2022

1(a) Define vector space → X

1(b) $S = \{(x, y, z) : x^2 + y^2 = z^2\}$ is not a subspace

Let us assume two vectors $\alpha, \beta \in S$ and two scalar $a, b \in F$

Now, by defn $a\alpha + b\beta \in S$

Let $\alpha = (x_1, y_1, z_1)$ where $x_1, y_1, z_1, x_2, y_2, z_2 \in R$

$\beta = (x_2, y_2, z_2)$

we have

$$x_1^2 + y_1^2 = z_1^2$$

$$x_2^2 + y_2^2 = z_2^2$$

for any two real numbers $a, b \in F$

$$a\alpha + b\beta = \{a(x_1, y_1, z_1) + b(x_2, y_2, z_2)\}$$

$$= \{ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2\}$$

Now, if $a\alpha + b\beta \in S$, then

$$(ax_1 + bx_2)^2 + (ay_1 + by_2)^2$$

$$= ax_1^2 + bx_2^2 + 2abx_1x_2 + ay_1^2 + by_2^2 + 2aby_1y_2$$

which is clearly not equal to

$$(az_1 + az_2)^2$$

$\therefore$ S is not a subspace of V_3.

(c) Let W_1 and W_2 be two subspaces of $V(F)$.

Let us assume a null vector $(\theta) \in V(F)$ which is present in W_1 and W_2

$\therefore \theta \in W_1 \cap W_2$, Let us also assume $a, b \in F$

Now, $\alpha \in W_1 \cap W_2 \Rightarrow \alpha \in W_1$ and $\alpha \in W_2$

$\beta \in W_1 \cap W_2 \Rightarrow \beta \in W_1$ and $\beta \in W_2$.

Now since both W_1 and W_2 are subspaces itself
$\therefore$ for any $a, b \in F$
$a\alpha + b\beta \in W_1$ and $a\alpha + b\beta \in W_2$
$\therefore a\alpha + b\beta \in W_1 \cap W_2$, $\forall$ $\underline{a, b \in F}$ and $\underline{\alpha, \beta \in W_1 \cap W_2}$

Let us assume
$W_1 = \{ (x, 0), x \in \mathbb{R} \}$ $\quad W_1 \cap W_2$
$W_2 = \{ (0, y), y \in \mathbb{R} \}$ $\quad = \{ (0,0), x, y \in \mathbb{R} \}$
clearly the null vector is indeed an element
and $\therefore$ $W_1 \cap W_2$ is a subspace. $\left(\underline{a = 0;\ b = 0} \right)$

(a) $T: \mathbb{R}^3 \to \mathbb{R}^3$ $\quad (1,2,1), (2,1,0)$ and $(1,-1,-2)$
$(1,0,0)$; $(0,1,0)$ and $(1,1,1)$.

$T(1,2,1) = (1,0,0)$
$T(2,1,0) = (0,1,0)$
$T(1,-1,-2) = (1,1,1)$

Let us assume a vector $\alpha = \{x, y, z\} \in \mathbb{R}^3$, domain of T.

Let us express α in terms of basis vectors $\underline{a, b, c \in \mathbb{R}}$
$\alpha\{x,y,z\} = a(1,2,1) + b(2,1,0) + c(1,-1,-2)$
$\alpha\{x,y,z\} = \{a+2b+c,\ 2a+b-c,\ a-2c\}$

$x = a+2b+c$ $\quad z = a-2c$
$y = 2a+b-c.$ $\quad \underline{a = z+2c}$

~~$x = z + 2b + 3c$~~ $\quad$ ~~$x = z + 2b + 3c$~~
~~$y = 2z + 4c + b + c$~~ $\quad$ ~~$y = 2z + b - 3c$~~

$\boxed{\cancel{x + y = 3z + 3b}}$ $\quad$ $\boxed{\cancel{y = 2z + b - 3c}}$

$\boxed{x + y = 3a + 3b}$ $\quad$ $\boxed{x - y = -a + b + 2c}$

$2x = 3a + 3b - a + b + 2c = 2a + 4b + 2c$
$\boxed{x = a + 2b + c}$

$x = a + 2b + c \qquad z = a - 2c$

$y = 2a + b - c$

$a = ?$

$b = ?$

$c = ?$

$$a = \frac{4y - 3z - 2x}{3} \qquad c = \frac{2y - 3z - x}{3}$$

$$b = x + z - y$$

$$T(\alpha) = a\,T_1(1,2,1) + b\,T_2(2,1,0) + c\,T_3(1,-1,-2)$$

$$= \frac{4y-3z-2x}{3}(1,0,0) + (x+z-y)(0,1,0) + \left(\frac{2y-3z-x}{3}\right)(1,1,1)$$

