DEFINING NEW TOPOLOGICAL SPACES - II : QUOTIENT TOPOLOGY

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1. Introduction

Today we will see an introduction to *quotient topology*. The quotient topology is one of the most ubiquitous constructions in algebraic, combinatorial, and differential topology. It could also be among some of the more difficult concepts in point-set topology to master.

2. Quotient topology

Recall that each injective function $f : X \to Y$ factors as a bijection $g : X \to A$ followed by an inclusion $i : A \to Y$ where $A = f(X)$ is the image of f. When Y is a topological space, we defined the subspace topology on A as the coarsest topology on A making i continuous, and we may also give X the unique topology that makes g a homeomorphism and then f is an embedding. The notion of a quotient topology is more or less dual to this. If we have a surjective function $f : X \to Y$, and X is a topological space, we shall explain how to give Y the *finest* topology making f continuous.

Definition 2.1. Recall that an equivalence relation on a set A is a subset $R \subset A \times A$ which satisfies -

(1) $(a, a) \in R$ for all $a \in A$.
(2) $(a, b) \in R \implies (b, a) \in R$.
(3) $(a, b) \in R, (b, c) \in R \implies (a, c) \in R$.

Note that an equivalence relation R divides A into a disjoint union of subsets called equivalence classes. We define $a \sim b$ to mean that $(a, b) \in R$. Then the equivalence classes are defined as :

$$[x] := \{y \in X : x \sim y\}.$$

We define

$$X/\sim := \{[x] : x \in X\}$$

to be the set of equivalence classes of X. The canonical surjection $\pi : X \to X/\sim$ is given by $\pi(x) = [x]$.

Lemma 2.2. *Let $f : X \to Y$ be a surjective function. Define an equivalence relation $\sim$ on X by declaring $x \sim x'$ if and only if $f(x) = f(x')$. There is an induced bijection*

$$(X/\sim) \to Y$$

given by $h([x]) = f(x)$. Its inverse h^{-1} takes y to $f^{-1}(y)$, which equals $[x]$ for any choice of x such that $f(x) = y$. The surjection $f : X \to Y$ thus factors as the canonical surjection $\pi : X \to (X/\sim)$ and a bijection $h : (X/\sim) \to Y$.

Proof. **Exercise.** □

In this way we can go back and forth between equivalence relations on X and surjective functions $X \to Y$, up to a bijection.

One can think of equivalence relations as "collapsing" some points on X. The question then is, can we give a topology on the space of equivalence classes so that this operation of collapsing is a continuous transformation.

Definition 2.3. Let $f : X \to Y$ be a surjective function, where X is a topological space. The quotient topology on Y (induced from X) is the collection of subsets $V \subseteq Y$ such that $f^{-1}(V)$ is open in X.

Lemma 2.4. *The above definition gives a topology on Y.*

Proof. (1) $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}(Y) = X$ are open in X.
(2) $f^{-1}(\cup_{\alpha\in J} V_\alpha) = \cup_{\alpha\in J} f^{-1}(V_\alpha)$.
(3) $f^{-1}(V_1 \cap V_2 \cap \cdots \cap V_n) = f^{-1}(V_1) \cap f^{-1}(V_2) \cap \cdots \cap f^{-1}(V_n)$.

□

Definition 2.5. A surjective function $f : X \to Y$ between topological spaces is called a *quotient map* if Y has the quotient topology from X, i.e., if $V \subseteq Y$ if and only if $f^{-1}(V)$ is open in X.

Lemma 2.6. *A quotient map $f : X \to Y$ is continuous.*

Proof. This follows from definition. □

Another interpretation of a quotient map is as follows : for any subset $C \subseteq X$, we say C is *saturated* with respect to a surjective function $p : X \to Y$ if C contains every set $p^{-1}(y)$ that it intersects for $y \in Y$. ($p^{-1}(y)$ is also known as the 'fiber' of the map p at y.) Thus C is saturated if it equals the complete inverse image of a subset of Y.

Then to say that p is a quotient map is equivalent of saying that p is continuous, and p maps *saturated* open sets of X to open sets of Y (or saturated closed sets of X to closed sets of Y.)

