METRIC TOPOLOGY AND FIRST COUNTABILITY

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1. Introduction

This section should not need much in the way of motivation. We have talked about $\mathbb{R}^n$ and how relatively easy it is to prove things about it due to the fact that the topology is defined by a distance function. In paticular, ϵ-balls generate this topology. The same is true for general metric spaces. In this section we will formally define them and explore their properties. As you will see, they are about as well-behaved as we could hope.

We will also explore how we can tell if a given topological space is a metric space. Of course, if we are given a basis for a topology made of ϵ-balls for some metric, we will know it is a metric space, but what about when we do not have such a convenient description? We will toy with this problem, and make some explorations but not answer it completely for now. In this process, we will encounter our first notion of 'countability'.

This being a vast but relatively familiar subject, the students are expected to read and assimilate particularly this topic (Sections 20 and 21) well from Munkres's book to supplement what will be covered in class and these notes.

2. Metric Topology

Definition 2.1. A metric d on a set X is a function

$$d : X \times X \to \mathbb{R}$$

having the following properties :

(1) $d(x, y) \geq 0$ for all $x, y \in X$, with equality holding if and only if $x = y$.
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(3) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

If (X, d) is a metric space, $x \in X$, and $\epsilon > 0$ then the set

$$B_\epsilon(x) := \{y \in x : d(x, y) < \epsilon\}$$

is called the ϵ-ball centered at x.

Example 2.2. (1) Let X be any nonempty set. Then $d(x, y) := 0$ if $x = y$ and $d(x, y) := 1$ otherwise is a metric on X. This is called the discrete metric on X.

(2) For $n \geq 1$, the standard Euclidean metric on $\mathbb{R}^n$ given by

$$d((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n)) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

is a metric on $\mathbb{R}^n$.

(3) The Square metric on $\mathbb{R}^2$ is given by $d(x, y) := \max\{|x_1 - y_1|, |x_2 - y_2|\}$ is a metric on $\mathbb{R}^2$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

(4) The Taxicab metric on $\mathbb{R}^2$ is given by $d(x, y) := |x_1 - y_1| + |x_2 - y_2|$ is a metric on $\mathbb{R}^2$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

(5) Let X be the space of continuous functions from $[0, 1]$ to $\mathbb{R}$. This is usually denoted $\mathcal{C}([0, 1])$. We define three metrics d_1, d_2, d_∞ on X as follows :

$$d_1(f, g) := \int_0^1 |f(x) - g(x)| dx,$$

$$d_2(f,g) := \sqrt{\int_0^1 |f(x)-g(x)|^2 dx},$$

and

$$d_\infty(f,g) := \max\{|f(x)-g(x)| : x \in [0,1]\}.$$

These metrics, and many others like these, (you can define d_p for any $p \in (0,\infty]$) are studied in detail in functional analysis. We won't be studying these in this course much, but it is worth seeing them to realise how spaces of functions can be given different metrics. Do these definitions seem analogous to the previous three?

Definition 2.3. Let (X,d) be a metric space. The metric topology $\mathscr{T}_d$ on X is the topology generated by the basis $\mathcal{B}$, which is simply the collection of all ϵ-balls $B_\epsilon(x)$ for $x \in X$ and $\epsilon > 0$.

Note that we have to check that this is indeed a basis on X. This is easy to check. Using this description, we can characterise the metric topology as :

Definition 2.4. A set U is open in the metric topology induced by d if and only if for each $y \in U$, there is a $\delta > 0$ such that $B_\delta(x) \subseteq U$. (Note that it is vacuously true that $\varnothing$ is open under this definition.)

Note that since δ can always be reduced, (i.e. if $\delta' < \delta$, then $B_{\delta'}(x) \subseteq B_\delta(x)$) it is possible to restrict $\delta < 1$, say, and get the same metric topology.

Definition 2.5. If X is a topological space, X is said to be *metrizable* if there exists some metric d on the set X that induces the topology of X.

We will come to the question of metrizability later. But let us first spend some time noting that it is important to be somewhat pedantic. Metrizability of a space $(X,\mathscr{T})$ depends only on the topology $\mathscr{T}$ of the space. But note that a metric (X,d) on X is not the same object as $(X,\mathscr{T})$ even when $\mathscr{T}$ is induced by the metric d. There can be many such choices for d. We will come to this shortly. And similarly, a metric d comes up with more properties than the underlying topology. For instance, the next definition makes sense in a metric space :

Definition 2.6. Let (X,d) be a metric space. A subset A of X is *bounded* if there is some number M such that

$$d(a,a') \leq M$$

for every $a, a' \in A$.

Boundedness is not a topological property. In fact, the topology $\mathscr{T}_d$ induced by any metric d can be induced by a bounded metric.

Proposition 2.7. *Let (X,d) be a metric space. Define $\bar{d} : X \times X \to \mathbb{R}$ as*

$$\bar{d}(x,y) = \min\{d(x,y), 1\}.$$

Then $\bar{d}$ is a metric that induces the same topology as d.

