SECOND COUNTABILITY AND CONNECTEDNESS

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First, we see another notion of countability that will be quite useful later.

1. Second countability

Definition 1.1. A topological space X is called 'second countable' if there exists a countable basis on X that generates $\mathscr{T}_X$.

Proposition 1.2. *Suppose X is second countable. Then X is separable.*

Proof. **Exercise.** $\square$

Note that this is quite a strong condition and not even every metric space satisfies it. (So do not get carried away by your intuition with $\mathbb{R}$!)

Exercise: Find an example of a second countable space X which has a quotient that is not second countable.

2. Connectedness

Connectedness is a nice topological property. Its definition is intuitive and easy to understand, and it is a powerful tool in proofs of well-known results. Roughly speaking, a connected topological space is one that is "in one piece". The way we will define this is by giving a very concrete notion of what it means for a space to be "in two or more pieces", and then say a space is connected when this is not the case. Along the way we will see some novel proof techniques.

3. Definitions

Definition 3.1. Let X be a nonempty topological space. A *separation* of X is a pair U, V of disjoint, nonempty open subsets of X whose union is X. The space X is said to be *connected* if there does not exist a separation of X. Otherwise it is *disconnected*. We say a space X is *totally disconnected* if the only connected subspaces of X are the singleton subsets $\{x\}$ for $x \in X$.

Proposition 3.2. *The following are equivalent for a topological space X.*

(1) X is disconnected.
(2) There exist nonempty, disjoint, closed sets $A, B \subseteq X$ such that $X = A \sqcup B$.
(3) There exist nonempty, disjoint sets $A, B \subseteq X$ such that $X = A \sqcup B$ and $\mathrm{Cl}\, A \cap B = \varnothing = A \cap \mathrm{Cl}\, B$.
(4) There is a nontrivial clopen subset of X. That is, there is a subset $A \subseteq X$ that is both open and closed, and A is not X or $\varnothing$.

Proof. **Exercise.** $\square$

A pair of sets $A, B \subseteq X$ witnessing that X is disconnected is often also called a *disconnection* of X.

The following can be a useful lemma. Also the ideas in this proof are quite simple but useful.

Lemma 3.3. *If $A \subseteq X$ is a connected subspace, then for any $A \subseteq B \subseteq \mathrm{Cl}\, A$, B is connected.*

Proof. Suppose that $B = C \sqcup D$ is a separation. Then A has to lie in C or D, since otherwise $(A \cap C) \sqcup (A \cap D)$ is a separation of B. Suppose that $A \subseteq C$ without loss of generality. Then $B \subset \mathrm{Cl}\, A \subseteq \mathrm{Cl}\, C$. But then $B \cap D = \varnothing$ by the previous proposition, since $\mathrm{Cl}\, C$ and D do not intersect. So B is connected, since this forces $D = \varnothing$. $\square$

Theorem 3.4. $\mathbb{R}$ *is connected.*

Proof. Assume there is an open subset X such that $\mathbb{R} \setminus X$ is also open, and both are nonempty. Let $a \in X$ and $b \in \mathbb{R} \setminus X$, and suppose without loss of generality that $a < b$.

Define $A := X \cap (-\infty, b]$. Note that A is nonempty since $a \in A$, so it has a least upper bound since it is also bounded above (by b). Call this least upper bound p.

Then, if $p \in X$, then by openness of X, there exists an open interval $(p - \epsilon, p + \epsilon)$ also contained in X, where we can make $\epsilon < b - p$ if necessary; but then any $t \in (p, p + \epsilon)$ is in A, contradicting the fact that p is the least upper bound of A.

But if $p \in \mathbb{R} \setminus X$, which is also open by definition, then there exists $\epsilon > 0$ such that $(p - \epsilon, p + \epsilon) \subseteq \mathbb{R} \setminus X$, but then any $t \in (p - \epsilon, p)$ would be a least upper bound for A, which is a contradiction, too. So p being in X or $\mathbb{R} \setminus X$ is a contradiction, and we are done. $\square$

Remark 3.5. In fact, one could rerun the same argument with $\mathbb{R}$ replaced by $[a, b]$ or (a, b). In fact, this is generalized in Munkres's book for a linearly ordered set L that satisfies certain order properties of $\mathbb{R}$.

