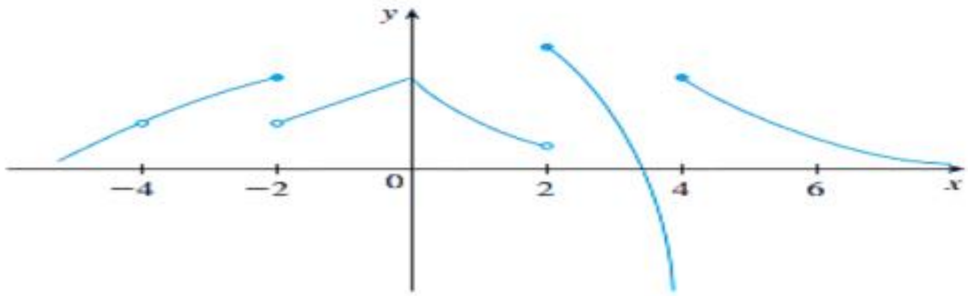
Question 01: **[CLO-03]** $[6 + 4 = 10]$

a) Let $f(x)$ be the function whose graph is shown below. State which condition of continuity does not hold at x = {-4, -2, 4} and also state type of discontinuity.



x =-4

f(-4) is not defined.

Hole discontinuity

x =-2

$$\lim_{x\to -2^-} f(x) \neq \lim_{x\to -2^+} f(x)$$

there is jump discontinuity

At x=4

$$\lim_{x\to 4^-} f(x) \neq \lim_{x\to 4^+} f(x)$$

infinite discontinuity

Q1(b)

$2x^3+x+7$ $\longleftrightarrow$ $m(x+1)+k$

$x=-1$

① $f(x) = 2x^3+x+7 \longrightarrow f(-1) = 2(-1)^3+(-1)+7$
$= -2-1+7 = 4$

② $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$

$\lim_{x\to -1^-} 2x^3+x+7 = \lim_{x\to -1^+} m(x+1)+k$

$+4 = m(-1+1)+k$

$\boxed{4 = k}$

$m(x+1)+k$ $\longleftrightarrow$ x^2+5

2

① $f(x) = m(x+1)+k$

② $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$

$\lim_{x\to 2^-} m(x+1)+k\} = \lim_{x\to 2^+} x^2+5$

$3m+4 = 9$

$\boxed{m = 5/3}$

Q.2a

$$\int_{-\infty}^{+\infty} \frac{e^{4x}dx}{1+e^{8x}}$$

$$\int_{-\infty}^{+\infty} \frac{e^{4x}dx}{1+e^{8x}} = \int_{-\infty}^{0} \frac{e^{4x}dx}{1+e^{8x}} + \int_{0}^{+\infty} \frac{e^{4x}dx}{1+e^{8x}}$$

$$\int_{-\infty}^{0} \frac{e^{4x}dx}{1+e^{8x}} = \left[\frac{1}{4}\tan^{-1}\left(e^{4x}\right)\right]_{-\infty}^{0} = \frac{1}{4}\left(\tan^{-1}\left(e^{0}\right) - \lim_{\ell \to -\infty} \tan^{-1}\left(e^{4\ell}\right)\right) =$$

$$\frac{1}{4}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{16}$$

$$\int_{0}^{+\infty} \frac{e^{4x}dx}{1+e^{8x}} = \left[\frac{1}{4}\tan^{-1}\left(e^{4x}\right)\right]_{0}^{+\infty} = \frac{1}{4}\left(\lim_{\ell \to +\infty} \tan^{-1}\left(e^{4\ell}\right) - \tan^{-1}\left(e^{0}\right)\right) =$$

$$\frac{1}{4}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{16}$$

$$\int_{-\infty}^{+\infty} \frac{e^{4x}dx}{1+e^{8x}} = \frac{\pi}{16} + \frac{\pi}{16} = \frac{\pi}{8}$$

b)

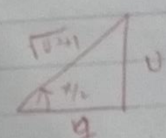
Q.2(b) $\int \frac{dt}{\sin t - \cos t}$

