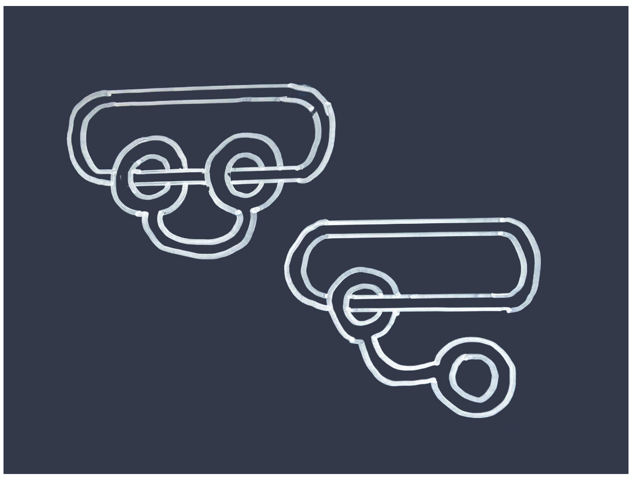
FUNDAMENTAL GROUPS OF SURFACES AND SIEFERT-VAN KAMPEN THEOREM

ADITYA KARNATAKI

1. Introduction



How would you prove or see that these two shapes are 'the same'?

Recall that we arrived at the following example at the end of the previous class.

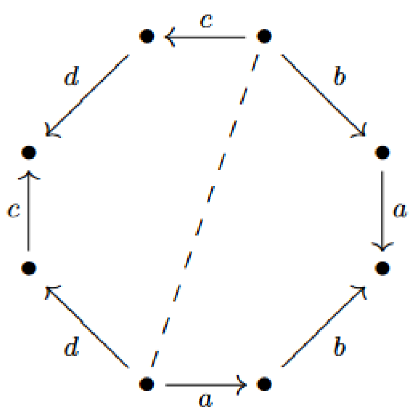
Example 1.1. Let X be the figure eight space. Then we can cover X by two open subsets U and V which deformation retract to S^1 and such that the intersection $U \cap V$ is simply connected. So a theorem from previous class implies that $\pi_1(X)$ is generated by the image of two maps from $\pi_1(S^1) \cong \mathbb{Z}$, i.e. we can express every loop in terms of $[a]$ and $[b]$, where $[a]$ and $[b]$ are loops that go around each copy of S^1 once, i.e. they generate $\pi_1(U)$ and $\pi_1(V)$. So every element in $\pi_1(X)$ can be written as $[h_1]^{n_1} * \cdots * [h_k]^{n_k}$ where each $h_i \in \{a, b\}$. But we don't know what relations are satisfied by $[a]$ and $[b]$.

For now, we can prove that this group is not abelian. To see this, look at a particular cover (figure 60.1 in Munkres). The lift of $[a] * [b]$ starting at e_0 ends at $(1, 0)$ while the lift of $[b] * [a]$ starting at e_0 ends at $(0, 1)$, so by uniqueness of liftings, we see that $[a] * [b] \neq [b] * [a]$.

This example actually leads to a nice example of a surface whose fundamental group is not abelian. (These examples are necessary to further our understanding because we have only seen abelian fundamental groups otherwise!)

2. Fundamental groups of surfaces

A 2-torus is the surface obtained by taking two copies of the torus, deleting a small open disc from each of them, and pasting the remaining pieces together along the deleted disc. This can be identified with a quotient space of the octagon.



Cutting along the dashed line, each part becomes a torus with a disc removed. The dashed line becomes the boundary circle in each part. Gluing together again along the dashed line, we get the 'connected sum' of the two tori.

Theorem 2.1. *The fundamental group of* $T^2\#T^2 = 2$*-torus is not abelian.*

Proof. The subspace $A \subset T^2\#T^2$ given by taking only the edges labeled a and c is homeomorphic to the figure eight space X. Furthermore, there is a retraction $r : X \to A$ given by first collapsing the dashed line, to get the 'one-point union' $T^2 \vee T^2$ of two tori that are joined at a single point, and then retracting each torus onto a circle, to get the one-point union $S^1 \vee S^1$ of two circles.

So there is an injective homomorphism $i_* : \pi_1(X) \to \pi_1(T^2\#T^2)$, so that $\pi_1(T^2\#T^2)$ has a non-abelian subgroup, so it can't be abelian. □

Another way of dealing with surfaces is to see if they can be formed from smaller subspaces. An obvious construction in this regard is the product topology. Let (X, x_0) and (Y, y_0) be pointed topological spaces, with cartesian product $X \times Y$ with basepoint (x_0, y_0). The projection maps $p = \pi_X : (X \times Y, (x_0, y_0)) \to (X, x_0)$ and $q = \pi_Y : (X \times Y, (x_0, y_0)) \to (Y, y_0)$ induce group homomorphisms $p_* : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0)$ and $q_* : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(Y, y_0)$. These are the components of a group homomorphism

$$\Phi = (p_*, q_*) : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

taking $[h]$ to $\Phi([h]) = (p_*([h]), q_*([h])) = ([ph], [qh])$ for any loop $h : I \to X \times Y$ based at (x_0, y_0).

Theorem 2.2. $\Phi = (p_*, q_*) : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ *is a group isomorphism.*

Proof. Any element of $\pi_1(X, x_0) \times \pi_1(Y, y_0)$ has the form $([f], [g])$, where $f : I \to X$ is a loop at x_0 and $g : I \to Y$ is a loop at y_0. Let $h : I \to X \times Y$ be given by $h(s) = (f(s), g(s))$. Then $ph = f$ and $qh = g$, so $\Phi([h]) = ([f], [g])$ and Φ is surjective.

