DEFORMATION RETRACTS AND HOMOTOPY TYPE

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1. Introduction

As we have seen, one way to obtain information about the fundamental group of a (path connected) space is to study the covering spaces of X. Another is to study the *homotopy type* of a space. It provides a method for reducing the problem of computing the fundamental group of a space to that of computing the fundamental group of some other space - which seems like we are not doing much in theory, but in practice this can be quite useful. We will explore this concept today.

2. Deformation retracts

We have seen that a retraction is a function that map a topological space into a subspace that preserves the position of all points in that subspace. The subspace is then called a *retract* of that space. A deformation retraction is a mapping that captures the idea of continuously shrinking a space into a subspace. The best way to understand the difference might be to think of some examples.

Example 2.1. (1) The constant map from S^1 to a point p on S^1 is a retraction onto p.
(2) The map $(x, y, z) \to (x, y, |z|)$ is a retraction of the sphere S^2 to upper hemisphere S^2_+.
(3) The map $x \to \frac{x}{|x|}$ is a retract of the space $\mathbb{R}^2 \setminus \{(0, 0)\}$ to S^1.
(4) The map that slowly shrinks the mobius strip to its central circle. (If we draw the mobius strip as the usual quotient of the square $I \times I$, where I deotes the closed interval $[0, 1]$, then this circle is given by the segment $y = 1/2$.)

The first two examples in this list are different from the last two: in the latter ones, we can *deform* the *identity map* id_X continuously in X to get the retraction.

For instance: the mobius strip is: $X = I \times I/(0, y) \sim (1, 1 - y)$ (as we said earlier). Then consider the map $H : X \times I \to X\}$ given by the formula $H((x, y), t) := (x, \frac{t}{2} + (1 - t)y)$ is a homotopy that gives the retraction map at $t = 1$ beginning from the identity map at $t = 0$. But this is not available to us in the first two examples, for instance: we know that id_{S^1} is NOT null homotopic, so the retraction onto a point p cannot be gotten this way.

Deformation retractions are thus retractions gotten by deformations of the identity map on X.

Definition 2.2. Let A be a subspace of X. We say that A is a *deformation retract* of X if the identity map of X is homotopic to a map that carries all of X into A, such that each point of A remains fixed during the homotopy. This means that there is a continuous map $H : X \times I \to X$ such that $H(x, 0) = x$ and $H(x, 1) \in A$ for all $x \in X$, and $H(a, t) = a$ for all $a \in A$ for all $t \in [0, 1]$. The homotopy H in this case is called a *deformation retraction* of X onto A. The map $r : X \to A$ given by $r(x) := H(x, 1)$ is a retraction of X onto A, and H is a homotopy between the identity map id_X of X and the map $i \circ r$ where $i : A \to X$ is the inclusion map.

Let us see this in action for another of the examples above: the map $x \to \frac{x}{|x|}$ is, we claim, a deformation retract of $X = \mathbb{R}^n \setminus \{0\}$ to S^{n-1}. What is the homotopy H? $H : X \times I \to X$ is defined by

$$H(x, t) := t\frac{x}{|x|} + (1 - t)x,$$

which we know is our friend the straight line homotopy. Note that the straight line segment joining x and $\frac{x}{|x|}$ can never pass through the origin. For a point $a \in S^{n-1}$, $H(a,t) = t\frac{a}{1} + (1-t)a = a$ for all t, so H is a deformation retract indeed.

We want to investigate how fundamental groups of spaces and their deformation retracts behave. We recall that if $r : X \to A$ is *any* retract, then $i_* : \pi_1(A) \to \pi_1(X)$ is injective. (Informally we are taking X and A to be path connected here.) Why is this?- We have $r \circ i = id_A$, so that by functoriality, we have $r_* \circ i_* = id_{\pi_1(A)}$. So if $[f]$ and $[g]$ in $\pi_1(A)$ satisfy $i_*([f]) = i_*([g])$, we have $[f] = r_*(i_*([f])) = r_*(i_*([g])) = [g]$. Now we will see what happens to i_* under this extra hypothesis of being a deformation retraction. Let us begin with a lemma.

Lemma 2.3. *Let $h, k : (X, x_0) \to (Y, y_0)$ be continuous maps of pointed topological spaces. If h and k are homotopic with $x_0 \to y_0$ throughout the homotopy (say H), i.e. the image of the point x_0 remains fixed at y_0 for each function $H|_{X\times\{t\}}$ for each t, then the homomorphisms h_* and k_* are equal to each other.*

Proof. To show that $h_* = k_*$, we have to show that for any loop f in X based at x_0, the classes $[h \circ f]$ and $k \circ f$ are the same, i.e. that there exists a homotopy between $h \circ f$ and $k \circ f$. (Note that these are both loops based at y_0 in Y.) We have

$$I \times I \xrightarrow{f\times id} X \times I \xrightarrow{H} Y$$

and so $H \circ (f \times id)$ is a homotopy between $h \circ f$ and $k \circ f$; this is a path homotopy because f is a loop at x_0 and H maps $\{x_0\} \times I$ to Y. □

In fact, this generalizes to:

Lemma 2.4. *Let $h, k : X \to Y$ be continuous maps of topological spaces. Let h and k be such that they are homotopic to each other under homotopy H, but the endpoint does not necessarily stay fixed, say $h(x_0) = y_0$ and $k(x_0) = y_1$. Then if $\alpha(t) := H(x_0, t)$ is the path defined by H from y_0 to y_1 and $\hat{\alpha} : \pi_1(Y, y_0) \xrightarrow{\sim} \pi_1(Y, y_1)$ is induced by α, then $k_* = \hat{\alpha}(h_*)$.*

