# COMPACTNESS

ADITYA KARNATAKI

## 1. Introduction

Compactness is probably a topologist's most favourite topological property. Note that this might feel like the least intuitive so far - it's hard to visualize what it means for a space to be compact, but we will try to understand some important examples and properties of such spaces and understand what it means for a space to be compact through them. In particular, we know that 'boundedness' is not a topological invariant. On some occasions, compact spaces will be an appropriate generalization of this property as a topological invariant. Another important property is that although compact spaces can be huge, in a strong sense that can be made precise, every compact space acts like a finite space. This allows us to think of some novel techniques in the context of these spaces which are powerful and instructive.

## 2. Definitions and examples

Compactness is defined in terms of open covers.

**Definition 2.1.** Let $X$ be a topological space and let $\mathcal{U}$ be a collection of open subsets of $X$. We say that $\mathcal{U}$ is an *open cover* of $X$ if $\bigcup_{U\in\mathcal{U}} U = X$. If $\mathcal{U}$ is an open cover of $X$ and $\mathcal{V} \subseteq \mathcal{U}$ is a subcollection of open sets such that $\mathcal{V}$ is also an open cover of $X$, we say that $\mathcal{V}$ is an open *subcover* of $X$.

*Remark* 2.2. We will often drop the word 'open' from covers and subcovers, since those are the only types of covers that we will talk about.

**Definition 2.3.** A topological space $X$ is said to be *compact* if every open cover of $X$ has a finite subcover, i.e. if every open cover contains a finite subcollection that also covers $X$.

Note that in order to show that a space is compact, you have to check *every* open cover has a finite subcover. On the other hand, we need only to exhibit one open cover without a finite subcover to show that a space is not compact.

**Example 2.4.**
(1) $X = \mathbb{R}$ is not compact. $\bigcup_n (n, n+2)$ is an open cover of $\mathbb{R}$ but removing any of these sets will exclude the integer $n+1$ from the subcover.
(2) $X = (0, 1]$ is not compact, since $(0, 1] = \bigcup_n (1/n, 1]$ is an open cover that cannot have a finite subcover. Similar argument shows that $X = (0, 1)$ is not compact.
(3) $X = \{0\} \bigcup \{1/n : n \in \mathbb{N}\}$ is a compact set. This is because given any open cover of $X$ there must be some element in the open cover which contains $\{0\}$. Any such open set must contain all but finitely many of the points $\{1/n\}$. Then for each of the remaining finitely many points, one can choose one open set of the open cover that contains them. This finite subcollection of the given cover will cover $X$, and thus we have shown compactness.

The following is a related definition of a similar form. This comes much later in Munkres's book, but we see no reason to delay it as a definition.

**Definition 2.5.** A topological space $X$ is said to be *Lindelöf* if every open cover of $X$ has a countable subcover.

Obviously every compact space is Lindelöf, but the converse is not true.

## 3. Basic results

**Definition 3.1.** If $A$ is a subspace of $X$, a collection $\mathcal{C}$ of subsets of $X$ covers $A$ if $\bigcup_{U\in\mathcal{C}} U$ contains A.

**Proposition 3.2.** *Let $A$ be a subspace of $X$. Then $A$ is compact (Lindelöf) if and only if each cover of $A$ by open subsets of $X$ contains a finite (countable) subcollection that covers $A$.*

*Proof.* Suppose that $A$ is compact, and that $\mathcal{C}$ is a collection of open sets of $X$ that covers $X$. Then $\mathcal{D} := \{U \cap A : U \in \mathcal{C}\}$ is a collection of open subsets of $A$ that covers $A$. So there exists a finite subcollection $\{U_1 \cap A, U_2 \cap A, \cdots, U_n \cap A$ that covers $A$. Then $\{U_1, U_2, \ldots, U_n\}$ is a finite subcollection of $\mathcal{C}$ that covers $A$. Running this argument in reverse will give the other direction.

Lindelöf case is exactly similar. □

The following formalizes the notion that a 'closed' interval should be compact.

**Lemma 3.3.** *Every closed subspace of a compact (Lindelöf) space is compact (Lindelöf).*

*Proof.* Let $Y$ be a closed subspace of a compact space $X$. Given an open cover $\mathcal{C}$ of $Y$, we can extend it to an open cover $\mathcal{D}$ of $X$ by taking unions of members of $\mathcal{C}$ with the open set $X \setminus Y$. Then extract a finite subcollection $U_1, \ldots, U_n$ out of $\mathcal{D}$ that covers $X$. If any of the $U_i = X \setminus Y$, just forget that particular $U_i$, and the rest of the subcollection will give a cover of $Y$ by open sets in $X$, so we are done by the previous lemma. Lindelöf case is exactly similar. □

Unfortunately, it is not always true that the every compact subspace of any topological space is closed. **Exercise:** Find a counterexample that shows this.

