INTRODUCTION

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1. Motivation and foreshadowing

What is topology as a subject? Very loosely, it can be described as *the study of topological properties of topological spaces*. This, of course, is almost a tautological statement that doesn't have much meaning in itself. So let us explore what these terms might mean and try to make sense of it.

The term *topological properties* loosely refers to properties that can be expressed in terms of continuity alone. So now we have something to work with. Namely, let X and Y be two sets and f be a function between them. In order to make sense of the assertion that f is a continuous function, we usually need to specify some extra data. After all, continuity roughly embodies the notion that if x and x' are two elements of the set X that are 'close together' or 'near each other', then the function values $f(x)$ and $f(x')$ are also values that are 'close together'.

This tells us two things - topological properties are properties that can be expressed in terms of continuity of functions, and *topological spaces* should be sets in which some sense of a notion of nearness can be given.

In many cases, this can be achieved by specifying a real number $d(x, x')$ for each pair of elements x and x' for any pair of elements $x, x' \in X$. We call this number $d(x, x')$ the distance between x and x', and we say that x and x' are close together if $d(x, x')$ is *sufficiently small*. This leads to the notion of a *metric space* (X, d) when the distance function d (called the 'metric') satisfies some reasonable properties.

It might be worth asking why we as mathematicians might be interested in properties that are expressible in terms of continuity. When we considered the *general* set X, the only information that is available about two elements x, x' of X is whether they are equal or not. So a general set X appears as an unorganized collection of some elements. When X is equipped with a metric d though, it acquires a *shape* or *form*. This is why we call it a *space* rather than a set and we call its collection *points* rather than elements. Then topological properties of such a space tell us about the qualitative properties of its shape.

Example 1.1 (Intermediate Value Theorem)**.** Let $X = [0, 1]$ and let $Y = [0, 1] \cup [2, 3]$. Let f be a function from X to $\mathbb{R}$ such that $f(X) \subseteq \{0, 1\}$, i.e. f only takes values 0 of 1. If f is now assumed to be *continuous*, then the intermediate value theorem tells us that f must be a constant function.

But on the other hand, we define g be a function from Y to $\mathbb{R}$ such that $g(Y) \subseteq \{0, 1\}$ by $g([0, 1]) = 0$ and $g([2, 3]) = 1$. Then this function is continuous! (**Exercise:** Prove using the epsilon-delta definition that this function is continuous.) But of course $g(y)$ cannot be equal to $1/2$ for any $y \in Y$. So intermediate value theorem fails to hold for this Y! Why might this happen? Intuitively we can see that this happens because Y is not 'connected' while X is. So continuous functions can tell us about such qualitative properties of the shape of an object under consideration.

However, metric spaces are somewhat special among shapes that appear in Mathematics. There are cases where one can usefully make sense of a notion of nearness, even if there does not exist a metric function that expresses this notion. An example of this is given by the notion of *pointwise convergence* for real-valued functions. Recall that a sequence of functions f_n for $n = 1, 2, \ldots$ converges pointwise to a function g if, for each point t in the domain, the sequence $f_n(t)$ of real numbers converges to the real number $g(t)$. But as it turns out, if we consider $X =$ the set of real-valued functions on a general domain, then there can be no good metric d on X that captures this notion! (We will see during this course how to prove this fact.)

To handle this, and such general examples, the theory of topological spaces and their topological properties, i.e. **topology** was developed, which is a more general concept than metric spaces. The notion of 'nearness' can be termed in a metric space in terms of open balls of radius r of the metric space, as we recalled earlier. So, rather than specifying the distance between any two elements x and y of a set X, we shall instead give a meaning to which subsets $U \subseteq X$ are 'open'. Open sets then (approximately) encode 'nearness' as the following statement -

If U is open in X and $x \in U$, then all y in X that are "sufficiently close" to x also satisfy $y \in U$.

The shape of X is thus not determined by some distance function d, but rather by the specification of which subsets U of X we consider open. When this specification satisfies some reasonable conditions, we call X together with the collection of all its open subsets a "topological space". The collection of all open subsets will be called the 'topology' on X and is usually denoted $\mathscr{T}$. This notion, as it turns out, includes examples that we want (subsets of $\mathbb{R}^n$) and also other constructions of interesting objects (such as the Möbius band) and have good continuity properties to study. Thus, this notion is now considered to be in the very foundations of mathematics, and permeates a wide array of other fields of mathematics.

Note: Clearly, studying this notion involves not just elements and functions for X, but also subsets and even collections of subsets for X. It is hence required to have some familiarity with the basic notions of set theory before studying topology.

