 = P_3 \in H.$

$P_1 * P_5^{-1} = P_1 * P_3 = P_3 \in H.$

$P_5 * P_1^{-1} = P_5 * P_1 = P_5 \in H.$

$P_3^{-1} * P_1 = P_5 * P_1 = P_5 \in H.$

$P_1^{-1} * P_3 = P_1 * P_3 = P_3 \in H$

$P_5^{-1} * P_3 = P_3 * P_3 = P_5 \in H$

$P_3^{-1} * P_5 = P_5 * P_5 = P_3 \in H.$

$P_5^{-1} * P_1 = P_3 * P_1 = P_3 \in H$

$P_1^{-1} * P_5 = P_1 * P_5 = P_5 \in H$

so, $\{H, *\}$ is subgroup of group $\{S_3, *\}$

⇒ **Cosets** — If $\{H, *\}$ is a sub-group of a group $\{G, *\}$ then the set aH where $a \in G$ defined by $aH = \{a * h \mid h \in H\}$ is called left coset of H in G.
Similarly $Ha = \{h * a \mid h \in H\}$ is called Right coset of H in G.

3. ⇒ **Normal Subgroup** —

A subgroup $\{H, *\}$ of a group $\{G, *\}$ is called a Normal subgroup if for any $a \in G$, $aH = Ha$ (That is the left and Right cosets of H in G generated by a are the same).

Note: $aH = Ha$ doesn't mean that $a * h = h * a$ for any $h \in H$ but it means that $a * h_1 = h_2 * a$ for some $h_1, h_2 \in H$.

Exa → If $\{H, *\}$ is a subgroup of the Symmetric group $\{S_3, *\}$ of degree 3 where $H = \{P_1, P_2\}$. Then find left coset of H in G

Here, $H = \{P_1, P_2\}$ and $\{S_3, *\} = \{P_1, P_2, P_3, P_4, P_5, P_6\}$ ($n!$)

$P_1 H = \{P_1 * P_1, P_1 * P_2\} = \{P_1, P_2\}$
$P_2 H = \{P_2 * P_1, P_2 * P_2\} = \{P_2, P_1\}$
$P_3 H = \{P_3 * P_1, P_3 * P_2\} = \{P_3, P_6\}$
$P_4 H = \{P_4 * P_1, P_4 * P_2\} = \{P_4, P_5\}$
$P_5 H = \{P_5 * P_1, P_5 * P_2\} = \{P_5, P_5\}$
$P_6 H = \{P_6 * P_1, P_6 * P_2\} = \{P_6, P_3\}$

Exa. If G is the addition group of integers and H is a subgroup of G obtained by multiplying each elements of G by 3 find distinct Right cosets of H in G.

→ Here, $\{G, *\} = \{Z, +\}$

$$Z = \{\ldots -4, -3, -2, -1, 0, 1, 2, 3, 4 \ldots\}$$

$$H = \{\ldots -12, -9, -6, -3, 0, 3, 6, 9, 12 \ldots\}$$

→ Right cosets –

$$H_0 = \{\ldots -12, -9, -6, -3, 0, 3, 6, 9, 12 \ldots\}$$
$$H_1 = \{\ldots -11, -8, -5, -2, 1, 4, 7, 10, 13 \ldots\}$$
$$H_2 = \{\ldots -10, -7, -4, -1, 2, 5, 8, 11, 14 \ldots\}$$
$$H_{-1} = \{\ldots -13, -10, -7, -4, -1, 2, 5, 8, 11 \ldots\}$$
$$H_{-2} = \{\ldots -14, -11, -8, -5, -2, 1, 4, 7, 10 \ldots\}$$

Theorem: Normal Subgroup –

A subgroup $\{H, *\}$ of a group $\{G, *\}$ is a Normal subgroup if and only if $a^{-1} * h * a \in H$ for every $a \in G$ and $h \in H$.

Here, $aH = Ha$.

$\therefore \cancel{a * h = h_1}$

$\therefore h * a = a * h_1$

$\therefore a^{-1} * h * a = a^{-1} * a * h_1$

$\therefore a^{-1} * h * a = e * h_1 = h_1 \in H$.

$\therefore \bar{a}^1 * h * a \in H$. —— (1)

so, ~~H is~~

→ Now, Multiply by a.

$a * \bar{a}^1 * h * a \in \cancel{a*h}.\ aH$

$\therefore e * h * a \in \cancel{Ha}\ aH$

$\therefore aHa \in \cancel{Hd}$.

