CONSTRUCTING CONTINUOUS MAPS AND PRODUCT TOPOLOGY

ADITYA KARNATAKI

Today we will construct various kinds of continuous maps and motivate the definition of product topology on an arbitrary product. We are slowly graduating out of discussing only the basic principles of pointset topology and going towards using these principles to derive somewhat more complicated statements. So by this time you should have a good grasp of everything that we have covered so far as fundamental definitions and properties.

1. Constructing Continuous Maps

Proposition 1.1. *Let X, Y, Z be topological spaces.*

1. *(Constant Functions) If $f : X \to Y$ is a constant function, then f is continuous.*
2. *(Inclusion) If A is a subspace of X, then the inclusion function $i : A \to X$ is continuous. In fact, the subspace topology is the coarsest topology such that this function is continuous.*
3. *(Composition) If $f : X \to Y$ and $g : Y \to Z$ is continuous, then the map $g \circ f : X \to Z$ is continuous.*
4. *(Restriction of domain) If $f : X \to Y$ is continuous, and if A is a subspace of X, then $f|_A : A \to Y$ is continuous.*
5. *(Restriction or Expansion of range) Let $f : X \to Y$ be continuous. If Z is a subspace of Y containing the image $f(X)$, then the function $g : X \to Z$ obtained by restricting the range of f is continuous. If Z is a space containing Y, then the function $h : X \to Z$ obtained by expanding the range of f is continuous.*
6. *(Local formulation of continuity) The map $f : X \to Y$ is continuous if $X = \bigcup U_\alpha$ for open sets U_α such that $f|_{U_\alpha}$ is continuous for each α.*

Proof.
1. This is easy since $f^{-1}(V) = \varnothing$ or X for all V open in Y.
2. $i^{-1}(U) = U \cap A$ for any open U in X. In particular, any topology in which i is continuous must contain all sets of the form $U \cap A$ and thus the subspace topology.
3. $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ for all U, so we are done.
4. Restriction of domain is composition of inclusion with the original function, so we are done.
5. **Exercise.**
6. For an open set V in Y, we have $f^{-1}(V) \cap U_\alpha = f|_\alpha^{-1}(V)$ by set theory. So that the LHS is open by hypothesis. But then
$$f^{-1}(V) = \bigcup_\alpha \left(f^{-1}(V) \cap U_\alpha\right)$$
so that $f^{-1}(V)$ is open.

□

The last property seems trivial, but in fact it led to Grothendieck generalising even the notion of topology in a way that is immensely useful in algebraic geometry.

Definition 1.2. We call $f : X \to Y$ an *embedding* if the corestriction $f : X \to f(X)$ is a homeomorphism.

Lemma 1.3 (Pasting Lemma)**.** *Let $X = A \cup B$ where A and B are closed (or open) in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous functions. If $f(x) = g(x)$ for all $x \in A \cap B$, then f and g 'glue together' to give a continuous function $h : X \to Y$ obtained by setting $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$.*

Proof. Note that $h(x)$ is well-defined. The case for open sets is a special case of local formulation of continuity. So we do the case for the closed sets. Now, $h^{-1}(K) = f^{-1}(K) \cup g^{-1}(K)$ for any closed set K in Y. Then $f^{-1}(K)$ is closed in A and A is closed in X. So $f^{-1}(K)$ is closed in X. Similarly, $g^{-1}(K)$ is closed in X and hence $h^{-1}(K)$ is closed in X. □

Exercise: Give an example to show that the infinite analogue of this statement for closed sets is not necessarily true.

1.1. **Maps into products.** Let X, Y be topological spaces and give $X \times Y$ the product topology. We have the two maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$. Then **Exercise:** π_1 and π_2 are continuous.

Proposition 1.4. *Let Z be any topological space. A function $f : Z \to X \times Y$ is continuous if and only if both of its components $f_1 = \pi_1 \circ f : Z \to X$ and $f_2 = \pi_2 \circ f : Z \to Y$ are continuous.*

Proof. If f is continuous, both of its components are continuous, since these are compositions of continuous maps.

Now assume f_1 and f_2 are continuous. Let U and V be open sets in X and Y respectively. Then

$$f^{-1}(U \times V) = f^{-1}(U \times Y \cap X \times V) = f^{-1}(U \times Y) \cap f^{-1}(X \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$$

and hence is open by hypothesis. □

Corollary 1.5. *The product topology is the coarsest topology on $X \times Y$ such that both projections π_1 and π_2 are continuous.*

1.2. **Maps out of products.** Let Z be any topological space. Let us ignore topologies for a second. Then there is a bijective correspondence between functions $f : X \times Y \to Z$ and functions $H_f : X \to \mathrm{Func}(Y, Z)$ given by

$$f(x, y) = H_f(x)(y).$$

Here, $\mathrm{Func}(Y, Z)$ is simply the set of all set-theoretical functions from Y to Z. We can write this as

$$\mathrm{Func}(X \times Y, Z) \cong \mathrm{Func}(X, \mathrm{Func}(Y, Z)).$$

Note that this construction is completely general for ANY X, Y, Z. So this is really a property of the construction of taking the direct product. This is an example of the 'Hom-Tensor product duality' from category theory, but we have no reason to go there. But it is good to note that such accessible instances of it exist.

