BASES FOR A TOPOLOGY

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1. Motivation

We have now seen some examples of topologies. We described each of them by explicitly specifying all of the open sets in each one. This is not be a feasible strategy for all —or even most— topologies we may wish to describe in the future. Even when we can explicitly specify all of the open sets in a topology, there is usually another way to describe the topology that will be easier to understand.

For instance, we have -

$$\mathscr{T}_{\mathbb{R}} = \{U \subseteq R : \forall x \in U, \exists \delta > 0 \text{ such that } (x-\delta, x+\delta) \subseteq U\}.$$

We know that the open sets in this topology are precisely the usual open intervals and their unions. So it seems like the entire collection of sets in $\mathscr{T}_{\mathbb{R}}$ can be specified by declaring *just* the usual open intervals to be open and allowing the condition of arbitrary unions to *generate* the rest of the open sets for us.

As an another instance, let X be a nonempty set and endow it with the discrete topology $\mathscr{T}_{\text{disc}}$. Then, since every set $U \in \mathcal{P}(X)$ is trivially a union of singleton sets - $U = \cup_{x \in U}\{x\}$ - we can just declare all of the singletons $\{x\}$ to be open for $x \in X$, and let the process of taking unions of these special open sets generate the rest of the open sets for us.

These special collections of sets are called *bases* of topologies.

2. Definition

Definition 2.1. Let X be a set. A collection of sets $\mathcal{B} \subseteq \mathcal{P}(X)$ is called a *basis* on X if the following two properties are satisfied -

(1) Covering property - For all $x \in X$, there is at least one $B \in \mathcal{B}$ such that $x \in B$. In more succinct terms, $X = \cup_{B \in \mathcal{B}} B$.
(2) Gluing property - For all $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

In words, the second property says: given a point x in the intersection of two elements of the basis $\mathcal{B}$, there is some element of the basis containing x and contained in this intersection.

If these two conditions are satisfied, we can use such a collection $\mathcal{B}$ to generate a topology by taking unions.

Definition 2.2. Let X be a set and $\mathcal{B}$ be a basis on X. Then we define

$$\mathscr{T}_{\mathcal{B}} := \left\{ \bigcup_{C \in \mathcal{C}} C : \mathcal{C} \subseteq \mathcal{B} \right\}.$$

This is called the topology generated by $\mathcal{B}$.

Of course, it is not clear immediately that this is a topology first of all! And we also have to justify the definite article 'the' for this topology. In fact, we take care of this first - since a topology must be closed under unions, every element of the collection $\mathscr{T}_{\mathcal{B}}$ we just described *must* be in any other topology that contains $\mathcal{B}$. So once we prove that $\mathscr{T}_{\mathcal{B}}$ is a topology, it is the 'smallest' one containing $\mathcal{B}$, which justifies the terminology.

Note that X is in $\mathscr{T}_{\mathcal{B}}$ by the covering property. Note also that $\varnothing$ is in $\mathscr{T}_{\mathcal{B}}$! Because $\varnothing \subset \mathcal{B}$ is an empty subcollection and the union as above over the empty subcollection will give you the empty set $\varnothing$ as an element of $\mathscr{T}_{\mathcal{B}}$.

Note also that this is not the definition in Munkres's book 'Topology'. It will be an exercise later to show that the two are in fact equivalent. For now, we show that $\mathscr{T}_{\mathcal{B}}$ is a topology.

Proof. We have already seen that $\varnothing$ and X are both in $\mathscr{T}_{\mathcal{B}}$, so that the condition (1) for it to be a topology is satisfied.

Now we check the condition on arbitrary unions. The approximate proof in words is simple, almost tautological - 'a union of union of elements of $\mathcal{B}$ is a union of elements of $\mathcal{B}$'. But we have to write it rigorously. To this end, let $\{V_\alpha\}_{\alpha\in I}$ be a collection of elements of $\mathscr{T}_{\mathcal{B}}$, where I is some index set. We want to show that

$$\bigcup_{\alpha\in I} V_\alpha \in \mathscr{T}_{\mathcal{B}}.$$

By definition of $\mathscr{T}_{\mathcal{B}}$, each element

$$V_\alpha = \bigcup_{\beta\in J} B_{\alpha\beta}$$

for some basis elements $B_{\alpha\beta}$ indexed by some index set J. So we have

$$\bigcup_{\alpha} V_\alpha = \bigcup_{\alpha\in I}\left(\bigcup_{\beta\in J} B_{\alpha\beta}\right) = \bigcup_{\alpha\in I}\bigcup_{\beta\in J} B_{\alpha\beta}$$

so that $U_{\alpha\in I}$ is expressed as a union of basis elements $B_{\alpha\beta}$ and we have checked condition (2).