Lemma 2.7. *Let $f : X \to Y$ be a surjective function where X is a topological space. Then the quotient topology is the finest topology on Y such that $f : X \to Y$ is continuous.*

Proof. This follows from definition. □

Lemma 2.8. *A surjective function $f : X \to Y$ is a quotient map if and only if the following condition holds : a subset $K \subseteq Y$ is closed if and only if $f^{-1}(K)$ is closed in X.*

Proof. This follows from definition. □

We recall the following definitions.

Definition 2.9. Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be two topological spaces and $f : X \to Y$ be a function between them.

- f is said to be an *open* function if $f(U) \in \mathscr{T}_Y$ for all $U \in \mathscr{T}_X$.
- f is said to be a *closed* function if $f(C)$ is a closed subset of Y whenever C is a closed subset of X.

Lemma 2.10. *(1) Each surjective, open map $f : X \to Y$ is a quotient map.*
(2) Each surjective, closed map $f : X \to Y$ is a quotient map.

Proof. (1) Let $V \subseteq Y$. If V is open then $f^{-1}(V)$ is open since f is continuous. Conversely, if $f^{-1}(V)$ is open in X then $V = f(f^{-1}(V))$ because f is surjective, and this is an open subset of Y since f is an open map.
(2) Let $K \subseteq Y$. If K is closed then $f^{-1}(K)$ is closed since f is continuous. Conversely, if $f^{-1}(K)$ is closed in X then $K = f(f^{-1}(K))$ because f is surjective, and this is a closed subset of Y since f is a closed map.

□

Example 2.11. Endow $\mathbb{R}$ with the standard topology. Define an equivalence relation $\sim$ on $\mathbb{R}$ by $x \sim y$ if and only if $x - y \in \mathbb{Z}$. Then we show that $\mathbb{R}/\sim$ is homeomorphic to the subspace S^1 of the plane $\mathbb{R}^2$.

For this purpose, we need to construct a continuous function $\mathbb{R} \to S^1$. We will identify S^1 with the subset $S := \{|z| = 1\}$ of the complex plane. So an **Exercise** is to show that topologically the real plane and the complex plane are the same, so that one can identify the subspace topologies on the two subsets.

Then $\tilde{f} : \mathbb{R} \to S^1$ is defined simply as $\tilde{f}(t) = e^{2\pi i t}$. Then we know from calculus that this map is continuous. Furthermore, if $t \sim t'$, then $\tilde{f}(t) = \tilde{f}(t')$ for any t, t'. So the induced map $f : \mathbb{R}/\sim \to S^1$ is well-defined and continuous.

This evidently is the quotient map. For constructing the inverse, we have to construct an inverse function on what is known as a 'chart' for S^1, Namely, we define two subsets of S^1 as follows :

$$A_1 := \{z \in S^1 : \mathrm{Im}(z) \geq 0\},$$

$$A_2 := \{z \in S^1 : \mathrm{Im}(z) \leq 0\}.$$

Then both A_1 and A_2 are closed in S^1 and $A_1 \cup A_2 = S^1$, while $A_1 \cap A_2 = \{-1, +1\}$.

Given $z \in A_1$, this can be uniquely written as $z = e^{2\pi i t}$ for some $0 \leq t \leq 1/2$. Then define $\tilde{g_1} : A_1 \to \mathbb{R}$ by $\tilde{g_1}(z) = t$. Similarly, Given $z \in A_2$, this can be uniquely written as $z = e^{2\pi i t'}$ for some $1/2 \leq t' \leq 1$. Then define $\tilde{g_2} : A_2 \to \mathbb{R}$ by $\tilde{g_2}(z) = t'$. Note that these two functions are continuous by our knowledge of calculus, but don't agree on the intersection $A_1 \cap A_2$, so do not paste. But their compositions with $\mathbb{R} \to \mathbb{R}/\sim$, which we define to be $g_1 = \pi \circ \tilde{g_1} : \mathbb{R} \to \mathbb{R}/\sim$ and $g_2 = \pi \circ \tilde{g_2} : \mathbb{R} \to \mathbb{R}/\sim$, are continuous, and $g_1(-1) = [1/2] = g_2(-1)$ and $g_1(1) = [0] = g_2(1)$, so these two can be pasted together to give a function $g : S^1 \to \mathbb{R}/\sim$. It follows from definition that f and g are mutual inverses of each other, and are continuous functions. So we are done.

