DEFINING NEW TOPOLOGICAL SPACES - I

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1. Motivation

Today we will learn some ways of creating new topological spaces. Perhaps our most useful tool for defining new topological spaces, and for analyzing properties of the ones we define is to learn how to define a topology on a subset of a topological space in a way that "agrees" with the topology of the larger space. More importantly for the study of topology itself, subspaces give us another important way to analyze how properties "move around". There is also a very natural way to consider product of two (and hence, finitely many) topological spaces as a topological space. The case of a product of infinitely many topological spaces, on the other hand, requires some care to handle, at least initially. The 'correct' definition that handles both these cases together is less intuitive at first. So we will keep it for later. We will also see a natural topology that can be put on any simply ordered set. We won't spend too much time on it in class, but you should study it well, for it can provide us with intriguing topological spaces with a combination of desirable properties.

2. Subspace Topology

Definition 2.1. Let X be a topological space, and $Y \subseteq X$ be a subset. We define the subspace topology $\mathscr{T}_Y$ on Y (we might sometimes write $\mathscr{T}_{\text{subspace}}$ for clarity) by :

$$\mathscr{T}_Y := \{U \cap Y : U \in \mathscr{T}_X\}.$$

We may call this the topology "induced by" or "inherited from" the topology on X. The topological space $(Y, \mathscr{T}_Y)$ is called a 'subspace' of X.

We have to check that this is a topology. This is immediate - $\mathscr{T}_Y$ contains $\varnothing$ and Y, because $\varnothing = \varnothing \cap Y$ and $Y = X \cap Y$. That it is closed under finite intersection and arbitrary unions follows from

$$(U_1 \cap Y) \cap (U_2 \cap Y) \cap \ldots \cap (U_n \cap Y) = (U_1 \cap U_2 \cap \ldots U_n) \cap Y$$

and

$$\bigcup_{\alpha \in J} (U_\alpha \cap Y) = \left(\bigcup_{\alpha \in J} U_\alpha \right) \cap Y.$$

Since the definition is so simple and direct, we don't really define this topology by way of giving a basis. If we do have a basis for the topology of X at hand, it is simple to get a basis for the topology of Y.

Lemma 2.2. *If $\mathcal{B}_X$ is a basis for the topology of X, then the collection*

$$\mathcal{B}_Y := \{B \cap Y : B \in \mathcal{B}_X\}$$

is a basis for the subspace topology of Y.

Proof. Given $U \subseteq X$ open and given $y \in U \cap Y$, we can choose an element $B \in \mathcal{B}_X$ such that $y \in B \subseteq U$. Then $y \in B \cap Y \subseteq U \cap Y$. Then it follows that $\mathcal{B}_Y$ is a basis for $\mathscr{T}_Y$. □

Example 2.3.
(1) Any subspace of a discrete space is discrete.
(2) The subspace topology on $(a, b) \subseteq \mathbb{R}$ induced by the usual topology on $\mathbb{R}$ is the topology generated by the basis $\mathcal{B}_{(a,b)} = \{(c, d) : a \leq c < d \leq b\}$.
(3) The subspace topology on $[a, b) \subseteq \mathbb{R}$ induced by the usual topology on $\mathbb{R}$ is the topology generated by the basis consisting of the intervals $[a, d)$ and (c, d) for $a < c < d \leq b$.

(4) Consider $\mathbb{Z} \subseteq \mathbb{R}$. Then for any $n \in \mathbb{Z}$, $\{n\} = (n - \frac{1}{2}, n + \frac{1}{2}) \cap \mathbb{Z}$. Hence, $\{n\}$ is open, and thus Z has the discrete topology as a subspace of $\mathbb{R}$. In general, a subspace of a topological space whose subspace topology is discrete is called a *discrete subspace.*

(5) The space $Y = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is not discrete as a subspace of $\mathbb{R}$. Indeed, for any open set $U \subset \mathbb{R}$ containing 0, U has to contain infinitely many elements of Y other than 0, so that the singleton $\{0\}$ cannot be written as an open set of Y.

(6) Let $Y = [a, b] \subset \mathbb{R}_\ell$ in the lower limit topology on $\mathbb{R}$. Then $\{b\}$ is open in the subspace topology on Y, since $\{b\} = [a, b] \cap [b, b + 1)$. No other singleton is open.

Exercise: Are the rationals $\mathbb{Q}$ discrete in $\mathbb{R}$?

2.1. **Basic properties.** We will say "Y is a subspace of X" when the topologies are understood easily from context.

