5. (a) Verify Gauss's divergence theorem for the vector function $\vec{F} = 2xz\hat{i} + y^2\hat{j} + yz\hat{k}$, taken over the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$. [6]

(b) For the function,

$$f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

show that $f'(0)$ does not exist though Cauchy-Riemann equations are satisfied at the origin. [2+2=4]

6. (a) Show that the function $u(x, y) = \cos x \cosh y$ is harmonic and find its harmonic conjugate. [1+2=3]

(b) Evaluate $\int_0^{1+i} (x - y - ix^2)dz$, along the real axis from $z = 0$ to $z = 1$ and then along the line parallel to the imaginary axis from $z = 1$ to $z = 1 + i$. [3]

(c) Evaluate $\frac{1}{2\pi i} \int_C \frac{e^{zt}}{z^2 + 1} dz$, where $t > 0$ and C is the circle: $|z| = \pi$. [4]

7. (a) Find the Laurent's expansion of the function $f(z) = \frac{z^2 - 1}{(z+2)(z+3)}$, when $2 < |z| < 3$. [3]

(b) Find the singularities and determine their natures of the following functions:

i) $\frac{z-2}{z^2} \sin \frac{1}{z-1}$ ii) $\sec \frac{1}{z}$ [2+2=4]

(c) Using Cauchy's Residue theorem, evaluate $\oint_C f(z)\, dz$, where $f(z) = \frac{z}{(z-1)(z-2)^2}$ and C is $|z - 2| = \frac{1}{2}$. [3]

2024 (End sem)

5. (b) repeat of 2022 qn.

6. (a) $u(x,y) = \cos x \cosh y$

$$\frac{\partial^2 u}{\partial x^2} = \cancel{-\sin x \cosh y} \quad -\cos x \cosh y$$

$$\frac{\partial^2 u}{\partial y^2} = \cos x \cosh y.$$

$$\therefore \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = 0$$

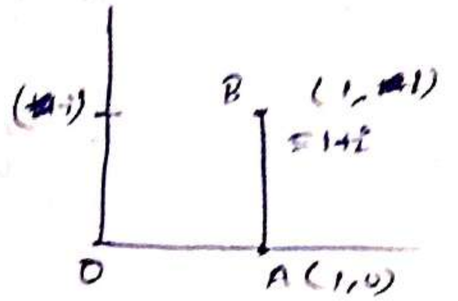
$\therefore$ the function $u(x,y)$ is obeying Laplace's eqn, hence it's a harmonic function.

Now, $$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$= (\cos x \sinh y) dx + (-\sin x \cosh y) dy$$

$$\Rightarrow v = \sin x \sinh y - \sin x \sinh y + c.$$

6. (b) ~~Also~~ the path segment of the integration is first OA then AB.



~~where~~, for segment OA,

$y = 0$, ∴ $z = x \Rightarrow dz = dx$, x varies from 0 to 1

for segment AB,

$x = 1$ ∴ $z = 1 + iy \Rightarrow dz = i\,dy$, y varies from 0 to 1

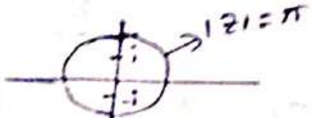
$$\therefore \int_0^{1+i} (x - y - ix^2)\,dz = \int_{OA} (x - ix^2)\,dx + \int_{AB} (1 - y - i)\,i\,dy$$

$$= \left[\frac{x^2}{2} - \frac{ix^3}{3}\right]_0^1 + i\left[y - \frac{y^2}{2} - iy\right]_0^1$$

$$= \frac{1}{2} - \frac{i}{3} + i - \frac{i}{2} + 1$$

$$= \frac{3}{2} + \frac{i}{6}$$ [Ans]

6. (c) $$\frac{1}{2\pi i} \int \frac{e^{zt}}{(z+i)(z-i)}\,dz$$



$$= \frac{1}{2\pi i} \int \frac{e^{zt}}{2i} \left(\frac{1}{z-i} - \frac{1}{z+i}\right) dz$$

$$= \frac{1}{2i} \left[\frac{1}{2\pi i} \int \frac{e^{zt}}{z-i}\,dz - \frac{1}{2\pi i} \int \frac{e^{zt}}{z+i}\,dz\right]$$

According to Cauchy integral theorem as $z = i$ and $z = -i$ lying inside the circle......

