

# A STUDY OF COUNTABILITY

A PROJECT REPORT

*Submitted by*

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**BANKURA CHRISTIAN COLLEGE**

*for*

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*in*

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*of*

**BANKURA UNIVERSITY**

# **A STUDY OF COUNTABILITY**

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**Name of the Supervisor: Dr. Arup Mukhopadhyay**

## **DEPARTMENT OF MATHEMATICS**



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### **PROJECT COMPLETION CERTIFICATE**

**To Whom It May Concern**

This is to certify that **Subham Dangar (UID: 22013121007, Registration No.: 00375 of 2022-23)** of **Department of Mathematics, Bankura Christian College, Bankura**, has successfully carried out this project work entitled "**A STUDY OF COUNTABILITY**" under my supervision and guidance.

This project has been undertaken as a part of the undergraduate CBCS (New) curriculum of **Mathematics (Honours), Semester: VI, Paper: DSE – 4, Course Title: Dissertation of any topic in Mathematics (Project Work), Course ID: 62127** and for the partial fulfilment of the degree of **Bachelor of Science (Honours) in Mathematics of Bankura University** under CBCS (New) Curriculum in **2024 – 25**.

**Signature of the Supervisor**

Name : **Dr. Arup Mukhopadhyay**

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# **DECLARATION**

I hereby declare that my project, titled '*A STUDY OF COUNTABILITY*', has been submitted by me to Bankura University for the DSE-4 paper in Semester VI, under the guidance of my professor of Mathematics at Bankura Christian College, Dr. Arup Mukhopadhyay.

I also declare that the project has not been submitted here by any other student.

**Name:** SUBHAM DANGAR

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**Semester:** VI

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# 1 Introduction

Countability is a cornerstone of modern mathematical thought, particularly within set theory and real analysis. It offers a framework for understanding the nature and size of infinite sets by categorizing them based on whether or not their elements can be put into a one-to-one correspondence with the natural numbers. Sets that can be matched with the natural numbers are termed *countable*, while those that cannot are termed *uncountable*.

The concept emerged from the pioneering work of Georg Cantor, who formalized ideas about different sizes of infinity. Cantor's diagonal argument, for instance, elegantly proves the uncountability of the real numbers, revealing that some infinities are larger than others. This counterintuitive notion has profound implications, challenging our intuitive understanding of size and quantity.

This project aims to delve into the foundational principles of countability, examining rigorous definitions, exploring significant theorems, and applying the concepts in areas such as real analysis, and. By analyzing examples like the set of rational numbers ( $\mathbb{Q}$ ), the natural numbers ( $\mathbb{N}$ ), and the real numbers ( $\mathbb{R}$ ), the study seeks to illuminate how different infinite sets behave and relate to one another.

Through this exploration, we aim not only to grasp the formal properties of countable and uncountable sets but also to appreciate their impact on broader mathematical reasoning and structures. Understanding countability enhances our insight into the continuum, the nature of mathematical infinity, and the limits of computability, offering a bridge between abstract theory and practical applications in analysis.

## Motivation of Counting

Counting compares the sizes of sets. For finite sets, we can simply count elements. For example, if there are 7 people and 7 hats, we confirm equality by counting.

But if you don't know numbers, you can still compare sizes. Put one hat on each person:

- If no hats or people are left over, the sets are the same size.
- If hats are left, there are more hats.
- If people are left, there are more people.

This approach compares sets *without counting*. It works even for infinite sets.

Mathematically, this corresponds to defining a function:

$$g(\text{Person X}) = \text{Hat assigned to Person X}$$

If  $g$  is injective (no person shares a hat) and surjective (every hat is used), then the sets are the same size.

This method generalizes to comparing sizes of all sets.

## 2 Rigorous Definitions

In this section we give precise definitions of finite, countable, and uncountable sets, explain how to compare sizes (cardinalities) of sets, and illustrate with basic examples.

### 2.1 Functions, Injections, Surjections, and Bijections

First, we recall basic definitions about functions, since cardinality comparisons rely on them.

**Definition 1** (Function). A *function*  $f : A \rightarrow B$  between sets  $A$  and  $B$  is a rule that assigns to each element  $a \in A$  exactly one element  $f(a) \in B$ .