Lemma 2.4 ("Open in open is open")**.** *Let X be a topological space and Y be a subspace of X. If U is a open set in Y and Y is an open set in X, then U is an open subset of X.*

Proof. Since U is open in Y, $U = V \cap Y$ for some open set $V \subseteq X$. Since $Y \subseteq X$ is also open, the finite intersection $U = V \cap Y$ is also open in X. □

Lemma 2.5 ("Subspace of a subspace is a subspace")**.** *Let X be a topological space, let Y be a subspace of X, and let A be a subset of Y. Then the subspace topology A inherits from Y is equal to the subspace topology it inherits from X.*

Proof. **Exercise**. □

3. Finite Products

Definition 3.1. Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces. Then the *product topology* on the set $X \times Y$ is the topology generated by the basis

$$\mathcal{B}_{X\times Y} := \{U \times V : U \in \mathscr{T}_X, V \in \mathscr{T}_Y\}.$$

More generally, the product topology on a finite product $X_1 \times X_2 \times \cdots \times X_n$ of topological spaces is the topology generated by the basis $\{\prod_{i=1}^n U_i : U_i \in \mathscr{T}_{X_i}\}$.

Remark: Note that unlike the previous example, this basis is not necessarily a topology in itself! See figure 15.1 in Munkres's 'Topology'.

We have to show that the above defines a basis on $X \times Y$. (The general case follows from induction.) The covering property is trivial, since $X \times Y$ is an element of $\mathcal{B}_{X\times Y}$. The gluing property follows from the fact that the intersection of any two $U_1 \times V_1$ and $U_2 \times V_2$ is itself an element of the same form :

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2).$$

As you would expect, bases play well with this definition, similarly to previous case.

Lemma 3.2. *Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces with $\mathcal{B}_X$ and $\mathcal{B}_Y$ bases on X and Y respectively, such that they generate the respective topologies on X and Y. Then*

$$\mathcal{B} := \{U \times V : U \in \mathcal{B}_X, V \in \mathcal{B}_Y\}$$

is a basis for the product topology on $X \times Y$.

Proof. **Exercise**. □

Remark: We will write X^2 instead of $X \times X$, X^3 instead of $X \times X \times X$ and so on. This agrees with the notation with $\mathbb{R}^n$.

Example 3.3. (1) A finite product of discrete spaces is discrete.

(2) $(\mathbb{R}_{\text{standard}})^n = \mathbb{R}^n_{\text{standard}}$ as topological spaces.

3.1. **Projections.** You have seen projection functions (say, from $\mathbb{R}^2$ to $\mathbb{R}$) in your mathematical education before. It turns out that they have an important relationship with product topologies, that is quite useful in practice.

Definition 3.4. Let $\{(X_i, \mathscr{T}_{X_i})\}_{i=1,\dots,n}$ be finitely many topological spaces. The k-th projection is defined

$$\pi_k : \prod_{i=1}^{n} X_i \to X_k$$

for $k = 1, \dots, n$ by $\pi_k(x_1, \dots, x_k, \dots, x_n) := x_k$.

For the purpose of writing proofs in this section, we will restrict our discussion to products of two spaces to avoid unnecessary complication and indexing. Everything we say here will extend to all finite products in the obvious ways.

So for two topological spaces $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$, there are two projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ given by $\pi_1((x,y)) = x$ and $\pi_2((x,y)) = y$. We state the following fact whose proof should be obvious:

Fact: Let $A \subseteq X$ and $B \subseteq Y$. Then $\pi_1^{-1}(A) = A \times Y$ and $\pi_2^{-1}(B) = X \times B$. Hence, $\pi_1^{-1}(A) \cap \pi_2^{-1}(B) = A \times B$.

Note that this is purely a set-theoretical fact, and there is no mention of topologies involved. We can use this fact to give a subbasis for the product topology. This fact will be useful later when we discuss arbitrary products (as opposed to finite products) of topological spaces.

Theorem 3.5. *The collection*

$$\mathcal{S} := \{\pi_1^{-1}(U) : U \in \mathscr{T}_X\} \cup \{\pi_2^{-1}(V) : V \in \mathscr{T}_Y\}$$

is a subbasis for the product topology $\mathscr{T}_{X \times Y}$.

Proof. Let $\mathscr{T}'$ be the topology generated by the subbasis $\mathcal{S}$. Then since every element of $\mathcal{S}$ is in $\mathscr{T}_{X\times Y}$ by the above fact, $\mathscr{T}' \subseteq \mathscr{T}_{X\times Y}$. On the other hand, by the same fact, every basis element $U \times V$ for $\mathscr{T}_{X\times Y}$ is a finite intersection $\pi_1^{-1}(U) \cap \pi_2^{-1}(V)$ of elements in $\mathcal{S}$, so $\mathscr{T}_{X\times Y} \subseteq \mathscr{T}'$, and we are done. $\square$

3.2. **Product of subspaces is the same as subspace of the product.** This gives a useful relation between the subsapce and product topology, which serves as a sanity check of sorts.

Theorem 3.6. *If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology that the set $A \times B$ inherits as a subspace of the topological space $X \times Y$.*

Proof. The general basis element for the subspace topology on $A \times B$ is given by $(U \times V) \cap (A \times B)$ where $U \in \mathscr{T}_X, V \in \mathscr{T}_Y$. But

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B).$$

But the right hand side is the general basis element for the product topology on the product of subspaces $A \times B$. Thus, the bases for the subspace topology on the set-theoretical product $A \times B$ and the product topology on $A \times B$ are the same. Hence, the topologies are the same. $\square$

4. Order Topology

Definition 4.1. Recall that a binary relation $<$ on a set A is called a *linear order* or a *simple order* if it has the following properties :

(1) (Comparability) For every $x, y \in A$ for which $x \neq y$, either $x < y$ or $y < x$.
(2) (Nonreflexivity) For no $x \in A$ does the relation $x < x$ hold.
(3) (Transitivity) If $x < y$ and $y < z$, then $x < z$.

