TIETZE EXTENSION THEOREM AND TOPOLOGICAL GROUPS

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1. Introduction

Today we will first see another useful consequence of Urysohn's Lemma. This property is one of the most useful properties of normal spaces.

Theorem 1.1 (Tietze Extension Theorem)**.** *Let X be a normal space, let $A \subseteq X$ be a closed subspace, and let $f : A \to [a, b]$ be a continuous function for some $[a, b] \subseteq R$. There exists a continuous function $\bar{f} : X \to [a, b]$ such that $\bar{f}|_A = f$.*

After we see these applications, we will see a very important topic - namely, topological groups.

2. Tietze Extension Theorem

Recall the following definition :

Definition 2.1. Let X be a topological space, let (Y, d) be a metric space, and let $\{f_n : X \to Y\}$ be a sequence of functions. We say that the sequence $\{f_n\}$ converges uniformly to a function $f : X \to Y$ if, for every $\epsilon > 0$ there exists $N > 0$ such that $d(f(x), f_n(x)) < \epsilon$ for all $x \in X$ and for all $n \geq N$.

Recall also the following lemma :

Lemma 2.2. *Let X be a topological space and let (Y, d) be a metric space. Assume that $f_n : X \to Y$ is a sequence of functions that converges uniformly to $f : X \to Y$. If all functions f_n are continuous then f is also a continuous function.*

Now we have the following key proposition.

Proposition 2.3. *Let X be a normal space, $A \subseteq X$ be a closed subspace, and let $f : A \to R$ be a continuous function such that for some $C > 0$ we have $|f(x)| \leq C$ for all $x \in A$. There exists a continuous function $g : X \to \mathbb{R}$ such that $|g(x)| \leq \frac{1}{3}C$ for all $x \in X$ and $|f(x) - g(x)| \leq \frac{2}{3}C$ for all $x \in A$.*

Proof. Define $Y := f^{-1}([-C, -\frac{1}{3}C]), Z := f^{-1}([\frac{1}{3}C, C])$. Since $f : A \to \mathbb{R}$ is a continuous function, these sets are closed in A, but since A is closed in X the sets Y and Z are also closed in X . Since $Y \cap Z = \varnothing$, by Urysohn's lemma, there is a continuous function $h : X \to [0, 1]$ such that $h(Y) \subseteq \{0\}$ and $h(Z) \subseteq \{1\}$. Define $g : X \to \mathbb{R}$ by

$$g(x) := \frac{2C}{3}\left(h(x) - \frac{1}{2}\right).$$

Then since $h(x) \in [0, 1]$, we have $|g(x)| \leq \frac{1}{3}C$. Finally, in each of the cases $x \in Y$, $x \in Z$, or $x \in X \setminus (Y \cup Z)$, it is straightforward to check that both $f(x)$ and $g(x)$ fall in the same intervals $[-C, -\frac{1}{3}C]$, $[\frac{1}{3}C, C]$, or $[-\frac{1}{3}C, \frac{1}{3}C]$ respectively. So that $|f(x) - g(x)| \leq \frac{2}{3}C$ always holds. □

Proof of Tietze Extension Theorem. Without loss of generality, we can assume that $[a, b] = [0, 1]$. For $n = 1, 2, \ldots$, we will construct continuous functions $g_n : X \to \mathbb{R}$ such that

(1) $|g_n(x)| \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$ for all $x \in X$;
(2) $|f(x) - \sum_{i=1}^{n} g_i(x)| \leq \left(\frac{2}{3}\right)^n$ for all $x \in A$.

We argue by induction. This is a repeated application of the previous lemma. Existence of g_1 follows directly from previous proposition. Assume that for some $n \geq 1$, we already have functions $g_1, \ldots, g_n$ satisfying (1) and (2).

Then in the previous proposition, take f to be $f - \sum_{i=1}^{n} g_i$ and take C to be $\frac{2}{3}^n$. Then we can take $g_{n+1} := g$ where g is defined by that proposition for this choice of f and C.

