FILTERS

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We will discuss a different characterization of compactness in terms of *filters* and *ultrafilters*. The proof using these ideas is fairly straightforward, which may not be the case with the proof in the textbook. Ultrafilters are seeing a surge in their usage in many areas of mathematics such as algebraic geometry and number theory, so this proof feels better from a modern viewpoint. Also, filters give a notion of convergence that is more general than sequences, so this is worth our time to learn.

We recall the characterization of compactness in terms of finite intersection property of closed sets.

Definition 0.1. A collection $\mathcal{E}$ of subsets of X has the *finite intersection property* if for each finite subcollection $\{C_1, ..., C_n\} \subseteq \mathcal{E}$ the intersection

$$C_1 \cap \cdots \cap C_n$$

is nonempty.

Proposition 0.2. *A space X is compact if and only if for each collection $\mathcal{E}$ of closed subsets of X having the finite intersection property, the intersection $\bigcap_{K \in \mathcal{E}} K$ is nonempty.*

This characterization will help us when we learn more about filters and ultrafilters. We now give the definition.

Definition 0.3. Let X be a set. A nonempty collection $\mathcal{F} \subset P(X)$ is called a *filter* on X if the following three properties are satisfied :

1. $\varnothing \notin \mathcal{F}$.
2. F is closed upwards : if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.
3. F is closed under finite intersections: if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

A filter $\mathcal{F}$ on a set X is called an *ultrafilter* if it is not properly contained in any other filter on X. A subset $\mathcal{F}'$ of a filter $\mathcal{F}$ that is itself a filter is called a *subfilter* of $\mathcal{F}$.

Example 0.4. (1) Given a nonempty set X, $\mathcal{F} = X$ is a filter on X.

(2) If $(X, \mathscr{T})$ is a topological space and $x \in X$, the collection

$$\mathcal{F}_x := \{A \subseteq X : \exists U \in \mathscr{T} \text{ such that } x \in U \subseteq A\}$$

is a filter on X. This is called *neighbourhood filter of x*.

(3) More generally, the collection $\mathcal{U}_x := \{A \subseteq X : x \in A\}$ is a filter on X, and in fact an ultrafilter on X. We can see this because if A is any subset of X not in the filter, then $x \notin A$. But $\{x\}$ is in $\mathcal{U}_x$. So if $\mathcal{F}$ is a filter that properly contains $\mathcal{U}_x$ and $A \in \mathcal{F}$, the intersection property of a filter would imply that $A \cap \{x\} = \varnothing \in \mathcal{F}$, which is a contradiction. Ultrafilters of this form (all the sets that contain x for some fixed x) are called *principle ultrafilters.*

(4) More generally, if X is an infinite set, the collection $\mathcal{F} = \{A \subseteq X : X \setminus A \text{ is finite}\}$ of co-finite subsets of X is a filter, usually called the *Fréchet filter*. An important property of the Fréchet filter on an infinite set is that any other filter containing it (in particular any ultrafilter containing it) cannot contain any finite sets. (**Exercise:** Check this!)

The first hint of the relation of this notion with compactness is given by the following proposition.

Proposition 0.5. *Any collection $S \subseteq \mathcal{P}(X)$ with the finite intersection property generates a unique smallest filter that contains it.*

Proof. **Exercise:**. (Hint: FIRST add finite intersections, then add supersets. Show that the resulting collection is a filter.) $\square$

Example 0.6. If $(X, \mathscr{T})$ is a topological space and $x \in X$, the collection $\{U \in \mathscr{T} : x \in U\}$ has the finite intersection property, since it is actually closed under finite intersections by definition of a topology. The filter it generates is the neighbourhood filter we saw earlier.

You should always think of a filter $\mathcal{F}$ on a set X as a definition of "largeness". That is, a subset $A \subseteq X$ is "large according to $\mathcal{F}$" if $A \in \mathcal{F}$. Ultrafilters are particularly good at this due to the following property, which says that according to an ultrafilter, either a set A or its complement must be 'large'.

Proposition 0.7. *Let X be a set and $\mathcal{U}$ a filter on X. Then $\mathcal{U}$ is an ultrafilter if and only if for any subset $A \subseteq X$, either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.*

Proof. Let $\mathcal{U}$ be an ultrafilter on X, and suppose A is a nonempty subset of X such that $A \notin U$. We want to show that $X \setminus A \in \mathcal{U}$.

