CATEGORY LANGUAGE

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1. Introduction

A category might be described as a system of related objects. The objects do not live in isolation from each other: there is some notion of relation between them. Typical examples of such are 'sets', 'groups' and 'topological spaces', and typical examples of 'relations' between them are 'function', 'homomorphism' and 'continuous map', respectively. But some categories can be very different from these, though. In fact, the 'maps' in categories need not be anything like maps that you are most familiar with. Category theory takes an eagle's eye view of mathematics, where specific details disappear, but it becomes possible to spot broad patterns that were hitherto difficult to see. We will not go into too much detail, but it is useful to have a basic familiarity with category theory.

Categories are themselves mathematical objects, of course, and so it seems reasonable to guess that there is a way to define a good notion of a 'map between categories'. Such maps are called as functors. What is perhaps more surprising is that functors themselves are well-behaved mathematical objects that admit a further level: we can talk about maps between functors! These are called natural transformations. In fact, algebraic topologists had already begun to grope around the notion of a natural transformation. Then Eileenberg and Maclane realised that a precise notion was needed - but for that a precise notion of a functor was first needed - but for that a precise notion of a category was first needed - and thus the subject came to be.

Since we are interested in learning algebraic topology, it is thus beneficial to study some basic notions in category theory. We won't go into too much details, but just introduce the subject.

2. Basic definitions

Definition 2.1. A category $\mathcal{C}$ consists of:

(1) a collection $\mathrm{ob}(\mathcal{C})$ of *objects*;

(2) for each $A, B \in \mathrm{ob}(\mathcal{C})$, a collection $\mathrm{Hom}(A, B)$ of *morphisms from A to B*;

(3) for each $A, B, C \in \mathrm{ob}(\mathcal{C})$, a *function*

$$\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$$

denoted $(g, f) \to g \circ f$ called *composition*;

(4) for each $A \in \mathrm{ob}(\mathcal{C})$, an element 1_A of $\mathrm{Hom}(A, A)$, called the *identity* on A;

such that these data satisfy the following axioms:

(i) associativity: for any $f \in \mathrm{Hom}(A, B), g \in \mathrm{Hom}(B, C)$ and $h \in \mathrm{Hom}(C, D)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$;

(ii) identity laws: for any $f \in \mathrm{Hom}(A, B)$, we have $f \circ 1_A = f = 1_B \circ f$.

We often write $A \in \mathcal{C}$ to mean that $A \in \mathrm{ob}(\mathcal{C})$, $f : A \to B$ to mean $f \in \mathrm{Hom}(A, B)$, and gf to mean $g \circ f$.

Remark 2.2. if $f \in \mathrm{Hom}(A, B)$, we say A is *domain* of f and B is *codomain* of f. Every map f in a category has a definite domain and codomain. Note that $\mathrm{Hom}(A, B)$ are treated as abstract sets here, and there can be no intersection $\mathrm{Hom}(A, B) \cap \mathrm{Hom}(A', B')$ of such abstract sets (unless $A = A'$ and $B = B'$) which manifest two different universes, so to speak.

Remark 2.3. Note that 1_A for any object A is unique. If e_A is any such object, then $e_A = e_A \circ 1_A = 1_A$.

Example 2.4. Below are well-known examples of categories.

(1) There is a category Sets described as follows. Its objects are sets. Given sets A and B, a map from A to B in the category Sets is exactly a set theoretical function. Composition in the category is ordinary composition of functions, and the identity maps are again what you would expect.

(2) There is a category Groups of groups, whose objects are groups and whose maps are group homomorphisms.

(3) Similarly, there is a category Rings of rings and ring homomorphisms.

(4) For each field k, there is a category $\underline{Vect_k}$ of vector spaces over k and linear maps between them.

(5) There is a category Top of topological spaces and continuous maps.

These examples may suggest to you that all categories are 'sets with some enhanced structure', but this is hardly the case. In general, there is nothing forcing a category to be 'sets with extra structure'. Thus, in a general category, it does not make sense to talk about the 'elements' of an object.

Definition 2.5. A map $f : A \to B$ in a category $\mathcal{C}$ is an isomorphism if there exists a map $g : B \to A$ in $\mathcal{C}$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. We can call g as 'the' inverse isomorphism of f in this case and denote it as f^{-1}. We sometimes write $A \cong B$ and say that A and B are isomorphic.

