SOME CONSEQUENCES OF $\pi_1(S^1) \cong \mathbb{Z}$

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1. Introduction

We will now explore some neat topological implications of $\pi_1(S^1) \cong \mathbb{Z}$. We will see Brouwer's fixed point theorem. In some sense, this is a generalization of an analogue of Intermediate Value Theorem, since applying the intermediate value theorem to $g(x) = f(x) - x$ for any continuous function $f : I \to I$, where I denotes the closed interval $[0, 1]$ we see that any continuous function $f : I \to I$ has a fixed point. We will explore such generalizations today. Another such generalisation is the Borsuk-Ulam theorem.

2. Brouwer fixed point theorem

Recall the following definition from one of the homework problems-

Definition 2.1. For $A \subseteq X$, a *retraction* $r : X \to A$ is a continuous map such that $r(a) = a$ for all $a \in A$. If such an r exists, we say that A is a retract of X.

Lemma 2.2. *if A is a retract of X, the homomorphism of fundamental groups $j_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$ induced by the inclusion $j : A \to X$ is an injective homomorphism.*

Proof. If $r : X \to A$ is the retraction, then $r \circ j : A \to A$ is the identity map on A. This implies that $r_* \circ j_* : \pi_1(A, a_0) \to \pi_1(A, a_0)$ is the identity homomorphism of groups since π_1 is a functor. This implies that j_* must be injective. (What does it say about r_*?) □

Theorem 2.3. *There is no retraction of B^2 to S^1. Here, B^2 denotes the closed unit disk $\{(x, y) : x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$.*

Proof. Assume S^1 is a retract of B^2. Then we have an injection $\pi_1(S^1) \to \pi_1(B^2)$ by the previous lemma. But $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(B^2) = \{1\}$ since it is a convex subset of $\mathbb{R}^2$. This is a contradiction. □

Theorem 2.4 (Brouwer's fixed point theorem in 2 dimensions)**.** *If $f : B^2 \to B^2$ is continuous, then there exists $x \in B^2$ such that $f(x) = x$.*

Proof. Assume that $f : B^2 \to B^2$ is continuous and $f(x) \neq x$ for all $x \in B^2$. Then we can define $h : B^2 \to S^1$ by mapping each $x \in B^2$ to the point where the ray from $f(x)$ to x hits S^1 which is the boundary of B^2. This map is only well-defined when x does not equal $f(x)$, which we are assuming is the case for all x. The explicit formula for h is given by-

$$h(x) = x + t(x - f(x)) \text{ where } t > 0 \text{ is such that } |h(x)| = 1.$$

We can solve for t by quadratic formula, so t does depend continuously on x, and the function $h(x)$ is continuous. So this gives a continuous map $h : B^2 \to S^1$ and further, $h(x) = x$ for all $x \in S^1$. This gives a retraction $B^2 \to S^1$, which is a contradiction. □

Let us see a more 'conceptual' proof of Brouwer's fixed point theorem in 2 dimensions. First, we will see a couple of useful lemmas.

Lemma 2.5. *Let $h : S^1 \to X$ be a continuous map. Then the following are equivalent.*

(1) h is nullhomotopic.
(2) h extends to a continuous map $k : B^2 \to X$, i.e. $k|_{\partial B^2 = S^1} = h$.
(3) $h_* : \pi_1(S^1) \to \pi_1(X)$ is the trivial homomorphism.

Proof. We will prove the first two implications, leaving the last as an exercise.

(1) *implies* (2): Let $H : S^1 \times I \to X$ be a homotopy between h and a constant map. Define a map $\Pi : S^1 \times I \to B^2$ by the formula

$$\Pi(x,t) := (1-t)x.$$

Then we can check that Π is a quotient map, collapsing the circle $S^1 \times \{1\}$ to the point at origin in B^2 and it is a homeomorphism on $S^1 \times [0,1) \to B^2 \setminus \{(0,0)\}$. Let $\sim$ denote the equivalence relation given by Π. Since H is constant on the (collapsed) space $S^1 \times \{1\}$, H induces a continuous map $(S^1 \times I/\sim) \to X$, i.e. there exists $k : B^2 \to X$ such that $H = k \circ \Pi$. Moreover, since Π maps $S^1 \times \{0\}$ to $S^1 = \partial B^2$, $k|S^1 = \partial B^2$ agrees with $H|_{S^1 \times \{0\}} = h$.

(2) *implies* (3): If $h = k|_{S^1}$, then we can write $h = k \circ i$ where $i : S^1 \hookrightarrow B^2$ is the inclusion. By functoriality of π_1, we have $h_* = k_* \circ i_*$, but $\pi_1(B^2) = \{1\}$, so k_* is trivial, and hence h_* is trivial as well.