**Definition 2** (Injection, Surjection, Bijection). Let  $f : A \rightarrow B$  be a function.

- $f$  is called an *injection* (or one-to-one) if for all  $x, y \in A$ ,  $f(x) = f(y)$  implies  $x = y$ . Equivalently, distinct elements in  $A$  map to distinct elements in  $B$ .
- $f$  is called a *surjection* (or onto) if for every  $b \in B$ , there exists at least one  $a \in A$  such that  $f(a) = b$ .
- $f$  is called a *bijection* if it is both injective and surjective. A bijection establishes a perfect one-to-one correspondence between  $A$  and  $B$ .

**Remark 1.** *Bijections are the key to comparing sizes of sets, because if there is a bijection  $A \leftrightarrow B$ , we say  $A$  and  $B$  have the same cardinality. In the finite case, this matches our usual notion of “same number of elements.” For infinite sets, this generalizes the concept of “same size” in a meaningful way.*

### 2.2 Cardinality Comparison

**Definition 3** (Cardinality Equal). For sets  $A$  and  $B$ , we write

$$|A| = |B|$$

**Definition 4** (Cardinality Less Than or Equal). We write

$$|A| \leq |B|$$

**Definition 5** (Strict Inequality of Cardinalities). we write

$$|A| < |B|.$$

### 2.3 Definitions of Finite, Countable, Uncountable

**Definition 6** (Finite Set). A set  $A$  is *finite* if there exists a bijection between  $A$  and the set  $\{1, 2, \dots, n\}$  for some nonnegative integer  $n$ . Equivalently,  $A$  has exactly  $n$  elements.

**Example 1.** • The empty set  $\emptyset$  is finite, with  $n = 0$ .

- The set  $\{a, b, c\}$  is finite of size 3.

**Definition 7** (Countably Infinite (Denumerable) Set). A set  $A$  is *countably infinite* (or *denumerable*) if there exists a bijection

$$f : \mathbb{N} \longrightarrow A,$$

where  $\mathbb{N} = \{1, 2, 3, \dots\}$ . In this case we say  $|A| = \aleph_0$ , the cardinality of the natural numbers.

**Example 2.** •  $\mathbb{N}$  is countably infinite by definition (identity map).

- The set of even naturals  $E = \{2, 4, 6, \dots\}$  is countably infinite: e.g. the bijection  $f(n) = 2n$  maps  $\mathbb{N} \rightarrow E$ .
- The set of integers  $\mathbb{Z}$  is countably infinite: see explicit bijection in the next section.

**Definition 8** (Countable Set). A set  $A$  is *countable* if it is either finite or countably infinite. Equivalently, there exists an injection  $A \hookrightarrow \mathbb{N}$ , i.e.  $|A| \leq \aleph_0$ .

**Remark 2.** Every finite set is trivially countable. An infinite set that is countable has exactly the “size” of  $\mathbb{N}$ .

**Definition 9** (Uncountable Set). A set  $A$  is *uncountable* if it is not countable; equivalently, there is no bijection  $A \rightarrow \mathbb{N}$ . We write  $|A| > \aleph_0$ .

**Example 3.** The real numbers  $\mathbb{R}$  (or the interval  $(0, 1)$ ) will be shown to be uncountable via Cantor’s diagonal argument.

## 2.4 Basic Properties and Immediate Consequences

We list some fundamental properties that follow directly from these definitions.

**Proposition 1.** Any finite set is countable.

**Proposition 2.** Any subset of a countable set is either finite or countably infinite.

**Proposition 3.** If  $A$  is countably infinite and  $B$  is finite, then  $A \cup B$  is countably infinite.

**Proposition 4.** If  $A$  and  $C$  are countable, then  $A \times C$  is countable.

**Remark 3.** These properties justify why “countable” is stable under taking subsets (yielding finite or countable), finite unions, countable unions (shown later), and finite products.

### 3 Fundamental Theorems and Proofs

In this section we address the key questions and theorems listed in the project.

#### 3.1 Countability of $\mathbb{N}$ and $\mathbb{Z}$

**Theorem 1.** *The set of natural numbers  $\mathbb{N}$  is countably infinite.*

*Proof.* By definition, a set  $A$  is countably infinite if there exists a bijection  $f : \mathbb{N} \rightarrow A$ . Taking  $A = \mathbb{N}$ , the identity map  $\text{id} : \mathbb{N} \rightarrow \mathbb{N}$  is a bijection. Hence  $\mathbb{N}$  is countably infinite.

**Theorem 2.** *The set of integers  $\mathbb{Z}$  is countably infinite.*

*Proof.* We construct an explicit bijection  $f : \mathbb{N} \rightarrow \mathbb{Z}$ . One common enumeration is:

$$f(1) = 0, \quad f(2n) = +n, \quad f(2n+1) = -n \quad (\text{for } n \geq 1).$$

Thus the sequence lists integers as  $0, 1, -1, 2, -2, 3, -3, \dots$ . Every integer appears exactly once in this sequence, so  $f$  is bijective. Therefore  $\mathbb{Z}$  is countably infinite.

