HOMEWORK 6

(1) Give an example to show that the infinite analogue of the pasting lemma for closed sets is not necessarily true.

(2) Let A_α be a subspace of X_α for each α. Then show that the product topology on $\prod_\alpha A_\alpha$ equals the subspace topology induced from $\prod_\alpha X_\alpha$.

(3) If X_α is a Hausdorff space for each X_α, then show that $\prod_\alpha X_\alpha$ is Hausdorff.

(4) Let A_α be a subspace of X_α for each α. Then show that the closure of the product of A_α is equal to the product of the closures.

(5) Prove or disprove - If $\{X_\alpha\}_{\alpha \in J}$ is such that each X_α has at least two points and the product $X := \prod_\alpha X_\alpha$ is Hausdorff, then X_α is Hausdorff for each α.

(6) Show that the product $\mathbb{R}^\mathbb{N}$ is separable in the product topology. (Note that $\mathbb{Q}^\mathbb{N}$ will not work, so you have to apply a fix to that idea.)

(7) Show that the product $\mathbb{R}^\mathbb{R}$ is separable in the product topology. (This is a bit surprising!) (Hint : Go about this in 'steps'.)

(8) Let J be an uncountable index set and let $\{X_\alpha\}$ be a collection of topological spaces X_α, each of which is T_1 and has at least two points, indexed by J. Let $X := \prod_{\alpha \in J} X_\alpha$. Then show that X has a non-separable subspace. (So that the esotericism of the previous exercise might feel salvaged a bit.) You can use the following steps for this -

(1) Fix a point $f \in X$. (Remember that f can be thought of as a function.) Define a subset
$$A \subseteq X := \{g \in X : g(\alpha) = f(\alpha) \text{ for all but countably many } \alpha \in J\}.$$
Then the aim is to show that A is not separable in its subspace topology.

(2) Fix any countable subset $C = \{g_1, g_2, \ldots\} \subseteq A$. Then show that there exists an $\alpha \in J$ such that $g_n(\alpha) = f(\alpha)$ for all n.

(3) Find an open subset $U \subseteq X$ such that $g_n \notin U \cap A$ for all n.

(4) Conclude that A is not separable.

(9) let $\{X_\alpha\}_{\alpha \in J}$ be a collection of separable Hausdorff topological spaces X_α, each of which has at least two points, indexed by J. Assume that $X := \prod_{\alpha \in J} X_\alpha$ is separable. Then show that $|J| \leq |\mathbb{R}|$, i.e. there is an injection from J to $\mathbb{R}$. (This provides some more psychological sense; you cannot have 'too large' products of separable spaces be separable.) You can use the following steps for this -

(1) For each $\alpha \in J$, find disjoint nonempty open subsets U_α and V_α in X_α.

(2) Let D be a countable dense set in X. Define for each $\alpha \in J$, $D_\alpha := \pi_\alpha^{-1}(U_\alpha) \cap D$. Show that $D_\alpha \neq 0$ for each $\alpha \in J$.

(3) Show that if $\alpha \neq \beta$, $D_\alpha \neq D_\beta$.

(4) Conclude the existence of the required injection.

(10) Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}^\mathbb{N}$ (i. e. a sequence of sequences) such that each x_n (thought of as a sequence in $\mathbb{R}$) has infinitely many nonzero terms. Show that the sequence $\{x_n\}_{n \in \mathbb{N}}$ does not converge to the constant zero sequence $(0, 0, 0, \ldots)$ in the box topology on $\mathbb{R}^\mathbb{N}$. (So that almost any interesting sequence in $\mathbb{R}^\mathbb{N}$ fails to converge in the box topology.)