**Proposition 3.4.** *Every compact subspace of a Hausdorff space is closed.*

*Proof.* We will show first the following lemma -

**Lemma 3.5.** *Let $Y$ be a compact subspace of a Hausdorff space $X$ and $x_0$ a point of $X$ not in $Y$. Then there exist disjoint open sets $U$ and $V$ of $X$ containing $x_0$ and $Y$ respectively.*

*Proof.* For each point $y \in Y$, let us choose disjoint neighbourhoods $U_y$ of $x_0$ and $V_y$ of $y$ respectively, using the Hausdorff hypothesis. Then, the collection $\{V_y\}$ is a collection of open sets in $X$ covering $Y$, so that we can extract a finite subcollection $V_{y_1}, \ldots, V_{y_n}$. Then the set $V = V_{y_1} \cup \cdots \cup V_{y_n}$ covers $Y$ and is disjoint from the open set $U = U_{y_1} \cap \cdots \cap U_{y_n}$. (Note the role of *finite intersections*!) □

From the lemma, it is clear that given a point $x_0$ of $X \setminus Y$, we have shown an open neighbourhood of $x_0$ that is completely contained in $X \setminus Y$, i.e. $X \setminus Y$ is open, and we are done. □

**Theorem 3.6.** *The image of a compact space under a continuous map is compact.*

*Proof.* Let $f : X \to Y$ be a continuous map and $X$ a compact space. For any cover $\mathcal{C}$ of $f(X)$ by open sets in $Y$, the collection $\{f^{-1}(C) : C \in \mathcal{C}\}$ is a cover of $X$, so that we can extract a finite subcollection $f^{-1}(C_1), \ldots, f^{-1}(C_n)$ out of it that covers $X$. Then $C_1, \ldots, C_n$ is a finite subcollection of $\mathcal{C}$ that covers $f(X)$. □

**Theorem 3.7.** *Let $f : X \to Y$ be a bijective continuous function. If $X$ is compact and $Y$ is Hausdorff, then $f$ is a homeomorphism.*

*Proof.* If $A$ is closed in $X$, then $A$ is compact. Then $f(A)$ is compact. Hence $f(A)$ is closed in $Y$. Hence $f$ is a homeomorphism. □

The idea is that Hausdorffness is about having "enough" open sets (as we have seen before) while compactness is about having "not too many" open sets (think about this!). This theorem, finely balanced between the tense thread that connects these notions, says that this is the limit that you can hope for : if we try to add more open sets, we lose compactness, if we try to remove open sets, we lose Hausdorffness. In a formal sense, of course. Such a statement should be appreciated for its internal beauty apart from its myriad uses!

In fact, the previous statement actually can be put in a more useful form.

**Proposition 3.8.** *Let $f : X \to Y$ be a map from a compact space $X$ to a Hausdorff space $Y$.*

*(1) $f$ is a closed map.*
*(2) If $f$ is surjective, then $f$ is a quotient map.*
*(3) If $f$ is bijective, then $f$ is a homeomorphism.*
*(4) If $f$ is injective, then $f$ is an embedding.*

**Example 3.9.** The constant map shows that the continuous preimage of a compact space need not be compact.

**Definition 3.10.** A map $f : X \to Y$ is said to be *proper* if for every compact subspace $L$ of $Y$, the preimage $f^{-1}(L)$ is compact in $X$.

We will come back to proper morphisms later. For now we note an explicit easy consequence of the discussion above and the definition.

**Corollary 3.11.** *If $X$ is compact and $Y$ is Hausdorff, then every continuous map $f : X \to Y$ is proper (and closed).*

## 4. Finite products of compact spaces

**Theorem 4.1.** *If $X$ and $Y$ are compact spaces, the product $X \times Y$ is compact.*

*Proof.* We will first prove the lemma below.

**Lemma 4.2** (Tube Lemma)**.** *Consider the product $X \times Y$ , let $p \in X$, and assume that $Y$ is compact. If $N \subseteq X \times Y$ is open, with $\{p\} \times Y \subseteq N$, then there exists a neighborhood $U \subseteq X$ of $p$ with $U \times Y \subseteq N$.*

*Proof.* The idea is essentially what we saw earlier. For each $(p, q) \in X \times Y$, we have $(p, q) \in \{p\} \times Y \subseteq N$. Since $N$ is open, there is a basis element $U_q \times V_q \subseteq N$ for the product topology on $X \times Y$ , with $p \in U_q$ open in $X$ and $q \in V_q$ open in $Y$. The collection $\{V_q\}_q$ is an open cover of $Y$. By compactness of $Y$, we can extract a finite subcover $\{V_{q_1}, \dots, V_{q_n}\}$. Then take $U := U_{q_1} \cap \dots \cap U_{q_n}$ for the corresponding indices $q_1, \dots, q_n$. Then $p \in U$ and $U$ is open in $X$. For any $(x, y) \in U \times Y$, there is some index $q_i$ for which $y \in V_{q_i}$ and then $x \in U$ implies $x \in U_{q_i}$ so that $(x, y) \in U_{q_i} \times V_{q_i} \subseteq N$ and we are done. □

**Example 4.3.** The tube lemma fails if $Y$ is not compact. Consider the neighborhood

$$N = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |xy| \leq 1\}$$

of $\{0\} \times \mathbb{R}$.

