Matrix 2021 pyq.

(2) (a) Diagonalize $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ and hence find A^{50}

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)^2 - (-1)^2 = 0$

$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$

$\Rightarrow (\lambda-1)(\lambda-3) = 0$

$\lambda_1 = 1, \; \lambda_2 = 3.$

(8) $\lambda = 1$:-

$(A - I)X = 0$

$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\left.\begin{array}{l} \Rightarrow x_1 - x_2 = 0 \\ -x_1 + x_2 = 0 \end{array}\right\} x_1 = x_2$

$X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 3$:-

$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$-x_1 - x_2 = 0$

$x_1 = -x_2$

$X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$P = [X_1 \;\; X_3]$

$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$P^{-1} = \frac{1}{|P|} adj(P)$ $\quad |P| = (-1) - 1 = -2$

$adj(P) = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$

We know, diagonal matrix $D = P^{-1}AP$

$= \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ (Ans) ✓

$A^{50} = ?$

$D = P^{-1} A P$

$A = P D P^{-1}$

$A^{50} = (P D P^{-1})^{50}$

$= P D \underbrace{P^{-1} \cdot P}_{I} D P^{-1} \cdots \cdots P D P^{-1}$

$= P D^{50} P^{-1}$

$D^{50} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}^{50} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \cdots 50$

$= \begin{bmatrix} 1 & 0 \\ 0 & 3^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \cdots$

$= \begin{bmatrix} 1 & 0 \\ 0 & 3^{50} \end{bmatrix}$

$P^{-1} = \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$

$A^{50} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^{50} \end{bmatrix} P^{-1}$

$= \begin{bmatrix} 1 & 3^{50} \\ 1 & -3^{50} \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$

$= \frac{1}{2} \begin{bmatrix} 1 & 3^{50} \\ 1 & -3^{50} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$\underbrace{= \frac{1}{2} \begin{bmatrix} 1 + 3^{50} & 1 - 3^{50} \\ 1 - 3^{50} & 1 + 3^{50} \end{bmatrix}}_{\text{(Ans)}}$$

[7+3]

4. a) Give the definition of rank of a matrix.

b) Find the rank of the rectangular matrix by changing it to echelon matrix form

$$\begin{bmatrix} 0 & 0 & 5 & -3 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix}.$$

c) Given the system of equations:

$$x_1 + 4x_2 + 2x_3 = 1$$
$$2x_1 + 7x_2 + 5x_3 = 2k$$
$$4x_1 + mx_2 + 10x_3 = 2k + 1$$

Find for what values of k and m the system has (i) a unique solution (ii) no solution (iii) many solutions.

[1+3+(2+2+2)]

5. a) Find the eigen values of the matrix A, where

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}.$$

9/11

Hence find an eigen vector corresponding to least eigen value.

b) Find the rank of the following matrix by changing it to normal form.

$$\begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 1 & 6 & 3 & 2 \end{bmatrix}.$$

[(2+4)+4]

$3-\lambda$

0

Maths (2022)

(4) (a) Rank of a matrix is the largest order of any non-vanishing minor of the matrix.
It is the maximum number of linearly independent rows or columns in the matrix.

(b) $\begin{bmatrix} 0 & 0 & 5 & -3 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix} \rightarrow \text{Rank} = ?$

Converting it into echelon matrix form :-

$R_3 \leftrightarrow R_1 \Rightarrow \begin{bmatrix} -1 & -2 & 6 & -7 \\ 2 & 4 & 3 & 5 \\ 0 & 0 & 5 & -3 \end{bmatrix}$

$\downarrow R_2 \to 2R_1 + R_2$

$\begin{array}{cccc} -2 & -4 & 12 & -14 \\ 2 & 4 & 3 & 5 \\ \hline 0 & 0 & 15 & -9 \end{array}$

$\Rightarrow \begin{bmatrix} -1 & -2 & 6 & -7 \\ 0 & 0 & 15 & -9 \\ 0 & 0 & 5 & -3 \end{bmatrix}$

$\downarrow R_3 \to 3R_3 - R_2$

$\Rightarrow \begin{bmatrix} -1 & -2 & 6 & -7 \\ 0 & 0 & 15 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

echelon form

Hence Rank $\rho(A) = 2$ as there are two non-zeros rows in the echelon form

(c) Equations ✓ $AX = B$

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 7 & 5 \\ 4 & m & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2k \\ 2k+1 \end{bmatrix}$$

Non-homogeneous equations.

