# Relations & Lattices.

→ Relations - Let A and B are two non-empty sets then the relations ~~and~~ R is defined as the set.

$$R = \{(a,b) \mid a \in R,\ b \in R\}, \text{ i.e. } R \subseteq A \times B.$$

→ Here, R is said to be Relation from A TO B also, if $(a,b) \in R$ then $aRb$. OR $\langle a,b \rangle \in R$.
(Read : a is related to b)

Exa. Let, $A = \{1, 2, 3\}$, $B = \{a, b, c\}$.

$$R = \{(1,a), (2,b), (3,c)\}.$$

Exa. Let, $A = \{1, 2, 3, 4, 5, 6\}$.
Let $R \subseteq A \times A$, such that $(\exists)$
$(a,b) \in R \Rightarrow$ a devides b.

$$R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (5,5), (6,6)\}.$$

Exa. How many relations are possible from set A to B, if A has m elements and B has n elements.

→ In $A \times B$, elements $m \times n$.
so, $2^{m \times n}$ max. relations are possible.

Exa Consider a non-empty set A defined as
$A = \{2, 4, 6, 8 \ldots\}$
Let the relations are defined as
$R : \leq$

→ $R = \{(2,2), (2,4), (2,6) \ldots$
$(4,4), (4,6), (4,8) \ldots \}$.

⇒ Domain: If Relation R is defined from non-empty set A to non-empty set B then domain of R is denoted by "Dom(R)" and is defined as the set.

$$Dom(R) = \{a \mid a \in A \text{ \& for some } b \in B, (a,b) \in R\}$$

→ $Dom(R) \subseteq A$.

⇒ Range: If Relation R is defined from non empty set A to non-empty set B then ~~domain~~ Range of R is denoted by Range (R),

$$Range(R) = \{b \mid b \in B \text{ \& for some } a \in A, (a,b) \in R$$

Types of Relations —



⇒ Void Relation —

→ If A is non-empty sets, R is relation defined from A → A, R is said to be void relation.

$R = \phi$.

⇒ Identity Relation — ($I_A$)

→ I has relation on non-empty set A, then $I = \{(a,a) \mid \forall a \in A\}$.

⇒ Complement of R ($\bar{R}$) —

→ If R is from $A \to A$, then complement of R is defined as,

$\bar{R} = \{(a,b) \mid (a,b) \notin R\}$

⇒ Universal — $R = A \times A$.

⇒ Inverse Relation — ($R^{-1}$).

$R^{-1} = \{(b,a) \mid (a,b) \in R\}$

Exa  Let A is a non empty set $A = \{1, 2, 3, 4, 5\}$ give examples of relation R → inverse, identity, complement.

(i) Identity — $R = \{(1,1), (2,2), (3,3), (4,4), (5,5)\}$.

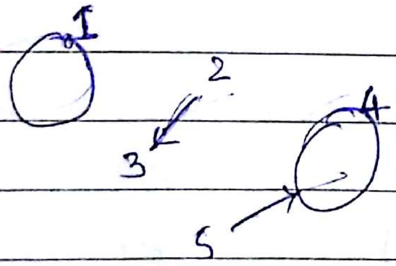
→ for exa we have relation,

$R = \{(1,1), (2,3), (4,4), (5,4)$

(ii) complement $\bar{R} = \{(1,2), (1,3), (1,4), (1,5), (2,1), (2,2), (2,4), (2,5), (3,1), (3,2), (3,3), (3,4), (3,5), (4,1), (4,2), (4,3), (4,5), (5,1), (5,2), (5,3), (5,5)\}$

(iii) Inverse — $R^{-1} = \{(1,1), (3,2), (4,4), (4,5)\}$

Graph representation




Matrix representation

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 0 & 1 & 0 \end{array}$$

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⇒ Properties of relation –

(1) Reflexive –

→ For a given non-empty set A, if R is a relation on set A → A then R is said to be reflexive if $\forall a \in A$, $(a, a) \in R$.

→ If R is reflexive then $I_A \subset R$.

(2) Irreflexive –

→ For a given non-empty set A, at least one a relation on set A → A ~~(a, a)~~ $\in R$. then R is said to be irreflexive if at least one $a \in A$, $(a, a) \in R$.
( Read from book).
for $\forall a \in A$, $(a, a) \notin R$.

(3) Symmetricity –

→ if for some $a, b \in A$, if aRb the bRa or if $(a, b) \in R$ then $(b, a) \in R$.

(4) Anti symmetric Relation –

→ A Relation is defineds on Not null set is said to be anti-symmetric if $\forall a, b \in A$ for $(a, b) \in R$ & $(b, a) \in R$
⇒ $a = b$

Exa. Consider $A = \{1,2,3,4,5,6\}$ give example of ~~Relative~~ Relations on A, which are Reflexive, Irreflexive, Symmetric, Anti-symmetric.

(i) Reflexive – $R = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(1,2),(1,5),(4,5)\}$.

(ii) Irreflexive – $R = \{(3,3),(4,4),(1,5),(2,6)\}$.

(iii) Symmetric – $R = \{(1,2),(3,2),(4,5),(5,4),(2,3),(2,1)\}$.

(iv) Anti-symmetric – $R = \{(1,1),(2,2),(1,2),(5,6)\}$.

⇒ Consider the set of int 'z' define Relation 'R' on 'z' such that $R = \{(a,b) / a,b \in z$ & $(a+b)$ is divisible by $5\}$. check whether R is reflexive, symmetric and anti-symmetric.

~~R = {(5,5)}~~, $(a \equiv b \pmod 5)$

$R = \{\ldots(0,5),(5,0),(5,10),(10,5),(10,15),(15,10)\ldots\}$.

It is Symmetric and

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Sol.$^n$:- Let $a, b \in Z$ and R is defined by,

$R = \{(a,b) / a, b \in Z$ & $(a-b)$ is divisible by $5\}$.

(i) We know that zero is divisible by 5
Hence $\forall a \in Z$, $(a-a)$ " "

$\Rightarrow (a,a) \in R$.
$\Rightarrow$ R is reflexive

(ii) $a, b \in Z$ if $(a,b) \in R \Rightarrow (a-b)$ is divisible by 5. let $(a-b) = 5k$ ($k$ = integer)

$\Rightarrow (b-a) = -5k$.
$\Rightarrow (b,a) \in R$.
$\Rightarrow$ R is symmetric.

(iii) $a, b \in Z$ if $(a,b) \in R = (a-b)$ is divisible by 5. but there exist $\forall a, b \in Z$ $(b,a) \in R$, $(a-b)$ is divisible by 5.

$\Rightarrow$ R is not anti-symmetric.

Property

(5) Transitivity – If R is a relation on a non-empty A. then R is Said to be transitive ~~if~~ for $a, b, c \in A$
if $(a,b)$ & $(b,c) \in R$.
$\Rightarrow (a,c) \in R$.

Example. Check wether previous exa. gives transitive relation or not?

soln :- ~~(a, b)~~ let $a, b, c \in Z$
and $(a,b), (b,c) \in R$.

$\Rightarrow (a-b) = 5m$ where, $m, n \in Z$
$(b-c) = 5n$
$\Rightarrow (a-b) + (b-c) = 5(m+n)$
$\Rightarrow a-c = 5(m+1) \in R$.

$\Rightarrow$ R is transitive.

$\Rightarrow$ Equivalence Relation – A relation 'R' define on non-empty set A is Said to be equivalent if R is reflexive, symmetric and Transitive

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Exa Let the relation R is define on Non-empty set A.

R = { a / string of binary no. 0 & 1 } such that $(a,b) \in R$ if string length of a = string length of b.

$a = \{101001\}$
$b = \{001110\}$.

Solution: ~~R~~ { (i) Let $a = \{1,01001\}$
$b = \{001110\}$.

Here, length of a = b if ~~b~~ $c \in C$.
then length a = c. b = c.

$\Rightarrow$ R is transitive.

(ii) if $a = \{0\phi0\phi\}$
$b = \{0\phi0\phi\}$.
$\Rightarrow (a,a) \in R \quad \forall a$.
$\Rightarrow$ R is reflaxive.

