HOMOTOPY OF PATHS AND FUNDAMENTAL GROUP

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1. Introduction

We have seen in pointset topology the various properties of 'shape' - these include connectedness, compactness, Hausdorffness, regularness, normalness, metrizability etc. These tell us a lot about spaces and their nature and these provide us with useful topological invariants. It is however still quite difficult to distinguish spaces based on these topological invariants. An example of this is Brouwer's 'invariance of domain' theorem, which approximately says, among other things, that $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic to each other when $m \neq n$. It is still quite difficult to prove this very intuitive-looking result with the tools we have at our disposal till now. Algebraic topology evolved to handle such questions - it puts more invariants at our disposal and these are invariants that are more 'computable' in some sense than just being abstract properties. The trade-off is that we have to weaken the notion of two spaces being homeomorphic to a different notion that gives us more flexibility and working space, so to speak. This notion is known as 'homotopy equivalence'. Today we will begin to study this idea and define a useful invariant, namely, the fundamental group(oid), of a topological space under this weaker concept of equivalence.

2. Homotopy of paths

Homotopy is the concept of continuous deformation, parametrized by the unit interval $I = [0, 1]$.

Definition 2.1. Let X, Y be two topological spaces. Let $f, g : X \to Y$ be two continuous maps between them. A *homotopy* between f and g is a continuous map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$. If such an F exists, we say that f and g are *homotopic* and write $f \simeq g$. If f is homotopic to a constant map, we say it is *nullhomotopic*.

We want to study *paths* in topological spaces, i.e. continuous $f : [0, 1] \to X, f(0) = x_0, f(1) = x_1$. The above notion of homotopy is not very useful if we don't fix the endpoints x_0 and x_1 of these paths, because then every path is homotopic to a constant path. (**Exercise!**) A better notion of 'homotopy of paths' only considers homotopies which keep the end points in place.

Definition 2.2. Two paths $f, g : [0, 1] \to X$ from x_0 to x_1 are *path homotopic* if there exists $F : [0, 1] \times [0, 1] \to X$ such that $F(s, 0) = f(s)$, $F(s, 1) = g(s)$, (so that it is a homotopy) and $F(0, t) = x_0$, $F(1, t) = x_1$ (so that endpoints are fixed). In particular, for all $t \in [0, 1]$, $f_t = F|_{[0,1]\times\{t\}}$ is a path from x_0 to x_1. Such an F is called a *path homotopy* and we write $f \simeq_p g$ for this relation between f and g.

We can think of a homotopy as giving a path between the paths f and g, that is, we can think of it as giving a function $\Gamma : [0, 1] \to \{\text{Paths in } X\}$, where $\Gamma(t) = f_t$. It is possible, with some mild assumptions on X, to make the set of paths in into a topological space in such a way that this means of producing a Γ from an F gives a bijection between homotopies as defined above and "paths between paths", but we will avoid that kind of point-set topological technicality. Instead, we will define homotopies as map from a rectangle $[0, 1] \times [0, 1]$ which gives a nice intuition as paths of paths.

Lemma 2.3. *The relations $\simeq$ and $\simeq_p$ are equivalence relations on paths.*

Proof.
(1) Clearly $f \simeq f$ since the constant homotopy $F(s, t) = f(s)$ for all $t \in [0, 1]$ gives a homotopy from f to itself.
(2) If $f \simeq g$ with homotopy $F(s, t)$, then the reverse homotopy $G(s, t) = F(s, 1 - t)$ gives a homotopy from g to f, so $g \simeq f$.

(3) If $f \simeq g$ and $g \simeq h$ with homotopies $F(s,t)$ and $G(s,t)$, then we can *concatenate* the homotopies : define $H(s,t) : [0,1] \times [0,1] \to X$ such that $H(s,t) = F(s, 2t)$ if $0 \leq t \leq 1/2$, and $H(s,t) = G(s, 2t-1)$ if $1/2 \leq t \leq 1$. These two formulae agree at $t = 1/2$, since $F(s,1) = g(s) = g(s,0)$, and so $H(s,t)$ is well-defined and continuous by the pasting lemma, and gives a homotopy $f \simeq h$.

In the case of path homotopies, we can check that the above constructions preserve the requirements $F(0,t) = x_0$ and $F(1,t) = x_1$ for all t, so yield path homotopies, and thus $\simeq_p$ is also an equivalence relation. ☐

We denote the (path) homotopy equivalence class of f by $[f]$.

Example 2.4. Below are some natural examples and non-examples of path homotopies.

(1) If f and g are paths in $\mathbb{R}^2$ (or any convex subset of $\mathbb{R}^n$ for any $n \geq 1$), we can define the *straight line homotopy* between f and g as-

$$F(s,t) = (1-t)f(s) + tg(s).$$

For each fixed value of s, this connects $f(s)$ with $g(s)$ by a straight line, and this defines a path homotopy by convexity hypothesis. So f and g are homotopic always!

(2) In the punctured plane $X = \mathbb{R}^2 \setminus \{(0,0)\}$, let f, g be paths from $(-1,0)$ to $(1,0)$ such that f stays in the upper half plane $\{(x,y) : y \geq 0\}$ and g stays in the lower half plane $\{(x,y) : y \leq 0\}$. Then there is no homotopy between f and g. We will prove this rigorously later.

3. Fundamental groupoid

Now we prove that the homotopy classes of paths in any space X form a category, in fact, a groupoid. The key operation is what we have seen earlier in a similar context- composition (concatenation) of paths.

Definition 3.1. If f is a path from x to y in a topological space X, and g is a path from y to z, define a path $f * g$ from x to z by running through f first and then through g, twice as fast-

$$(f * g)(s) = f(2s), 0 \leq s \leq 1/2$$

and then

$$(f * g)(s) = g(2s-1), 1/2 \leq s \leq 1.$$

As before, this gives a well-defined path by the pasting lemma.

