HOMOTOPY EQUIVALENCES AND SOME CONSTRUCTIONS

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1. Introduction

We have seen the notion of homotopy equivalences, which is a weakening of the notion of a homeomorphism. This allows us to understand the 'homotopy type' of a space better. To this end, we reached the following theorem in the previous class.

Theorem 1.1. *Let $f : (X, x_0) \to (Y, y_0)$ be a continuous map. If f is a homotopy equivalence, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.*

Today we will see a proof of this. We will also see some more concepts related to contractible spaces.

2. Homotopy equivalences give isomorphic fundamental groups

Recall that we proved the following lemma :

Lemma 2.1. *Let $h, k : X \to Y$ be continuous maps of topological spaces. Let h and k be such that they are homotopic to each other under homotopy H, but the endpoint does not necessarily stay fixed, say $h(x_0) = y_0$ and $k(x_0) = y_1$. Then if $\alpha(t) := H(x_0, t)$ is the path defined by H from y_0 to y_1 and $\hat{\alpha} : \pi_1(Y, y_0) \xrightarrow{\sim} \pi_1(Y, y_1)$ is induced by α, then $k_* = \hat{\alpha}(h_*)$.*

The reason we stated it in this generality is because homotopy inverse of a homotopy equivalence may not keep the base point fixed. Now we use this to prove the theorem about fundamental groups.

Proof of Theorem 1.1. Let $g : Y \to X$ be the homotopy inverse of f, and let $g(y_0) = x_1$. Let $f(x_1) = y_1$. Consider the sequence of maps

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (X, x_1) \xrightarrow{f} (Y, y_1).$$

This gives rise to a sequence of group homomorphisms

$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, y_0) \xrightarrow{g_*} \pi_1(X, x_1) \xrightarrow{(f_*)'} \pi_1(Y, y_1)$$

where we are putting a $'$ on the latter f_* to denote that its domain and codomain are different than the former f_*.

Now, we have $g \circ f : (X, x_0) \to (X, x_1)$ homotopic to the identity map on X. So by the previous lemma applied to $h = g \circ f$ and $k = id_X$, we have

$$(g \circ f)_* = \hat{\alpha} \circ id_{\pi_1(X)} = \hat{\alpha}$$

which is an isomorphism. Hence, g_* must be surjective, since $(g \circ f)_* = g_* \circ f_*$.

Similarly,

$$(f_*)' \circ g_* : \pi_1(Y, y_0) \to \pi_1(Y, y_1)$$

is an isomorphism. Hence, g_* must be injective.

Hence, g_*, which is a group homomorphism that is both injective and surjective, must be an isomorphism. Therefore, $f_* = (g_*)^{-1} \circ \hat{\alpha}$ is also an isomorphism. $\square$

Note that even if g is a homotopy inverse of f, g_* is not an inverse for the homomorphism f_*.

We define a basic but very useful notion. This says that any space can be embedded into a contractible space.

Definition 2.2. The (unreduced) cone $\tilde{C}X$ is defined to be the quotient space

$$\tilde{C}X := \frac{X \times I}{X \times \{1\}}$$

and let $j_X : X \hookrightarrow \tilde{C}X$ by sending $x \to [(x, 0)]$ where we use [] to denote equivalence classes in the quotient space $\tilde{C}X$.

Lemma 2.3. *$\tilde{C}X$ is a contractible space.*

Proof. We can contract $\tilde{C}X$ to its vertex via the (what else?) straight line homotopy-

$$F_s([x, t]) := [x, (1 - s)t + s].$$

□

Lemma 2.4. *j_X is a closed embedding.*

Proof. **Exercise.** □

A consequence of this construction is a very useful fact.

Lemma 2.5. *A map $f : X \to Y$ is null homotopic if and only if it extends over $\tilde{C}X$.*

Proof. If f is null homotopic, then there exists a homotopy $H : X \times I \to Y$ to a constant map $c(x) := y_0$. Then $H_1(X) \subseteq \{y_0\}$, so that the homotopy factors through the quotient $\frac{X \times I}{X \times \{1\}}$ to give a map $\tilde{f} : \tilde{C}X \to Y$ that satisfies $\tilde{f} \circ j_X = H_0 = f$. Note that the continuity of $\tilde{f}$ follows from the continuity of H and the fact that $\tilde{C}X$ has the quotient topology. (Check this!)

The reverse direction is also similar: if F is an extension of f to $\tilde{C}X$, then the homotopy $H(x, t)$ is defined to be

$$H(x, t) := F([x, t]).$$

This is a homotopy between f (at $t = 0$) and the constant map to $F([x, 1])$ (at $t = 1$). (ANY x , since they are all equivalent.) □

Another thing to note is that the cone construction is functorial. That is, say we have $f : X \to Y$ a continuous map. Then, we have $f : X \xrightarrow{f} Y \xrightarrow{'} \tilde{C}Y$ such that the composition $j_Y \circ f$ is null homotopic, since $\tilde{C}Y$ is contractible. So, by previous lemma, $j_Y \circ f$ induces a continuous map $\tilde{C}f : \tilde{C}X \to \tilde{C}Y$ that is an extension of $j_Y \circ f$, i.e. we have the following commutative diagram-

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "j_X"'] & Y \arrow[d, "j_Y"] \\ \tilde{C}X \arrow[r, "\tilde{C}(f)"] & \tilde{C}Y \end{tikzcd}$$

Lemma 2.6. *Cone $\tilde{C}$ is a functor from $\underline{Top}$ to $\underline{Top}$.*

Proof. **Exercisse.** You have to check that- (i) Firstly, $\tilde{C}(id) = id_{\tilde{C}X}$ and (ii) if $g : Y \to Z$ is a continuous map, $\tilde{C}(g \circ f) = \tilde{C}(g) \circ \tilde{C}(f)$ as maps from $\tilde{C}X$ to $\tilde{C}Z$. These are both straightforward from the definitions. □

Note that cones can defy intuition sometimes. An exercise today will make this precise.

A related notion is that of a suspension.

Definition 2.7. The (unreduced) suspension $\tilde{S}X$ is the quotient space

$$\tilde{S}X := \frac{X \times I}{X \times \{0\}, X \times \{1\}}.$$

Suspensions are more difficult to understand, since they may not be contractible or anything. In fact, this is a good way of understanding more complex spaces by seeing them as suspensions of some other spaces. For example, the unreduced suspension of S^1 is the sphere S^2! In fact, the unreduced suspension of S^n is S^{n+1}. It is possible, at least in some cases, to understand the relationship between homotopy groups of pointed spaces and their suspensions. Freudenthal suspension theorem is one such statement. We won't study it in this course. But cones and suspensions and other such constructions are a great tool to study algebraic topology. We will, however, study fundamental groups of spheres using another powerful method later.