COMPACTNESS IN METRIC SPACES

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1. Introduction

There is a story about Sir Michael Atiyah and Graeme Segal giving an oral exam to a student. The poor student was a nervous wreck, and it got to a point where he could hardly answer any questions at all. At one point, Atiyah (endeavouring to be kind) asked the student to give an example of a compact set. The student said: "The real line." Trying to play along, Segal said: "In what topology?"

Today we will first explore compactness in metric spaces. This will lead to some interesting results and concepts that we can then generalise.

2. Compactness in metric spaces

Theorem 2.1. *Each closed interval $[a, b]$ in $\mathbb{R}$ is compact.*

Proof. We will give a proof a bit different in flavour from the usual one. You should also look up the proof in Munkres's book and understand it well.

We can assume $[a, b] = [0, 1]$ since compactness is a topological invariant. Let $\mathcal{C} := \{U_\alpha\}_{\alpha \in J}$ be an open cover of $[0, 1]$. Then define the set $A := \{x \in [0, 1] : [0, x]$ has a finite subcover by members of $\mathcal{C}\}$. A is nonempty since $0 \in A$. We will show that A is both open and closed. This will imply that $A = [0, 1]$ by connectedness of A.

If a finite subcover works for $[0, x]$, then $x \in U_\alpha$ for some α, where U_α is a member of the finite subcover. U_α is open, so there exists $B_r(x) \subseteq U_\alpha$ for some $r > 0$. Then $x \in B_r(x) \subseteq A$, since the same finite subcover will work for members of $B_r(x)$. So A is open.

To show that A is closed, suppose y is a limit point of A. Then y is in some U_β and so $B_r(y) \subseteq U_\beta$ for some $r > 0$. There exists a $z \in [y - \frac{r}{2}, y)$ such that $[0, z]$ admits a finite subcover, since y is a limit point of A. Then if we just append U_β to this finite subcover, we get that $[0, y]$ admits a finite subcover too, and thus $y \in A$. Since A contains all its limit points, it is closed, and we are done. $\square$

Corollary 2.2. *Any finite product of closed intervals $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ is compact.*

Theorem 2.3 (Heine-Borel Theorem)**.** *A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and bounded in any of the (equivalent) metrics coming from the euclidean norm.*

Proof. Suppose $A \subset \mathbb{R}^n$ is compact. Then it is closed, since $\mathbb{R}^n$ is Hausdorff. Cover A with open balls $\{B_r(0) : r \in \mathbb{N}\}$. Then there is finite subcover $B_{r_1}(0), B_{r_2}(0), \ldots, B_{r_k}(0)$ for some r_i, $1 \leq i \leq k$. Then A is contained in some $B_r(0)$, i.e. it is bounded.

Now assume A is closed and bounded. Since A is bounded, it is contained in some product $[-r, r]^n$ for some r. This product is compact by the corollary. A is a closed subset of this product, and hence compact. $\square$

Exercise: Show that Heine-Borel theorem is not necessarily true for a general metric space X.

Exercise: Does there exist a metric d on $\mathbb{R}^n$ such that it induces the standard topology but Heine-Borel Theorem is not true in it?

Theorem 2.4. *Let $f : X \to \mathbb{R}$ be continuous, with X compact. Then there exist points $c, d \in X$ with $f(c) \leq f(x) \leq f(d)$ for all $x \in X$.*

Proof. The continuous image $f(X) \subseteq R$ is compact, hence closed and bounded, so contains both its infimum and its supremum. Writing these values as $f(c)$ and $f(d)$, we are done. ☐

Definition 2.5. Let (X, d) be a metric space and let $A \subseteq X$ be a nonempty subset. For each $x \in X$ the *distance from x to A* is $d(x, A) := \inf\{d(x, y) : y \in A\}$. The *diameter* of A is $d(A) := \sup\{d(a, b) : a, b \in A\}$.