Proof. Assuming $\bar{d}$ is a metric, it is clear that δ-balls with $\delta < 1$ are the same in either of the metric topologies induced by d or $\bar{d}$, and thus the condition of being open is the same in either topologies. We leave the work of checking $\bar{d}$ is a metric as an **Exercise**. $\square$

It feels desirable to have a criterion when two metric functions induce the same topology.

Lemma 2.8. *Let d and d' be two metrics on the set X. Let $\mathscr{T}$ and $\mathscr{T}'$ be the topologies they induce respectively. Then $\mathscr{T}'$ is finer than $\mathscr{T}$ if and only if for each $x \in X$ and each $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$B_{\delta,d'}(x) \subseteq B_{\epsilon,d}(x)$$

where the d and d' in the subscript denote the δ-balls or ϵ-balls in those metrics respectively.

Proof. **Exercise.** $\square$

Proposition 2.9. *The topologies on $\mathbb{R}^n$ induced by the Euclidean metric and the square metric are the same as the product topology on $\mathbb{R}^n$.*

Proof. Standard inequalities like Cauchy-Schwarz and geometry show, along with the previous lemma, that conditions of previous lemma are satisfied in either direction for both the Euclidean metric and the square metric., so that they are each finer than the other, i.e. they are equal.

On the other hand, it is easy to show that the square metric induces the product topology from first principles. □

The case of defining a metric on arbitrary products (the *uniform metric*) is left as an exercise for self-study and tutorials. The fact that $\mathbb{R}^\omega$ is metrizable is also left for self-study and tutorials.

3. First Countability Axiom

Definition 3.1. A topological space X has a *Countable basis at a point* $x \in X$ if there is a countable collection $\{B_n\}_{n=1}^\infty$ of neighbourhoods of x in X such that each neighbourhood U of x in X contains (at least) one of the B_n. A space X with a countable basis at each of its points x is said to satisfy the *first countability axiom* or be *first countable*. Note that we can take B_n to be nested by replacing a given B_n by the intersection $B_1 \cap B_2 \cap \ldots \cap B_n$.

Lemma 3.2. *Every metric space is first countable, i.e. satisfies the first countability axiom.*

Proof. $\{B_{1/n}(x)\}_{n=1}^\infty$ give the required system B_n at each point x. □

Lemma 3.3 (The Sequence Lemma)**.** *Let X be a topological space and $A \subseteq X$. Let x be a point in X.*

(1) If there is a sequence $(x_n)_{n=1}^\infty$ of points in A that converge to x, then $x \in$ Cl A.
(2) If X is metrizable (more generally, if X is first countable), and $x \in$ Cl A, then there is a sequence of points x_n in A that converges to x.

Proof. The proof is simple enough, and yet presents some points of interest.

(1) Suppose $x_n \to x$ as $n \to \infty$. Then for any neighbourhood U of x, there exists N such that x_n for $n \geq N$ are all in U. In particular, $A \cap U \neq \varnothing$, so that $x \in \text{Cl } A$.
(2) Let d be a metric generating the topology on X. We will fix this choice of d. Then, for $x \in \text{Cl } A$, we have that the neighbourhood $B_{1/n}(x)$ meets A nontrivially, so that we can pick a point x_n in $B_{1/n}(x) \cap A$. Then $x_n \to x$ as each neighbourhood U of x contains some $B_{1/N}(x)$ and hence also all $B_{1/n}(x)$ for all $n \geq N$, and hence all of x_n for all $n \geq N$. The proof for X first countable is similar.

□

Lemma 3.4. *Let $f : X \to Y$ be a function of topological spaces X and Y.*

(1) If f is continuous, then for every convergent sequence $x_n \to x$ in X, the image sequence $f(x_n) \to f(x)$ in Y.
(2) If X is metrizable (more generally, if X is first countable), and for every convergent sequence $x_n \to x$ in X, the image sequence $f(x_n) \to f(x)$ in Y, then f is continuous at x.

Proof. Again the proof is simple enough, and yet presents some points of interest.

(1) Let V be any neighbourhood of $f(x)$ in Y. Then, since f is continuous, $f^{-1}(V)$ is an open neighbourhood of x in X and hence contains all x_n for $n \geq N$ for some N. Then by definition, V contains all function values $f(x_n)$ for $n \geq N$, and we are done.
(2) Let A be any subset of X. We will show that $f(\text{Cl } A) \subseteq \text{Cl } f(A)$. Any point in $f(\text{Cl } A)$ has the form $f(x)$ for some $x \in \text{Cl } A$. Then there exists a sequence of points $(x_n)_{n=1}^\infty$ of A such that $x_n \to x$, since X is metrizable, by second part of previous lemma. By hypothesis, $f(x_n)$ converges to $f(x)$ in Y and $\{f(x_n)\}_{n=1}^\infty$ is a sequence of points of $f(A)$. Therefore, by the first part of the previous lemma, $f(x) \in \text{Cl } f(A)$. The proof for X first countable is similar.

□

Exercise: Show that the pointwise convergence topology on functions $\{f : \mathbb{R} \to \mathbb{R}\}$ is not metrizable.