CLOSED SETS, LIMIT POINTS, SEPARATION AXIOMS

ADITYA KARNATAKI

1. Motivation

Up to this point, what we have done is : define what topologies are, define a way of comparing two topologies, define a method for more easily specifying a topology (as a collection of sets generated by a basis or a subbasis), investigated some simple properties of bases, and saw how to define 'new' topological spaces from existing topological spaces or order relations.

At this point, we will start introducing some more interesting definitions and phenomena one might encounter in a topological space, starting with the notions of *closed sets* and *closures*. Our approach, which parallels the approach in Munkres's book, might feel unintuitive at first, since we are used to seeing limit points in the definition of closed sets in Analysis, but the relation to that approach will become clear. For now, the fact that 'closed' sets can be defined only from the notion of open sets, and not much else, is alone quite intriguing.

In relation to closed sets, we will also explore the notion of limit points and convergence of sequences in topological spaces, motivated by their analogues in analysis. In general topological spaces, this necessitates the study of separation axioms - something that is available for free in metric spaces, but not so in general. We will see the basic idea behind these axioms and their definitions.

2. Closed Sets

Definition 2.1. A subset A of a topological space X is said to be *closed* if and only if its complement $X \setminus A$ is open.

This definition is so fundamental and elegant that we can immediately see some examples.

Example 2.2. (1) In $(X, \mathscr{T}_{\text{trivial}})$, the only closed sets are $\varnothing$ and X.

(2) In $(X, \mathscr{T}_{\text{discrete}})$, every subset $A \subseteq X$ is closed. Note that this already shows that a set can be both open and closed as a subset of a topological space.
(3) In $\mathbb{R}$, let $a < b$. Then $\{a\}$ is closed. Similarly, $[a, b], (-\infty, b], [a, +\infty)$ are open. Note that (a, b) is not closed, since $\mathbb{R} \setminus (a, b) = (-\infty, a] \cup [b, +\infty)$ points a and b, around which no open interval exists that is entirely contained in $\mathbb{R} \setminus (a, b)$. Similarly, the set $S := \{\frac{1}{n} : n \in \mathbb{N}\}$ is not closed, since $0 \in \mathbb{R} \setminus S$, and there is no open interval around 0 that is contained entirely in $\mathbb{R} \setminus S$.
(4) Consider $\mathbb{R}_\ell$, the lower limit topology on $\mathbb{R}$. $a < b \in \mathbb{R}$. Then $[a, b]$ is closed. Since the complement
$$\mathbb{R}_\ell \setminus [a,b] = \left(\bigcup_{\substack{n \in \mathbb{Z} \\ n < a}} [n, a) \right) \cup \left(\bigcup_{r=2}^{\infty} [b + \frac{1}{r}, b + r) \right)$$
is an arbitrary union of basic open sets, and hence open. What is interesting is that the basic open set $[a, b)$ is also closed, since its complement
$$\mathbb{R}_\ell \setminus [a,b) = \left(\bigcup_{\substack{n \in \mathbb{Z} \\ n < a}} [n, a) \right) \cup \left(\bigcup_{r \in \mathbb{N}} [b, b + r) \right)$$
is open. So this is much different from the standard topology on $\mathbb{R}$.
(5) In the finite complement topology $\mathscr{T}_f$ on $\mathbb{R}$, the closed sets are either $\mathbb{R}$ or finite sets.

Theorem 2.3. *Let X be a topological space.*

(1) The subsets $\varnothing$ and X are closed.
(2) A finite union of closed sets of X is again closed.
(3) An arbitrary intersection of closed sets of X is again closed.

Proof. The first assertion holds because $\varnothing$ and X are both open in $\mathscr{T}_X$ and are complements of each other. The second assertion holds since by de Morgan's law,

$$X \setminus (K_1 \cup K_2 \cup \cdots \cup K_n) = (X \setminus K_1) \cap (X \setminus K_2) \cap \cdots \cap (X \setminus K_n).$$

The third assertion holds since by de Morgan's law,

$$X \setminus \left(\bigcap_{\alpha \in J} K_\alpha \right) = \bigcup_{\alpha \in J} (X \setminus K_\alpha) .$$

□

If $Y \subseteq X$ is a subspace and $A \subseteq Y$, the sentence 'A is closed' could carry two meanings : either A is closed in X, or A is closed in Y. *These two are not always the same!* (We have seen an analogue of this for the subspace topology before.)

Definition 2.4. We say that A is closed in Y, if $A \subseteq Y$ and A is closed in the subspace topology of Y.

Lemma 2.5. *Let Y be a subspace and $A \subseteq Y$ a subset. Then A is closed in Y if and only if A equals the intersection of a closed set in X with Y.*

Proof. Assume that $A = C \cap Y$, where C is closed in Y. Then $(X \setminus C) \cap Y$ is open in Y by definition. But $(X \setminus C) \cap Y = Y \setminus A$, and hence A is closed in Y by definition.

