SEPARATION AXIOMS AND PROPERTIES

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1. Introduction

While studying sequence convergence, we isolated three properties of topological spaces that are called separation axioms or T-axioms. These were called T_0, T_1, and T_2 (or Hausdorff). All three of these properties are called "separation axioms" because they codify the extent to which a topology can distinguish between points in the underlying set, or in other words how well it can separate points with open sets.

Today we will define and study two stronger separation properties, and perhaps some variations on them. Since Hausdorff spaces can already separate points with open sets, both of our properties will involve separating more complicated things with open sets.

2. Regular Spaces

Definition 2.1. A topological space X is *regular* if

(1) the singleton set $\{x\}$ is closed in X for each $x \in X$, and
(2) for each point $x \in X$ and each closed subset $B \subseteq X$, with $x \notin B$, there exist disjoint open subsets $U, V \subseteq X$ with $x \in U$ and $B \subseteq V$.

We then say that U and V separate x and B.

Note that the space X with trivial topology (trivially!) satisfies property (2), so we should not really omit the first condition. That is, T_1 condition is necessary for this stronger separation property to make sense 'in the real world'.

Lemma 2.2. *Let X be a T_1 space. Then X is regular if and only if for each point $x \in X$ and neighborhood W of x there is a neighborhood U of x with $Cl\, U \subseteq W$.*

Proof. If X is regular and $x \in W \subseteq X$, consider $B = X \setminus W$. Let $x \in U$ and $B \subseteq V$ with U and V open and disjoint. Then $X \setminus V$ is closed and contains U , so $\mathrm{Cl}\, U \subseteq X \setminus V \subseteq X \setminus B = W$.

Conversely, if $x \in X$ and B closed in X are given, with $x \notin B$, consider $W = X \setminus B$. Then W is a neighborhood of x. If U is a neighborhood of x with $\mathrm{Cl}\, U \subseteq W$ then U and $V = X \setminus \mathrm{Cl}\, U$ separate x and B. □

Lemma 2.3. *Any subspace of a regular space X is regular. Any product of regular spaces X_α is regular.*

Proof. **Exercise.** (Use the above lemma.) (Note that you will have to use results from earlier, such as 'any product of closures of sets U_α is equal to the closure of the product of sets U_α', so that the complexity of these proofs is gradually increasing.) □

Lemma 2.4. *Let X be a topological space that admits a basis of clopen sets, i.e. a basis $\mathcal{B}$ whose members are closed as well as (by definition) open. Then X is regular.*

Proof. Suppose $\mathcal{B}$ is a basis for X consisting entirely of clopen sets. Fix a point $x \in X$ and a closed set $C \subseteq X$ not containing x. Then $x \in X \setminus C$ which is open, and therefore there is a basic clopen set $B \in \mathcal{B}$ such that $x \in B \subseteq X \setminus C$. But then $C \subseteq X \setminus B$, and therefore B and $X \setminus B$ are the desired open sets separating x and C. □

Corollary 2.5. *$\mathbb{R}_\ell$ is regular.*

Example 2.6. Recall the space $\mathbb{R}_K$ which denote the set of real numbers with topology generated by the basis of all open intervals (a,b) and all the sets of the form $(a,b) \setminus K$ where $K = \{1/n : n \in \mathbb{N}\}$. This space is Hausdorff, since any two distinct points in $\mathbb{R}$ have open intervals separating them.

This space is not regular, however. Note that the set K is closed in $\mathbb{R}_K$. Suppose there exist open sets U and V containing 0 and K respectively. Then there is a basis element B contained in U. This has to be of the form $(a,b) \setminus K$ for some (a,b), since every (x,y) containing 0 will intersect K. Now choose n large enough that $1/n \in (a,b)$. Then choose a basis element around $1/n$ that is entirely contained in V. This must be of the form (c,d) since it contains $1/n$. Finally, choose x such that $x > \max\{c, 1/(n+1)\}$ and $x < 1/n$. But then $x \in (c, 1/n) \subseteq (c,d)$ and $x \in (a,b) \setminus K$ since $1/n \in (a,b)$ and $1/(n+1) < x < 1/n$ by construction. So x is contained in both U and V, which means they can never be disjoint.

