ONE-POINT COMPACTIFICATIONS

ADITYA KARNATAKI

1. Introduction

Recall the definition from the end of the last class :

Definition 1.1. Let X be a locally compact Hausdorff space that is not compact. Let $Y = X \cup \{\infty\}$ where ∞ is a formal symbol that is not in X. Give Y the topology $\mathscr{T}_\infty$ consisting of :

1. The open subsets $U \subseteq X$, and
2. The complements $Y \setminus C$ of compact subsets $C \subseteq X$.

We call Y the *one-point compactification of* X.

2. One-point compactifications

Theorem 2.1. *Let X be a locally compact Hausdorff space that is not compact. The one-point compactification $Y = X \cup \{\infty\}$ is a compact Hausdorff space, $X \subseteq Y$ is a subspace, and $Y \setminus X$ consists of a single point.*

Proof. Note that we have to show that $\mathscr{T}_\infty$ is a topology first. $\varnothing$ is a set of type (1) while Y is a set of type (2). So the first condition for a topology is satisfied.

To check that the intersections of two open sets is open, there are three cases :

$$U_1 \cap U_2 \subseteq X$$

is of type (1),

$$(Y \setminus C_1) \cap (Y \setminus C_2) = Y \setminus (C_1 \cup C_2)$$

is of type (2), and

$$U_1 \cap (Y \setminus C_1) = U_1 \cap (X \setminus C_2)$$

is of type (1), since X is Hausdorff (so that compact subsets are closed).

To check that arbitrary unions are open, again there are three cases :

$$\bigcup_\alpha U_\alpha =: U \subseteq X$$

is of type (1),

$$\bigcup_\alpha (Y \setminus C_\alpha) = Y \setminus \bigcap_\alpha C_\alpha =: Y \setminus C$$

is of type (2), and

$$U \cup (Y \setminus C) = Y \setminus (C \setminus U)$$

where U and C are as above, is of type (2), since C is compact and $C \setminus U$ is a closed subset of C, hence compact.

Next, we show that X is a subspace. The open sets in the subspace topology are of the form $X \cap V$ where V is open in Y. If $V = U \subseteq X$ is of type (1),then $X \cap V = U$ is open in X. If $V = Y \setminus C$ is of type (2), then $X \cap V = X \setminus C$ is open in X since $C \subseteq X$ is compact, hence closed, in the Hausdorff space X. Conversely, if $U \subseteq X$ is open, then U is open of type (1) in Y.

Next, we show that Y is compact. Let $\mathcal{C}$ be a cover of Y. Then some $U \in \mathcal{C}$ contains ∞, so U is of the form $X \setminus C$ for some C. Now, $\mathcal{C}$ covers C, so there is some finite subcover $\{U_1, U_2, \ldots, U_n\}$ that covers C.

Then $\{U, U_1, U_2, \ldots, U_n\}$ is a finite subcover of $\mathcal{C}$ that covers Y.

Finally, we show that Y is Hausdorff. Let $x, y \in Y$. If $x, y \in X$, then there are open sets U and V of type (1), i.e. $U, V \subseteq X$ such that $x \in U$, $y \in V$, $U \cap V = \varnothing$. So we may assume $x \in X$ and $y = \infty$. Since X is locally compact at x there exists a compact $C \subseteq X$ containing a neighborhood U of x. Let $V = X \setminus C$. Then $\infty \in V$, $x \in U$, and $U \cap V = \varnothing$. So we are done.

We leave the claim that X is dense in Y as an exercise. Proving this would show that a one-point compactification as defined above is indeed a compactification(!) as we defined in the previous class. $\square$

Remark 2.2. Above proof suggests, in fact, that one can modify the definition of one-point compactification for any general non-compact topological space X. Namely :

Let X be a topological space. Let $Y = X \cup \{\infty\}$ where ∞ is a formal symbol that is not in X. Give Y the topology $\mathscr{T}_\infty$ consisting of :

(1) The open subsets $U \subseteq X$, and
(2) The complements $Y \setminus C$ of closed and compact subsets $C \subseteq X$.