**Corollary 1.**  $|\mathbb{N}| = |\mathbb{Z}| = \aleph_0$ .

*Proof.* Since  $\mathbb{N}$  is countably infinite by definition and  $\mathbb{Z}$  is shown countably infinite, both have cardinality  $\aleph_0$ .

#### 3.2 Countability of $\mathbb{Q}$

**Theorem 3.** *The set of rational numbers  $\mathbb{Q}$  is countably infinite.*

*Proof.* Every nonzero rational number can be written uniquely in lowest terms as  $p/q$  where  $p \in \mathbb{Z}$  and  $q \in \mathbb{N}$ ,  $\gcd(p, q) = 1$ . Consider the set

$$S = \{(p, q) \in \mathbb{Z} \times \mathbb{N} \mid \gcd(p, q) = 1\}.$$

There is a surjection  $\phi : S \rightarrow \mathbb{Q} \setminus \{0\}$  given by  $\phi(p, q) = p/q$ . Also include 0 separately as one more element. Since  $\mathbb{Z}$  and  $\mathbb{N}$  are countable,  $\mathbb{Z} \times \mathbb{N}$  is countable;  $S \subseteq \mathbb{Z} \times \mathbb{N}$  is a subset, hence at most countable. Removing those  $(p, q)$  not in lowest terms leaves at most countably many pairs, so  $S$  is countable. A surjection from a countable set onto  $\mathbb{Q} \setminus \{0\}$  implies  $\mathbb{Q} \setminus \{0\}$  is countable; adding the single element 0 preserves countability. Therefore  $\mathbb{Q}$  is countably infinite. Similarly we can do same for the negative rational numbers.

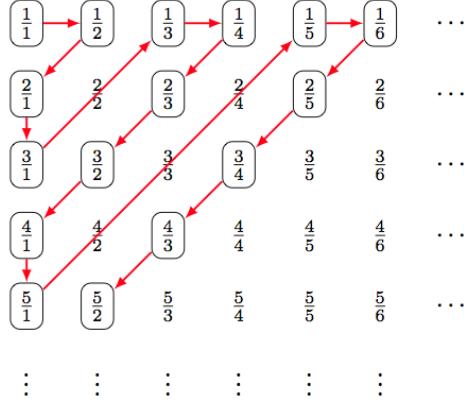


Figure 1: Illustration:  $\mathbb{Q}$  is countable

### 3.3 Uncountability of $\mathbb{R}$

**Theorem 4** (Uncountability of  $(0, 1)$ ). *The open interval  $(0, 1)$  is uncountable.*

*Proof.* Assume, for contradiction, that there is a surjection

$$f: \mathbb{N} \longrightarrow (0, 1).$$

Write

$$f(1) = 0.a_{11}a_{12}a_{13}\dots,$$

$$f(2) = 0.a_{21}a_{22}a_{23}\dots,$$

$$f(3) = 0.a_{31}a_{32}a_{33}\dots,$$

$\vdots$

in non-terminating decimal form (never ending in all 9's). Define

$$x = 0.d_1d_2d_3\dots, \quad d_n = \begin{cases} a_{nn} + 1, & a_{nn} \in \{0, 1, \dots, 8\}, \\ 8, & a_{nn} = 9. \end{cases}$$

Then  $x \in (0, 1)$  but  $x$  differs from  $f(n)$  in the  $n$ th decimal place for every  $n$ , contradicting surjectivity. Hence  $(0, 1)$  is uncountable.

**Theorem 5** (Uncountability of  $\mathbb{R}$ ). *The set of all real numbers  $\mathbb{R}$  is uncountable.*

*Proof.* Define

$$g: (0, 1) \longrightarrow \mathbb{R}, \quad g(x) = \tan(\pi(x - \frac{1}{2})).$$

Since  $g$  is a bijection and  $(0, 1)$  is uncountable, it follows that  $\mathbb{R}$  is uncountable.