If $h : I \to X \times Y$ is a loop at (x_0, y_0) with $\Phi([h]) = 0$ then $ph : I \to X$ is path homotopic to the constant loop e_{x_0} at x_0 and $qh : I \to Y$ is path homotopic to the constant loop e_{y_0} at y_0. Let $F : I \times I \to X$ and $G : I \times I \to Y$ be such path homotopies. Then $H : I \times I \to X \times Y$ given by $H(s, t) = (F(s, t), G(s, t))$ is a path homotopy from $h = (ph, qh)$ to $e(x_0, y_0) = (e_{x_0}, e_{y_0})$. Hence $[h] = e$ and Φ is injective. □

Corollary 2.3. *The fundamental group of the torus is* $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$.

Corollary 2.4. *The compact, connected surfaces* S^2, T^2, *and* $T^2\#T^2$ *are all topologically distinct.*

Proof. $\pi_1(S^2) \cong \{1\}$ while $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$, while $\pi_1(T^2\#T^2)$ is not abelian. Since these groups are not pairwise isomorphic, these surfaces are topologically distinct. □

3. Strong form of Siefert-van Kampen theorem

We have seen that if a space X is written as the union of two open subsets U and V whose intersection $U \cap V$ is path connected, then the images of the two groups $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$ under the homomorphisms induced by inclusion generate the group $\pi_1(X, x_0)$ for $x_0 \in U \cap V$. The strong form of Siefert-van Kampen theorem allows us to know the kernel. We state the strong form here but we won't prove them.

Theorem 3.1. *Let $X = U \cup V$, where U and V are open in X. Assume that U, V and $U \cap V$ are path connected and let $x_0 \in U \cap V$. Let $j_1 : U \hookrightarrow X$ and $j_2 : V \hookrightarrow X$ be the respective inclusions and let $j_* : \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$ be the homomorphism of the free product that extends the homomorphisms $j_{1,*}$ and $j_{2,*}$. Similarly, let $i_1 : U \cap V \hookrightarrow U$ and $i_2 : U \cap V \hookrightarrow V$ be the inclusions of the intersection $U \cap V$ in the respective subsets.*

Then j_* is surjective, and its kernel is the least normal subgroup N of the free product that contains all elements represented by words of the form $(i_{1,*}(g)^{-1}, i_{2,*}(g)$ for g varying in $\pi_1(U \cap V, x_0)$.

That is to say, that the kernel of j_* is generated by all elements of the free product of the form $i_{1,*}(g)^{-1} i_{2,*}(g)$, their inverses, and their conjugates.

Geometrically, the loops in $\pi_1(U \cap V, x_0)$ live in the intersection $U \cap V$, and i_1 and i_2 give two ways of identifying these loops as loops in X, The kernel N is then telling you that you can think of these loops as living in U, or you can think of them living in V; it makes no difference.

Corollary 3.2. *Assume the hypotheses above. If $U \cap V$ is in addition simply connected, then there is an isomorphism*

$$j_* : \pi_1(U, x_0) * \pi_1(V, x_0) \xrightarrow{\sim} \pi_1(X, x_0).$$

Example 3.3. The corollary above shows that the fundamental group of the theta space, and hence the figure eight space, is the free group on two generators. (Example 1 following corollary 70.4 in Munkres.)

4. Epilogue

The fundamental group in fact is a much more powerful invariant. Thurston's work on his geometrization conjecture and Perelman's subsequent work on Poincaré conjecture (along with the work of a lot of other people..) showed that, except a class of compact, connected 3-manifolds called Lens spaces which have a finite fundamental group, we know that the fundamental group is a complete invariant for irreducible 3-manifolds.

In particular, it is clear that the fundamental group needs to be studied even more deeply. Covering spaces offer such a possibility. One thing we did not do in this course was to study morphisms between covering spaces of the same base space. It turns out that there is a notion of 'equivalence of two covering spaces'. Any two equivalent covering spaces identified essentially by a subgroup of the fundamental group, to which they both correspond in a natural way. This leads to a 'Galois correspondence' between subgroups of the fundamental group and covering spaces. This idea led Grothendieck to define fundamental groups in situations other than topological spaces and their coverings. This has proved very fruitful in a lot of mathematics.

But of course even the fundamental group has its limits. (Pun intended? Maybe not.. even I have my limits!) Which is why algebraic topology for more complicated spaces needs to study either higher homotopy groups, or more commonly, what are known as homology and cohomology groups. Those who are interested should take a look. Tools such as higher homotopy and homology and cohomology allow us to prove higher dimensional analogues of familiar theorems such as Brouwer's fixed point theorem and Borsuk-Ulam theorem. Of course, with added power comes added complexity- the nature of these objects is still a matter of study and deliberation. But in cases where one can compute them, there has been great wealth of knowledge made possible. In fact, J.-P. Serre got his fields medal for developing new techniques to compute homotopy groups of spheres. These groups describe how spheres of various dimensions can wrap around each other. Unlike homology groups, these are surprisingly complex and difficult to compute.

It is also possible to study these objects using category theory and higher category theory. We have barely touched category theory in our course. But it is far desirable to acquire a working knowledge of the subject for any mathematician. We have already seen how category theory language can simplify difficult-looking proofs in algebraic topology. Both these subjects have fed each other in terms of new ideas and their development and both have become considerably richer for it.

Topology in general is a great subject, studied by many great intellectuals over many years. This first course in topology can only show you so much in it. By solving problems, working out examples, and exploring for yourself, you can engage further with it and enjoy your journey through it.