(iii) if $a = \{0101\}$
$b = \{1010\}$.

$(a,b) \in R, \quad (b,a) \in R$
$\Rightarrow$ R is ~~transio~~ symmetric.

Theorem: Consider a non-empty set A. if R is defined on A, then

(i) If R is reflexive; so, is $R^{-1}$
(ii) If R is symmetric: if and only if R is same as $R^{-1}$ ($R \equiv R^{-1}$)
(iii) If R is anti-symmetric: if and only if $R \cap R^{-1} \subseteq I_A$

(i) Here, R is reflexive
$\Rightarrow (a, a) \in R$ $(\because b = a)$
$\Rightarrow (a, a) \in R^{-1}$
$\Rightarrow R^{-1}$ is reflexive

(ii) Let, R is symmetric,
let, $(a, b) \in R$, $a, b \in A$
$\Rightarrow (b, a) \in R$ ($\because$ R is Symmetric)
$\Rightarrow (b, a) \in R^{-1}$
$\Rightarrow R \subseteq R^{-1}$.

Now, $(b, a) \in R^{-1}$
$\Rightarrow (a, b) \in R$
$\Rightarrow R^{-1} \subseteq R$

$\Rightarrow R = R^{-1}$.

Let, $R = R^{-1}$.
Now for $(a, b) \in R$
$\Rightarrow (b, a) \in R^{-1}$
$\Rightarrow (b, a) \in R$.

$\Rightarrow$ If $(a, b) \in R$
$\Rightarrow (b, a) \in R$

(iii) $(a,b) \in R$.
$(b,a) \in R$ ($\because$ by definition of Anti-symmetric)
$\Rightarrow a = b$.
$\Rightarrow (a,a) \in R$, $\exists a \in R$ / $\forall a \in R$.
$\Rightarrow (a,a) \in R^{-1}$ ($\because$ If $R$ is reflexive $\Rightarrow R^{-1}$ is also reflexive)

$\Rightarrow R \cap R^{-1} \subseteq I_A$.

<u>OR</u> $(a,b) \in R \cap R^{-1}$.
$\Rightarrow (a,b) \in R$ & $(a,b) \in R^{-1}$.
$\Rightarrow (b,a) \in R^{-1}$ & $(b,a) \in R$
$\Rightarrow (a,b), (b,a) \in R$ ($\because$ R is Anti-symmetric).
$\Rightarrow a = b$.
$\Rightarrow (a,a) \in R \cap R^{-1}$.
$\Rightarrow R \cap R^{-1} \subseteq I_A$,

$\Rightarrow$ Equivalence classes –

$\rightarrow$ If $R$ is an equivalence relation on a non-empty set $A$, then equivalence class of $x \in A$ is ~~no~~ denoted by $[x]_R$

$$[x]_R = \{ y \mid y \in A, \ \& \ (x,y) \in R \}.$$

e.g. $[x]_R = \{ y / y \in A, \ \& \ (x-y)/5 \}$.

$[5]_R =$

$y$ is called as "representative".

**Theorem:** Let $R$ is an equivalence relation on non-empty set $A$ then prove that,

(i) $a \in [a]_R$ , $a \in R$.
(ii) $b \in [a]_R \Rightarrow [a]_R = [b]_R$
(iii) $[a] = [b] \Leftrightarrow (a,b) \in R$.
(iv) either $[a] = [b]$ or $[a] \cap [b] = \phi$

(i) $\cancel{[a]_R}$ $R$ is equivalence
$\Rightarrow$ $R$ is reflexive.
$\Rightarrow$ so. $\forall a \in A$, $(a,a) \in R$.
$\Rightarrow$ $\cancel{a \in [a]_R}$, $[a_R] = \{b / b \in A, (a,b) \in R\}$
$\Rightarrow$ $a \in [a]_R$.

(ii) Let, $b \in [a]_R \Rightarrow (a,b) \in R$,
$(b,a) \in R$ ($\because$ Symmetric)

Now, $x \in [a_R]_R$.
$\Rightarrow (a,x) \in R$.
$\Rightarrow \cancel{(x,a) \in R}$ $\Rightarrow (b,a) \in R \;\&\; (a,x) \in R$
$\cancel{\Rightarrow (x,a), (a,x) \in R}$.
$\Rightarrow (b,x) \in R$. ($\because$ Transitivity)
$\Rightarrow \cancel{(b,x) \in R}$
$\Rightarrow x \in [b]_R$.

$\Rightarrow [a] \subseteq [b]$ —— (1).

Now, $y \in [a]_R$. $y \in [b]_R$
$\Rightarrow (a,y) \in R$.
$\Rightarrow (y,a) \in R$.
$\Rightarrow (a,y), (y,a) \in R$
$\Rightarrow (y,a) \in R$. $\Rightarrow b \in [a]_R$.

(iv) If $a, b \in A$ either $[a] = [b]$ or $[a] \cap [b] = \phi$.

Let, $x \in [a] \cap [b]$ ($\because$ contradiction)

$\Rightarrow x \in [a]$ & $x \in [b]$.

$\Rightarrow xRa$ & $xRb$

$\Rightarrow aRx$ & $xRb$ ($\because$ symmetricity and R is an equivalence relation).

$\Rightarrow aRb$. (Transitivity).

$\Rightarrow [a] = [b]$ ($\because$ from (iii))

$\Rightarrow$ either $[a] = [b]$ or $[a] \cap [b] = \phi$.

$\Rightarrow$ Partition of a set —

Let A is ~~an~~ a non-empty set. then $A_1, A_2 \dots A_n$ are said to ~~be~~ give partition of set A, if $A_i \subseteq A$, $1 \leq i \leq n$ &

Union $\rightarrow$ $\bigcup_{i=1}^{A} A_i = A$ & $A_i \cap A_j = \phi$, $i \neq j$ & $1 \leq i \leq n$, $1 \leq j \leq n$.

Theorem: Prove that, if A is a non-empty set and R is an equivalence relation on A, then the equivalence classes (distinguish) with respect to R form partition of set A.

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Que. If all set of integers $Z$ in equivalence Relation are defined as,

$R = \{ (a,b) / a, b \in Z, (a-b)$ is divisible by $6 \}$.

find partitioned of $Z$ with respect to this relation $R$.

$R = \{ (1,7), (2,8), (7,1), (8,2), (3,9), (9,3), (4,10), (5,11), (6,12), (0,6), (6,0) \}$.

$= \{ [0], [1], [2], [3], [4], [5], \cancel{[6]} \}$.

Here, $[6] = [0]$

$[7] = [1]$

$\vdots$

$=$ Eq. classes set $= \{ [0], [1], [2], [3], [4], [5] \}$

$=$ quotient set $= Z/R$.

( Means relation on $Z$ ).

Que Let, $A = \{0, 1, 2, 3 \ldots 9, 10\}$.

Let, $A_1 = \{1, 2, 3 \ldots 8\}$, $A_2 = \{1, 3, 5, 7, 9\}$,

$A_3 = \{0, 2, 4\}$, $A_4 = \{6, 8, 10\}$.

from the given subsets $A_1, A_2, A_3, A_4$ find partitioned of set $A$.

Ans. $= \{ A_2, A_3, A_4 \}$.

$A_1 \cap A_2 \neq \phi$
$A_2 \cap A_3 = \phi$
$A_3 \cap A_4 = \phi$
$A_2 \cap A_4 = \phi$

$A_2 \cup A_3 \cup A_4 = A$.

Que Get the equivalence relation R from the quotient set $= \{A_2, A_3, A_4\}$.

$A = \{0, 1, 2, 3 \ldots 10\}$.

$R = \{(0,0), (1,1), (2,2) \ldots (10,10) \ldots\}$

⇒ Partial Ordering (POset)

→ If A is a non-empty set, the relation R defined on A is said to be partial ordering R is reflexive, Anti-sym and transitive. The set A along with the partial ordering defined on A is said to be "partially Ordered set" (PO set)

→ Generally poset is denoted by $(A, \leq)$

< > Ordered.

