$x = 0,\ y = 0,\ x + y = 1.$

[2+(2+6)]

11. Prove that the function $f(z) = u + iv$, where $f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

satisfies the Cauchy-Riemann equations at the origin but $f'(0)$ does not exist.

[5+5]

12. a) Prove that $U = y^3 - 3x^2y$ is a harmonic function. Find the analytic function $f(z)$ of which the real part is U.

b) Evaluate $\oint_C \frac{(z^2+1)dz}{z(2z+1)}$, where C is $|z| = 1$, by using Cauchy's integral formula.

[(2+3+2)+3]

13. a) Expand $f(z) = \frac{1}{(z-1)(z-2)}$ in the regions

i) $|z| < 1$, ii) $1 < |z| < 2$

b) Find the nature of singularities of the following functions:

i) $\frac{z - \sin z}{z^2}$, ii) $\frac{e^{1/z}}{z^2}$

$+\frac{z'}{6s} - \frac{z}{4L}$ $+\frac{1}{6}$

[(3+3)+(2+2)]

14. a) State Residue theorem.

b) Find the poles of $f(z) = \frac{z^2-2z}{(z+1)^2(z^2-4)}$ and residues at its poles. Hence evaluate $\oint_C f(z)dz$, where C is the circle $|z| = \frac{7}{2}$.

(i) $n\pi$

[2+(2+4+2)]

□ August, 2022

12(a) P.T $U = y^3 - 3x^2 y$ is a harmonic funcⁿ

Find $f(z)$.

Ans.)

$$\frac{\partial^2 U}{\partial x^2} = \cancel{6y} - 6y$$

$$\frac{\partial^2 U}{\partial y^2} = 6y$$

$$\therefore \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

$\therefore$ $\boxed{\text{U is a harmonic function}}$

$U(x,y) = y^3 - 3x^2 y$

$\cancel{U(z,0)=}$ $U_x = -6xy = \phi_1(x,y)$

$U_y = +3y^2 - 3x^2 = \phi_2(x,y)$

$\phi_1(z,0) = 0$

$\phi_2(z,0) = -3z^2$

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$$

$$= -i\int(-3z^2)\,dz$$

$$= iz^3 + c \quad \text{(Ans.)}$$

---

b) $\oint_c \frac{z^2+1}{z(2z+1)}\,dz \qquad c = |z| = 1$

sing :- $z = 0,\ z = -\frac{1}{2}$

$f(z) = z^2 + 1$

$$\frac{1}{z(2z+1)} = \frac{(2z+1) - 2z}{z(2z+1)} = \frac{1}{z} - \frac{2}{2z+1}$$

$$\therefore \text{Ans} = \oint_c \frac{f(z)}{z}\,dz + \oint_c \frac{2f(z)}{2z+1}\,dz$$

$$= 2\pi i \times f(0) + 2\pi i \times 2 \times f\left(-\frac{1}{2}\right)$$

$$= 2\pi i + 4\pi i\left(\frac{1}{4} + 1\right)$$

$= 2\pi i + 4\pi i \times \frac{5}{4}$

$= 7\pi i$ (Ans)

---

11 (a) $f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$

$U(x,y) = \frac{x^3 - y^3}{x^2+y^2} \quad (x,y) \neq (0,0)$

$= 0 \quad x,y = (0,0$

$V(x,y) = \frac{x^3+y^3}{x^2+y^2} \quad (x,y) \neq (0,0)$

$= 0 \quad (x,y) = (0,0)$

□ C-R Equations :-

$U_x(0,0) = \lim_{h \to 0} \frac{u(0+h,0) - u(0,0)}{h} = 1$

$U_y(0,0) = \lim_{k \to 0} \frac{u(0,k) - u(0,0)}{k} = \frac{-k}{k} = -1$

$V_x(0,0) = \lim_{h \to 0} \frac{v(0+h,0) - v(0,0)}{h} = \frac{h}{h} = 1$

$V_y(0,0) = \lim_{k \to 0} \frac{v(0,k) - v(0,0)}{k} = \frac{k}{k} = 1$

at the origin

$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$

$\boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$

$$f'(0) = \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{(x^3 - y^3) + i(x^3 + y^3)}{(x^2 + y^2)(x + iy)}$$

$y = mx$, so $z \to 0$, $x \to 0$

$$f'(0) = \lim_{x \to 0} \frac{x^3 - m^3 x^3 + i(x^3 + m^3 x^3)}{(x^2 + m^2 x^2)(x + imx)}$$

$$= \lim_{x \to 0} \frac{1 - m^3 + i(1 + m^3)}{(1 + m^2)(1 + im)}$$

diff values depending on m

$\therefore f'(z)$ is not unique at $(0,0)$.

---

13 (a) $f(z) = \dfrac{1}{(z-1)(z-2)}$

(i) $|z| < 1$

$$f(z) = \frac{1}{(z-1)(z-2)}$$

$$= \frac{(z-1) - (z-2)}{(z-1)(z-2)}$$

$$= \frac{1}{z-2} - \frac{1}{z-1}$$

$$= \frac{1}{-2\left(1 - \frac{z}{2}\right)} + \frac{1}{1 - z}$$

$$= -\frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1} + 1(1 - z)^{-1}$$

$$= -\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \dots\right) + \left(1 + z + z^2 + \dots\right)$$

$$= \frac{1}{2} + \frac{3z}{4} + \frac{7z^2}{8} + \dots \quad \text{(Ans.)}$$

ii) $1 < |z| < 2$

$\Rightarrow |z| > 1 \quad \Big| \quad |z| < 2$

$\frac{1}{|z|} < 1 \quad \Big| \Rightarrow \frac{|z|}{2} < 1$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

$$= \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}$$

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\cdots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)$$

---

b) (i) $\frac{z-\sin z}{z^2}$

$z=0$ is a singularity of

$$f(z) = \frac{z-\sin z}{z^2}$$

$$f(z) = \frac{1}{z^2}\left\{\cancel{z} - \left(\cancel{z} - \frac{z^3}{3!} + \frac{z^5}{5!}\cdots\right)\right\}$$

$$= \frac{1}{z^2}\left\{\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \cdots\right\}$$

$$= \left\{\frac{z}{3!} - \frac{z^3}{5!} + \frac{z^5}{7!}\cdots\right\}$$

$\therefore$ No terms with negative power of $(z-0)$

$\therefore$ It is a removable singularity.

(ii) $\frac{e^{1/z}}{z^2}$ singularities $z=0$

$$f(z) = \frac{e^{1/z}}{z^2} = \frac{1 + \frac{1}{z} + \frac{1}{z^2 2!} + \frac{1}{z^3 3!} + \cdots}{z^2}$$

$$= \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4 2!} + \frac{1}{z^5 3!} + \cdots$$

$\therefore$ Infinite no. of Terms with negative power.

$\therefore$ essential singularity.

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14. a) **Residue Theorem**

Let $z = z_0$ be an isolated singularity of the fu[nction] $f(z)$, since z_0 is an isolated singularity, there exists a deleted neighbourhood (of z_0)

$$0 < |z - z_0| < r$$

in which $f(z)$ is analytic.

$$\therefore f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

$$a_n = \frac{1}{2\pi i} \oint_c \frac{f(\xi)\, d\xi}{(\xi - z_0)^{n+1}}$$

$$b_n = \frac{1}{2\pi i} \oint_c \frac{f(\xi)\, d\xi}{(\xi - z_0)^{-n+1}}$$

(b)