5. (a) Show that the shortest distance between the straight lines $\vec{r} = \vec{a} + t\vec{\alpha}$ and $\vec{r} = \vec{b} + s\vec{\beta}$, where $\vec{a} = 6\hat{i} + 2\hat{j} + 2\hat{k}$, $\vec{b} = -4\hat{i} - \hat{k}$, $\vec{\alpha} = \hat{i} - 2\hat{j} + 2\hat{k}$ and $\vec{\beta} = 3\hat{i} - 2\hat{j} - 2\hat{k}$ is 9 units. [3]

(b) Show that the directional derivative of $f = x^2y^2z^2$ at the point $(2, 1, -1)$ is maximum in the direction $4\hat{i} + 8\hat{j} - 8\hat{k}$, its magnitude being 12. [3]

6. (a) Find the moment about the point $(\hat{i} + 2\hat{j} - \hat{k})$ of a force represented by $(3\hat{i} + \hat{k})$ acting through the point $2\hat{i} - \hat{j} + 3\hat{k}$. [3]

(b) If α, β, γ are independent vectors in a real vector space, then show that the set $\{\beta + \gamma, \alpha + \gamma, \alpha + \beta\}$ is also independent. [3]

7. Let $\vec{r}$ be the position vector of a point and $r = |\vec{r}|$. Show that

(a) $\text{grad}\left(\frac{1}{r}\right) = -\frac{1}{r^3}\,\vec{r}$

(b) $\text{div}(r^n\,\vec{r}) = (n+3)\,r^n$

(c) $\text{curl}(r^n\,\vec{r}) = \vec{0}$

for any natural number n. [2+2+2]

Answer any FIVE questions

8. a) Find the unit normal to the surface $2x^2y + 3yz = 4$ at the point $(1, -1, -2)$.

b) Find, in terms of k, the shortest distance between the lines $r = \alpha + t\beta$ and $r = \gamma + s\delta$, where $\alpha = (1,2,3)$, $\beta = (2,3,4)$, $\gamma = (k,3,4)$ and $\delta = (3,4,5)$. For what value of k are the lines coplanar?

c) A particle acted on by a force of 15 units, is displaced from the point (1,1,1) to the point (2,1,3) and the line of action of the force is the vector $(i + 2j + 2k)$. Find the work done by the force.

[3+(3+1)+3]

9. a) A force of 15 units acts in the direction of the vector $(i - 2j + 2k)$ and passes through the point $(2i - 2j + 2k)$. Find the moment of the force about the point $(i + j + k)$.

b) If $F = (3x^2 + 6y)i - 14yzj + 20zx^2k$, then evaluate $\int_C F.dr$, from (0,0,0) to (1,1,1), along the curve $x = t,\ y = t^2,\ z = t^3$.

c) Use divergence theorem to evaluate $\iint_S F.n\,ds$, where $F = 2xzi + y^2j + yzk$ and S is the surface of the cube bounded by $x = 0,\ y = 0,\ z = 0,\ x = 1,\ y = 1,\ z = 1$.

[3+3+4]

10. a) Find the value of a for which the vector $F = (x + 3y)i + (y + az)j + (x + az)k$ is solenoidal.

b) State Green's theorem in a plane. Verify Green's theorem in a plane for

$\oint_C \{(3x^2 - 6y^2)dx + (y - 3xy)dy\}$, where C is the boundary of the region

$x = 0,\ y = 0,\ x + y = 1$.

[2+(2+6)]

$\boxed{2022 \text{ (Midsem)}}$ (June)

5)(a) $\vec{r} = \vec{a} + t\vec{\alpha}$

$\vec{r} = \vec{b} + s\vec{\beta}$

$$d = \left| \frac{(\vec{b}-\vec{a}).(\vec{\alpha}\times\vec{\beta})}{|\vec{\alpha}\times\vec{\beta}|} \right| \qquad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix}$$

$$= \left| \frac{(-10\hat{i}-2\hat{j}-3\hat{k}).(8\hat{i}+8\hat{j}+4\hat{k})}{\sqrt{8^2+8^2+4^2}} \right| \qquad = \hat{i}(4+4)+\hat{j}(6+2) + \hat{k}(-2+6)$$

$$= \frac{|-80-16-12|}{\sqrt{144}} \qquad = 8\hat{i}+8\hat{j}+4\hat{k}$$

$$= \frac{108}{12} = 9 \text{ units}$$

$$\begin{array}{r} 128 \\ +16 \\ \hline 144 \end{array} \qquad \begin{array}{r} 80 \\ 16 \\ +12 \\ \hline 108 \end{array}$$

