Solution Mid1: Differential Equations MT1006

Q1(a)

$y = c_1e^{-3x} + c_2e^{2x} \quad \rightarrow \text{equation (1)}$

$y' = -3c_1e^{-3x} + 2c_2e^{2x} \quad \rightarrow \text{equation (2)}$

$y'' = 9c_1e^{-3x} + 4c_2e^{2x} \quad \rightarrow \text{equation (3)}$

3 × equation (1) + equation (2)

$3y + y' = 5c_2e^{2x} \quad \rightarrow \text{equation (4)}$

3 × equation (2) + equation (3)

$3y' + y'' = 10c_2e^{2x} \quad \rightarrow \text{equation (5)}$

2 × equation (4) - equation (5)

$2(3y + y') - (3y' + y'') = 0$

$6y + 2y' - 3y' - y'' = 0$

$6y - y' - y'' = 0$

Q1(b)

$$\frac{d}{dx}\left[e^x y\right] = 1$$

$$e^x y = x + c_1$$

$$y = xe^{-x} + c_1e^{-x}$$

Using $y(0) = 5$, we have $c_1 = 5$. Therefore $y = xe^{-x} + 5e^{-x}$. Then for $x \geq 1$,

$$\frac{d}{dx}\left[e^x y\right] = 0$$

$$e^x y = c_2$$

$$y = c_2e^{-x}$$

Requiring that $y(x)$ be continuous at $x = 1$ yields

$$c_2e^{-1} = e^{-1} + 5e^{-1}$$

$$c_2 = 6$$

Therefore

$$y(x) = \begin{cases} xe^{-x} + 5e^{-x}, & 0 \leq x < 1 \\ 6e^{-x}, & x \geq 1 \end{cases}$$

Q1(c)

Assume $R\,dq/dt + (1/C)q = E(t)$, $R = 200$, $C = 10^{-4}$, and $E(t) = 100$
$q = 1/100 + ce^{-50t}$. If $q(0) = 0$ then $c = -1/100$ and $i = \frac{1}{2}e^{-50t}$.

Q1(d)

I) Non Exact form

$(M_y - N_x)/N = 3$, so an integrating factor is $e^{\int 3\,dx} = e^{3x}$.

$M = (10 - 6y + e^{-3x})e^{3x} = 10e^{3x} - 6ye^{3x} + 1$ and $N = -2e^{3x}$,

so that $M_y = -6e^{3x} = N_x$.

From $f_x = 10e^{3x} - 6ye^{3x} + 1$ we obtain $f = \frac{10}{3}e^{3x} - 2ye^{3x} + x + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution of the differential equation is $\frac{10}{3}e^{3x} - 2ye^{3x} + x = c$.

II) Homogeneous form

Let $y = ux$

$$(x^3 - u^3x^3)\,dx + u^2x^3(u\,dx + x\,du) = 0$$

$$dx + u^2x\,du = 0$$

$$\frac{dx}{x} + u^2\,du = 0$$

$$\ln|x| + \frac{1}{3}u^3 = c$$

$$3x^3\ln|x| + y^3 = c_1x^3.$$

III) Bernoulli form

$$y' - \left(1 + \frac{1}{x}\right)y = y^2 \text{ and } w = y^{-1}$$

$$\frac{dw}{dx} + \left(1 + \frac{1}{x}\right)w = -1.$$

An integrating factor is xe^x

so that $xe^xw = -xe^x + e^x + c$

or $y^{-1} = -1 + \frac{1}{x} + \frac{c}{x}e^{-x}$.

Q2-MCQS

1(b) 2(c) 3(d) 4(c) 5(c) 6(b) 7(b) 8(b) 9(c) 10(d)