$\therefore aHa \subseteq \cancel{Hd}$.

→ Now, $a \in G$, $\bar{a}^1 \in G$.

$\Rightarrow b = \bar{a}^1 \in G$.

so, $\bar{b}^1 * h * b \in H$

$\therefore (\bar{a}^1)^{-1} * h * \bar{a}^1 \in H$

$\therefore a * h * \bar{a}^1 \in H$

$\therefore (a * h * \bar{a}^1) * a \in Ha$.

$\therefore aH \in Ha$.

$\therefore aH \subseteq Ha. \Rightarrow aH = Ha$

H Exa. Show that intersection of two normal subgroup of a group G is also a normal subgroup of G.

$a \in G$ and $h \in H_1 \cap H_2$.

$\therefore a \in G$ and $h \in H_1$, $h \in H_2$

$\therefore \bar{a}^1 * h * a \in H_1$, $\bar{a}^1 * h * a \in H_2$.

$\therefore \bar{a}^1 * h * a \in H_1 \cap H_2$.

Theorem: The Order of a Subgroup of a finite group is a devison of the Order of the group.

Example. If H is a Normal Subgroup of G and K is a Subgroup of G Such that $H \subseteq K \subseteq G$. so that, H is a normal sub-group of K.

→ Here, H is a Normal Subgroup of G Therefore H is a Subgroup of G. Since, ~~$a \in G$ and~~ $H \subseteq K \subseteq G$ and K is a Subgroup of G. That means h is subgroup of K also. Let x be any element of K then x is an element of G, too.

→ Since H is an Normal Subgroup G, we have $xH = Hx$ for $\forall x \in G$. Since, H is a Subgroup of K and $x \in K$, Hence $xH = Hx$ $\forall x \in K$. Therefore h is a normal Subgroup of K.

Example. Show that $\{P_1, P_2\}$, $\{P_1, P_4\}$, $\{P_1, P_6\}$ are subgroup of Symmetric group of $\{S_3, *\}$ of degree 3. Are they normal Subgroup ?

$\{S_3, *\} = \{P_1, P_2, P_3, P_4, P_5, P_6\}$.

$$H = \{P_1, P_2\}$$

$$P_1 H = \{P_1 * P_1, P_1 * P_2\} = \{P_1, P_2\}$$
$$P_2 H = \{P_2 * P_1, P_2 * P_2\} = \{P_2, P_1\}$$
$$P_3 H = \{P_3 * P_1, P_3 * P_2\} = \{P_3, P_6\}$$
$$P_4 H = \{P_4 * P_1, P_4 * P_2\} = \{P_4, P_5\}$$
$$P_5 H = \{P_5 * P_1, P_5 * P_2\} = \{P_5, P_5\}$$
$$P_6 H = \{P_6 * P_1, P_6 * P_2\} = \{P_6, P_3\}.$$

$$HP_1 = \{P_1 * P_1, P_2 * P_1\} = \{P_1, P_2\}$$
$$HP_2 = \{P_1 * P_2, P_2 * P_2\} = \{P_2, P_1\}$$
$$HP_3 = \{P_1 * P_3, P_2 * P_3\} = \{P_3, P_4\}$$
$$HP_4 = \{P_1 * P_4, P_2 * P_4\} = \{P_4, P_3\}$$
$$HP_5 = \{P_1 * P_5, P_2 * P_5\} = \{P_5, P_6\}$$
$$HP_6 = \{P_1 * P_6, P_2 * P_6\} = \{P_6, P_5\}$$

$$H = \{P_1, P_4\}.$$

$$P_1 H = \{P_1 * P_1, P_1 * P_4\} = \{P_1, P_4\}$$
$$P_2 H = \{P_2 * P_1, P_2 * P_4\} = \{P_2, P_3\}$$
$$P_3 H = \{P_3 * P_1, P_3 * P_4\} = \{P_3, P_2\}$$
$$P_4 H = \{P_4 * P_1, P_4 * P_4\} = \{P_4, P_1\}$$
$$P_5 H = \{P_5 * P_1, P_5 * P_4\} = \{P_5, P_6\}$$
$$P_6 H = \{P_6 * P_1, P_6 * P_4\} = \{P_6, P_5\}.$$

$$HP_1 = \{P_1 * P_1, P_4 * P_1\} = \{P_1, P_4\}$$
$$HP_2 = \{P_1 * P_2, P_4 * P_2\} = \{P_2, P_5\}$$
$$HP_3 = \{P_1 * P_3, P_4 * P_3\} = \{P_3, P_6\}$$
$$HP_4 = \{P_1 * P_4, P_4 * P_4\} = \{P_4, P_1\}$$
$$HP_5 = \{P_1 * P_5, P_4 * P_5\} = \{P_5, P_2\}$$
$$HP_6 = \{P_1 * P_6, P_4 * P_6\} = \{P_6, P_3\}$$