$= \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

$= \frac{1}{2i}(\cos\theta + i\sin\theta - \cos\theta + i\sin\theta)$

$= \sin\theta$ [Ans]

7. (a) $$f(z) = \frac{z^2-1}{(z+2)(z+3)}$$

$$\frac{z^2-1}{(z+2)(z+3)} = \frac{(z^2+5z+6)-5z-7}{(z+2)(z+3)} = 1 - \frac{5z+7}{(z+2)(z+3)}$$

Now, $$\frac{5z+7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$$

$\therefore A = -3 \;,\; B = 8$

$$\therefore f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

when $2 < |z| < 3$

$$f(z) = 1 + \frac{3}{z}\left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1 + \frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

$$= 1 + 3\sum_{n=0}^{\infty}(-1)^n 2^n z^{-(n+1)} - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{3^n}$$

(Ans)

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7. (b) (i) $\frac{z-2}{z^2}\sin\frac{1}{(z-1)} = f(z)$

here the singularities are $z = 0$, $z = 1$

~~case~~ for $z = 0$,

It's a pole of order 2.

for $z = 1$,

$$\frac{z-2}{z^2}\left(\frac{1}{z-1} - \frac{1}{3!(z-1)^3} + \frac{1}{5!(z-1)^5} - \frac{1}{7!(z-1)^7} + \dots\infty\right)$$

$$= \frac{1}{z}\left(\frac{1}{z-1} - \frac{1}{3!(z-1)^3} + \frac{1}{5!(z-1)^5}\right) - \frac{1}{z^2}\left(\frac{2}{z-1} - \frac{2!}{3(z-1)^3} + \frac{2}{60(z-1)^5} \dots\infty\right)$$

∴ There are infinite terms of negative powers of $(z-1)$,

hence $z = 1$ is an isolated essential singularity of $f(z)$

(ii) $\sec\frac{1}{z} = \frac{1}{\cos\frac{1}{z}}$

Here $z=0$, is a singularity of $f(z)$.

Poles of $f(z)$ are given by $\cos 1/z = 0$ or

$$1/z = (2n+1)\pi/2 \Rightarrow z = \frac{2}{(2n+1)\pi}, \quad n = \pm 1, \pm 2, \pm 3 \ldots$$

These are simple poles.

Obviously $z=0$ is a limit point of these poles, so $z=0$ is a non-isolated essential singularity of $f(z) = \sec 1/z$.

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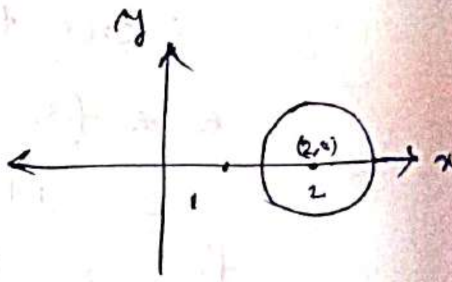
7. (c) $f(z) = \dfrac{z}{(z-1)(z-2)^2}$



Now, The poles are $z = 1, 2, 2$

Only the pole ~~at~~ $z=2$ of order 2 is lying inside the circle $|z-2| = 1/2$.

Now, $\operatorname{Res}(f;2) = \dfrac{1}{1!} \lim\limits_{z\to 2} \dfrac{d}{dz}\{(z-2)^2 f(z)\}$

$$= \lim_{z\to 2} \frac{d}{dz}\left\{\cancel{(z^2-4z+4)}\,\frac{z}{z-1}\right\}$$

$$= \lim_{z\to 2} \frac{(z-1)-z}{(z-1)^2} = \lim_{z\to 2} \frac{-1}{(z-1)^2}$$

$$= -1$$

$\therefore$ clearly $f(z)$ ~~x~~ is analytic on $|z-2| = 1/2$ and at all points inside $|z-2| = 1/2$ except at the pole $z=2$.

By Cauchy Residue theorem,

$$\int_{|z-2|=1/2} f(z)\,dz = 2\pi i\, \mathrm{Res}(f;2) = 2\pi i(-1)$$

$$= \boxed{-2\pi i} \text{ [Ans]}$$