URYSOHN'S LEMMA AND METRIZATION THEOREM

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1. Introduction

Recall the definitions:

Definition 1.1. A topological space X is *regular* if

(1) the singleton set $\{x\}$ is closed in X for each $x \in X$, and
(2) for each point $x \in X$ and each closed subset $B \subseteq X$, with $x \notin B$, there exist disjoint open subsets $U, V \subseteq X$ with $x \in U$ and $B \subseteq V$.

We then say that U and V separate x and B.

Definition 1.2. A topological space X is *normal* if

(1) the singleton set $\{x\}$ is closed in X for each $x \in X$, and
(2) for each pair of disjoint closed subsets $A, B \subseteq X$ there exist disjoint open subsets $U, V \subseteq X$ with $A \subseteq U$ and $B \subseteq V$.

We then say that U and V separate A and B.

We are going to see the following theorem in the next lecture.

Theorem 1.3. *Every regular second countable space X is metrizable.*

Recall that we have already shown that such a space is normal and so Urysohn metrization theorem is a strengthening of this statement. The key ingredient in this statement is Urysohn's Lemma. It roughly says that for normal spaces, we can separate disjoint closed subsets by real-valued functions, in the following sense:

Theorem 1.4 (Urysohn's Lemma)**.** *Let A and B be disjoint closed subsets of a normal space X. Then There exists a continuous map $f : X \to [0,1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.*

This fits in well with the philosophy that we have been seeing lately, namely, let continuous functions on a space help us in describing the properties of shape of the space. We will first see how to prove this lemma, and then move on to the metrization theorem.

2. Urysohn's Lemma

First recall the following characterization of normal spaces :

Lemma 2.1. *Let X be a T_1 space. Then X is normal if and only if for each closed subset $A \subset X$ and neighborhood W containing A there is a neighborhood U containing A such that $A \subseteq U \subseteq Cl\, U \subseteq W$.*

Now we prove Urysohn's lemma as stated above.

Proof of Urysohn's Lemma. We will give a proof that is a slight variation of the proof in Munkres's book. (He remarks in a footnote about the possibility of such a proof.) But the core idea is the same, namely use normality to construct a certain family U_p of open sets of X, then use these sets to define the required continuous function f. We simply use a different index set which in our opinion is easier to deal with.

We define a *dyadic number* to be a rational number of the form $r = a/2^n$, where a and n integers with $n \geq 0$. Then it follows from the archimedean property that the dyadic numbers are dense in $\mathbb{R}$.

For each dyadic number $0 \leq r \leq 1$ we shall construct an open subset $U_r \subset X$, with $A \subset U_r \subset X \setminus B$, such that for each pair of dyadic numbers $0 \leq p < q \leq 1$ we have $\mathrm{Cl}\ U_p \subseteq U_q$.

Let $U_1 = X \setminus B$. Then $A \subseteq U_1$, so by the above characterization of normal spaces, there exists an open subset U_0 such that $A \subset U_0 \subset \mathrm{Cl}\ U_0 \subset U_1$. Again by the characterization of normal spaces applied to the closed set $\mathrm{Cl}\ U_0$, we can find an open set U_1 such that $U_0 \subset U_{1/2} \subset \mathrm{Cl}\ U_{1/2} \subset U_1$. This is the base case of our induction.

So now let $n \geq 2$ and assume that we have constructed the U_r for all $0 \leq r \leq 1$ of the form $b/2^{n-1} = 2b/2^n$. So all we have to do is to construct U_r for r of the form $(2b+1)/2^n$. By induction we have constructed $U_{2b/2^n}$ and $U_{(2b+2)/2^n}$. Using the above characterisation of normality, we can find an open set, which we name $U_{(2b+1)/2^n}$, such that $\mathrm{Cl}\ U_{2b/2^n} \subset U_{(2b+1)/2^n} \subset \mathrm{Cl}\ U_{(2b+1)/2^n} \subset U_{(2b+2)/2^n}$. Hence, by induction, we have constructed U_r for all dyadic numbers r in $[0, 1]$.

Extend the definition of U_r to all dyadic numbers in r by defining $U_r = \varnothing$ for all $r < 0$ and $U_r = X$ for all $r > 1$. Note that the key property that $\mathrm{Cl}\ U_p \subset U_q$ for all $p < q$ dyadic numbers is still satisfied.