Calculating rank of matrix $A = \begin{Bmatrix} 1 & 4 & 2 \\ 2 & 7 & 5 \\ 4 & m & 10 \end{Bmatrix}$

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 7 & 5 \\ 4 & m & 10 \end{bmatrix} \xrightarrow[R_3 \to 4R_1 - R_3]{R_2 \to 2R_1 - R_2} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & -1 \\ 0 & 16-m & -2 \end{bmatrix}$$

$$\Big\downarrow R_3 \to 2R_2 - R_3$$

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & -1 \\ 0 & m-14 & 0 \end{bmatrix}$$

2 8 4
2 7 5
0 1 -1
4 16 8
4 m 10

2 -2
16-m +2
2-16+m 0
m-14

Now Augment matrix A:B

$$\Rightarrow \begin{bmatrix} 1 & 4 & 2 & 1 \\ 2 & 7 & 5 & 2K \\ 4 & m & 10 & 2K+1 \end{bmatrix}$$

$$\Big\downarrow \begin{matrix} R_2 \to 2R_1 - R_2 \\ R_3 \to 4R_1 - R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 1 & -1 & 2-2K \\ 0 & 16-m & -2 & 3-2K \end{bmatrix}$$

2
-2K
2-2K

4-2K-1
3-2K

$$\Big\downarrow R_3 \to 2R_2 - R_3$$

$$\begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 1 & -1 & 2-2K \\ 0 & m-14 & 0 & 1-2K \end{bmatrix}$$

4-4K
-3+2K
1-2K

$\rho(A) = \rho(A:B) =$ No. of unknowns $\rightarrow$ Unique solutions

i.e for $m \neq 14$ $\rho(A) = \rho(A:B) = 3 \rightarrow$ Unique solutions any value of K.

for $m = 14$ $\rho(A) < 3$

But if $\rho(A:B) = \rho(A)$ i.e $1-2K = 0$ or $K = 1/2$
$= 2$

$m = 14, K = 1/2 \leftarrow$ then ∞ solutions with $(3-2) = 1$ arbitrary constant

if $K \neq 1/2$ $\rho(A) \neq \rho(A:B)$
$m = 14$ Hence No solution.

for ∞ soln:-
$x_1 + 4x_2 + 2x_3 = 1$ $\quad$ Let $x_3 = t$ $\quad t \in \mathbb{R}$
$x_2 - x_3 = 1$ $\quad \boxed{x_2 = 1 + x_3}$ $\quad$ then $x_1 = 3 - 6t$ $\quad x_2 = t + 1$

5 (a). Eigen value of $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$

$|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 1 & -1 \\ 2 & 2-\lambda & -1 \\ 2 & 2 & -\lambda \end{vmatrix} = 0$$

$(3-\lambda)((2-\lambda)(-\lambda)+2) - 1(-2\lambda+2) - 1(4 - 2(2-\lambda))$

$(3-\lambda)(-2\lambda+\lambda^2+2) + 2\lambda - 2 - 4 + 2(2-\lambda) = 0$

$-6\lambda + 3\lambda^2 + \cancel{6} + 2\lambda^2 - \lambda^3 - \cancel{2\lambda} + \cancel{2\lambda} - \cancel{6} + 4 - 2\lambda = 0$

$5\lambda^2 - \lambda^3 - 8\lambda + 4 = 0$

$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

$\lambda = 1$ ✓ $\quad \rightarrow (\lambda-1)\underbrace{(\lambda^2 - 4\lambda + 4)}_{} = 0$

$\lambda = 2$ ✓

$1 - 5 + 8 - 4$
$-4 + 4 = 0$

$\cancel{x} \times 8 - 20 + 16 - 4$
$-12 + 12$

$\underline{\lambda = 2, 2}$

$\therefore$ Eigen values

$\underline{\lambda = 1, 2, 2}$

Eigen vector $\underline{\lambda = 1}$ $\Rightarrow$ $\begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$2x_1 + x_2 - x_3 = 0$

$2x_1 + 2x_2 - x_3 = 0$

$\therefore$ $x_2 = 0$ $\qquad$ $\underline{x_3 = 2x_1}$

Let $\underline{x_1 = t}$ $\qquad \Longrightarrow \quad t\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ is eigen vector

$\underline{t \in R}$

5.