Example 2.12. Let $X = [0,1] \times [0,1]$ be a square in the subspace topology of $\mathbb{R}^2$ and let $Y = S^1 \times S^1$ be the product as a subspace of $\mathbb{R}^4$. This is a *torus*. Define $g : X \to Y$ given by $g(s,t) = (\cos 2\pi s, \sin 2\pi s, \cos 2\pi t, \sin 2\pi t)$. Then the function g is continuous and surjective. It is also closed. So it is a quotient map, and thus the torus Y is realized, up to a homeomorphism, as a quotient of the unit square $[0,1] \times [0,1]$ under the equivalence relation induced by g, which is simply the equivalence relation generated by the relations $(s,0) \sim (s,1)$ for all $s \in [0,1]$ and $(0,t) \sim (1,t)$ for all $t \in [0,1]$.

Theorem 2.13. *Let $f : X \to Y$ be a quotient map. Let $B \subseteq Y$ and $A = f^{-1}(B) \subseteq X$ be subspaces and let $g : A \to B$ the restricted map.*

1. *If A is open or closed, then g is a quotient map.*
2. *If f is an open map or a closed map, then g is a quotient map.*

Proof. Look at the diagram :

$$\begin{tikzcd}
g^{-1}(V) \arrow[r] & A \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] & U \arrow[l] \\
V \arrow[r] & B \arrow[r] & Y & f(U) \arrow[l]
\end{tikzcd}$$

Then complete the proof via the folliowing steps :

1. Show that $g^{-1}(V) = f^{-1}(V)$ for any $V \subseteq B$ and $g(A \cap U) = B \cap f(U)$ for any $U \subseteq X$, using the above diagram or otherwise.
2. Using Step (1), show that g is a quotient map under the hypotheses given in each case.

□

Quotient maps satisfy a 'universal' property somewhat dual to the case of a subspace or a product.

Theorem 2.14. *Let $f : X \to Y$ be a quotient map, and Z be any topological space. Let $h : X \to Z$ be a function such that $h(x) = h(x')$ whenever $f(x) = f(x')$. Then, h induces a unique function $g : Y \to Z$ such that $h = g \circ f$. The function g is continuous if and only if h is continuous. The function g is a quotient map if and only if h is a quotient map.*

Proof. Since f is surjective, every y in Y is of the form $f(x)$ for some $x \in X$. Then we don't have much option but to define $g(y) := h(x)$. Note that this is well defined, since $h(x) = h(x')$ whenever $f(x) = f(x')$ so it doesn't matter which preimage we choose.

Now, if g is continuous, then $h = g \circ f$ is continuous. If h is continuous on the other hand, $h^{-1}(V) = f^{-1}(g^{-1}(V))$ is open for any open $V \subseteq Z$, and since f is a quotient map, this implies $g^{-1}(V)$ is open, proving continuity of g.

If g is a quotient map, then $h = g \circ f$ is too, since $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$. If h is a quotient map, then note that g is surjective, since $h(x) = g(f(x))$. Then, suppose that $g^{-1}(V)$ is open for some $V \subseteq Z$. We want to show that V is open. But $f^{-1}(g^{-1}(V)) = h^{-1}(V)$ is open since f is continuous. Then since h is a quotient map, V is open too, and we are done. $\square$

Corollary 2.15. *Let $h : X \to Z$ be a continuous surjective map. Let $\sim$ be the equivalence relation on X given by $x \sim y$ if and only if $h(x) = h(y)$ and let $X/\sim$ be the set of equivalence classes: $(X/\sim) = \{h^{-1}(z) | z \in Z\}$. Give $X/\sim$ the quotient topology from the canonical surjection $\pi : X \to (X/\sim)$.*

(1) The map h induces a continuous bijective map $g : (X/\sim) \to Z$.
(2) The map $g : (X/\sim) \to Z$ is a homeomorphism if and only if h is a quotient map.
(3) If Z is Hausdorff, so is $(X/\sim)$.

Quotient maps do not always behave nicely as a general notion. For instance, above we saw that if the quotient map is 'nice' or the subspace is 'nice', then the restrictied quotient map is also a quotient map. But usually we need some such condition. Similarly, Example 7 from Section 22 in Munkres's book shows that even finite products do not behave well. In general, care should be taken to use properties of quotient maps. Some extra hypothesis is usually needed. For instance, we will see a condition called 'local compactness' that will be sufficient in most cases to ensure we are not in the nasty landscape of pathological examples where intuition is routinely eaten for breakfast.