Note that this way of writing the proof avoids the use of an argument using contradiction. You should try to avoid arguments involving contradiction if you can help it.

3. Normal Spaces

At this point, this property should seem like the natural next step.

Definition 3.1. A topological space X is *normal* if

(1) the singleton set $\{x\}$ is closed in X for each $x \in X$, and
(2) for each pair of disjoint closed subsets $A, B \subseteq X$ there exist disjoint open subsets $U, V \subseteq X$ with $A \subseteq U$ and $B \subseteq V$.

We then say that U and V separate A and B.

Let us quickly note the following characterization of normal spaces in much the same vein as that of regular spaces.

Lemma 3.2. *Let X be a T_1 space. Then X is normal if and only if for each closed subset $A \subset X$ and neighborhood W of x there is a neighborhood U containing A such that $A \subseteq U \subseteq \mathit{Cl}\, U \subseteq W$.*

Proof. **Exercise.** □

Example 3.3. The space $\mathbb{R}_\ell$ is normal. It is immediate that it is T_1. For any sets A and B that are disjoint and closed in $\mathbb{R}_\ell$, choose a basis element $[a, x_a)$ around each point $a \in A$ that does not intersect B (this is possible, since otherwise a would be in $\mathrm{Cl}\ B = B$), and similarly choose $[b, x_b)$ around each point b in B that does not intersect A. Then $U := \cup_{a\in A}[a, x_a)$ and $V := \cup_{b\in B}[b, x_b)$ are sets that separate A and B.

Example 3.4. $\mathbb{R}_\ell^2$ is not normal. Let $A \subseteq \mathbb{R}_\ell^2$ be the antidiagonal of the plane. That is, the graph of the function $y = -x$, or in other words $A = \{(x, -x) : x \in R\}$. A is a closed, discrete subspace of $\mathbb{R}_\ell^2$. Now consider the two subsets of $A :\ A_1 := \{(x, -x) : x \in \mathbb{Q}\}$ and $A_2 := \{(x, -x) : x \notin \mathbb{Q}\}$. Then A_1 and A_2 are closed (and hence open) disjoint subspaces of $\mathbb{R}_\ell^2$, but they cannot be separated by open subsets of $\mathbb{R}_\ell^2$. Showing this is somewhat tricky, and you should take a look at this example (example 3 in section 31) in Munkres's textbook to understand the argument.

Note that this shows that even finite products of normal spaces need not be normal. It is also true that subspaces of normal spaces need not be normal. An example of this is a bit tricky to get, so we will come back to this issue a little later in the course. What we will see is another fun result, that every non-normal space can be embedded into a space that satisfies the second condition in the definition of a normal space (but sadly may not be T_1). The proof is quite illustrative.

Proposition 3.5. *Let Y be a topological space that is not normal. Then there exists a topological space X containing Y as a subspace such that X satisfies the second condition in the definition of a normal space.*

Proof. Let $(Y, \mathscr{T}_Y)$ be any non-normal topological space, and let ∞ be a symbol that is not an element of Y. The underlying set for the new space we are constructing is $X := Y \cup \{\infty\}$. We define a topology $\mathscr{T}_X$ on X by $\mathscr{T}_X = \mathscr{T}_Y \cup \{X\}$. Do note that this actually is a topology on X.