Now we have to check the condition on finite intersections. In fact, we will show that for any two sets $V, W \in \mathscr{T}_{\mathcal{B}}$, $V \cap W \in \mathscr{T}_{\mathcal{B}}$. Then the general result follows from this by induction, since $V_1 \cap V_2 \cap \cdots \cap V_n = (V_1 \cap V_2 \cap \cdots \cap V_{n-1}) \cap V_n$. So we have to show that : if $V = \cup_{\alpha\in I} B_\alpha$ and $W = \cup_{\beta\in J} B_\beta$, then $V \cap W \in \mathscr{T}_{\mathcal{B}}$. Now,

$$V \cap W = \left(\bigcup_{\alpha\in I} B_\alpha\right) \cap \left(\bigcup_{\beta\in J} B_\beta\right) = \bigcup_{\alpha\in I,\beta\in J} (B_\alpha \cap B_\beta)$$

so by the previous condition on arbitrary unions, it suffices to check that $B_\alpha \cap B_\beta$ is in the topology $\mathscr{T}_{\mathcal{B}}$ for each pair (α, β) of indices. Thus, it suffices to prove that if B_1 and B_2 are two elements of the basis $\mathcal{B}$, $B_1 \cap B_2 \in \mathscr{T}_{\mathcal{B}}$. If $B_1 \cap B_2 = \varnothing$, we are done. Otherwise, Let $x \in B_1 \cap B_2$. Then by the second condition in the definition of a basis, there exists an element $B_x \in \mathscr{T}_{\mathcal{B}}$ such that

$$x \in B_x \subseteq B_1 \cap B_2.$$

Doing this for every element x and taking the union over all x, we get

$$B_1 \cap B_2 \subseteq \left(\bigcup_{x\in B_1\cap B_2} B_x\right) \subseteq B_1 \cap B_2.$$

Thus,

$$B_1 \cap B_2 = \left(\bigcup_{x\in B_1\cap B_2} B_x\right)$$

and we are done.

□

It is good to go through this proof carefully. It is not a very 'heavy machinery' proof, but it has some notations and layers worth getting to know - there were points, sets of points, collections of sets, and even collections of collections of sets!

Now we see some examples of bases on familiar spaces. You should check that these are indeed bases by checking the covering property and the gluing property.

Example 2.3.
(1) Let X be a nonempty set. Then the collection $\mathcal{B} = \{\{x\} : x \in X\}$ is a basis on X.
(2) Let X be a nonempty set and $\mathscr{T}_X$ a topology on X. Then $\mathscr{T}_X$ is a basis on X.
(3) The collection $\mathcal{B} = \{(a, b) \subseteq \mathbb{R} : a < b\}$ of open intervals on $\mathbb{R}$ is a basis on $\mathbb{R}$.
(4) The collection $\mathcal{B} = \{[a, b) \subseteq \mathbb{R} : a < b\}$ of 'half-open intervals' is a basis on $\mathbb{R}$. This is interesting!
(5) The collection $\mathcal{B} = \{B_\epsilon(x) : x \in \mathbb{R}^2, \epsilon > 0\}$ is a basis on $\mathbb{R}^2$.

The last example or even the second one reminds us of our motivating definition. Namely, we had

$$\mathscr{T}_{\mathbb{R}^n} := \{U \subseteq X : \forall x \in U, \exists \epsilon \text{ such that } B_\epsilon(x) \subseteq U\}.$$

This is *a priori* different from the definition of topology generated by the basis $\mathcal{B}$. But we would want these to match! Is something missing? First let us write down the general analogue of this definition.

Definition 2.4. Let X be a nonempty set and let $\mathcal{B}$ be a basis on X. Define

$$\mathscr{T}'_{\mathcal{B}} := \{U \subseteq X : \forall x \in U, \exists B \in \mathcal{B} \text{ such that } x \in B \subseteq U\}.$$

This is the definition found in the textbook 'Topology' by Munkres.

Exercise: Show that $\mathscr{T}_{\mathcal{B}}$ and $\mathscr{T}'_{\mathcal{B}}$ define the same topology.

Now we will look at the topologies generated by these bases.

3. Topologies

Example 3.1.
(1) The basis consisting of all the singletons in a nonempty set X generates the discrete topology on X.
(2) The basis $\mathscr{T}_X$ which is a topology on a nonempty set X generates $\mathscr{T}_X$, i.e. a topology generates itself as a basis.
(3) The basis consisting of open intervals of $\mathbb{R}$ generates the standard topology $\mathscr{T}_{\mathbb{R}}$ on $\mathbb{R}$.
(4) The basis consisting of half-open intervals of $\mathbb{R}$ generates a very interesting topology on $\mathbb{R}$. This is called the *lower limit topology on* $\mathbb{R}$. This space is denoted $\mathbb{R}_\ell$ in the textbook.
(5) The basis consisting of open balls $B_\epsilon(x) \subseteq \mathbb{R}^2$ generates the standard topology $\mathscr{T}^2_{\mathbb{R}}$ on $\mathbb{R}^2$.

