NOMURA QUANT CHALLENGE

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Pricing Problem Statement Solution

1. Introduction

This document outlines the design, implementation, and mathematical derivations for a pricing system for ValueNotes and DeliveryContracts, as per the problem statement. The solution prioritizes accuracy, robust design, clear documentation, and analytical derivations for all required financial metrics and sensitivities.

Assumptions Made:

- **ValueNote Coupon Timing:** For simplicity in cash flow generation, the first coupon payment for a ValueNote is assumed to occur at $1/\text{PF}$ years from the valuation date, the second at $2/\text{PF}$ years, and so on, up to M (Maturity). The i-th payment occurs at $t_i = i/\text{PF}$. The total number of payments n is $\lfloor M \cdot \text{PF} \rfloor$.
- **Effective Rate (ER_0) for Cumulative/Recursive Conventions:** Solving for ER_0 given VP_0 for Cumulative and Recursive rate conventions generally does not have a closed-form solution. Newton-Raphson numerical root-finding is proposed and used.
- **Recursive Rate VP_0 Formula Interpretation:** The formula $\text{VP}_0 = (N + \text{FV}_n)/(1 + \text{ER}_0 \cdot M/100)$ (from section C, page 2, implicitly using overall maturity M) is used, where FV_n is calculated via the recursion $\text{FV}_i = (\text{FV}_{i-1} + \text{VF}_i)(1 + \text{ER}_0 \cdot m_i/100)$, with $m_i = 1/\text{PF}$ (time between payments) and $m_n = 0$ as stated.
- **DeliveryContract Pricing - GBM Path:** The primary method for pricing the DeliveryContract follows the Geometric Brownian Motion (GBM) model for $\text{ER}^i_{T_{ED}}$ and the subsequent quadratic approximation of the Price-to-RelativeFactor ratio. The quadratic equation for $\text{ER}^i_{T_{ED}}$ presented earlier is considered an alternative or component of a different framework not fully utilized in the specified pricing approach for the DeliveryContract.
- **ValueNote Pricing at T_{ED}:** When pricing a ValueNote at a future time T_{ED} (e.g., for DeliveryContract valuation), its characteristics are adjusted: Maturity becomes $M' = M - T_{ED}$. The number of remaining payments n' and their timings t'_i are recalculated relative to T_{ED}. The coupon amount VF per period remains $\text{VR} \cdot N/(100 \cdot \text{PF})$. The "Cumulative Rate" convention is used for VP(ER) calculations at T_{ED} as per basket parameters.
- **Numerical Integration for DC Price:** The expectation for the DeliveryContract price and its sensitivities will be calculated using numerical integration (e.g., Simpson's rule or Gaussian Quadrature) over the range of z (e.g., [-5, 5] or as specified [-3,3] for quadratic fit).
- **Risk-Free Rate r:** Used as given for forward price calculation.

2. Question 1: ValueNote Pricing and Sensitivities

A ValueNote class would encapsulate the following properties:

- N: Notional (double)
- M: Maturity in years (double)
- VR: Value Rate (double, e.g., 5 for 5%)
- PF: Payment Frequency (int)
- VP_0: Current Market Price (double, optional input)
- ER_0: Effective Rate (double, optional input)

Helper property:

- VF = $(\text{VR} \cdot N)/(100.0 \cdot \text{PF})$: Periodic coupon payment.
- $n = \lfloor M \cdot \text{PF} \rfloor$: Number of payments.
- $t_i = i/\text{PF}$: Time to i-th payment (for $i = 1, \ldots, n$).

2.1. ValueNote Calculations (Price to Rate, Rate to Price)

2.1.1. Price from Effective Rate ($\mathbf{VP}_0(\mathbf{ER}_0)$)

A) Linear Rate Convention:

$$\text{VP}_0 = N \cdot (1 - \text{ER}_0 \cdot M/100)$$

B) Cumulative Rate Convention: Let $d = \text{ER}_0/(100.0 \cdot \text{PF})$.

$$\text{VP}_0 = \sum_{i=1}^{n-1} \frac{\text{VF}}{(1+d)^{\text{PF}\cdot t_i}} + \frac{\text{VF}+N}{(1+d)^{\text{PF}\cdot t_n}}$$

(Note: $\text{PF} \cdot t_i = i$)

$$\text{VP}_0 = \sum_{i=1}^{n-1} \frac{\text{VF}}{(1+d)^{i}} + \frac{\text{VF}+N}{(1+d)^{n}}$$

