HOMEWORK 8

(1) Show the lemma that we mentioned in class : Namely, Let $f : X \to Y$ be a surjective function. Define an equivalence relation $\sim$ on X by declaring $x \sim x'$ if and only if $f(x) = f(x')$. There is an induced bijection

$$(X/\sim) \to Y$$

given by $h([x]) = f(x)$. Its inverse is h^{-1} takes y to $f^{-1}(y)$, which equals $[x]$ for any choice of x such that $f(x) = y$. The surjection $f : X \to Y$ thus factors as the canonical surjection $\pi : X \to (X/\sim)$ and a bijection $h : (X/\sim) \to Y$.

(2) Show that topologically the real plane and the complex plane are the same.

(3) Show that if X is second countable, then X is separable.

(4) Show that a countable topological space X (that is, X is countable as a set) is second countable if and only if it is first countable.

(5) Find an example of a second countable space X which has a quotient that is not second countable.

(6) Show that there exists a topological space X that is countable but not first countable.

(7) Let $X = \mathbb{R} \times \{1, 2\}$, where $\{1, 2\}$ is equipped with the discrete topology, and consider the equivalence relation given by $(x, 1) \sim (x, 2)$ for all $x \neq 0$ (but $(0, 1) \not\sim (0, 2)$). Show that the quotient topology on $X/\sim$ is not Hausdorff.

(8) Let R denote an equivalence relation on X and let S denote an equivalence relation on Y. Show that $R \times S$ is an equivalence relation on $X \times Y$. Show that it is not necessary that $(X/R) \times (Y/S)$ is homeomorphic to $(X \times Y)/(R \times S)$.

(9) Topology (Munkres), Chapter 2, Section 22, Exercise (2).

(10) Topology (Munkres), Chapter 2, Section 22, Exercise (3).

(11) Topology (Munkres), Chapter 4, Section 31, Exercise (2).

(12) Topology (Munkres), Chapter 4, Section 31, Exercise (12).

(13) Topology (Munkres), Chapter 4, Section 31, Exercise (13).