Que Consider the natural nos from 1 to 10. find Poset for relation R,

R : { ~~(a R b)~~ (a, b) / a devides b }.

Ans: R = { (1,1), (1,2), (1,3), (1,4), (1,5), (1,6) ... (1,10), (2,2), (3,3), (4,4) ... (10,10), (2,4), (2,6), (2,8), (2,10) ... }.

→ Here, R is not equivalence relation. (∵ Not symmetric)

Que Show that the relation R : ⊆ : Inclusion is a partial ordering only powerset $\wp(A)$, of $A = \{\alpha, \beta, \gamma\}$

$\wp(A) = \{\phi, \alpha, \beta, \gamma, \{\alpha, \beta\}, \{\beta, \gamma\}, \{\alpha, \gamma\}, \{\alpha, \beta, \gamma\}\}$.

(1) Reflexity : We know that every set $\beta \subseteq \beta$ it self.

$\Rightarrow \forall \beta \in \wp(A)$.

$\Rightarrow \beta \subseteq \beta$.

$\Rightarrow \beta R \beta$.

(2) Anti-symmetric: $\beta, \gamma \in \wp(A)$.

$\beta \subseteq \gamma$. and $\gamma \subseteq \beta$

$\Rightarrow \beta = \gamma$.

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(3) Transitivity: $\alpha, \beta, \gamma \in \wp(A)$.

$\alpha \subseteq \beta, \quad \beta \subseteq \gamma.$
$\Rightarrow \alpha \subseteq \gamma.$
$\Rightarrow \alpha R \gamma.$

<u>Note</u>: If A is a non-empty set R $\leq$ such that $(A, \leq)$ is poset than,

(1) for $a, b \in A$, if $a \leq b$, a precedes b

$a < b$.
a strongly preceds b.
$a \nleq b$ & $a \ngeq b$ ($a \not\leq b$ are incomparable)

(2) cover:- If $a, b, c \in A$, $a < b$ and b is said to be cover of a if $a < b$ & whenever there $c \in A$ such that $a \leq c$ and $c \leq b$ then c must be $a = c$ or $c = b$.

($\therefore$ i.e. there exist no c in A such that $a < c < b$).

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⇒ Hasse diagramme —

(1) Diagramatic (graphical) way to represent Poset -

→ Draw the reflexivity by dropping self loops.

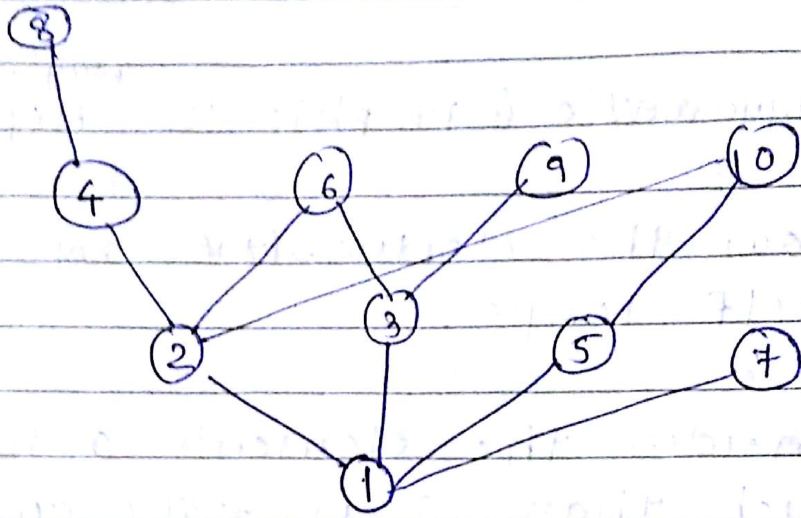
→ Consider the element a to be bottom such that a is not a cover of any element in set A. Consider the covers of a in upward direction from a at same level. Join them by straight line and repeat this process untill all vertices are included.

→ Thus, Hasse is a directed graph which grows in upward direction.
( Hasse Diagram will never have any edge going in downward direction. Also no horizontal connections).

e.g $A = \{1, 2, 3, 4, 5, \ldots 10\}$.

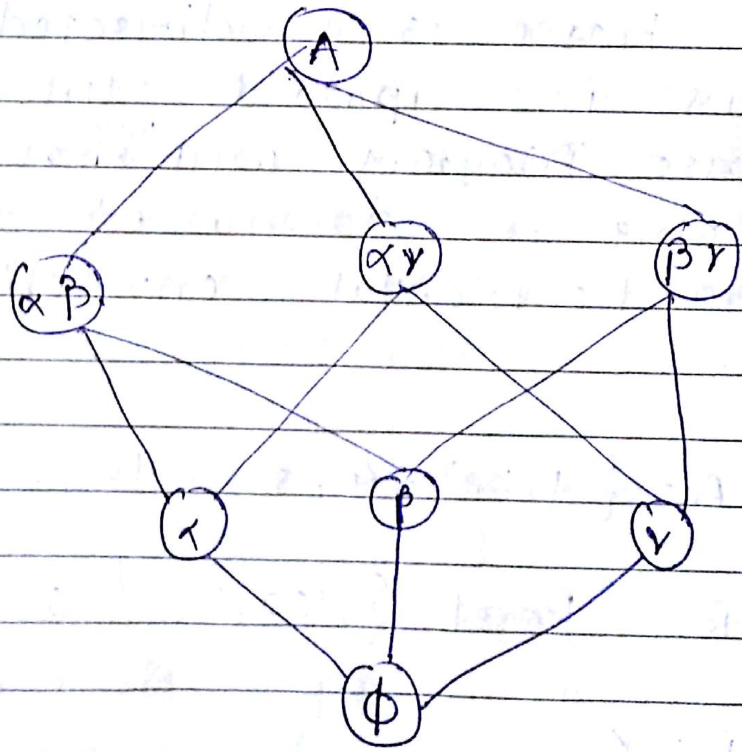
$R =$ ~~Poset~~ $\{ (a, b) \mid a \text{ divides } b \}$.

$R : \leq$




2, 3, 5, 7
are cover of 1.
4 is not cover.
because (1, 2) is there.

Qu. Plot the hasse diagram for powerset.



⇒ Special elements in Poset —

(1) Maximal elements - $a \in A$ is Said to be maximal element in Poset $(A, \leq)$ if there exist no such $b \in A$, such that $a < b$ then a is Said to be maximal element in a poset.

→ It may have so many maximal elements

(2) Minimal elements - $c \in A$ is Said to be minimal element in poset $(A, \leq)$ if there exist no such $b \in A$, Such that $b < c$ then c is Said to be minimal.

(Here we are comparing c with all b $\nexists b \in A$ ($\because$ no some b))

(3) Greatest element - $g \in A$ is Said to be greatest in the poset $(A, \leq)$ if $\forall a \in A$ Such that $\forall a \in A$, $a \leq g$

if it exist that it is unique.

(4) Least element - $L \in A$ is Said to be least in the poset $(A, \leq)$ if $\forall a \in A$, $a \geq l$.

it will also be unique

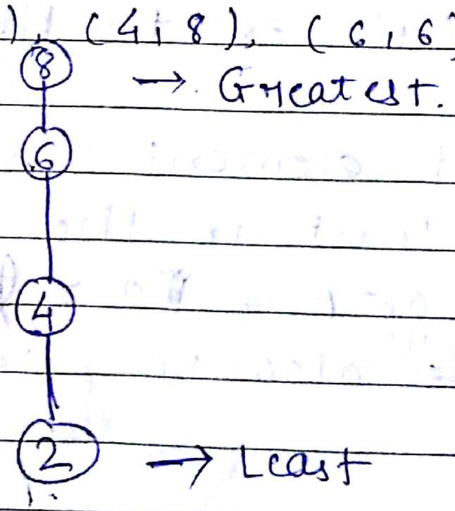
(5) Dual Order — If $(A, \leq)$ is a poset for given partial ordering, $S : \leq$ then the inverse $S^{-1} : \geq$ always exist on set A. This is known as dual ordering on set A.