$$= \left(\frac{4y-3z-2x+2y-3z-x}{3}, \frac{3x+3z-3y+2y-3z-x}{3}, \frac{2y-3z-x}{3}\right)$$

$$\left(\frac{6y-6z-3x}{3}, \frac{2x-y}{3}, \frac{2y-3z-x}{3}\right)$$

$$= \left(2y-2z-x, \frac{2x-y}{3}, \frac{2y-3z-x}{3}\right)$$

$$T(3,-3,3) = \left(-6-6-3, 9/3, -6-6-3/3\right)$$

$$= (-15, 3, -5)$$

$T(x) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} x - x \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

Let $\alpha_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ $\quad \beta = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ $\forall\ \alpha, \beta \in$ 2×2 matrix

$\alpha + \beta = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$

$T(\alpha+\beta) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \alpha + \beta - (\alpha+\beta) \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} =$

$= \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix} - \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

$T(\alpha+\beta) = A(\alpha+\beta) - (\alpha+\beta)A$

$= A\alpha + A\beta - \alpha A - \beta A$

$= (A\alpha - \alpha A) + A\beta - \beta A$

$= T(\alpha) + T(\beta)$

$T(c\alpha)$

$= A(c\alpha) - (c\alpha)A$

$= cA\alpha - c\alpha A$

$= c(A\alpha - \alpha A)$

$= cT(\alpha)$

$\therefore$ T is linear

$T(\alpha) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} - \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

$= \begin{bmatrix} a_1 + 2c_1 & b_1 + 2d_1 \\ 3c_1 & 3d_1 \end{bmatrix} - \begin{bmatrix} a_1 & 2a_1 + 3b_1 \\ c_1 & 2c_1 + 3d_1 \end{bmatrix}$

$= \begin{bmatrix} 2c_1 & 2d_1 - 2a_1 - 2b_1 \\ 2c_1 & -2c_1 \end{bmatrix}$

For $\alpha \in \ker(T)$

Let $T(\alpha) = 0$

$2c = 0 \Rightarrow c = 0$

$-2a - 2b + 2d = 0 \quad a + b = d$

$\therefore X = \begin{bmatrix} a & b \\ 0 & a+b \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$

dim (Ker T) = 2 = Nullity of T

3(a)

3(b) $R^2 \to R^3$ $\quad F(x,y) = (x-y, x+y, y)$

Let us assume

$\alpha = (x_1, y_1) \quad \beta = (x_2, y_2)$

$\forall \; \alpha, \beta \in \mathbb{R}^2$ (domain F)

$\alpha + \beta = (x_1 + x_2, y_1 + y_2)$

$F(\alpha + \beta) = F(x_1 + x_2, y_1 + y_2)$

$= F(x_1 + x_2 - y_1 - y_2, x_1 + x_2 + y_1 + y_2, y_1 + y_2)$

$= F(x_1 - y_1, x_1 + y_1, y_1) + F(x_2 - y_2 + x_2 + y_2, y_2)$

$= F(\alpha) + F(\beta) \quad (*)$

Let $c \in \mathbb{R}$.

$c\alpha = c(x_1, y_1) = (cx_1, cy_1)$

$F(c\alpha) = F(cx_1, cy_1) = (cx_1 - cy_1, cx_1 + cy_1, cy_1)$

$= c(x_1 - y_1, x_1 + y_1, y_1)$

$= cT(\alpha) \quad (*)$

June 2023

$S_1 = \{(x,y,z) \in \mathbb{R}^3 ; z^2 = xy\}$

Let, $\alpha, \beta \in S$ and $a, b \in R$

Now, by definition if $\alpha, \beta \in S$, then

$a\alpha + b\beta \in S$ (vector space)

Let $\alpha = (x_1, y_1, z_1)$ we have

$\beta = (x_2, y_2, z_1)$ $z_1^2 = x_1 y_1$

$z_2^2 = x_2 y_2$

Now $a\alpha + b\beta = \{a(x_1, y_1, z_1) + b(x_2, y_2, z_2)\}$

$= \{ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2\}$

$(az_1 + bz_2)^2 = (az_1)^2 + (bz_1)^2 + 2ab\, z_1 + z_2$

which is not equal to

$(ax_1 + bx_2)(ay_1 + by_2) = a^2 x_1 y_1 + b^2 x_2 y_2$

$= ab x_1 y_2 + ay_1 bx_2$

∴ not a subspace.

