FAST REVIEW OF BASIC NOTIONS

You should review Chapter 1 of the Munkres's Topology (Second Edition) book quite well. This document only briefly lists concepts that you will need to know well for this course. In some places, we offer some very brief explanation or recollection. In particular, we have given in this document the definition of an arbitrary direct product of sets that occurs later in Munkres's book in Chapter 2, but we feel it is better to have this definition at the beginning itself. So you should make sure that you know this definition well. In general it is your responsibility to thoroughly study and be familiar with the concepts listed below (and in general Chapter 1). These are basic concepts in Set Theory and Logic which are not only useful for this course, but are ingrained within the very foundations of mathematics. If you have any difficulties or questions, you should use the tutorials in the first week (or later, but not too late!) to get them resolved from the course TAs.

1. Set Theory and Logic

Definition 1.1. Sets, elements, (arbitrary) unions, (arbitrary) intersections, empty set, set difference, Cartesian products of two sets.

Definition 1.2. If P, then Q means that if statement P is true, Q is true also; but if P is false, Q may either be true or false.

Here are two examples.

Example 1.3. Let x be a real number.

(1) If $x > 0$, then $x^3 \neq 0$.
(2) If $x^2 < 0$, then $x = 23$.

Both of these are true statements! Whenever $x > 0$, x^3 can't be 0. The second one is more tricky. In every case for which the hypothesis holds, which is to say NEVER, the conclusion holds as well, which is to say NEVER. A statement of this sort is sometimes said to be vacuously true. Think about this - to falsify this statement, you first will need to exhibit x a real number such that $x^2 < 0$ and then show that $x \neq 23$, but you can never begin this process of falsification.

Definition 1.4. Contrapositive, Converse, Negation.

Example 1.5. Contrapositive of (1) is - if $x^3 = 0$, then $x \leq 0$. This is also true. Contrapositive of (2) is - if $x \neq 23$, then $x^2 \geq 0$. This is also true. In fact, a statement and its contrapositive are both logically equivalent - they are two different ways of stating precisely the same assertion. So, if you are sometimes worried about vacuously true statement (such as above), it can be a good idea to convert it into its contrapositive form and see if it remains true (such as above).

Converse of (1) is - if $x^3 \neq 0$, then $x > 0$. This is false. Converse of (2) is - If $x = 23$, then $x^2 < 0$. This is false. A statement might be true without its converse being true. Sometimes a statement $P \implies Q$ and its converse $Q \implies P$ are both true. In this case we say that P and Q are equivalent.

Definition 1.6. Functions on sets, domain, codomain, range, injective functions, surjective functions, bijective functions, composition of functions.

What is a function $f : A \to B$? Formally, it is a subset F of the Cartesian product $A \times B$ such that

- For all $a \in A$, there exists a $b \in B$ such that $(a, b) \in F$.
- If (a, b) and (a, c) are both in F, then $b = c$.

In other words, this gives a rule assigning to each element in A a unique element in B.

Definition 1.7. Relations on a set, equivalence relations.

A relation on a set A is simply any subset $R \subset A \times A$.

An equivalence relation on a set A is a subset $R \subset A \times A$ which satisfies -

(1) $(a, a) \in R$ for all $a \in A$.
(2) $(a, b) \in R \implies (b, a) \in R$.
(3) $(a, b) \in R, (b, c) \in R \implies (a, c) \in R$.

Note that an equivalence relation R divides A into a disjoint union of subsets called equivalence classes. We define $a \sim b$ to mean that $(a, b) \in R$.

An order relation (we will refer to this as a linear order or a simple order in class) on a set A is a relation R on a set A which satisfies -

(1) For $a, b \in A$ such that $a \neq b$, either $(a, b) \in R$ or $(b, a) \in R$.
(2) $(a, a) \notin R$ for all $a \in A$.
(3) $(a, b) \in R, (b, c) \in R \implies (a, c) \in R$.

We define $a < b$ to mean $(a, b) \in R$. $a \leq b$ means either $a < b$ or $a = b$.