(b) $\begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 1 & 6 & 3 & 2 \end{bmatrix}$

$$\xrightarrow[R_3 \to R_3 + R_1]{R_2 \to 2R_1 + R_2} \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 8 & 2 & 2 \\ 0 & 8 & 2 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 8 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

echelon form

Now changing it to normal form:-
by column transformations.

$C_1 \leftrightarrow -C_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 8 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[C_3 \to C_3 + C_1]{C_2 \to C_2 - 2C_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\cancel{C_3 \to C_3 - C_4}$

$C_4 \to C_4 - C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{C_3 \to 4C_3 - C_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{C_2 \to \frac{C_2}{8}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, the rank of matrix is 2 Ans)

Indian Institute of Engineering Science and Technology, Shibpur

B.Tech. 2nd Semester Final Examination, June 2023

Subject: Mathematics - II (MA 1201)

Time : 3 hours **Full Marks : 50**

Answer any FIVE questions.

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1. (a) Prove that the set $S_1 = \{(x, y, z) \in \mathbb{R}^3 : z^2 = xy\}$ does not form a subspace of the vector space $\mathbb{R}^3$ over $\mathbb{R}$. [2]

   (b) Find the coordinate vector of $\alpha = (0, 3, 1) \in \mathbb{R}^3$ relative to the ordered basis $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ of $\mathbb{R}^3$. [2]

   (c) Let $\mathbb{R}^{2\times 2}$ denotes the vector space of all 2×2 matrices with real entries. Prove that the set S is a subspace of $\mathbb{R}^{2\times 2}$ where

   $$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2} : a + b = 0,\ c + d = 0 \right\}.$$

   Find a basis and determine the dimension of the subspace S. [2+2+1=5]

   (d) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined as $T(x, y) = (x, x - y, 2y)$, $(x, y) \in \mathbb{R}^2$. Check whether T is a linear mapping or not. [1]

2. (a) Use elementary row operations on A to obtain A^{-1} where

   $$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & 4 \\ 3 & 3 & 7 \end{pmatrix}.$$ [3]

   (b) If λ be an eigenvalue of a real orthogonal matrix A, then prove that $\frac{1}{\lambda}$ is also an eigenvalue of A. [2]

   (c) Reduce the matrix $A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$ to the diagonal form. [5]

3. (a) Find a Fourier series to represent $f(x) = x - x^2$ for $x \in [-\pi, \pi]$. Hence deduce that

   $$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}.$$ [5+1=6]

   (b) Solve the system of equations

   $$\begin{aligned} x + 2y + z - 3w &= 1 \\ 2x + 4y + 3z + w &= 3 \\ 3x + 6y + 4z - 2w &= 4. \end{aligned}$$ [4]

Maths (2023)

(2) (a) Row operations on A to get A^{-1}

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & 4 \\ 3 & 3 & 7 \end{pmatrix}$$

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(b) $AX = \lambda X$

A is orthogonal i.e $AA^T = I$

$$\frac{1}{\lambda} X = \frac{A^{-1} X}{\lambda}$$

Now $A^T = A^{-1}$

$$\therefore \frac{X}{\lambda} = A^T X$$

Now, as we know that eigen value of A and A^T is equal.

therefore $\frac{1}{\lambda}$ is also an eigen value of A

(c) $A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$ $|A - \lambda I| = 0$

$$\left|\begin{pmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{pmatrix}\right| = 0 \Rightarrow (4-\lambda)\left[(4-\lambda)^2 - 4\right] - 2\left(2(4-\lambda) - 4\right) + 2\left(4 - 2(4-\lambda)\right) = 0$$

$$(4-\lambda)\left[\lambda^2 + 16 - 8\lambda - 4\right] - 2\left(8 - 2\lambda - 4\right) + 2\left(4 - 8 + 2\lambda\right)$$

$$(4-\lambda)\left(\lambda^2 - 8\lambda + 12\right) - 2\left(4 - 2\lambda\right) + 2\left(-4 + 2\lambda\right)$$

$$(4-\lambda)\left(\lambda^2 - 8\lambda + 12\right) + 4\lambda - 8 + 4\lambda - 8$$

$$4\lambda^2 - 32\lambda + 48 - \lambda^3 + 8\lambda^2 - 12\lambda + 8\lambda - 16$$

$$12\lambda^2 - \lambda^3 + 32 \cancel{+ 4\lambda} \cancel{+ 8\lambda} - 36\lambda = 0$$

