HOMEWORK 22

Note: $*$ marked problems might be slightly more difficult or interesting than the unmarked ones.

(1)* Let $X = \mathbb{R}^{n+1} \setminus \{0\}$ where 0 denotes the origin. Define an equivalence relation $\sim$ on X by $x \sim y$ if and only if there exists $\alpha \in \mathbb{R}, \alpha \neq 0$, such that $x = \alpha y$. (You should mentally check that this is indeed an equivalence relation.) The quotient space $X/\sim$ is called the n-dimensional real projective space $\mathbb{RP}^n$.

(i) Show that $\mathbb{RP}^n$ is homeomorphic to the quotient space $S^n/\cong$ where $a \cong b$ if and only if $a = \pm b$ in S^n, i.e. $\cong$ is the equivalence relation that identifies antipodal points of S^n.

(ii) Show that the map $S^n \to \mathbb{RP}^n$ is a covering map.

(2)* Let $p : E \to B$ is a covering map. Define the *fiber product* as a topological subspace of the product $E \times E$ by the following definition

$$E \times_B E := \{(e_1, e_2) \in E \times E : p(e_1) = p(e_2)\}.$$

- Show that the image of E under the diagonal map

  $$\Delta : E \to E \times_B E$$

  given by

  $$\Delta(e) := (e, e)$$

  is both open and closed as a subset of $E \times_B E$.
- Using the above assertion as an input, give another proof of the following assertion on unique liftings- Let Y be a connected space and $f : Y \to X$ is a continuous map, with $\tilde{f}_1$ and $\tilde{f}_2$ two lifts of f to E. Then if there exists a point y such that $\tilde{f}_1(y) = \tilde{f}_2(y)$, then $\tilde{f}_1 = \tilde{f}_2$. (Unique liftings of paths and path homotopies is a special case of this.)

(3) Topology (Munkres), Chapter 9, Section 54, Exercise (3).

(4) Topology (Munkres), Chapter 9, Section 54, Exercise (6).

(5) Topology (Munkres), Chapter 9, Section 54, Exercise (8).