HOMEWORK 9

(1) Show that the following are equivalent for a topological space X.

- X is disconnected.
- There exist nonempty, disjoint, closed sets $A, B \subseteq X$ such that $X = A \sqcup B$.
- There exist nonempty, disjoint sets $A, B \subseteq X$ such that $X = A \sqcup B$ and $\mathrm{Cl}\, A \cap B = \varnothing = A \cap \mathrm{Cl}\, B$.
- There is a nontrivial clopen subset of X. That is, there is a subset $A \subseteq X$ that is both open and closed, and A is not X or $\varnothing$.

(2) Show that if L is a linear continuum, then L is connected in the order topology and so are intervals and rays in L.

(3) Complete the proof that an arbitrary product of connected spaces is connected.

(4) Show that the connected components of X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.

(5) If X is a connected space, we call a point $p \in X$ a *cut point* if $X \setminus \{p\}$ is disconnected.

- Show that homeomorphisms $f : X \to Y$ send cut points to cut points.
- Show that if p is a cut point, and $f : X \to Y$ is a homeomorphism, then $f|_{X\setminus\{p\}} : X\setminus\{p\} \to Y\setminus\{f(p)\}$ is also a homeomorphism.
- Show that for $n \in \mathbb{N}$, having n cut points is a topological invariant.
- Show that $\mathbb{R}$ is not homeomorphic to any $\mathbb{R}^n$ for $n > 1$.

(6) Think why we cannot generalize the above argument to *cut lines* for showing that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n > 2$. (That is, if you were to write the analogous proof, where would it fail?)

(7) Let C be a countable subset of $\mathbb{R}^2$. Show that $\mathbb{R}^2 \setminus C$ is connected as a subspace of $\mathbb{R}^2$.

(8) Topology (Munkres), Chapter 3, Section 23, Exercise (5).

(9) Topology (Munkres), Chapter 3, Section 23, Exercise (9).

(10) Topology (Munkres), Chapter 3, Section 23, Exercise (11).