TYCHONOFF'S THEOREM

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1. Introduction

We have seen filters and ultrafilters. Now we will see their relation to compactness and prove Tychonoff's theorem. Tychonoff's theorem says that arbitrary products of compact spaces are compact. This is an absolutely extraordinary result. It feels like this should not be true. The intuition that we have developed thus far for compact topological spaces is that they feel small. A space can be very large in terms of cardinality, but if it is compact, it acts like a finite set in a number of ways. But taking large products can produce very 'large' spaces that can destroy such properties. We know that arbitrarily large products of separable spaces are almost never separable. What's more, even metrizability gets broken under arbitrary large products. $\mathbb{R}^{\mathbb{R}}$ is not metrizable. In fact, even 'finiteness' behaves differently as a property under taking arbitrarily large products. $\{0,1\}$ is a finite set but even taking a countable product $\{0,1\}^{\omega}$ produces an uncountable set. In this way, Tychonoff's theorem is remarkable. In practice almost anything that you might want to prove compact will most likely use Tychonoff's theorem.

2. Ultrafilter characterisation of compactness

Recall the following definition from the previous class.

Definition 2.1. Let $(X, \mathscr{T})$ be a topological space, $\mathcal{F} \subseteq \mathcal{P}(X)$ a filter on X, and $x \in X$. Then $\mathcal{F}$ *converges* to x if $\mathcal{F}_x \subseteq \mathcal{F}$ where $\mathcal{F}_x$ is the neighbourhood filter we saw earlier. In this case we write $\mathcal{F} \to x$.

The point is that ultrafilters capture convergence of sequences in a 'correct' way. Before spelling it out, we recall the following useful fact.

Lemma 2.2. *Let X be a set and let $\mathcal{U}$ be an ultrafilter on X. Suppose X is written as a union of finitely many sets $X = X_1 \cup \ldots \cup X_n$. Then there is a k such that $X_k \in \mathcal{U}$.*

Theorem 2.3. *A topological space X is compact if and only if every ultrafilter on X converges.*

Proof. Suppose X is compact, and suppose for the sake of contradiction that $\mathcal{U}$ is an ultrafilter on X that does not converge to any point. This means that for every $x \in X$, there is an open set U_x containing x such that $U_x \notin \mathcal{U}$. Then$\{U_x : x \in X\}$ is an open cover of X,and so it has a finite subcover $\{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\}$. So $X = U_{x_1} \cup \ldots \cup U_{x_n}$. But then by the previous lemma, $U_{x_k} \in \mathcal{U}$ for some $k = 1, \ldots, n$, a contradiction.

Suppose every ultrafilter on X converges, and let $\mathcal{A}$ be a collection of closed subsets of X with the Finite Intersection Property. We want to show that $\cap_{A \in \mathcal{A}} A \neq \varnothing$. Now, $\mathcal{A}$ generates a filter on X, by first adding all finite intersections of elements of $\mathcal{A}$ (which does not add $\varnothing$ since $\mathcal{A}$ has the Finite Intersection Property!), and then adding all supersets of the resulting collection, as we saw in the previous class. Call this filter $\mathcal{F}$. Then by Zorn's Lemma $\mathcal{F}$ is contained in an ultrafilter $\mathcal{U}$ on X.

By assumption, $U \to x$ for some $x \in X$. We show that $x \in \cap_{A \in \mathcal{A}} A$. Fix any $A \in \mathcal{A}$, and let U be any open set containing x. Then $U \in \mathcal{U}$ since $\mathcal{U} \to x$. Now we have that A and U are both elements of $\mathcal{U}$, and therefore $A \cap U \neq \varnothing$. Since we can do this for any such U, we have shown that $x \in \mathrm{Cl}\ A = A$! Repeating this argument for each $A \in \mathcal{A}$ shows that $x \in A$ for all $A \in \mathcal{A}$, or in other words $x \in \cap_{A \in \mathcal{A}} A$, and we are done. □

Now we just need one more tool, namely, how filters and ultrafilters behave under continuous maps, i.e. 'transport of structure'. This is a routine result, 'what you would expect'. But only equipped with these results, we can deduce Tychonoff's theorem. The magic is hidden in the formalism itself.

Proposition 2.4. *Let X and Y be sets, let $\mathcal{F}$ be a filter on X, and let $f : X \to Y$ be a function. Then the collection*

$$f_*(\mathcal{F}) := \{B \subseteq Y : f^{-1}(B) \in \mathcal{F}\}$$

is a filter on Y. If $\mathcal{F}$ is an ultrafilter on X, then $f_*(\mathcal{F})$ is an ultrafilter on Y.

Proof. The proof is entirely routine set theory. But let us go through it. Note that $Y \in f_*(\mathcal{F})$ since $X \in \mathcal{F}$ so $f_*(\mathcal{F})$ is not empty.

(1) $f^{-1}(\varnothing) = \varnothing \notin \mathcal{F}$, so $\varnothing \notin f_*(\mathcal{F})$.

(2) $f_*(\mathcal{F})$ is closed upwards: Let $A \in f_*(\mathcal{F})$, and let $B \supseteq A$ be a superset of A. Then $f^{-1}(B) \supseteq f^{-1}(A)$. Now $f^{-1}(A) \in \mathcal{F}$ since $A \in f_*(\mathcal{F})$, and so $f^{-1}(B) \in \mathcal{F}$ as well, since $\mathcal{F}$ is closed upwards. Then $B \in f_*(\mathcal{F})$ by definition.

(3) $f_*(\mathcal{F})$ is closed under finite intersections. Let $A, B \in f_*(\mathcal{F})$. Then by definition $f^{-1}(A)$ and $f^{-1}(B)$ are elements of $\mathcal{F}$. But $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$, the latter of which is in $\mathcal{F}$ since $\mathcal{F}$ is closed under finite intersections.