(6) Totally Ordered / Linearly Ordered set

A partial ordering "$\leq$" on a non-empty set A is said to be linear ordering / total ordering if $\forall a, b \in A$ either $a \leq b$ or $b \leq a$.

e.g. consider the set $A = \{2, 4, 6, 8 \ldots\}$ along with the partial ordering "$\leq$" draw the hasse diagram. also show that this is a linear ordering on set A.

$R = \{(2,2), (2,4), (2,6), (2,8), (4,4), (4,6), (4,8), (6,6), (6,8), (8,8)\}$



⇒ Representation of Relation by Matrices –

→ consider a non-empty set A along with relation R on set A, the matrix representation of R is given by,

$M = [m_{ij}]_{n \times n}$ , $n$ = No. of elements in A.

$m_{ij} = 1$ , if $(a_i, a_j) \in R$.
$= 0$, otherwise.

$A = \{1, 2, 3, a, b, c\}$.

$R = \{(1,1), (3,a), (2,c), (b,1)\}$.

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & a & b & c \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

⇒ Ordered Subset –

If $A \leq$ is a poset, then $B \subseteq A$ is said to be ordered subset, if $a, b \in B$,
$a \leq b$ or $b \leq a$ in $(A, \leq)$
⇒ $a \leq b$ or $b \leq a$ in $(B, \leq)$

⇒ Upper bounds of ~~an~~ ordered subset –

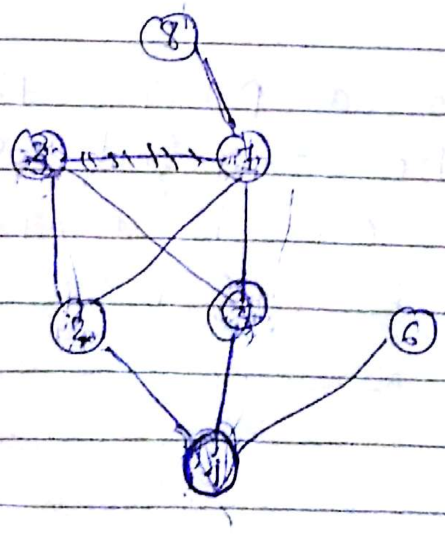
if $(A, \leq)$ is a poset having $(B, \leq)$ as a ordered subset then $a \in A$ is said to be upper bound of B if $\forall b \in B$, $b \leq a$.

Note: (1) The upper bound u may belongs to subset B.
(2) B may have so many upper bounds or ~~th~~ it may not have a single bound.

⇒ Lower bounds –

if $(A, \leq)$ is a poset having $(B, \leq)$ as a ordered subset then $a \in A$ is said to be ~~upr~~ lower bound of B if $\forall b \in B$, $b \geq a$

Exa for the given poset, find upper and lower bounds for subsets given below.



$S_1 : \{1, 5\}$. $\Rightarrow$ U.B : 5, 7, 3, 8
L.B : 1.

$S_2 = \{2, 5\}$. $\Rightarrow$ L.B : 1
U.B. : 3, 7, 8.

$S_3 = \{2, 3, 4\}$ $\Rightarrow$ L.B : 2, 1.
U.B : doesn't exist.

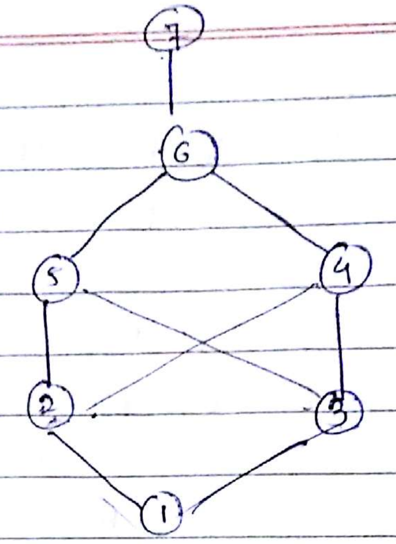
$\Rightarrow$ Suprimum / least upper bound —

for a ordered subset $(B, \leq)$ an element $u \in A$ is said to be least upper bound or lub or suprimum if $u$ is upper bound of $B$ & $u \leq b$ where b's are upper bounds of $B$.

$\Rightarrow$ Greatest ~~upper~~ lower bound — (infimum)-

for a ordered subset $(B, \leq)$ an element $u \in A$ is said to be greatest upper bound if $u$ is ~~upper~~ lower bound of $B$ and $u \geq b$ where b's are ~~upper~~ lower bounds of $B$.

e.g.




(1) $S_1 = \{2, 4\}$.

2 and 4 have relation.

L.B : 2, 1.
U.B : 4, 6, 7.

glb (∵ L.B. are 2, 1 , $2 \geq 1$

$\Rightarrow$ glb $S_1 = 2$.

lub $S_1 = 4$.

(2) $S_2 = \{2, 3\}$.

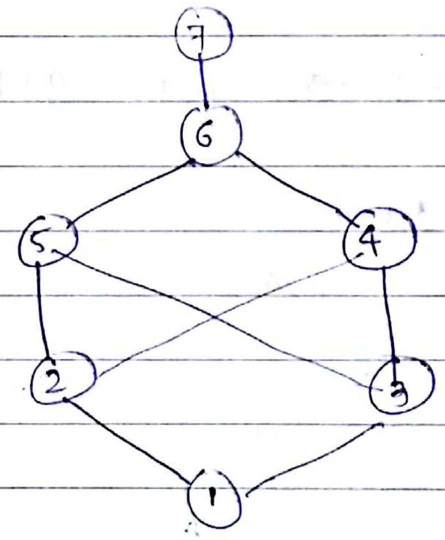
L.B. 1.
U.B : 5, 4, 6, 7 (Here, b/w 5 & 4 there is no relation so, we can't decide which is lub.
glb : 1
lub : doesn't exist.

# Lattice.

⇒ Lattice: A poset $(L, \leq)$ is said to be lattice if there exist glb and lub for every pair in $L$.

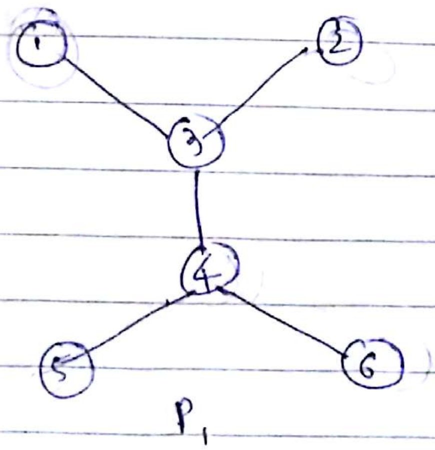
(i.e. if $(L, \leq)$ is lattice then $\forall a, b \in L$. glb $\{a, b\}$ & lub $\{a, b\}$ should exist

<u>Example</u> Check whether the given poset P is a lattice? Justify answer.



→ This is not a lattice because for (2, 3) lub doesn't exist.

(2)

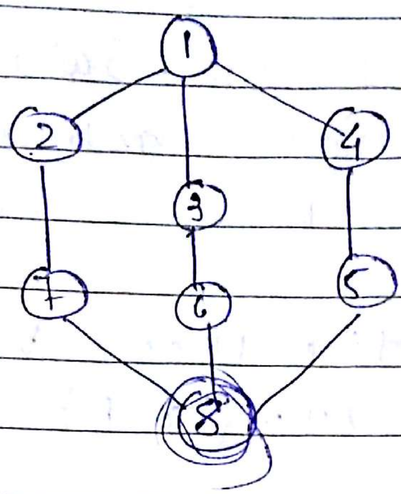


$P_1$

→ for pair (1, 2) & (5, 6) glb and lub ~~glb~~ doesn't exist.

→ so, $P_1$ is not a lattice

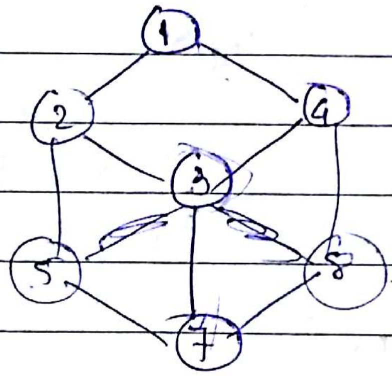
(3)




→ ~~For~~ This is a lattice. For every pair we can get glb, lub.