Now let $x \in X$. We define

$$D(x) := \{r \text{ dyadic } : x \in U_r\}.$$

Since $x \notin U_r$ for $r < 0$ and $x \in U_r$ for $r > 1$, $D(x)$ is nonempty and bounded below by 0. So the greatest lowed bound

$$f(x) := \inf D(x)$$

exists and lies in the interval $[0, 1]$.

We have to show that f defined above is the required function. Note that $f(x) = 0$ if $x \in A$ since $x \in U_p$ for every $p > 0$, and $f(x) = 1$ if $x \in B$, since $x \in U_p$ for no $p \leq 1$. So it remains to show that f is continuous.

We first prove that if $x \in \mathrm{Cl}\ U_r$, then $f(x) \leq r$. If $x \in \mathrm{Cl}\ U_r$ then $x \in U_q$ for all $r < q$, so $D(x)$ contains all dyadic numbers greater than r. The dyadic numbers are dense in the reals, so $f(x) \leq r$.

Similarly, we prove that if $x \notin U_r$, then $f(x) \geq r$. If $x \notin U_r$ then $x \notin U_p$ for all $p < r$, so $D(x)$ contains no dyadic numbers less than r. So r is a lower bound for $D(x)$, and $r \leq f(x)$.

Now we prove continuity of f. Let $x \in X$ and consider any neighborhood (c, d) in $\mathbb{R}$ of $f(x)$. We will find a neighborhood U of x with $f(U) \subset (c, d)$. So, pick dyadic numbers p and q with $c < p < f(x) < q < d$. Then $x \notin \mathrm{Cl}\ U_p$, and $x \in U_q$ by the two claims above, so $U = U_q \setminus \mathrm{Cl}\ U_p$ is a neighborhood of x! But then for any $y \in U$, we have $y \notin U_p$ by definition, so $c < p \leq f(y)$, and on the other hand $y \in U_q$ by definition, so $f(y) \leq q < d$. Hence $f(U) \subset (c, d)$ and thus f is continuous. $\square$

3. Hilbert Cube

We briefly recall a homework problem from the time when we studied metric spaces. This will be useful in the proof of Urysohn's metrization theorem.

Definition 3.1. The Hilbert cube is the product

$$H := \prod_{n=1}^{\infty} [0, \frac{1}{n}]$$

in the product topology.

A point $x \in H$ can be viewed as a sequence $(x_n)_{n=1}^{\infty}$ with $0 \leq x_n \leq 1/n$ for each $n \geq 1$. We consider two different metrics on H. The uniform metric $\rho(= d_\infty)$ is given by

$$\rho(x, y) := \sup_{n \geq 1} |x_n - y_n|$$

and the ℓ^2 metric d_2 given by

$$d_2(x,y) := \left(\sum_{n=1}^{\infty}(x_n - y_n)^2\right)^{1/2}.$$

Note that these give well-defined metric functions since $1/n \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} 1/n^2 < \infty$. Then the result we recall is the following :

Proposition 3.2. *The uniform metric ρ and the ℓ^2-metric d_2 define the same topology on the Hilbert cube H as the product topology. In particular, H is metrizable.*

Proof. **Exercise.** (If you have not done it already, please do it now.) (Hint: It is easy to show just from the definitions that a basis element for the product topology contains a basis element for the uniform metric topology, and similarly easy to show that a basis element for the uniform metric contains a basis element for the ℓ^2-metric topology. So it remains to show that a basis element for the ℓ^2-metric topology contains a basis element for the product topology.) □

Corollary 3.3. *The countably infinite product*

$$[0,1]^\omega = \prod_{n=1}^{\infty}[0,1]$$

is metrizable.

Proof. We have the obvious homeomorphisms $[0,1] \xrightarrow{\sim} [0,1/n]$ sending x to x/n. Taking their product we have a homeomorphism

$$[0,1]^\omega \cong H.$$

Since the product topology on H is metrizable, $[0,1]^\omega$ is metrizable. □

In fact, the uniform metric on $[0,1]^\omega$ given by

$$M(x,y) := \sup_{n\geq 1} \frac{|x_n - y_n|}{n}$$

is simply the uniform metric on the Hilbert cube pulled back via this homeomorphism.