Lemma 2.6. *The function $x \to d(x, A)$ is continuous.*

Proof. **Exercise.** ☐

Lemma 2.7 (Lesbegue Number Lemma)**.** *Let $\mathcal{C}$ be an open cover of a compact metric space (X, d). There exists a $\delta > 0$ such that for each subset $B \subseteq X$ of diameter $< \delta$, there exists an element $U \in \mathcal{C}$ with $B \subseteq U$. The number δ is called a Lebesgue number of $\mathcal{C}$.*

Proof. If $X \in \mathcal{C}$ then any positive number is a Lebesgue number for $\mathcal{C}$. Otherwise, by compactness there is a finite subcollection $\{U_1, \dots, U_n\}$ of $\mathcal{C}$ that covers X. Let $C_i = X \setminus U_i$ be the (closed) complement; each C_i is nonempty. To say that $x \in U_i$ is equivalent to saying $d(x, C_i) > 0$, since $d(x, C_i) = 0$ if and only if $x \in C_i$.

Define $f : X \to \mathbb{R}$ as the average

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i).$$

Then $f(x) > 0$ for all x, since at least one of the summands has to be nonzero.

Since $f(x)$ is continuous, it has to have a positive minimum value δ. We claim that this δ is the Lesbegue number for $\{U_1, \dots, U_n\}$ and hence for $\mathcal{C}$. Indeed, let B be a set with diameter less than δ. If B is empty, there is nothing to prove. Assume there exists p in B, so that $B \subseteq B_\delta(p)$. Consider the numbers $d(p, C_i)$ for $1 \leq i \leq n$. Choose m so that $d(p, C_m)$ is the largest of these numbers. Then

$$\delta \leq f(p) \leq d(p, C_m).$$

Hence, $B_\delta(p) \cap C_m = \varnothing$. Hence $B \subseteq B_\delta(p) \subseteq U_m$, and we are done.

☐

Definition 2.8. Let $f : (X, d) \to (Y, d')$ be a function between metric spaces. We say that f is *uniformly continuous* if given $\epsilon > 0$, there exists a $\delta > 0$ such that for any two points $x, x' \in X$ with $d(x, x') < \delta$, we have $d'(f(x), f(x')) < \epsilon$, or equivalently, if for any $x \in X$ we have $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$.

Theorem 2.9. *Let $f : (X, d) \to (Y, d')$ be a continuous map on metric spaces. If X is compact then f is uniformly continuous.*

Proof. Given $\epsilon > 0$, take the open cover of Y by open balls $B_{\epsilon/2}(y)$ of radius $\epsilon/2$ for $y \in Y$. Let $\mathcal{A}$ be the open cover of X by the inverse images of this cover. Then δ a Lebesgue number for this cover $\mathcal{A}$ will have the property that if $d_X(x_1, x_2) < \delta$ for any pair of points $x_1, x_2 \in X$, then by definition of the cover, $d_Y(f(x_1), f(x_2)) < \epsilon$, so f is uniformly continuous. ☐

Finally we note the following theorem.

Theorem 2.10. *Let X be a nonempty compact Hausdorff space. If X has no isolated point, i.e. no point x such that $\{x\}$ is open, then X is uncountable.*

Proof. **Exercise.** (Theorem 27.7 from Munkres.) ☐

3. Sequential Compactness

Definition 3.1. Let $(x_n)_{n=1}^{\infty}$ be a sequence of points in X. If $n_1 < n_2 < \cdots < n_k < \cdots$ is a strictly increasing sequence of natural numbers, the sequence $x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots$ is called a subsequence of $(x_n)_{n=1}^{\infty}$. It is a convergent subsequence if $x_{n_k} \to p$ as $k \to \infty$, for some $p \in X$.

Definition 3.2. A space X is sequentially compact if every sequence $(x_n)_{n=1}^{\infty}$ in X has a convergent subsequence $(x_{n_k})_{k=1}^{\infty}$.

Theorem 3.3. *Let X be a metrizable space. Then X is compact if and only if X is sequentially compact.*

Proof. We will give a proof that does not explicitly involve limit point compactness. These are all different definitions that try to capture the essence of compactness. They are all equivalent for metric spaces. Their equivalence will be an **Exercise** for you. For general topological spaces, our definition of compactness is the one that works best, as it turns out.