2. Topological spaces

Definition 2.1. Let X be a set. A *topology* on X is a collection $\mathscr{T}$ of subsets of X, such that :

(1) $\varnothing$ and X are both in $\mathscr{T}$.
(2) For any subcollection $\{U_i\}_{i \in I}$, the union $\bigcup_{i \in I} U_i$ is also in $\mathscr{T}$.
(3) For any finite subcollection $\{U_1, \ldots, U_n\}$, the intersection $U_1 \cap U_2 \cap \cdots U_n$ is also in $\mathscr{T}$.

A *topological space* $(X, \mathscr{T})$ is a set X with a chosen topology $\mathscr{T}$.

The subsets $U \in \mathscr{T}$ are said to be *open*. Note that this *defines* the property of being open, i.e. a set A is open in X if and only if $A \in \mathscr{T}$. Thus, we can rewrite the above axioms as -

(1) $\varnothing$ and X are both open as subsets of X.
(2) The union of any collection of open subsets of X is open.
(3) The intersection of any finite collection of open subsets of X is open.

We usually refer to the condition (2) by saying $\mathscr{T}$ is closed under arbitrary unions, and to the condition (3) by saying $\mathscr{T}$ is closed under finite intersections.

Example 2.2. Recall the notion of an ϵ-ball around a point x in $\mathbb{R}^n$ -

$$B_\epsilon(x) := \{y \in \mathbb{R}^n : d(x, y) < \epsilon\}$$

where $d(x, y)$ is the usual Euclidean distance function on $\mathbb{R}^n$. Then let $X = \mathbb{R}^n$ and define

$$\mathscr{T}_{\mathbb{R}^n} := \{U \subseteq X : \forall x \in U, \exists \epsilon \text{ such that } B_\epsilon(x) \subseteq U\}.$$

This gives the usual topology on $\mathbb{R}^n$ as seen in real analysis, since this is precisely the definition of open sets seen there. When it is unambiguous to do so, we will drop the $\mathscr{T}_{\mathbb{R}^n}$ from the notation $(\mathbb{R}^n, \mathscr{T}_{\mathbb{R}^n})$ and simply refer to $\mathbb{R}^n$ as this topological space. This topology will also be referred to as the *standard topology* on $\mathbb{R}^n$.

Exercise: Show explicitly that (x, ∞) is an open set in $\mathbb{R}$ for any real number x. Prove explicitly that $[a, b)$ is not open in $\mathbb{R}$.

Note that this topology on $\mathbb{R}$ contains sets such as $(-20, -3) \cup (5, 17) \cup (25, 167894)$, we can describe all of such sets by simply specifying the usual open intervals and then allowing all unions of them. The idea that an entire topology can be specified by some smaller collection of special open sets (open intervals in this case) along with arbitrary unions of them is an important one. We will explore it in the next lecture.

The following two topologies can be defined over any nonempty set.

Example 2.3. Let X be any nonempty set. Define $\mathscr{T}_{\text{disc}} := \mathcal{P}(X)$, that is, the collection of *all* subsets of X. This is called the *discrete topology* on X.

Example 2.4. Let X be any nonempty set. Define $\mathscr{T}_{\text{triv}} := \{\varnothing, X\}$, that is, the *only* open sets are $\varnothing$ and X. This is called the *trivial topology* on X.

These two topologies are extreme examples of topologies that you can have on any nonempty set. However, they differ from each other substantially. The discrete topology comes up sometimes. It is even a metric space. It has some strange properties; for instance, every function on such a topological space is automatically 'continuous', as we will see later, once we have defined continuous functions on a topological space. But it occurs naturally sometimes. On the other hand, the trivial topology does not come up much while doing mathematics, and we only mention it when we have to do so.

Remark 2.5. In fact, one could extend the same definitions to the empty set to give it a unique topology, since both definitions trivially match for the empty set (and also a singleton set, incidentally). Note that the axioms for a topology do not require that the set X has an element. So the empty set is also a topological space.

Example 2.6. Let $X = \{a, b\}$ a two-element set. There are four different possible topologies on X.

(1) The trivial topology $\mathscr{T}_{\text{triv}} = \{\varnothing, X\}$.
(2) An intermediate topology $\mathscr{T}_a = \{\varnothing, \{a\}, X\}$.
(3) An intermediate topology $\mathscr{T}_b = \{\varnothing, \{b\}, X\}$.
(4) The discrete topology $\mathscr{T}_{\text{disc}} = \{\varnothing, \{a\}, \{b\}, X\}$.

(1) and (4) are topologies as we have already seen. To see that $\mathscr{T}_a$ is a topology, observe that $\{b\}$ does not occur as the union or the intersection of any collection of sets in $\mathscr{T}_a$. Interchanging the role of a and b, we see that $\mathscr{T}_b$ is also a topology.