$\sin t = \frac{2u}{1+u^2}$

$\cos t = \frac{1-u^2}{1+u^2}$

$dt = \frac{2}{1+u^2}\,du$

$u = \tan t/2$



$$\int \frac{2/(1+u^2)\;du}{\left(\frac{2u}{1+u^2}\right)-\left(\frac{1-u^2}{1+u^2}\right)}$$

$$\int \frac{2\;du}{2u-1+u^2}$$

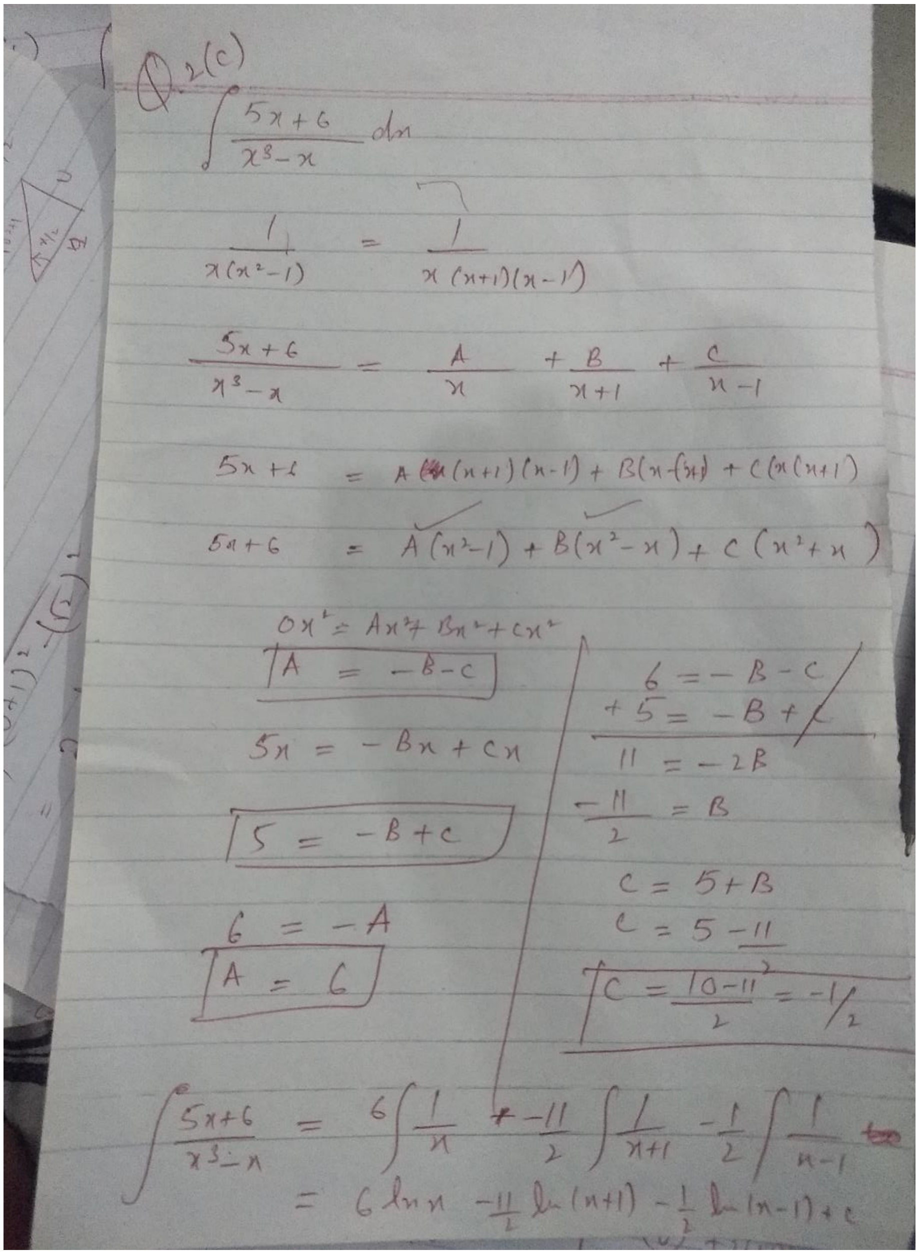
$$\int \frac{2}{(u+1)^2-2}$$

$$\int \frac{2}{(u+1)^2-(\sqrt{2})^2}$$

$$= \frac{2\ln(u-\sqrt{2}+1)-2\ln(x+\sqrt{2}+1)}{2^{3/2}}$$

$$= \frac{2\ln\left(\tan t/2-\sqrt{2}+1\right)-2\ln\left(\tan x/2+\sqrt{2}+1\right)}{2^{3/2}}$$

c).



Q.2(c)

$$\int \frac{5x+6}{x^3-x}\,dx$$

$$\frac{1}{x(x^2-1)} = \frac{1}{x(x+1)(x-1)}$$

$$\frac{5x+6}{x^3-x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

$$5x+6 = A(x+1)(x-1) + B(x(x-1)) + C(x(x+1))$$

$$5x+6 = A(x^2-1) + B(x^2-x) + C(x^2+x)$$

$$0x^2 = Ax^2 + Bx^2 + Cx^2$$

$$\boxed{A = -B-C}$$

$$5x = -Bx + Cx$$

$$\boxed{5 = -B+C}$$

$$6 = -A$$

$$\boxed{A = 6}$$

$$6 = -B - C$$
$$+5 = -B + C$$
$$11 = -2B$$
$$-\frac{11}{2} = B$$
$$C = 5 + B$$
$$C = 5 - \frac{11}{2}$$
$$\boxed{C = \frac{10-11}{2} = -\frac{1}{2}}$$

$$\int \frac{5x+6}{x^3-x} = 6\int \frac{1}{x} + -\frac{11}{2}\int \frac{1}{x+1} - \frac{1}{2}\int \frac{1}{x-1}$$
$$= 6\ln x - \frac{11}{2}\ln(x+1) - \frac{1}{2}\ln(x-1) + c$$

Q.3(a)