**Proof of the theorem (continued):** Let $\mathcal{C} = \{W_\alpha\}_{\alpha \in J}$ be a cover of $X \times Y$. For each point $p \in X$, the subspace $\{p\} \times Y$ is compact, and is therefore covered by a finite subcollection $\mathcal{F}_p = \{W_{\alpha_1,p}, \dots, W_{\alpha_n,p}\}$ of $\mathcal{C}$. Let $N_p = W_{\alpha_1,p} \cup \ldots \cup W_{\alpha_n,p}$. Then $N_p$ is an open set in $X \times Y$ containing $\{p\} \times Y$. By the Tube lemma, there exists neighbourhood $U_p \subseteq X$ of $p$ such that $U_p \times Y \subseteq N_p$. Note that $U_p \times Y$ is thus covered by the finite subcollection $\mathcal{F}_p$ of $\mathcal{C}$.

Now let $p$ vary. The collection $\{U_p\}$ is a cover of $X$, so that we can extract a finite subcover $U_{p_1}, \dots, U_{p_m}$. Then for $1 \leq i \leq m$, the subspace $U_{p_i} \times Y$ is covered by $\mathcal{F}_{p_i}$. Hence,

$$X \times Y = \bigcup_i U_{p_i} \times Y$$

is covered by the finite subcollection $\mathcal{S} = \mathcal{F}_{p_1} \cup \dots \cup \mathcal{F}_{p_m}$. Since $\mathcal{C}$ was arbitrary, we are done. □

## 5. The finite intersection property

Let $\mathcal{C}$ be a collection of open subsets of a space $X$. Let $\mathcal{E} = \{X \setminus U : U \in \mathcal{C}\}$ be the collection of closed complements. To say that $\mathcal{C}$ is a cover of $X$ is equivalent to saying that $\mathcal{E}$ has empty intersection:

$$\bigcap_{K \in \mathcal{E}} K = \bigcap_{U \in \mathcal{C}} (X \setminus U) = X \setminus \bigcup_{U \in \mathcal{C}} U$$

is empty if and only if $\bigcup_{U \in \mathcal{C}} U = X$.

**Definition 5.1.** A collection $\mathcal{E}$ of subsets of $X$ has the *finite intersection property* if for each finite subcollection $\{C_1, ..., C_n\} \subseteq \mathcal{E}$ the intersection

$$C_1 \cap \cdots \cap C_n$$

is nonempty.

**Proposition 5.2.** *A space $X$ is compact if and only if for each collection $\mathcal{E}$ of closed subsets of $X$ having the finite intersection property, the intersection $\bigcap_{K \in \mathcal{E}} K$ is nonempty.*

*Proof.* This is actually an **exercise** in logic. □

## 6. Some other variations on compactness

Just for some extra flavour, we are going to spend a bit of time talking about some topological properties that are very similar to compactness. We have seen in analysis one such property.

**Definition 6.1.** A topological space $X$ is called *sequentially compact* if every sequence in $X$ has a convergent subsequence.

As the name implies, this is an attempt at characterizating of compactness using only sequences. The nature of such a definition and the fact that it comes from analysis points to the fact that this definition may have something to do with first countability. We won't investigate this connection too much, but this leads to another definition.

**Definition 6.2.** A topological space $X$ is called *countably compact* if every countable open cover of X has a finite subcover.

The connection is underlined, for instance, by the following proposition.

**Proposition 6.3.** *A first countable topological space is countably compact if and only if it is sequentially compact.*

*Proof.* **Exercise.** □

In general, if $X$ is sequentially compact, then it is countably compact.

Another approach to the first countability issue is to define something a little more general than sequential compactness.

**Definition 6.4.** A topological space $X$ is called *limit point compact* if every infinite subset of $X$ has a limit point. That is, for every infinite $S \subseteq X$, there is a point $x \in X$ such that for every open $U$ containing $x$, $(U \cap S) \setminus \{x\} \neq \varnothing$.

There are many more propositions involving these concepts that we can prove. However, we next examine these in the context of metric spaces which are well-behaved and familiar, so that we don't get lost in the murky marshlands of different flavours of compactness.