[redacted]

4. (a) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$, then prove that

   (i) $\operatorname{div}(r^n \vec{r}) = (n+3)r^n$.

   (ii) $\nabla^2\left(\frac{1}{r}\right) = 0$. [2+2]

   (b) Verify Stokes' theorem for the vector function $\vec{F} = x^2\hat{i} + xy\hat{j}$ integrated round the square in the plane $z = 0$ whose sides are along the straight lines $x = 0, x = a, y = 0, y = b$. [3+3]

5. (a) A force of 10 units acts in the direction of the vector $(3\hat{i} + \hat{j} + \hat{k})$ and passes through $(2\hat{i} - \hat{j} + 3\hat{k})$. Find the moment of the force about the point $(\hat{i} + 2\hat{j} - \hat{k})$. [4]

   (b) If $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$, then evaluate

   (i) $\iiint\limits_V \vec{\nabla} \cdot \vec{F}\, dV$ and

   (ii) $\iiint\limits_V \vec{\nabla} \times \vec{F}\, dV$

   where $V$ is the closed region bounded by the planes $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$. [3+3]

6. (a) Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at origin although the Cauchy-Riemann equations are satisfied at the origin. [2+2]

   (b) Define harmonic conjugate of a function. Show that $u(x, y) = \frac{1}{2}\log(x^2 + y^2)$ is harmonic function and find the corresponding analytic function $f(z)$ in terms of $z$. [1+2+3]

7. (a) Find the Laurent's expansion of

$$f(z) = \frac{7z - 2}{z(z+1)(z-2)}$$

   in the region $1 < |z + 1| < 3$. [4]

   (b) Determine the poles of the function

$$f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z^2 + z - 6)}$$

   and the residue at each pole. Hence evaluate $\oint_C f(z)\, dz$, where $C$ is the circle $|z| = 2.5$. [1+3+2]

5. (a) Reduce the matrix $A$ to row-reduced echelon form and hence find its rank, where

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}.$$

(b) Investigate for what values of $\lambda$ and $\mu$ the following equations

$$\begin{aligned} x + y + z &= 6, \\ x + 2y + 3z &= 10 \\ x + 2y + \lambda z &= \mu \end{aligned}$$

have (i) no solution, (ii) a unique solution and (iii) an infinite number of solutions.

[3+3]

6. (a) Define the shortest distance between two skew lines and hence find the shortest distance between two skew lines $r = r_1 + t\alpha$ and $r = r_2 + t\beta$, where $t$ is a scalar and $r_1, \alpha, r_2, \beta$ are vectors with coordinates $(1, -2, 3), (2, 1, 1), (-2, 2, -1)$ and $(-3, 1, 2)$ respectively.

(b) Find $div\,(\vec{F})$ and $curl\,(\vec{F})$, where $\vec{F} = grad(x^3 + y^3 + z^3 - 3xyz)$.

[(1+2)+(1+2)]

7. (a) Define directional derivative of a scalar point function.

(b) Show that $\vec{A} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational. Find the scalar function $\phi$ such that $\vec{A} = \vec{\nabla}\phi$.

[1+(2+3)]

.....

$\begin{vmatrix} 0 & 1 & a \\ 1 & a & 1 \\ a & 1 & 0 \end{vmatrix} = 0(\ ) - 1(0-a) + a(1-a^2)$

$\Rightarrow a + a - a^3$

$\Rightarrow a^3 - 2a$

$\Rightarrow a(a^2 - \ldots$

$\Rightarrow a = 0$

$\Rightarrow a = 0$

Page 2 / 8

$(\ ,3,0)$

$(1,1,1)$

$3a + b = 0.$

2023 (Midsem) [Manya]

(6) $r = r_1 + t\alpha$ $\qquad r_1 = (1,-2,3)$

$r = r_2 + t\beta$ $\qquad \alpha = (2,1,1)$

$r_2 = (-2,2,-1)$

$\beta = (-3,1,2)$

Shortest dist$^n$ (d) $= \left| \dfrac{(r_2 - r_1)\cdot(\alpha \times \beta)}{|\alpha \times \beta|} \right|$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ -3 & 1 & 2 \end{vmatrix}$$

$\hat{i}(1) + \hat{j}(-3-4) + \hat{k}(5)$

$= (\hat{i} - 7\hat{j} + 5\hat{k})$

$= \left| \dfrac{(-3,4,-4)\cdot(1,-7,5)}{\sqrt{1^2+7^2+5^2}} \right|$

$= \left| \dfrac{-3-28-20}{\sqrt{75}} \right|$

$= \dfrac{51}{5\sqrt{3}}$ unit.