(a) $f'(z)$ if $f(z) = \sin(\cos(\tan z))$

$$\frac{d}{dz}\left(\sin\left(\cos\left(\tan(z)\right)\right)\right) = -\sec^2(z)\cos\left(\cos\left(\tan(z)\right)\right)\sin\left(\tan(z)\right)$$

(b)

$y \sin 2x = x \cos 2y, \quad (\pi/2, \pi/4)$

$$y \sin 2x = x \cos 2y \qquad (\pi/2, \pi/4)$$

$$y \cos 2x \cdot (2) + \frac{dy}{dx} \sin 2x = x(-\sin 2y)\cdot\left(2\frac{dy}{dx}\right) + \cos 2y$$

$$\frac{dy}{dx}\left[\sin 2x + 2x \sin 2y\right] = \cos 2y - 2y \cos 2x$$

$$\frac{dy}{dx}\left[\sin \cancel{2}\left(\frac{\pi}{\cancel{2}}\right) + \cancel{2}\left(\frac{\pi}{\cancel{2}}\right)\sin \cancel{2}\left(\frac{\pi}{4}\right)\right] = \cos \cancel{2}\left(\frac{\pi}{4}\right) - \cancel{2}\left(\frac{\pi}{4}\right)\cos \cancel{2}\left(\frac{\pi}{\cancel{2}}\right)$$

$$\frac{dy}{dx}\left[\cancel{\sin \pi}^{\,0} + \pi \sin \frac{\pi}{2}\right] = \cancel{\cos \frac{\pi}{2}}^{\,0} - \frac{\pi}{2}\cos \pi$$

$$\frac{dy}{dx}[\pi] = \frac{\pi}{2}$$

$$\boxed{\frac{dy}{dx} = 1/2}$$

(c) $f'(t)$ if $f(t) = \frac{1-te^t}{t+e^t}$

$$\frac{d}{dt}\left(\frac{1-te^t}{t+e^t}\right) = \frac{-e^t t^2 - e^{2t} - e^t - 1}{\left(t+e^t\right)^2}$$

Q.4(a)(I)

EXAMPLE 4 Find $\lim_{x \to 0} \frac{\tan x - x}{x^3}$. (See Exercise 44 in Section 2.2.)