Assume (X, d) is compact and $(x_n)_{n=1}^{\infty}$ be a sequence of points of X. Then define, for every n, the set

$$A_n := \{x_m : m \geq n\}$$

which is a 'tail' of the sequence. Let $C_n : \mathrm{Cl}\ A_n$ be the closure.

Then $C_1 \supseteq C_2 \supseteq \cdots \supseteq C_n \supseteq \cdots$ and each $C_n \neq \varnothing$ so no finite intersection of the collection $\mathcal{C} = (C_n)_{n=1}^{\infty}$ is empty. By the finite intersection property of the compact space X, we can choose a point

$$p \in \bigcap_{n=1}^{\infty} C_n.$$

Then, for each k, we have $p \in \mathrm{Cl}\ A_k$, so that $B_{1/k}(p) \cap A_k \neq \varnothing$, and we can choose an $n_k \geq k$ such that $x_{n_k} \in B_{1/k}(p)$. Then the subsequence x_{n_k} converges to p as k goes to ∞. Note that n_k as chosen may not be a strictly increasing sequence but only an increasing one. But each n_k can appear only finitely many times, so deleting finitely many terms for each k and renumbering if necessary, we get a genuine subsequene x_{n_k} that converges to p.

So now assume X is sequentially compact. First we show that sequentially compact spaces satisfy the Lesbegue number lemma.

Lemma 3.4. *Let $\mathcal{C}$ be an open cover of a sequentially compact metric space (X, d). Then there exists a $\delta > 0$ such that for each subset $B \subseteq X$ of diameter $< \delta$ there is an element $U \in C$ with $B \subseteq U$.*

Proof. We assume that no such δ exists, and achieve a contradiction. For each $n \in \mathbb{N}$ there is a set B_n of diameter $< 1/n$ that is not contained in any element of $\mathcal{C}$. Choose $x_n \in B_n$. By the assumed sequential compactness, the sequence $(x_n)_{n=1}^{\infty}$ has a convergent subsequence $(x_{n_k})_{k=1}^{\infty}$, with $(n_k)_{k=1}^{\infty}$ a strictly increasing sequence. Let $p \in X$ be its limit.

Now, there is some $U \in \mathcal{C}$ such that $p \in U$. Since U is open, there exists an $\epsilon > 0$ such that $B_\epsilon(p) \in U$. Now, we can choose n_k sufficiently large, since the sequence $(n_k)_{k=1}^{\infty}$ is strictly increasing, so that $1/n_k < \epsilon/2$ and $d(x_{n_k}, p) < \epsilon/2$. Then

$$B_{n_k} \subseteq B_{\epsilon/2}(x_{n_k})$$

by choice of B_{n_k} and thus $B_{n_k} \subseteq B_\epsilon(p) \subseteq U$, which is a contradiction. □

Now we show that if X is sequentially compact then it is *totally bounded*:

Lemma 3.5. *Let (X, d) be sequentially compact. For each $\epsilon > 0$ there exists a finite covering of X by ϵ-balls.*

Proof. Assume that for some $\epsilon > 0$ there is no finite covering of X by ϵ-balls, to reach a contradiction. Construct a sequence $(x_n)_{n=1}^{\infty}$ by induction as follows. Choose any point $x_1 \in X$. Having chosen $x_1, \ldots, x_n$, note that the finite union

$$B_\epsilon(x_1) \cup B_\epsilon(x_2) \cup \cdots \cup B_\epsilon(x_n)$$

is not the whole of X, so there exists some x_{n+1} in the complement that we can choose and continue by induction.

