HOMEWORK 18

Note: $*$ marked problems might be slightly more difficult or interesting than the unmarked ones.

(1) Recall that a map $f : A \to B$ in a category $\mathcal{C}$ is an isomorphism if there exists a map $g : B \to A$ in $\mathcal{C}$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$.

(1) Check that g as above, if it exists, is unique.
(2) Check that id_A is an isomorphism for any A.
(3) Check that if f is an isomorphism, f^{-1} is also an isomorphism.
(4) Check that if h and f are isomorphisms, $h \circ f$ is an isomorphism.

(2) Show that if $A \cong B$ in a category, then $\mathrm{Aut}(A) \cong \mathrm{Aut}(B)$ as groups.

(3) Let X be a topological space. Define $\pi_0(X)$ to be the set of path components of X. Show that $\pi_0(X)$ is a functor from <u>Top</u> to <u>Sets</u>.

(4) Show that the process of abelianization induces a functor ab : <u>*Groups*</u> $\to$ <u>*Groups*</u> and that the projection map $\pi_G : G \to G^{\mathrm{ab}}$ gives a natural transformation π from the Id functor to ab functor. (Id denotes the identity functor.)

(5)* Let p be a fixed prime number. For $n \geq 1$, consider the finite groups $X_n = \mathbb{Z}/p^n\mathbb{Z}$ with the discrete topology on each of them. Show that the space

$$\mathbb{Z}_p := \left\{ (a_n)_{n=1}^{\infty} \in \prod_n X_n : a_{n+1} \cong a_n \mod p^n \text{ for } n = 1, 2, \ldots \right\}$$

is compact, Hausdorff, and totally disconnected.