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DE-Solution (Final)

Q1 (a)

(i) $$\frac{dy}{dx} = x^2 + y \,, \quad y(1) = 3$$

Analytical solution

$$\frac{dy}{dx} - y = x^2 \quad (\text{Linear})$$

$$I.F = e^{-\int dx} = e^{-x}$$ integrating factor

$$e^{-x}\frac{dy}{dx} - e^{-x} y = x^2 e^{-x}$$

$$e^{-x} dy - e^{-x} y \, dx = x^2 e^{-x} dx$$

$$\int d(y e^{-x}) = \int x^2 e^{-x} dx$$

Tabular by Parts

x^2	e^{-x}
$2x$	$-e^{-x}$
2	e^{-x}
0	$-e^{-x}$

$$y e^{-x} = -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C$$

General solution

$$y = \cancel{e}(-x^2 - 2x - 2) + Ce^{x}$$

applying intial condition

$$3 = (-1 - 2 - 2) + Ce$$

$$3 = -5 + Ce$$

$$8 = Ce \Rightarrow C = 8e^{-1}$$

$$y = -(x^2 + 2x + 2) + 8e^{-1} e^{x}$$

$$\boxed{y = -x^2 - 2x - 2 + 8e^{x-1}}$$ (Exact solution)

ii) Euler's method:

first iteration

$$y_1 = y_0 + h\,f(x_0, y_0)$$
$$= 3 + 0.05\left[x_0^2 + y_0\right]$$
$$= 3 + 0.05\left[1 + 3\right]$$
$$= 3.2$$

iii)

n	x_n	y_n (App)	y_n (Exact)	AE (Error)
0	1	3	3	0
1	1.05	3.2	3.207669	0.00767
2	1.1	3.415125	3.431367	0.01624
3	1.15	3.646381	3.672174	0.02579
4	1.2	3.894825	3.931222	0.03640
5	1.25	4.161567	4.209703	0.04814

Q1(b) $\dfrac{dP}{dt} = P(10^{-1} - 10^{-7}P)$, $P(0) = 5000$ IVP

using Papulation model

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}$$

where $a = 10^{-1}$, $b = 10^{-7}$, $P_0 = 5000$

$$P(t) = \frac{10^{-1}(5000)}{0.0005 + 0.0995\,e^{-0.1t}}$$

$$\boxed{P(t) = \frac{500}{0.0005 + 0.0995 e^{-0.1t}}}$$

required modal of Papulation

ii) If $t \to \infty$.

$$P(t \to \infty) = \frac{500}{0.0005} = \boxed{1,000,000}$$

iii) $$P(t=15) = \frac{500}{0.0005 + 0.0995 e^{-1.5}}$$

$$\boxed{P(t=15) = 22025.0239}$$ =22025 (approx)

Q2(a) Varition of Parameter (Cauchy Euler)

$$x^2 y'' - 3xy' + 3y = 2x^4 e^x$$

Sol: Auxilary Eq.

$$x^2\left[m(m-1)x^{m-2}\right] - 3x\left[mx^{m-1}\right] + 3x^m = 0$$

$y = x^m$
$y' = mx^{m-1}$
$y'' = m(m-1)x^{m-2}$

$$x^m\left[m(m-1) - 3m + 3\right] = 0$$

$$m^2 - m - 3m + 3 = 0$$

Auxilary equation

$$m^2-4m+3=0$$
$$m^2-3m-m+3=0$$
$$m(m-3)-1(m-3)=0$$
$$(m-3)(m-1)=0$$
$$m=1,3$$

Distinicit Roots

$$\boxed{y_c = c_1 x + c_2 x^3} = c_1 y_1 + c_2 y_2$$

$$y_1 = x, \quad y_2 = x^3$$

Now

standard form

$$y'' + P(x) y' + Q(x) y = f(x)$$

Divide by x^2

$$y'' - \frac{3}{x} y' + \frac{3}{x^2} y = 2x^2 e^x$$

$$f(x) = 2x^2 e^x, \quad P(x) = -\frac{3}{x}$$

$$v_1 = \int \frac{w_1}{w} dx, \quad v_2 = \int \frac{w_2}{w} dx$$

$$W_1 = \begin{vmatrix} 0 & x^3 \\ 2x^2 e^x & 3x^2 \end{vmatrix} = -2x^5 e^x$$

wronskian

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & 2x^2 e^x \end{vmatrix} = 2x^3 e^x$$