Now, note that the subspace topology on Y in this space just equals $\mathscr{T}_Y$, so our original non-normal space $(Y, \mathscr{T}_Y)$ is a subspace of $(X, \mathscr{T}_X)$. To see that $(X, \mathscr{T}_X)$ is normal, note that the only open set in this space that contains the new point ∞ is X itself. In other words, every non-empty closed set in this space contains ∞. This means that there are NO pairs of disjoint nonempty closed sets, and so the space is vacuously normal! $\square$

What is worth doing is figuring out why the "obvious proof" that a subspace of a normal space "should be" normal fails. So let X be normal and let $A \subseteq X$ be a subspace. Let $C_1, C_2 \subseteq A$ be disjoint, nonempty closed subsets in the subspace topology. Then there exist two sets D_1 and D_2 which are closed in X such that $C_i = A \cap D_i$ for $i = 1, 2$. We want to use the normality of X to separate D_1 and D_2, but here is where it breaks down. Can you see why?

Once you see it, the next should be an easy exercise, which is a salvage.

Proposition 3.6. *A closed subspace of a normal space is normal.*

Proof. **Exercise.** $\square$

Now we see a nice result involving normality. This sort of a statement on how some combination of topological properties implies some other topological property has been a major area of study in point-set topology. This result might seem somewhat odd, in the sense that it is not obvious why the properties should interact this way, but the proof is very constructive and easy to follow.

Theorem 3.7. *Every regular second countable topological space is normal.*

Proof. Suppose $(X, \mathscr{T}_X)$ is regular and second countable with $\mathcal{B}$ a countable basis for $\mathscr{T}$, and let C and D be disjoint, nonempty, closed subsets of X. We wish to separate these two closed sets with open sets.

For each point $x \in C$, by the characterization of regularity in lemma 2.2, we can find a basic open set $U_x \in \mathcal{B}$ that contains x and such that $U_x \cap D = \varnothing$. Similarly for each $y \in D$ find an open set $V_y \in \mathcal{B}$ that contains y and $V_y \cap C = \varnothing$. We thus have :

$$C \subseteq \bigcup_{x \in C} U_x, \; D \subseteq \bigcup_{y \in D} V_y.$$

Note that since $\mathcal{B}$ is countable by the second countability hypothesis, both of these unions are actually countable unions of sets of the form U_x and V_y respectively, even if $|C|$ and/or $|D|$ might be uncountable. Re-indexing these sets, we can write :

$$C \subseteq \bigcup_{n \in \mathbb{N}} U_n, \; D \subseteq \bigcup_{n \in \mathbb{N}} V_n.$$

These unions are open, but the obstacle in them being the open sets *that we want* is that they might intersect. But this is easily remedied. For each $n \in \mathbb{N}$, define :

$$U_n' := U_n \setminus \left(\bigcup_{k=1}^{n} \mathrm{Cl}\ V_k \right), \; V_n' := V_n \setminus \left(\bigcup_{k=1}^{n} \mathrm{Cl}\ U_k \right).$$

Each of these newly defined sets is open since we are taking an open set and removing a *closed* set. This is the point at which second countability works, since the fact that we could rewrite each of the unions written earlier as *countable* unions allows us to only have to take the union of *finitely many closed sets* at each inductive step.

By construction, none of the $\mathrm{Cl}\ V_k$ intersect C and none of the $\mathrm{Cl}\ U_k$ intersect D. Thus we still have

$$C \subseteq U := \bigcup_{n \in \mathbb{N}} U_n', \; D \subseteq V := \bigcup_{n \in \mathbb{N}} V_n'.$$

Finally, these two open sets U and V are disjoint. Indeed, let $x \in U$. Then $x \in U_k'$ for some $k \in \mathbb{N}$. By construction of the U_n''s, this means that $x \notin V_i'$ for all $i = 1, \ldots, k$. On the other hand, if $i > k$, $V_i' \cap \mathrm{Cl}\ U_k = \varnothing$, so $x \notin V_i'$ for $i > k$ either. Therefore $x \notin V$, and so $U \cap V = \varnothing$, as required. $\square$