Assume that A is closed in Y, so that $Y \setminus A$ is open in Y. Hence it equals the intersection of an open set U of X with Y. But then $(X \setminus U) \cap Y = A$ and $(X \setminus U)$ is a closed set of X by definition. □

Theorem 2.6 ("Closed in closed is closed")**.** *Let Y be a subspace of X. If Y is closed in X and A is closed in Y, then A is closed in X.*

Proof. **Exercise.** □

2.1. **Closure and Interior.**

Definition 2.7. Let X be a topological space and $A \subseteq X$ a subset. The closure $\mathrm{Cl}\ A = \overline{A}$ of A is the intersection of all the closed subsets of X that contain A. The interior $\mathrm{Int}\ A$ of A is the union of all the open subsets of X that are contained in A.

It is immediate that $\mathrm{Int}\ A$ is an open set, and $\mathrm{Cl}\ A$ is a closed set, and that $\mathrm{Int}\ A \subseteq A \subseteq \mathrm{Cl}\ A$. Similarly, it is immediate that :

Lemma 2.8. *(1) If $U \subseteq A \subseteq X$ with U open, then $U \subseteq Int\ A$.*
(2) If $A \subseteq K \subseteq X$ with K closed, then $Cl\ A \subseteq K$.

That is, the closure is the minimal closed set containing A while the interior is the maximal open set contained in A.

Lemma 2.9. *The complement of the closure is the interior of the complement, and the complement of the interior is the closure of the complement :*

$$X \setminus Cl\ A = Int\ (X \setminus A).$$
$$X \setminus Int\ A = Cl\ (X \setminus A).$$

Proof. **Exercise.** □

Definition 2.10. We say that a set $A \subseteq X$ is dense in X if $\mathrm{Cl}\ A = X$.

Example 2.11. $\mathbb{Q}$ has empty interior in $\mathbb{R}$ but its closure is whole of $\mathbb{R}$. **Exercise:** Show this using the archimedean property of real numbers.

Definition 2.12. A topological space X is called *separable* if it contains a countable dense subset.

To emphasize the role of the ambient space, we might write $\mathrm{Cl}_X(A)$ to denote the closure of A in X.

Theorem 2.13. *Let X be a topological space, $Y \subseteq X$ a subspace, and $A \subseteq Y$ a subset. Then*

$$Cl_Y(A) = Cl_X(A) \cap Y.$$

Proof. $\mathrm{Cl}_X(A) \cap Y$ is a closed set of Y by lemma 2.5 and it contains A. Hence, $\mathrm{Cl}_Y(A) \subseteq \mathrm{Cl}_X(A) \cap Y$ by definition.

On the other hand, $\mathrm{Cl}_Y(A)$ is a closed set in Y and hence $\mathrm{Cl}_Y(A) = C \cap Y$ for some $C \subseteq X$ closed. Moreover, C (contains $\mathrm{Cl}_Y(A)$ which) contains A. Then $\mathrm{Cl}_X(A) \cap Y \subseteq C \cap Y (= \mathrm{Cl}_Y(A))$ by definition. □

Definition 2.14. Let X be a topological space, $x \in X$ a point. We say that U a subset of X is a *open neighbourhood* of x if $x \in U$ and U is open in X.

We shall say that a set A *intersects* another set B if the intersection $A \cap B$ is nonempty.

Theorem 2.15. *Let A be a subset of the topological space X and x a point in X.*

(1) $x \in Cl\ A$ if and only if every open neighbourhood of x intersects A.
(2) $x \in Cl\ A$ if and only if every basis element B that is an open neighbourhood of x intersects A.

Proof. Assume that there exists U open set containing x such that $U \cap A = \varnothing$. Then $K := X \setminus U$ is a closed set that contains A, hence $\mathrm{Cl}\ A \subset K$ but K does not contain x, so $x \notin \mathrm{Cl}\ A$.

Conversely, if $x \notin \mathrm{Cl}\ A$, then $(X \setminus \mathrm{Cl}\ A)$ is an open set of X that contains X and does not intersect A since $A \subset \mathrm{Cl}\ A$.

The claim about basis elements is left as an exercise. □

3. Limit Points

The previous theorem, of course, relates to the concept of a limit point.

Definition 3.1. If A is a subset of a topological space X and $x \in X$ a point, we say that x is a *limit point* of A, if every open neighbourhood of x intersects A in some point *other than x itself.*

Theorem 3.2. *Let A be a subset of a topological space X and A' be the set of all limit points of A. Then*

$$Cl\ A = A \cup A'.$$

Proof. If $x \in A'$, then by previous theorem, $x \in \mathrm{Cl}\ A$ by definition. So $A \cup A' \subseteq \mathrm{Cl}\ A$ since $A \subseteq \mathrm{Cl}\ A$.