$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$\lambda = 2$

$8 - 48 + 72 - 32$

$-40 \quad 40$

$$(\lambda-2)(\lambda^2 - 10\lambda + 16) = 0$$

$+20\lambda + 16$

$$(\lambda-2)(\lambda^2 - 8\lambda - 2\lambda + 16)$$

$$(\lambda-2)((\lambda-8)(\lambda-2)) = 0$$

$$(\lambda-2)^2(\lambda-8) = 0$$

$\lambda = 2, 2, 8$

$\underline{D = P^{-1}AP}$ where $P = [X_2 \ X_2 \ X_8]$ (Done previously)

$\therefore$ Diagonal form $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ (Ans)

3 b.

5 Solve $(AX = B)$

$$\begin{bmatrix} 1 & 2 & 1 & -3 \\ 2 & 4 & 3 & 1 \\ 3 & 6 & 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$\rho(A) = ?$ and $\rho(A:B) = ?$

$$A:B = \begin{bmatrix} 1 & 2 & 1 & -3 & 1 \\ 2 & 4 & 3 & 1 & 3 \\ 3 & 6 & 4 & -2 & 4 \end{bmatrix}$$

$2 \ 4 \ 2 \ -6$

$-2 \ 4 \ 3 \ -1$

$0 \ 0 \ -1 \ -7$

$3 \ 6 \ 4 \ -2$

$3 \ 6 \ 3 \ -9$

$0 \ 0 \ 1 \ 7$

$\underline{\rho(A) = ?}$ $\therefore A = \begin{bmatrix} 1 & 2 & 1 & -3 \\ 2 & 4 & 3 & 1 \\ 3 & 6 & 4 & -2 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to 2R_1 - R_2} \begin{bmatrix} 1 & 2 & 1 & -3 \\ 0 & 0 & -1 & -7 \\ 0 & 0 & 1 & 7 \end{bmatrix}$

$\downarrow R_3 \to R_3 + R_2$

$\rho(A) = 2$

$$\begin{bmatrix} 1 & \cancel{2} & 1 & -3 \\ 0 & 0 & -1 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now $\rho(A:B)$

$2-3=-1$

$③$

$4-3=1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 & -3 & 1 \\ 2 & 4 & 3 & 1 & 3 \\ 3 & 6 & 4 & -2 & 4 \end{bmatrix}$$

$$\Big\downarrow \begin{array}{l} R_2 \to 2R_1 - R_2 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 1 & -3 & 1 \\ 0 & 0 & -1 & -7 & -1 \\ 0 & 0 & 1 & 7 & 1 \end{bmatrix}$$

$$\Big\downarrow R_3 \to R_3 + R_2$$

$$\begin{bmatrix} 1 & 2 & 1 & -3 & 1 \\ 0 & 0 & -1 & -7 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \rho(A:B) = 2$$

As $\rho(A) = \rho(A:B)$ but $<$ no. of unknowns (4)

$\therefore$ It will have $\underline{\underline{\infty}}$ solutions with $(4-2) = 2$ arbitary constants.

Let arbitary contants be p, q.

$w = p$

$-z - 7w = -1$

$-z = -1 + 7p$

$z = 1 - 7p$.

$x + 2y +$

Indian Institute of Engineering Science and Technology, Shibpur
B.Tech. (All groups) 2nd Semester End Semester Examination, April 2024
Subject: Mathematics - II (MA-1201)

Full Marks : 50 **Time : 3 hours**

Answer any FIVE questions.

[$5 \times 10 = 50$]

1. (a) Let $S = (x, y, z) \in \mathbb{R}^3 : 101x - 102y + 103z = 0$. Show that S is a subspace of $\mathbb{R}^3$. Find a basis for S. [1+1=2]

   (b) Let V be the vector space of all $n \times n$ matrices over the field $\mathbf{F}$ and let B be a fixed $n \times n$ matrix. If $T(A) = AB - BA, \forall A \in V$, then verify whether T is a linear transformation from V into V or not. [2]

   (c) Determine the space spanned by the vectors $\alpha = (0, 3, 1)$ and $\beta = (2, 1, -2)$. Verify whether the vector $(4, 7, -2)$ belongs to this space or not. [1+1=2]

