HOMEWORK 5

(1) Show that $\mathcal{B} := \{U_p\}_{p(x)}$ (defined in notes) is a basis on X.

(2) Is the Zariski topology Hausdorff?

(3) Prove or disprove : If X is a topological space such that every sequence of points x_n converges to at most one point, then X is Hausdorff.

(4) Check that the projection functions π_1 and π_2 from $\mathbb{R}^2$ to $\mathbb{R}$ are continuous.

(5) Let X be a topological space with two topologies $\mathscr{T}$ and $\mathscr{T}'$ on it. Then show that the identity function $(X, \mathscr{T}) \to (X, \mathscr{T}')$ is continuous if and only if $\mathscr{T}' \subseteq \mathscr{T}$, i.e. $\mathscr{T}$ refines $\mathscr{T}'$.

(6) Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be two topological spaces and $f : X \to Y$ be a function between them. Let $\mathcal{B}$ and $\mathcal{S}$ be a basis and subbasis of $\mathscr{T}_Y$ respectively. Note that we are not assuming that $\mathcal{S}$ generates $\mathcal{B}$. Then show that the following are equivalent :

- (1) Preimages of open sets are open, i.e. $f^{-1}(V) \in \mathscr{T}_X$ for all $V \in \mathscr{T}_Y$.
- (2) Preimages of basic open sets are open, i.e. $f^{-1}(V) \in \mathscr{T}_X$ for all $V \in \mathcal{B}$.
- (3) Preimages of subbasic open sets are open, i.e. $f^{-1}(V) \in \mathscr{T}_X$ for all $V \in \mathcal{S}$.

(7) Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be two topological spaces and $f : X \to Y$ be a function between them. Then show that the following are equivalent (part of this is already done in class):

- (1) f is continuous.
- (2) For every closed set $C \subseteq Y$, the preimage $f^{-1}(C)$ is closed in X.
- (3) For every $x \in X$, f is continuous at x.
- (4) For every subset $A \subseteq X$, $f(\mathrm{Cl}\ A) \subseteq \mathrm{Cl}\ f(A)$, i.e. image of closure is contained in closure of image.

(8) Check that addition and multiplication, thought of as functions $\mathbb{R}^2 \to \mathbb{R}$ with their usual topologies, are both continuous. This can be done from the original definition of continuity alone, but it is much easier with one of the four equivalent ones above.

(9) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous when the domain and codomain have the usual topology, but not when both of them have the ray+ topology.

(10) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous when the domain and codomain have the usual topology, but not when both of them have the lower limit topology.

(11) Show that the projection functions π_1 and π_2 from $\mathbb{R}^2 \to \mathbb{R}$ are continuous and open, but not closed.

(12) Find an example of a function $f : \mathbb{R} \to \mathbb{R}$, if it exists, such that :

- (1) f is continuous precisely at n points $a_1, a_2, \dots, a_n$ and nowhere else.
- (2) f is continuous precisely at the set of integers and nowhere else.
- (3) f is continuous precisely at the set of rational numbers and nowhere else.
- (4) f is continuous precisely at the set of irrational numbers and nowhere else.

(13) Show that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic with their usual topologies. This is a bit challenging at this point of the course, knowing what you know. Later we will see easier ways of proving this, but at this point you will have to be creative.