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3$$

$$U_1 = -\int x^2 e^x dx$$

$$= -\left[x^2 e^x - 2x e^x + 2e^x\right]$$

Tabular by Parts

x^2	e^x
$2x$	e^x
2	e^x
0	e^x

$$U_2 = \int e^x dx = e^x$$

$$y_p = U_1 y_1 + U_2 y_2$$

$$= \left(-x^2 e^x + 2x e^x - 2e^x\right)x + e^x\left(x^3\right)$$

$$\boxed{= 2x^2 e^x - 2x e^x}$$

General sol:

$$y = y_c + y_p$$

$$\boxed{y = c_1 x + c_2 x^3 + 2x^2 e^x - 2x e^x}$$

Jamilusmani

Q2(b)(i) $y'' - 4y = x - 7e^{2x}$

$y(0) = 1, \ y'(0) = 3$

Sol:

$m^2 - 4 = 0$ Auxilary eqn

$m = \pm 2$ Real Roots

$$\boxed{y_c = c_1 e^{2x} + c_2 e^{-2x}}$$

$$y_p = Ax + B + Cxe^{2x}$$

$$y' = A + C\left[2xe^{2x} + e^{2x}\right]$$

$$y'' = C\left[4e^{2x}x + 2e^{2x} + 2e^{2x}\right]$$

$$= C\left[4xe^{2x} + 4e^{2x}\right]$$

Now

$$\cancel{4xCe^{2x}} + 4Ce^{2x} - 4Ax \mp 4B - \cancel{4Cxe^{2x}} = x - 7e^{2x}$$

Compare coefficient

$4C = -7 \Rightarrow C = -7/4$

$-4A = 1 \Rightarrow A = -1/4$

$-4B = 0 \Rightarrow B = 0$

$$\boxed{y_p = -\frac{1}{4}x - \frac{7}{4}xe^{2x}}$$

General Solution

2(b) ii)

$$y = y_c + y_p$$

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4}x - \frac{7}{4}x e^{2x}$$ eqn---1

$$y' = 2c_1 e^{2x} - 2c_2 e^{-2x} - \frac{1}{4} - \frac{7}{4}\left(e^{2x} + 2x e^{2x}\right)$$

eqn---2

Apply initial Condition

$$y(0) = 1, \quad y'(0) = 3$$

$$c_1 + c_2 = 1$$

$$c_1 - c_2 = 5/2$$

Use Calculator

$$c_1 = \frac{7}{4}, \quad c_2 = -3/4$$

Now particular Solution is

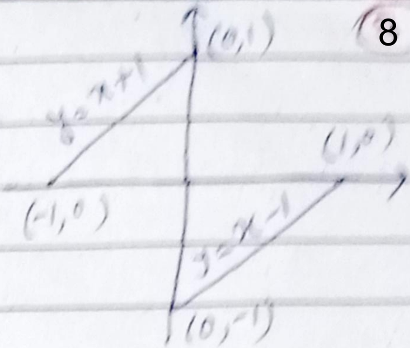
$$\boxed{y = \frac{7}{4}e^{2x} - \frac{3}{4}e^{-2x} - \frac{1}{4}x - \frac{7}{4}x e^{2x}}$$

required Particular solution

Q3 (a) (i) let $f(x) = x+1$

$$f(-x) = -x+1$$
$$= -(x-1)$$
$$f(-x) = -f(x)$$

$\boxed{f(x) \text{ is odd.}}$



ii) $a_0 = a_n = 0$ (Sine Series)

$$b_n = \frac{2}{1}\int_0^1 (x-1)\sin n\pi x\,dx \quad \text{(by parts)}$$

$$= 2\left[\int_0^1 x\sin n\pi x\,dx - \int_0^1 \sin n\pi x\,dx\right]$$

$$= 2\left[\frac{1}{n^2\pi^2}\sin n\pi x - \frac{x}{n\pi}\cos n\pi x + \frac{1}{n\pi}\cos n\pi x\right]_0^1$$

$$= 2\left[0 - \cancel{\frac{1}{n\pi}(-1)^n} + \cancel{\frac{1}{n\pi}(-1)^n} - 0 - 0 - \frac{1}{n\pi}\right]$$

$$\boxed{b_n = -2/n\pi}$$

Now Fourier Sine Series

$$f(x) = -\sum_{n=1}^{\infty} \frac{2}{n\pi}\sin n\pi x$$

sine series

$$= -\frac{1}{\pi}\left[2\sin \pi x + \frac{2}{2}\sin 2\pi x + \frac{2}{3}\sin 3\pi x + \frac{2}{4}\sin 4\pi x + \cdots\right]$$

iii)

let $x = 1/2$

$$f\left(\frac{1}{2}\right) = -\frac{1}{\pi}\left[2\sin\frac{\pi}{2} + \sin\pi + \frac{2}{3}\sin\frac{3\pi}{2} + \frac{1}{2}\sin 2\pi + \cdots\right]$$

$$\frac{1}{2} - 1 = \frac{-1}{\pi}\left[2 + 0 + \frac{2}{3}(-1) + 0 + \cdots\right]$$