For (2, 7) → 7 is glb, 2 is lub.

(4)

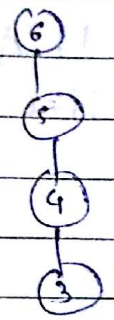



For {3, 7},

U.B: 1, 2, 3, 4.
L.B: 7
lub: 3.
glb: 7.

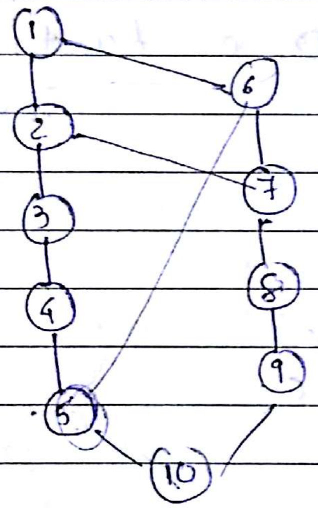
→ This is lattice.

(5)




→ This is totally ordered. so, It is lattice

(6)




for {5, 8}.

U.B: 2, 6, 1 ;

l.u.b : doesn't exist.

This is not a lattice.

⇒ Notations of glb and lub in boolean Algebra —

$\inf\{a,b\} = glb\,\{a,b\} = a * b\ (a \cdot b) = a \text{ meet } b = a \cap b = a \wedge b$

$\sup\{a,b\} = lub\,\{a,b\} = a + b = a \text{ join } b = a \cup b = a \vee b$

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Example. Consider the poset $(\wp(A), \subseteq)$, $A = \{a, b, c\}$ show that this is a lattice.

$glb\{\{a\}, \{b\}\} = \phi$.
$lub\{\{a\}, \{b\}\} = \{a, b\}$
$lub\{\{a\}, \{a, b\}\} = \{a, b\}$.

(2) Is the poset $(Z^+, /)$ a lattice?

(3) $a \vee a = a$
$a \wedge a = a$.

(4) $a \vee b = b \vee a$
$a \wedge b = b \wedge a$.

(5) $a \vee (b \vee c) = (a \vee b) \vee c$
$a \wedge (b \wedge c) = (a \wedge b) \wedge c$.

(6) $a \vee (a \wedge b) = a$.
$a \wedge (a \vee b) = a$.

⇒ Idempotent law —

→ $a \wedge a = a$

↳ $glb\{a, a\} = a$. ($\because$ reflexivity $a \leq a$).

⇒ Commutative law —

$a \wedge b = b \wedge a$

↳ $glb\{a, b\} = glb\{b, a\}$.

⇒ ~~Absor~~ Associativity —

$a \vee (b \vee c) = (a \vee b) \vee c$.

⇒ Absorption law - (M. Imp.)

$a \vee (a \wedge b)$

$= lub\{a, glb\{a,b\}\}$

$= lub\{a, K\}$. where, $K \leq a, K \leq b$.

$= a$.

(ii) $a \wedge (a \vee b) = a$

→ $a \leq a$ & $a, b \leq a \vee b$

→ $a \leq a$ & $a \leq a \vee b$

→ a is lower bound of a and $a \vee b$.

→ $a \leq glb\{a, a \vee b\}$.

→ $a \leq a \wedge (a \vee b)$.

Now, $a \geq a$ & , $a, b \geq a \wedge b$.

⇒ $a \geq a$ & $a \geq a \wedge b$.

Now ⇒ $a \wedge (a \vee b) = glb\{a, lub\{a,b\}\}$

$= glb\{a, K\}$, $K \geq a$, $K \geq b$.

which can be $\leq a$.

$\therefore a \wedge (a \vee b) \leq a$.

(ii) $a \vee (a \wedge b) = a$

(i) $\text{lub}\{a, \text{glb}\{a, b\}\}$.

$= \text{lub}\{a, k\}$, $a \geq k, b \geq k$.

which $\geq a$.

$\therefore a \vee (a \wedge b) \geq a$.

(ii)

$a \wedge b \leq a \wedge c$.

$a \geq a$ & $(a \wedge b) \leq a, b$

$a, b \geq a \wedge b$. | $a, c \geq a \wedge c$

$b \geq a \wedge b$ | $c \geq a \wedge c$

$a \geq a$ & $a \geq a \wedge b$

so, $\boxed{a \wedge c \geq a \wedge b}$

a is ub of a & $a \wedge b$

$a \geq \text{lub}\{a, a \wedge b\}$

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⇒ Other properties —

(i) If $L$ is a lattice then,
(A) $a \vee b = b$ iff $a \leq b$.
(B) $a \wedge b = a$ iff $a \leq b$.
(C) $a \wedge b = a$ iff $a \vee b = b$.

(ii) If $(L, \leq)$ is lattice & $a, b, c \in L$,
if $b \leq c$ then,
(A) $a \wedge b \leq a \wedge c$ (B) $a \vee b \leq a \vee c$.

(iii) Distributive ~~in~~ Inequalities: for lattice $(L, \leq$
$a, b, c \in L$.

(i) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$
(ii) $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$.

(i) $a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

Now, $a \wedge b \leq a, b$
→ $a \wedge b \leq a$ & $b \vee c$
→ $a \wedge b$ is lower bound of $a$ and $b \vee c$.

→ $a \wedge c \leq a, c$.
$a \wedge c \leq a, b \vee c$
$a \wedge c$ is lower bound of $a$ and $b \vee$

→ $a \wedge b \leq a \wedge (b \vee c)$
$a \wedge c \leq a \wedge (b \vee c)$

→ $a \wedge (b \vee c)$ is upper bound of $a \wedge b$ and $a \wedge c$.

$\therefore \quad a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

⇒ Duality - for a given lattice, any property true for given binary operations $\wedge, \vee$ & $\leq, \geq$.

→ The same property holds if these are change to $\vee, \wedge$ & $\geq, \leq$

⇒ Axiomatic Definition —
Let, L be a Non-empty set closed under two binary operations $\wedge$ & $\vee$ (called as "meet" and "join"). Then L is said to be a lattice if following axioms hold true.

($L_1$) commutative law —

$$a \wedge b = b \wedge a \quad \& \quad a \vee b = b \vee a.$$

($L_2$) associative law —

$$a \vee (b \vee c) = (a \vee b) \vee c$$
$$a \wedge (b \wedge c) = (a \wedge b) \wedge c.$$

($L_3$) Absorption -

$$a \wedge (a \vee b) = a \quad \& \quad a \vee (a \wedge b) = a.$$

($L_4$) Idempotent law - $a \wedge a = a$ & $a \vee a = a$

## ⇒ Distributive lattice –

→ A lattice $(L, \wedge, \vee)$ is known as distributive lattice, if for $a, b, c \in L$

(1) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(2) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

M.IMP

**Theorem:** Every chain is a Distributive lattice.

for any $a, b, c \in L$.
L is a chain. Hence, each pair is comparable.

(i) $a \leq b \leq c \Rightarrow a \leq c$.
$a \wedge (b \vee c) = a \wedge c = a$
$(a \wedge b) \vee (a \wedge c) = a \vee a = a$.

so, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

(ii) $a \leq b$ & $b \geq c$.

| Case-I $a \leq c$ | Case-II $a \geq c$. |
|---|---|
| $a \wedge (b \vee c) = a \wedge b = a$ | $a \wedge (b \vee c) = a \wedge b = a$. |
| $(a \wedge b) \vee (a \wedge c) = a \vee a = a$. | $(a \wedge b) \vee (a \wedge c) = a \vee c = a$. |

⇒ Modular lattice –

A lattice is said to be Moduler if, $a \vee (b \wedge c) = (a \vee b) \wedge c$, for $a, b, c \in L$

wherever $a \leq c$

Eg. Prove that every distributive lattice is Modular.