   (d) Let $\mathbb{R}$ be the field of real numbers and let $\mathbf{P_n}$ be the set of all polynomials (of degree at most n) over the field $\mathbb{R}$. Prove that $\mathbf{P_n}$ is a vector space over the field $\mathbb{R}$. [4]

2. (a) If λ is an eigen value of a non-singular matrix A, then show that $\dfrac{|A|}{\lambda}$ is an eigen value of $adj(A)$. [2]

   (b) Solve the following homogeneous system of equations:

$$2x + y + 2z = 0, x + y + 3z = 0, 4x + 3y + \lambda z = 0$$

   for different values of λ. [3]

   (c) Diagonalise, if possible, the matrix $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$. [5]

3. (a) If A and B are orthogonal matrices of same order and $|A| + |B| = 0$, prove that $(A + B)$ is a singular matrix. [2]

   (b) If A and B are both Hermitian matrices, then show that $(AB + BA)$ is Hermitian and $(AB - BA)$ is Skew-Hermitian. [2]

   (c) Expand $f(x) = |x|$ in Fourier series in the interval $-\pi \leq x \leq \pi$ and hence show that $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots\infty = \dfrac{\pi^2}{8}$. [5+1=6]

4. (a) A function $\phi(x, y, z)$ is such that $\vec{\nabla}\phi = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz\hat{k}$. If $\phi(1, -2, 2) = 4$, then find $\phi(x, y, z)$. [3]

   (b) Prove that $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2\vec{A}$. [3]

   (c) Evaluate $\int\int_S \vec{F}.\hat{n}dS$, where $\vec{F} = 6z\hat{i} - 4\hat{j} + y\hat{k}$, S is the part of the plane $2x + 6y + 3z = 10$ which is located in the first octant and $\hat{n}$ is the unit outer normal vector of the surface S. [4]

PTO.

end sem, April 2024

(Q)

(2) (a) If $\lambda \to$ e.v of Non-singular matrix A.

$$|A| \neq 0$$

To show: $\frac{|A|}{\lambda}$ is ev. of adj(A)

$\Rightarrow AX = \lambda X \longrightarrow \frac{1}{\lambda} X = \frac{1}{A} X \Rightarrow \frac{1}{\lambda} X = A^{-1} X$ — (1)

$\therefore \frac{1}{\lambda}$ is e.v of A^{-1}.

$A^{-1} = \frac{1}{|A|} adj(A)$

$A^{-1}|A| = adj(A)$

postmultiplying with X :- $|A| A^{-1} X = adj(A) X$

$$|A| A^{-1} X = adj(A) X$$

from (1) $|A| \frac{1}{\lambda} X = adj(A) X$

$\therefore$ e.v of adj $(A) = \frac{|A|}{\lambda}$ (Proved)

(b) Homogeneous system of equations:-

$2x + y + 2z = 0$
$x + y + 3z = 0$
$4x + 3y + \lambda z = 0$

} Solve for differ. values of λ

AX = B

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since $AX = 0$

The system has ∞ soln if $\rho(A) = r < n$

$$\det(A) = \begin{vmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & \lambda \end{vmatrix} = 2(\lambda - 9) - 1(\lambda - 12) + 2(3 - 4)$$

$$= 2\lambda - 18 - \lambda + 12 - 2$$

$$= \lambda - 8$$

If $\lambda - 8 = 0$ i.e $\Rightarrow \lambda = 8$

$\det(A) = 0$ i.e $\rho(A) < 3$

hence Non-trivial ∞ solutions.

Therefore, for $\lambda \neq 8$, system has only trivial solutions i.e $x = y = z = 0$

for $\lambda = 8$, system has ∞ solutions.

$\therefore \frac{x+y}{3} = -z$

$2x + y + 2\left(-\frac{x-y}{3}\right) = 0$

$6x + 3y - 2x - 2y = 0$

$4x + y = 0$

$4x = -y$

Now, $-y + 3y + \lambda z = 0$

$8 \times z = -2y$

where $\lambda = 8$ $\boxed{z = -\frac{y}{4}}$

Let $x = t$ $t \in R$.

