PATH CONNECTEDNESS

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We have seen connectedness property of spaces and how it can tell us more about the shape of the space. Today We will explore a stronger property called path-connectedness. A path-connected space is one in which you can essentially walk continuously from any point to any other point.

1. Path Connectedness

Definition 1.1. Given points $x, y \in X$ a *path* in X from x to y is a continuous map $f : [a, b] \to X$ with $f(a) = x$ and $f(b) = y$, where $[a, b] \subseteq R$. A space X is *path connected* if for any two points x and y of X there exists a path in X from x to y.

Remark 1.2. Note that by a linear change of variables, we can take $[a, b] = [0, 1]$ without loss of generality.

Definition 1.3. We can define an equivalence relation $x \sim y$ if and only if x and y there exists a path between x and y. The equivalence classes are called 'path comoponents' of X. (We will return to these in more detail in algebraic topology.) Note that this is an equivalence relation because : (1) a point is connected to itself by the constant path $f(t) = x$ for $t \in [0, 1]$, and (2) paths can always be reversed via the variable change $t \to 1 - t$, and (3) two paths which have a common endpoint can always be concatenated by a linear variable change. (Make sure you know how to write this down!)

Proposition 1.4. *If X is path connected, X is connected.*

Proof. Assume that X admits a separation $X = U \sqcup V$ where U and V are disjoint open subsets. Let $x \in U$. Then, given any other point $y \in X$ and a path $f : [a, b] \to X$ which connects x to y, we have that $f([a, b]) \subseteq U$ since $f([a, b])$ must be connected. This implies that $y \in U$. Since y was an arbitrary point in X, this implies that $U = X$ and thus $V = \varnothing$ is the only possibility, and we are done. $\square$

The converse is not true. The canonical example is the 'topologist's sine curve'.

Example 1.5. Let $S := \{(x, y) \in \mathbb{R}^2 : y = \sin(1/x), x > 0\}$ and $\bar{S} := S \cup \{(0, 0)\}$. Note that this definition differs a bit from the definition in Munkres's book.

S is connected, since it is the continuous image of the interval $(0, \infty)$, and $(0, 0)$ is a limit point of S, so $\bar{S}$ is connected.

But $\bar{S}$ is not path-connected since there is no path that can connect, say $(\frac{1}{\pi}, 0) \in S$ to the point $(0, 0)$. Indeed, if there were such a path $f : [a, b] \to \bar{S}$, then by the intermediate value theorem, the x-coordinate would need to take all the values between $1/\pi$ and 0, so that there exist $t_1, t_2, \dots, t_n, \dots \in [a, b]$ such that

$$f(t_n) = \left(\frac{1}{2n\pi + \pi/2}, 1\right)$$

and thus $f(t_n) \to (0, 1)$ as $n \to \infty$. We can then find a convergent subsequence of $\{t_n\}$ which converges to some t_∞. (Remember this fact from analysis course!) Then by continuity of f, we have $f(t_{\infty)} = (0, 1) \notin \bar{S}$, which is a contradiction.

However, for "well-behaved" spaces, path connectedness is the same as connectedness.

Proposition 1.6. *If $A \subseteq \mathbb{R}^n$ is open, then A is connected if and only if it is path connected.*

Proof. We need to show that A as above is path connected if it is connected. The key claim in this proof then is the following :

Claim: If A is open in $\mathbb{R}^n$, then the path components of A are open in $\mathbb{R}^n$.

Indeed, if $x \in A$, then there exists $r > 0$ such that $B_r(x) \subseteq A$, and any two points of $B_r(x)$ can be connected by a straight line segment. So all of $B_r(x)$ is in the same path component as x, since if there is a path from some $a \in A$ to x, then we can add the corresponding line segment to any other point in $B_r(x)$ to get a parth to that point - and vice versa. Hence, path component of x is open. Since x was arbitrary, the claim is proven.

Equipped with the claim, we can take U to be the path component of some $a \in A$, and V to be the union of all the other path components. Then, $X = U \sqcup V$ as a union of open sets, which by hypothesis of connectedness of X must mean that $U = X$ and $V = \varnothing$ and thus X has only one path component, i.e. X is path connected. $\square$

The claim in the above proof is evidently an important property. This leads to the following definition (the relation will be clear after the proposition that will follow the definition).

Definition 1.7. X is said to be *locally connected at x* if for every neighbourhood U of x, there is a connected neighbourhood V of x contained in U. If X is locally connected at each of its points, it is said to be locally connected.

X is said to be *locally path connected at x* if for every neighbourhood U of x, there is a path connected neighbourhood V of x contained in U. If X is locally path connected at each of its points, it is said to be locally path connected.

Exercise: Show that there is no implication relationship between connectedness and local connectedness. In fact, find four topological spaces which are respectively:

(1) Not connected or locally connected.
(2) Connected but not locally connected.
(3) Locally connected but not connected.
(4) Connected and locally connected.

Proposition 1.8. *(1) X is locally connected if and only if for every open set U of X, each connected component of U is open in X.*
(2) X is locally path connected if and only if for every open set U of X, each path component of U is open in X.

Proof. We will show the first claim, leaving the second one as an **Exercise.** Suppose X is locally connected, and take an open set U and a connected component C of U. If x is in C, we can choose a connected neighbourhood V of x such that $V \subseteq U$ by definition. Since V is connected, it must lie entirely in the connected component C of U. Hence C is open in X by definition.

Conversely, suppose components of open sets in X are open in X. Given a point x and a neighbourhood U of x, let C be the component of U containing X. C is connected, and it is open in X by hypothesis, so by definition X is locally connected at x. $\square$

You can explore the notions of locally connectedness or locally path connectedness more through self-study or tutorials (and certainly the homework problems).