Coming back to topological spaces, we would like, if possible, similar description of the subset

$$\mathrm{Cont}(X \times Y, Z) \subseteq \mathrm{Func}(X \times Y, Z)$$

when we take topologies into account. This, unfortunately, is not so simple.

Suppose $f : X \times Y \to Z$ is a continuous function. Now, for each x, we have

$$i_x : Y \to X \times Y$$

an inclusion given by

$$i_x(y) = (x, y)$$

is an embedding of Y in $X \times Y$, identifying it with the copy $\{x\} \times Y$. In particular, the map $H_f(x) := f \circ i_x : Y \to Z$ is a composition of two continuous maps, and hence continuous. So, we get a function

$$\mathrm{Cont}(X \times Y, Z) \to \mathrm{Func}(X, \mathrm{Cont}(Y, Z))$$

$$f \to H_f : H_f(x) := f \circ i_x.$$

However, this function is not surjective. What this means is that, a function may be continuous in the second variable y for each value of the first variable x without itself being continuous.

What is more surprising that a function can be continuous in each variable without being continuous.

$$f(x, y) = \frac{2xy}{x^2 + y^2}$$

for $(x, y) \neq (0, 0)$ and 0 if $(x, y) = (0, 0)$ is not continuous as $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, although the functions $f_{y=c} : \mathbb{R} \to \mathbb{R}$ and $f_{x=d} : \mathbb{R} \to \mathbb{R}$ sending $x \to f(x, c)$ and $y \to f(d, y)$, for any fixed constants c and d, are continuous as functions of single variable.

What we in particular may want to do to rectify this situation is to impose more conditions on the correspondence

$$H_f : X \to \text{Cont}(Y, Z)$$

that sends

$$x \to H_f(x).$$

In particular, we may want H_f to be 'continuous'. But we can only make sense of this if the space $\text{Cont}(Y, Z)$ is made into a topological space!

There are many ways of specifying a topology on spaces of functions such as $\text{Cont}(Y, Z)$ and indeed even $\text{Func}(Y, Z)$. The former can be given a topology called 'compact-open' topology which suffices for many natural purposes. We will not be studying that during this course. In the latter case, the space $\text{Func}(Y, Z)$ can also be thought of as a product of one copy of Z for each element of Y.

$$\text{Func}(Y, Z) \cong \prod_{y \in Y} Z$$

since

$$h \to (h(y))_{y \in Y}$$

is clearly a bijective correspondence, since a function is defined and determined by all its values. When $Y = \{y_1, y_2\}$, we have already discussed what the product topology on $Z \times Z$ looks like. Now we extend this to arbitrary products.

2. Product Topology

Definition 2.1. Recall that for $\{X_\alpha\}_{\alpha \in J}$ a collection of sets indexed by a set J, the cartesian product $\prod_{\alpha \in J} X_\alpha$ is the set of J-indexed sequences $(x_\alpha)_{\alpha \in J}$ with $x_\alpha \in X_\alpha$ for each α.

Also recall that there are projection maps $\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ sending $(x_\alpha)_{\alpha \in J}$ to x_β.

Now suppose that each X_α is a topological space. What topology should we give for the product? The corollary we saw earlier today provides us with a direction. We wish to equip $\prod_{\alpha \in J} X_\alpha$ with the coarsest topology possible such that the projection maps π_β are continuous for each β. In other words, for every β and U_β an open set in X_β, we want $\pi_\beta^{-1}(U_\beta)$ to be contained in the topology on $\prod_{\alpha \in J} X_\alpha$.

