HOMEWORK 10

(1) The space of $n \times n$ real-valued matrices $M_n(\mathbb{R})$ can be viewed as the space of order n-tuples of column vectors in $\mathbb{R}^n$, and thus we can identify $M_n(\mathbb{R}) \cong (\mathbb{R}^n)^n \cong \mathbb{R}^{n^2}$.

- Show that the determinant function $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is continuous.
- Let $\mathrm{GL}_n(\mathbb{R})$ denote the set of invertible $n \times n$ real-valued matrices. Show that this is not connected as a subspace of $M_n(\mathbb{R})$.
- Let $\mathrm{GL}_n^+(\mathbb{R}) \subseteq \mathrm{GL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{R})$ denote the subgroup of $n \times n$ matrices A with $\det A > 0$ and $\det A = 1$ respectively. Show that $\mathrm{GL}_n^+(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{R})$ are path connected. (Hint : Use the three standard row operations used in Gaussian elimination method to draw a path from any matrix A in $\mathrm{GL}_n^+(\mathbb{R})$ to the identity $n \times n$ matrix.)

(2) Prove that $\mathrm{GL_n}(\mathbb{C})$ and $\mathrm{SL_n}(\mathbb{C})$ are path connected. (This is far easier to prove than the previous example.)

(3) Let $(W, \leq)$ be a well-ordered set endowed with the order topology. If W has more than one point, show that it is disconnected.

(4) Show the following properties of locally connected spaces :

- Show that a continuous image of a locally connected space need not be continuous by finding a counterexample.
- Show that a subspace of a locally connected space need not be locally connected by finding a counterexample.
- Show that local connectedness is a topological invariant.

(5) Show the following properties of totally disconnected spaces:

- Show that a topological space X is totally disconnected if for every pair of distinct points $a, b \in X$ there are disjoint open sets U, V such that $a \in U$, $b \in V$ and $X = U \sqcup V$.
- Show that a subspace of a totally disconnected space is totally disconnected.
- Show that an arbitrary product of totally disconnected spaces X_α is totally disconnected.
- Show that any discrete space is totally disconnected.
- Show that $\mathbb{Q}$ (in the subspace topology from $\mathbb{R}$) is totally disconnected but not discrete.
- Show that every discrete space is locally connected.
- Show that a totally disconnected locally connected space X must be discrete.

(6) Topology (Munkres), Chapter 3, Section 24, Exercise (1).

(7) Topology (Munkres), Chapter 3, Section 24, Exercise (2).

(8) Topology (Munkres), Chapter 3, Section 24, Exercise (3).

(9) Topology (Munkres), Chapter 3, Section 24, Exercise (8).

(10) Topology (Munkres), Chapter 3, Section 24, Exercise (12).

(11) Topology (Munkres), Chapter 3, Section 25, Exercise (3).

(12) Topology (Munkres), Chapter 3, Section 25, Exercise (4).