COVERING SPACES AND LIFTING CORRESPONDENCE

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We have seen that convex subsets of $\mathbb{R}^n$ have a trivial fundamental group at any basepoint. At some point we would like to compute more examples, specially where the fundamental groups are not trivial. In particular, we would like to compute $\pi_1(S^1)$. We will do this by introducing a key tool for the study of π_1 : the notion of a *covering space*.

Definition 0.1. Let $p : E \to B$ be a continuous surjective map. We say p *evenly covers* an open subset $U \subseteq B$ if

$$p^{-1}(U) = \bigsqcup_{\alpha \in A} V_\alpha,$$

where A is some indexing set, $V_\alpha \subseteq E$ are disjoint open subsets, and for each $\alpha \in A$, $p|_{V_\alpha} : V_\alpha \in U$ is a homeomorphism. The sets V_α are called *slices*.

Equivalently, there exists a 'commutative diagram'

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\sim", "\varphi"'] \arrow[dr, "p|_U"'] & & U \times A \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

where A carries the discrete topology, and $p|_U = \pi_1 \circ \varphi$.

Definition 0.2. Let $p : E \to B$ be a continuous surjective map. If every point of B has a neighbourhood that is evenly covered by p, then we say E is a *covering space* of B and p is a *covering map*. B is called the *base* of the covering.

Note that the most trivial examples of covering spaces can be simply constructed from the definition: $id : X \to X$ is trivially a covering map. Similarly, we can take finitely many disjoint copies of X indexed by say $\{1, 2, \dots, n\}$ and take $E = X \times \{1, 2, \dots, n\}$ and the map $p(x, i) = x$ for all $i = 1, 2, \dots, n$ is again trivially a covering map. Munkres calls these examples as 'a stack of pancakes over X'. In practice, we often restrict ourselves to covering spaces that are path connected, under which hypothesis such examples cannot occur.

Example 0.3. Let $p : \mathbb{R} \to S^1$ be the map $t \to (\cos 2\pi t, \sin 2\pi t)$. Then p is a covering map.

This follows from basic properties of cosine and sine functions. We illustrate it briefly by way of an example here, inviting the student to take a look at Theorem 53.1 in Munkres's book for a complete argument. For instance, consider $(1, 0) \in S^1$ and consider its neighbourhood $U := \{(x, y) \in S^1 : x > 0\}$. Then

$$p^{-1}(U) = \bigsqcup_{n \in \mathbb{Z}} \left(n - \frac{1}{4}, n + \frac{1}{4} \right).$$

Then it is not too difficult to show that if we denote $(n - \frac{1}{4}, n + \frac{1}{4})$ as V_n, then $p|_{V_n}$ is a homeomorphism of V_n with U.

We now record some basic properties of covering maps.

Proposition 0.4. *If $p : E \to B$ is a covering map, then p is an open map.*

Proof. By definition of a covering space, there exists an open cover $\{U_i\}_{i \in I} \subseteq B$ such that $p^{-1}(U_i) \cong U_i \times A$. Since $p|_{U_i}$ is given by the projection π_1 under this identification and projections out of a product are open maps, it follows that p is an open map when restricted to any of the $p^{-1}(U_i)$. But any general open set $V \subseteq E$ is a union of its restrictions to these subspaces :

$$V = \bigcup_{i \in I} (V \cap p^{-1}(U_i)).$$

It follows that

$$p(V) = \bigcup_{i \in I} p(V \cap p^{-1}(U_i))$$

is a union of open sets and therefore open. $\square$

Proposition 0.5. *Let $p : E \to B$ a covering map. if B_0 is a subspace of B, and if $E_0 = p^{-1}(B_0)$, then the map $p_0 := p|_{E_0} : E_0 \to B_0$ is a covering map.*

Proof. **Exercise.** $\square$

Proposition 0.6. *If $p : E \to B$ and $p' : E' \to B'$ are covering maps, then $p \times p' : E \times E' \to B \times B'$ is a covering map.*

Proof. Given $b \in B, b' \in B'$ with $U \ni b, U' \ni b'$, such that $p^{-1}(U) = \sqcup_{\alpha \in J} V_\alpha$ and $(p')^{-1}(U) = \sqcup_{\beta \in J'} V'_\beta$, we have $(p \times p')^{-1}(U \times U') = p^{-1}(U) \times (p')^{-1}(U') = \sqcup_{\alpha,\beta}(V_\alpha \times V'_\beta)$. $\square$

Example 0.7. This example explicitly shows how preceding properties work for us. Consider the 'torus' $T := S^1 \times S^1$. Let $p : \mathbb{R} \to S^1$ denote the covering map $t \to (\cos 2\pi t, \sin 2\pi t)$ that we saw earlier. Then we know from previous proposition that $\mathbb{R}^2$ is a covering space for $T = S^1 \times S^1$. Each square $[n, n+1] \times [m, m+1] \subseteq \mathbb{R}^2$ gets wrapped by $p \times p$ entirely around the torus.