4. Urysohn's Metrization Theorem

Theorem 4.1. *Every second-countable regular space is metrizable.*

Proof. Recall that every second-countable regular space is normal. So we can take X to be a normal space with a chosen countable basis $\mathcal{B} = \{B_k\}_{k=1}^{\infty}$.

Claim 1**:** There is a countable collection $\{f_n\}_{n=1}^{\infty}$ of maps $f_n : X \to [0,1]$, such that for any $p \in U \subseteq X$ with U open there is an f_n in the collection, such that $f_n(p) = 1$ and $f_n \equiv 0$ on $X \setminus U$.

Proof of Claim 1*:* This follows from Urysohn's lemma. For each pair (i,j) of indices with $i,j \in \mathbb{N}$ with $B_i \subseteq \mathrm{Cl}\ B_j$, use Urysohn's lemma to choose a map $g_{i,j} : X \to [0,1]$ with $g_{i,j}(B_i) \subseteq 1$ and $g_{i,j}(X \setminus B_j) \subseteq 0$. Then the collection $\{g_{i,j}\}$ satisfies the claim. To see this, consider $p \in U$ open in X. Since $\mathcal{B}$ is a basis, there is a basis element B_j with $p \in B_j \subseteq U$. By regularity, there is an open V with $p \in V \subseteq \mathrm{Cl}\ V \subseteq B_j$, and by the basis property there is a basis element B_i with $p \in B_i \subseteq V$. Then $B_i \subseteq \mathrm{Cl}\ V \subseteq B_j$, so $p \in B_i \subseteq B_j \subseteq U$. Then $g_{i,j}$ as defined above satisfies $g_{i,j}(p) = 1$ and $g_{i,j}(X \setminus U) \subseteq 0$. We reindex $(g_{i,j})_{i,j\in\mathbb{N}}$ to $(f_n)_{n=1}^{\infty}$ by choosing some bijection $\mathbb{N} \times \mathbb{N} \cong \mathbb{N}$.

Now define a map

$$F : X \to [0,1]^\omega$$

simply by the formula

$$F(x) := (f_1(x), f_2(x), \ldots, f_n(x), \ldots).$$

In other words, $\pi_n \circ F = f_n : X \to [0,1]$.

Claim 2: F is an embedding of X into $[0,1]^\omega$.

Proof of Claim 2: F is continuous for the product topology on $[0,1]^\omega$, because each of its component f_n is continuous. F is also injective, for each $x \neq y$, the set $U_y = X \setminus \{y\}$ is open (since X is T_1!) so that there exists an index r such that $f_r(x) = 1$ and f_r is identically zero on $X \setminus U_y$, i.e. $f_r(y) = 0$. Therefore, r-th components of $F(x)$ and $F(y)$ are different and hence $F_r(x) \neq F(y)$.

It remains to be shown that $F : X \to Z := F(X)$ is a homeomorphism. Since we have shown it is a continuous bijection $X \to Z$, what we need to prove is that if $U \subset X$ is open, then $F(U) \subset Z$ is open.

So let $U \subset X$ be any open set and let $x_0 \in U$ be any element. Then there exists index n such that $f_n(x_0) > 0$ and $f_n \equiv 0$ on $X \setminus U$. Let $V_n := \pi_n^{-1}((0,1]) \cap Z$, that is,

$$V_n := \{z = (z_1, z_2, \ldots) \in Z : z_n > 0\}$$

and thus V_n is an open subset of Z. Then $x_0 \in F^{-1}(V_n) \subseteq U$, since $f_n(x_0) > 0$ and since $f_n(x) > 0 \implies x \in U$ by construction. Therefore, $F(X_0) \in V_n \subseteq F(U)$, so what we have shown is that for any element $F(x_0)$ of the set $F(U)$, there exists an open subset V_n around $F(x_0)$ that is entirely contained in $F(U)$, that is, $F(U)$ is open. Since U was arbitrary, this finishes the proof.

$\square$

Note that the last construction can be used to show that if normal space X has a basis indexed by a set J, we can embed X as a subspace of $\mathbb{R}^J$. But for uncountable J, $\mathbb{R}^J$ is not metrizable.