⇒ Sublattice – If $(L, \wedge, \vee)$ is a lattice, then the subset $S \subseteq L$ is said to be sublattice, if for $a, b \in S$, $a \vee b, a \wedge b \in S$

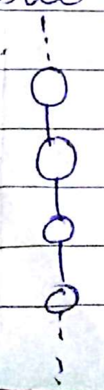
M. IMP.

→ Always take elements from same branch to make sublattice.

⇒ Complete lattice – A lattice is said to be complete if each of its non-empty subsets has a least upper bound and a greatest lower bound.

⇒ A lattice said to be bounded if it has 0 and 1
glb   lub.

e.x. complete but not bounded



Bounded lattice –
It is an Algebric structure, If $(L, \wedge, \vee)$ is a lattice then for any $x \in L$
$x \vee 1 = 1$, $x \wedge 1 = x$ and $x \vee 0 = x$, $x \wedge 0 = 0$.

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⇒ complemented lattice - A Bounded lattice $(L, \wedge, \vee)$ is said to be complemented if for $\forall a \in L$, $\forall b \in L$ such that $a \vee b = 1$ & $a \wedge b = 0$ then $b$ is a complement of $a$. $(b = a')$

⇒ In any lattice, if complement of lattice exist then it is always unique.

e.g. $(\wp(A), \subseteq)$
$a = \{\alpha\}$, $b = \{\beta, \gamma\}$

$a \wedge b = \phi = 0$
$a \vee b = \{\alpha, \beta, \gamma\} = A = 1$.

⇒ If $a \vee c = 1$ & $a \wedge c = 0$ then $b = c$. and $b = a'$

~~$a \vee c = 1$~~
~~$a \vee c \geq a, c$. and $a \wedge c \leq a, c$~~

~~$a \wedge c \leq a, c \leq a \vee c$.~~

~~if $b \neq c$ and $b > c$ then $a \vee c = b$~~

Here, b is complemented lattice of a

so, $b = b \wedge 1$
$= b \wedge (a \vee c)$
$= (b \wedge a) \vee c$
$= 0 \vee c$
$= c$

$\therefore b = c$.



⇒ Show that in a complemented distributive lattice, the following are equivalent.

(i) $a \leq b$.  (ii) $a \wedge b' = 0$.

$a \vee b = b = 1$
$a \wedge b = a = 0$.

(iii) $a' \vee b' = 1$  (iv) $b' \leq a'$.

# Mathematical Logic.

Statement — $x + 3 = 4$ (This is not a statement because we don't know the value of $x$).

(1) P: Sun rises in West.

~P: Sun ~~rises~~ doesn't rise in west

OR It is not the case that sun rises in west.

| P | ~P |
|---|---|
| T | F |
| F | T |

(2) $\vee$ : (disjunction) (+)

Truth value —

| p | q | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

(3) $\wedge$ : (conjunction) (*)

| p | q | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

e.g. - 1 Make truth table for,

(1) $(\sim p \wedge q) \wedge \sim r$.

| $p$ | $q$ | $r$ | $\sim p \wedge q$ | $(\sim p \wedge q) \wedge \sim r$ |
|---|---|---|---|---|
| T | T | T | F | F |
| T | T | F | F | F |
| T | F | T | F | F |
| T | F | F | F | F |
| F | T | T | T | F |
| F | T | F | T | T |
| F | F | T | F | F |
| F | F | F | F | F |

e.g. - 2 Find truth value for,

$(p \wedge (q \wedge r)) \vee \sim((p \vee q) \wedge (r \vee s))$

| $p$ | $q$ | $r$ | $s$ |
|---|---|---|---|
| T | T | F | F |

$p = q = T$

$r = s = F$

$= (T \wedge F) \vee \sim(T \wedge F)$

$= (T \wedge F) \vee \sim(\sim T \wedge \sim F)$

$= F \vee \sim F$

$= F \vee T$

$= T$

De Morgan's law —

(1) $\sim(p \vee q) = \sim p \wedge \sim q$.
(2) $\sim(p \wedge q) = \sim p \vee \sim q$



⇒ Conditional Statements —

$p \Rightarrow q = \sim p \vee q$
If "p" then "q".

| p | q | $p \Rightarrow q$ | |
|---|---|---|---|
| T | T | T | |
| T | F | F | (incoming is True but outgoint in False = False) |
| F | T | T | (input = F, output T = T) |
| F | F | T | (input = F, output F = F) |

⇒ Biconditional —

$p \Leftrightarrow q$.

| p | q | $p \Leftrightarrow q$ | |
|---|---|---|---|
| T | T | T | |
| T | F | F | (one side T other side F intersection = F) |
| F | T | F | |
| F | F | T | |

Exa. Find the truth value,
$5 > 4$ iff $0 < 5 - 4$

p q

P: $5 > 4$. (True).
Q: $0 < 5 - 4$ (True).

$p = T$.
$Q = T$.

$p \Leftrightarrow q = T$.

⇒ Rules for the propositional Algebra — (Prove them).

(1) Idempotent Law —

$p \vee p = p$

$p \wedge p = p$.

(2) Associativity Law —

$(p \vee q) \vee r = p \vee (q \vee r)$

$(p \wedge q) \wedge r = p \wedge (q \wedge r)$.

(3) Commutativity —

$p \vee q = q \vee p$

$p \wedge q = q \wedge p$

(4) Distribution law —

$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$.

(5) Identity law —

$p \vee T = T$

$p \wedge F = F$

(6) Complement Law —

$p \vee \sim p = T$.

$p \wedge \sim p = F$.

(7) Involution law —

$\sim(\sim p) = p$.

(8) De-Morgan's law —

$\sim(p \vee q) = \sim p \wedge \sim q$.

$\sim(p \wedge q) = \sim p \vee \sim q$.

→ What is POS / SOP ?

⇒ Conjunctive Normal form –

e.g. (i) $p \wedge (p \to q) = p \wedge (\sim p \vee q)$
$= (p \wedge \sim p) \vee (p \wedge q)$
$= 0 \vee p \wedge q$
$= p \wedge q.$

(ii) $\sim (p \vee q) \leftrightarrow (p \wedge q)$

⇒ Disjunctive Normal form –

e.g. 1. $p \to \{ (p \to q) \wedge (\sim(\sim q \vee \sim p)) \}$

$= p \to \{ (p \to q) \wedge (q \wedge p) \}.$

$= p \to \{ (\sim p \vee q) \wedge (q \wedge p) \}$ $= p \to \{ (\sim p \wedge q \wedge p) \vee (q \wedge q \wedge p) \}$

$= \sim p \vee \{ (\sim p \vee q) \wedge (q \wedge p) \}.$ $= p \to \{ 0 \vee p \wedge q \}$

$= p \to (p \wedge q)$

$= \sim p \vee \{ (\sim p \vee q) \vee (p \wedge q) \}.$ $= \sim p \vee (p \wedge q)$

$= \sim p \vee \{ (\sim p \vee q) \vee \bar{p} \vee \bar{q} \}.$

$\sim(\sim p \vee \sim q)$

(2) $(p \to q \wedge r) \vee (r \leftrightarrow s)$

(3) $p \wedge \sim (q \wedge r) \vee (p \to q)$

Tautology — (T)

⇒ ~~Topology~~

e.g. Show that (i) $p \wedge q \to q = \sim(p \wedge q) \vee q$

(ii) $p \vee \sim p$ are tautology

(i)

| $p$ | $q$ | $\sim(p \wedge q)$ | $\sim(p\wedge q) \vee q$ |
|---|---|---|---|
| T | T | F | T |
| T | F | T | T |
| F | T | T | T |
| F | F | T | T |

(ii)

| $p$ | $\sim p$ | $p \vee \sim p$. |
|---|---|---|
| T | F | T |
| F | T | T |

⇒ Contradiction-

If its truth is false regardless of truth value of its components statement

e.g. $(p \wedge q) \wedge \sim(p \vee q)$ is a contradiction

| $p$ | $q$ | $p \wedge q$ | $\sim(p \vee q)$ | $(p \wedge q) \wedge \sim(p \vee q)$ |
|---|---|---|---|---|
| T | T | T | F | F |
| T | F | F | F | F |
| F | T | F | F | F |
| F | F | F | T | F |

$\sim$(contradiction) = tautology.