**Example 4** (Set of all sequences with 0s and 1s). *The set of all infinite sequences of 0's and 1's,  $\{0, 1\}^{\mathbb{N}}$ , is uncountable.*

*Proof.* Suppose, for contradiction, that the set is countable. Then we can list all such sequences as

$$a_1, a_2, a_3, a_4, \dots$$

where each  $a_i$  is an infinite sequence of 0s and 1s:

$$a_1 = (a_{11}, a_{12}, a_{13}, \dots)$$

$$a_2 = (a_{21}, a_{22}, a_{23}, \dots)$$

$$a_3 = (a_{31}, a_{32}, a_{33}, \dots)$$

⋮

Now construct a new sequence  $b = (b_1, b_2, b_3, \dots)$  by flipping the diagonal elements:

$$b_n = \begin{cases} 1 & \text{if } a_{nn} = 0 \\ 0 & \text{if } a_{nn} = 1 \end{cases}$$

By construction,  $b$  differs from each  $a_n$  in the  $n$ th position. Hence,  $b$  is not in the list, which contradicts the assumption that all such sequences were listed. Therefore, the set  $\{0, 1\}^{\mathbb{N}}$  is uncountable.

### 3.4 Uncountability of $\mathbb{R} \setminus \mathbb{Q}$

We know that the set of real numbers  $\mathbb{R}$  is uncountable, while the set of rational numbers  $\mathbb{Q}$  is countable.

Since removing a countable subset from an uncountable set cannot make it countable, the set  $\mathbb{R} \setminus \mathbb{Q}$ , which is the set of irrational numbers, must be uncountable.

*Proof Outline.* Suppose, for contradiction, that  $\mathbb{R} \setminus \mathbb{Q}$  is countable. Then both  $\mathbb{Q}$  and  $\mathbb{R} \setminus \mathbb{Q}$  would be countable, and their union  $\mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}$  would be countable. But this contradicts the known fact that  $\mathbb{R}$  is uncountable. Hence,  $\mathbb{R} \setminus \mathbb{Q}$  must be uncountable.

### 3.5 Set of All Algebraic Numbers

**Theorem 6.** *The set of all algebraic numbers is countable.*

*Proof.* A real number is said to be an algebraic number if it is a root of a polynomial in  $\mathbb{Q}[x]$ , the set of all finite-degree polynomials with rational coefficients.

By definition, the set of algebraic numbers  $A$  is the subset of the real numbers consisting of roots of such polynomials.

We can prove the theorem by a cardinality argument. First, the set  $\mathbb{Q}[x]$  is countable because the set of polynomials over a countable field is itself countable.

Next, note that the set of algebraic numbers is the union of the set of roots of each polynomial. By the fact that a polynomial over a field has only finitely many roots, this is a union of countably many finite sets.

Therefore, by the theorem that a countable union of countable sets is countable, the set of all algebraic numbers is countable.

### 3.6 Cantor–Schröder–Bernstein Theorem

**Theorem 7** (Cantor–Schröder–Bernstein). *Let  $A$  and  $B$  be sets. If  $|A| \leq |B|$  and  $|B| \leq |A|$ , then  $|A| = |B|$ . Equivalently, if there exist injections  $f : A \hookrightarrow B$  and  $g : B \hookrightarrow A$ , then there exists a bijection  $h : A \rightarrow B$ .*

### 3.7 Some theorems without their proofs

**Theorem 8.** *An infinite subset of an enumerable (countably infinite) set is itself enumerable (countably infinite).*

**Theorem 9.** *The union of a finite set and an enumerable (countably infinite) set is enumerable.*

**Theorem 10.** *The union of two enumerable (countably infinite) sets is enumerable.*

**Theorem 11.** *The union of a countable family of enumerable (countably infinite) sets  $\bigcup_{n=1}^{\infty} A_n$  is enumerable.*

**Theorem 12.** *Let  $A$  be a nonempty set. The following are equivalent:*

1.  *$A$  is countable (i.e., there is a bijection  $h : A \rightarrow \mathbb{N}$  or  $h : A \rightarrow I_n$ , for some  $n \in \mathbb{N}$ ).*
2. *There exists a surjection  $f : \mathbb{N} \rightarrow A$ .*
3. *There exists an injection  $g : A \rightarrow \mathbb{N}$ .*

**Corollary 2.** *Let  $S$  and  $T$  be sets.*

1. *If there is an injective function  $f : S \rightarrow T$  and  $T$  is countable, then  $S$  is countable.*
2. *If there is a surjective function  $g : S \rightarrow T$  and  $S$  is countable, then  $T$  is countable.*

## 4 Illustrative Examples and Thought Experiments

### 4.1 Hilbert's Grand Hotel Paradox

Hilbert's Hotel is a famous thought experiment showing counterintuitive properties of countably infinite sets. Imagine a hotel with rooms numbered  $1, 2, 3, \dots$ , and suppose every room is occupied (i.e., there is exactly one guest in each room). We explore how the hotel can still accommodate additional guests.