$$\cancel{-}\frac{1}{2} = \cancel{-}\frac{1}{\pi}\left[2 - \frac{2}{3} + \cdots\right]$$

$$\frac{\pi}{2} = 2\left(1 - \frac{1}{3} + \frac{1}{5} \cdots\right)$$

$$\boxed{\pi/4 = 1 - \frac{1}{3} + \frac{1}{5} + \cdots}$$

Proved

Q3(b) let $f_1 = x$, $f_2 = \cos 2x$, $[-\pi/2, \pi/2]$ (1)

$$\langle f_1, f_2 \rangle = \int_{-\pi/2}^{\pi/2} x \cos 2x \, dx$$

inner product

$$= \frac{1}{2}\left(\frac{1}{2}\cos 2x + x\sin 2x\right)_{-\pi/2}^{\pi/2} \quad \text{(by parts)}$$

$$= \frac{1}{2}\left[\cancel{\frac{1}{2}\cos\pi} + \cancel{\frac{\pi}{2}\sin\pi} - \cancel{\frac{1}{2}\cos\pi} + \cancel{\frac{\pi}{2}\sin\pi}\right]$$

$$\boxed{\langle f_1, f_2 \rangle = 0}$$ $\therefore$ f_1, f_2 are orthogonal.

Norm: $\|f_2\| = \sqrt{\langle f_2, f_2 \rangle} = \sqrt{\int_a^b f_2^2(x)\, dx}$ — (1)

$$\int \cos^2 2x\, dx = \int\left(\frac{1+\cos 4x}{2}\right)dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos 4x\, dx$$

$$= \frac{1}{2}x + \frac{1}{8}\sin 4x$$

$$\int_{-\pi/2}^{\pi/2} \cos^2 2x\, dx = \left[\frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{8}\sin 4\left(\frac{\pi}{2}\right)\right] - \left[\frac{1}{2}\left(\frac{-\pi}{2}\right) + \frac{1}{8}\sin 2\pi\right]$$

$$= \left(\frac{\pi}{4} + 0\right) - \left(\frac{-\pi}{4} - 0\right) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

Eq (1) $\Rightarrow$ $\boxed{\|f_2\| = \sqrt{\frac{\pi}{2}}}$ required norm of f_2

Assistant prof: Jamil

Q4 (a) $Z = a e^{-b^2 t} \cos bx$

elimination of arb.const

$$\frac{\partial z}{\partial t} = -ab^2 e^{-b^2 t} \cos bx$$

$$\frac{\partial z}{\partial x} = -ab e^{-b^2 t} \sin bx$$

$$\frac{\partial^2 z}{\partial x^2} = -ab^2 e^{-b^2 t} \cos bx$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial t}}$$ is required PDEs.

let $A = 1,\ B = 0 = C$

$B^2 - 4AC = (0) - 4(1)(0) = 0$ $\boxed{\text{Parabolic PDEs}}$

4(b) $U = e^{-t} \cos(x/c)$

$$\frac{\partial U}{\partial t} = -e^{-t} \cos(x/c)$$

$$\frac{\partial U}{\partial x} = e^{-t}\left(\frac{-1}{c}\right)\sin(x/c)$$

$$\frac{\partial^2 U}{\partial x^2} = \frac{-1}{c^2} e^{-t} \cos(x/c)$$

$$= \frac{1}{c^2}\left[-e^{-t}\cos(x/c)\right] = \frac{1}{c^2}\left[\frac{\partial U}{\partial t}\right]$$

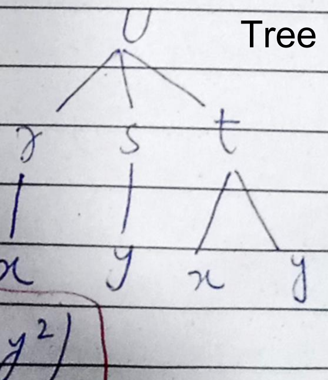
or $\boxed{\frac{\partial U}{\partial t} = c^2 \frac{\partial^2 U}{\partial x^2}}$ (Satisfied Heat Eqn)

4(c) Chain rule

$$\frac{\partial U}{\partial y} = \frac{\partial U}{\partial s} \cdot \frac{ds}{dy} + \frac{\partial U}{\partial t} \cdot \frac{\partial t}{\partial y}$$

$$= (2rs \ln t)(4) + \left(\frac{rs^2}{t}\right)(3xy^2)$$

Tree



$$\boxed{\frac{\partial U}{\partial y} = 8(x^2)(4y+1)\ln(xy^3) + \frac{x^2(4y+1)^2}{xy^3}(3xy^2)}$$