(3) *implies* (1): This is left as an **exercise**. □

We record a few useful corollaries.

Corollary 2.6. *The inclusion $i : S^1 \to \mathbb{R}^2 \setminus \{(0,0)\}$ is not nullhomotopic.*

Proof. The function $r(X) := \frac{x}{|x|}$ is a retraction of $\mathbb{R}^2 \setminus \{(0,0)\}$ to S^1. This means that $r \circ i = id_{S^1}$. So $r_* \circ i_* = id_{\pi_1(S^1)}$. In particular, i_* is not trivial, hence i is not nullhomotopic. □

Corollary 2.7. *The identity map $id : S^1 \to S^1$ is not nullhomotopic.*

Proof. $id_* = id : \mathbb{Z} \to \mathbb{Z}$ is not trivial. □

Another proof of Brouwer's fixed point theorem: Assume $f : B^2 \to B^2$ is a continuous function that has no fixed point. Then define $g : B^2 \to \backslash\mathbb{R}^2\{(0,0)\}$ by the formula

$$g(p) := p - f(p).$$

Then for $p \in S^1$, the points p and $g(p)$ lie in a convex subset of $\mathbb{R}^2 \setminus \{(0,0)\}$. So we can apply the straight line homotopy between these two points p and $g(p)$: the definition of $H : S^1 \times I \to$ by the formula

$$H(p,t) := (1-t)g(p) + tp$$

gives a homotopy between the maps

$$g|_{S^1} \to \mathbb{R}^2 \setminus \{(0,0)\}$$

(this is H at $t = 0$) and the inclusion map $i \to \mathbb{R}^2 \setminus \{(0,0)\}$ (this is H at $t = 1$). By the corollary, this implies that $g|_{S^1}$ is not nullhomotopic, but it clearly has an extension $g : B^2 \to \mathbb{R}^2 \setminus \{(0,0)\}$. This is a contradiction.

Similar statement for B^n for $n \geq 3$ is also true, but it requires more algebraic topology than what we will learn in this course. This will be the theme of the next result too.

3. Borsuk-Ulam theorem

We want to prove the following theorem.

Theorem 3.1 (Borsuk-Ulam Theorem)**.** *If $f : S^2 \to \mathbb{R}^2$ is a continuous map, then there exists a pair of antipodal points $\pm x \in S^2$ such that $f(x) = f(-x)$.*

Note that the case $n = 1$ for any function $S^1 \to \mathbb{R}^1$ follows from looking at the function $g(x) = f(x) - f(-x)$ and using connectedness of S^1, which is an analogue of intermediate value theorem again. As above, similar statement for $n \geq 3$ is also true, but it requires more algebraic topology.

First we will record some simple definitions and statements.

Definition 3.2. If $x \in S^n$, its *antipode* is $-x$. A map $h : S^n \to S^m$ is *antipode-preserving* if it maps antipodes to antipodes, i.e. $h(-x) = -h(x)$ for all $x \in S^n$.

Example 3.3. Rotation of S^1 by an angle θ is antipode-preserving- $r_\theta(z) = e^{\iota\theta} z = -(e^{\iota\theta}(-z)) = -r_\theta(-z)$.

Remark 3.4. The following easy corollaries of the definition and this example will be helpful. Proofs of these are left as an (easy) **exercise**.

- Composition of two antipode-preserving maps is antipode-preserving.
- If $h : S^1 \to S^1$ is null homotopic, then $r_\theta \circ h$ is null homotopic for any value of θ.

Then we have the following theorem.

Theorem 3.5. *If $h : S^1 \to S^1$ is continuous and antipode-preserving, then h is not null homotopic.*

Proof. In fact, more generally the statement is true for $n \geq 1$. But that needs more techniques from algebraic topology. For $n = 1$, one can do things 'by hand' as we have done before. We show that the map $h_* : \pi_1(S^1) \to \pi_1(S^1)$ is a nontrivial homomorphism, so that by the earlier lemma h can't be null homotopic.

Let $b_0 = (1, 0)$ and let $h(b_0) = a_0$. If $b_0 = a_0$, then we proceed with the map h. If not, then choose a rotation $r_\theta : S^1 \to S^1$ such that $r_\theta(a_0) = b_0$. Then redefine the map $h := r_\theta \circ h$. In light of the previous remark, it is sufficient to prove that h is not null homotopic in either the old definition or the new definition. We show that the map $h_* : \pi_1(S^1) \to \pi_1(S^1)$ is a nontrivial homomorphism, so that by the earlier lemma h can't be null homotopic.