#### 4.1.1 One New Guest

If exactly one new guest arrives, we can still accommodate them:

- Move the guest currently in room  $n$  to room  $n + 1$  for every  $n \in \mathbb{N}$ . Formally, if the original guest in room  $n$  is moved to room  $n + 1$ , then room 1 becomes vacant.
- Assign the new guest to room 1.

This defines a bijection between the old occupants plus the new guest and the rooms  $\mathbb{N}$ . Even though the hotel was “full” (every room occupied), it can fit one more guest.

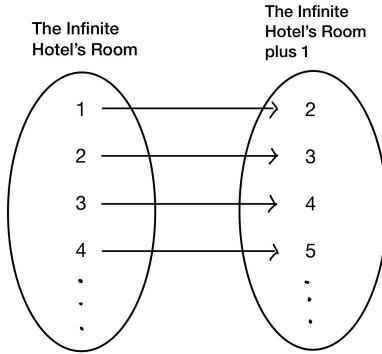


Figure 2: Illustration: Hilbert's Hotel accommodates one new guest by shifting each guest from room  $n$  to room  $n + 1$ , freeing room 1.

#### 4.1.2 Finitely Many New Guests

If  $k$  new guests arrive (for some finite  $k$ ), we can still accommodate all:

- Shift each current guest from room  $n$  to room  $n + k$ . That is, the occupant of room  $n$  moves to room  $n + k$ .
- Rooms  $1, 2, \dots, k$  become vacant.
- Assign the  $k$  new guests to rooms  $1, 2, \dots, k$ .

Again, this is a bijection between the union of old and new guests and the rooms. The hotel remains “full” yet accommodates  $k$  more.

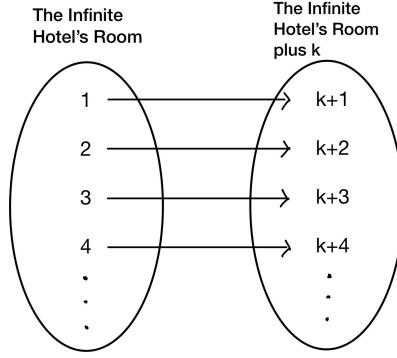


Figure 3: Illustration: To accommodate  $k$  new guests, shift each current guest from room  $n$  to room  $n + k$ , freeing rooms 1 through  $k$  for the newcomers.

#### 4.1.3 Countably Infinite New Guests

If countably infinitely many new guests arrive (e.g., an infinite bus with guests labeled  $1, 2, 3, \dots$ ), the hotel can still accommodate them:

- Move each existing guest from room  $n$  to room  $2n$ . This shifts occupants of room 1 to room 2, room 2 to room 4, room 3 to room 6, and so on.
- All odd-numbered rooms  $1, 3, 5, \dots$  become vacant.
- Assign the new guests to the odd-numbered rooms in order: guest 1 to room 1, guest 2 to room 3, guest 3 to room 5, etc.

This again produces a bijection between the combined set of old and new guests and the rooms  $\mathbb{N}$ , illustrating that adding a countable infinity to a countable infinity yields a countable infinity.

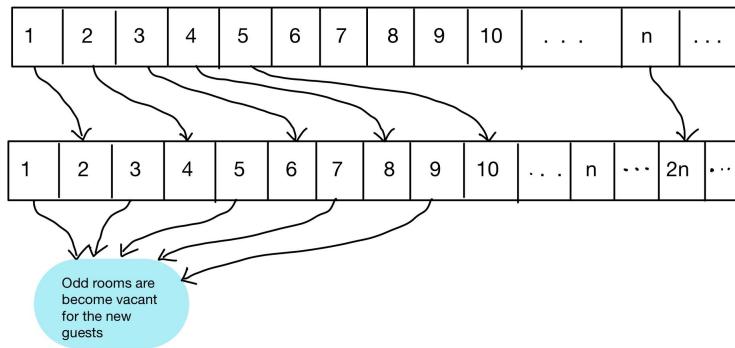


Figure 4: Shifting old guests to even rooms to make space for countably infinite new ones

#### 4.1.4 Countably Many Buses Each with Countably Many Guests

Suppose infinitely many buses arrive, bus  $i$  carrying infinitely many guests labeled  $(i, 1), (i, 2), (i, 3), \dots$ . The hotel's rooms  $1, 2, 3, \dots$  are already all occupied, so we must reassign both old and new guests to obtain a bijection with  $\mathbb{N}$ .

