Linear Algebra- Final Solution FALL 2022

Question 1 [CLO-2] **Marks (2+1+2=05)**

Let $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

$R_2 \leftarrow R_2 - 2 \times R_1$ $\quad R_3 \leftarrow R_3 + R_1$ $\quad R_2 \leftarrow R_2 \div -7$

$$= \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ -1 & 3 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 7 & 7 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 7 & 7 & 4 \end{bmatrix}$$

$R_1 \leftarrow R_1 - 4 \times R_2$ $\quad R_3 \leftarrow R_3 - 7 \times R_2$

$$= \begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 7 & 7 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of a matrix is the number of non all-zeros rows

$\therefore Rank = 2$

Parametric solution

$$x_1 = -s + \tfrac{2}{7}t\,,\ x_2 = -s - \tfrac{4}{7}t\,,\ x_3 = s\,,\ x_4 = t$$

Rank(A)+ nullity(A) $= n$, **The Rank(A) is 2 and Nullity (A) is 2**

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{bmatrix}$$

basis for the null space of A. $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{bmatrix}$

A basis for the row space is

$\begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 1 & \frac{4}{7} \end{bmatrix}$.

Column Space :

The matrix has 2 pivots and Pivots are in the columns 1 and 2. $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$

Question 2 [CLO-3] **Marks (2+2+6=10)**

Consider $Q = 5x_1^2 + 2x_2^2 + 4x_3^2 + 4x_1x_2$

$$Q = \mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 5 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix};$$

the characteristic polynomial of the matrix A is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-5 & -2 & 0 \\ -2 & \lambda-2 & 0 \\ 0 & 0 & \lambda-4 \end{vmatrix} = (\lambda-1)(\lambda-4)(\lambda-6)$$

eigenvalues of A are 1, 4, and 6.

forms a basis for this eigenspace. at the eigenvalues of A are 1, 4, and 6.

$$\mathbf{p}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \qquad \mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{p}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Normalize the Eigen vectors

$$P = \begin{bmatrix} -\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix}$$

an orthogonal change of variables $\mathbf{x} = P\mathbf{y}$

that eliminates the cross product terms in Q is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

In terms of the new variables, we have

$$Q = \mathbf{x}^T A\mathbf{x} = \mathbf{y}^T (P^T AP)\mathbf{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1^2 + 4y_2^2 + 6y_3^2.$$

Question 3 [CLO-3] **Marks (5+3+2=10)**

Consider $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ and eigenvectors $u_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Gram-schmidt process

for this eigenspace: $\mathbf{v}_1 = \mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \mathbf{p}_2 - \frac{\langle \mathbf{p}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$, then

$\mathbf{p}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

normalize the two vectors to yield an orthonormal basis: $\mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$ and

$\mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$.

A matrix $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ orthogonally diagonalizes A

$$P^{-1}AP = P^TAP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

spectral decomposition of A.

$$\boxed{A = \lambda_1 \mathbf{u}_1\mathbf{u}_1^T + \lambda_2 \mathbf{u}_2\mathbf{u}_2^T + \cdots + \lambda_n \mathbf{u}_n\mathbf{u}_n^T}$$

Question 4 [CLO-3] Marks (3+2=05)

Q4(a)

If $\mathbf{u} = U$ and $\mathbf{v} = V$ then $\| U \| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = \sqrt{\operatorname{tr}\left(U^T U\right)}$

$$= \sqrt{\operatorname{tr}\left(\begin{bmatrix} 25 & 26 \\ 26 & 68 \end{bmatrix}\right)} = \sqrt{93} \text{ and}$$

$$d(U,V) = \| U - V \| = \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle^{1/2} = \sqrt{\operatorname{tr}\left((U-V)^T (U-V)\right)}$$

$$= \sqrt{\operatorname{tr}\left(\begin{bmatrix} 25 & 1 \\ 1 & 74 \end{bmatrix}\right)} = \sqrt{99} = 3\sqrt{11}.$$

Q4(b)

$$\| \mathbf{u} \| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = \left[2(-3)(-3) + 3(2)(2)\right]^{1/2} = \sqrt{30}$$

$$d(\mathbf{u}, \mathbf{v}) = \| \mathbf{u} - \mathbf{v} \| = \langle (-4,-5), (-4,-5) \rangle^{1/2}$$

$$= \left[2(-4)(-4) + 3(-5)(-5)\right]^{1/2} = \sqrt{107}$$

$$||v|| = < v, v >^{\frac{1}{2}} = (2(1)(1) + 3(7)(7))^{\frac{1}{2}} = \sqrt{149}$$

$$< u, v > = 2(-3)(1) + 3(2)(7) = 36$$

$$\boxed{cos\theta = \frac{<u,v>}{||u||\,||v||} = \frac{36}{\sqrt{30}\,\sqrt{149}}}$$

Question 5 [CLO-3] **Marks (3+7=10)**

$$Let \quad A=\begin{bmatrix}1&0&2\\0&1&1\\1&2&0\end{bmatrix}, Q=\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{3}}&\frac{1}{\sqrt{6}}\\0&\frac{1}{\sqrt{3}}&\frac{2}{\sqrt{6}}\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{3}}&\frac{-1}{\sqrt{6}}\end{bmatrix}$$

Let $\mathbf{u}_1=(1,0,1)$, $\mathbf{u}_2=(0,1,2)$, $\mathbf{u}_3=(2,1,0)$, $\mathbf{q}_1=\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)$, $\mathbf{q}_2=\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$, and

$\mathbf{q}_3=\left(\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)$.