Then, we define

$$\bar{f} := \sum_{n=1}^{\infty} g_n.$$

This series converges uniformly because of condition (1). By the lemma we recalled, this implies that $\bar{f}$ is continuous, since all the partial sums are continuous. Finally, $\bar{f}(x) = f(x)$ for all $x \in A$ because of condition (2). $\square$

Here is a useful reformulation of Tietze Extension Theorem:

Theorem 2.4. *Let X be a normal space, let $A \subseteq X$ be a closed subspace, and let $f : A \to \mathbb{R}$ be a continuous function. There exists a continuous function $\bar{f} : X \to \mathbb{R}$ such that $\bar{f}|_A = f$.*

Proof. It is enough to show that for any continuous function $g : A \to (-1, 1)$ there exists a continuous function $\bar{g} : X \to (-1, 1)$ such that $\bar{g}|_A = g$, since $(-1, 1)$ is homeomorphic to $\mathbb{R}$.

Assume then that $g : A \to (-1, 1)$ is a continuous function. Then we know by Tietze extension theorem that we can extend this to a function $g' : X \to [-1, 1]$. The question then is how to find $\bar{g}$ that maps X into the *open* interval?

Let $B := g'^{-1}(\{-1, 1\})$. The set B is closed in X and it does not intersect A since $g'(A) = g(A) \subseteq (-1, 1)$. Therefore by Urysohn's lemma, there is a continuous function $k : X \to [0; 1]$ such that $B \subseteq k^{-1}(\{0\})$ and $A \subseteq k^{-1}(\{1\})$. Then we define

$$\bar{g}(x) = k(x).g'(x).$$

This is a continuous function. We claim this is the required function.

Indeed, if $g'(x) \in (-1, 1)$, i.e. $x \in A$, then $\bar{g}(x) \in (-1, 1)$. And if $g'(x) \in \{-1, 1\}$, then $\bar{g}(x) = 0 \in (-1, 1)$. Hence, $\bar{g}(x) \in (-1, 1)$ for all $x \in X$. Finally, $\bar{g}(x) = g'(x) = g(x)$ for all $x \in A$. $\square$

This ends our discussion of Urysohn's lemma and its consequences. There is another very interesting application of it, which is constructing imbeddings of manifolds. That is left for self-study and tutorials.

3. Topological Groups

Definition 3.1. A topological group G is a group endowed with a topology $\mathscr{T}_G$ such that multiplication $(x, y) \to xy : G \times G \to G$ and inversion $x \to x^{-1}$ are continuous in this topology.

If G is a topological group and X a topological space, then a topological group action of G on X is a continuous action $(g, x) \to gx : G \times X \to X$.

Lemma 3.2. *Let H be a group that is also a topological space satisfying T_1 condition, i.e. points of H are closed. Then H is a topological group if and only if the map $H \times H \to H$ sending (x, y) to xy^{-1} is continuous.*

Proof. If H is a topological group, then the map $(x, y) \to xy^{-1}$ is a composite of two continuous functions and hence continuous. On the other hand, if this function is continuous, then it is continuous in both components, and hence the functions $f((1, y))$ and $f(x, f((1, y)))$ are both continuous. $\square$

In fact, we will implicitly assume that all our topological groups are T_1. This is a harmless assumption, even more than visible, because for topological groups it is in fact true that a topological group G is T_0 if and only if it is T_1 if and only if it is T_2! This will be an **exercise** for you to prove.

Example 3.3.
(1) Any group can be a topological group if it is endowed with the discrete topology. In particular, $(\mathbb{Z}, +)$ is a topological group.
(2) $(\mathbb{R}, +)$ is a topological group. To see this, let $(x, y) \in \mathbb{R}^2$ and $\epsilon > 0$. Then, choose $B_{\epsilon/2} := \{x' : |x' - x| < \epsilon/2\} \times \{y' : |y' - y| < \epsilon/2\}$ and then triangle inequality shows that $|(x + y) - (x' + y')| < \epsilon$ for $(x', y') \in B_{\epsilon/2}$, i.e. addition is continuous at (x, y). Similarly, inversion is continuous, since for any $x \in \mathbb{R}$ and $\epsilon > 0$, if $B_\epsilon := \{x' : |x' - x| < \epsilon\}$, then $|(-x') - (x)| = |x - x'| < \epsilon$ so inversion is continuous at x.
(3) $(\mathbb{R}_+, \times)$ is a topological group. **Exercise:** Prove this.
(4) $(S^1, \times)$ is a topological group where we identify S^1 with the complex unit circle. **Exercise:** Prove this.
(5) The groups $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$ are topological groups when considered as subspaces of $\mathbb{R}^{n^2}$ and $\mathbb{C}^{n^2}$. **Exercise:** Prove this. (This might feel difficult, but it really is not, since matrix multiplication and inversion are functions of matrix entries in a direct way. It is just more complicated to write down.)