A partial order relation on a set A is a relation R that satisfies the last 2 properties satisfied by a linear order, but not necessarily the first property. (Not all elements might be comparable to each other.)

Definition 1.8. Bounded above, upper bound, least upper bound property.

Definition 1.9. Cartesian products of finitely many sets, collection of sets indexed by a set J, ω-tuple of elements of a set A.

Given a set A, we define an ω-tuple of elements of A to be a function $x : \mathbb{Z}_+ \to A$. This is also known as a(n infinite) sequence of elements of A, and is completely determined by its coordinates $x = (x_1, x_2, \ldots)$ or $(x_n)_{n \in \mathbb{Z}_+}$.

Definition 1.10. If $(A_1, A_2, \ldots)$ is a family of sets indexed by $\mathbb{Z}_+$, we denote by A their union. Then the Cartesian product

$$\prod_{i \in \mathbb{Z}_+} A_i$$

is defined to be the set of all ω-tuples $(x_1, x_2, \ldots)$ of elements of A such that $x_i \in A_i$ for each i.

Definition 1.11. Finite sets, Countable sets, Uncountable sets, their unions, their products, power sets.

Definition 1.12. In fact, let $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ be a family of sets indexed by an arbitrary set J. Recall that this just means there is a surjective function $J \to \mathcal{A}$.

Note that for any set A, we can define a J-tuple of elements of A as a function $x : J \to A$ and this can be denoted as $(x_{\alpha)_{\alpha \in J}}$ as before.

Then, by a similar logic as above, let $A = \bigcup_{\alpha \in J} A_\alpha$, and we can define the cartesian product

$$\prod_{\alpha \in J} A_\alpha := \{(x_\alpha)_{\alpha \in J} : (x_\alpha) \text{ is a } J\text{-tuple in } A \text{ and } x_\alpha \in A_\alpha \text{ for each } \alpha\}.$$

For each $\beta \in J$, there is a natural projection map

$$\pi_\beta : \prod_{\alpha \in J} A_\alpha \to A_\beta$$

given by

$$\pi_\beta((x_\alpha)) = x_\beta.$$

In particular, the case of countable or uncountable products of arbitrary sets should be studied well. In particular, the result showing that ‘a countable product of countable sets is not necessarily countable’ is really significant.

Axiom 1.13 (Axiom of Choice)**.** *Given an arbitrary collection $\{A_\alpha\}_{\alpha\in J}$ of nonempty, disjoint sets, there exists a set C having exactly one element in common with each element A_α of $\{A_\alpha\}_{\alpha\in J}$. That is, there exists a set C such that $C\cap A_\alpha$ is a singleton for each $\alpha\in J$.*

Lemma 1.14 (Zorn's Lemma)**.** *Let A be any nonempty partially ordered set. Assume that every totally ordered nonempty subset of A has an upper bound in A. Then in fact A has a maximal element m. That is, there exists an element m such that there exists no element a in A such that $m<a$.*

Definition 1.15. An ordered set A is 'well-ordered' if every nonempty subset of A has a smallest element.

Proposition 1.16 (Well-ordering Principle)**.** *If A is any set, there exists an order relation on A that is a well-ordering.*

Theorem 1.17. *Axiom of Choice $\Leftrightarrow$ Zorn's Lemma $\Leftrightarrow$ Well-ordering principle.*

In particular, there exists an uncountable well-ordered set!

Definition 1.18. Let X be a well-ordered set. Given $\alpha\in X$, we denote by S_α the set

$$S_\alpha := \{x\in X : x<\alpha\}.$$

This is called a section of X by α.

Lemma 1.19. *There exists a well-ordered set A having a largest element Ω such that the section S_Ω of A is uncountable, but every other section of A is countable.*

Note that S_Ω is an uncountable well-ordered set, every section of which is countable. It is in fact uniquely determined as a well-ordered set by this condition. This will be denoted as a 'minimal uncountable well-ordered set'. In fact, we will denote $A=S_\Omega\cup\{\Omega\}$ as $\tilde{S}_\Omega$.

Proposition 1.20. *If B is a countable subset of S_Ω, then B has an upper bound in S_Ω.*