Theorem 3.8. *Every metrizable space is normal.*

Proof. Let (X, d) be a metric space. A metric space is T_1 so we have to show that disjoint nonempty closed sets can be separated by open sets. Let $A, B \subseteq X$ be disjoint, closed subsets. Recall the definition

$$d(x, B) = \inf\{d(x, b) : b \in B\}$$

of the distance from x to B. It is continuous as a function of x, and $d(x, B) = 0$ if and only if $x \in B$, since B is closed. Let $U = \{x \in X : d(x, A) < d(x, B)\}$ and $V = \{x \in X : d(x, A) > d(x, B)\}$. Then U and V are disjoint open subsets of X, with $A \subseteq U$ and $B \subseteq V$. $\square$

Theorem 3.9. *Every compact Hausdorff space is normal.*

Proof. Let X be compact Hausdorff. It is clearly T_1. So we must show that disjoint nonempty closed subsets $A, B \subseteq X$ can be separated by disjoint open subsets.

Being closed subsets of a compact space, A and B are themselves compact. Recall that in proving that a closed subspace of a Hausdorff space is closed, we have already considered the case $A = \{x\}$. Recall that argument first : For each $y \in B$, we have open sets $x \in U_y$ and $y \in V_y$ with $U_y \cap V_y = \varnothing$ by Hausdorff hypothesis. The collection $\{V_y : y \in B\}$ covers B, which is compact, so there is a finite subcollection $\{V_{y_1}, \ldots, V_{y_n}\}$ that also covers B. Then $U = U_{y_1} \cap \cdots \cap U_{y_n}$ and $V = V_{y_1} \cup \cdots \cup V_{y_n}$ are disjoint open sets with $x \in U$ and $B \subseteq V$.

Now consider the case of a general compact A. For each $x \in A$, we can choose disjoint open sets $x \in U_x$ and $B \subseteq V_x$. The collection $\{U_x : x \in A\}$ covers A, which is compact, so there is a finite subcollection $\{U_{x_1}, \ldots, U_{x_m}\}$ that also covers A. Then $U = U_{x_1} \cap \cdots \cap U_{x_n}$ and $V = V_{x_1} \cup \cdots \cup V_{x_n}$ are disjoint open sets with $A \subseteq U$ and $B \subseteq V$. $\square$

Example 3.10. The space $\bar{S}_\Omega \times \bar{S}_\Omega$ is normal, because it is compact Hausdorff. It is true that its subspace $S_\Omega \times \bar{S}_\Omega$ is not normal. Proving this is quite tricky. You should take a look at Example 2, section 32 in Munkres's book. So this gives an example of a subspace of a normal space that is not normal itself. Later when we see Tychonoff's theorem, we will see more 'natural' examples of this.

In particular, it is in general true that :

Theorem 3.11. *Every well-ordered set is normal in the order topology.*

Proof. This is Theorem 32.4 in Munkres's book. It is proven using similar techniques as above, namely, take a basic open set, i.e. an interval, around each point in the disjoint closed sets $A, B \subseteq X$, and adjust lengths of the intervals to show that the unions of these can be made to be disjoint from each other. If A or B contains the smallest element of X (if it exists), one has to be slightly careful, but a small trick will give you the result there too. $\square$

These theorems give some sufficient conditions for spaces to be normal. As we have seen, metrizable implies first countable and also normal. A theme of pointset topology has been to give (at least partial) converses - these are known as 'metrization theorems'. We will focus on one that is particularly simple to state but still requires some clever ideas in order to prove. This is known as the Urysohn metrization theorem.

Theorem 3.12. *Every regular second countable space X is metrizable.*

Note that we have already shown today that such a space is normal. So Urysohn metrization theorem is a strengthening of this statement. Regularity of course is necessary, but we know that not all metric spaces are second countable, so the second condition is stronger than necessary. Nagata-Smirnov metrization theorem gives a necessary and sufficient condition for a space X to be metrizable. But we will not study it in this course. We will study Urysohn's theorem and the clever ideas required for it next time.