Lemma 3.4. *On a topological group, the following maps are homeomorphisms.*
- *Left multiplication by an element.*
- *Right multiplication by an element.*
- *Inversion.*
- *Conjugation by an element.*

Proof. From the definition, these are all continuous maps, and they are obviously bijections. Since their inverse maps are also given by the same types of maps, they are also continuous. □

These innocuous statements imply that to check what happens near any point g, it suffices to check near the identity e. Such spaces X, where for any pair of points x, y there is a homeomorphism of X onto itself that carries x to y, are called *homogeneous spaces*.

Lemma 3.5. *Let H be a subspace of a topological group G. Then if H is also a subgroup of G, then both H and Cl H are topological groups.*

Proof. Note that both H and Cl H are T_1 spaces. Then let $f : H \times H \to H$ be the map $(x, y) \to xy^{-1}$. Cl H is a subgroup if and only if $f(\text{Cl } H \times \text{Cl } H) \subseteq \text{Cl } H$. This is true since $f(\text{Cl } H \times \text{Cl } H) = f(\text{Cl } (H \times H))$ (recall this!) and thus

$$f(\text{Cl } H \times \text{Cl } H) = f(\text{Cl } (H \times H)) \subseteq \text{Cl } f(H \times H) \subseteq \text{Cl } H$$

since f is continuous and H is a subgroup. Now since H and Cl H are subspaces, the restriction of f to H or Cl H is continuous, and hence they are topological groups. □

Note that this shows that the subgroups $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ of $\mathrm{GL}_n(\mathbb{R})$ are topological groups.

3.1. Quotient groups.

For any group G and a subgroup H, we have the collection of left cosets of H in G denoted G/H. This is a partition of G, so we can give it a quotient topology. If this is a group (as is the case when H is normal), is it a topological group?

Proposition 3.6.
(1) G/H is a homogeneous space.
(2) If H is closed in G, then G/H is T_1.
(3) Show that the quotient map $p : G \to G/H$ is open.
(4) Show that if H is closed in G, and H is a normal subgroup of G, then G/H is a topological group.

Proof.
(1) For any $\alpha \in G$, denote the left multiplication by α as a map f_α. Then, for any α, $p_\alpha = p \circ f_\alpha : G \to G/H$ is a quotient map, since it is a composite of a homeomorphism with a quotient map. It maps $x \in G$ to $\alpha x H \in G/H$. Then $\alpha x H = \alpha y H$ if and only if $xH = yH$ by elementary group theory, so that p_α is constant on the sets $p^{-1}(xH)$, and hence it induces a map $G/H \to G/H$ sending xH to $\alpha x H$. Taking $\alpha = yx^{-1}$ for any pair of elements x, y, we get the result.

(2) We know H is closed. Since f_α is a homeomorphism, $xH = f_x(H)$ is closed, and by definition of quotient topology, $\{xH\}$ is closed in G/H.
(3) Let U be an open set in G. Then $p(U) = \bigcup_{x \in U} xH$ is simply the union

$$p(U) = \bigcup_{x \in U} xH = \bigcup_{x \in U} \bigcup_{h \in H} x.h = \bigcup_{h \in H} U.h$$

and this is open since right multiplication by h is a homeomorphism for each h.
(4) **Exercise.** (You should use the fact that p is an open quotient map, that we just proved.)

□

We end with a definition that we won't explore further in class.

Definition 3.7. Let G and H be topological groups. A map of topological groups $f : G \to H$ is a continuous map of topological spaces which is also a group homomorphism.