The point is that this gives a well-defined composition operation on path-homotopy classes as well, as long as $f(1) = g(0)$: If $f \simeq_p f'$ and $g \simeq_p g'$ with homotopies $F(s,t)$ and $G(s,t)$, then $f * g \simeq_p f' * g'$ using the homotopy

$$(F * G)(s,t) = F(2s,t), 0 \leq s \leq 1/2$$

and then

$$(F * G)(s,t) = G(2s-1,t), 1/2 \leq s \leq 1.$$

So we can define

$$[f] * [g] := [f * g].$$

Note that this is still assuming $f(1) = g(0)$, and since all paths in a path-homotopy class have the same endpoints, it doesn't matter which representatives f and g we choose.

Proposition 3.2. *The operation $*$ is associative on path-homotopy classes, and has identity and inverses.*

Proof. As it turns out, associativity is the trickiest, but not very difficult.

(1) **Associativity:** Given paths f, g, h with $f(1) = g(0)$ and $g(1) = h(0)$, the claim is that $(f * g) * h \simeq_p f * (g * h)$. Both these paths run along f, then g, then h, but the parametrizations are different, i.e. the speeds are different. In the LHS, f and g are traversed at constant speed and h is traversed at double that speed, and similarly for RHS, f is traversed at double the speed of g and h. The homotopy comes from adjusting for this: define

$$F(s,t) := f\left(\frac{4s}{1+t}\right), 0 \leq s \leq \frac{1+t}{4}$$

and

$$F(s,t) := g(4s-(1+t)), \frac{1+t}{4} \leq s \leq \frac{2+t}{4}$$

and finally,

$$F(s,t) := h\left(\frac{4s-(2+t)}{2-t}\right), \frac{2+t}{4} \leq s \leq 1.$$

This may feel like a lot, but this is really not much. If you think of what happens at $t=0$ and $t=1$ for $s = 1/4, 1/2, 3/4$, you will be able to combine this information with straight-line homotopies in the variable t that we have kind of seen earlier.

(2) **Identity:** Given $x, y \in X$, and f any path from x to y, we claim that $id_x := [e_x]$ is the identity for this operation, where $e_x : I \to X$ defined as $e_x(s) = x$ for all s is the constant path for the point x. That is, we want to show $[f] * id_y = id_x * [f] = f$. Indeed, there is an explicit homotopy $f \simeq_p f * id_y$ by defining

$$F(s,t) := f\left(\frac{s}{1-t/2}\right), s \in [0, 1-t/2]$$

and

$$F(s,t) = y, s \in [1-t/2, 1].$$

Similarly, there is an explicit homotopy $f \simeq_p (f * id_x)$ by defining

$$G(s,t) := x, s \in [0, t/2]$$

and

$$G(s,t) := f\left(\frac{s-t/2}{1-t/2}\right), s \in [t/2, 1].$$

(3) **Inverses:** Given f a path from x to y, define the reverse path $\bar{f}(s) := f(1-s)$ from y to x. Then we claim that $[\bar{f}]$ is an inverse for $[f]$, i.e. $e_x \simeq_p f * \bar{f}$ and $e_y \simeq_p \bar{f} * f$. We will only do the first case here, leaving the second as an **exercise**. The homotopy in this case is quite easy to write: $F(s,t) := f(2ts)$ for $s \in [0, 1/2]$ and $F(s,t) := f(2t(1-s))$ for $s \in [1/2, 1]$. For any given t, this runs forward in the direction of f from $f(0) = x$ to $f(t)$ at $s = 1/2$, then runs backward from $f(t)$ to $f(0) = x$ at $s = 1$. For $t = 0$, we have the path e_x, and for $t = 1$, we have the path $f * \bar{f}$.

□

We can package all this information in a category $\mathcal{C}$: the objects of $\mathcal{C}$ are points of X, and $\operatorname{Hom}(x, y)$ in $\mathcal{C}$ are simply {homotopy classes of paths from x to y}. This set is empty if x and y are not in the same path-component of X. With the above discussion, it is clear that this is a category, because the equivalence class of the concatenation of two paths only depends on the equivalence classes of the individual paths. This is in fact a groupoid, since all morphisms are invertible and hence isomorphisms. This is called the 'fundamental groupoid' of X.

4. Fundamental Group

Since the fundamental groupoid of any non-empty non-singleton space X has more than one objects, the inability to multiply every pair of paths prevents us from having an actual group structure. To address this, one usually restricts to a single object of the category, i.e. we fix a base point $x_0 \in X$, and only consider paths that go from x_0 to itself - i.e. *loops*, based at x_0.

Definition 4.1. The set of path-homotopy classes of loops based at x_0, with the concatenation operation $*$ is called the *fundamental group* of X for the base point x_0, denoted $\pi_1(X, x_0)$.

Example 4.2. In any convex subset X of $\mathbb{R}^n$ and a point $x_0 \in X$, every loop at x_0 is path-homotopic to the constant path (i.e. the identity) at x_0 by the straight-line homotopy : $F(S,t) = (1-t)f(s) + tx_0$. So $\pi_1(X, x_0) = \{id_{x_0}\}$.

Definition 4.3. A nonempty topological space X is *simply connected* if X is path connected, and for any $x_0 \in X$, $\pi_1(X, x_0) = \{1\}$.

Example 4.4. $\mathbb{R}^n$, convex subspaces of $\mathbb{R}^n$, the n-sphere S^n for $n \geq 2$ (not completely obvious).

We will study the fundamental group and examples of it in detail in the next few lectures.