STUDYING FUNDAMENTAL GROUP YET ANOTHER WAY

ADITYA KARNATAKI

1. Introduction

We have seen a couple of powerful ways to approach computations of fundamental groups. First, we saw covering spaces. These allowed us to compute $\pi_1(S^1)$ and gain some insights on how loops in a space behave. Later we saw the idea of a deformation retract which allowed us to study the relation between the fundamental group of a space X and that of a subspace A that exists very specially inside the space. This naturally led to the idea of homotopy equivalence, which is a very important area of research even now.

Today, we are going to see a third invaluable tool to study fundamental groups. Namely, given any *arbitrary* subspaces A and B that together make X, we can often say something about the fundamental group of X in terms of the fundamental groups of A and B. The most general statement of this type is known as the 'Seifert-van Kampen theorem'. It is most naturally expressed by saying that 'the fundamental groupoid functor preserves certain colimits' but of course in this course we have not reached a stage where we can make much sense of it. If you are interested, you should try to find out how to make sense of this statement and its meaning.

2. Fundamental groups of S^n

The goal is to prove the following theorem.

Theorem 2.1. *Suppose that $X = U \cup V$ is covered by two open, simply-connected subspaces U and V, and that $U \cap V$ is path connected. Then X is simply-connected.*

In fact, what we will end up proving is a stronger version of this statement.

Theorem 2.2. *Suppose that $X = U \cup V$, where U and V are open, path connected subsets of X. Suppose that $U \cap V$ is path connected and that $x_0 \in U \cap V$. Let i and j be the two inclusion maps of U and V into X, respectively. Then the images of the induced homomorphisms*

$$i_* : \pi_1(U, x_0) \to \pi_1(X, x_0), j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$$

generate $\pi_1(X, x_0)$ as a group.

Proof. If we unravel all the words and definitions, this theorem is saying in concrete terms that given any loop f in X based at x_0, it is path homotopic to a product of the form $(g_1 * (g_2 * (\cdots g_n)))$, where each g_i is a loop in X based at x_0 that lies either in U or in V.

First of all, note that under the hypotheses, X is path connected. Let f be a loop in X based at x_0. We show that there is a finite subdivision $a_0 < a_1 < \cdots < a_n$ of the unit interval such that $f(a_i) \in U \cap V$ and $f([a_{i-1}, a_i])$ is contained in either U or V for each i. For this, choose a subdivision $b_0 < b_1 < \cdots b_m$ of $[0, 1]$ such that for each i, the set $f([b_{i-1}, b_i])$ is contained in either U or V. This exists because of the Lesbegue number lemma. Namely, the sets $f^{-1}(U)$ and $f^{-1}(V)$ cover the compact set $[0, 1]$. So by the Lesbegue number lemma, there exists a finite partition $b_0 < b_1 < \cdots b_m$ of $[0, 1]$ such that $[b_{i-1}, b_i]$ is contained in either $f^{-1}(U)$ or $f^{-1}(V)$ (or both), i.e. f maps $[b_{i-1}, b_i]$ into either U or V. If $f(b_i)$ belongs to $U \cap V$ for each i, we are done, as $\{b_i\}$ is then the required subdivision $\{a_j\}$. Otherwise, let i be an index such that $f(b_i) \notin U \cap V$. Then, each of the sets $f([b_{i-1}, b_i])$ and $f([b_i, b_{i+1}])$ lies either in U or V by definition. If $f(b_i) \in U$, then both of these must lie in U, and if $f(b_i) \in V$, then both of these must lie in V. In either case, deleting b_i gives a new subdivision $c_0 < c_1 < \cdots c_{m-1}$ which still has the property that $f[(c_{i-1}, c_i])$ is contained in either U or V for each i. This process must terminate after finite number of repetitions, and in the end we would have deleted all points causing the issue, so we get the desired subdivision $a_0 < a_1 < \cdots < a_n$.

Given this subdivision, we can now prove the theorem. Define f_i to be the path in X that is given by mapping $[0,1]$ to $[a_{i-1}, a_i]$ and then applying f. Then f_i is a path that lies in either U or V for each i. Further, by simple concatenation of paths, we have

$$[f] = [f_1] * [f_2] * \cdots * [f_n].$$

Since $U \cap V$ is path connected, we can and do choose a path $\alpha_i \in U \cap V$ from x_0 to α_i. For $i = 0$ and n, we use the constant path at x_0, since $f(a_0) = f(a_n) = x_0$ as f is a loop based at x_0.

Then we define

$$g_i = (\alpha_{i-1} * f_i) * \bar{\alpha}_i$$

for each i. Then each g_i is a loop in X that is based at x_0 whose image lies either in U or V. Then we see that the expression

$$[g_1] * [g_2] * \cdots * [g_n]$$

telescopes to

$$[f_1] * [f_2] * \cdots * [f_n] = [f]$$

and so the LHS gives the required construction. □

Earlier theorem is a corollary of this.