⇒ contingency –

e.g. $q \vee (\sim q \wedge p)$

| $p$ | $q$ | $\sim q \wedge p$ | $q \vee (\sim q \wedge p)$ |
|---|---|---|---|
| T | T | F | T |
| T | F | T | T |
| F | T | F | T |
| F | F | F | F |

⇒ Contrapositive, Converse, Inverse –

If $p$ & $q$ are statements such that $p \Rightarrow q$, then contrapositive of conditional statement is $\sim q \to \sim p$.

Converse : $q \to p$.
Inverse : $\sim p \to \sim q$.

| $p$ | $q$ | $p \to q$ | $\sim q \to \sim p$ | $q \to p$ | $\sim p \to \sim q$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | F | F | T | T |
| F | T | T | T | F | F |
| F | F | T | T | T | T |

$\therefore \; p \to q \equiv \sim q \to \sim p$

$\therefore \; q \to p \equiv \sim p \to \sim q$

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Que. If $x$ is a prime even integer, then $x \not> 4$.

⇒ Contrapositive Statement:

If $x > 4$, then $x$ is not a prime even integer

⇒ Converse:

If $x \not> 4$, then $x$ is a prime even integer

⇒ Inverse:

If $x$ is not a prime even integer, then $x > 4$.

Que. This comp. program is correct, if it produces correct output for all possible inputs.

Que. If I will work hard, then I'll be successful.

M.IMP.

⇒ Proof Techniques —

(1) Theorem: It is a statement that can be shown true.

(2) Proof: Proof is a sequence of statements that forms an argument to prove the theorem as conclusion.

(3) Lemma: It is a simple theorem used in the proof of other theorems.

(4) Corollary: Corollary is a proposition that can be established directly from a theorem which has been proved.

(5) Conjecture: It is a statement whose ~~tr~~ truth value is unknown.

⇒ Rules of Inference —

The rules which are used to draw conclusion from other assertions together in the steps of a proof.

⇒ fallacies —

Some common forms of incorrect ~~reg~~ reasoning is known as fallacies.

## ⇒ Rules of Inference —

(1) Modus Ponens

The tautology $(p \wedge (p \to q)) \to q$

$$\begin{array}{l} p \\ p \to q \\ \hline \therefore q \end{array}$$

→ ($\because$ if $p$ is true, $p \to q$ is true ⇒ $q$ is also true (use truth-table))

(2) Modus Tollens : $(\sim q \wedge (p \to q)) \to \sim p$.

$$\begin{array}{l} \sim q \\ p \to q \\ \hline \therefore \sim p \end{array}$$

(3) Addition : $\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}$

(4) Simplification : $\begin{array}{l} p \wedge q \\ \hline \therefore p \end{array}$

(5) conjunction : $\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$

(6) Hypothetical syllogism : $\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$

(7) Disjunctive syllogism : $\begin{array}{l} p \vee q \\ \sim p \\ \hline \therefore q \end{array}$

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(8) Resolution: $p \vee q$

$\sim p \vee r$

$\therefore q \vee r$.

(9) Constructive dilemma: $(p \to q) \wedge (r \to s)$

$p \vee r$

$\therefore q \vee s$.

(10) Distructive dilemma: $(p \to q) \wedge (r \to s)$

$\sim q \vee \sim s$

$\therefore \sim p \vee \sim r$

(11) Absorption: $p \to q$

$\therefore p \to (p \wedge q)$

⇒ Direct Proof: A chain of statements beginning with premises/argument or hypothesis of a theorem are considered to show the ~~truth~~ proof of theorem.

Exa. Let, the following statements are true, hence conclude t statement is true.

$p \to q,\ q \to r,\ r \to s,\ \sim s\ \&\ p \vee t$

$p \to q$ is true

$q \to r$

$\therefore p \to r$ (∵ Hypothetical syllogism)

$$\begin{array}{l} p \to r \\ r \to s \\ \hline p \to s \end{array}$$ ($\because$ Hypothetical syllogism)

$$\begin{array}{l} \sim s \\ p \to s \\ \hline \therefore \sim p \end{array}$$ ($\because$ modus tollens)

$$\begin{array}{l} \sim p \\ p \vee t \\ \hline \therefore t \end{array}$$ (Disjunctive syllogism).

Example. following statements are true, conclude that t is true.

$\sim p \wedge q,\ r \to p,\ \sim r \to s,\ s \to t.$

$$\begin{array}{l} \sim p \wedge q \\ \hline \therefore \sim p \end{array}$$ ($\because$ Simplification)

$$\begin{array}{l} \sim p \\ r \to p \\ \hline \therefore \sim r \end{array}$$ ($\because$ Modus tollens).

$$\begin{array}{l} \sim r \\ \sim r \to s \\ \hline \therefore s \end{array}$$ ($\because$ Modus Ponens)

$$\begin{array}{l} s \\ s \to t \\ \hline \therefore t \end{array}$$ ($\because$ Modus Ponens)

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Ex. If Karan will complete b-tech on MCA then I will be assure of good job.

If karan is assure of good job, then karan will be happy.

Karan is not a happy.

Show Karan has not completed the B-Tech.

Ans
p: $b \rightarrow J$.
q: $J \rightarrow h$
r: $\sim h$.

$b \rightarrow J$, $J \rightarrow h$, $\sim h$

$$\begin{array}{l} b \rightarrow J \\ J \rightarrow h \\ \hline \therefore b \rightarrow h \end{array}$$ (hypothetical syllogism).

and $$\begin{array}{l} \sim h \\ b \rightarrow h \\ \hline \therefore \sim b \end{array}$$ (Modus tollens).

$\rightarrow$ $\sim b$ is true.

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⇒ Method of contradiction –

Example Show that $5+\sqrt{2}$ is irrational number.

→ Suppose that $5+\sqrt{2}$ is rational number.
but $\sqrt{2}$ is irr

$5+\sqrt{2}$ = rational NO.

⇒ $5+\sqrt{2} = x$

⇒ $\sqrt{2} = x-5$

⇒ $\sqrt{2}$ = some rational no.

⇒ $\sqrt{2}$ is rational no.

but $\sqrt{2}$ is not a rational no.
Hence Our Assumption is wrong
that $5+\sqrt{2}$ is irrational no.

Example Consider the following statements.

(1) If my checkbook is on my office then I paid my Phone-bill.
(2) I was looking at the phone-bill for payment at breakfast or I was looking at the phone-bill for payment at my office.
(3) If I was looking at the phone-bill at breakfast then the checkbook is on breakfast table.
(4) I did n't pay my phone-bill.
(5) If I was looking at the phone-bill in my office then the checkbook is on my office-table.

Where was my check-book

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p: checkbook on my office-table.
q: I paid my phone-bill.
r: looking at phone-bill at breakfast for Payment
s: " " at Office.
t: checkbook on my breakfast table.

$$\cancel{\begin{array}{l} p: \\ p \to q \\ \hline q \end{array}}$$

$p \to q, \; r \vee s, \; r \to t, \; \sim q, \; s \to p$

$$\begin{array}{l} s \to p \\ p \to q \\ \hline \therefore s \to q \end{array}$$ (Hypothetical Syllogism)

$$\begin{array}{l} s \to q \\ \sim q \\ \hline \therefore \sim s \end{array}$$ (Modus tollens)

$$\begin{array}{l} \sim s \\ r \vee s \\ \hline \therefore r \end{array}$$ (Disjunctive Syllogism)

$$\begin{array}{l} r \\ r \to t \\ \hline \therefore t \end{array}$$ (Modus Ponens).

⇒ Test of Validity:-

Example. (a) If my brother passes the examination of CAT of first attempt. I will give him a prize.
Either he passes the examination or I was out of station.
(b) I did not give him a prize, Although he passed the exam. Therefore I was out of station.

p: My brother passes the exam.
q: I will give him a prize.
r: out of station.

