PYQ

[2024] ~~Mid~~ (End-sem) Math

3. (c) $f(x) = |x|$

~~as~~ $f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |x|\, dx = \frac{2}{\pi} [x^2/2]_0^{\pi} = \pi$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos nx\, dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx\, dx$

$= \frac{2}{\pi} \left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{\pi}$

$= \frac{2}{\pi n^2} (\cos n\pi - 1)$

$= \begin{cases} -4/n^2\pi & \text{when } n = \text{odd} \\ 0 & \text{when } n = \text{even} \end{cases}$

$b_n = 1/\pi \int_{-\pi}^{\pi} |x| \sin nx\, dx = 0$

~~$|x| \approx \pi/2 + \sum_{n=1}^{\infty} [-4/n^2\pi \cos n\pi]$~~

$\therefore |x| \approx \pi/2 + \sum_{n=1}^{\infty} \left[\frac{2}{n\pi^2} (\cos n\pi - 1) \right]$

$\Rightarrow |x| \approx \pi/2 - 4/\pi - 4/9\pi - 4/25\pi - \cdots$ — (i)

Now, $f(x) = |x|$ is continuous at $x = 0$ ~~at~~ and also $f'(x) = -1 < 0$ in the interval $-\pi \le x < 0$ ~~and $f'(x) = 1$~~ hence monotonic decreasing and $f'(x) = 1 > 0$ in the interval $0 < x \le \pi$ hence monotonic increasing.

$\therefore f(x) = |x|$ is bounded in the range $-\pi \leq x \leq \pi$, hence it's following Dirichlet's condition.

$\therefore$ putting $x=0$ in (i),

$$\pi/2 = 4/\pi \left(1 + 1/3^2 + 1/5^2 + \ldots \infty\right)$$

$$\Rightarrow 1/1^2 + 1/3^2 + 1/5^2 + 1/7^2 + \ldots \infty = \pi^2/8$$

[proved]

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3. (a) $\rightarrow$ ~~C~~ $\boxed{\text{C.W}}$ (2023) End-Sem

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2022 (End-Sem)

6. (b) $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_0^{\pi} \sin t\,dt$$

$$= 2/\pi$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} \sin t \cos nt\,dt$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\sin(t+nt) + \sin(t-nt)\right]dt$$

$$= \frac{1}{2\pi}\left[\frac{\cos(t+nt)}{1+n} + \frac{\cos(t-nt)}{1-n}\right]_{\pi}^{0}$$

$$= \frac{1}{2\pi}\left[\frac{1}{1+n} - \frac{\cos(n+1)\pi}{1+n} + \frac{1}{1-n} - \frac{\cos(n-1)\pi}{1-n}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{1+n} - \frac{1}{n-1} - \frac{(-1)^{n+1}}{1+n} + \frac{(-1)^{n-1}}{n-1}\right]$$

$$= \begin{cases} 0 & \text{when } n = \text{odd} \\ \frac{1}{\pi}\left(\frac{1}{1+n} - \frac{1}{n-1}\right) = \frac{-2}{\pi(n^2-1)} & \text{when } n = \text{even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin t \sin nt \, dt$$

$$= \frac{1}{2\pi}\int_0^{\pi} (\cos(n-1)t - \cos(n+1)t)\, dt$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)t}{n-1} - \frac{\sin(n+1)t}{n+1}\right]_0^{\pi}$$

$$= 0$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{(4n^2-1)}$$

$$= \frac{1}{\pi}\left[1 - 2\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}\right]$$

(Ans)

[2021] (end-sem)

3. $$a_0 = \frac{1}{\pi}\int_{-\pi}^{0}(\pi+x)\,dx = (\pi^2 - \pi^2/2)\,1/\pi = \pi/2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(\pi+x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{(\pi+x)\sin nx}{n} - \int \frac{\sin nx}{n}\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{(\pi+x)\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{1-(-1)^n}{n^2}\right] = \begin{cases} 0, & n = \text{even} \\ \frac{2}{\pi n^2}, & n = \text{odd}\end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0}(\pi+x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[-\frac{(\pi+x)}{n}\cos nx + \int\frac{\cos nx}{n}\right]$$

$$= \frac{1}{\pi}\left[-\frac{(\pi+x)\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\right] = -\frac{1}{n}$$

$$\therefore f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{2\cos(2n-1)x}{\pi(2n-1)^2} - \frac{1}{n}\sin nx\right)$$

$$= \frac{\pi}{4} + \frac{2}{1^2\pi}\cos x + \frac{2}{3^2\pi}\cos 3x + \frac{2}{5^2\pi}\cos 5x$$

$$+ \frac{2}{7^2\pi}\cos 7x + \dots\infty - \frac{\sin x}{1} - \frac{\sin 2x}{2}$$

$$- \frac{\sin 3x}{3} - \frac{\sin 4x}{4} - \frac{\sin 5x}{5} - \dots\infty \quad \text{(i)}$$

(ii) Now, $f(x)$ has an ordinary discontinuity at $x=0$,
hence, $f(x)$ will converge at

$$\frac{1}{2}\{f(0+0)+f(0-0)\} = \frac{\pi}{2}.$$

$$\therefore \frac{\pi}{2} = \frac{\pi}{4} + \frac{2}{1^2\pi} + \frac{2}{3^2\pi} + \frac{2}{5^2\pi} + \frac{2}{7^2\pi} + \dots\infty$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} = \frac{\pi^2}{8} \text{ [proved]}$$

(i) putting ~~$x=\pi$~~ $x=(-\pi/2)$ in (i),

$$\pi - \frac{\pi}{2} = \frac{\pi}{4} + \frac{\sin \pi/2}{1} + \frac{\sin 2\pi/2}{2} + \frac{\sin 3\pi/2}{3}$$

$$+ \frac{\sin 4\pi/2}{4} + \frac{\sin 5\pi/2}{5} + \dots\infty$$

$\Rightarrow \pi/4 = 1 - 1/3 + 1/5 - 1/7 + \dots \infty$ [proved]

<u>2022 (End-Sem)</u>

7. <u>cosine</u>

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$\cancel{= \frac{2}{\pi}\left[-\frac{1}{n^2} - \frac{1}{n^2}\right] =}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right] = \begin{cases} 0, & n = \text{even} \\ -\frac{4}{n^2\pi}, & n = \text{odd} \end{cases}$$

$\therefore$ half-range cosine series,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[-\frac{4}{(2n-1)^2\pi}\cos(2n-1)x\right]$$

$$= \frac{a_0}{2} - \frac{4}{\pi}\left\{\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right\}$$

<u>sine</u>

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx\,dx$$

$$= \frac{2}{\pi}\left[-\frac{x \cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi \cos \pi x}{n}\right]$$

$$= -\frac{2(-1)^n}{n}$$

∴ half range sine series

$$\sum_{n=1}^{\infty}\left[-\frac{2(-1)^n}{n}\sin nx\right]$$

$$= -2\left[-\frac{\sin x}{1}+\frac{\sin 2x}{2}-\frac{\sin 3x}{3}+\frac{\sin 4x}{4}-\cdots\infty\right]$$

[Ans]