$\boxed{\text{Vector}}$ $\boxed{\text{2024 (Midsem)}}$

(4) (b) $r^n \vec{r}$ is Solenoidal for $(n+3) = 0$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ $\quad \to r^2 = (x^2 + y^2 + z^2)$

$V = r^n r = r^n(x\hat{i} + y\hat{j} + z\hat{k})$

$V_1 = r^n x$ $\qquad \frac{\partial v_1}{\partial x} = r^n + n x^2 r^{(n-2)}$

$V_2 = r^n y$ $\qquad \frac{\partial v_2}{\partial y} = r^n + n y^2 r^{(n-2)}$

$V_3 = r^n z$ $\qquad \frac{\partial v_3}{\partial z} = r^n + n z^2 r^{(n-2)}$

$$\vec{\nabla} \times \vec{V} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{i} - \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right)\hat{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}$$

$$= 3r^n + n(x^2 + y^2 + z^2) \times r^{(n-2)}$$

$$\nabla \cdot V = 3r^n + n(r^2) \cdot r^{(n-2)}$$

$$= 3r^n + nr^n$$

$$= r^n(n+3)$$

For solenoidal $\nabla \cdot V = 0$

$n + 3 = 0$

Proved

5) (a)

$$d = \frac{(\alpha_2-\alpha_1).(\beta_1\times\beta_2)}{|\beta_1\times\beta_2|}$$

$\vec{r} = \vec{\alpha} + t\vec{\beta}$

$\vec{r} = \vec{\gamma} + s\vec{\delta}$

$\alpha \equiv (1,2,3)$

$\beta \equiv (2,3,4)$

$\gamma \equiv (k,3,4)$

$\delta \equiv (3,4,5)$

$$= \frac{(\gamma-\alpha).(\beta\times\delta)}{|\beta\times\delta|}$$

$$= \frac{((k-1)\hat{i} + \hat{j} + \hat{k}).(-\hat{i}+2\hat{j}-\hat{k})}{\sqrt{1^2+2^2+1^2}}$$

$\beta\times\delta$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$= \hat{i}(-1) + \hat{j}(2) + \ldots$

$= -\hat{i} + 2\hat{j} - \hat{k}$

$|\beta\times\delta| =$

$$= \left|\frac{((1-k)+2-1)}{\sqrt{6}}\right|$$

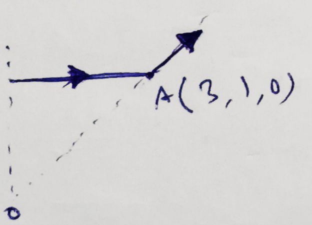
$$= \left|\frac{2-k}{\sqrt{6}}\right|$$

For Coplanarity —

$$\frac{2-k}{\sqrt{6}} = 0$$

$\Rightarrow \boxed{k=2}$

(b)



$\vec{P} = (\hat{i}+3\hat{j}+2\hat{k})$

$\vec{Q} = (3\hat{i}+4\hat{j}+3\hat{k})$

$\vec{PQ} = (2\hat{i}+\hat{j}+\hat{k})$

$\vec{OA} = (3\hat{i}+\hat{j})$

Moment $= \vec{OA}\times\vec{PQ}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= \hat{i}(1) + \hat{j}(-1) + \hat{k}(1)$$

$$= (\hat{i} - \hat{j} + \hat{k})$$

mag. of moment $= \sqrt{1+1+1} = \sqrt{3}$ units.

(6) (a) $u = x+y+z$, $v = x^2+y^2+z^2$, $w = xy+yz+zx$

$$\vec{\nabla} u = (\hat{i} + \hat{j} + \hat{k})$$

$$\vec{\nabla} v = (2x\hat{i} + 2y\hat{j} + 2z\hat{k})$$

$$\vec{\nabla} w = ((y+z)\hat{i} + (x+z)\hat{j} + (x+y)\hat{k})$$

$$(\vec{\nabla} u \times \vec{\nabla} v) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2x & 2y & 2z \end{vmatrix}$$

$$= \hat{i}(2z - 2y) + \hat{j}(2x - 2z) + \hat{k}(2y - 2x)$$

$$= 2(z-y)\hat{i} + 2(x-z)\hat{j} + 2(y-x)\hat{k}$$

$$(\vec{\nabla} u \times \vec{\nabla} v) \cdot \vec{\nabla} w$$

$$\cancel{= 2(yz - y^2) + 2(xz - z^2) + 2(2xy - x^2)}$$

$$= 2(z-y)\cdot(y+z) + 2(x-z)(x+z) + 2(y-x)(x+y)$$

$$= 2(z^2 - y^2) + 2(x^2 - z^2) + 2(y^2 - x^2)$$

$$= 2(\cancel{z^2} - \cancel{y^2} + \cancel{x^2} - \cancel{z^2} + \cancel{y^2} - \cancel{x^2}) = 0$$

$\therefore$ They are Co-Planners.