$p \to q$

$p \vee q$

$\sim q$

$p \to \sim q$

$q$

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⇒ fallacies: (sometime True / sometimes false)

(1) Affirming the conclusion:

$$\begin{array}{l} p \to q \\ q \\ \hline \therefore p \end{array}$$ (false)

(2) Denying the hypothesis:

$$\begin{array}{l} p \to q \\ \sim p \\ \hline \therefore \sim q \end{array}$$ (false)

Example Prove that ~~the~~ following arguments are Invalid.

$$\begin{array}{l} p \to q \\ q \\ \hline p \end{array}$$

If I will solve 7 problems correctly, then I will get grade A. I got grade A. Therefore I solved 7 problems correctly.

⇒ Predicate Calculus :-

A statement given as. $P(x)$ requires the analysis of sentence structure to conclude terms quantifiers and predicate where $x$ is a term and the predicate is the word in the sentence which express the nature of the term.

terms: They are the individual variables and constants either in the form of proper name or given description.

Quantifiers: In predicate calculus, the word/phrase of the type ∀ for "all", for "every", for "atleast one", "there exist atleast one Such that etc. are known as quantifiers

→ There are two types of quantification.
(1) Universal. (2) existential quantifiers

(1) Universal quantifier - If statement is true for all values of variable $x$ in particular domain than this domain is known as "Universe of discourse and the functional predicate $P(x)$ creates a proposition or statement of the form for $\boxed{\forall x \, P(x)}$, $x \in$ Domain

become a statement.

(2) Existential qualifier: For a given propositional function $P(x)$, if there exist at least one $x$ in the Universe of discourse then the proposition is created as "There exist atleast one $x$, such that $P(x)$." i.e. $\exists x\, P(x)$.

Example Let $P(x)$: $x^2 = x$, $x \in R$.

Let $x = 3$ (Predicate: "=" and square)

$P(3)$ : $3^2 = 3$

$\therefore P(3) = F$.

For $x = 1$

$P(1)$ : $1^2 = 1$

$\therefore P(1) = T$.

$\exists x\, P(x)$ : True.

$\forall x\, P(x)$ : False.

Example. Let the set of integers $Z$ is a Universal discourse. Let $P(x)$: $x > 3$.
What is the truth value of statement $\exists x\, P(x)$.

Here $P(x)$ : $x > 3$

Let $x = 2$,

$P(2)$ : $2 > 3$, false.

$\therefore P(2) = F$.

So, for $\forall x$, $P(x)$ is not true.

let. $x = 4$,

$P(4)$ : $4 > 3$ , true

so. $P(4)$ : T.

so. $\exists x P(x)$ : true.

<u>Note</u>: $\sim \forall x P(x) = \exists x \sim P(x)$. (There exist at least one $x$, for that $P(x)$ is not true).

$\sim \exists x P(x) = \forall x \sim P(x)$ (There exist no $x$. $\Rightarrow$ for all $x$ $P(x)$ is not true)

<u>Example</u>: Negate the Universal and existential quantification from ex. (1) and (2) respectively.

(i) $P(x): x | x$, $x \in N$.

Here, $\forall x P(x)$ = True.

$\sim \forall x P(x)$ : ~~There exist~~ for all $x$ $P(x)$ is not true.

$\therefore$ $\sim \forall x P(x) = \exists x \sim P(x)$

means there exist at least one $x$ for which $P(x)$ is false. (not true)

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(ii) $P(x): x^2 = x, \quad x \in R$.

$\exists x P(x) = $ True.

$\sim \exists x P(x) = \forall x \sim P(x)$.

means there exist no $x$ for which $P(x)$ is true so, There exist all $x$ for which $P(x)$ is false.

Exa. Negate the statement, $\forall$ real no. $x$, if $x > 3$ then $x^2 > 9$.

$P(x): x > 3$.
$Q(x): x^2 > 9$.

~~$P(x)$: $\exists x P(x) = $ true.~~

$\forall x \left( P(x) \rightarrow Q(x) \right)$

$= \sim \forall x \left( \sim P(x) \vee Q(x) \right)$

$= \exists x \sim \left( \sim P(x) \vee Q(x) \right)$

$= \exists x \left( P(x) \wedge \sim Q(x) \right)$

means there exist ~~some~~ at least one $x$ for $P(x)$ is true and $Q(x)$ is false.

Exa. Translate in English logic using term, predicate and quantifiers.

P(x) (i) All humming bird are richly coloured.
q(x) (ii) No Large bird lives on honey.
r(x) (iii) Birds that do not live on honey are dull in colour.

Therefore humming birds are Small.

~~(i) $\forall x\, P(x)$ : $x$ = humming birds.~~
~~$\forall y\, q(x)$ : $y$ = Large birds.~~

~~$x$ : humming bird.~~

$x$ : Any bird.
$P(x)$ : $x$ is a humming bird.
$q(x)$ : $x$ is richly coloured.
$r(x)$ : $x$ lives on honey
$s(x)$ : $x$ is large bird

(i) $\forall x\,(P(x) \rightarrow q(x))$
(ii) $\sim \forall x\,(s(x) \rightarrow \sim r(x))$

(iii) $\forall x\,(\sim r(x) \rightarrow \sim q(x))$

$\therefore \quad \forall x\,(P(x) \rightarrow \sim s(x))$

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⇒ Rules of Inference: (for quantified statements).

(1) Universal Instantiation: (US) (specification)

$$\frac{\forall x\, P(x)}{\therefore P(c)}$$

c = Any element from Universe of discourse.

(2) Universal Generalisation: (UG).

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x\, P(x)}$$

(3) Existential specification: (ES)

$$\frac{\exists x\, P(x)}{\therefore P(c) \text{ for some element } c}$$

(4) Existential generalisation: (EG)

$$\frac{P(c) \text{ for some } c}{\therefore \exists x\, P(x)}$$

⇒ Converse Error - (fallecies)

$$\frac{\begin{array}{l}\forall x\,(P(x) \rightarrow Q(x)) \\ Q(a)\end{array}}{\therefore P(x)}$$

⇒ Inverse Error -

$$\frac{\begin{array}{l}\forall x\,(P(x) \rightarrow Q(x)) \\ \sim P(a)\end{array}}{\therefore \sim Q(a).}$$

⇒ Nested Quantifiers —

In predicate calculus if a quantified statement has more variable, then it is known as "nested quantified statement"

Eg. $x, y \in R$

$\forall x . \forall y \ (x+y = y+x)$

$\forall x . \forall y \ (xy = yx)$.

(i) $\forall x . \exists y \ (x > y)$ (∵ ~~There ex~~ for all $x$ there exist atleast one $y$ such that $x > y$

(ii) $\forall x \ \forall y \ \exists z \ (x+y=z)$

(iii) $\sim \forall x \ \forall y \ \exists z \ (x+y=z)$

$= \exists x \ \exists y \ \forall z \ (x+y \neq z)$.

Exa. Every student in SUNIT have a lapton or has a friend who has a laptop.

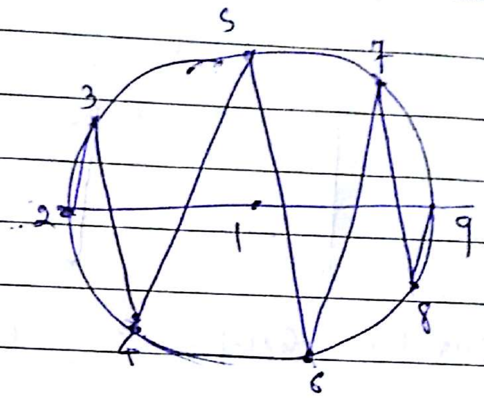
Exa. The Sum of two positive int is positive

Exa. For any real no. $x$, there must be a real no. $y$ such that $x \cdot y = 1$.

Exa. Negate all above nested quatifier statements

⇒ 2's comp. Arithmatic –

(1) Addition :- Add the two nos including their sign bits and discard any carry out of the sign bit position.




→ 1 2 3 4 5 6 7 8 9 |

→ 1 4 2 6 3 8 5 9 7 |

→