SOLUTION Noting that both $\tan x - x \to 0$ and $x^3 \to 0$ as $x \to 0$, we use l'Hospital's Rule:

$$\lim_{x \to 0} \frac{\tan x - x}{x^3} = \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2}$$

Since the limit on the right side is still indeterminate of type $\frac{0}{0}$, we apply l'Hospital's Rule again:

$$\lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \to 0} \frac{2 \sec^2 x \tan x}{6x}$$

Because $\lim_{x \to 0} \sec^2 x = 1$, we simplify the calculation by writing

$$\lim_{x \to 0} \frac{2 \sec^2 x \tan x}{6x} = \frac{1}{3} \lim_{x \to 0} \sec^2 x \cdot \lim_{x \to 0} \frac{\tan x}{x} = \frac{1}{3} \lim_{x \to 0} \frac{\tan x}{x}$$

We can evaluate this last limit either by using l'Hospital's Rule a third time or by writing $\tan x$ as $(\sin x)/(\cos x)$ and making use of our knowledge of trigonometric limits. Putting together all the steps, we get

$$\lim_{x \to 0} \frac{\tan x - x}{x^3} = \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \to 0} \frac{2 \sec^2 x \tan x}{6x}$$

$$= \frac{1}{3} \lim_{x \to 0} \frac{\tan x}{x} = \frac{1}{3} \lim_{x \to 0} \frac{\sec^2 x}{1} = \frac{1}{3}$$

(II)

V EXAMPLE 6 Evaluate $\lim_{x \to 0^+} x \ln x$.

SOLUTION The given limit is indeterminate because, as $x \to 0^+$, the first factor (x) approaches 0 while the second factor ($\ln x$) approaches $-\infty$. Writing $x = 1/(1/x)$, we have $1/x \to \infty$ as $x \to 0^+$, so l'Hospital's Rule gives

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0$$

(b). The minute hand of a certain clock is 4 in long. Starting from the moment when the hand is pointing straight up, how fast is the area of the sector that is swept out by the hand increasing at any instant during the next revolution of the hand?

Let A be the area swept out, and θ the angle through which the minute hand has rotated. Find $\frac{dA}{dt}$ given that $\frac{d\theta}{dt} = \frac{\pi}{30}$ rad/min; $A = \frac{1}{2}r^2\theta = 8\theta$, so $\frac{dA}{dt} = 8\frac{d\theta}{dt} = \frac{4\pi}{15}$ in^2/min.

Q#5.

FIRST DERIVATIVE:

$$\frac{d}{dx}[f(x)] = f'(x) =$$

$$\frac{2\left(\frac{5}{2}-x\right)}{3\sqrt[3]{x}} - x^{\frac{2}{3}}$$

Simplify/rewrite:

$$-\frac{5x-5}{3\sqrt[3]{x}}$$

SECOND DERIVATIVE:

$$\frac{d^2}{dx^2}[f(x)] = f''(x) =$$

$$\frac{5x-5}{9x^{\frac{4}{3}}} - \frac{5}{3\sqrt[3]{x}}$$

Simplify/rewrite:

$$-\frac{10x+5}{9x^{\frac{4}{3}}}$$

s.p: x=1

C.p: x=0,1,

Inflection: x=-1/2

Increasing: $(0,1]$

Decreasing : $(-\infty, 0) \cup [1, \infty)$

Concave up: $(-\infty, -\frac{1}{2})$

Concave down: $(-\frac{1}{2}, 0) \cup (0, \infty)$

Max:x=1,y=1.5

Min:no relative minima

Q#6. a) ()