Note one interesting phenomenon! Namely, in the space $(\{a, b\}, \mathscr{T}_a)$, the element a is 'separated away' from the other point b by the open set $\{a\}$ that contains a but not b, while the only open set that contains b also contains a, since it is the set $\{a, b\}$. This means that point a is 'arbitrarily close' to point b, while point b is NOT arbitrarily close to point a! This kind of asymmetry of "nearness" in such topological spaces is not seen in metric spaces, where two points are necessarily mutually close to each other.

Example 2.7. Let X be a set. Let the *cofinite topology* or *finite complement topology* $\mathscr{T}_f$ denote the collection of subets $U \subseteq X$ whose complement $X \setminus U$ is finite or all of X.

Lemma 2.8. *The collection $\mathscr{T}_f$ is a topology.*

Proof. (1) The subset $\varnothing$ is in $\mathscr{T}_f$ by definition. The subset X is in $\mathscr{T}_f$, because its complement $\varnothing$ is finite.

(2) Let $(U_\alpha)_{\alpha \in J}$ be a subcollection of $\mathscr{T}_f$. Then we have to show that $V = \bigcup_{\alpha \in J} U_\alpha$ is in $\mathscr{T}_f$.
If each U_α is empty, then V is empty and we are done. Otherwise, there is some $\beta \in J$, such that U_β is nonempty, and thus $X \setminus U_\beta$ is finite. Then we have $X \setminus V \subset X \setminus U_\beta$ and a subset of a finite set is necessarily finite, so we are done.

(3) Let $\{U_1, \ldots, U_n\}$ be a finite subcollection of $\mathscr{T}_f$. Then we have to show that $W = \bigcap_{i=1,2,\ldots,n} U_i$ is in $\mathscr{T}_f$.
If some U_i is empty, then $W \subseteq U_i$ is empty, and we are done. Otherwise, $X \setminus U_i$ is finite for each $i = 1, 2, \ldots, n$. Further we have

$$X \setminus W = (X \setminus U_1) \cup (X \setminus U_2) \cup \cdots (X \setminus U_n)$$

and a finite union of sets of finite cardinality is finite, so we are done.

$\square$

Remark 2.9. In fact, the same argument shows that the collection $\mathscr{T}_c$ of subets $U \subseteq X$ whose complement $X \setminus U$ is countable or all of X is also a topology.

Example 2.10. Let $X = \mathbb{R}$. Then we define the *ray+ topology* on X as $(\mathbb{R}, \mathscr{T}_{\text{ray}+}) := \{(a, \infty) : a \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$. It is easily checked that this is a topology, since an arbitrary union or a finite intersection of any such 'rays' (extending to $+\infty$) is another such ray. We will see generalizations of such a definition later.

3. Comparing topologies

Definition 3.1. Let $\mathscr{T}$ and $\mathscr{T}'$ be two topologies on the same set X. We say that $\mathscr{T}$ is *coarser* than $\mathscr{T}'$, or equivalently, $\mathscr{T}'$ is *finer* than $\mathscr{T}$, if $\mathscr{T} \subseteq \mathscr{T}'$. This means that each subset $U \subseteq X$ that is open in $(X, \mathscr{T})$ is also open in $(X, \mathscr{T}')$. (So $\mathscr{T}'$ has at least as many open sets as $\mathscr{T}$.)

Lemma 3.2. *The trivial topology is coarser than any other topology, and the discrete topology is finer than any other topology.*

Proof. For any topology $\mathscr{T}$ on X, we have

$$\mathscr{T}_{\text{triv}} = \{\varnothing, X\} \subseteq \mathscr{T} \subseteq \mathcal{P}(X) = \mathscr{T}_{\text{disc}}.$$

□

The set of topologies on X becomes partially ordered by the "coarser than"-relation. Note that two topologies need not be comparable under this relation. (Hence 'partially ordered', of course.) For example, neither one of the two topologies $\mathscr{T}_a$ and $\mathscr{T}_b$ on $X = \{a, b\}$ is coarser or finer than the other. Note that two topologies $\mathscr{T}$ and $\mathscr{T}'$ on set X are equal, if they are both finer or coarser than each other.

Exercise: When X is an infinite set, show that the finite complement topology $\mathscr{T}_f$ is strictly coarser than the discrete topology $\mathscr{T}_{\text{disc}}$, i.e. $\mathscr{T}_f \subsetneq \mathscr{T}_{\text{disc}}$.

Exercise: What can you say about the standard topology $\mathscr{T}_{\mathbb{R}}$ on $\mathbb{R}$ and the ray+ topology $\mathscr{T}_{\text{ray}+}$ on $\mathbb{R}$? Is one of them coarser or finer than the other? Justify your assertion.