C) Recursive Rate Convention: $m_i = 1/\text{PF}$ for $i < n$, $m_n = 0$.

$$\begin{aligned}
&\text{FV}_0 = 0\\
&\text{FV}_i = (\text{FV}_{i-1} + \text{VF}) \cdot (1 + \text{ER}_0 \cdot m_i/100) \quad \text{for } i = 1, \ldots, n.\\
&\text{VP}_0 = \frac{N + \text{FV}_n}{1 + \text{ER}_0 \cdot M/100}
\end{aligned}$$

(Here M is total maturity in years from today)

2.1.2. Effective Rate from Price ($\mathbf{ER}_0(\mathbf{VP}_0)$)

A) Linear Rate Convention:

$$\text{ER}_0 = (1 - \text{VP}_0/N) \cdot 100/M$$

(Requires $M > 0$. If $M = 0$, VP_0 should be N, ER_0 is ill-defined or can be taken as 0 if consistent).

B) Cumulative Rate Convention: Solve $f(\text{ER}_0) = \text{VP}_0^{\text{calc}}(\text{ER}_0) - \text{VP}_0^{\text{target}} = 0$ for ER_0 using Newton-Raphson.

$$\text{ER}_{0,k+1} = \text{ER}_{0,k} - \frac{f(\text{ER}_{0,k})}{f'(\text{ER}_{0,k})}$$

where $f'(\text{ER}_{0,k}) = \frac{\partial \text{VP}_0^{\text{calc}}}{\partial \text{ER}_0}$ (derived in 2.2.1.B).

C) Recursive Rate Convention: Solve $f(\text{ER}_0) = \text{VP}_0^{\text{calc}}(\text{ER}_0) - \text{VP}_0^{\text{target}} = 0$ for ER_0 using Newton-Raphson. $f'(\text{ER}_{0,k}) = \frac{\partial \text{VP}_0^{\text{calc}}}{\partial \text{ER}_0}$ (derived in 2.2.1.C).

2.2. Q1.2: First-Order Sensitivities

2.2.1. Price Sensitivity to Effective Rate ($\frac{\partial \mathbf{VP}_0}{\partial \mathbf{ER}_0}$)

A) Linear Rate Convention:

$$\frac{\partial \text{VP}_0}{\partial \text{ER}_0} = N \cdot (-M/100) = -\frac{NM}{100}$$

B) Cumulative Rate Convention: Let $d = \text{ER}_0/(100.0 \cdot \text{PF})$. Then $\frac{\partial d}{\partial \text{ER}_0} = \frac{1}{100.0 \cdot \text{PF}}$.

$$\text{VP}_0 = \sum_{i=1}^{n-1} \text{VF} \cdot (1+d)^{-i} + (\text{VF}+N) \cdot (1+d)^{-n}$$

$$\frac{\partial \text{VP}_0}{\partial \text{ER}_0} = \left[\sum_{i=1}^{n-1} \text{VF} \cdot (-i) \cdot (1+d)^{-(i+1)} + (\text{VF}+N) \cdot (-n) \cdot (1+d)^{-(n+1)}\right] \cdot \frac{1}{100.0 \cdot \text{PF}}$$

$$\frac{\partial \text{VP}_0}{\partial \text{ER}_0} = -\frac{1}{100.0 \cdot \text{PF}} \cdot \left[\sum_{i=1}^{n-1} i \cdot \text{VF} \cdot (1+d)^{-(i+1)} + n \cdot (\text{VF}+N) \cdot (1+d)^{-(n+1)}\right]$$

C) Recursive Rate Convention:

$$\text{VP}_0 = \frac{N + \text{FV}_n(\text{ER}_0)}{1 + \text{ER}_0 \cdot M/100}$$

Let $u(\text{ER}_0) = N + \text{FV}_n(\text{ER}_0)$ and $v(\text{ER}_0) = 1 + \text{ER}_0 \cdot M/100$.

$$\frac{\partial u}{\partial \text{ER}_0} = \frac{\partial \text{FV}_n}{\partial \text{ER}_0}$$
$$\frac{\partial v}{\partial \text{ER}_0} = \frac{M}{100}$$
$$\frac{\partial \text{VP}_0}{\partial \text{ER}_0} = \frac{\frac{\partial \text{FV}_n}{\partial \text{ER}_0} \cdot v(\text{ER}_0) - u(\text{ER}_0) \cdot \frac{M}{100}}{v(\text{ER}_0)^2}$$