Let b_0 denote the point $(1, 0)$. Let B_0 denote the subspace $(S^1 \times b_0) \cup (b_0 \times S^1) \subset S^1 \times S^1$. Then B_0 is the union of two circles with a common point (b, b). This is called the 'figure-eight space'. Then the space $E_0 = p^{-1}(B_0)$ is the 'infinite grid' $(\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R})$. The map $p_0 : E_0 \to B_0$ obtained by restricting $p \times p$ is thus a covering map. But it would not be as straightforward to think of the infinite grid as a covering space for the figure eight space just out of the blue!

What do covering spaces have to do with homotopy of paths and fundamental groups? The connection is given by the idea of *lifts*. We give the general definition first.

Definition 0.8. Given $p : E \to B$ a continuous map, a *lifting* of a continuous map $f : X \to B$ is a map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$.

The point is that if $p : E \to B$ is a covering map, you can always *locally lift* any f, namely, if f is such that $f(X) \subseteq U \subseteq B$ and U is evenly covered by p, then we can lift f to one of the slices in E above U. What is interesting is that for a covering space, *paths* can always be lifted; what's more, even *path homotopies* can always be lifted.

Example 0.9. Recall the covering map $p : \mathbb{R} \to S^1, t \to (\cos 2\pi t, \sin 2\pi t)$. Consider the path $f(s) = (\cos 2\pi s, \sin 2\pi s) : [0, 1] \to S^1$. Then this path has many lifts to $\mathbb{R}$ corresponding to each interval $[n, n+1] \in \mathbb{R}$, if 0 is mapped to n in $\mathbb{R}$.

The next theorem says that this is the only way in which one can have many lifts.

Theorem 0.10. *Let $p : E \to B$ be a covering map. Let $f : [0, 1] \to B$ be a path with $f(0) = b$. Let $e \in p^{-1}(b)$. Then there exists a unique lift $\tilde{f} : [0, 1] \to E$ of the path f such that $\tilde{f}(0) = e$.*

1. Lifting properties of covering spaces

Recall that we arrived at the following lifting proprty in the previous class.

Theorem 1.1. *Let $p : E \to B$ be a covering map. Let $f : [0, 1] \to B$ be a path with $f(0) = b$. Let $e \in p^{-1}(b)$. Then there exists a unique lift $\tilde{f} : [0, 1] \to E$ of the path f such that $\tilde{f}(0) = e$.*

Proof. As we have seen previously, there is an open covering $\{U_\alpha\}_{\alpha \in J}$ of B such that each U_α is evenly covered by p. Then the preimages $\{f^{-1}(U_\alpha)\}_{\alpha \in J}$ yield an open covering of $[0,1]$, so by the Lesbegue number lemma, there exists $\delta > 0$ such that for all x, we have $(x, x+\delta) \subseteq f^{-1}(U_\alpha)$ for some α. Hence we can find a finite subdivision $0 = s_0 < s_1 < \cdots < s_n = 1$ such that each $f([s_i, s_{i+1}])$ lies in one of the U_α.

Define $\tilde{f}(0) = \tilde{f}(s_0) = e$. Assume we have defined $\tilde{f}(s)$ for $s \in [0, s_i]$. We define $\tilde{f}(s)$ for $s \in [s_i, s_{i+1}]$ as follows. We know $f([s_i, s_{i+1}]) \subseteq U$ for some U that is evenly covered by p, so that $p^{-1}(U) = \sqcup_{j \in I} V_j$. Take the slice V out of these V_j which contains $\tilde{f}(s_i)$. Then $p|_V : V \to U$ is a homeomorphism. So it has a continuous inverse and thus we can define (and indeed don't have much choice in defining) $\tilde{f}(s) := p|_V^{-1}(f(s))$ for $s \in [s_i, s_{i+1}]$. This extends $\tilde{f}$ continuously over $[s_i, s_{i+1}]$. Repeating this argument finitely many times, we get a lift $\tilde{f} : [0,1] \to E$ of f. This lift is unique, since for each s_i there was a unique slice containing it and a unique way of lifting $f|_{[s_i, s_{i+1}]}$ into it. (For a rigorous argument saying this, take a look at Lemma 54.1 in Munkres's book.) □

As we said in the previous lecture, path homotopies lift under covering maps as well.

Theorem 1.2. *Let $p : E \to B$ be a covering map. Let $F : [0,1] \times [0,1] \to B$ be a continuous map with $F((0,0)) = b$. Let $e \in p^{-1}(b)$. Then there exists a unique lift $\tilde{F} : [0,1] \times [0,1] \to E$ such that $\tilde{F}((0,0)) = e$.*

Proof. The proof is exactly in the same fashion as the previous proof, subdividing $[0,1] \times [0,1]$ into squares of side length $< \delta$ which map into open subsets of B that are evenly covered by p, then calculating the lift $\tilde{F}$ one square at a time. You are invited to take a look at Lemma 54.2 in Munkres's book to see the complete rigorous argument.

