6. (a) Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at origin although the Cauchy-Riemann equations are satisfied at the origin. [2+2]

(b) Define harmonic conjugate of a function. Show that $u(x, y) = \frac{1}{2}\log(x^2 + y^2)$ is harmonic function and find the corresponding analytic function $f(z)$ in terms of z. [1+2+3]

7. (a) Find the Laurent's expansion of

$$f(z) = \frac{7z - 2}{z(z+1)(z-2)}$$

in the region $1 < |z+1| < 3$. [4]

(b) Determine the poles of the function

$$f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z^2 + z - 6)}$$

and the residue at each pole. Hence evaluate $\oint_C f(z)\, dz$, where C is the circle $|z| = 2.5$. [1+3+2]

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6. (a) $f(z) = \sqrt{|xy|}$

$u(x,y) = \sqrt{|xy|}$, $v(x,y) = 0$

$$u_x(0,0) = \lim_{h\to 0} \frac{u(h,0) - u(0,0)}{h} = 0$$

$$u_y(0,0) = \lim_{k\to 0} \frac{u(0,k) - u(0,0)}{k} = 0$$

$$v_x(0,0) = \lim_{h\to 0} \frac{v(h,0) - v(0,0)}{h} = 0$$

$$v_y(0,0) = \lim_{k\to 0} \frac{v(0,k) - v(0,0)}{k} = 0$$

$\therefore$ $\boxed{u_x = v_y}$ & $\boxed{u_y = -v_x}$

$\therefore$ at origin Cauchy-Reiman Eq^{n}s are satisfied.

<u>Though</u>

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0}$$

$$= \lim_{z\to 0} \frac{\sqrt{xy} - 0}{x+iy}$$ [take $z \to 0$ along the line $y = mx$ $\therefore x \to 0$

$$= \lim_{x\to 0} \frac{\sqrt{|m|x^2}}{x(1+im)}$$

$$\therefore \lim_{x\to 0^+} = \frac{\cancel{x}\sqrt{|m|}}{\cancel{x}(1+im)} = \frac{\sqrt{|m|}}{1+im}$$

$$\lim_{x\to 0^-} = \frac{-\sqrt{|m|}}{1+im}$$

$\therefore$ $f'(0)$ does not exist, but C-R equations are satisfied.

(b) ☐ Harmonic Conjugate

A function $u(x,y)$ which possess cont partial derivatives of 1^{st} and 2^{nd} order and satisfies the Laplace's Eqn $\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\right)$ is called as a harmonic func^n

— If 2 harmonic function $u(x,y)$ and $v(x,y)$ satisfy the cauchy reiman eq^{n}s $u_x = v_y$, $u_y = -v_x$ then they are known as conjugate harmonic function.

■ $u(x,y) = \frac{1}{2}\log(x^2+y^2)$

$$\frac{\partial u}{\partial x} = \frac{1}{\cancel{2}} \times \frac{\cancel{2}x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad \text{—— (i)}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\cancel{2}} \times \frac{\cancel{2}y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)-y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \quad \text{—— (ii)}$$

(i) + (ii)

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

∴ It satisfies Laplace's Equation

∴ It is a harmonic function.

$$u_x = \frac{x}{x^2+y^2} = \phi_1(x,y)$$

$$u_y = \frac{y}{x^2+y^2} = \phi_2(x,y)$$

$$\phi_1(z,0) = \frac{z}{z^2+0^2} = \frac{1}{z}$$

$$\phi_2(z,0) = \frac{0}{z^2+0^2} = 0$$

According to Milne Thomson's relation :-

$$f(z) = \int \phi_1(z,0)\,dz \;\; i\int \phi_2(z,0)\,dz$$

$$= \int \frac{1}{z}\,dz$$

$$= \log z + c \quad (\text{Ans.})$$

---

7 (a) $f(z) = \dfrac{7z-2}{z(z+1)(z-2)} = \dfrac{A}{z} + \dfrac{B}{z+1} + \dfrac{C}{z-2}$

$\Rightarrow 7z - 2 = A(z+1)(z-2) + Bz(z-2) + C\times z(z+1)$

at $z = 0$

$-2 = A \times 1 \times (-2)$

$\Rightarrow \boxed{A = 1}$

at $z = -1$

$\cancel{-}9 = B \times (\cancel{-}1)(-3)$

$\boxed{B = -3}$

at $z = 2$

$\overset{6}{\cancel{12}} = C \times \cancel{2} \times 3$

$\Rightarrow \boxed{C = 2}$

$\therefore f(z) = \dfrac{1}{z} + \dfrac{-3}{z+1} + \dfrac{2}{z-2}$

$1 < |z+1| < 3$

$\boxed{\dfrac{1}{z+1} < 1}$ $\quad$ $\boxed{\dfrac{z+1}{3} < 1}$

$$\therefore f(z) = \frac{1}{z} - \frac{3}{z+1} + \frac{2}{z-2}$$

$$= \frac{1}{(z+1)-1} - 3(z+1)^{-1} + \frac{2}{(z+1)-3}$$

$$= -\frac{1}{1 \mp (z+1)} - 3(z+1)^{-1} - \frac{2}{3\left(1 - \frac{z+1}{3}\right)}$$

$$= \frac{1}{z+1}\left(\frac{1}{1 - \frac{1}{z+1}}\right) - \frac{3}{(z+1)} - \frac{2}{3}\left(1 \mp \frac{z+1}{3}\right)^{-1}$$

$$= \frac{1}{z+1}\left(1 - \frac{1}{z+1}\right)^{-1} - \frac{3}{z+1} - \frac{2}{3}\left(1 - \frac{z+1}{3}\right)^{-1}$$

$$= (z+1)^{-1}\left\{1 + \frac{1}{z+1} + \frac{1}{(z+1)^2} + \cdots - 3\right\} - \frac{2}{3}\left(1 + \frac{z+1}{3} + \frac{(z+1)^2}{9} + \cdots\right),$$

$$= \left(\frac{-2}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \cdots\right) - \frac{2}{3}\left(1 + \frac{z+1}{3} + \frac{(z+1)^2}{9} + \cdots\right)$$

[Ans.]

(b) $$f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z^2 + z - 6)}$$

$z^2 + z - 6$
$= (z-2)(z+3)$

$$= \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)(z+3)}$$

$z = 1 \rightarrow$ pole of order 2

$z = 2 \longrightarrow$ simple pole

$z = -3 \longrightarrow$ simple pole

- **At $z=1$**

$$\text{Res}\,(f(z)) = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}\left[\frac{\sin\pi z^2+\cos\pi z^2}{z^2+z-6}\right] = \underline{\qquad}$$

- **at $z=2$**

$$\text{Res}\,(f(z)) = \lim_{z\to 2}(z-2)f(z)$$

$$= \lim_{z\to 2}\frac{\sin\pi z^2+\cos\pi z^2}{(z-1)^2(z+3)}$$

$$= \frac{\sin 4\pi+\cos 4\pi}{1\times 5} = \frac{1}{5}$$

- **at $z=-3$**

$$\text{Res}\, f(z) = \lim_{z\to -3}(z+3)f(z)$$

$$= \frac{\sin 9\pi+\cos 9\pi}{(-4)^2(-5)} = \frac{1}{80}$$