To find $\frac{\partial \text{FV}_n}{\partial \text{ER}_0}$:

$$\text{FV}_i = (\text{FV}_{i-1} + \text{VF}) \cdot (1 + \text{ER}_0 \cdot m_i/100)$$
$$\frac{\partial \text{FV}_i}{\partial \text{ER}_0} = \frac{\partial \text{FV}_{i-1}}{\partial \text{ER}_0} \cdot (1 + \text{ER}_0 \cdot m_i/100) + (\text{FV}_{i-1} + \text{VF}) \cdot \frac{m_i}{100}$$

Base case: $\frac{\partial \text{FV}_0}{\partial \text{ER}_0} = 0$. Iterate to find $\frac{\partial \text{FV}_n}{\partial \text{ER}_0}$.

2.2.2. Effective Rate Sensitivity to Price ($\frac{\partial \textbf{ER}_0}{\partial \textbf{VP}_0}$)

Using the inverse function theorem:

$$\frac{\partial \text{ER}_0}{\partial \text{VP}_0} = \frac{1}{\frac{\partial \text{VP}_0}{\partial \text{ER}_0}}$$

This applies to all three conventions, using the respective $\frac{\partial \text{VP}_0}{\partial \text{ER}_0}$ derived above. (Requires $\frac{\partial \text{VP}_0}{\partial \text{ER}_0} \neq 0$).

2.3. Q1.3: Second-Order Sensitivities

2.3.1. Second Derivative of Price wrt Effective Rate ($\frac{\partial^2 \textbf{VP}_0}{\partial \textbf{ER}_0^2}$)

A) Linear Rate Convention:

$$\frac{\partial \text{VP}_0}{\partial \text{ER}_0} = -\frac{NM}{100} \quad (\text{constant wrt ER}_0)$$
$$\frac{\partial^2 \text{VP}_0}{\partial \text{ER}_0^2} = 0$$

B) Cumulative Rate Convention: Differentiate $\frac{\partial \text{VP}_0}{\partial \text{ER}_0}$ from 2.2.1.B wrt ER_0. Let $K = -1/(100.0 \cdot \text{PF})$.

$$\frac{\partial \text{VP}_0}{\partial \text{ER}_0} = K \cdot \left[\sum_{i=1}^{n-1} i \cdot \text{VF} \cdot (1+d)^{-(i+1)} + n \cdot (\text{VF} + N) \cdot (1+d)^{-(n+1)} \right]$$

$$\frac{\partial^2 \text{VP}_0}{\partial \text{ER}_0^2} = K \cdot \left[\sum_{i=1}^{n-1} i \cdot \text{VF} \cdot (-(i+1)) \cdot (1+d)^{-(i+2)} \right.$$
$$\left. + n \cdot (\text{VF} + N) \cdot (-(n+1)) \cdot (1+d)^{-(n+2)} \right] \cdot \frac{1}{100.0 \cdot \text{PF}}$$
$$\frac{\partial^2 \text{VP}_0}{\partial \text{ER}_0^2} = \frac{1}{(100.0 \cdot \text{PF})^2} \cdot \left[\sum_{i=1}^{n-1} i(i+1) \cdot \text{VF} \cdot (1+d)^{-(i+2)} \right.$$
$$\left. + n(n+1) \cdot (\text{VF} + N) \cdot (1+d)^{-(n+2)} \right]$$

C) Recursive Rate Convention: Let $X = \frac{\partial \text{VP}_0}{\partial \text{ER}_0} = \frac{u'v - uv'}{v^2}$ where u, v, u', v' are functions of ER_0. $u' = \frac{\partial \text{FV}_n}{\partial \text{ER}_0}$, $v' = M/100$.

$$\frac{\partial X}{\partial \text{ER}_0} = \frac{(u''v + u'v' - u'v' - uv'')v^2 - (u'v - uv')2vv'}{v^4}$$

$$\frac{\partial X}{\partial \mathrm{ER}_0} = \frac{(u''v - uv'')v - 2v'(u'v - uv')}{v^3}$$