$\therefore y = -4t$

$z = t$

for $\lambda = 8$

hence ∞ many solutions.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \text{ (Ans)}$$

(c) Diagonalise matrix $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(2-\lambda)(-1-\lambda) - 1\right] - 1\left((-1)(-1-\lambda) - 0\right) - 2(-1)$$

$$(1-\lambda)\left[-(2-\lambda)(1+\lambda) - 1\right] - (1+\lambda) + 2$$

$$(1-\lambda)\left[-\{2 - \lambda^2 - \lambda + 2\lambda\} - 1\right] - 1 - \lambda + 2$$

$$(\lambda - 1)\left[2 - \lambda^2 + \lambda + 1\right] - \lambda + 1$$

$$(\lambda - 1)\left[-\lambda^2 + \lambda + 3\right] - \lambda + 1$$

$$-\lambda^3 + \lambda^2 + 3\lambda + \lambda^2 - \lambda - 3 - \lambda + 1$$

$$-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$$

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

$\cancel{1 - 2 - 1 + 2 = 0}$ $\quad \lambda = 1$ ✓ $\quad \lambda = 2$ ✓

$-1 - 2 + 1 + 2$ $\quad \lambda = -1$ ✓

$\cancel{8} - \cancel{8} - \cancel{2} + \cancel{2}$

$\therefore$ E.V $= 1, -1, 2$. Yes, diagonalisable.

$$\therefore [A - \lambda I]X = 0$$

$\lambda = 1$ $\quad [A - I]X = 0 \Rightarrow \begin{bmatrix} 0 & 1 & -2 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$x_2 - 2x_3 = 0 \Longrightarrow x_2 = 2x_3$$

$$-x_1 + x_2 + x_3 = 0$$

$$-x_1 + 3x_3 = 0 \Longrightarrow x_1 = 3x_3$$

Eigen vector corresponding to $\lambda = 1$

$$X_1 = t\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Let $x_3 = t$
$x_2 = 2t$
$x_1 = 3t$

$\underline{\lambda = 2}$:-
$$\begin{bmatrix} -1 & 1 & -2 \\ -1 & 0 & 1 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_3 = 0$ $\qquad$ $\underline{x_1 = x_3}$

$x_2 - 3x_3 = 0$ $\qquad$ $x_2 = 3x_3$ $\qquad$ Let $x_3 = t$

$$\therefore X_2 = t\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

$\underline{\underline{\lambda = -1}}$:-
$$\begin{bmatrix} 2 & 1 & -2 \\ -1 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_2 = 0.$

$-x_1 + 3x_2 + x_3 = 0$

$\underline{x_1 = x_3}$

$$\therefore X_{-1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence $P = [X_1 \; X_2 \; X_{-1}] = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Now $D = P^{-1}AP$

or $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\underline{\underline{\text{(Ans)}}}$

or

$$\underline{\underline{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}}}$$

(3) (a) A and B $\rightarrow$ orthogonal matrices of same order

$$|A| + |B| = 0$$

To prove $(A+B)$ is singular

i.e $|A+B| = 0$

Orthogonal : $AA^T = I$.

$$\det |A^t (A+B) B^t| = |A^t A B^t + A^t B B^t|$$
$$= |B^t + A^t|$$
$$= |(B+A)^t|$$
$$= |(A+B)|$$

$$\therefore |A^t| \, |(A+B)| \, |B^t| = |(A+B)|$$

$$\Rightarrow |A| \, |A+B| \, |B| = |A+B|$$

$$|A+B| \left[1 - |A||B|\right] = 0 \qquad \text{as } |A| = -|B|$$

$$\Rightarrow |A+B| \left[1 + |A|^2\right] = 0 \qquad 1 + |A|^2 = 0$$

$$\underline{|A|^2 \neq -1}$$

$\therefore |A+B| = 0$

Hence it is singular

(Proved)

$$\cancel{(|A| + |B|)^2 = 0}$$

$$\cancel{|A|^2 + |B|^2 + 2|A||B| = 0}$$

$$\cancel{\frac{|A|^2 + |B|^2}{2} = |A||B|}$$

3) b) A and B are Hermitian

To show : (AB+BA) is Hermition
and (AB−BA) is skew−Hermition

$\Rightarrow (AB+BA)^\theta = (AB)^\theta + (BA)^\theta$

$= B^\theta A^\theta + A^\theta B^\theta$

$= BA + AB$

$= AB + BA$

$\therefore$ It is Hermitian

$\Rightarrow (AB-BA)^\theta = (AB)^\theta - (BA)^\theta$

$= B^\theta A^\theta - A^\theta B^\theta$

$= BA - AB$

$= -(AB-BA)$

Hence it is skew hermitian.