FUNDAMENTAL GROUP AND FUNDAMENTAL PROPERTIES

ADITYA KARNATAKI

1. Introduction

Recall that we defined the fundamental group $\pi_1(X, x_0)$ of a space X based at x_0 as-

Definition 1.1. The set of path-homotopy classes of loops based at x_0, with the concatenation operation $*$ is called the *fundamental group* of X for the base point x_0, denoted $\pi_1(X, x_0)$.

Today we will explore its properties and related concepts.

2. Fundamental properties of fundamental groups

First we need to figure out how much the group $\pi_1(X, x_0)$ depends on the basepoint x_0. To this end, let us observe that if x_0 and x_1 are in the same path component of X, by definition there exists a path α from x_0 to x_1. Then, given any loop f based at x_0, we can get a loop based at x_1, by taking $\bar{\alpha} * f * \alpha$, i.e. traversing from x_1 to x_0 first via the reverse path $\bar{\alpha}$, looping at x_0 via f, and then returning back to x_1 via α. Thus, we get a map

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$$

given by

$$[f] \to [\bar{\alpha} * f * \alpha] = [\alpha]^{-1} * [f] * [\alpha].$$

(Recall that $*$ is well-defined on homotopy classes.)

Proposition 2.1. *$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is an isomorphism of groups.*

Proof. This actually follows formally- in the fundamental groupoid we defined earleir, the objects x_0 and x_1 are isomorphic, with the isomorphism given by $[\alpha]$. A homework problem last week then would show us (if we have indeed solved it!) that $\mathrm{Aut}(x_0) \cong \mathrm{Aut}(x_1)$ as groups, and the statement follows. Let us work this out explicitly in this case.

First of all, we need to show that $\hat{\alpha}$ is a group homomorphism. This is a usual trick. If we have $a, b \in \pi_1(X, x_0)$, we have

$$\hat{\alpha}(a * b) = [\alpha]^{-1} * (a * b) * [\alpha] = [\alpha]^{-1} * a * [\alpha] * [\alpha]^{-1} * b * [\alpha]$$

and thus $\hat{\alpha}(a * b) = \hat{\alpha}(a) * \hat{\alpha}(b)$.

Let us denote $\beta = \bar{\alpha}$. Then we have $\hat{\beta} : \pi_1(X, x_1) \to \pi_1(X, x_0)$ which is also a group homomorphism. We claim that $\hat{\alpha}$ and $\hat{\beta}$ are inverses of each other. Indeed, for any $a \in \pi_1(X, x_0)$, we have

$$\hat{\beta}(\hat{\alpha}(a)) = \hat{\beta}([\alpha]^{-1} * a * [\alpha]) = [\beta]^{-1} * [\alpha]^{-1} * a * [\alpha] * \beta$$

and since $[\beta] = [\alpha]^{-1}$ in the groupoid, we get that $\hat{\beta}(\hat{\alpha}(a)) = a$. Similarly, we have $\hat{\alpha}(\hat{\beta}(b)) = b$ for any $b \in \pi_1(X, x_1)$, so we are done.

Corollary 2.2. *If X is path connected, then $\pi_1(X, x_0)$ is independent of the base point x_0 up to an isomorphism. Sometimes in such a situation, we will denote this object simply as $\pi_1(X)$.*

Corollary 2.3. *A loop f at x_0 induces an automorphism $\hat{f}$ of $\pi_1(X, x_0)$. This is simply of the form $a \to [f]^{-1} * a * [f]$, called an 'inner automorphism' ("conjugation by f").*

□

This illustrates how category language can be useful in simplifying proofs and ideas in our minds.

3. π_1 as a functor

Consider the category of *pointed topological spaces*: this has objects $\{(X, x_0)\}$ ordered pairs of topological spaces X and the choice of a base point $x_0 \in X$. The morphisms are given by continuous maps that take base points to base points- $f : (X, x_0) \to (Y, y_0)$ means that $f : X \to Y$ is continuous, and $f(x_0) = y_0$.

Definition 3.1. Any morphism of pointed topological spaces $h : (X, x_0) \to (Y, y_0)$ induces a group homomoprhism $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined by

$$h_*([f]) := [h \circ f].$$

Note that h_* is well-defined. If $f \simeq_p f'$ with homotopy $F(s,t)$, then $h \circ f \simeq_p h \circ f'$ with homotopy $h \circ F(s,t)$. It is also a group homomorphism because composition with h is compactible with concatenation of loops: $h \circ (f * g) = (h \circ f) * (h \circ g)$. Thus, $h_*([f] * [g]) = h_*([f]) * h_*([g])$.

Lemma 3.2. *π_1 is a functor from the category of pointed topological spaces to Groups.*

Proof. On objects, π_1 takes (X, x_0) to $\pi_1(X, x_0)$. On morphisms, we have $\pi_1(h) = h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$. We have to check the two required properties:

(i) **Composition:** Let $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ be two morphisms of pointed topological spaces. Then we have to show that $\pi_1(k \circ h) = \pi_1(k) \circ \pi_1(h)$, i.e. for $(k \circ h)_* = k_* \circ h_*$. For $[f] \in \pi_1(X, x_0)$, we have $(k \circ h)_*([f]) = [(k \circ h) \circ f] = [k \circ (h \circ f)] = k_*([h \circ f]) = k_*(h_*(f))$.

(ii) **Identity:** We have $(id_{x_0})_*([f]) = [id_{x_0} \circ f] = [f]$ so that the identity element in the category of pointed topological spaces induces the identity on the group $\pi_1(X, x_0)$.

$\square$

Since functors carry isomorphisms to isomorphisms (**exercise!**), we get the following corollary:

Corollary 3.3. *If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism, then h_* is an isomorphism of groups.*

In fact there are much more general statements of this kind, once we get to homotopy equivalence for topological spaces.