HOMEWORK 18

Note: $*$ marked problems might be slightly more difficult or interesting than the unmarked ones.

(1) Let X and Y be topological spaces. Show that a function $f : X \to Y$ is continuous if and only if it respects the convergence of all filters on X.

(2) Let X be a set. Define a filter $\mathcal{F}$ on S to be *free* if the intersection

$$\bigcap_{F \in \mathcal{F}} F = \varnothing$$

. Otherwise call it *fixed* (by any element $x \in \cap_{F \in \mathcal{F}}$).

Show that for an ultrafilter $\mathcal{U}$ on S, the following are equivalent.

(i) $\mathcal{U}$ is fixed.
(ii) $\mathcal{U} = \mathcal{U}_x$ for some element $x \in X$, where $\mathcal{U}_x$ denotes the principal ultrafilter.
(iii) There is some $x \in X$ such that $\{x\} \in \mathcal{U}$.
(iv) $\mathcal{U}$ contains a finite subset of X.

As a corollary, prove that if X is an infinite set, then an ultrafilter $\mathcal{U}$ on X is free if and only if every set in $\mathcal{U}$ is infinite if and only if $\mathcal{U}$ contains the Fréchet filter on S.

(3) Show that a space X is Hausdorff if and only if every ultrafilter $\mathcal{U}$ on X converges to at most one point. Since we showed today in class that a space Y is compact if and only if every ultrafilter $\mathcal{U}$ on Y converges to at least one point, this is yet another way of explicitly seeing the delicate tension between compactness and Hausdorffness.

(4) Identify $\mathcal{P}(\mathbb{N})$ with the set $\{0,1\}^\omega$ naturally. The latter has the product topology. Consider a principal ultrafilter $\mathcal{U} \subseteq \mathcal{P}(\mathbb{N})$ as a subset in this topology.

(i) Is $\mathcal{U}$ open?
(ii) Is $\mathcal{U}$ closed?

(5) Topology (Munkres), Chapter 4, Section 37, Exercise (4).

(6)* Topology (Munkres), Chapter 4, Section 37, Exercise (5).