HOMEWORK 11

(1) Show that it need not be true that every compact subspace of any topological space is closed by finding a counterexample.

(2) Show that a space X is compact if and only if for each collection $\mathcal{E}$ of closed subsets of X having the finite intersection property, the intersection $\bigcap_{K\in\mathcal{E}} K$ is nonempty.

(3) Show that a first countable topological space is countably compact if and only if it is sequentially compact.

(4) Topology (Munkres), Chapter 3, Section 26, Exercise (2).

(5) Topology (Munkres), Chapter 3, Section 26, Exercise (5).

(6) Topology (Munkres), Chapter 3, Section 26, Exercise (7).

(7) Topology (Munkres), Chapter 3, Section 26, Exercise (8).

(8) Topology (Munkres), Chapter 3, Section 26, Exercise (10)(a).

(9) Let X be a topological space, and let $\mathcal{B}$ be a basis for the topology. Prove that X is compact if and only if every open cover of X consisting of basic open sets has a finite subcover. (This is in fact true if 'basic' is replaced by 'subbasic', but that is a more challenging question. Try it if you wish!)

(10) Show that every compact metrizable space is second countable.

(11) Define for $n \geq 1$ the group $\mathrm{O}(n)$ to be

$$\mathrm{O}(n) := \{A \in M_n(\mathbb{R}) : A.A^t = I_n\}$$

where ^{t} denotes the transpose and I_n is the identity $n \times n$ matrix. Define also the group $\mathrm{SO}(n) \subseteq O(n)$ as

$$\mathrm{SO}(n) := \{A \in O(n) : \det A = 1\}.$$

- Show that $\mathrm{O}(n)$ is not connected. (The topology for both $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ is the subspace topology gotten from $M_n(\mathbb{R})$.)
- Show that $\mathrm{O}(n)$ is closed in $M_n(\mathbb{R})$. Hence show that $\mathrm{SO}(n)$ is closed in $M_n(\mathbb{R})$.
- Show that
$$\mathrm{O}(n) \subseteq S^{n-1} \times \cdots \times S^{n-1}$$
where the product is taken of n copies of S^{n-1}, where $S^{n-1} \subset \mathbb{R}^n$ is the subspace of unit vectors. Conclude that both $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are compact.
- Recall the Gram-Schmidt orthogonalization process which sends a matrix $A = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathrm{GL}_n(\mathbb{R})$ to a matrix $B = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathrm{O}(n)$, where α_i and β_j are column vectors of dimension n, and $\beta_1 = \alpha_1/\|\alpha_1\|$, β_2 is the normalization of $\alpha_2 - (\alpha_2.\beta_1)\beta_1$, etc. Show that this defines a continuous surjective map $\mathrm{GS} : \mathrm{SL}_n(\mathbb{R}) \to \mathrm{SO}(n)$. Conclude that $\mathrm{SO}(n)$ is connected.

The groups $\mathrm{SO}(n)$ are fundamental examples of connected compact Lie groups. We won't study Lie groups in this course, but it is nice to know that elementary properties of these are quite accessible to us.