(b)

$$\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2xy+z^3) & x^2 & 3xz^2 \end{vmatrix}$$

$$= \hat{i}(0-0) + \hat{j}(3z^2 - 3z^2) + \hat{k}(2\ldots$$

$$= 0$$

As curl of a ~~vector~~ field is zero so the field is conservative.

$$\vec{F} = -\nabla\phi$$

$$= \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

$$\phi(x,y,z) = -\int F_x\,dx + c$$

$$= -\int (2xy + z^3)\,dx + c$$

$$= -(x^2y + xz^3) + c(y,z)$$

$$\phi(x,y,z) = -\int F_y\,dy + c(x,z)$$

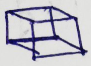
$$= -\int x^2\,dy + c(x,z)$$

$$= -x^2y + c(x,z)$$

$$\phi(x,y,z) = -\int F_z \, dz + c$$

$$= -\int 3xz^2 \, dz + c(x,y)$$

$$= -xz^3 + C(x,y)$$



$$W = \vec{F} \cdot d\vec{r}$$

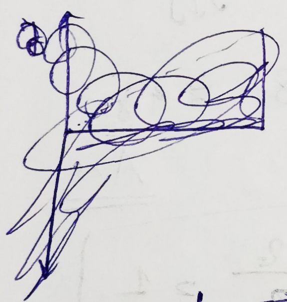
$$\cancel{= (2xy+z^3).2 + 3x^2 + 3xz^2 \times 3}$$

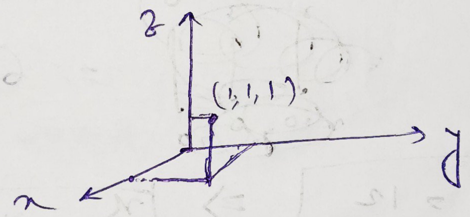
$$\cancel{= (4xy+2z^3+3x^2+9xz^2)}$$

P.T.O

7)(a)

$$\vec{F} = (5x^2+6y)\hat{i} - (3x+2y^2)\hat{j} + 2xz^2\hat{k}$$





$$\int_C \vec{F} \cdot d\vec{r} = \int_{x=0}^{1} \vec{F} \cdot d\vec{x} + \int_{y=0}^{1} \vec{F} \cdot d\vec{y} + \int_{z=1}^{1} \vec{F} \cdot d\vec{z}$$

$$= \int_0^1 (5x^2+6y)\,dx - \int_0^1 (3x+2y^2)\,dy + \int_{z=0}^{1} 2xz^2\,dz$$

$$= \left(\frac{5x^2}{2} + 6\cdot 1x\right)_0^1 - \left(\frac{3x^2}{2} + 2\cdot x\right)_0^1 + \left(2\cdot 1\cdot \frac{z^3}{3}\right)_0^1$$

$$= \left(\frac{5}{2}+6\right) - \left(\frac{3}{2}+2\right) + \frac{2}{3}$$

$$= 5 + \frac{2}{3} = \frac{17}{3} \text{ Ans.}$$

$= \iint (18 \ldots \hat{k})$.

(b) $\boxed{\vec{F} = 18z\,\hat{i} - 12\,\hat{j} + 3y\,\hat{k}}$ Wrong.

should be : $(18x\,\hat{i} - 12y\,\hat{j} + 3y$

$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(18\cancel{z}) - \frac{\partial}{\partial y}(12y) + \frac{\partial}{\partial z}(3y)$

$= 18 - 12 = 6$

<u>By applying div thm :-</u>

$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V (\nabla\cdot\vec{F})\, dv$

$= (\nabla\cdot\vec{F}) \iiint dv$

$= 6 \times 8 = 48$

<u>Ans</u>

$\boxed{2x + 3y + 6z = 12} \Rightarrow \boxed{\frac{x}{6} + \frac{y}{4} + \frac{z}{2} = 1}$

1^{st} octant :

$V = \frac{1}{6} \times x \times y \times z$

$= \frac{1}{6} \times (6) \times (4) \times (2)$

$= 8$

(*) $W = \int \vec{F}\cdot d\vec{r}$

$= \int_1^3 (2xy + z^3)\, dx + \int_{-2}^{1} x^2\, dy + \int_1^4 3xz^2\, dz$

$= (x^2 y + xz^3)_1^3 + x^2[y]_{-2}^{1} + 3x[z^3]_1^4$

$= 9(-1) + 3(1) - 1(-2) - 1(1) + 1(1+2)$
$+ 1(64-1)$

$\phi = -18+3 + 2 - 1 + 3 + 63$

$= 71 - 19$

$= 52$ units.