. A QR-decomposition of the matrix A is formed by the given matrix Q

$$R=\begin{bmatrix}\langle\mathbf{u}_1,\mathbf{q}_1\rangle&\langle\mathbf{u}_2,\mathbf{q}_1\rangle&\langle\mathbf{u}_3,\mathbf{q}_1\rangle\\0&\langle\mathbf{u}_2,\mathbf{q}_2\rangle&\langle\mathbf{u}_3,\mathbf{q}_2\rangle\\0&0&\langle\mathbf{u}_3,\mathbf{q}_3\rangle\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{2}}+0+\frac{1}{\sqrt{2}}&0+0+\frac{2}{\sqrt{2}}&\frac{2}{\sqrt{2}}+0+0\\0&0+\frac{1}{\sqrt{3}}+\frac{2}{\sqrt{3}}&-\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}+0\\0&0&\frac{2}{\sqrt{6}}+\frac{2}{\sqrt{6}}+0\end{bmatrix}=\begin{bmatrix}\sqrt{2}&\sqrt{2}&\sqrt{2}\\0&\sqrt{3}&-\frac{1}{\sqrt{3}}\\0&0&\frac{4}{\sqrt{6}}\end{bmatrix}.$$

Q5(b)

$\det(\lambda I-A)=\begin{vmatrix}\lambda-3&i\\-i&\lambda-3\end{vmatrix}=(\lambda-2)(\lambda-4)$ thus A has eigenvalues $\lambda=2$ and $\lambda=4$.

The reduced row echelon form of $2I-A$ is $\begin{bmatrix}1&-i\\0&0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda=2$ consists of vectors $\begin{bmatrix}x\\y\end{bmatrix}$ where $x=(i)t$, $y=t$. A vector $\begin{bmatrix}i\\1\end{bmatrix}$ forms a basis for this eigenspace.

The reduced row echelon form of $4I-A$ is $\begin{bmatrix}1&i\\0&0\end{bmatrix}$ so that the eigenspace corresponding to $\lambda=4$ consists of vectors $\begin{bmatrix}x\\y\end{bmatrix}$ where $x=(-i)t$, $y=t$. A vector $\begin{bmatrix}-i\\1\end{bmatrix}$ forms a basis for this eigenspace.

Applying the Gram-Schmidt process to both bases amounts to simply normalizing the respective vectors.

Therefore A is unitarily diagonalized by $P=\begin{bmatrix}\frac{1}{\sqrt{2}}i&-\frac{1}{\sqrt{2}}i\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}$. Since P is unitary,

$P^{-1}=P^*=\begin{bmatrix}-\frac{1}{\sqrt{2}}i&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}i&\frac{1}{\sqrt{2}}\end{bmatrix}$. It follows that $P^{-1}AP=\begin{bmatrix}-\frac{1}{\sqrt{2}}i&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}i&\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}3&-i\\i&3\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}i&-\frac{1}{\sqrt{2}}i\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}2&0\\0&4\end{bmatrix}$.

Question 6 [CLO-1] Marks (2+2+6=10)

Q6(a)

$$u_1 = (2, -1, 3),\ \ u_2 = (4,1,2)\ \ and\ \ u_3 = (8, -1, 8)$$

The given vectors span R^3 if an arbitrary vector $\mathbf{b} = (b_1, b_2, b_3)$ can be expressed as a linear combination

$$(b_1, b_2, b_3) = k_1(2, -1, 3) + k_2(4, 1, 2) + k_3(8, -1, 8)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcl} 2k_1 & + & 4k_2 & + & 8k_3 & = & b_1 \\ -1k_1 & + & 1k_2 & - & 1k_3 & = & b_2 \\ 3k_1 & + & 2k_2 & + & 8k_3 & = & b_3 \end{array}$$

The determinant of the coefficient matrix of this system is $\begin{vmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{vmatrix} = 0$, therefore by

We conclude that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ do not span R^3.

Q6(b)

Let $A = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and Let $P = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$$D = PAP^{-1} = PAP^{\mathrm{T}}$$

$$A^{2301} = PD^{2301}P^{-1} = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} (-1)^{2301} & 0 & 0 \\ 0 & (-1)^{2301} & 0 \\ 0 & 0 & 1^{2301} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Question 6 [CLO-1] Marks (2+2+6=10)

Q6(c)

$$\begin{bmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 1 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} \leftarrow \text{multiplier} = -\frac{1}{3} \\ \\ \\ \end{matrix} \qquad \begin{bmatrix} 1 & -4 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{multiplier} = -1 \\ \leftarrow \text{multiplier} = 0 \end{matrix}$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{multiplier} = \frac{1}{2} \\ \\ \end{matrix} \qquad \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \text{multiplier} = -1 \end{matrix}$$

$$U = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \text{multiplier} = 1 \end{matrix} \qquad L = \begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Step 1. Rewrite the system as $\underbrace{\begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}}_{\mathbf{b}}$

Step 2. Define y_1, y_2, and y_3 by $\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}}$ and solve $\underbrace{\begin{bmatrix} -3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}}_{\mathbf{b}}$ by forward substitution to obtain $y_1 = 11$, $y_2 = -2$, $y_3 = 1$.

Step 3. Solve $\underbrace{\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 11 \\ -2 \\ 1 \end{bmatrix}}_{\mathbf{y}}$ by back substitution to find $x_1 = 1$, $x_2 = -2$, $x_3 = 1$.

Happy new year

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