Here, $u'' = \frac{\partial^2 \mathrm{FV}_n}{\partial \mathrm{ER}_0^2}$ and $v'' = \frac{\partial^2}{\partial \mathrm{ER}_0^2}(1 + \mathrm{ER}_0 \cdot M/100) = 0$. So,

$$\frac{\partial^2 \mathrm{VP}_0}{\partial \mathrm{ER}_0^2} = \frac{(\frac{\partial^2 \mathrm{FV}_n}{\partial \mathrm{ER}_0^2}) \cdot v(\mathrm{ER}_0) \cdot v(\mathrm{ER}_0) - 2\frac{M}{100}\left(\frac{\partial \mathrm{FV}_n}{\partial \mathrm{ER}_0} \cdot v(\mathrm{ER}_0) - u(\mathrm{ER}_0) \cdot \frac{M}{100}\right)}{v(\mathrm{ER}_0)^3}$$

$$\frac{\partial^2 \mathrm{VP}_0}{\partial \mathrm{ER}_0^2} = \frac{v \cdot \frac{\partial^2 \mathrm{FV}_n}{\partial \mathrm{ER}_0^2} - 2 \cdot \frac{M}{100} \cdot \frac{\partial \mathrm{VP}_0}{\partial \mathrm{ER}_0} \cdot v}{v^2} \quad \text{This looks wrong.}$$

Correct formula: $\frac{(u''v-uv'')v-2v'(u'v-uv')}{v^3} = \frac{u''v^2-2v'(u'v-uv')}{v^3}$ (since $v'' = 0$).

To find $\frac{\partial^2 \mathrm{FV}_n}{\partial \mathrm{ER}_0^2}$: From $\frac{\partial \mathrm{FV}_i}{\partial \mathrm{ER}_0} = \frac{\partial \mathrm{FV}_{i-1}}{\partial \mathrm{ER}_0} \cdot (1 + \mathrm{ER}_0 \cdot m_i/100) + (\mathrm{FV}_{i-1} + \mathrm{VF}) \cdot \frac{m_i}{100}$. Differentiate wrt ER_0:

$$\begin{aligned}\frac{\partial^2 \mathrm{FV}_i}{\partial \mathrm{ER}_0^2} &= \frac{\partial^2 \mathrm{FV}_{i-1}}{\partial \mathrm{ER}_0^2} \cdot (1 + \mathrm{ER}_0 \cdot m_i/100) + \frac{\partial \mathrm{FV}_{i-1}}{\partial \mathrm{ER}_0} \cdot \frac{m_i}{100} \\ &\quad + \frac{\partial \mathrm{FV}_{i-1}}{\partial \mathrm{ER}_0} \cdot \frac{m_i}{100} + 0 \\ \frac{\partial^2 \mathrm{FV}_i}{\partial \mathrm{ER}_0^2} &= \frac{\partial^2 \mathrm{FV}_{i-1}}{\partial \mathrm{ER}_0^2} \cdot (1 + \mathrm{ER}_0 \cdot m_i/100) + 2 \cdot \frac{\partial \mathrm{FV}_{i-1}}{\partial \mathrm{ER}_0} \cdot \frac{m_i}{100}\end{aligned}$$

Base case: $\frac{\partial^2 \mathrm{FV}_0}{\partial \mathrm{ER}_0^2} = 0$. Iterate to find $\frac{\partial^2 \mathrm{FV}_n}{\partial \mathrm{ER}_0^2}$.

2.3.2. Second Derivative of Effective Rate wrt Price ($\frac{\partial^2 \mathbf{ER}_0}{\partial \mathbf{VP}_0^2}$)

Let $g(y) = f^{-1}(y)$, so $\mathrm{ER}_0 = g(\mathrm{VP}_0)$. We know $g'(y) = 1/f'(g(y))$.

$$\begin{aligned}g''(y) &= \frac{d}{dy}\left[\frac{1}{f'(g(y))}\right] = -[f'(g(y))]^{-2} \cdot f''(g(y)) \cdot g'(y) \\ g''(y) &= -\frac{f''(g(y))}{[f'(g(y))]^3}\end{aligned}$$

So,

$$\frac{\partial^2 \mathrm{ER}_0}{\partial \mathrm{VP}_0^2} = -\frac{\frac{\partial^2 \mathrm{VP}_0}{\partial \mathrm{ER}_0^2}}{\left(\frac{\partial \mathrm{VP}_0}{\partial \mathrm{ER}_0}\right)^3}$$

This applies to all three conventions, using the respective first and second derivatives of VP_0 wrt ER_0. (Requires $\frac{\partial \mathrm{VP}_0}{\partial \mathrm{ER}_0} \neq 0$).