Note the syntactic definition of a basis on X - the condition for $\mathcal{B}$, a collection of subsets of X, being a basis on X is independent of the topology that it generates. So now we want to ask - given a topology $\mathscr{T}_X$ and a collection $\mathcal{C}$ of subsets on X, how do we know if $\mathscr{T}_X$ is generated by $\mathcal{C}$ as a basis? A necessary condition is that all elements of $\mathcal{C}$ must be open in $\mathscr{T}_X$. What else do we need to add?

Lemma 3.2. *Let $(X, \mathscr{T}_X)$ be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets on X such that, for each open set U of X and each x in U, there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis on X that generates the topology $\mathscr{T}_X$.*

Proof. The proof is in two steps. First, show that $\mathcal{C}$ is a basis on X. Second, show that the topology $\mathscr{T}_{\mathcal{C}}$ generated by $\mathcal{C}$ as a basis is the same as the topology $\mathscr{T}_X$ on X.

Why is $\mathcal{C}$ a basis on X? Checking the covering property is easy - X is an open set, so for each $x \in X$, there exists $C \in \mathcal{C}$ by hypothesis such that $x \in C \subseteq X$. To check the gluing property, let $x \in C_1 \cap C_2$ for $C_1, C_2 \in \mathcal{C}$. Since C_1 and C_2 are open sets, so is $C_1 \cap C_2$. So, by hypothesis, there exists $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$. Thus $\mathcal{C}$ is a basis on X.

Exercise: Show that the topology generated by $\mathcal{C}$ is the same as $\mathscr{T}_X$. □

This also leads to another natural question - when can we say that two bases on X generate the same topology?

Lemma 3.3. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two bases on X. Then, $\mathscr{T}_{\mathcal{B}_1} \subseteq \mathscr{T}_{\mathcal{B}_2}$ if and only if for every $x \in X$ and for $x \in B_1 \in \mathcal{B}_1$, there exists $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1$.*

Proof. Assume that for every $x \in X$ and for $x \in B_1 \in \mathcal{B}_1$, there exists $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1$. Then given $U \in \mathscr{T}_{\mathcal{B}_1}$, we need to show that $U \in \mathscr{T}_{\mathcal{B}_2}$. Let $x \in U$. Since $\mathcal{B}_1$ generates $\mathscr{T}_{\mathcal{B}_1}$, there exists $B_1 \in \mathcal{B}_1$ such that $x \in B_1 \subseteq U$. Then by hypothesis, there exists $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1 \subset U$. So by definition, $U \in \mathscr{T}_{\mathcal{B}_2}$.

Now assume that $\mathscr{T}_{\mathcal{B}_1} \subseteq \mathscr{T}_{\mathcal{B}_2}$. Now we are given $x \in X$ and $B_1 \in \mathcal{B}_1$ with $x \in B_1$. Now, $B_1 \in \mathscr{T}_{\mathcal{B}_1} \subseteq T_{\mathcal{B}_2}$ by hypothesis. Since $T_{\mathcal{B}_2}$ is generated by $\mathcal{B}_2$, we get, by definition, $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq B_1$.

□

By now we have achieved a reasonable state of knowledge regarding bases and topologies generated by them. Can we generate a topology out of an arbitrary collection of subsets of X? Since the conditions for a topology involve being closed under the operations of finite intersections and arbitrary unions, you might expect that we might want to perform these operations on a given collection. This leads to the notion of a *subbasis* on X.

Definition 3.4. Let X be a nonempty set. A subbasis $\mathcal{S}$ on X is a collection of subsets of X whose union equals X. (In particular the collection cannot be empty.) The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection $\mathscr{T}_{\mathcal{S}}$ of all unions of finite intersections of elements of $\mathcal{S}$.

As before, we have to show that $\mathscr{T}_{\mathcal{S}}$ is actually a topology. This will follow from the lemma below and the definition of the topology generated by a basis.

Lemma 3.5. *The collection $\mathcal{B}$ consisting of all finite intersections*

$$B = S_1 \cap S_2 \cap \cdots \cap S_n$$

of elements $S_1, S_2, \ldots, S_n \in \mathcal{S}$ for $n \geq 1$ is a basis on X. (Note that in this way we are avoiding any ambiguity about the empty intersection, which might creep in your mind from the definition.)

Proof. Note first that by taking $n = 1$, we have $\mathcal{S} \subseteq \mathcal{B}$. Now we check the covering property. Let $x \in X$. Then since union of elements of $\mathcal{S}$ is X, x belongs to an element of $\mathcal{S}$ and hence to an element of $\mathcal{B}$, so that this property is satisfied.

Now we check the gluing property. Let

$$B_1 = S_1 \cap S_2 \cap \cdots \cap S_r \text{ and } B_2 = S_1' \cap S_2' \cap \cdots \cap S_t'$$

be two elements of $\mathcal{B}$. Then their intersection

$$B_1 \cap B_2 = (S_1 \cap S_2 \cap \cdots \cap S_r) \bigcap (S_1' \cap S_2' \cap \cdots \cap S_t')$$

is also a finite intersection of elements of $\mathcal{S}$, so it belongs to $\mathcal{B}$. Thus, this property is satisfied.

□