(b) $\vec{F} = \text{grad}(x^3+y^3+z^3-3xyz)$

$= \underline{(3x^2-3yz)\hat{i} + (3y^2-3xz)\hat{j} + (3z^2-3xy)\hat{k}}$

$= 3(x^2-yz)\hat{i} + 3(y^2-xz)\hat{j} + 3(z^2-xy)\hat{k}$

Curl $\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3(x^2-yz) & 3(y^2-xz) & 3(z^2-xy) \end{vmatrix}$

$= \hat{i}(-3x+3x) + \hat{j}(-3y+3y) + \hat{k}(-3z+3z)$

$= (0,0,0)$

$$\text{div } \vec{F} = \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy)$$

$$= (6x + 6y + 6z)$$

7) (a) **Define: Dir$^n$ derivative of a Scalar pt. function**

Ans: The dir$^n$ derivative of a scalar pt $f = \phi(x, y, z)$ at a point $P(x_0, y_0, z_0)$ in the dir$^n$ of a vector $v = (a\hat{i} + b\hat{j} + c\hat{k})$ is the rate at which $\phi$ changes at that point in the given dir$^n$

$$\nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)$$

$$\hat{u} = \frac{v}{|v|} = \left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right)$$

$$D_v \phi = \nabla \phi \cdot \hat{u}$$

$$= \left(\frac{\partial \phi}{\partial x} \cdot \frac{a}{|v|} + \frac{\partial \phi}{\partial y} \cdot \frac{b}{|v|} + \frac{\partial \phi}{\partial z} \cdot \frac{c}{|v|}\right)$$

(b) $\vec{A} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$

~~$\text{div } \vec{A} = \frac{\partial}{\partial x}(6xy + z^3) + \frac{\partial}{\partial y}(3x^2 - z) + \frac{\partial}{\partial z}($~~

$$\text{Curl } \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (6xy + z^3) & (3x^2 - z) & (3xz^2 - y) \end{vmatrix}$$

$$= \hat{i}(-1 + 1) + \hat{j}(3z^2 - 3z^2) + \hat{k}(6x - 6x)$$

$$= 0$$

$$\vec{A} = \vec{\nabla}\phi$$

$$\phi_x = \int (6xy + z^3)\, dx$$

$$= 3x^2y + xz^3 + C_1(y, z)$$

$$\phi_y = \int (3x^2 - z)\, dy$$

$$= (3x^2y - yz) + C_2(x, z)$$

$$\phi_z = \int (3xz^2 - y)\, dz$$

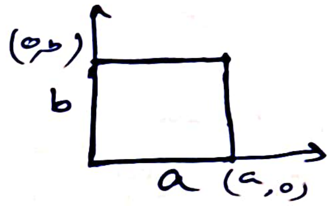
$$= (xz^3 - yz) + C_3(x, y)$$

$$\boxed{\phi = (3x^2y - yz + xz^3) + C}$$

2023 (Ensem) [June]

(4) (a) done in 2024

(b) $\vec{F} = x^2\hat{i} + xy\hat{j}$




$\int_C \vec{F}\cdot d\vec{r}$

$= \int_{x=0}^{a} (x^2\hat{i} + xy\hat{j})\cdot dx + \int_{y=0}^{b} (x^2\hat{i} + xy\hat{j})\, dy + \int^{0} (x^2\hat{i} + xy)\, dx$

$+ \int_{y=b} (x^2\hat{i} + xy\hat{j})\cdot dy$

$= \left[\frac{x^3}{3} + 0\right]_0^a + \left[0 + \frac{ay^2}{2}\right]_0^b + \left[\frac{x^3}{3} + 0\right]_a^0 - 0$