Note that if F is a path homotopy, then $\tilde{F}$ is a path homotopy. To see this, note that F carries the entire left edge $\{0\} \times [0,1]$ of the square into a single point $b_0 \in B$. Since $\tilde{F}$ is a lifting, this edge is carried to $p^{-1}(b_0)$. This set has the discrete topology, so in particular it is totally disconnected. Since $\{0\} \times [0,1]$ is connected, and $\tilde{F}$ is continuous, this implies that $\tilde{F}(\{0\} \times [0,1])$ must equal a singleton set. The same argument for $\{1\} \times [0,1]$ shows that $\tilde{F}$ is a homotopy. □

Remark 1.3. Note that, on the other hand, loops don't always lift to loops! For instance, the loop $f : [0,1] \to S^1$ given by $f(s) = (\cos 2\pi s, \sin 2\pi s)$ does not lift to a loop in $\mathbb{R}$ under the covering map p. But this is not a bug of the system, but a feature, as we will shortly see.

Loops don't necessarily always lift to loops, but given a starting point $e_0 \in p^{-1}(b_0)$, there is a uniquely determined endpoint of any lift, since such liftings are unique. So we have a map

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$$

given by $\phi([f]) := \tilde{f}(1)$ where $\tilde{f}$ is the lift of f such that $\tilde{f}(0) = e_0$. Note that we have to justify that it is well-defined! This amounts to saying that if $f \simeq_p g$, then $\tilde{f}(1) = \tilde{g}(1)$ and there is a path homotopy between f and g. This follows from the previous discussion: if F is a path homotopy between f and g, then its lifting $\tilde{F}$ is a path homotopy between them, and thus $\tilde{f}(1) = \tilde{g}(1)$.

Definition 1.4. Let $p : E \to B$ be a covering map, and let $b_0 \in B$. Let $e_0 \in p^{-1}(b_0)$. Then the function of sets $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ given by $\phi([f]) = \tilde{f}(1)$ is called the *lifting correspondence* derived from the covering map p. Note that this depends on the choice of e_0.

Theorem 1.5. *Let $p : E \to B$ be a covering map, and let $b_0 \in B$. Let $e_0 \in p^{-1}(b_0)$. if E is path connected, then the lifting correspondence*

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$$

is surjective. If E is simply connected, it is bijective.

Example 1.6. For the covering $p : \mathbb{R} \to S^1$, taking $b_0 = (1,0)$ and $e_0 = 0 \in \mathbb{R}$, if a path f loops around the circle for k times, then its lift $\tilde{f}$ ends at k. This gives a map $\varphi : \pi_1(S^1, b_0) \to \mathbb{Z}$ that is surjective.

Proof. If E is path connected, for any $e_1 \in p^{-1}(b_0)$, there is a path $\tilde{f}$ in E from e_0 to e_1. Then $f := p \circ \tilde{f}$ is a loop in B based at b_0 and $\phi([f]) = e_1$ by definition.

If E is simply connected, let $[f]$ and $[g]$ be two elements of $\pi_1(B, b_0)$ such that $\phi([f]) = \phi([g])$. Let $\tilde{f}$ and $\tilde{g}$ be the unique liftings of f and g respectively to paths in E that begin at the point e_0. Then $\tilde{f}(1) = \tilde{g}(1)$. Since E is simply connected, there exists a path homotopy $\tilde{F}$ between $\tilde{f}$ and $\tilde{g}$ since their endpoints are the same. Then $p \circ \tilde{F}$ is a path homotopy between the paths f and g in B, i.e. $[f] = [g]$ and so the map is bijective. □

Theorem 1.7. *The fundamental group $\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$ as a group under addition.*

Proof. Let $p : \mathbb{R} \to S^1$ be the covering map we have defined earlier, let $e_0 = 0$, $b_0 = p(e_0)$. Then $p^{-1}(b_0)$ is the set of integers $\mathbb{Z}$. Since $E = \mathbb{R}$ is simply connected, the lifting correspondence $\phi : \pi_1(S^1, b_0) \to \mathbb{Z}$ is bijective. So we need to show that it is a homomorphism, and we will be done.

Given $[f], [g] \in \pi_1(S^1, b_0)$, let $\tilde{f}$ and $\tilde{g}$ denote their respective liftings to paths in $\mathbb{R}$ beginning at 0. Let $\tilde{f}(1) = n$ and $\tilde{g}(1) = m$, so that $\phi([f]) = n$ and $\phi([g]) = m$. Then define a new path $h(s) := n + \tilde{g}(s)$ on $\mathbb{R}$. This is a lift of the path g starting at $n = \tilde{f}(1)$. Then $\tilde{f} * h$ is a well-defined path in $\mathbb{R}$ and it goes from $0 = \tilde{f}(0)$ to $n+m = n+\tilde{g}(1)$. Thus it is the (unique) lifting of the path $f * g$ that begins at 0. Then since the endpoint of this path is $n + m$, we have $\phi([f] * [g]) = n + m = \phi([f]) + \phi([g])$, which finishes the proof. □

Next time we will explore consequences of this computation.