$\alpha = (0,3,1)$

basis $\rightarrow \{(1,1,0), (1,0,1), (0,1,1)\}$

Let $a, b, c \in R$.

Now, since basis is LI.

$\therefore \{a(1,1,0) + b(1,0,1) + c(0,1,1)\} = \{0,3,1\}$

$= \{a+b, a+c, b+c\} = \{0,3,1\}$

$\boxed{a+b=0}$

$\boxed{a=-b}$

$a+c=3$

$b+c=1$

$a-b=2$

$-2b=2$

$\boxed{b=-1}$

$c=2$

$\boxed{a=1}$

∴ coordinate of $\alpha = (1,-1,2)$

(c) $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \subset \mathbb{R}^{2\times 2} : a+b=0,\ c+d=0 \right\}$

Let $\alpha, \beta \in S$ and $x, y \in \mathbb{R}$.

- $x\alpha + y\beta \in S$ (definition of vector space)

Let

$\alpha = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ $\beta = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$

$a_1 + b_1 = 0$ $a_2 + b_2 = 0$

$c_1 + d_1 = 0$ $c_2 + d_2 = 0$

Now

$x\alpha + y\beta = \begin{pmatrix} xa_1 + xb_1 \\ xc_1 + xd_1 \end{pmatrix} + \begin{pmatrix} ya_2 + yb_2 \\ yc_2 + yd_2 \end{pmatrix}$

$= \begin{pmatrix} xa_1 + ya_2 & xb_1 + yb_2 \\ xc_1 + yc_2 & xd_1 + yd_2 \end{pmatrix}$

Now, here

$xa_1 + ya_2 + xb_1 + yb_2$ $xc_1 + yc_2 + xd_1 + yd_2$

$x(\cancel{a_1 + b_1})^{0} + y(\cancel{a_2 + b_2})^{0}$ $x(\cancel{c_1 + d_1})^{0} + y(\cancel{c_2 + d_2})^{0}$

$= 0$ $= 0$

$\therefore$ clearly S is a vector subspace (P)

Now

general element of $S = \left\{ \begin{matrix} a & b \\ c & d \end{matrix} \right\}$

$a = -b$

$c = -d$ $S = \left\{ \begin{matrix} -b & b \\ -d & d \end{matrix} \right\}$

$S = b \left\{ \begin{matrix} -1 & +1 \\ 0 & 0 \end{matrix} \right\} + d \left\{ \begin{matrix} 0 & 0 \\ -1 & 1 \end{matrix} \right\}$

$$\left[\begin{pmatrix}-1 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ -1 & 1\end{pmatrix}\right]$$

lets test for L.I., let $p, q \in \mathbb{R}$

$$p \cdot B_1 + q \cdot B_2 = p\begin{pmatrix}-1 & 1\\ 0 & 0\end{pmatrix} + q\begin{pmatrix}0 & 0\\ -1 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}-p & p\\ q & q\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \Rightarrow x = 0, y = 0$$

only trivial sol^n $\therefore$ Basis

dimension = 2

(d) $T(x,y) = (x, x-y, 2y)$, $(x,y) \in \mathbb{R}^2$

let $\alpha = (x_1, y_1)$ $\beta = (x_2, y_2)$ $\forall$ $\alpha, \beta \in \mathbb{R}^2$

