CONTINUOUS FUNCTIONS

ADITYA KARNATAKI

1. Continuous Functions

Up to now we have defined just a few topological properties, like the first three T-axioms. Recall that we said in the first class that topological properties are those properties of topological spaces that can be studied in terms of continuity alone. We have seen some examples and constructions on topological spaces. Now it is time to see how these spaces relate to one another, and how well they are preserved through different constructions like taking continuous images, subspaces, products, etc. We will find that a relatively small number of new definitions are needed in this but it will open up quite a lot of new territory for us to explore. We are now beginning to study topological properties themselves, rather than just particular topological spaces.

1.1. **Continuous Functions.**

Definition 1.1. Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces and $f : X \to Y$ be a function between them. We say that f is continuous if $f^{-1}(V) \in \mathscr{T}_X$ for every $V \in \mathscr{T}_Y$.

Remark 1.2. Note that this is a 'global' definition. We will later define the 'local' definition of continuity at a point and see how they are related.

Example 1.3.

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^3$. Then we know by the calculus definition of continuity that f is continuous by that definition. Let us prove that it is continuous by this definition. Let U be an open set in $\mathbb{R}$. We want to prove that $f^{-1}(U)$ is open.

Fix a point $x \in f^{-1}(U)$. Then $f(x) = x^3 \in U$. So by definition, there exists an $\epsilon > 0$ such that $(x^3 - \epsilon, x^3 + \epsilon) \subseteq U$. Then by the calculus definition of continuity, there exists a $\delta > 0$ such that if $y \in (x - \delta, x + \delta)$, then $f(y) \in (x^3 - \epsilon, x^3 + \epsilon)$. In other words, $(x - \delta, x + \delta) \subseteq f^{-1}(U)$ and $x \in (x - \delta, x + \delta)$. Thus, $f^{-1}(U)$ is open.

More generally, let $f : \mathbb{R}^n \to \mathbb{R}^k$ be any function continuous in the calculus sense. Then f is continuous.

(2) The projection functions π_1 and π_2 from $\mathbb{R}^2$ to $\mathbb{R}$ are continuous. **Exercise:** Check this.

(3) Any function from a discrete space to any other topological space is continuous.

(4) Any function from any topological space to a trivial topological space is continuous.

(5) Any constant function from any topological space to any topological space is continuous.

(6) If $f : X \to Y$ and $g : Y \to Z$ are continuous functions, then $g \circ f : X \to Z$ is a continuous function.

(7) The identity function $\mathrm{id} : \mathbb{R} \to \mathbb{R}_\ell$ is not a continuous function. But the identity function $\mathrm{id} : \mathbb{R}_\ell \to \mathbb{R}$ is a continuous function.

Lemma 1.4. *Let X be a topological space with two topologies $\mathscr{T}$ and $\mathscr{T}'$ on it. Then the identity function $(X, \mathscr{T}) \to (X, \mathscr{T}')$ is continuous if and only if $\mathscr{T}' \subseteq \mathscr{T}$, i.e. $\mathscr{T}$ refines $\mathscr{T}'$.*

Proof. **Exercise.** □

1.2. **Equivalent conditions.**

Lemma 1.5. *Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be two topological spaces and $f : X \to Y$ be a function between them. Let $\mathcal{B}$ and $\mathcal{S}$ be a basis and subbasis of $\mathscr{T}_Y$ respectively. Note that we are not assuming that $\mathcal{S}$ generates $\mathcal{B}$. Then the following are equivalent :*

(1) Preimages of open sets are open, i.e. $f^{-1}(V) \in \mathscr{T}_X$ for all $V \in \mathscr{T}_Y$.

(2) Preimages of basic open sets are open, i.e. $f^{-1}(V) \in \mathscr{T}_X$ for all $V \in \mathcal{B}$.
(3) Preimages of subbasic open sets are open, i.e. $f^{-1}(V) \in \mathscr{T}_X$ for all $V \in \mathcal{S}$.

Proof. **Exercise.** □

Definition 1.6. Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be two topological spaces and $f : X \to Y$ be a function between them. Let $x \in X$. Then f is said to be *continuous at x* if for every open set $V \in \mathscr{T}_Y$ containing $f(x)$, there exists an open set $U \in \mathscr{T}_X$ containing x such that $f(U) \subseteq V$.

Lemma 1.7. *Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be two topological spaces and $f : X \to Y$ be a function between them. Then the following are equivalent :*

(1) f is continuous.
(2) For every closed set $C \subseteq Y$, the preimage $f^{-1}(C)$ is closed in X.
(3) For every $x \in X$, f is continuous at x.
(4) For every subset $A \subseteq X$, $f(Cl\ A) \subseteq Cl\ f(A)$, i.e. image of closure is contained in closure of image.

Proof. We will prove the assertion (1) $\implies$ (4), leaving the rest as an **exercise**. Assume f is continuous, and fix a set $A \subseteq X$. We want to show $f(\mathrm{Cl}\ A) \subseteq \mathrm{Cl}\ f(A)$. So pick $x \in \mathrm{Cl}\ A$ and an open set V of Y containing $f(x)$. Then, by assumption, $f^{-1}(V)$ is an open set containing x and so by definition of closure, $f^{-1}(V) \cap A \neq \varnothing$. Let $y \in f^{-1}(V) \cap A$. Then $f(y) \in V \cap f(A)$ so that $V \cap f(A) \neq \varnothing$ so that $f(x)$ is in $\mathrm{Cl}\ f(A)$, which was to be shown. □

Having access to all of these different characterizations of continuity allows us to much more easily prove that certain functions are continuous, and to prove facts about continuous functions.