Proof of Theorem 2.1. Any homomorphism from the trivial group to any group has trivial image. □

Now finally we can compute the fundamental groups of spheres.

Theorem 2.3. *If $n \geq 2$, the n-sphere S^n is simply connected.*

Proof. Most of you might know the stereographic projection, but let us define it anyway for the sake of completeness. Let $N = (0, 0, \ldots, +1)$ and $S = (0, 0, \ldots, -1)$ be the north and south pole of S^n, respectively. Let $U = S^n \setminus \{S\}$ and $V = S^n \setminus \{N\}$ be open neighborhoods of the upper and lower hemisphere, respectively. There is a homeomorphism $h : U \cong R^n$ given by straight-line projection from the south pole, called 'stereographic projection'. In coordinates,

$$h(x_1, x_2, \ldots, x_{n+1}) = \frac{1}{1 + x_{n+1}}(x_1, x_2, \ldots, x_n)$$

for $(x_1, x_2, \ldots, x_n) \in U$. The inverse h^{-1} is given by

$$h^{-1}(y_1, y_2, \ldots, y_n) = (0, 0, \ldots, -1) + t(y)(y_1, y_2, \ldots, y_n, 1)$$

for $y = (y_1, y_2) \in \mathbb{R}^2, t(y) = \frac{2}{1+\|y\|^2}$.

Similarly, there is a homeomorphism $V \cong \mathbb{R}^n$. The intersection $U \cap V$ corresponds under h to $\mathbb{R}^n \setminus \{0\}$, which is path connected. Therefore, the previous corollary applies, since U and V are both simply connected, and we get $\pi_1(S^n) \cong \{1\}$.

□

Now we see the fundamental group of the real projective space as a corollary.

Theorem 2.4. $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2$ *for $n \geq 2$.*

Proof. Let us recall from a past homework that $p : S^n \to \mathbb{RP}^n$ is a covering map. This map is a quotient map, given by identifying each point x of S^n with its antipodal point $-x$. This is an open map, since if $U \subseteq S^n$ is an open in S^n. The map $a : x \to -x$ is a homeomorphism of S^n. Thus, $a(U)$ is open in S^n. Then $p^{-1}(p(U)) = U \cup a(U)$ implies that $p^{-1}(p(U))$ is open, hence by definition of a quotient map, $p(U)$ is open, too. So p is an open map. (Note that a similar argument shows that p is a closed map.) Now, for any point $y \in \mathbb{RP}^n$, choose $x \in p^{-1}(y)$. Then choose a small ϵ-neighbourhood U of x in $\mathbb{RP}^n$ such that it does not contain any pair of antipodal points $z, a(z)$. This is possible since $d(z, a(z)) = 2$ in the Euclidean metric, so taking $\epsilon < 1$ sufficiently small, we can do this. Therefore, $p : U \to p(U)$ is bijective. As it is continuous and open, it is a homeomorphism. Similarly, $p : a(U) \to p(a(U)) = p(U)$ is also a homeomorphism. Then $p(U)$ is a neighbourhood of $p(x) = y$ that is evenly covered by p and since y was arbitrary, p is a covering map.

Since S^n is simply connected, we can apply a previous theorem to see that there is a bijective correspondence between $\pi_1(\mathbb{RP}^n, y)$ and the preimage $p^{-1}(y)$. But the latter only has 2 elements by construction. Since $\pi_1(\mathbb{RP}^n, y)$ is a group, it is a group with 2 elements in it. But there is only one such group, namely, $\mathbb{Z}/2$. $\square$

We can have even finer applications of this theorem.

Example 2.5. Let X be the figure eight space. Then we can cover X by two open subsets U and V which deformation retract to S^1 and such that the intersection $U \cap V$ is simply connected. So the theorem implies that $\pi_1(X)$ is generated by the image of two maps from $\pi_1(S^1) \cong \mathbb{Z}$, i.e. we can express every loop in terms of $[a]$ and $[b]$, where $[a]$ and $[b]$ are loops that go around each copy of S^1 once, i.e. they generate $\pi_1(U)$ and $\pi_1(V)$. So every element in $\pi_1(X)$ can be written as $[h_1]^{n_1} * \cdots * [h_k]^{n_k}$ where each $h_i \in \{a, b\}$. But we don't know what relations are satisfied by $[a]$ and $[b]$.

The full strength of Seifert-van Kampen theorem allows us to answer this question. For now, we can prove that this group is not abelian. To see this, look at a particular cover (figure 60.1 in Munkres). The lift of $[a] * [b]$ starting at e_0 ends at $(1, 0)$ while the lift of $[b] * [a]$ starting at e_0 ends at $(0, 1)$, so by uniqueness of liftings, we see that $[a] * [b] \neq [b] * [a]$.

We will further explore fundamental groups in the next lecture.