1. **Reassign existing guests.** Label the current occupants  $g_n$  in room  $n$ . Move each  $g_n$  to room

$$2^n.$$

Thus all old guests occupy exactly the powers of 2.

2. **Assign new guests by prime powers.**

- Assign to each bus  $i$  a distinct prime  $p_i$ , starting  $p_1 = 3, p_2 = 5, p_3 = 7, \dots$
- Send passenger  $(i, j)$  (bus  $i$ , seat  $j$ ) to room

$$p_i^j.$$

### Examples:

- Old guest  $g_1$  (from room 1) moves to room  $2^1 = 2$ .
- New guest  $(1, 1)$  (bus 1, seat 1) goes to room  $3^1 = 3$ .
- New guest  $(2, 2)$  (bus 2, seat 2) goes to room  $5^2 = 25$ .
- New guest  $(3, 3)$  (bus 3, seat 3) goes to room  $7^3 = 343$ .

**Why no collisions?** By the Fundamental Theorem of Arithmetic, every natural number has a unique prime factorization. Hence if  $p \neq q$  are primes or  $m \neq n$  are exponents, then

$$p^m = q^n \implies p = q \text{ and } m = n.$$

Therefore the sets  $\{2^n\}_{n \geq 1}$  (old guests) and  $\{p_i^j\}_{i,j \geq 1}$  (new guests) are pairwise disjoint, giving a bijection to  $\mathbb{N}$ .

	Old Guests(powers of 2)	Bus1(powers of 3)	Bus2(powers of 5)	Bus3(powers of 7)
Guest1	$2^1 = 2$	$3^1 = 3$	$5^1 = 5$	$7^1 = 7$
Guest2	$2^2 = 4$	$3^2 = 9$	$5^2 = 25$	$7^2 = 49$
Guest3	$2^3 = 8$	$3^3 = 27$	$5^3 = 125$	$7^3 = 343$
Guest4	$2^4 = 16$	$3^4 = 81$	$5^4 = 625$	$7^4 = 2401$
⋮	⋮	⋮	⋮	⋮

Figure 5: Prime-power encoding for infinite guests from infinite buses

#### 4.1.5 Discussion

Hilbert's Hotel vividly demonstrates:

- A fully occupied countably infinite set can still “fit” additional elements, whether finitely many or countably infinitely many.
- The notion of “size” for infinite sets differs from finite intuition:  $\aleph_0 + k = \aleph_0$  and  $\aleph_0 + \aleph_0 = \aleph_0$ .
- More generally, countable unions of countable sets remain countable.

## 5 Cardinalities and Comparisons

In this section we discuss how to compare the “sizes” of sets, especially infinite sets, using simple language and step-by-step explanations. We introduce the basic infinite sizes  $\aleph_0$  and the continuum  $c$ , explain why some infinities are larger than others, and address the question of whether any size lies strictly between  $\aleph_0$  and  $c$ . We also summarize how finite, countable, and uncountable sets relate.

### 5.1 Comparing Sizes via Bijections and Injections

Before comparing infinite sizes, recall how we compare finite sets:

- Two finite sets  $A$  and  $B$  have the same size if there is a one-to-one correspondence (bijection) between them.
- If there is an injection (one-to-one map) from  $A$  into  $B$ , we say  $|A| \leq |B|$ . For finite sets, this matches the usual “number of elements” comparison.

For infinite sets, we use the same language:

- $|A| = |B|$  if a bijection  $A \rightarrow B$  exists.
- $|A| \leq |B|$  if an injection  $A \hookrightarrow B$  exists.
- If  $|A| \leq |B|$  but no bijection  $A \rightarrow B$  exists, we write  $|A| < |B|$ .

This approach generalizes “size comparison” to all sets.

### 5.2 The Smallest Infinite Size: $\aleph_0$

**Definition 10.** We denote by  $\aleph_0$  (read “aleph-null”) the cardinality of the natural numbers  $\mathbb{N}$ . Any set that can be listed in a sequence (i.e. put into a bijection with  $\mathbb{N}$ ) is said to have size  $\aleph_0$ ; we also call such sets “countably infinite.”

**Examples of size  $\aleph_0$ :**

- $\mathbb{N}$  itself.
- The integers  $\mathbb{Z}$ , via a standard enumeration like  $0, 1, -1, 2, -2, \dots$ .
- The rationals  $\mathbb{Q}$ , by enumerating integer–natural pairs in lowest terms.
- Any infinite subset of a countable set (e.g. the even naturals) also has size  $\aleph_0$ .