Finally, let $\mathcal{F}$ be an ultrafilter on X. Let $A \subseteq Y$ be a set such that $A \notin f_*(\mathcal{F})$. Then we want to show that $Y \setminus A$ belongs to $f_*(\mathcal{F})$. Since, $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$ and since $f^{-1}(A)$ is not in $\mathcal{F}$ by assumption, RHS must be in $\mathcal{F}$ since $\mathcal{F}$ is an ultrafilter. Hence, we are done. □

In the setting of topological spaces, we want to find out if continuous functions respect filter convergence.

Proposition 2.5. *Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous map. Suppose $\mathcal{F}$ is a filter on X converging to a point x. Then $f_*(\mathcal{F}) \to f(x)$.*

Proof. Suppose $\mathcal{F} \to x$, and let U be any open subset of Y containing $f(x)$. We want to show that $U \in f_*(\mathcal{F})$, or in other words that $f^{-1}(U) \in \mathcal{F}$. But this is immediate, since $f^{-1}(U)$ is an open set containing x, and therefore is in $\mathcal{F}$ since $F \to x$.

In fact, this property characterizes continuity. That is, a function $f : X \to Y$ is continuous if and only if it respects the convergence of all filters. This will be an **Exercise** for you. □

All these basic facts combine to allow us to prove the following powerful result that we will use to prove Tychonoff's theorem. This result is the analogue, in "filter world", of the results that say a sequence converges in a product if and only if all of the component sequences converge, or that a function to a product is continuous if and only if all the component functions are continuous.

Lemma 2.6. *Let I be a nonempty indexing set, and let $\{X_\alpha : \alpha \in I\}$ be a collection of nonempty topological spaces indexed by I. Let $X = \prod_{\alpha \in I} X_\alpha$ be their product equipped with the product topology. Let $\mathcal{F}$ be a filter on X, and let $x \in X$. Then $\mathcal{F} \to x$ if and only if $(\pi_\alpha)_*(\mathcal{F}) \to \pi_\alpha(x)$ for all $\alpha \in I$.*

Proof. Suppose $\mathcal{F} \to x$ and fix an index α. Then $\pi_\alpha : X \to X_\alpha$ is continuous and so the result follows from the previous proposition.

Now suppose that $(\pi_\alpha)_*(\mathcal{F}) \to \pi_\alpha(x)$ for all $\alpha \in I$. We want to show that $\mathcal{F} \to x$, or in other words, that for every open set U containing x in the product topology on X, that $U \in \mathcal{F}$. Note that it suffices to prove this for basic open sets since $\mathcal{F}$ is closed upwards. We will show this now. (You may find some similarity with the proof we did that the product topology on space of real functions was the topology of pointwise convergence.)

So, let U be a basic open set containing x. By definition of the product topology, U is of the form

$$U = \pi_{\alpha_1}^{-1}(V_1) \cap \cdots \cap \pi_{\alpha_n}^{-1}(V_n)$$

where for each $k = 1, \dots, n, V_k$ is an open subset of X_{α_k}. Now for each $k = 1, \dots, n$, since $x \in U$, we also have $\pi_{\alpha_k}(x) \in V_k$, and since by assumption $(\pi_{\alpha_k})_*(\mathcal{F}) \to \pi_{\alpha_k}(x)$, this means $V_k \in (\pi_{\alpha_k})_*(\mathcal{F})$. By definition, this implies that $\pi_{\alpha_k}^{-1}(V_k) \in \mathcal{F}$. Since this is true for every $k = 1, \dots, n$, the finite intersection

$$U = \pi_{\alpha_1}^{-1}(V_1) \cap \cdots \cap \pi_{\alpha_n}^{-1}(V_n)$$

also lies in $\mathcal{F}$, which was to be shown. $\square$

We end the discussion of this beautiful topic by showing you the promised application, namely, Tychonoff's theorem.

Theorem 2.7. *Let I be a nonempty indexing set, and let $\{X_\alpha : \alpha \in I\}$ be a collection of nonempty compact topological spaces indexed by I. Let $X = \prod_{\alpha \in I} X_\alpha$ be their product equipped with the product topology. Then X is compact.*

Proof. We will show that every ultrafilter on X converges. So fix an ultrafilter $\mathcal{U}$ on X. Fix $\alpha \in I$. Now $(\pi_\alpha)_*(\mathcal{U})$ is an ultrafilter on X_α by proposition 2.4. Since X_α is compact, $(\pi_\alpha)_*(\mathcal{U}) \to x_\alpha$ for some $x_\alpha \in X_\alpha$ by theorem 2.3. Then simply take x to be the point in X such that $\pi_\alpha(x) = x_\alpha$ for all $\alpha \in I$. Then $(\pi_\alpha)_*(\mathcal{U}) \to \pi_\alpha(x)$ for all $\alpha \in I$, and hence, by Lemma 2.6, $\mathcal{U} \to x$. $\square$

Tychonoff's theorem is actually equivalent of the axiom of choice. We end by mentioning some striking applications of Tychonoff's theorem.

Example 2.8. Tychonoff's theorem is widely applicable to many areas of mathematics.

(1) **Theorem(de Bruijn-Erdös):** Let G be any graph (where V could be infinite of any cardinality), and $k \in \mathbb{N}$. If any finite subgraph of G is k-colourable, i.e. there exists a colouring of G by k colours such that the endpoints of any edge are not coloured by the same colour, then G is k-colourable.
(2) **Theorem(Van der Waerden):** For any partition $\mathbb{Z} = S_1 \cup \cdots \cup S_k$, there exists j such that S_j contains arbitrary long arithmetic progressions.