HOMEWORK 25

Note: $*$ marked problems might be slightly more difficult or interesting than the unmarked ones.

(1) Show that every homomorphism $\pi_1(S^1) \to \pi_1(S^1)$ can be realized as the induced homomorphism ϕ_* of a map $\phi : S^1 \to S^1$.
(2)* Let X be any topological space. Show that the cone $\tilde{C}X$ is locally (path-)connected if and only if X is.
(3) Show that $j_X : X \to \tilde{C}X$ is a closed embedding.

(4) Complete the proof that cone $\tilde{C}$ is a functor from $\underline{Top}$ to $\underline{Top}$.

(5)* Let X be the space $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \subseteq \mathbb{R}$. Form the cone $\tilde{C}X$. We know from class that this is contractible.

(i) Try to draw a picture of $\tilde{C}X$ and get a feel for it.
(ii) Prove or disprove: $\tilde{C}X$ cannot be embedded in any $\mathbb{R}^n$.
(iii) Consider $0 \in X \subseteq \tilde{C}X$. Show that $\{0\} \subseteq \tilde{C}X$ is a deformation retract in a weak sense as we have discussed in class, that is, there is a homotopy $H : \tilde{C}X \times I \to \{0\}$ such that $H(x,0) = x$ and $H(x,1) \in \{0\}$ for all $x \in X$, but that it is NOT a deformation retract as we have defined it.
(iv) Form the space
$$\tilde{C}^2X := \frac{\tilde{C}X \cup \tilde{C}X}{0 \sim 0}$$
by identifying the two respective $0 \in \tilde{C}X$. Show that even if each copy of $\tilde{C}X$ is contractible, $\tilde{C}^2X$ is not contractible. (You cannot simply contract one cone first and then the other!) (Note: This part might be more difficult than the others. But at least try to get a feel for what goes wrong.)

(6) Show that the inclusion of groups $O(n) \subseteq \mathrm{GL}_n(\mathbb{R})$ is a deformation retract for $n \geq 1$.