Definition 3.6. A linearly ordered set L having more than one element is called a *linear continuum* if the following hold :

(1) L has the least upper bound property, i.e. if every nonempty subset of L that is bounded above has a least upper bound.
(2) If $x < y$, there exists z such that $x < z < y$.

Theorem 3.7. *If L is a linear continuum, then L is connected in the order topology and so are intervals and rays in L.*

Proof. **Exercise.** $\square$

Example 3.8. In particular, the least upper bound property and density property apply to show that the ordered square $[0, 1] \times [0, 1]$ is connected in the lexicographic or dictionary order topology.

Theorem 3.9 (Intermediate Value Theorem)**.** *Let $f : X \to \mathbb{R}$ be a continuous map, where X is a connected space. If $a, b \in X$ are points, and $r \in R$ lies between $f(a)$ and $f(b)$, then there exists a point $c \in X$ with $f(c) = r$.*

Proof. Suppose that $f(X) \subseteq \mathbb{R} \setminus \{r\} = (-\infty, r) \cup (r, \infty)$. Then X is the union of the disjoint, nonempty subsets $U = f^{-1}((-\infty, r))$ and $V = f^{-1}((r, \infty))$, each of which is open in X since f is continuous. $\square$

Essentially the same argument gives us :

Theorem 3.10. *The continuous image of a connected space is connected.*

In a similar vein, there is another useful criterion to check connectedness.

Proposition 3.11. *A topological space X is connected if and only if every continuous function $f : X \to \{0, 1\}$ is constant (where $\{0, 1\}$ has the discrete topology).*

Proof. Use $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ as above. $\square$

4. Constructions and Connectedness

In general, connectedness does not behave well with unions or intersections. Singletons are connected, but any $\{a, b\} \subseteq \mathbb{R}$ with $a < b$ is disconnected. Similarly, consider graphs of functions $f(x) = x^2 - 1$ and $g(x) = 1 - x^2$. These are connected, being continuous images of $\mathbb{R}$, but their intersection is a discrete 2-element set, hence disconnected. However, we have a salvage for unions.

Proposition 4.1. *The union of a collection of connected subspaces of X, that all have a point in common, is connected.*

Proof. Let A_α be a collection of connected subspaces of X and let $a \in X$ be a common point to all A_α. But then for any

$$f : \bigcup_\alpha A_\alpha \to \{0,1\},$$

we know that $f(x) = f(a)$ for all $x \in A_\beta$ for a fixed A_β, since $f|_{A_\beta}$ is constant since A_β is connected. And for any $y \in A_\gamma$ for any A_γ, we also have $f(y) = f(a)$ for the same reason. So that f is constant on $\cup_\alpha A_\alpha$. □

Theorem 4.2. *The product of two (and hence finitely many) connected spaces is connected.*

Proof. We want to show that any continuous

$$f : X \times Y \to \{0,1\}$$

is constant, i.e. $f((x_1, y_1)) = f((x_2, y_2))$ for any (x_1, y_1) and (x_2, y_2). Fix some (x_1, y_1) and (x_2, y_2). Then, simply note that $(x_1, y_1) \in \{x_1\} \times Y$ and $f|\{x_1\} \times Y$ is constant since $Y \equiv \{x_1\} \times Y$ is connected, so that $f((x_1, y_1)) = f((x_1, y_2))$. Then, interchanging the roles of X and Y, we see that $f((x_1, y_2)) = f((x_2, y_2))$ by the same logic, and we are done. □

Theorem 4.3. *An arbitrary product of connected spaces is connected.*

Proof. In fact, the proof follows almost the same argument as the previous one, it is just more bookkeeping. So it is an **Exercise.** □

Proposition 4.4 (Quotient of connected is connected)**.** *If X is connected, then any quotient of X is connected.*

Proof. It is the continuous image of a connected space. □

Definition 4.5. Define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing x and y. The equivalence classes of this are called the *connected components* of X.

Note that connected components of any space X are always closed, but not necessarily open.

The following characterisation of connected components is left as an exercise.

Proposition 4.6. *The connected components of X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.*

Proof. **Exercise.** □