Proof. Now consider

$$I \times I \xrightarrow{F} X \times I$$

given by concatenating: the path $(x_0, 1) \to (x_0, t)$, the loop f in $X \times \{t\}$, and the path $(x_0, t) \to (x_0, 1)$ for each $t \in [0, 1]$. Then as before, we have

$$I \times I \xrightarrow{F} X \times I \xrightarrow{H} Y$$

and $H \circ F$ is a path homotopy from $\bar{\alpha} * (h \circ f) * \alpha$ to $e * (k \circ f) * e$. □

Finally we can say what happens at the level of fundamental groups of deformation retracts:

Theorem 2.5. *if $A \subseteq X$ is a deformation retract, then $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ is an isomorphism.*

Proof. We will suppress the basepoints from the notation. Let H be the homotopy defined by the deformation retract, so that $H(x, 0) = x$, $H(x, 1) \in A$, $H(a, t) = a$ for all $a \in A$. Then if $r : X \to A$ is the retraction and $i : A \to X$ the inclusion as before, then we already know $r_* \circ i_* = id_{\pi_1(A)}$. But now, we have the extra information that $i \circ r = H|_{X\times\{1\}}$ is homotopic via H to $id_X = H|_{X\times\{0\}}$. Therefore, $i_* \circ r_* = id_{\pi_1(X)}$, so that r_* and i_* are inverses of each other, and we are done. □

Again, examples are helpful to look at.

Example 2.6. (1) S^1 has the same π_1 as the various spaces in which it is a deformation retract: $S^1 \times I$, i.e. a cyllinder, mobius band, $B^2 \setminus \{0\}, \mathbb{R}^2 \setminus \{0\}$, $S^1 \times B^2$ the 'solid torus'.

(2) Figure 8 space, obtained by joining two disjoint circles at a single point, is a deformation retract of $\mathbb{R}^2 \setminus \{\text{two points}\}$ or $B^2 \setminus \{\text{two points}\}$, and also of $S^1 \times S^1 \setminus \{\text{point}\}$. (This last one takes a little time to visualise. The best way might be to show that $I \times I \setminus \{0\}$ deformation retracts to $\partial(I \times I)$.) So all of these spaces have the same π_1. Another deformation retract of $\mathbb{R}^2 \setminus \{\text{two points}\}$ is the 'theta graph' $\theta := S^1 \cup \{0 \times [-1, 1]\} \subseteq \mathbb{R}^2$. As a result, the figure eight space and the theta graph have isomorphic fundamental groups, yet neither of them is a deformation retract of each other.

This last example suggests that there is a more general situation under which π_1 (and "all of homotopy theory") remains unchanged. Namely, under this situation, two spaces that can be continuously deformed into each other should be considered the same.

3. Homotopy equivalence

Definition 3.1. Let $f : X \to Y$ and $g : Y \to X$ be continuous maps. If $g \circ f : X \to X$ is homotopic to id_X, *and* $f \circ g : Y \to Y$ is homotopic to id_Y, then we say that f and g are *homotopy equivalences*, and we say X and Y have the same homotopy type. f and g are also called homotopy inverses of each other.

Example 3.2. If A is a deformation retract of X, with maps i and r as before, then $r \circ i$ is homotopic to id_X as we have seen, and $i \circ r$ is the identity id_A, so that A and X have the same homotopy type.

Example 3.3. If A is the figure eight space with $i : A \to X = \mathbb{R}^2 \setminus \{\textbf{two points}\}$ and $r' : \mathbb{R}^2 \setminus \{\textbf{two points}\} \to A'$ where A' is the theta graph $S^1 \cup \{0 \times [-1, 1]$, and vice versa we have maps i' and r, then

$$A \xrightarrow{i} X \xrightarrow{r'} A' \xrightarrow{i'} X \xrightarrow{r} A$$

is homotopic to $A \xrightarrow{i} X \xrightarrow{r} A = id_A$, because $r' \circ i'$ is homotopic to id_X, and vice versa in the other direction. Therefore, A and A' have the same homotopy type. This is true more generally, as the following result says.

Proposition 3.4. *If $f : X \to Y$ and $g : Y \to Z$ are homotopy equivalences, then $g \circ f : X \to Z$ is a homotopy equivalence. Thus, having the same homotopy type is an equivalence relation.*

Proof. This is exactly the same argument as in the example, so this is an **exercise**. □

Example 3.5. We define X to be *contractible* if id_X is homotopic to a constant map $X \to \{p\}$ for some point p in X. (This includes $X = I, B^n, \mathbb{R}^n$ etc.) Then $X \to \{p\} \hookrightarrow X$ is homotopic to id_X and $\{p\} \hookrightarrow X \to \{p\}$ is the identity, so $\{p\} \hookrightarrow X$ is a homotopy equivalence.

Warning: The given homotopy between id_X and the constant map does not necessarily define a deformation retraction, since p might not be fixed by the homotopy.

We will see a proof of the following theorem the next time. In fact, we have all the ingredients necessary to do this, but since these are new concepts that require some time and efforts to get familiar, we will push it to the next class.

Theorem 3.6. *Let $f : (X, x_0) \to (Y, y_0)$ be a continuous map. If f is a homotopy equivalence, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.*