The following is an easy **Exercise.** But in this generality, you should do it properly.

(1) Check that g as above, if it exists, is unique.
(2) Check that id_A is an isomorphism for any A.
(3) Check that if f is an isomorphism, f^{-1} is also an isomorphism.
(4) Check that if h and f are isomorphisms, $h \circ f$ is an isomorphism.

Corollary 2.6. *For any object A in a category $\mathcal{C}$, the subset $\{f \in \operatorname{Hom}(A, A) : f \text{ is an isomorphism}\} \subseteq \operatorname{Hom}(A, A)$ is a group. We denote this subset as* $\operatorname{Aut}(A)$.

Lemma 2.7. *If $A \cong B$ in a category, then* $\operatorname{Aut}(A) \cong \operatorname{Aut}(B)$ *as groups.*

Proof. **Exercise.** □

Now we look at relations between categories, i.e. functors.

Definition 2.8. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ consists of:

(1) a rule that assigns an object $F(A)$ of $\mathcal{D}$ to any object A of $\mathcal{C}$, written as $A \to F(A)$;

(2) for each $A, A' \in \mathcal{C}$, a function $\operatorname{Hom}_{\mathcal{C}}(A, A') \to \operatorname{Hom}_{\mathcal{D}}(F(A), F(A'))$, written as $f \to F(f)$;

such that the following axioms are satisfied:

(i) $F(f' \circ f) = F(f') \circ F(f)$ whenever $f : A \to A'$ and $f' : A' \to A''$ in $\mathcal{C}$; and

(ii) $F(1_A) = 1_{F(A)}$ for any A an object in $\mathcal{C}$.

Example 2.9. There are some familiar functors that we know already.

(1) 'Forgetful functors' are functors that forget all or some of the structure carried by the objects and morphisms in the category. For example, Groups, Rings, Top, all have a functor to Sets that simply carries an object of these categories to its underlying set. Any morphisms are then carried to the same morphisms now understood as set-theoretic functions. One can also consider the functor from Rings to Groups that forgets the multiplicative structure on any ring and only keeps the group structure.

(2) There is a functor F from Sets to Groups that sends a set A to the free group formally generated by elements of A.

Definition 2.10. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Let $F_1 : \mathcal{C} \to \mathcal{D}$ and $F_2 : \mathcal{C} \to \mathcal{D}$ be two functors from $\mathcal{C}$ to $\mathcal{D}$. A *natural transformation* α from F_1 to F_2 consists of the following data: consists of:

(1) For each object A of $\mathcal{C}$, a morphism $\alpha_A : F_1 A \to F_2 A$ in $\mathcal{D}$;

(2) for each morphism $f : A \to A'$ of objects $A, A' \in \mathcal{C}$, the following diagram has to commute:

$$\begin{tikzcd} F_1 A \arrow[r, "F_1 f"] \arrow[d, "\alpha_A"'] & F_1 A' \arrow[d, "\alpha_{A'}"] \\ F_2 A \arrow[r, "F_2 f"'] & F_2 A' \end{tikzcd}$$

This condition is called 'naturality'.

We end with an example of the power of categories that will be illustrative later.

Example 2.11. Fix a group G. Then we claim that this is essentially the same as a category $\mathcal{G}$ that has that has only one object and in which all the maps are isomorphisms.

So let $\mathcal{G}$ be a category with only one object in which all maps are isomorphisms. It is not important what we call this object. It is like choosing a variable name for a polynomial ring. Let us call this object A. Then we have seen that $\mathrm{Aut}(A) = \mathrm{Hom}(A, A)$ is a group. Conversely, given a group G, we can simply define our category to be a category with a single object A such that $\mathrm{Hom}(A, A) = G$ and the $\circ$ operation is simply equal to the group multiplication opearation.

This prompts the following definition.

Definition 2.12. A *groupoid* is a category in which all maps are isomorphisms.

In algebraic topology, we will associate to a topological space X an algebraic object $A(X)$ (a set or a group). So functors show up quite naturally there. As a matter of fact, groupoids also occur quite naturally in algebraic topology. We will see all this from the next time.