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3. Question 2: DeliveryContract Pricing

A DeliveryContract class would have properties:

- T_{ED}: Expiration Date in years from today (double)
- BoVN: Basket of ValueNotes (list/vector of ValueNote objects)
- SVR: Standardized Value Rate (double, e.g., 5 for 5%)
- r: Risk-free interest rate (double, e.g., 0.04 for 4%)

And methods to calculate:

- Relative Factors for each ValueNote in BoVN.
- DeliveryContract Price.
- Delivery Probabilities.
- Sensitivities.

3.1. Q2.1: Basket of ValueNotes and RelativeFactor (RF)

a) Create a basket of n ValueNotes:
This involves instantiating n ValueNote objects with their specific N, M, VR, PF, σ_i (volatility of ER) and VP_0^i (today's price). The initial ER_0^i for each ValueNote must be calculated from its VP_0^i using the "Cumulative Rate".

b) Calculate RelativeFactor (RF) for each deliverable ValueNote:
The problem specifies "Relative Factor convention: Cumulative".

$$\text{RF}_i = \text{VP}_i(\text{ER}_0 = \text{SVR})/100$$

Where $\text{VP}_i(\text{ER}_0 = \text{SVR})$ is the price of ValueNote i calculated using the "Cumulative Rate" convention with its N, M, VR, PF but setting ER_0 to the contract's SVR.

3.2. Q2.2: Calculate the price of the DeliveryContract

The pricing approach involves these steps:

1. For each ValueNote i in BoVN:
   - Determine its current ER_0^i from its market price VP_0^i using Q1.1.b (Cumulative Rate convention).
   - Store its specific volatility σ_i.
2. Define the Price-to-RelativeFactor function $f_i(z)$ for each ValueNote i:

   $$f_i(z) = \text{VP}_i(\text{ER}_{T_{ED}}^i(z))/\text{RF}_i$$

   Where:
   - z is a standard normal random variable $N(0, 1)$.
   - $\text{ER}_{T_{ED}}^i(z) = \text{ER}_0^i \cdot \exp(\sigma_i \cdot \sqrt{T_{ED}} \cdot z - 0.5 \cdot \sigma_i^2 \cdot T_{ED})$.
   - $\text{VP}_i(\text{ER}_{T_{ED}}^i(z))$ is the price of ValueNote i at time T_{ED}, given the effective rate $\text{ER}_{T_{ED}}^i(z)$. This price is calculated using the "Cumulative Rate" convention (Q1.1.A.B). The ValueNote i used for this calculation must be adjusted for T_{ED}:
     - New Maturity: $M_i' = M_i - T_{ED}$ (if $M_i' < 0$, the bond has matured; handle this case, e.g., it's not deliverable or its price is N).
     - New set of cash flows: Only coupons scheduled after T_{ED} are included. Timings are relative to T_{ED}. For example, an original payment at t_k becomes $t_k' = t_k - T_{ED}$.
     - Number of payments $n_i' = \lfloor M_i' \cdot \text{PF}_i \rfloor$.
3. Quadratic Approximation of $f_i(z)$:
   For each ValueNote i:
   - Generate K (e.g., 2000) equally spaced points z_j in a range (e.g., [-3, 3]).
   - Calculate $f_i(z_j)$ for each z_j.
   - Fit $f_i(z) \approx Q_i(z) = a_i z^2 + b_i z + c_i$ by minimizing the normally weighted mean squared error:

     $$\min_{a_i, b_i, c_i} \sum_{j=1}^{K} \left[(f_i(z_j) - (a_i z_j^2 + b_i z_j + c_i))^2 \cdot w(z_j) \right]$$

     where $w(z_j) = (1/\sqrt{2\pi}) \cdot \exp(-z_j^2/2)$ (normal PDF value). This is a weighted least squares problem.
4. Determine MEV Intervals:
   - For every pair of ValueNotes (i, k), find intersection points of their quadratic approximations by solving $Q_i(z) = Q_k(z)$:

     $$(a_i - a_k)z^2 + (b_i - b_k)z + (c_i - c_k) = 0.$$

     This gives up to two roots for each pair.

- Collect all unique real roots from all pairs and sort them. These sorted roots, along with the boundaries of the z range (e.g., -3, 3 or effectively $-\infty, +\infty$ for integration), define a set of intervals $[z_{\text{lower}}, z_{\text{upper}}]$.
- In each interval, determine the Most Economical ValueNote (MEV) by picking a test point (e.g., midpoint) and finding j such that $Q_j(z_{\text{test}})$ is minimized. $Q_{\text{MEV}_J}(z)$ is the minimum quadratic in that interval.