### 5.3 Power Sets and a Larger Size

To find a larger infinity, we consider the power set construction:

**Theorem 13** (Cantor’s Theorem). *For any set  $A$ , the power set  $\mathcal{P}(A)$  (the set of all subsets of  $A$ ) has strictly larger cardinality than  $A$ . In symbols,  $|A| < |\mathcal{P}(A)|$ .*

In particular, taking  $A = \mathbb{N}$ , we get  $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$ . Since  $|\mathbb{N}| = \aleph_0$ , the power set  $\mathcal{P}(\mathbb{N})$  is strictly larger than  $\aleph_0$ . Thus  $\mathcal{P}(\mathbb{N})$  is uncountable.

## 5.4 The Continuum and Its Size

The real numbers  $\mathbb{R}$  also form an uncountable set. In fact, one shows:

**Theorem 14.** *There is a bijection between the real numbers in the interval  $(0, 1)$  and the power set  $\mathcal{P}(\mathbb{N})$ . Consequently,  $|(0, 1)| = |\mathcal{P}(\mathbb{N})|$ , and so  $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$ . We denote this cardinality by  $2^{\aleph_0}$  or  $c$  (the continuum).*

Thus the continuum  $c = |\mathbb{R}|$  equals  $2^{\aleph_0}$ , strictly larger than  $\aleph_0$ .

## 5.5 Is There an Infinity Between $\aleph_0$ and $2^{\aleph_0}$ ?

Cantor himself asked: “Is there a set whose size is strictly between that of the naturals and that of the real numbers?” This is the content of the Continuum Hypothesis (CH).

**Definition 11** (Continuum Hypothesis). The Continuum Hypothesis asserts: *There is no cardinal strictly between  $\aleph_0$  and  $2^{\aleph_0}$ .* Equivalently: every infinite subset of  $\mathbb{R}$  is either countable (size  $\aleph_0$ ) or has size equal to  $|\mathbb{R}|$  (the continuum).

## 5.6 Hierarchy of Infinities Beyond the Continuum

By repeatedly taking power sets, one obtains ever larger sizes:

$$\aleph_0 < 2^{\aleph_0} < 2^{2^{\aleph_0}} < 2^{2^{2^{\aleph_0}}} < \dots$$

Each time Cantor’s Theorem ensures  $|A| < |\mathcal{P}(A)|$ . Thus there is no “largest” infinity.

## 5.7 Cardinal Arithmetic (Intuitive Approach)

While full cardinal arithmetic can be advanced, a few intuitive points:

- For infinite cardinals  $\kappa$ ,  $\kappa + n = \kappa$  for any finite  $n$ . For example,  $\aleph_0 + 1 = \aleph_0$ , reflecting that adding finitely many elements to a countably infinite set leaves it countably infinite.
- Also  $\aleph_0 + \aleph_0 = \aleph_0$ : the union of two disjoint countably infinite sets is countably infinite (e.g. interleaving two lists).
- However,  $\aleph_0 \times \aleph_0 = \aleph_0$ : the Cartesian product  $\mathbb{N} \times \mathbb{N}$  is countable (e.g. diagonal enumeration).
- But  $2^{\aleph_0} > \aleph_0$ , so “powers” of infinite cardinals can jump to strictly larger sizes.
- In general, cardinal arithmetic rules like  $\kappa \times 2 = \kappa$  hold for infinite  $\kappa$ , but  $\kappa^{\aleph_0}$  or  $2^\kappa$  can be larger.

## 5.8 Equipotency and Cardinality Relations

We now show several fundamental equipotencies and cardinality relations between common sets. In each case, one exhibits an explicit bijection; where helpful, a diagram can illustrate the correspondence.

1.  $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$ .
2.  $\mathbb{R} \setminus \mathbb{Q} \sim \mathbb{R}$ .
3.  $(0, 1) \sim \mathbb{R}$ , and more generally  $\mathbb{R}^n \sim \mathbb{R}$  for each finite  $n$ .
4.  $(a, b) \sim (c, d)$ .

$$f: (a, b) \longrightarrow (c, d), \quad f(x) = \frac{d - c}{b - a}(x - a) + c$$

is a bijection.

5.  $(0, 1) \sim (0, 2)$ .

$$g: (0, 1) \longrightarrow (0, 2), \quad g(x) = 2x$$

is a bijection.

