HOMEWORK 17

Note: $*$ marked problems might be slightly more difficult or interesting than the unmarked ones.

(1) Let X be an infinite set and $\mathcal{F}$ be the Fréchet filter on it. Show that any other filter containing $\mathcal{F}$ (in particular any ultrafilter containing it) cannot contain any finite sets.

(2) Show that for a nonempty set X, any collection $S \subseteq \mathcal{P}(X)$ with the finite intersection property generates a unique smallest filter on X that contains it.

(3) Let X be a nonempty set and $\mathcal{U}$ a filter on X. Then show that $\mathcal{U}$ is an ultrafilter if for any subset $A \subseteq X$, either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$. (We did the converse implication of this in class.)

(4)* Let X be a nonempty set and $\mathcal{F}$ be a filter on X. Then show that $\mathcal{F}$ can be extended to an ultrafilter $\mathcal{U}$ on X. (Hint: Use Zorn's Lemma.)

(5) Let X be a topological space. Then x is accumulation point of a filter $\mathcal{F}$ if and only if there exists a filter $\mathcal{G} \supseteq \mathcal{F}$ such that $\mathcal{G} \to x$.