Note that

$$\pi_\beta^{-1}(U_\beta) = \prod_{\alpha \in J} A_\alpha$$

where $A_\alpha = X_\alpha$ if $\alpha \neq \beta$ and $A_\beta = U_\beta$. So, the collection

$$\mathcal{S} := \{\pi_\beta^{-1}(U_\beta) : \beta \in J, U_\beta \text{ open in } X_\beta\}$$

which we want to be contained in the product topology, forms a subbasis on X. So we simply define -

Definition 2.2. The *Product topology* on $\prod_{\alpha \in J} X_\alpha$ is the topology generated by the subbasis

$$\mathcal{S} := \{\pi_\beta^{-1}(U_\beta) : \beta \in J, U_\beta \text{ open in } X_\beta\}.$$

Lemma 2.3. *A basis for the product topology is given by*

$$\mathcal{B} := \{\pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \pi_{\beta_2}^{-1}(U_{\beta_2}) \cap \cdots \cap \pi_{\beta_n}^{-1}(U_{\beta_n}) : \beta_i \in J, U_{\beta_i} \text{ open in } X_{\beta_i}, 1 \leq i \leq n\}.$$

Proof. This is straightforward from the definitions. ☐

Let us go back to the perspective of functions. Let us consider the case where all X_α are the same topological space Z and let us write the index set as Y, where Y is a topological space too. Then $\prod_{\alpha \in Y} Z$ is the set of functions $\text{Func}(Y, Z)$ with f corresponding to the tuple $(f(y))_{y \in Y}$. Note that $\pi_y(f) = f(y)$. What does the product topology look like in terms of this representation?

Proposition 2.4. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions $f_n : Y \to Z$ and let $f : Y \to Z$ be another such function. Then*

$$f_n \to f \text{ as } n \to \infty$$

in $\mathrm{Func}(Y, Z)$ *if and only if the functions f_n converge pointwise to f, i.e. if and only if for each $y \in Y$, we have*

$$f_n(y) \to f(y) \text{ as } n \to \infty$$

in Z.

Proof. Let us first assume that $f_n \to f$ in $\mathrm{Func}(Y, Z)$. Fix $y \in Y$. We want to show that $f_n(y) \to f(y)$. That is, we want to show that, for any neighbourhood U of $f(y)$ in Z, there exists an N such that $f_n(y) \in U$ for all $n \geq N$. Now, we know by definition that $\pi_y^{-1}(U)$ is a neighbourhood of f in the product topology. By hypothesis, there exists an N such that $f_n \in \pi_y^{-1}(U)$ for all $n \geq N$. But this is equivalent to saying that $f_n(y) = \pi_y(f_n) \in U$ for all $n \geq N$, and hence we are done.

Now let us assume that for each $y \in Y$, we have $f_n(y) \to f(y)$ as $n \to \infty$. We want to show that $f_n \to f$. That is, we need to show that given any neighbourhood V of f in the product topology, there exists an N such that $f_n \in V$ for all $n \geq N$.

Given f and V, by definition there exists a basis element

$$B = \pi_{y_1}^{-1}(U_{y_1}) \cap \pi_{y_2}^{-1}(U_{y_2}) \cap \cdots \cap \pi_{y_m}^{-1}(U_{y_m})$$

such that $f \in B \subseteq V$. Then, since $f \in B \subseteq \pi_{y_1}^{-1}(U_{y_1})$, we have $f(y_1) \in U_{y_1}$. Then, since $f_n(y_1) \to f(y_1)$, we have that there exists some N_1 such that $f_n(y_1) \in U_{y_1}$ for all $n \geq N_1$. Similarly, there exists some N_2 such that $f_n(y_2) \in U_{y_2}$ for all $n \geq N_2$, and so on, till there exists some N_m such that $f_n(y_m) \in U_{y_m}$ for all $n \geq N_m$. Then, let $N = \max(N_1, N_2, \ldots, N_m)$. Then $f_n(y_i) = \pi_{y_i}(f_n) \in U_{y_i}$ for $1 \geq i \geq m$ and for all $n \geq N$. This is equivalent of saying that

$$f_n \in \pi_{y_1}^{-1}(U_{y_1}) \cap \pi_{y_2}^{-1}(U_{y_2}) \cap \cdots \cap \pi_{y_m}^{-1}(U_{y_m)} = B \subseteq V$$

for all $n \geq N$. Thus, we are done. □

3. Compatibility with constructions and properties

Proposition 3.1. *Let A_α be a subspace of X_α for each α. Then the product topology on $\prod_\alpha A_\alpha$ equals the subspace topology induced from $\prod_\alpha X_\alpha$.*

Proof. **Exercise.** □

Proposition 3.2. *If X_α is a Hausdorff space for each X_α, then $\prod_\alpha X_\alpha$ is Hausdorff.*

Proof. **Exercise.** □

Exercise: Prove or disprove - If $\{X_\alpha\}_{\alpha \in J}$ is such that each X_α has at least two points and the product $X := \prod_\alpha X_\alpha$ is Hausdorff, then each X_α is Hausdorff.

Proposition 3.3. *Let A_α be a subspace of X_α for each α. Then closure of the product of A_α is equal to the product of the closures.*

Proof. **Exercise.** □