We will use the notation $x \leq y$ to mean that "either $x < y$ or $x = y$" and $y > x$ to mean that $x < y$ and $y \geq x$ to mean that $x \leq y$.

Your favourite linear order is probably the real numbers. Your favourite topology is probably also the usual topology on the real numbers. These two things are intimately connected. The topology on the real numbers is defined in terms of the usual ordering on the real numbers. In this same way, we can define a topology on any linear order.

Suppose that X is a set having a simple order relation $<$. We define the familiar intervals and rays as follows :

Definition 4.2 (Intervals and Rays)**.** Given elements $a, b \in X$ such that $a < b$, we define :

(1) $(a, b) = \{x : a < x < b\}$.
(2) $(a, b] = \{x : a < x \leq b\}$.
(3) $[a, b) = \{x : a \leq x < b\}$.
(4) $[a, b] = \{x : a \leq x \leq b\}$.
(5) $(a, +\infty) = \{x : x > a\}$.
(6) $(-\infty, a) = \{x : x < a\}$.
(7) $[a, +\infty) = \{x : x \geq a\}$.
(8) $(-\infty, a] = \{x : x \leq a\}$.

The first four are called 'intervals' while the last four are called 'rays'.

Definition 4.3. Let X be a set having more than one element and a simple order relation $<$. Let $\mathcal{B}$ be the collection of all sets of the following types :

(1) All open intervals (a, b) in X.
(2) All intervals of the form $[a_0, b)$ where a_0 is the smallest element (if any) of X.
(3) All intervals of the form $(a, b_0]$ where b_0 is the largest element (if any) of X.

The collection $\mathcal{B}$ is a basis on X, and the topology generated by it is called the order topology on X. (Note that if X does not have a smallest element, there are no sets of type (2), and if X does not have a largest element, there are no sets of type (3).)

We have to check that $\mathcal{B}$ is a basis on X. Note that covering property is satisfied : the smallest element, if any, is contained in all sets of type (2), the largest element, if any, is contained in all sets of type (3), and every other element is contained in a set of type (1). **Exercise:** Check the gluing property.

Lemma 4.4. *The open rays $(a, +\infty)$ and $(-\infty, b)$ form a subbasis for the order topology.*

Proof. Note that the open rays are indeed open sets in the order topology : if the set X contains a largest element b_0, $(a, +\infty)$ is equal to $(a, b_0]$ and hence open, otherwise it is the union of all basis elements (a, x) for $x > a$, and hence open. Likewise for the ray $(-\infty, b)$. Hence, the topology generated by the open rays as a subbasis is contained in the order topology.

On the other hand, each basis element for the order topology equals a finite intersection of open rays : $(a, b) = (-\infty, b) \cap (a, +\infty)$ while $[a_0, b)$ and $(a, b_0]$, if they exist, are themselves open rays as seen above. So the order topology is contained in the topology generated by the open rays as a subbasis. □

4.1. **Subspace of order topology need not be the same as the order topology on the subspace.** Now, let X be a linearly ordered set with $\mathscr{T}_X$ the order topology, and let Y be a subset of X. We can restrict the linear order $<$ to Y and make it into a linearly ordered set. However the order topology that this gives on Y need not be the same as the subspace topology Y inherits from X! Sometimes they can be equal, sometimes not. We leave the following examples to check as an exercise :

Example 4.5. (1) If $X = \mathbb{R}$, and $Y = [0, 1]$, the subspace topology on Y is the same as the order topology on Y by direct inspection of bases of the topologies.
(2) If $X = \mathbb{R}$, and $Y = [0, 1) \cup \{2\}$, the subspace topology on Y has the singleton $\{2\}$ as an open set, but $\{2\}$ is not open in the ordered topology.

This example seems to happen because there are "missing" points between $[0, 1)$ and $\{2\}$ in $\mathbb{R}$. This intuition is correct, so we can salvage the situation taking this into mind.

Definition 4.6. For a linearly ordered set X, we say that a subset Y of X is *convex* in X if for each pair of points $a < b$ of Y, the entire interval (a, b) of points of X lies in Y.

Theorem 4.7. *Let X be a linearly ordered set with $\mathscr{T}_X$ being the order topology, and let Y be a convex subset of X. Then the order topology on Y obtained by restriction of the order relation $<$ is the same as the subspace topology on Y inherited from X.*

Proof. **Exercise.** $\square$

To avoid ambiguity, whenever X is a linearly ordered set with the ordered topology and Y is a subspace, we will assume that Y is given the subspace topology $\mathscr{T}_{\text{subspace}}$ unless specifically mentioned otherwise.