$\alpha + \beta = (x_1 + x_2, y_1 + y_2)$

$T(\alpha + \beta) = T(x_1 + x_2, y_1 + y_2)$

$= (x_1 + x_2, x_1 + x_2 - y_1 - y_2, 2y_1 + y_2)$

$= (x_1, x_1 - y_1, 2y_1) + (x_2, x_2 - y_2, 2y_2)$

$= T(\alpha) + T(\beta)$

let $c \in \mathbb{R}$

$c\alpha = c(x_1, y_1) = (cx_1, cy_1)$

$T(c\alpha) = T(cx_1, cy_1) = (cx_1, cx_1 - cy_1, 2cy_1)$

$= c(x_1, x_1 - y_1, 2y_1)$

$= cT(\alpha)$ Linear mapping

April 2024

1(a) $S = \{(x, y, z) \in \mathbb{R}^3 : 101x - 102y + 103z = 0\}$

Let $\alpha, \beta \in S$ and $a, b \in R$

By definition of VS $\underline{a\alpha + b\beta \in S}$

$\alpha = (x_1, y_1, z_1) \quad \beta = (x_2, y_2, z_2)$

$a\alpha + b\beta = \{ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2\}$

$101x_1 - 102y_1 + 103z_1 = 0$

$101x_2 - 102y_2 + 103z_1 = 0$

$101(ax_1 + bx_2) - 102(ay_1 + by_2) + 103(az_1 + bz_2)$

$a(\cancel{101x_1 - 102y_1 + 103z_1})^{0} + b(\cancel{101x_2 - 102y_2 + 103z_2})^{0} = 0$

(b) $T(A) = AB - BA, \ \forall A \in V$.

Let α and β $\in$ to $n \times n$ matrix vector space.

Now, B is a fixed matrix

$B \rightarrow$ fixed $n \times n$ matrix

$\alpha, \beta \rightarrow$ 2 arbitrary $\underline{n \times n \text{ matrix}}$

$T(\alpha + \beta) = (\alpha + \beta)B - B(\alpha + \beta)$

$= \alpha B + \beta B - B\alpha - B\beta$

$= (\alpha B - B\alpha) + (\beta B - B\beta)$

$= T(\alpha) + T(\beta)$ $\longrightarrow$ Linear Transformation

Let c be any constant

$T(c\alpha) = (c\alpha)B - B(c\alpha)$

$= c\alpha B - Bc\alpha = c(\alpha B - B\alpha)$

$= cT(\alpha)$

c) $\alpha = (0,3,1)$ $\beta = (2,1,-2)$ → space spanned.
Let $x, y \in R$ to find out span

$$x(\alpha) + y(\beta) = x(0,3,1) + y(2,1,-2)$$
$$= (2y, 3x+y, x-2y)$$

To check whether $\gamma = (4,7,-2)$ in span

$2y, 3x+y, x-2y$

$\boxed{2y = 4}$ $y = 2$ $3x + y = 7$
$x = 5/3$

clearly $5/3 - 4 \neq -2$

$\therefore (4,7,-2)$ not in span

d) $P_n = \{ a_0 + a_1 x + a_2 x^2 + \dots a_n x^n \mid$ where $a_0 \dots n \in R \}$

Vector space over R

i) Closure under vector addition $a_i, b_i \in R$

$S_n = \{ a_0 + a_1 x + a_2 x^2 \dots a_n x^n \}$
$R_n = \{ b_0 + b_1 x + b_2 x^2 + \dots b_n x^n \}$

$S_n + R_n = \{ a_0 + b_0 + (a_1 + b_1) x + (a_2 + b_2) x^2 + \dots (a_n + b_n) x^n \}$

Let $a_0 + b_0 = c_0$; $a_1 + b_1 = c_1$
$\vdots$

$S_n + R_n = \{ c_0 + c_1 x + c_2 x^2 + \dots c_n x^n \} \in P_n$ $c_n \in IR$

2) Closure under scalar multiplication

Let $\alpha \in R$

Now $a_i \in R$

$S_n = \{ a_0 + a_1 x + a_2 x^2 \dots \dots a_n x^n \}$

$cS_n = \{ \alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 \dots \dots + \alpha a_n x^n \}$

$\alpha a_0 = P_0 \quad \alpha a_1 = P_1$

$cS_n = \{ P_0 + P_1 x + P_2 x^2 \dots \dots P_n x^n \}$
$\in P_n$

$\therefore$ closed

3) Presence of zero element

$a_0 = a_1 = a_2 \dots \dots = a_n = 0$

$0 + 0x + 0x^2 + \dots 0x^n \in P_n$

$0 \in P_n$

4) Presence of Additive inverse

$a_i \in R$

$S_n^{(x)} = \{ a_0 + a_1 x + a_2 x^2 \dots \dots a_n x^n \}$

$a_i \in R$

$-S_n^{(x)} = \{ -a_0 + (-a_1 x) + (-a_2 x^2) \dots \dots -a_n x^n \}$

Clearly $-S_n^{(x)} \in P_n$.

Since field is $R \rightarrow$ other axioms are inherited from R and hold true.

$\therefore P_n$ is a Vector Space