Q4(d) let $U = \tan^{-1}(y/x)$

$$U_x = \frac{1}{1+\left(\frac{y}{x}\right)^2} \frac{\partial}{\partial x}\left(y/x\right)$$

$$= \frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$U_{xx} = -y \frac{\partial}{\partial x}(x^2+y^2)^{-1}$$

$$= -y(-1)(x^2+y^2)^{-2}(2x)$$

$$U_{xx} = \frac{2xy}{(x^2+y^2)^2} \quad \text{--- } (1)$$

Similarly $U_y = x/(x^2+y^2)$

$$U_{yy} = x(-1)(x^2+y^2)^{-2}(2y)$$

$$= \frac{-2xy}{(x^2+y^2)^2} \quad \text{--- } (2)$$

Adding (1) & (2) $\boxed{U_{xx} + U_{yy} = 0}$ verified

A: Prof: Jamil Usman

Q5(a) Given $\frac{\partial^2 U}{\partial x^2} = 9 \frac{\partial U}{\partial t}$

let $U = XT$ Product form

$U_{xx} = X''T$

$U_t = XT'$

Now

$X''T = 9XT'$

$\frac{X''}{9X} = \frac{T'}{T} = -\lambda$ variable seprable

$X'' + 9\lambda X = 0, \quad T' + \lambda T = 0$

Consider all three Cases λ: zero, negative, Positive

that $\lambda = 0, \lambda = -\alpha^2 < 0, \lambda = \alpha^2 > 0$

Case-I if $\lambda = 0$

$X'' = 0, \quad T' = 0$

$\boxed{X = C_1 + C_2 x}, \quad \boxed{T = C_3}$

$U = XT = (C_1 + C_2 x)C_3 = \boxed{A_1 + B_1 x}$

Case-II if $\lambda = -\alpha^2$

$X'' - 9\alpha^2 X = 0, \quad T' - \alpha^2 T = 0$

Real root

$X = C_4 \cosh 3\alpha x + C_5 \sinh 3\alpha x, \quad T = C_6 e^{\alpha^2 t}$

$U = XT = (C_4 \cosh 3\alpha x + C_5 \sinh 3\alpha x) C_6 e^{\alpha^2 t}$

$= \boxed{A_2 e^{\alpha^2 t} \cosh 3\alpha x + B_2 e^{\alpha^2 t} \sinh 3\alpha x}$

Case-III if $\lambda = \alpha^2$

complex root

$X'' + 9\alpha^2 X = 0, \quad T' + \alpha^2 T = 0$

$X = C_7 \cos 3\alpha x + C_8 \sin 3\alpha x, \quad T = C_9 e^{-\alpha^2 t}$

$U = XT = (C_7 \cos 3\alpha x + C_8 \sin 3\alpha x) C_9 e^{-\alpha^2 t}$

$= \boxed{A_3 e^{-\alpha^2 t} \cos 3\alpha x + B_3 e^{-\alpha^2 t} \sin 3\alpha x}$

Q5(b) let $2U_{tt} = U_{xx}$

$$U_{tt} = \frac{1}{2} U_{xx} \quad \therefore a^2 = \frac{1}{2}$$

compare with standard form $L = \pi$

$$a = \pm \frac{1}{\sqrt{2}}$$

$$f(x) = \frac{10x}{\pi}, \quad g(x) = 0$$

$$B_n = 0$$

Now

using formula

$$A_n = \frac{2}{\pi} \int_0^{\pi} \frac{10x}{\pi} \sin\left(\frac{n\pi}{\pi}\right) x \, dx$$

$$= \frac{2 \times 10}{\pi^2} \int_0^{\pi} x \sin(nx) \, dx$$

$$= \frac{20}{\pi^2} \left[-\frac{x}{n} \cos nx + \frac{\sin nx}{n^2} \right]_0^{\pi}$$

$$= \frac{20}{\pi^2} \left[-\frac{\pi}{n} \cos n\pi + \frac{1}{n^2} \sin n\pi \right]$$

$$= \frac{20}{\pi^2} \left[\frac{-\pi}{n} (-1)^n + 0 \right]$$

$$= \frac{20}{n\pi} (-1)^{n+1}$$

Tabular by Parts

x	$\sin nx$
1 (+)	$-\frac{\cos nx}{n}$
0 (−)	$-\frac{\sin nx}{n^2}$

Now $$U(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi a}{L}\right) t \right] \sin\left(\frac{n\pi}{L}\right) x$$

$$= \sum_{n=1}^{\infty} \frac{20}{n\pi} (-1)^{n+1} \cos\left(\frac{n\pi}{\pi} \cdot \frac{1}{\sqrt{2}}\right) t \sin nx$$

$$\boxed{U(x,t) = \frac{20}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cos\left(\frac{n}{\sqrt{2}}\right) t \sin nx}$$

series solution

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