Then the above proof goes through to show that $\mathscr{T}_\infty$ is a topology. (We need the extra *closed* hypothesis on C now since X may not be Hausdorff.) Then the above proof goes through verbatim to show that X is a subspace, and that Y is compact. It is also a similar argument to show that X is dense in Y. But for a general non-compact space X that is not locally compact, Y would not be Hausdorff.

Exercise: Show that the above construction carried for the space $X = \mathbb{Q}$ produces Y that is not Hausdorff. In the case when Y is compact and Hausdorff, there is a converse.

Theorem 2.3. *Let $X \subseteq Y$ be a subspace of a compact Hausdorff space, such that $Y \setminus X$ consists of a single point. Then X is locally compact and Hausdorff.*

Proof. As a subspace of a Hausdorff space, it is clear that X is Hausdorff. We prove that it is locally compact. Let $x \in X$ and let y be the single point of $Y \setminus X$. Since Y is Hausdorff, there are open sets $U, V \subseteq Y$ with $x \in U, y \in V, U \cap V = \varnothing$. Let $C = Y \setminus V$. It is a closed subset of a compact space, hence compact. Thus $x \in U \subseteq C \subseteq X$, as required for local compactness at x. $\square$

There is also the following uniqueness statement, which justifies why we can say "the one-point compactification", not just "a one-point compactification".

Proposition 2.4. *Let X be locally compact Hausdorff, such that X is not compact, with one-point compactification $Y = X \sqcup \{\infty\}$. Suppose that Y' is a compact Hausdorff space such that $X \subseteq Y'$ is a subspace and $Y' \setminus X$ is a single point. Then the unique bijection $Y' \to Y$ that is the identity on X is a homeomorphism.*

Proof. It suffices to prove that the bijection $f : Y' \to Y$ is continuous, since Y' is compact and Y is Hausdorff. An open subset of Y is of the form U or $Y \setminus C$, with $U \subseteq X$ open and $C \subseteq X$ compact. The preimage $f^{-1}(U) = U$ is then open in X, hence also in Y', since X must be open in the Hausdorff space Y' because its complement is a single point. The preimage $f^{-1}(Y \setminus C) = Y' \setminus C$ will also be open in Y', because C is compact and Y' is Hausdorff, so $C \subseteq Y'$ is closed. Hence, we are done. $\square$

Example 2.5. The one-point compactification of the open interval $(0, 1)$ is homeomorphic to the circle S^1. This follows from the uniqueness statement above, and the homeomorphism

$$f : (0, 1) \to S^1 - \{(1, 0)\}$$

given by $f(t) = (\cos 2\pi t, \sin 2\pi t)$. Note that the closed interval $[0, 1]$ is a different compactification of $(0, 1)$, with $[0, 1] \setminus (0, 1) = \{0, 1\}$ consisting of two points.

Since $(0, 1) \cong \mathbb{R}$, we have that one-point compactification of $\mathbb{R}$ is also homeomorphic to S^1.

Example 2.6. The one-point compactification of the open unit ball in n dimensions

$$B(0,1) := \{x \in \mathbb{R}^n : d(x,0) < 1\}$$

is homeomoprhic to the n-sphere S^n. Another compactification is the closed unit ball

$$D^n := \{x \in \mathbb{R}^n : d(x,0) \leq 1\}.$$

Since $B(0,1) \cong \mathbb{R}^n$, the one-point compactification of $\mathbb{R}^n$ is homeomorphic to S^n, too.

Example 2.7. if X is a disjoint union of n open intervals in $\mathbb{R}$, then its one-point compactification is homeomorphic to n circles in $\mathbb{R}^2$ that are disjoint except for a single common point (of tangency, say).

Finally, as we said in the previous class, this notion helps characterizing locally compact Hausdorff spaces.

Corollary 2.8. *A space X is homeomorphic to an open subspace of a compact Hausdorff space if and only if X is locally compact Hausdorff.*

Proof. **Exercise.** □