$\boxed{\text{Endsem}}$

(4) $\vec{\nabla}\phi(x,y,z) = 2xyz^3\,\hat{i} + x^2z^3\,\hat{j} + 3x^2yz^2\,\hat{k}$

(a)

$\phi_x = -\int 2xyz^3\,dx + C_1$

$= -x^2yz^3 + c_1(y,z)$

$\phi_y = -\int x^2z^3\,dy + c_2$

$= -x^2yz^3 + c_2(x,z)$

$\phi_z = -\int 3x^2yz\,dz + c_3(x,y)$

$= -x^2yz^3 + c_3(x,y)$

$$\boxed{\phi(x,y,z) = -x^2yz^3}$$

(b) $\vec{\nabla}\times(\vec{\nabla}\times\vec{A})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) & \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) & \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) & \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) & \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) \end{vmatrix}$$

$$= \left\{ \frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\right\}\hat{i}$$

$$+ \left\{ \frac{\partial}{\partial z}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\right.$$

$$+ \left\{ \frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\right.$$

$$= -\left(\frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2}\right)\hat{i} - \left(\frac{\partial^2 A_2}{\partial x^2} + \frac{\partial^2 A_2}{\partial z^2}\right)\hat{j}$$

$$- \left(\frac{\partial^2 A_3}{\partial x^2} + \frac{\partial^2 A_3}{\partial y^2}\right)\hat{k}$$

$$+ \left(\frac{\partial^2 A_2}{\partial x \partial y} + \frac{\partial^2 A_3}{\partial z \partial x}\right)\hat{i} + \left(\frac{\partial^2 A_3}{\partial z \partial y} + \frac{\partial^2 A_1}{\partial x \partial y}\right)\hat{j}$$

$$+ \left(\frac{\partial^2 A_1}{\partial x \partial z} + \frac{\partial^2 A_2}{\partial y \partial z}\right)\hat{k}$$

$$= -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}\right)$$

$$= -\nabla^2 A + \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\vec{\nabla}\cdot\vec{F}$$

$$= -\nabla^2 A + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = \text{R.H.S.}$$

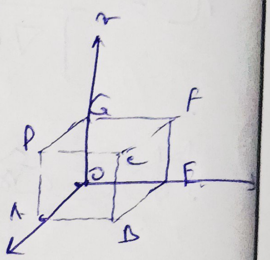
(c) $\vec{F} = 6xz\,\hat{i} - 4\hat{j} + y\,\hat{k}$

(5)
(a) $\vec{F} = 2xz\,\hat{i} + y^2\,\hat{j} + yz\,\hat{k}$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(2xz) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 2z + 2y + y$$

$$= (3y + 2z)$$



From divergence thm :-

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V (\nabla\cdot\vec{F})\,dV$$

$$= \int_{x=0}^{1}\int_{0}^{1}\int_{0}^{1} (3y + 2z)\,dx\,dy\,dz$$

$$= \int_{y=0}^{1}\int_{z=0}^{1} \left[3xy + 2xz\right]_0^1 dy\,dz$$

$$= \int_{y=0}^{1}\int_{0}^{1} \left[3y + 2z\right] dy\,dz$$

$$= \int_{0}^{1} \left[\frac{3}{2}y^2 + 2yz\right]_0^1 dz$$

$$= \int_{0}^{1} \left[\frac{3}{2} + 2z\right] dz$$

$$= \left[\frac{3}{2}z + z^2\right]_0^1$$

$$= \frac{3}{2} + 1 = \frac{5}{2}$$

L.H.S of

$$\int_S \vec{F}\cdot\hat{n}\,ds = \int_{S_1}\vec{F}\cdot\hat{n}\,ds + \int_{S_2}\vec{F}\cdot\hat{n}\,ds + \int_{S_3}\vec{F}\cdot\hat{n}\,ds + \int_{S_4}\vec{F}\cdot\hat{n}\,ds + \int_{S_5}\vec{F}\cdot\hat{n}\,ds + \int_{S_6}\vec{F}\cdot\hat{n}\,ds.$$

$$= \int_{ABCD} 2xz\,dy\,dz + \int_{DEFC} y^2\,dx\,dz$$

$$\int_{AOGD} y^2(-1)\,dx\,dz + \int_{BOFG} 2xz(-1)\,dy\,dz$$

$$+ \int_{ABEO} yz(-1)\,dx\,dy + \int_{DCFG} yz\,dx\,dy$$

$$= \frac{5}{2}$$