Proved

(b) $f = x^2y^2z^2$; pt $\equiv (2, 1, -1)$

$$\vec{\nabla}f = \frac{\partial}{\partial x}(x^2y^2z^2)\hat{i} + \frac{\partial}{\partial y}(x^2y^2z^2)\hat{j} + \frac{\partial}{\partial z}(x^2y^2z^2)\hat{k}$$

$$= 2xy^2z^2\hat{i} + 2x^2yz^2\hat{j} + 2x^2y^2z\,\hat{k}$$

$$(\vec{\nabla}f)_{(2,1,-1)} = (4\hat{i} + 8\hat{j} - 8\hat{k})$$

$$\left|(\vec{\nabla}f)_{(2,1,-1)}\right| = \sqrt{4^2+8^2+8^2} = 12 \text{ units}$$

6)(a) $F = (3\hat{i}+\hat{k}) \quad \vec{r} = (\vec{r_2}-\vec{r_1}) = (\hat{i}-3\hat{j}+4\hat{k})$

Moment $= \vec{r}\times\vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 4 \\ 3 & 0 & 1 \end{vmatrix} = \hat{i}(-3)+\hat{j}(12-1) + \hat{k}(+9)$$

$$= (-3\hat{i}+11\hat{j}+9\hat{k})$$

7)

(a) $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$

$$\frac{1}{|r|} = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$\text{grad}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}\left(\frac{1}{r}\right)\hat{i} + \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\hat{j} + \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\hat{k}$$

$$= -\frac{x}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\hat{i} - \frac{y\,\hat{j}}{\left(\sqrt{x^2+y^2+z^2}\right)^3} - \frac{z\,\hat{k}}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$$

$$= \frac{-(x\hat{i}+y\hat{j}+z\hat{k})}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$$

$$= \frac{-\vec{r}}{r^3}$$

Proved

Y28s 5G
Apr 17, 2025, 09:57

(b) done in 2024.

(c) Curl $(r^n \vec{r})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^{n+1} & y^{n+1} & z^{n+1} \end{vmatrix}$$

$= \hat{i}(0) + \hat{j}(0) + \hat{k}(0)$

$= \vec{0}$

<u>Endsem</u>

(8) (a) $2x^2y + 3yz = 4$

$\Rightarrow (2x^2y + 3yz - 4) = 0$

$\nabla S = \frac{\partial}{\partial x}(2x^2y + 3yz - 4)\hat{i} + \frac{\partial}{\partial y}(2x^2y + 3yz - 4)\hat{j}$
$+ \frac{\partial}{\partial z}(2x^2y + 3yz - 4)\hat{k}$

$= 4xy\,\hat{i} + (2x^2 + 3z)\hat{j} + 3y\,\hat{k}$

$\vec{n} = 4 \times 1(-1)\hat{i} + \{2(1)^2 + 3(-2)\}\hat{j} + 3 \times (-1)\hat{k}$

$= (-4\hat{i} - 4\hat{j} - 3\hat{k})$

$\hat{n} = \frac{(-4\hat{i} - 4\hat{j} - 3\hat{k})}{\sqrt{4^2 + 4^2 + 3^2}} = \frac{(-4\hat{i} - 4\hat{j} - 3\hat{k})}{\sqrt{41}}$

(b)

$$d = \left| \frac{(\gamma - \alpha) \cdot (\beta \times \delta)}{|(\beta \times \delta)|} \right|$$

$$= \frac{\left((k-1)\hat{i} + \hat{j} + \hat{k}\right) \cdot \left(-\hat{i} + 2\hat{j} - \hat{k}\right)}{\sqrt{1^2 + 2^2 + 1^2}}$$

$$= \left| \frac{(1-k) + 2 - 1}{\sqrt{6}} \right|$$

$$= \left| \frac{2-k}{\sqrt{6}} \right|$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$$= \hat{i}(15-16) + \hat{j}(12-1$$

$$+ \hat{k}(-1)$$

$$= (\hat{i} + 2\hat{j} - \hat{k})$$

$d = 0$; for Coplanarity.

$$\boxed{k = 2}$$

(c) $\vec{r} = (\hat{i} + 2\hat{k})$;

$$\vec{F} = 10\left(\frac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{9}}\right)$$

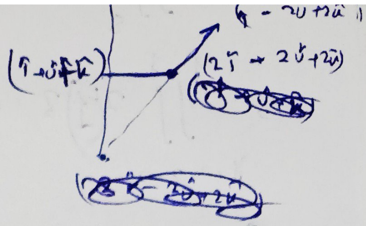
$$= \frac{10}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$$

$$W = \vec{F} \cdot \vec{r}$$

$$= \frac{10}{3}(1 + 4) = \frac{50}{3} \text{ units.}$$

$$\cancel{\vec{r} = (\hat{i} - 2\hat{j} + 2\hat{k}) - (2\hat{i} - 2\hat{j} + 2\hat{k})}$$