5. Calculate DeliveryContract Price:
The price is $E[\min_i f_i(z)] \approx E[\min_i Q_i(z)]$.

$$\text{DC Price} = \sum_{\text{intervals } J} \int_{z_{\text{lower}, J}}^{z_{\text{upper}, J}} Q_{\text{MEV}_J}(z) \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp(-z^2/2)dz$$

This integral can be computed analytically or numerically. Analytically, for $Q(z) = az^2 + bz + c$:

$$E[Q(Z) \cdot I(Z \in [z_1, z_2])] = \frac{1}{\sqrt{2\pi}} \cdot \int_{z_1}^{z_2} (az^2 + bz + c) \exp(-z^2/2)dz$$

This involves integrals of $\exp(-z^2/2)$, $z \exp(-z^2/2)$, $z^2 \exp(-z^2/2)$, which can be related to the Normal CDF ($\Phi(z)$) and PDF ($\phi(z)$):

- $\int \exp(-u^2/2)du = \sqrt{2\pi}\Phi(z)$
- $\int u \exp(-u^2/2)du = -\exp(-u^2/2)$
- $\int u^2 \exp(-u^2/2)du = \sqrt{2\pi}\Phi(z) - z\exp(-z^2/2)$ (using integration by parts)

A careful derivation of the integral of $(az^2 + bz + c)\phi(z)$:

$$\begin{aligned} &E[(aZ^2 + bZ + c)I(z_L \leq Z \leq z_U)] \\ &\quad = a(z_L\phi(z_L) - z_U\phi(z_U) + \Phi(z_U) - \Phi(z_L)) + b(\phi(z_L) - \phi(z_U)) + c(\Phi(z_U) - \Phi(z_L)) \end{aligned}$$

3.3. Q2.3: Implement calculation of delivery probabilities

For each ValueNote k in BoVN:

$$\text{Prob}(k \text{ is MEV}) = \sum_{\substack{\text{intervals } J \\ \text{where } k \text{ is MEV}}} \int_{z_{\text{lower}, J}}^{z_{\text{upper}, J}} \frac{1}{\sqrt{2\pi}} \cdot \exp(-z^2/2)dz$$

$$\text{Prob}(k \text{ is MEV}) = \sum_{\substack{\text{intervals } J \\ \text{where } k \text{ is MEV}}} [\Phi(z_{\text{upper}, J}) - \Phi(z_{\text{lower}, J})]$$

where Φ is the standard normal CDF.

3.4. Q2.4: Derive and implement analytical expressions for sensitivities

Let P_{DC} be the DeliveryContract Price. We use the "pathwise derivative" method, assuming interchangeability of expectation and differentiation, or differentiate the integrated quadratic approximation form. Differentiating the integral form is more robust given the min operator. The coefficients a_i, b_i, c_i of $Q_i(z)$ depend on the parameters σ_k or $\text{VP}_{0,k}$ because $f_i(z)$ depends on them. This means $\partial a_i/\partial\text{param}, \partial b_i/\partial\text{param}, \partial c_i/\partial\text{param}$ are needed. Also, the intersection points z_{cross} depend on these parameters. Let's assume the pathwise derivative for $E[\min_i f_i(z)]$:

$$\frac{\partial P_{DC}}{\partial \text{param}} = E\left[\frac{\partial}{\partial \text{param}}(\min_i f_i(z))\right]$$

If $j = \text{argmin}_i f_i(z)$, then $\frac{\partial}{\partial \text{param}}(\min_i f_i(z)) = \frac{\partial f_j(z)}{\partial \text{param}}$. So,

$$\frac{\partial P_{DC}}{\partial \text{param}} = E\left[\frac{\partial f_{\text{MEV}(z)}(z)}{\partial \text{param}}\right] = \int_{-\infty}^{\infty} \frac{\partial f_{\text{MEV}(z)}(z)}{\partial \text{param}} \cdot \phi(z)dz$$

where $\text{MEV}(z)$ is the index of the minimum $f_i(z)$ at point z.