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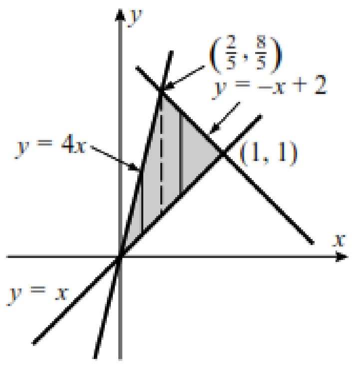
$$A = \int_0^{2/5} (4x - x)\,dx + \int_{2/5}^{1} (-x + 2 - x)\,dx = \int_0^{2/5} 3x\,dx + \int_{2/5}^{1} (2 - 2x)\,dx = 3/5.$$



(b)

6(b)

$y = x^2 + 1$, $y = -x + 3$

$-x + 3 = x^2 + 1$

$x^2 + x + 1 - 3 = 0$

$x^2 + x - 2 = 0$

$x = -2, 1$

$$V = \pi \int \left[f(x)\right]^2 - \left[g(x)\right]^2$$

$$= \pi \left[\int_{-2}^{1} (x^2+1)^2 - (-x+3)^2 \, dx \right]$$

-23.4

$$\boxed{V = +23.4\pi}$$

Q7 (a)

$$\lim_{n\to\infty} \left(\frac{\pi}{4}\right)^n$$

Let, $y = \lim_{n\to\infty} \left(\frac{\pi}{4}\right)^n$

$\ln y = \lim_{n\to\infty} \ln\left(\frac{\pi}{4}\right)^n$

$\ln y = \lim_{n\to\infty} n \ln\left(\frac{\pi}{4}\right)$

$= \lim_{n\to\infty} n\left[-0.241\right]$

Applying limit

$y = e^{-\infty}$

$y = \frac{1}{e^{\infty}}$

$y = \frac{1}{\infty} = 0$

Convergen

$\left\{\left(\frac{\pi}{4}\right)^n\right\}_{n=1}^{n=\infty}$

Converge

Dated:

7(b)

$$\lim_{n\to\infty} \sqrt{n^2+3n} - n \times \frac{\sqrt{n^2+3n}+n}{\sqrt{n^2+3n}+n}$$

$$\lim_{n\to\infty} \frac{\left(\sqrt{n^2+3n}\right)^2 - (n)^2}{\sqrt{n^2+3n}+n}$$

$$\lim_{n\to\infty} \frac{\cancel{n^2}+3n-\cancel{n^2}}{\sqrt{n^2+3n}+n}$$

$$\lim_{n\to\infty} \frac{+3\cancel{n}}{\cancel{n}\sqrt{1+\frac{3}{n}}+1}$$

Applying limit

$$\frac{3}{2}$$ Convergent

Q#8: a) limit comparision test.

Compare with the divergent series $\sum_{k=1}^{\infty}\frac{1}{k}$, $\rho = \lim_{k\to+\infty}\frac{k^2(k+3)}{(k+1)(k+2)(k+5)} = 1$, which is finite and positive, therefore the original series diverges.

Q.8(c)

Comparision test :-

$$\sum_{k=1}^{\infty} \frac{5}{4^k+1}$$

$$a_k = \frac{5}{4^k+1}$$

$$b_k = \frac{5}{4^k}$$

$$b_k = \frac{1}{4^k}$$

$$\sum \frac{1}{4^k} = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3}$$

Geometric Series

$|r| < 1$ Converge

$|r| \geq 1$ Diverge.

$r = \frac{1}{4} < 1$ Converge

b)

$a_k \leq b_k$

b_k is Converges than

a_k also Converges

$\sum_{k=1}^{\infty} \frac{5}{4^k+1}$ Converges.

c)

Ratio Test, $\rho = \lim_{k\to+\infty} \frac{k+5}{4(k+1)} = 1/4$, converges.

d)

Root Test, $\rho = \lim_{k\to+\infty} \left(\frac{k}{k+1}\right)^k = \lim_{k\to+\infty} \frac{1}{(1+1/k)^k} = 1/e$, converges.