By the maximality of $\mathcal{U}$, it must be the case that $\mathcal{U} \cup \{A\}$ is not a filter, and moreover that it does not have the finite intersection property (if it did, it would generate a filter that contains $\mathcal{U}$, again contradicting maximality). That means there is a $B \in \mathcal{U}$ such that $A \cap B = \varnothing$. But then $B \subseteq X \setminus A$, and therefore $X \setminus A \in \mathcal{U}$ since $\mathcal{U}$ is closed upwards.

The other side of the implication is left as an **Exercise**. (Hint: Suppose $\mathcal{U}$ is properly contained in another filter, so in particular there is some set A such that $\mathcal{U} \sqcup \{A\}$ is contained in this larger filter. Use this set to contradict the property you assumed.) ☐

Even better, ultrafilters in fact tell you that any subset of a large set is either large, or its complement in the large set is large.

Corollary 0.8. *Let $\mathcal{U}$ be an ultrafilter on X, and let $A \in \mathcal{U}$. Given a subset $B \subseteq A$, either $B \in \mathcal{U}$ or $A \setminus B \in \mathcal{U}$.*

Another useful fact is the following lemma.

Lemma 0.9. *Let X be a set and let $\mathcal{U}$ be an ultrafilter on X. Suppose X is written as a union of finitely many sets $X = X_1 \cup \ldots \cup X_n$. Then there is a k such that $X_k \in \mathcal{U}$.*

Proof. Suppose for the sake of contradiction that $X_k \notin \mathcal{U}$ for all $k = 1, \ldots, n$. Then by the previous proposition, $X \setminus X_k \in \mathcal{U}$ for all $k = 1, \ldots, n$. These are finitely many sets in $\mathcal{U}$, so their intersection must be in $\mathcal{U}$. But their intersection is empty, which is a contradiction since $\mathcal{U}$ is a filter. ☐

Definition 0.10. Let X be a topological space, $\mathcal{F} \subseteq \mathcal{P}(X)$ a filter on X, and $x \in X$. Then $\mathcal{F}$ *converges* to x if $\mathcal{F}_x \subseteq \mathcal{F}$ where $\mathcal{F}_x$ is the neighbourhood filter we saw earlier. In this case we write $\mathcal{F} \to x$.

The more intuitive way to think about this definition is that $\mathcal{F}$ converges to x if every open set containing x is an element of $\mathcal{F}$. This is equivalent to saying $\mathcal{F}_x \subseteq \mathcal{F}$, since $\mathcal{F}$ is closed upwards.

Definition 0.11. Let $\mathcal{F}$ be a filter on a topological space X, and let $x \in X$. Then x is called an *accumulation point* of $\mathcal{F}$ if for every $F \in \mathcal{F}$ and every open set U containing x, $F \cap U \neq \varnothing$. Equivalently, x is an accumulation point of F if $x \in \mathrm{Cl}\ F$ for every $F \in \mathcal{F}$.

Proposition 0.12. *Let X be a topological space. If $\mathcal{F}$ is a filter on X and $\mathcal{F} \to x$, then x is an accumulation point of $\mathcal{F}$. Conversely, if x is an accumulation point of an ultrafilter $\mathcal{U}$ on X, then $\mathcal{U} \to x$.*

Proof. First, suppose $\mathcal{F}$ is a filter and $\mathcal{F} \to x$. Then $\mathcal{F}_x \subseteq \mathcal{F}$, and so in particular for any open set $U \in \mathcal{F}_x$ and any $F \in \mathcal{F}, U \cap F \neq \varnothing$ since $\mathcal{F}$ is closed under finite intersections and $\varnothing \notin \mathcal{F}$.

Second, suppose x is an accumulation point of an ultrafilter $\mathcal{U}$ on X. We want to show that $\mathcal{F}_x \subseteq \mathcal{U}$, so fix $F \in \mathcal{F}_x$. Since x is an accumulation point of $\mathcal{U}$ and F contains an open set containing x, we have that $U \cap F \neq \varnothing$ for every $U \in \mathcal{U}$. But then the collection $\mathcal{U} \cup \{F\}$ has the finite intersection property, and so it must be contained in some filter. That filter must be $\mathcal{U}$ itself, since $\mathcal{U}$ is not properly contained in any other filter on X. ☐