Exercise: Check that addition and multiplication, thought of as functions $\mathbb{R}^2 \to \mathbb{R}$ with their usual topologies, are both continuous. This can be done from the original definition of continuity alone, but it is much easier with one of the four equivalent ones above.

Exercise: Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous when the domain and codomain have the usual topology, but not when both of them have the ray topology.

Exercise: Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous when the domain and codomain have the usual topology, but not when both of them have the lower limit topology.

1.3. **Open and Closed functions.** Just a brief interlude to mention another sort of function. We mention it for a few reasons. First, it is a useful enough property of functions to give a name, as we will soon see. Second, it is a natural property to consider. So natural, in fact, that a student new to the field might expect that this is the sort of function that "preserves topological structure". Thirdly, it is a concept students often conflate with continuity, so we would like to call specific attention to the differences between this and continuity.

Definition 1.8. Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be two topological spaces and $f : X \to Y$ be a function between them.

- f is said to be an *open* function if $f(U) \in \mathscr{T}_Y$ for all $U \in \mathscr{T}_X$.
- f is said to be a *closed* function if $f(C)$ is a closed subset of Y whenever C is a closed subset of X.

So a function is open if the images of open sets are open, whereas a function is continuous if the preimages of open sets are open. These sound similar, but open and continuous are very different.

Here is some intuition for why "continuity" has the definition it does. The idea is that the function $f : X \to Y$ should tell you about the topology on Y in terms of the topology on X. So we require that given some information about the topology on Y —an open subset of Y— the function returns an open subset of X that we know about. Very colloquially speaking, a continuous function gathers information about Y and brings it back for analysis, whereas an open function just shouts things at Y .

Also, caution! It seems like "open function" and "closed function" might be equivalent somehow (via taking complements, or something?), but they are not. The examples below illustrate this.

Example 1.9. (1) Consider the function $f(x) = 691$ from $\mathbb{R}$ to $\mathbb{R}$. Then we know f is continuous. Image of any set is $\{691\}$ so this is a closed function but not open.
(2) The projection functions π_1 and π_2 from $\mathbb{R}^2 \to \mathbb{R}$ are continuous and open, but not closed. **(Exercise.)**

1.4. **Homeomorphisms.** Arguably, this section is the ultimate payoff of studying continuous functions. Here we define a particularly nice sort of continuous function that provides a way of detecting whether two topological spaces are "the same" from the point of view of topological structure. These functions are the analogues of bijective linear transformations for vector spaces, isomorphisms between groups/ rings/ fields/ modules/ graphs/ linear orders/etc.

Definition 1.10. Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be two topological spaces and $f : X \to Y$ be a bijective function between them. We say that f is a *homeomorphism* if f is continuous and its inverse function f^{-1} is continuous.

In this case, we say that $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ are homeomorphic, and write $(X, \mathscr{T}_X) \cong (Y, \mathscr{T}_Y)$, or more often, simply $X \cong Y$ to denote this.

Lemma 1.11. *Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be two topological spaces and $f : X \to Y$ be a bijective function between them. Then the following are equivalent :*
(1) f is a homeomorphism.
(2) f is continuous and open.
(3) f is continuous and closed.
(4) $U \subseteq X$ is open if and only if $f(U) \subseteq Y$ is open.

This last property shows why homeomorphisms are the correct type of functions to preserve all topological properties, that is, properties that can be stated in terms of open sets alone.

Example 1.12. (1) $(-1, 1)$ is homeomorphic to $\mathbb{R}$, since the function f defined by
$$f(x) = \frac{x}{1 - x^2}$$
is a homeomorphism.
(2) The function
$$f(t) = (\cos 2\pi t, \sin 2\pi t)$$
is a continuous bijection from $[0, 1)$ to the circle S^1, but not a homeomorphism; since the image of the open set $U = [0, 1/4)$ (say) of the domain is not open in S^1, since the point $p = f(0)$ does not lie in any open set $V \subset \mathbb{R}^2$ such that $V \cap S^1 \subseteq f(U)$.

1.5. **Topological invariants.**

Definition 1.13. A property $\mathscr{P}$ of a topological space X is called a *topological invariant* if whenever $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ are two homeomorphic topological spaces, one has property $\mathscr{P}$ if and only if the other has property $\mathscr{P}$.

We can see that topological invariants furnish us with a quick series of checks on whether two spaces can be homeomorphic: if we can identify a topological invariant that one space has and another does not, they cannot be homeomorphic.

Lemma 1.14. *The following properties are topological invariants.*
(1) T_0.
(2) T_1.
(3) T_2, i.e. Hausdorff.
(4) Having a particular cardinality.
(5) Separable.

Exercise: Show that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic with their usual topologies. This is a bit challenging at this point of the course, knowing what you know. Later we will see easier ways of proving this, but at this point you will have to be creative.

It is even more difficult to prove that $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic for general $n \neq m$. Topological invariants provide us with a series of checks to solve such problems - if we can identify a topological invariant that one space has and another does not, they cannot be homeomorphic. So knowing more and more powerful topological invariants is desirable.