$H = \{ P_1, P_6 \}$

$P_1 H = \{ P_1 * P_1, P_1 * P_6 \}$
$P_2 H = \{ P_2 * P_1, P_2 * P_6 \}$
$P_3 H = \{ P_3 * P_1, P_3 * P_6 \}$
$P_4 H = \{ P_4 * P_1, P_4 * P_6 \}$
$P_5 H = \{ P_5 * P_1, P_5 * P_6 \}$
$P_6 H = \{ P_6 * P_1$

Exa. ~~Show~~ Verify that wether the subgroup $H = \{ P_1, P_2, P_3 \}$ of $\{ S_3, * \}$ is a normal subgroup of $\{ S_3, * \}$.

$\{ S_3, * \} = \{ P_1, P_2, P_3, P_4, P_5, P_6 \}$.

$P_1 H = \{ P_1 * P_1, P_1 * P_2, P_1 * P_3 \} = \{ P_1, P_2, P_3 \}$
$P_2 H = \{ P_2 * P_1, P_2 * P_2, P_2 * P_3 \} = \{ P_2, P_1, P_4 \}$
$P_3 H = \{ P_3 * P_1, P_3 * P_2, P_3 * P_3 \} = \{ P_3, P_6, P_5 \}$
$P_4 H = \{ P_4 * P_1, P_4 * P_2, P_4 * P_3 \} = \{ P_4, P_5, P_6 \}$
$P_5 H = \{ P_5 * P_1, P_5 * P_2, P_5 * P_3 \} = \{ P_5, P_4, P_1 \}$
$P_6 H = \{ P_6 * P_1, P_6 * P_2, P_6 * P_3 \} = \{ P_6, P_3, P_4 \}$

$HP_1 = \{P_1, P_2, P_3\}$
$HP_2 = \{P_2, P_1, P_6\}$
$HP_3 = \{P_3, P_4, P_5\}$
$HP_4 = \{P_4, P_3, P_2\}$
$HP_5 = \{P_5, P_6, P_1\}$
$HP_6 = \{P_6, P_5, P_4\}$

Exa. If G is an additive group of integers & H is Subgroup of G. $H = \{5x / x \in G\}$ find distinct left coset of H in G.

Exa. If H is a Subgroup of G and K is a normal subgroup of G. Verify $H \cap K$ is a normal Subgroup of H or not.

(1) $G\{*, +\}$ is a group of integers.

$G = \{ \ldots -4, -3, -2, -1, 0, 1, 2, 3, 4 \ldots \}$

$H = \{ \ldots -20, -15, -10, -5, 0, 5, 10, 15, 20 \ldots \}$

Left cosets:

$a_0 H = \{\ldots -20, -15, -10, -5, 0, 5, 10, 15 \ldots\}$
$a_1 H = \{\ldots -19, -14, -9, -4, 1, 6, 11, 16 \ldots\}$
$a_2 H = \{\ldots -18, -13, -8, -3, 2, 7, 12, 17 \ldots\}$
$a_{-1} H = \{\ldots -21, -16, -11, -6, -1, 4, 9, 14 \ldots\}$
$a_{-2} H = \{\ldots -22, -17, -12, -7, -2, 3, 8, 13 \ldots\}$

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(2) Here K is a normal subgroup of G.

$h \in K$, $x \in G$.

$\therefore xH = Hx$

and H is a Subgroup of G.
and $H \cap K \in K$.

means $h \in K$.

so, ~~$H \cap K \in h$~~ $h \in H \cap K$.

so. We can say that $xH = Hx$

so, $H \cap K$ is a normal subgroup of G
and H is a Subgroup of G.

$\therefore$ $H \cap K$ is a normal subgroup of H.

Exa. If G is the multiplicative group of all $(n \times n)$ non-singular matrices whose elements are real numbers and G' is the multiplicative group of all non-zero real nos. so that mapping $f: G \to G'$ where $f(A) = |A|$ for $A \in G$ is a homomorphism