Consider a map $q : S^1 \to S^1$ given by $q(z) = z^2$. Then q is a closed, continuous surjection and hence a quotient map. In fact, by arguments we have seen earlier, q is in fact a covering map. (This was an exercise on homework 20. The proof is very similar to the claim that the usual $p : \mathbb{R} \to S^1$ is a covering map. You can take a look at Munkres's book, Theorem 57.1 for an outline of the argument.)

The inverse image of any point w on S^1 under the map q consists of the antipodal points z and $-z$ of S^1 corresponding to the two possible squareroots. i.e., $w = q(z) = q(-z)$. Therefore, in particular, $q \circ h$ is a map that is constant on each preimage $q^{-1}(w)$, since

$$q(h(-z)) = q(-h(z)) = q(h(z)).$$

So by the property of quotient maps, this implies that h induces a map $k : S^1 \to S^1$ on the quotient such that $k \circ q = q \circ h$.

$$\begin{array}{ccc} S^1 & \xrightarrow{h} & S^1 \\ q\downarrow & & \downarrow q \\ S^1 & \xrightarrow{k} & S^1 \end{array}$$

Note that $q(b_0) = h(b_0) = b_0$, so that $k(b_0) = b_0$ as well. Also $h(-b_0) = -b_0$ since h is antipode preserving.

Now we show that $k_* : \pi_1(S^1) \to \pi_1(S^1)$ is a nontrivial homomorphism. Recall that q is a covering map. Secondly, note that if $\tilde{f}$ is any path in S^1 from b_0 to $-b_0$, then $f := q \circ \tilde{f}$ is a loop that corresponds to a nontrivial element in $\pi_1(S^1, b_0)$, since $\tilde{f}$ is a lifting of f to the covering space S^1 that begins at b_0 and does not end at b_0. Then we claim that $k_*([f])$ is non-trivial, since

$$k_*([f]) = [k \circ (q \circ \tilde{f})]$$

and since $k \circ q = q \circ h$, we have

$$k_*([f]) = [q \circ (h \circ \tilde{f})].$$

But $h \circ \tilde{f}$ is a path in S^1 from b_0 to $-b_0$, so the RHS can't be the trivial element.

It follows that h_* is nontrivial from this. Indeed, k_* being nontrivial has to be injective since $\pi_1(S^1) \cong \mathbb{Z}$. It can be seen that q_* corresponds to multiplication by 2, since $q : S^1 \to S^1$ is a degree 2 cover. (This can be seen by the same argument where we proved that $\pi_1(S^1) \cong \mathbb{Z}$.) Since $q_* \circ h_* = k_* \circ q_*$ is thus injective, it follows that h_* is injective as well, and we are done. $\square$

Corollary 3.6. *There is no continuous antipode-preserving map $g : S^2 \to S^1$.*

Proof. Suppose $g : S^2 \to S^1$ is a continuous antipode-preserving map. Take S^1 to be the equator of S^2. Then the restriction $h := g|_{S^1}$ to this equator is an antipode preserving continuous map of S^1 to S^1. So by the previous theorem, h is not null homotopic. But the upper hemisphere E of S^2 is homeomorphic to B^2 and $g|_{B^2}$ for this B^2 is an extension of h to B^2, which is a contradiction. $\square$

Corollary 3.7 (Borsuk-Ulam Theorem)**.** *If $f : S^2 \to \mathbb{R}^2$ is a continuous map, then there exists a pair of antipodal points $\pm x \in S^2$ such that $f(x) = f(-x)$.*

Proof. Suppose that $f(X) \neq f(-x)$ for all $x \in S^2$. Then the map

$$g(x) := \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$$

is a continuous map $S^2 \to S^1$ such that g is antipode preserving, i.e. $g(x) = -g(-x)$ for all x. $\square$

Corollary 3.8. *An open set in $\mathbb{R}^2$ cannot be homeomorphic to an open set in $\mathbb{R}^n$ for $n \geq 3$.*

Proof. Assume $U \subseteq \mathbb{R}^n$ is an open set ($n \geq 3$) and there exists a homeomorphism $f : U \xrightarrow{\sim} V$ where $V \subseteq \mathbb{R}^2$. Then there exist $x \in U$ and $r > 0$ such that Cl $B_r(x) \subseteq U$, which is homeomorphic to $B^n \supseteq B^3 \supset S^2$. So, by restriction, we get $f|_{S^2} : S^2 \to \mathbb{R}^2$ which is a continuous and injective map since f is a homeomorphism. This contradicts Borsuk-Ulam theorem. $\square$

Another nice and simple application of the Borsuk-Ulam theorem is the bisection theorem. (Theorem 57.4 in Munkres's book.) This is left for self-study and tutorials. As we have noted earlier, ALL of these statements generalise to $n \geq 3$, but the proofs requires more techniques from algebraic topology. This is just a glimpse of what can be done.