$= \frac{a^3}{3} + \frac{ab^2}{2} - \frac{a^3}{3} = \frac{ab^2}{2}$ Ans

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix}$$

$$= y\hat{k}$$

$$\iint_S (\nabla \times F) \cdot \hat{k} \, ds$$

$$= \int_{x=0}^{a}\int_{y=0}^{b} y \, dx \, dy$$

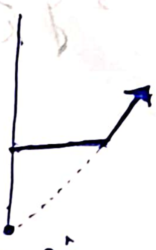
$$= \int_{y=0}^{b} [xy]_0^a \, dy$$

$$= \int_0^b ay \, dy$$

$$= \left[\frac{ay^2}{2}\right]_0^b = \frac{ab^2}{2}$$

$\therefore$ Stokes Thm is proved

5)(a) $\vec{p} =$



$\hat{i} + 2\hat{j}$

5)(b) $F = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(2x^2 - 3z) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(-4x)$$

$$= 4x - 2x$$

$$= 2x$$

$2x + 2y = 4$

$y = 4 - 2x$

$z = 4 - 2x$

(i) $\iiint 2x \, dx\,dy\,dz$

$$= \int_{x=0}^{2-y} \int_{y=0}^{2} \int_{z=0}^{4-2x-2y} 2\cancel{x} \, dx\,dy\,dz$$

$$= \int_{y=0}^{2} \int_{z=0}^{4-2x-2y} \left[x^2\right]_0^{2-y} dy\,dz$$

$$= \int_{y=0}^{2} \int_{z=0}^{4-2x-2y} (2-y)^2 \, dy\,dz$$

$$= \int_{y=0}^{2} \int_{z=0}^{4-2x-2y} (4 - 2y + y^2) \, dy\,dz$$

$$= \int_{z=0}^{4-2x-2y} \left[4y - y^2 + \frac{y^3}{3}\right]_0^2 dz$$

$$= \int_{z=0}^{4-2x-2y} \left(8 - 4 + \frac{8}{3}\right) dz$$

$$= \frac{20}{3}\left[z\right]_0^{4-2x-2y}$$

$\frac{9}{3} \times 2^3$

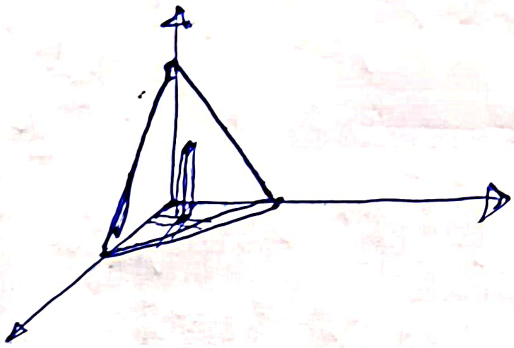
$$= \frac{20}{3} x$$

iii) $\iiint (\vec{\nabla} \times \vec{F})\, dv$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \end{vmatrix}$$

$= (\hat{j} - 2y\hat{k})$

$$\boxed{\begin{array}{c} x=0,\ y=0,\ z=0 \\ 2x+2y+z=4 \end{array}}$$



$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x^2-3z & -2xy & -4x \end{vmatrix}$$

$= i(0) - j(-4+3) + k(-2y)$

$= \hat{j} - 2y\hat{k}$

$2x+2y=4$

$x+y=2$

$$\int_0^2 \int_0^{2-y} \int_0^{4-2x-2y} \cancel{(\hat{j}-2y\hat{k})}$$

$$\int_0^2 \int_0^{2-y} \int_0^{4-2x-2y} 2x\, dx\, dy\, dz$$

$$\int_0^2 \int_0^{2-y} 2x(4-2x-2y)\, dx\, dy$$

$y^2-4y+4$

$-y^3+4y^2-4y$

$$\int_0^2 \int_0^{2-y} 8x - 4x^2 - 2xy\, dx\, dy$$

$$\int_0^2 4x^2 - \frac{4}{3}x^3 - x^2y \Big|_0^{2-y} dy$$

$$\int_0^2 4(2-y)^2 - \frac{4}{3}(2-y)^3 - y(2-y)^2\, dy$$

$$\int_0^2 4(y-2)^2 + \frac{4}{3}(y-2)^3 - y(y-2)^2\, dy$$

$$\frac{4(y-2)^3}{3}\Big|_0^2 + \frac{(y-2)^4}{3}\Big|_0^2 - \frac{y^4}{4}\Big|_0^2 + \frac{4}{3}y^3\Big|_0^2 - 2y^2\Big|_0^2$$

$$\frac{32}{3} - \frac{16}{3} - \frac{16}{4} + \frac{4\times 8}{3} - 2\times 4$$

$4 \quad 12/3$

$48-36$

$$\frac{64}{3} - \frac{16}{3} - 4 - 8$$

$$\frac{48}{3} - 12$$