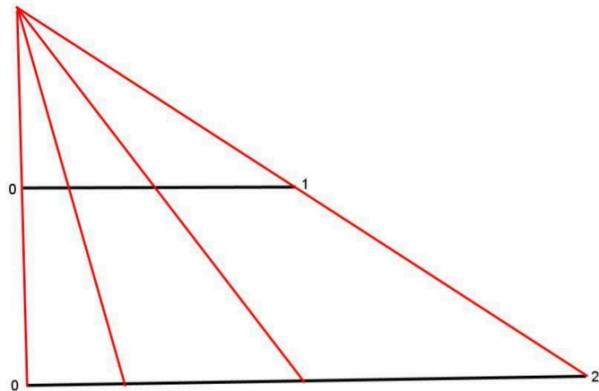


Figure 6: Illustration: Doubling map  $x \mapsto 2x$  from the interval  $(0, 1)$  to  $(0, 2)$

## 6 Conclusion

In this concluding section, we summarize the main insights gained throughout the project.

### 6.1 Summary of Key Insights

Throughout this project, we have seen:

- **Rigorous Definitions:** We defined finite sets, countably infinite (denumerable) sets, countable sets (finite or countably infinite), and uncountable sets. We learned to compare sizes via injections and bijections.
- **Fundamental Results:**
  - $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$  are countably infinite (size  $\aleph_0$ ), via explicit enumerations.
  - $(0, 1)$  and  $\mathbb{R}$  are uncountable, by Cantor's diagonal argument.
  - Closure properties: infinite subsets of countable sets are countable; unions of finitely or countably many countable sets remain countable; finite Cartesian products of countable sets are countable.
  - The set of algebraic numbers is countable; therefore, transcendental numbers are uncountable (abundant).
  - Cantor–Schröder–Bernstein theorem: mutual injections imply equal cardinality.
- **Illustrative Thought Experiments:**
  - Hilbert's Grand Hotel shows how a fully occupied countable “hotel” can still accommodate one, finitely many, or countably many new guests, illustrating that  $\aleph_0 + k = \aleph_0$  and  $\aleph_0 + \aleph_0 = \aleph_0$ .
- **Cardinalities and Comparisons:**
  - $\aleph_0$  is the smallest infinite cardinal (size of  $\mathbb{N}$ ).
  - Cantor's Theorem shows  $|A| < |\mathcal{P}(A)|$ ; applying to  $\mathbb{N}$  gives  $2^{\aleph_0}$  strictly larger than  $\aleph_0$ .
  - $|\mathbb{R}| = 2^{\aleph_0}$ , via bijection with  $\mathcal{P}(\mathbb{N})$  through binary expansions.
  - There is an infinite hierarchy of larger infinities by iterated power sets.
  - The Continuum Hypothesis asks whether any size lies strictly between  $\aleph_0$  and  $2^{\aleph_0}$ ; it is independent of standard axioms.

### 6.2 Final Remarks

Countability distinguishes between “listable” infinities and larger infinities that cannot be listed. Grasping these ideas enhances mathematical maturity, sharpens proof techniques (injections, bijections, diagonal arguments), and prepares students for deeper studies in analysis, logic, and beyond. Although infinite sets may seem abstract, thought experiments and explicit enumerations ground the concepts in tangible reasoning. We hope this project has provided a clear, detailed, and accessible treatment of countability and its rich consequences in mathematics.

## 7 Bibliography

### References

- [1] Apostol, Tom M., *Multi-Variable Calculus and Linear Algebra with Applications*, Calculus, Vol. 2, 2nd ed., John Wiley & Sons, 1969.
- [2] Rudin, Walter, *Principles of Mathematical Analysis*, McGraw-Hill, 1976.
- [3] Halmos, Paul R., *Naive Set Theory*, D. Van Nostrand Company, Inc., 1960; reprinted by Springer-Verlag, New York, 1974 (ISBN: 0-387-90092-6); reprinted by Martino Fine Books, 2011 (ISBN: 978-1-61427-131-4).
- [4] Cantor, Georg, “Ein Beitrag zur Mannigfaltigkeitslehre,” *Journal für die Reine und Angewandte Mathematik*, 1878 (84): 242–248. DOI: 10.1515/crelle-1878-18788413.
- [5] Mapa, S. K., *Real Analysis*, Academic Publishers.
- [6] Lang, Serge, *Real and Functional Analysis*, Springer-Verlag, 1993. ISBN: 0-387-94001-4.
- [7] Jech, Thomas, *Set Theory*, Springer Monographs in Mathematics, 3rd Millennium ed., Springer, 2002. ISBN: 3-540-44085-2.