By construction, $d(x_r, x_t) > \epsilon$, for any $r \neq t$ and hence (x_n) as above does not contain any convergent subsequence, which is a contradiction. □

Finally, we can finish the proof that a sequentially compact metrizable space X is compact : Consider any open cover $\mathcal{C}$ of X. It has a Lebesgue number $\delta > 0$. Let $\epsilon = \delta/3$. Choose a finite covering of X by ϵ-balls. Each ϵ-ball has diameter $\leq 2\epsilon < \delta$, hence is contained in an element of $\mathcal{C}$. Hence X is covered by finitely many of the elements of $\mathcal{C}$, so $\mathcal{C}$ has a finite subcover. □

4. Local Compactness

We know that compact spaces are 'nice' and 'well-behaved'. They allow us to prove theorems much more easily than usual. Of course, there are spaces that are not compact and the best we can hope for in general is that a given space can be embedded into a compact space. Note that a similar hope can be entertained for a moment for another type of 'nice' spaces - namely, metric spaces. But we know that a subspace of a metric space has to be metrizable itself. This can be a convenient way of proving a space is metrizable, but does not say anything new about how nice the space is. But with compact Hausdorff spaces, we can hope to embed interesting spaces into them and derive more information about these spaces. It turns out that these interesting spaces need to have some local features of compactness in order to be sufficiently well-behaved as to admit such embeddings into compact spaces.

Definition 4.1. A space X is *locally compact at x* if there is a compact subspace C of X that contains a neighborhood V of x:

$$x \in V \subseteq C \subseteq X.$$

A space is *locally compact* if it is locally compact at x for all $x \in X$.

Example 4.2.
(1) Any compact space is locally compact.
(2) $\mathbb{R}^n$ is locally compact.
(3) The set $\mathbb{Q}$ as a subspace of $\mathbb{R}$ is not locally compact.
(4) $\mathbb{R}^{\mathbb{N}}$ is not locally compact in the product topology, since any neighbourhood of $(0, 0, \ldots)$ will have $\mathbb{R}$ in all but finitely many coordinates.

For Hausdorff spaces, the property of being locally compact is a local property in the following sense.

Proposition 4.3. *Let X be a Hausdorff space. Then X is locally compact if and only if for each point $x \in X$ and each neighborhood U of x, there is a neighborhood V of x such that the closure $\mathit{Cl}\, V$ is compact and is contained in U :*

$$x \in V \subseteq \mathit{Cl}\, V \subseteq U.$$

Proof. **Exercise.** □

Proposition 4.4. *Let X be locally compact Hausdorff. If $A \subseteq X$ is an open or closed subspace, then A is locally compact.*

Proof. **Exercise.** □

Remark 4.5. Note that this does not extend to any general subset A of X.

The concept appearing in the next section will characterise locally compact Hausdorff spaces.

5. One-point compactifications

Definition 5.1. A compactification of a topological space X is an embedding of X as a dense subspace of a compact topological space. In other words, it is a compact topological space Y and a map $f : X \to Y$ such that $f : X \to f(X)$ is a homeomorphism, and $\mathrm{Cl}\ f(X) = Y$.

We will only care about compactifications up to homeomorphism, so we will say two compactifications $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$ are *equivalent* if there exists a homeomorphism $h : Y_1 \to Y_2$ that fixes the embedded elements of X, in the sense that $h(f_1(x)) = f_2(x)$ for all $x \in X$.

Note that we will implicitly assume for the remainder of this section, and in general when we speak about compactifications, that the locally compact spaces that appear are not compact, since denseness condition in the definition shows that a compactification of a compact space is the space itself. (There is nothing to compactify.)

Note also that the following definition can be made quite generally for any non-compact topological space X, and it is kind of a *minimal* compactification of any space. But for locally compact spaces, this goes through most smoothly.

Definition 5.2. Let X be a locally compact Hausdorff space, such that X is not compact. Let $Y = X \cup \{\infty\}$ where ∞ is a formal symbol that is not in X. Give Y the topology $\mathscr{T}_\infty$ consisting of :

(1) The open subsets $U \subseteq X$, and
(2) The complements $Y \setminus C$ of compact subsets $C \subseteq X$.

We call Y the *one-point compactification of* X.

We will explore the properties of one-point compactifications in the next lecture.