If $x \in \mathrm{Cl}\ A \setminus A$, we have to show $x \in A'$. Since $x \in \mathrm{Cl}\ A$, by previous theorem, every open neighbourhood U of x intersects A, but it has to do so in a point other than x since $x \notin A$. Hence $x \in A'$. □

Corollary 3.3. *A subset of a topological space is closed if and only if it contains all its limit points.*

4. Convergence of sequences and Separation axioms

We begin with a definition -

Definition 4.1. Let $(x_1, x_2, \ldots, x_n)_{n\in\mathbb{N}}$ be a sequence of points in a topological space X. We say that the sequence (x_n) converges to a point $y \in X$ if for each open neighbourhood U of y there is an $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. In this case, y is called *a limit* of (x_n) and we may write $x_n \to y$ as $n \to \infty$.

In the definition, we defined *a* and not *the* limit. Why is this?

Example 4.2. Consider the space $X = \{a, b\}$ with the topology $\mathscr{T}_a = \{\varnothing, \{a\}, X\}$. The constant sequence $(x_n) = a, a, a, \ldots$ converges to a since the only neighbourhoods of a are $\{a\}$ and $\{a, b\}$ and both of them contain x_n for all n. Hence, a is a limit of the sequence $a, a, a, \ldots$ (obviously).

However, *the same sequence also converges to* b, since the only open neighbourhood of b is $\{a, b\}$ which of course contains all the terms of the sequence $a, a, a, \ldots$! Hence, b is also a limit of the sequence $a, a, a, \ldots$ (not obviously).

To obtain unique limits for convergent sequences, and be able to talk about the limit of a sequence, we must assume that the topology is sufficiently fine to separate the individual points. Such additional hypotheses are called separation axioms (German: Trennungsaxiome). The most common separation axiom is known as the *Hausdorff property*.

5. Hausdorff Property

Definition 5.1. A topological space X is called a *Hausdorff space* if for each pair of points $x, y \in X$ with $x \neq y$, there exist open sets U, V such that $x \in U$, $y \in V$, and $U \cap V = \varnothing$.

Remark 5.2. If $(X, \mathscr{T})$ is Hausdorff, then for any $\mathscr{T}'$ that is finer than $\mathscr{T}$, $(X, \mathscr{T}')$ is also Hausdorff. The Hausdorff property roughly says that there are "enough" open sets, locally in X.

Theorem 5.3. *Let X be a Hausdorff space. Then a sequence of points (x_n) in X converges to at most one point in X.*

Proof. Suppose (x_n) converges to y and z in X. We must prove that $y = z$. Suppose $y \neq z$. Then y has a neighbourhood U and z has a neighbourhood V such that $U \cap V = \varnothing$. Then by definition, there exists $N_Y \in \mathbb{N}$ such that U contains all x_n for $n \geq N_y$ and similarly there exists $N_z \in \mathbb{N}$ such that V contains all x_n for $n \geq N_z$. But then $U \cap V$ must contain all points x_n for $n \geq \max(N_y, N_z)$, which is a contradiction. $\square$

Definition 5.4. If X is a Hausdorff space, and a sequence of points (x_n) of X converges to a point y of X, we say that y is *the limit* of (x_n) and write $y = \lim_{n\to\infty} x_n$.

In fact, there are weaker conditions of separability than the Hausdorff condition as well as stronger. We will mention the weaker ones here and leave you to study them in detail. They are not very useful "in practice" as they are weaker. But it is good to have seen them.

Definition 5.5. A topological space X is said to be T_0 (or less commonly, a Kolmogorov space), if for any pair of points x, y with $x \neq y$, there exists an open set U that contains one of them (say x) but not the other (say y).

This is a very weak property, and most spaces you will ever see will satisfy it. The trivial topology $\{\varnothing, X\}$ does not satisfy it, but that is about it.

Definition 5.6. A topological space X is said to be T_1 (or less commonly, a Fréchet space), if for any pair of points x, y with $x \neq y$, there exist open sets U, V such that U contains x but not y, and V contains y but not x.

This is slight strengthening of the T_0 condition, and Hausdorff is a strengthening of this by further requiring that U and V be disjoint. For this reason, Hausdorff is also called the T_2 property.

Example 5.7. On $X = \mathbb{R}^n$, define the *Zariski topology* as follows : For any $p : \mathbb{R}^n \to \mathbb{R}$ a polynomial in n variables, let

$$U_p := \{x \in \mathbb{R}^n : p(x) \neq 0\}.$$

Then, **Exercise:** $\mathcal{B} := \{U_p\}_{p(x)}$ is a basis on X. The topology generated by $\mathcal{B}$ is called the Zariski topology on $\mathbb{R}^n$. **Exercise:** Is the Zariski topology Hausdorff?

Exercise: Prove or disprove : If X is a topological space such that every sequence of points x_n converges to at most one point, then X is Hausdorff.