$$\cancel{= \hat{i}}$$



$$\vec{r} = (2\hat{i} - 2\hat{j} + 2\hat{k}) - (\hat{i} + \hat{j} + \hat{k})$$

$$= (\hat{i} - 3\hat{j} + \hat{k})$$

$$\hat{F} = \frac{(\hat{i} - 2\hat{j} + 2\hat{k})}{3}$$

$$|\vec{r} \times \vec{F}| = (\hat{i} - 3\hat{j} + \hat{k}) \times \frac{(\hat{i} - 2\hat{j} + 2\hat{k})}{\cancel{3}} \times \cancel{15}^{5}$$

$$= 5 \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 1 & -2 & 2 \end{vmatrix}$$

$$= 5 \{ \hat{i}(-6+2) + \hat{j}(1-2) + \hat{k}(-2+3) \}$$

$$= 5 \{ -4\hat{i} - \hat{j} + \hat{k} \}$$

$$= 5\sqrt{4^2 + 1^2 + 1^2} = 5\sqrt{18} = 15\sqrt{2} \text{ units}$$

$x = t$
$dx = dt$
$3t^2 dt$ $\quad 20\int t t^2$

(b) $\int_C F \cdot dr$

$$= \int (3x^2 + 6y) \cdot dx - 14 \int yz \cdot dy + 20 \int z x^2 dz$$

$$= \int_0^1 3t^2 dt - 28 \int t^6 dt + 60 \int t^7 dt$$

$$= 3[t^3]_0^1 - 4[t^7]_0^1 + \frac{\cancel{60}^{15}}{\cancel{8}_2} [t^8]_0^1$$

$$= 3 - 4 + \frac{15}{2} = 10.5 - 4 = 6.5 \text{ Ans}$$

(c) $\iint_S F\cdot n\, ds$ where $F = (2xz\,\hat{i} + y^2\hat{j} + yz\hat{k})$

$$= \int_{y=0}^{1}\int_{z=0}^{1} 2(1)z\, dy\, dz + \int_{x=0}^{1}\int_{z=0}^{1} (1)x\, dx\, dz + \int_{x=0}^{1}\int_{y=0}^{1} y(1)\, dx\, dy$$

$$= \int_{y=0}^{1} \left[(1)z^2\right]_0^1 dy + \int_{x=0}^{1} \left[(1)z\right]_0^1 dx + \frac{1}{2}\int_{x=0}^{1} \left[y^2(1)\right]_0^1 dx$$

$$= \int_{y=0}^{1} (1)\, dy + \int_{x=0}^{1} (1)\, dx + \frac{1}{2}\int_{x=0}^{1} (1)\, dx$$

$$= 1 + 1 + \frac{1}{2} = 2.5 \text{ unit}$$

10)(a) $\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y + az) + \frac{\partial}{\partial z}(x + cz)$

$$= (1 + 1 + a)$$

$(2+a) = 0$

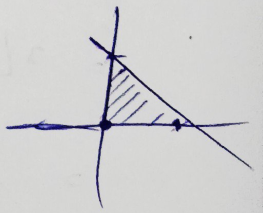
$\Rightarrow \boxed{a = -2}$

Ans

(b) $\oint_C (P\,dx + Q\,dy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$P = (3x^2 - 6y^2)$

$Q = (y - 3xy)$



$$\int_{x=0}^{1} (3x^2 - 6y^2)\, dx + \int_{y=0}^{1} (y - 3xy)\, dy$$

$$= \left[x^3\right]_0^1 - 6 + \left[\frac{y^2}{2}\right]_0^1 - 3\left[\frac{x^2}{2}\right]_0^1$$

$$= 1 - 6 + \frac{1}{2} - \frac{3}{2} = \frac{3}{2} - \frac{3}{2} - 6 = -6$$

$$\int_{x=0}^{1}\int_{y=1-x}^{1} \left(-3y + 12y\right) dx\, dy$$

$$= (9)\int_{x=0}^{1}\int_{y=1-x}^{1} y \, dy \, dx$$

$$= (9)\int_{x=0}^{1} \left[\frac{y^2}{2}\right]_{(1-x)}^{1} dx$$

$$= \left(\frac{9}{2}\right)\int_{x=0}^{1} \left(1^2 - (1-x)^2\right) dx$$

$$= \left(-\frac{9}{2}\right)\int_{x=0}^{1} \left(\cancel{1} - \cancel{1} + 2x - x^2\right) dx$$

$$= \left(-\frac{9}{2}\right)\int_{x=0}^{1} (x^2 + 2x)\, dx$$

$$= -\frac{9}{2}\left[\frac{x^3}{3} + \frac{x^2}{1}\right]_0^1$$

$$= -\frac{9}{2}\left(\frac{1}{3} + 1\right)$$

$$= -\frac{\cancel{9}^{\,3}}{\cancel{2}} \times \frac{\cancel{4}^{\,2}}{\cancel{3}} = -6$$