a) Sensitivity to volatility σ_k: $\partial P_{DC}/\partial\sigma_k$

$$f_i(z) = \text{VP}_i(\text{ER}^i_{T_{ED}}(z))/\text{RF}_i$$

$\frac{\partial f_i(z)}{\partial \sigma_k} = 0$ if $i \neq k$. For $i = k$:

$$\frac{\partial f_k(z)}{\partial \sigma_k} = \frac{1}{\text{RF}_k} \cdot \frac{\partial \text{VP}_k}{\partial \text{ER}^k_{T_{ED}}} \cdot \frac{\partial \text{ER}^k_{T_{ED}}}{\partial \sigma_k}$$

Where $\frac{\partial \text{VP}_k}{\partial \text{ER}^k_{T_{ED}}}$ is the ValueNote price sensitivity (Q1.2.A.B - Cumulative) for the adjusted ValueNote k at T_{ED}. And $\text{ER}^k_{T_{ED}}(z) = \text{ER}^k_0 \cdot \exp(\sigma_k \cdot \sqrt{T_{ED}} \cdot z - 0.5 \cdot \sigma_k^2 \cdot T_{ED})$.

$$\frac{\partial \text{ER}^k_{T_{ED}}}{\partial \sigma_k} = \text{ER}^k_{T_{ED}}(z) \cdot (\sqrt{T_{ED}} \cdot z - \sigma_k \cdot T_{ED}).$$

So,

$$\frac{\partial P_{DC}}{\partial \sigma_k} = E\left[I(k = \text{MEV}(z)) \cdot \frac{1}{\text{RF}_k} \cdot \frac{\partial \text{VP}_k}{\partial \text{ER}^k_{T_{ED}}} \cdot \text{ER}^k_{T_{ED}}(z) \cdot (\sqrt{T_{ED}} \cdot z - \sigma_k \cdot T_{ED})\right]$$

This expectation is computed by numerical integration over z, using the MEV determined by comparing $Q_i(z)$ values in each segment.

b) Sensitivity to today's price $\text{VP}_{0,k}$: $\partial P_{DC}/\partial\mathbf{VP}_{0,k}$ RF_i is calculated based on SVR, not $\text{VP}_{0,i}$, so $\partial \text{RF}_i/\partial \text{VP}_{0,k} = 0$. $\frac{\partial f_i(z)}{\partial \text{VP}_{0,k}} = 0$ if $i \neq k$. For $i = k$:

$$\frac{\partial f_k(z)}{\partial \text{VP}_{0,k}} = \frac{1}{\text{RF}_k} \cdot \frac{\partial \text{VP}_k}{\partial \text{ER}^k_{T_{ED}}} \cdot \frac{\partial \text{ER}^k_{T_{ED}}(z)}{\partial \text{VP}_{0,k}}$$

$\text{ER}^k_{T_{ED}}(z) = \text{ER}^k_0 \cdot \exp(\sigma_k \cdot \sqrt{T_{ED}} \cdot z - 0.5 \cdot \sigma_k^2 \cdot T_{ED})$.

$$\frac{\partial \text{ER}^k_{T_{ED}}(z)}{\partial \text{VP}_{0,k}} = \frac{\partial \text{ER}^k_0}{\partial \text{VP}_{0,k}} \cdot \exp(\sigma_k \cdot \sqrt{T_{ED}} \cdot z - 0.5 \cdot \sigma_k^2 \cdot T_{ED})$$
$$\frac{\partial \text{ER}^k_{T_{ED}}(z)}{\partial \text{VP}_{0,k}} = \frac{\partial \text{ER}^k_0}{\partial \text{VP}_{0,k}} \cdot \frac{\text{ER}^k_{T_{ED}}(z)}{\text{ER}^k_0} \quad (\text{if } \text{ER}^k_0 \neq 0)$$

$\frac{\partial \text{ER}^k_0}{\partial \text{VP}_{0,k}}$ is the ValueNote k's effective rate sensitivity to its own price (Q1.2.B - Cumulative). So,

$$\frac{\partial P_{DC}}{\partial \text{VP}_{0,k}} = E\left[I(k = \text{MEV}(z)) \cdot \frac{1}{\text{RF}_k} \cdot \frac{\partial \text{VP}_k}{\partial \text{ER}^k_{T_{ED}}} \cdot \frac{\partial \text{ER}^k_0}{\partial \text{VP}_{0,k}} \cdot \frac{\text{ER}^k_{T_{ED}}(z)}{\text{ER}^k_0}\right]$$

This expectation is computed by numerical integration.